# Derivations of Post Keplerian parameters

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# Contents

# 1 General Relativity

Here we provide the full derivations of the post Keplerian parameters that determine the timing model for how a pulsar in a binary with a companion star changes in time due to carious effects of gravity. The first section derives these equations using the Einsteins general relativity.

## 1.1 Periastron advance $\dot{\omega}$:

Einsteins field equations have an exact solution when considering a spherically symmetric star of mass M. This results in the Schwarzschild metric:

$$g = ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.1.1)$$

Then using the fact that the action of a relativistic point particle is:

$$S = -mc^2 \int d\tau. \quad (1.1.2)$$

With the definition of proper time $\tau$ as $d\tau^2 = -\frac{1}{c^2} g_{\mu\nu} x^\mu x^\nu$, integrating along the path of this point particle results in:

$$\tau = \frac{1}{c} \int_0^1 d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \quad (1.1.3)$$

Thus the action becomes:

$$S = -mc \int_0^1 d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \quad (1.1.4)$$

Letting $G = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$ and noting that if the Lagrangian is defined as $\mathcal{L} = G^2$ then varying the action results in the same geodesic equation , since an extremum of $G$ must be an extremum of $\mathcal{L}$, due to: $\delta\mathcal{L} = 2G\delta G$. This means for the Schwarzschild metric along with Restriction that this is a Binary system, i.e. we let $\theta = \pi/2$. The Lagrangian is:

$$\mathcal{L} = \left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2. \quad (1.1.5)$$

Here still over-dots denote derivative with respect to $\lambda$. Both t and $\phi$ are cyclic so the Lagrange equations are:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0 \implies E \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}. \quad (1.1.6)$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0 \implies L \equiv -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \dot{\phi}. \quad (1.1.7)$$

There is also the constraint that the metric $-g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = -u_\mu u^\mu = c^2$, (letting $\lambda \to \tau$), a constant. Thus since $\mathcal{L} = c^2$:

$$-E^2 + \dot{r}^2 + \left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} + c^2\right) = 0. \tag{1.1.8}$$

Then letting $\mathcal{E} = \frac{E^2}{2c^2}$ this takes the form:

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E} \tag{1.1.9}$$

. Where:

$$V(r) \equiv \frac{c^2}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{c^2 r^3} \tag{1.1.10}$$

Using:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{d\phi}{d\tau}\right)^2 \left(\frac{dr}{d\phi}\right)^2 = \frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2. \tag{1.1.11}$$

Then 1.1.9 becomes:

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{c^2 r^4}{L^2} - \frac{2GMr^3}{L^2} + r^2 + \frac{-2GMr}{c^2} = \frac{2\mathcal{E}r^4}{L^2}. \tag{1.1.12}$$

Using the substitutions $u \equiv \frac{L^2}{GMr} \implies \frac{du}{dr} = -\frac{L^2}{GMr^2} \implies \left(\frac{dr}{d\phi}\right)^2 = \left(\frac{du}{d\phi}\right)^2 \left(\frac{GMr^2}{L^2}\right)^2$. Thus:

$$\left(\frac{du}{d\phi}\right)^2 + \frac{c^2 L^2}{(GM)^2} - 2u + u^2 - \frac{2(GM)^2 u^3}{c^2 L^2} = \frac{2\mathcal{E}L^2}{(GM)^2}. \tag{1.1.13}$$

Differentiating with respect to $\phi$ results in the following after canceling all the factors of $\frac{du}{d\phi}$:

$$\frac{d^2 u}{d\phi^2} - 1 + u = \frac{3(GM)^2 u^2}{c^2 L^2}. \tag{1.1.14}$$

In Newtonian mechanics the term on the RHS is 0, so to solve this problem in GR we consider u to be the Newtonian solution plus a small deviation. Thus $u = u_0 + u_1$. Then:

$$\frac{d^2 u_0}{d\phi^2} - 1 + u_0 = 0. \tag{1.1.15}$$

Then letting $\alpha = \frac{3(GM)^2}{c^2 L^2}$:

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \alpha u_0 + \mathcal{O}(\alpha^2). \tag{1.1.16}$$

Where we have dropped the two terms on the RHS with $u_1$ in them as $u_1$ is small and so is the factor of $\alpha$, as for pulsars $L$ is very large. This assumption that $u_1 \approx \alpha$, is self consistent as can be seen in the equation below where the solution to $u_1$ is proportional to $\alpha$ The first ODE's solution is well known: $u_0 = 1 + e\cos\phi$. Subbing this in to he second ODE results in:

$$\frac{d^2 u_1^2}{d\phi^2} + u_1 = \alpha\left[1 + \frac{1}{2}e^2 + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right]. \tag{1.1.17}$$

