$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix}$$

■

Haar Transform

The Haar transform is one (of several) wavelet transforms that can be calculated with a formula. The formula (actually, two formulas) is given by

$$c_{00} = \int_0^1 v(t)\,\varphi_{00}(t)\,dt \tag{1.13a}$$

$$d_{kj} = \int_0^1 v(t)\,\psi_{kj}(t)\,dt \tag{1.13b}$$

where the basis functions φ_{00} and ψ_{kj} are as shown in Fig. 1.16. The function $\varphi_{00}(t)$ is called the *scaling function*, and the $\psi_{kj}(t)$ are called *wavelets*.

This diagram shows the level 0, 1, and 2 Haar basis functions, but these can continue to higher and higher levels. Level 0 contains the functions $\varphi_{00}(t)$ (called the *scaling function*) and $\psi_{00}(t)$ (called the *mother wavelet*). Level 1 contains the two functions $\psi_{10}(t)$ and $\psi_{11}(t)$. Level 2 contains the four functions $\psi_{20}(t)$, $\psi_{21}(t)$, $\psi_{22}(t)$, and $\psi_{23}(t)$. Each higher level contains twice as many wavelets $\{\psi_{jk}\}$. Thus, level 3 contains eight wavelets, each just half as long as the level 2 wavelets and scaled appropriately to have unit energy. To calculate the Haar transform, expand the time function in terms of these basis functions. Equation 1.13 gives the formulas for calculating the transform.

Example 1.10. Calculate the Haar transform of the signal in Fig. 1.17.

Solution: Applying Eq. 1.13 to the function v(t) gives the following numbers.

$$c_{00} = \int_0^1 v(t)\,\varphi_{00}(t)\,dt = 0$$

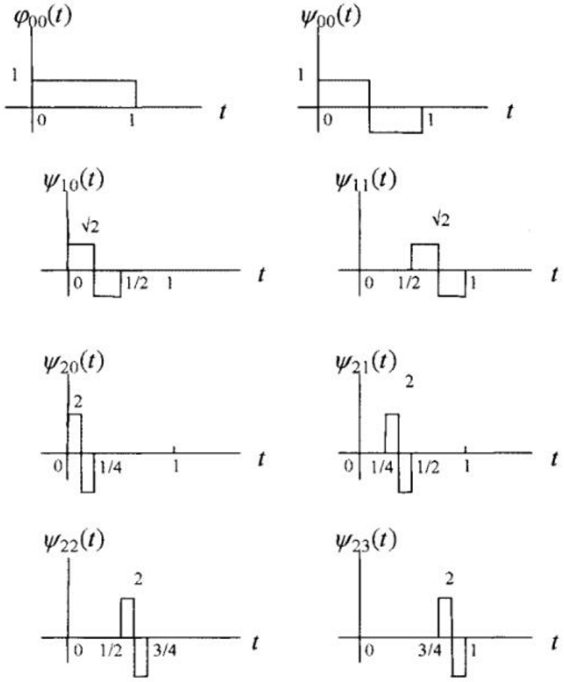

Figure 1.16. Haar basis functions to level 2.

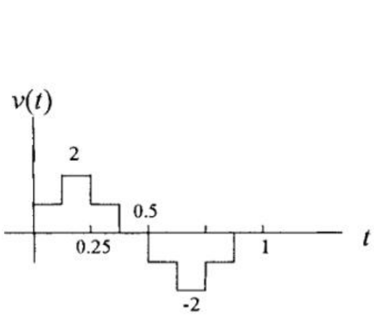

Figure 1.17. The function $v(t)$.

$$d_{00} = \int_0^1 v(t)\,\psi_{00}(t)\,dt = 1$$

$$d_{10} = \int_0^{0.5} v(t)\,\psi_{10}(t)\,dt = \frac{\sqrt{2}}{4}$$

$$d_{11} = \int_{0.5}^{1} v(t)\,\psi_{11}(t)\,dt = -\frac{\sqrt{2}}{4}$$

$$d_{20} = \int_0^{0.25} v(t)\,\psi_{20}(t)\,dt = -\frac{1}{4}$$

$$d_{21} = \int_{0.25}^{0.5} v(t)\,\psi_{21}(t)\,dt = \frac{1}{4}$$

$$d_{22} = \int_{0.5}^{0.75} v(t)\,\psi_{22}(t)\,dt = \frac{1}{4}$$

$$d_{23} = \int_{0.75}^{1} v(t)\,\psi_{23}(t)\,dt = -\frac{1}{4}$$

Thus, the first eight terms of the Haar transform of $v(t)$ consist of the eight values $\left\{0,\ 1,\ \frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4},\ -\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{4}\right\}$.

■

Equation 8.12 gives the equation for the inverse wavelet transform. When applied to the function in Fig. 1.17, it takes the form

$$v(t) = c_{00}\varphi_{00}(t) + d_{00}\psi_{00}(t) + d_{10}\psi_{10}(t) + \cdots + d_{23}\psi_{23}(t)$$

Example 1.11. Show that the inverse transform gives the function $v(t)$ in Fig. 1.17.

Solution: Multiplying the functions in Fig. 1.16 by the coefficients, $c_{00} = 0$, $d_{00} = 1$, and so on, results in the functions in Fig. 1.18. If these functions are summed, the result is $v(t)$ in Fig. 1.17. (Try it if you don't believe it.)

■

From these examples one can see many similarities between Fourier and wavelet transforms. Time functions are expanded in terms of basis functions in both transforms. These basis functions are exponentials for the Fourier transform, and they are square pulses for the Haar transform. The coefficients are calculated by an inner product operation [i.e., by integrating the product of $v(t)$ and basis function].

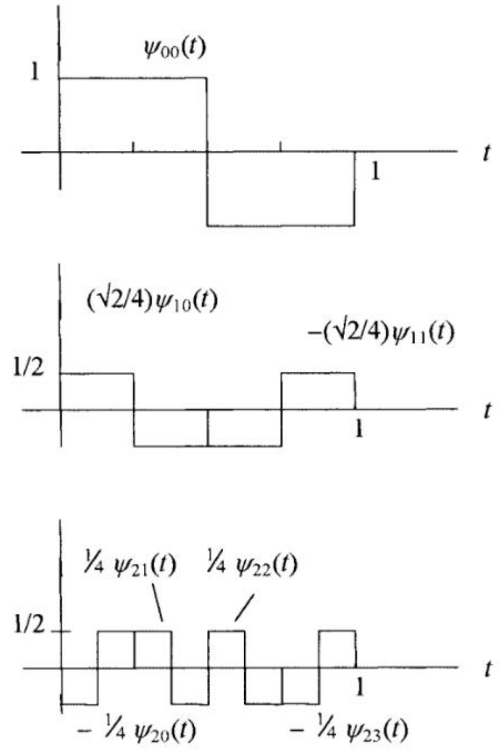

Figure 1.18. Sum these functions to obtain $v(t)$.

However, there are differences that are not apparent in these simple examples. There are many wavelet basis functions other than the Haar functions, and there is one wavelet transform for each set of basis functions. This is similar to the Fourier transforms, where there are four forms of the Fourier transform, but not similar in that there is a vast array of wavelet basis functions. Another important difference is in the way the coefficients for the wavelet transform are calculated. Note that in the above example using an inner product formulation, the formula for the basis function must be known. Most wavelet coefficients are calculated in a different way, using multirate sampling theory. There it is necessary to know only the filter coefficients. This method does not use the basis functions. We will devote much of this book to this procedure