3D Transformations

Lecture 5

3D Translation

- Point in 3D given by $(X_1\ Y_1\ Z_1)$
- Translated by $(dx\ dy\ dz)$

$X_2 = X_1 + dx$

$Y_2 = Y_1 + dy$

$Z_2 = Z_1 + dz$

Translation

- In matrix form

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix}$$

Inverse Transformation

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -dx \\ 0 & 1 & 0 & -dy \\ 0 & 0 & 1 & -dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}\,\mathbf{T}^{-1} = \mathbf{I}$$

Scaling

- Point in 3D given by $(X_1\ Y_1\ Z_1)$
- Scaled by $(Sx\ Sy\ Sz)$

$X_2 = X_1 * Sx$

$Y_2 = Y_1 * Sy$

$Z_2 = Z_1 * Sz$

Scaling

- In matrix form

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix}$$

Inverse Transformation

$$\mathbf{S} = \begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{Sx} & 0 & 0 & 0 \\ 0 & \frac{1}{Sy} & 0 & 0 \\ 0 & 0 & \frac{1}{Sz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S}\,\mathbf{S}^{-1} = \boldsymbol{I}$$

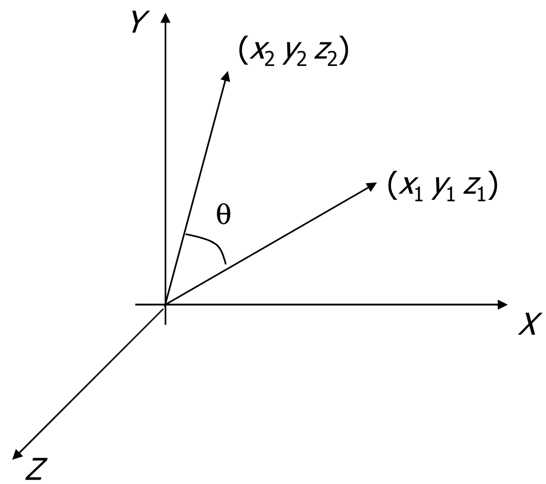
Shearing

- What will these do?

$$\begin{bmatrix} 1 & e & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & e & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & 0 & 0 \\ e & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Rotation

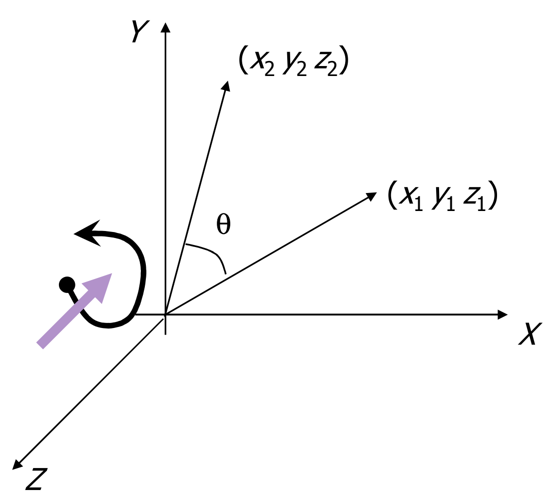
- Rotation about *Z*-axis
- Z-coordinate will not change
- Z' = Z
- If we ignore the Z-coordinate, it is 2-D transformation in XY plane



$$\begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

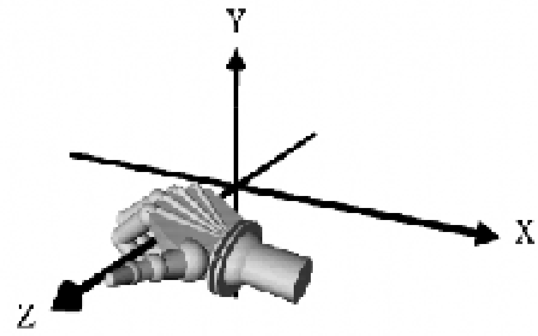
Rotation

- Positive rotation is counterclockwise when **looking down** the axis of rotation towards origin