To which one solution can be obtained from assuming $u_1$ takes the same form as the RHS of the above ODE but with different coefficients. Thus:

$$u_1 = \alpha \left[ 1 + \frac{1}{2}e^2 + e\phi\cos\phi + \frac{1}{6}e^2\cos 2\phi \right]. \tag{1.1.18}$$

The only non-periodic term here is $e\phi\cos\phi$,:

$$u = 1 + e\cos\phi + \alpha e\phi\sin\phi. \tag{1.1.19}$$

Then assuming $\alpha$ is small as it is for pulsars the orbit equation becomes:

$$r = \frac{L^2}{GM(1 + e\cos((1-\alpha)\phi))} + \mathcal{O}(\alpha^2). \tag{1.1.20}$$

As then $\alpha\phi \approx \sin\alpha\phi$ and $1 \approx \cos\alpha\phi$. Thus if we measure from periastron to periastron, r starts at a minimum when $\phi = 0$ and returns to this minimum when $\cos((1-\alpha)\phi) = 1 \implies (1-\alpha)\Delta\phi = 2\pi$. Thus the precession of the angle is:

$$\Delta\phi = \frac{2\pi}{1-\alpha} = 2\pi + 2\pi\alpha + \mathcal{O}(\alpha^2) = 2\pi\alpha + \mathcal{O}(\alpha^2) = \frac{6\pi(GM)^2}{c^2L^2} + \mathcal{O}(\alpha^2). \tag{1.1.21}$$

Since $L^2 \approx GM(1-e^2)a$ and $P_b^2 = \frac{4\pi^2a^3}{GM}$ and in the case that $\phi$ is the periastron angle $\omega$. Then $\dot{\omega} = \frac{\Delta\omega}{P_b}$ and thus:

$$\dot{\omega} = \frac{3(2\pi)^{5/3}(GM)^{2/3}}{P_b^{5/3}c^2(1-e^2)}. \tag{1.1.22}$$

Here $M = m_p + m_c$ i.e. is the total mass of the system and if we use mass units of the sun ($M_\odot$ ) and the variable $T_\odot = \frac{GM_\odot}{c^3}$, we get:

$$\dot{\omega} = \frac{3(2\pi)^{5/3}T_\odot^{2/3}(m_p + m_c)^{2/3}}{P_b^{5/3}(1-e^2)}. \qquad \blacksquare \tag{1.1.23}$$

### 1.2 Amplitude of Einstein delay $\gamma$:

Just as above we start with the Schwarzschild metric, choosing again to set $\theta = \pi/2$ and also choosing some constant value for $\phi$ as the equations we need have no $\phi$ dependence. Thus $d\phi = 0$ and the metric becomes:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2. \tag{1.2.1}$$

Seeing as we are dealing with the motion of a pulsar the gravitational field effect it feels comes from the companion star thus, $M \to m_c$ and $r$ in the denominator of the potential is the distance between the two pulsars, where as the $r$ in $dr$ is the distance from the pulsar to the barycenter $r_p$. Also for this system since all the masses are on the order of solar masses we can consider the quantity $\frac{GM}{rc^2} \approx \frac{v_p^2}{c^2} \approx 10^{-6}$ is small, where $v_p = \frac{dr_p}{dt}$, is the velocity of the pulsar in the binary. Then taylor expanding:

$$-c^2 d\tau^2 \approx -\left(1 - \frac{2Gm_c}{rc^2}\right) c^2 dt^2 + \left(1 + \frac{2Gm_c}{rc^2} + \mathcal{O}(\frac{v_p^4}{c^4})\right) dr_p^2. \tag{1.2.2}$$

Then using $v_p = \frac{dr_p}{dt}$:

$$c^2 d\tau^2 = \left(1 - \frac{2Gm_c}{rc^2}\right) c^2 dt^2 - \left(v_p^2 + \frac{2Gm_c v_p^2}{rc^2}\right) dt^2. \tag{1.2.3}$$

Here we can drop the last term in the last bracket as $\frac{GM}{rc^2} \approx \frac{v_p^2}{c^2} \approx 10^{-6}$. Thus:

$$c^2 d\tau^2 = dt^2 \left(c^2 - \frac{2Gm_c}{r} - v_p^2 + \mathcal{O}(\frac{v_p^4}{c^2})\right). \tag{1.2.4}$$

