Module 3 Recurrence Relations Assignment

Tyler Kim
tkj9ep

October 11, 2022

Response 1

1. The algorithm would take in two integer inputs, *int start* and *int end*, and would return an integer. The algorithm would check if $end - start == 2$, if so the algorithm would call f([0], [1]). If f([0], [1]) == -1, then the algorithm returns the parameter *start* and if f([0],[1]) == 1, then the algorithm returns the parameter *end*. Else, the algorithm would calculate mid by taking the size, n, and dividing it by 2. If n is even, the algorithm would check if the left or right side of the array is bigger by calling f([0..mid], [mid..end]). If f([0..mid], [mid..end]) == -1, then the algorithm would recursively call on itself with $start = 0$ and $end =$ mid. If the function resulted in 1, then the algorithm will call on [mid..end] of array instead. If n is odd, then the function would check if the left half and right half (excluding the exact middle value) of the array is bigger or not. If calling f([0..mid],[mid+1..end]) is 0, then *mid* must be the index. If not, then the algorithm will proceed the recursion calls like previously described.
2. The recurrence relation is $T(n) = T(\frac{n}{2}) + 2f()$ because the algorithm makes one recursive call and calls f() twice at most.

3. **Given**: $T(n) = T(\frac{n}{2}) + 2f()$

Use Master's Theorem

Let $a = 1, b = 2, k = \log_2(1) = 0, f(n) = 2f() = 2n$
Since $n^k = n^0 = 1 < 2n$, Case 3 applies.
Need to check regularity: $af(\frac{n}{b}) \leq cf(n)$. This means that $c < \frac{af(\frac{n}{b})}{f(n)} < 1$. This means that $c < \frac{\frac{2n}{2}}{2n} = \frac{1}{2}$ which proves the regularity theorem; thus, case 3 of the Master's Theorem can be used.

$T(n) \in \Theta(f(n))$
$\therefore \Theta()$

Response 2

Given: $T(n) = T(n-1) + n$

Unroll the Recurrence
Let d denote level of unrolling

$d = 0$: $T(n) = T(n-1) + n$
$d = 1$: $T(n) = [T(n-2) + (n-1)] + n = T(n-2) + 2n - 1$
$d = 2$: $T(n) = [T(n-3) + (n-2)] + 2n - 1 = T(n-3) + 3n - 3$
$d = 3$: $T(n) = [T(n-4) + (n-4)] + 3n - 3 = T(n-4) + 4n - 7$

General Pattern: $T(n) = T(n - (d+1)) + (d-1)n - \frac{d(d+1)}{2}$

The base case when $T(1)$ is reached when $n - d - 1 = 1$.
Solve for d:
$n - d = 2$
$-d = 2 - n$
$d = n - 2$

Plug d back in:
$T(n) = T(n - ((n-2)+1)) + ((n-2)-1)n - \frac{(n-2)(n-2+1)}{2}$
$T(n) = T(1) + (n-3)n - \frac{n^2-2n+2}{2}$
$T(n) = n^2 - 3n - \frac{n^2-2n+2}{2} = \Theta(n^2)$
$\therefore \Theta(n^2)$

Response 3

Proof. Claim: $T(n) = 4T(\frac{n}{3} + n \in O(n^{\log_3(4)})$

Guess: $O(n^{\log_3(4)})$

Prove: $T(n) \leq cn^{\log_3(4)}$ where c is a constant.

Base Case: $n = 3$

$T(3) \leq c \cdot 3^{\log_3(4)}$
$T(3) \leq c \cdot 4$
$4T(\frac{3}{3}) + 3 \leq 4c$
$4T(1) + 3 \leq 4c$
$4 \cdot 1 + 3 \leq 4c$
$7 \leq 4c$ when $c \geq \frac{7}{4}$
$\therefore$ Since $3 \geq \frac{7}{4}$, the base case holds.

Inductive Hypothesis: Let $k \leq n$ such that $T(k) \leq c \cdot n^{\log_3(4)} - dk$ where d is a constant.

Inductive Case:
$T(n) = 4T(\frac{n}{3}) + n$
$T(n) \leq 4[c \cdot (\frac{n}{3})^{\log_3(4)} - dn] + \frac{n}{3}$
$T(n) \leq 4[c \cdot \frac{n^{\log_3(4)}}{3^{\log_3(4)}} - dn] + \frac{n}{3}$
$T(n) \leq 4[c \cdot \frac{n^{\log_3(4)}}{4} - d] + \frac{n}{3}$
$T(n) \leq cn^{\log_3(4)} - 4dn + \frac{n}{3}$
Since $4dn$ is larger than $\frac{n}{3}$, we can transform the recurrence relation to $T(n) \leq cn^{\log_3(4)} - dn$
$\therefore$ the inequality was proven and the claim is true.

□

Response 4

Given: $T(n) = 2T(\frac{n}{4}) + 1$

Apply Master Theorem:
A = 2, B = 4, $f(n) = 1$
$k = \frac{\log 2}{\log 4} = \frac{1}{2}$

Compare $f(n) = 1$ to $n^{\frac{1}{2}}$
Since $f(n) = O(n^{\frac{1}{2}-\epsilon})$ where $\epsilon = \frac{1}{2}$, Case 1 applies:
$T(n) \in \Theta(n^{\frac{1}{2}})$
$\therefore \Theta(n^{\frac{1}{2}})$

Response 5

Given: $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$

Apply Master Theorem:
A = 2, B = 4, $f(n) = \sqrt{n}$
$k = \frac{\log 2}{\log 4} = \frac{1}{2}$
Compare $f(n) = \sqrt{n}$ to $n^{\frac{1}{2}}$
Since $f(n) = \sqrt{n}$ is equal to $n^k = n^{\frac{1}{2}}$, then we apply Case 2:
$T(n) = \Theta f(n) log(n) = \Theta(n^{\frac{1}{2}} log(n^{\frac{1}{2}}))$
$\therefore \Theta(nlog(n))$

Response 6

Given: $T(n) = 2T(\frac{n}{2}) + n$

Apply Master Theorem:
A = 2, B = 4, $f(n) = n$
$k = \frac{\log 2}{\log 4} = \frac{1}{2}$
Compare $f(n) = n$ to $n^{\frac{1}{2}}$
Since $n^{\frac{1}{2}-\epsilon}$ results in $\epsilon = \frac{1}{2}$, and $cf(n) \geq n^{\frac{1}{2}}$, apply Case 3:
$T(n) \in \Theta(f(n))$.
$\therefore \Theta(n)$

Response 7

Given: $T(n) = 2T(\frac{n}{4}) + n^2$

Apply Master Theorem:
A = 2, B = 4, $f(n) = n^2$
$k = \frac{\log 2}{\log 4} = \frac{1}{2}$
Compare $f(n) = n^2$ and $n^{\frac{1}{2}}$

Since $n^{\frac{1}{2}+\epsilon}$ results in $\epsilon = 1.5$ and $cf(n) \geq n^{\frac{1}{2}}$, apply Case 3:
$T(n) \in \Theta f(n)$.
$\therefore \Theta(f(n))$