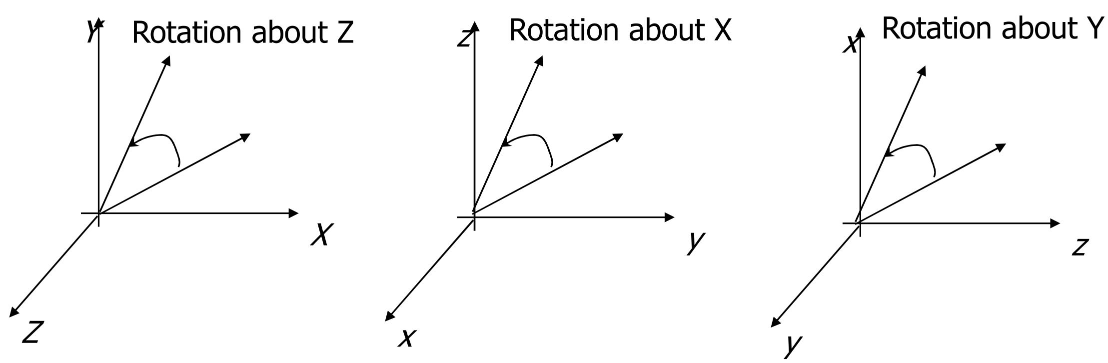


Rotation about Principal Axes

- Right hand rule
- XYZ is a right handed system if:
  - Place your right wrist at the origin, curl your finge from X to Y, thumb shoul point along +ve Z



Rotation about Principal Axes



$X' = X\cos\theta - Y\sin\theta$	$Y' = Y\cos\theta - Z\sin\theta$	$Z' = Z\cos\theta - X\sin\theta$
$Y' = X\sin\theta + Y\cos\theta$	$Z' = Y\sin\theta + Z\cos\theta$	$X' = Z\sin\theta + X\cos\theta$
$Z' = Z$	$X' = X$	$Y' = Y$

Rotation about Principal Axes

$$\begin{aligned} X' &= X\cos\theta - Y\sin\theta \\ Y' &= X\sin\theta + Y\cos\theta \\ Z' &= Z \end{aligned}$$

Rotation about Z

$$\begin{aligned} Y' &= Y\cos\theta - Z\sin\theta \\ Z' &= Y\sin\theta + Z\cos\theta \\ X' &= X \end{aligned}$$

Rotation about X

$$\begin{aligned} Z' &= Z\cos\theta - X\sin\theta \\ X' &= Z\sin\theta + X\cos\theta \\ Y' &= Y \end{aligned}$$

Rotation about Y

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation of Rotations

- Rotation around X by γ followed by rotation around Y by β followed by rotation around Z by α

$$R = R_\alpha^Z R_\beta^Y R_\gamma^X$$

$$R = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Small Angle Approximation

$$R = R_Z^{\alpha} R_Y^{\beta} R_X^{\gamma} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Small angle approximation $$R = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix}$$

Properties of Rotation Matrix

- Rotation Matrices are **orthonormal** and have a determinant of +1

  i.e. $R R^T = R^T R = I$
- The inverse of a rotation matrix is its transpose
- 3D rigid motion can be described by a rotation and translation

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_X \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Properties of Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- *R* has 9 unknowns, but orthonormality provides 6 constraints

$$\sum_{j=1}^{3} r_{ij} r_{kj} = \sum_{j=1}^{3} r_{ji} r_{jk} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

- Hence the number of degrees of freedom of a 3D rotation are 9 – 6 = 3

Properties of Rotation Matrices

- Any concatenation of rotation matrices also forms a rotation matrix i.e. the matrix remains orthonormal [Proof?]

Properties of Rotation Matrices

- A rotation matrix transforms its own rows onto the principal axes

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} = \ ?$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{31} \\ r_{32} \\ r_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

- Any complex rotation can be described by a single rotation around an axis **n** by an angle θ
- Therefore, any complex rotation can be described by $[n_1, n_2, n_3]^T$ and θ
- Still 3 degrees of freedom as **n** can be taken to be a unit vector without any loss of generality

$$\sqrt{n_1^2 + n_2^2 + n_3^2} = 1$$

Rotation about Arbitrary Axis

- To rotate about an axis **n** by an angle θ

1. Set up rotations such that **n** rotates onto one of the principal axis [How?]
2. Rotate about that axis by θ
3. Undo the transformations in step 1

Rotation about Arbitrary Axis

- Question: Given an arbitrary 3D rotation matrix, how can we find out the axis **n** and the angle ϑ that represents this rotation?

$$\begin{bmatrix} -0.8256 & 0.40388 & -0.39404 \\ -0.20084 & -0.86294 & -0.46367 \\ -0.5273 & -0.30367 & 0.79356 \end{bmatrix}$$

Given R on the left, how can we tell n and θ?

Eigenvectors and Values of a Rotation Matrix

- 3D rotation matrix has eigenvalues of 1, $\cos\vartheta + i\sin\vartheta$ and $\cos\vartheta - i\sin\vartheta$ [Proof?]
- The eigenvector associated with the real eigenvalue represents the axis of rotation [proof?]