Then:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}(\frac{2Gm_c}{r} + v_p^2)} = 1 - \frac{Gm_c}{rc^2} - \frac{v_p^2}{2c^2} + \mathcal{O}(\frac{v_p^4}{c^4}). \tag{1.2.5}$$

Then we use the fact that the total energy of a binary system is given by:

$$E = \frac{1}{2}\mu v^2 - \frac{Gm_p m_c}{r}. \tag{1.2.6}$$

Where $r$ and $v$ represent the relative displacement and velocity. There is no GR term in this equation as it is very small compared to the Then using the fact that $E = -\frac{Gm_p m_c}{2a}$, where a is the relative semi-major axis, we can express the velocity as:

$$v^2 = \frac{Gm_p m_c}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right) = G(m_p + m_c)\left(\frac{2}{r} - \frac{1}{a}\right). \tag{1.2.7}$$

Relations like $E = -\frac{Gm_p m_c}{2a}$ remain the same as in the Keplerian case as the contributions from GR are small enough to be ignored. This is equivalent to deriving these relations from equation 1.1.20. Then since $r_p = -\frac{\mu}{m_p} r$, then $v_p = -\frac{\mu}{m_p} v$. So:

$$v_p^2 = \frac{Gm_c^2}{m_p + m_c}\left(\frac{2}{r} - \frac{1}{a}\right). \tag{1.2.8}$$

This means equation 1.2.5 becomes the following:

$$\frac{d\tau}{dt} = 1 - \frac{Gm_c}{rc^2} - \frac{Gm_c^2}{c^2(m_p + m_c)r} = 1 - \frac{Gm_c}{c^2 r}\left(1 + \frac{m_c}{m_p + m_c}\right) = 1 - \frac{Gm_c}{c^2 r}\left(\frac{m_p + 2m_c}{m_p + m_c}\right). \tag{1.2.9}$$

Here we have dropped any constant terms as they end being proportional to t once we integrate and can then be dropped by the re-scaling of t. To integrate this expression we consider how r and t relate to the eccentric anomaly $E$. $r = a(1 - e\cos E)$, $\omega t = E - e\sin E$ and thus $\omega dt = (1 - e\cos E)dE$. This means the integral becomes:

$$\tau = \frac{1}{\omega}\int\left[1 - \frac{Gm_c}{ac^2(1 - e\cos E)}\frac{m_p + 2m_c}{m_p + m_c}\right](1 - e\cos E)dE, \tag{1.2.10}$$

$$= \frac{1}{\omega}\int\left[1 - e\cos E - \frac{Gm_c}{ac^2}\frac{m_p + 2m_c}{m_p + m_c}\right]dE \tag{1.2.11}$$

$$\implies \tau = \frac{1}{\omega}(E - e\sin E - \frac{Gm_c}{ac^2}\frac{m_p + 2m_c}{m_p + m_c}E) \tag{1.2.12}$$

$$\tau = t - \frac{Gm_c}{\omega ac^2}\frac{m_p + 2m_c}{m_p + m_c}E \to t - \frac{Gm_c}{\omega ac^2}\frac{m_p + 2m_c}{m_p + m_c}e\sin E \tag{1.2.13}$$

Where we have gotten rid of the last term as the time $t$ can be re-scaled to get rid of the factor that is introduced from subbing in $E = \omega t + e\sin E$ . Then seeing as $P_b^2 = \frac{4\pi^2 a^3}{GM}$ , $\omega = \frac{2\pi}{P_b}$, $(M = m_p + m_c)$ and once again taking the masses to be in units of solar masses. The amplitude of this change in the Einstein delay is:

$$\gamma = T_\odot^{\frac{2}{3}}\left(\frac{2\pi}{P_b}\right)^{-\frac{1}{3}}\frac{m_c(m_p + 2m_c)}{(m_p + m_c)^{\frac{4}{3}}}e. \qquad \blacksquare \tag{1.2.14}$$

### 1.3 Shapiro delay $\Delta S$:

Once again we begin with the Schwarzschild metric. Though this time as we are dealing with the delay in the travel time of light beam, thus the metric becomes light like meaning: $ds^2 = 0$. Then once again fixing $\phi$ and $\theta$ so that the solid angle is 0 the metric becomes:

$$-\left(1 - \frac{2GM}{|\mathbf{r}|c^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{|\mathbf{r}|c^2}\right)^{-1} d\mathbf{r}^2 = 0, \tag{1.3.1}$$

$$\implies \left|\frac{d\mathbf{r}}{dt}\right| = c\left(1 - \frac{2Gm_c}{c^2|\mathbf{r}|}\right). \tag{1.3.2}$$

