

Module 3 Recurrence Relations Assignment

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Response 1

1. The algorithm would take in two integer inputs, *int start* and *int end*, and would return an integer. The algorithm would check if $end - start == 2$, if so the algorithm would call $f([0], [1])$. If $f([0], [1]) == -1$, then the algorithm returns the parameter *start* and if $f([0], [1]) == 1$, then the algorithm returns the parameter *end*. Else, the algorithm would calculate mid by taking the size, *n*, and dividing it by 2. If *n* is even, the algorithm would check if the left or right side of the array is bigger by calling $f([0..mid], [mid..end])$. If $f([0..mid], [mid..end]) == -1$, then the algorithm would recursively call on itself with $start = 0$ and $end = mid$. If the function resulted in 1, then the algorithm will call on $[mid..end]$ of array instead. If *n* is odd, then the function would check if the left half and right half (excluding the exact middle value) of the array is bigger or not. If calling $f([0..mid], [mid+1..end])$ is 0, then *mid* must be the index. If not, then the algorithm will proceed the recursion calls like previously described.

2. The recurrence relation is $T(n) = T(\frac{n}{2}) + 2f()$ because the algorithm makes one recursive call and calls $f()$ twice at most.

3. **Given:** $T(n) = T(\frac{n}{2}) + 2f()$

Unroll the Recurrence

let *d* denotes level of unrolling and let $f()$ denote $\Theta(\max(|l_1|, |l_2|))$

$$d = 1: T(n) = T(\frac{n}{2}) + 2f()$$

$$d = 2: T(n) = [T(\frac{n}{4}) + 2f()] + 2f() = T(\frac{n}{4}) + 4f()$$

$$d = 3: T(n) = [T(\frac{n}{8}) + 2f()] + 4f() = T(\frac{n}{8}) + 6f()$$

$$d = 4: T(n) = [T(\frac{n}{16}) + 2f()] + 6f() = T(\frac{n}{16}) + 8f()$$

General pattern: $T(n) = T(\frac{n}{2^d}) + 2df()$

The base case when $T(1)$ is reached when $d = \log n$

Solve for d :

$$\frac{n}{2^d} = 1$$

$$n = 2^d$$

$$\log n = d$$

Plug d back in:

$$T(n) = T(\frac{N}{2^{\log n}}) + 2 \log n f() = T(1) + 2 \log n f() = 2 \log n [\max |l_1| |l_2|]$$

$$\therefore \Theta(\log n)$$

Response 2

Given: $T(n) = T(n - 1) + n$

Unroll the Recurrence

Let d denote level of unrolling

$$d = 1: T(n) = T(n - 1) + n$$

$$d = 2: T(n) = [T(n - 2) + (n - 1)] + n = T(n - 2) + 2n - 1$$

$$d = 3: T(n) = [T(n - 3) + (n - 2)] + 2n - 1 = T(n - 3) + 3n - 3$$

$$d = 4: T(n) = [T(n - 4) + (n - 4)] + 3n - 3 = T(n - 4) + 4n - 7$$

$$\text{General Pattern: } T(n) = T(n - d) + dn - (2^{d-1} - 1)$$

The base case when $T(1)$ is reached when $n - d = 1$.

Solve for d :

$$n - d = 1$$

$$-d = 1 - n$$

$$d = n - 1$$

Plug d back in:

$$T(n) = T(n - (n - 1)) + (n - 1)n - (2^{n-1-1} - 1)$$

$$T(n) = T(1) + n^2 - n - 2^{n-1} + 1$$

$$T(n) = n^2 - n - 2^{n-2} + 1 = \Theta(2^n)$$

$$\therefore \Theta(2^n)$$

Response 3

Response 4

Given: $T(n) = 2T(\frac{n}{4}) + 1$

Apply Master Theorem:

A = 2, B = 4, $f(n) = 1$

$$k = \frac{\log 2}{\log 4} = \frac{1}{2}$$

Compare $f(n) = 1$ to $n^{\frac{1}{2}}$

Since $f(n) = O(n^{\frac{1}{2}-\epsilon})$ where $\epsilon = \frac{1}{2}$, Case 1 applies:

$$T(n) \in \Theta(n^{\frac{1}{2}})$$

The solution must be $T(n) = \Theta(n)$ since $k = \frac{1}{2}$ and rounds up to 1

Response 5

Given: $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$

Apply Master Theorem:

A = 2, B = 4, $f(n) = \sqrt{n}$

$$k = \frac{\log 2}{\log 4} = \frac{1}{2}$$

Compare $f(n) = \sqrt{n}$ to $n^{\frac{1}{2}}$

Since $f(n) = \sqrt{n}$ is equal to $n^k = n^{\frac{1}{2}}$, then we apply Case 2:

$$T(n) = \Theta(f(n) \log(n) = \Theta(n^{\frac{1}{2}} \log(n^{\frac{1}{2}})))$$

$$\therefore \Theta(n \log(n))$$

Response 6

Given: $T(n) = 2T(\frac{n}{2}) + n$

Apply Master Theorem:

A = 2, B = 2, $f(n) = n$

$$k = \frac{\log 2}{\log 2} = 1$$

Compare $f(n) = n$ to $n^{\frac{1}{2}}$

Since $n^{\frac{1}{2}-\epsilon}$ results in $\epsilon = \frac{1}{2}$, and $cf(n) \geq n^{\frac{1}{2}}$, apply Case 3:

$$T(n) \in \Theta(f(n)).$$

$$\therefore \Theta(n)$$

Response 7

Given: $T(n) = 2T(\frac{n}{4}) + n^2$

Apply Master Theorem:

A = 2, B = 4, $f(n) = n^2$

$$k = \frac{\log 2}{\log 4} = \frac{1}{2}$$

Compare $f(n) = n^2$ and $n^{\frac{1}{2}}$

Since $n^{\frac{1}{2}+\epsilon}$ results in $\epsilon = 1.5$ and $cf(n) \geq n^{\frac{1}{2}}$, apply Case 3:

$T(n) \in \Theta f(n)$.

$\therefore \Theta(f(n))$