CS 3120 - Module 1 HW 2

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1 Prove Set Countability

To prove that the set $|\mathbf{Q} = \frac{a}{b}|$ such that $a \in \mathbf{Z}$ and $b \in \mathbf{Z}^+$ is countable, I must prove that $|\mathbf{Q}| \leq |\mathbf{N}|$.

Let the the set $q = \mathbf{Z} \times \mathbf{Z}^+$ where each element in q represents a possible a and b pair in $\mathbf{Q}$.

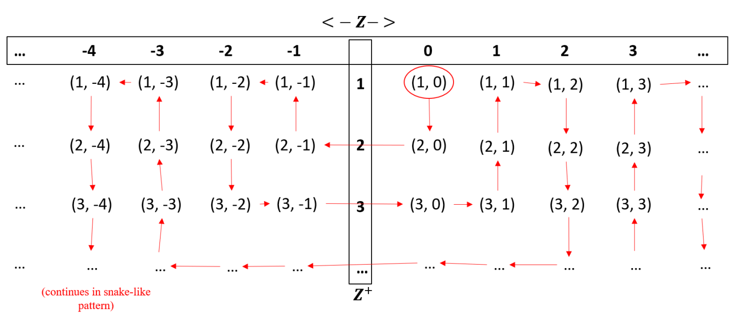

Figure 1: Mapping from $\mathbf{Z} \times \mathbf{Z}^+$ with $\mathbf{N}$

Essentially, we start at $(1, 0)$ and let that map to $x = 0$ and move in a snake-like pattern starting at the positive side and moving towards the negative side and vice versa, similar to that of a pendulum $\mathbf{Z}$. Everytime we move to a different element, we increase x by one. This is surjective because following this pattern means that we will hit every element in the co-domain of $\mathbf{Z} \times \mathbf{Z}^+$ to each value in $\mathbf{N}$. Since its surjective, this means that $|\mathbf{Q}| \leq |\mathbf{N}|$; therefore, $\mathbf{Q}$ is countable.

2 Prove through Diagonalization

To prove that the set $F = \{f : \mathbf{N} \to \mathbf{N} | (a \geq b) \to (f(a) \geq f(b))\}$ is uncountable, I need to prove that there is no bijection between any function in F and the $\mathbf{N}$.

I will prove it by proving that a subset is uncountable since a set is uncountable if its subset is uncountable. Let $F' = \{f : \mathbf{N} \to \mathbf{N} | \forall a \in \mathbf{N}, f(a) = a^2 + b | b \in \mathbf{N}\}$.

N	0	1	2	3	4	…
f(x) = x^2	0	1	4	9	16	…
f(x) = x^2+ 1	1	2	5	10	17	…
f(x) = x^2+ 2	2	3	6	11	18	…
f(x) = x^2+ 3	3	4	7	12	19	…
…	…	…	…	…	…	…

Figure 2: Mapping of all in F'

The Figure above represents all the possible functions in the set F' to $\mathbf{N}$. Let each digit i be called d_i and the transformed value to n_i. I will increment d_0 by 1 making n_0 to 1. For the following digits, n_i will be next highest value that is strictly greater than n_{i-1} and d_i. This will yield the new string to be $1, 3, 7, 13, ...$. I have created a new string that will not appear in the mapping because it will differ by at least one digit for each possible entry.

Since F' is not bijective to $\mathbf{N}$ and $F' \subset F$, F is not bijective to $\mathbf{N}$.

Therefore, $F = \{f : \mathbf{N} \to \mathbf{N} | (a \geq b) \to (f(a) \geq f(b))\}$ is uncountable.

3 Each Subset in $\mathbf{N}$ is Countable

To prove that every subset of $\mathbf{N}$ is countable, I must prove that the cardinality of each subset in $\mathbf{N}$ is less or equal to $|\mathbf{N}|$.

Given an arbitrary subset of $\mathbf{N}$ which we will call x, a bijective function can be made between x and $\mathbf{N}$. Let each element in x correspond to a particular index. For each subset of x, if a specific value exists, then the corresponding index will be 1, otherwise, it will be a 0.

For example: Let $x = \{0, 1, 2\}$
$f(\{\}) = 000$
$f(\{0\}) = 100$
$f(\{0, 1\} = 110$
$f(\{0, 1, 2\}) = 111$
etc.

The function is injective because given subsets of $\mathbf{N}$ which we will call X and Y where $X \neq Y$, $f(X) \neq f(Y)$ because there will be at least one bit that is different.

In addition, the function is surjective because I can find a subset of $\mathbf{N}$ such that it corresponds to appropriate bit configuration in $\mathbf{N}$.

4 Proving $A \cup B$ is countable where B is a finite set

Given that A is a countably infinite set and B is a finite set, I must prove that $A \cup B$ must also be countable.

To make $A \cup B$ where B is a finite set countable, I need to provide a bijective function from $A \cup B$ to $\mathbf{N}$. To make the function, you can simply first map B to $\mathbf{N}$ since B is a finite set. Then you can make map the rest of $\mathbf{N}$ to A. This is injective because each value in $A \cup B$ will map to a unique value in $\mathbf{N}$. The function is surjective because each value in $\mathbf{N}$ will be mapped by some value in $A \cup B$. Therefore, $A \cup B$ is bijective to $\mathbf{N}$ making it countable.

5 Proving $A \cup B$ is countable when B is countably infinite

Given that A and B are countably infinite sets, I must prove that $|A \cup B|$ is also countable. To prove that that $A \cup B$ is countable, I must argue how $A \cup B$ can be bijectively mapped to $\mathbf{N}$. You can simply make a function such that each value in A is mapped to each even number in $\mathbf{N}$ and that each value in B can be mapped to the odd values in $\mathbf{N}$. The function will be surjective because each value in $\mathbf{N}$ will be mapped to $A \cup B$ since A and B are both infinite in size. The function will be injective because each value in A will be mapped to a unique even number and each value in B mapped to a unique odd number.

Therefore, you can make a bijective function from $A \cup B$ to $\mathbf{N}$ proving that $A \cup B$ is countable.