Where $M \to m_c$ as the gravity well of the pulsar is just a constant and doesn't contribute to the spacing of the time of arrivals. Flipping this around and integrating from the time of emission $t_{em}$ to the time of arrival $t_{arr}$, results in the total time take for the light beam to travel from the pulsar to earth. Here the RHS of the above equation when divided across has been taylor expanded as $\frac{2Gm_c}{c^2r}$ is small.

$$t_{arr} - t_{em} = \int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})} \frac{1}{c}\left(1 + \frac{2Gm_c}{c^2|\mathbf{r}|}\right)|d\mathbf{r}|. \tag{1.3.3}$$

Here $r_e$ is the vector pointing from the origin which is the binary barycenter to the earth. Then:

$$t_{arr} - t_{em} = \frac{1}{c}(\mathbf{r}_p(t_{em}) - \mathbf{r}_e(t_{arr})) + \frac{2Gm_c}{c^3}\int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})}\left(\frac{1}{|\mathbf{r}|}\right)|d\mathbf{r}|. \tag{1.3.4}$$

The last term of the RHS of this equation corresponds to the Shapiro delay, which when considering $\mathbf{r}$ as a function of time and thus picking up an extra factor of c as $|d\mathbf{r}| = cdt$ becomes:

$$\Delta S = \frac{2Gm_c}{c^2}\int_{t_{em}}^{t_{arr}} \frac{dt}{|\mathbf{x}(t) - \mathbf{r}_c(t_{em})|} + const. \tag{1.3.5}$$

The $|\mathbf{r}|$ here has become the distance between the light beam and the companion mass. Here $x(t)$ is the straight line path that the light travels and is given by:

$$\mathbf{x}(t) = \mathbf{r}_p(t_{em}) + \frac{t - t_{em}}{t_{arr} - t_{em}}(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})). \tag{1.3.6}$$

Using the substitution $\theta = \frac{t - t_{em}}{t_{arr} - t_{em}}$ the the bounds of the integral now go from 0 to 1. Then:

$$\Delta S = \frac{2Gm_c}{c^2}\int_0^1 \frac{(t_{arr} - t_{em})d\theta}{|\mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em}) + \theta(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em}))|}. \tag{1.3.7}$$

Then seeing as $|\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})| \approx c(t_{arr} - t_{em})$ and $|\mathbf{r}_e(t_{arr})| \equiv |\mathbf{r}_e| >> |\mathbf{r}_p(t_{em})|$ :

$$\Delta S \approx \frac{2Gm_c}{c^3}\int_0^1 \frac{d\theta}{|\frac{\mathbf{r}}{|\mathbf{r}_e|} + \theta\hat{\mathbf{r}}_e|} = \frac{2Gm_c}{c^3}\int_0^1 \frac{d\theta}{\sqrt{\theta^2 + 2\theta\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2}}. \tag{1.3.8}$$

Where $\mathbf{r} = \mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em})$. Then:

$$\Delta S = \frac{2Gm_c}{c^3} \int_0^1 \frac{d\theta}{\sqrt{(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2 + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2}}. \tag{1.3.9}$$

Then letting $k\sinh(\psi) = \theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}$, where $k^2 = (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2$ . The integral then becomes:

$$\frac{2Gm_c}{c^3} \int \frac{k\cosh(\psi)d\psi}{k\sqrt{1+\sinh^2(\psi)}} = \frac{2Gm_c}{c^3} \sinh^{-1}\left(\frac{1}{k}(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})\right)\Bigg|_0^1. \tag{1.3.10}$$

Then using the fact that $\sinh^{-1}(u) = \ln(\sqrt{u^2+1} + u)$:

$$\Delta S = \frac{2Gm_c}{c^3}\left(\ln\left|\sqrt{(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2 + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2} + \theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}\right| - \ln|k|\right)\Bigg|_0^1. \tag{1.3.11}$$

Then expanding the inside of the square root and imposing the bounds:

$$\Delta S \approx \frac{2Gm_c}{c^3} \ln\left|\frac{\sqrt{1 + 2\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2} + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} + 1}{\frac{|\mathbf{r}|}{|\mathbf{r}_e|} + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}}\right|. \tag{1.3.12}$$

Here $k$ is a constant so we have lost the last term in 1.3.11. Then dropping the terms in the square root with $|\mathbf{r}_e|$ in the denominator and imposing once again that $|\mathbf{r}_e| >> |\mathbf{r}|$, results in:

$$\Delta S \approx \frac{2Gm_c}{c^3} \ln\left|\frac{2|\mathbf{r}_e|}{|\mathbf{r}| + \hat{\mathbf{r}}_e \cdot \mathbf{r}}\right| = \frac{2Gm_c}{c^3}\left(\ln\left|\frac{1}{|\mathbf{r}| - |\mathbf{r}|\sin(i)\sin(\omega+\phi)}\right| + \ln 2|\mathbf{r}_e|\right). \tag{1.3.13}$$

Here we can drop the constant term once again. Then using Kepler's ellipse equation to write $|\mathbf{r}|$ and once again dropping the constant factor of $\ln|a|$:

$$\Delta S \approx \frac{2Gm_c}{c^3} \ln\left|\frac{1 + e\cos(\phi)}{(1-e^2)(1 - \sin(i)\sin(\omega+\phi))}\right| \tag{1.3.14}$$

$$= -\frac{2Gm_c}{c^3} \ln\left|\frac{(1-e^2)(1 + \sin i(\sin\omega\cos\phi + \cos\omega\sin\phi))}{1 + e\cos\phi}\right|. \tag{1.3.15}$$

Where we have expanded $\sin(\omega + \phi)$ and flipped the fraction. Next we introduce again the eccentric anomaly $E$. Which is related to the phase angle $\phi$ by: $\cos E = \frac{e+\cos\phi}{1+e\cos\phi}$ and $\sin E = \frac{\sqrt{1-e^2}\sin\phi}{1+e\cos\phi}$, Thus:

$$1 - e\cos E = 1 - \frac{e^2 + e\cos\phi}{1 + e\cos\phi} = \frac{1-e^2}{1+e\cos\phi}, \tag{1.3.16}$$

$$\frac{(1-e^2)(\sin\phi)}{1+e\cos\phi} = \sqrt{1-e^2}\sin E, \tag{1.3.17}$$

and

$$\cos E - e = \frac{\cos\phi(1-e^2)}{1+\cos\phi} \implies \frac{1-e^2}{1+e\cos\phi} = \frac{\cos E - e}{\cos\phi}. \tag{1.3.18}$$

Thus plugging into 1.3.15:

$$\Delta S = -\frac{2Gm_c}{c^3} \ln \left| 1 - e\cos E - \sin i(\sin\omega(\cos E - e) + \sqrt{1-e^2}\sin E\cos\omega) \right| \quad \blacksquare \qquad (1.3.19)$$

## 1.4 Shapiro shape, range and the binary mass function:

The Shapiro delay is parameterised by the Shapiro range $r$ and the Shapiro shape $s$. Where:

$$r = \frac{Gm_c}{c^3} = T_\odot m_c, \tag{1.4.1}$$

$$s = \sin i. \tag{1.4.2}$$

To find the from of $s$ we derive the binary mass function $f(m_c, m_p)$. We start with the semi-major axis of the relative orbit $a$, which can be written in terms of the semi-major axis of the pulsar and it's companion by: $a = a_p + a_c$. The total mass of the system is $M = m_p + m_c$ and by definition of center of mass $m_p a_p = m_c a_c$. This leads to:

$$a = a_p(1 + \frac{m_p}{m_c}) = \frac{a_p M}{m_c}. \tag{1.4.3}$$

Inserting this into Kepler's law:

$$GM = \frac{a_p^3 M^3 4\pi^2}{m_c^3 P_b^2}. \tag{1.4.4}$$

Then defining the projected semi-major axis as $x = a_p \sin i$, The binary mass function becomes:

$$f(m_c, m_p) = \frac{(m_c \sin i)^3}{(m_c + m_p)^2} = \frac{4\pi^2 x^3}{T_\odot P_b^2}. \tag{1.4.5}$$

Thus:

$$s = \sin i = T_\odot^{-\frac{1}{3}} \left(\frac{2\pi}{P_b}\right)^{\frac{2}{3}} x \frac{(m_p + m_c)^{\frac{2}{3}}}{m_c} \qquad \blacksquare \tag{1.4.6}$$

## 2 Yukawa addition to General Relativity

The simplest way to modify GR is to generalize the Einstein - Hilbert action by changing from just the Ricci scalar to an arbitrary function of the Ricci scalar $f(R)$, with the simple condition that in the weak field limit the theory tends towards GR. This is preferred over alternate theories that have to produce ad-hoc mechanisms to explain the lack of evidence in Solar system environments. $f(R)$ gravity results in a Yukawa-like term being added to the gravitational potential $\delta e^{-\frac{r}{\lambda}}$. Here $\lambda$ is the scale length of the theory and must very large in order to reduce to GR in the weak field regime and $\delta$ the strength of this theory.

### 2.1 Periastron advance $\dot{\omega}$: