Lab 2: The Pendulum

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1 Introduction

The aim of this lab was to preform an investigation into dynamical systems and using python, solve coupled ordinary differential equations (ODE's). To do this we looked at the dynamical system of a pendulum in its various forms. Even the simplest form the simple pendulum has no analytical solution outside of approximating with small angles (linear pendulum). Thus to find solutions to more complicated dynamical systems like the pendulum, analytical methods are insufficient and numerical methods are needed.

The first two exercises were to solve the linear and then non-linear pendulum using the trapezoid method of approximating the differential equations. Then the Runge-Kutta method, another (more accurate) approximation method, was tested for the non-linear pendulum and the results were compared to the trapezoidal method. The last two exercises involved adding dampening (essentially air resistance) and then a driving force to the pendulum system. Both these last methods were carried out with the Runge-Kutta method.

This Lab was carried out using python 3 and was written in the Spyder application.

2 Methodology

2.1 Exercise 1: Linear pendulum

The Linear pendulum is an approximation of the non-linear pendulum where in the differential equation that governs the system the $sin(\theta)$ term is replaced by θ, this can be done for small angles of theta as the taylor expansion of $sin(x)$ around 0 is x.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) \implies \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \qquad (1)$$

The code for this reduced form of the pendulum differential equation was added along with the trapezoidal method of approximation. The code for the trapezoidal method was placed in a for loop so it could be applied as many times as needed. To examine the accuracy of the method plots of theta (the angle of pendulum) and omega (the angular speed of the pendulum) were made for several different initial values, vs time and nsteps. The trapezoidal approximation is of the order Δt, This is shown in figure 1.

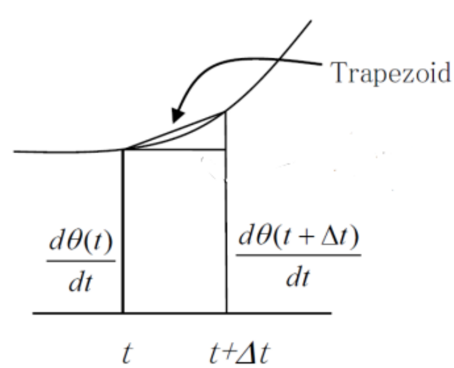

Figure 1: *Diagram showing method of trapezoidal approximation*

2.2 Exercise 2: Non-Linear pendulum

The Non-linear pendulum was essentially the exact same method for the Linear pendulum. The code from the linear pendulum was copied except the theta in the differential was replaced back to the $sin(\theta)$. Then just like before plots of theta and omega vs time were made and the graphs of non-linear and linear pendulum were compared against each other for different initial values.

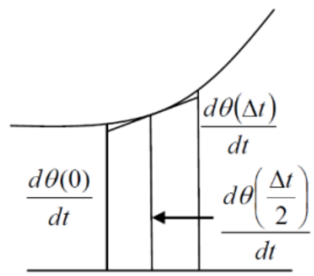

Figure 2: *Diagram showing Runge-Kutta method*

2.3 Exercise 3: Runge-Kutta method

The Runge-Kutta method is a more accurate method of approximation as it is of order Δt^2 this is shown in figure 2. This means that the graphs produced by the Runge-Kutta method should more accurately emulate the solutions to the differential equations and the cost of talking a negligible amount of extra time to compile. To show the difference between the Runge-Kutta method and the trapezoidal method, plots of exaggerated initial conditions were made and compared.

2.4 Exercise 4: The damped non-linear pendulum

Adding damped motion to the pendulum system more accurately simulates reality, as in real life the pendulum is not swung in a vacuum, but in a room with air , causing air resistance. This air resistance is proportional to the pendulums speed at any given moment, this then becomes part of the differential equation as shown below.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) - k\frac{d\theta}{dt}$$

In the code the Runge-Kutta method of approximation was used to find the solutions to this differential equation. Plots were then made of both theta and omega vs time.

2.5 Exercise 5: Damped, driven non-linear pendulum

In this exercise an external periodic force was added to the pendulum. The force acts continuously and may result in harmonic motion or chaotic depending on the frequency (ϕ) of the driving force. If the resulting motion is harmonic it may have a frequency that differs to that of the intrinsic frequency of the pendulum. The differential equation governing this motion is shown below.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) - k\frac{d\theta}{dt} + A\cos(\phi t)$$

Once again the Runge-Kutta method was used to calculate the solutions to this differential equation. This motion is often more chaotic and complex than even the damped motion and so in order to represent the symmetrical nature of the motion, plots of the phase (which is the angle theta vs omega) were made, for different values of the amplitude A. The phase diagrams only started after nsteps=5000, as at the start of every driven motion there is a transient phase before the pendulum enters a steady, repeating closed loop called a limit cycle.

3 Results

3.1 Exercise 1: Linear Pendulum

The Trapezoidal approximation was extremely effective at plotting the linear pendulum, when ran a large number of times (nsteps=1000). As can be seen in the figure 3 below, the plots strongly resembled simple harmonic motion, which is known to be the analytical solution.

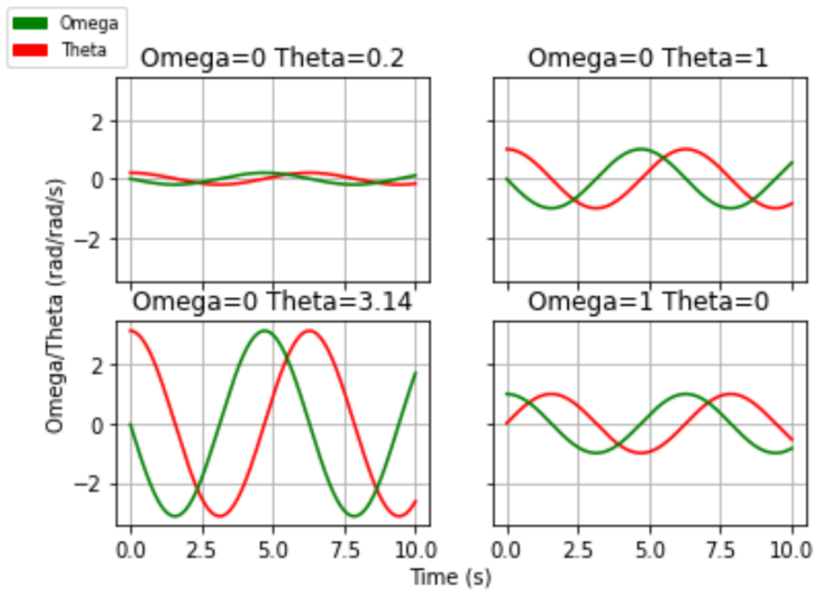

Figure 3: *Linear pendulum plots for different initial conditions*

3.2 Exercise 2: Non-Linear Pendulum

Once again the Trapezoidal method was extremely effective at plotting the pendulum motion. The results for the same 4 initial conditions are shown below in the figure 4.

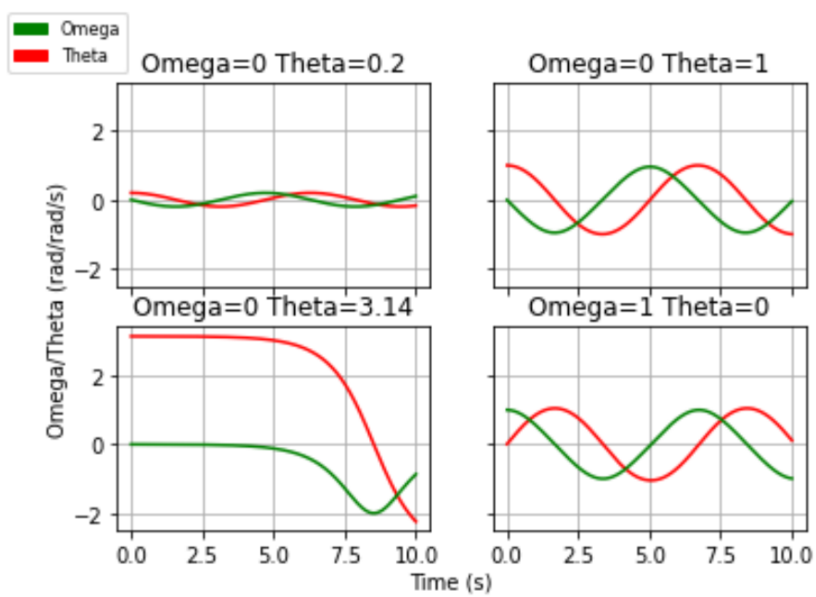

Figure 4: *Non-Linear pendulum plots for different initial conditions*

What is noticeable about the above figures is the similarity of plots 1,2 and 4 , but the stark difference in plot 3. This makes sense in the context of the initial conditions as in plots 1,2 and 4 the pendulum is starting close to or at $\theta_0 = 0$ and thus the approximation $\sin(\theta) = \theta$ is somewhat valid and the graphs resemble each other quite well. However for plot 3 the pendulum is given an initial value $\theta = 3.14$ which is almost vertical, so here the linear pendulum breaks down in describing the differential equation accurately, resulting in a drastically different plot to that of the non-linear motion

3.3 Exercise 3: Runge-Kutta method

The Runge-Kutta method of approximation was also extremely effective at getting a solution to the differential equation. It is hard to tell it apart from the trapezoidal method, so to show the difference a plot was made for the exaggerated initial conditions of $\theta = 3.14$, this is shown if figure 5.

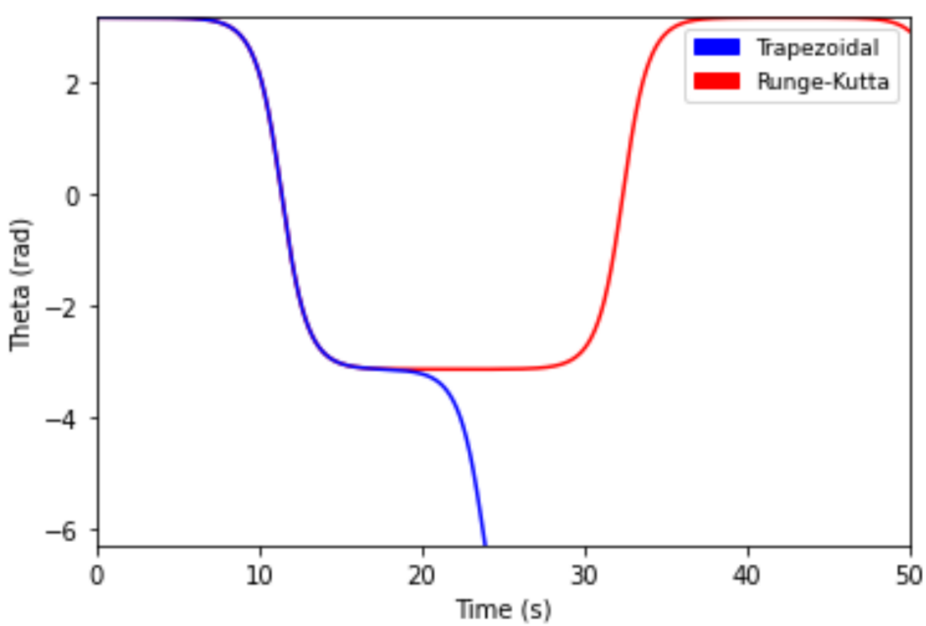

Figure 5: *Plot of Non-linear pendulum for* $\theta_0 = 3.14,\ \omega_0 = 0$

As is clear from the above figure 5 the trapezoidal method appears to breakdown after one rotation of theta, and starts to take values such as $\theta < -\pi$ which we know is impossible and can't be a solution to the differential equation for $\theta_0 = 3.14,\ \omega_0 = 0$ as that would violate the principle of conservation of energy. This tells us that the approximations made in the trapezoidal method, keeping it at an order of just Δt make it so that the method is not as accurate as the Runge-Kutta method which perfectly emulates the expected motion in the above figure 5.

3.4 Exercise 4: The damped nonlinear pendulum

Once again the Runge-Kutta method was effective in calculating the expected motion. This motion was certainly different from the un-damped non-linear pendulum, as is evident from the plot 6 below. For the damped motion, the dampening term is proportional to the speed of the particle at any given moment. In the case of the plot below 6 the constant of proportionality was $k = 0.5$.

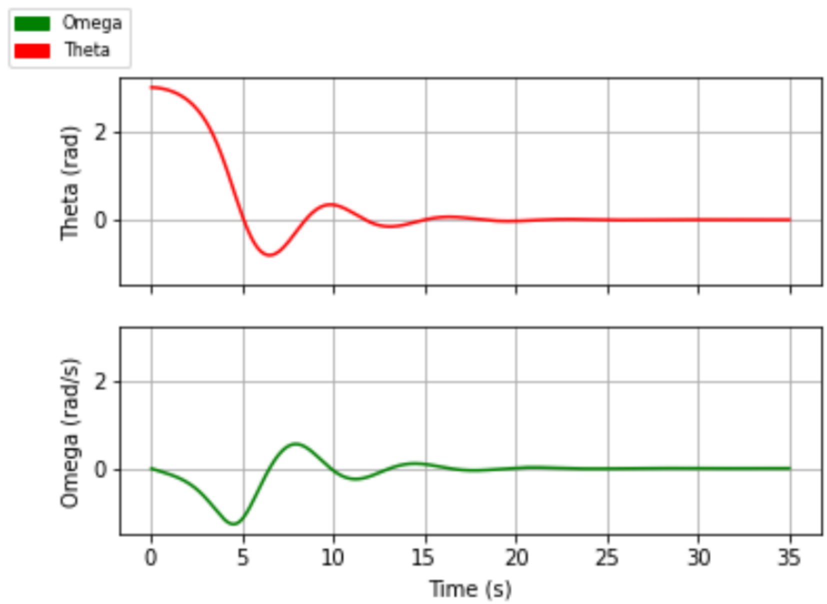

Figure 6: *Plot showing Omega and Theta for damped non-linear motion*

The plot shows how both Theta and Omega peter out to 0 over time, due to the added dampening term to the differential equation. This makes sense as it is expected that in the real world with air resistance, a pendulum will slow down and reduce in amplitude over time. Thus this damped pendulum is a better simulation of reality.

3.5 Exercise 5: Damped, driven non-linear pendulum

Finally, using the Runge-Kutta method once again, the driven motion was added. The various phase plot that were made are shown in the below figures 7,8. For each plot the frequency of the driving force (ϕ) was kept the same at $\phi = 0.66667$, only the amplitude A was changed.

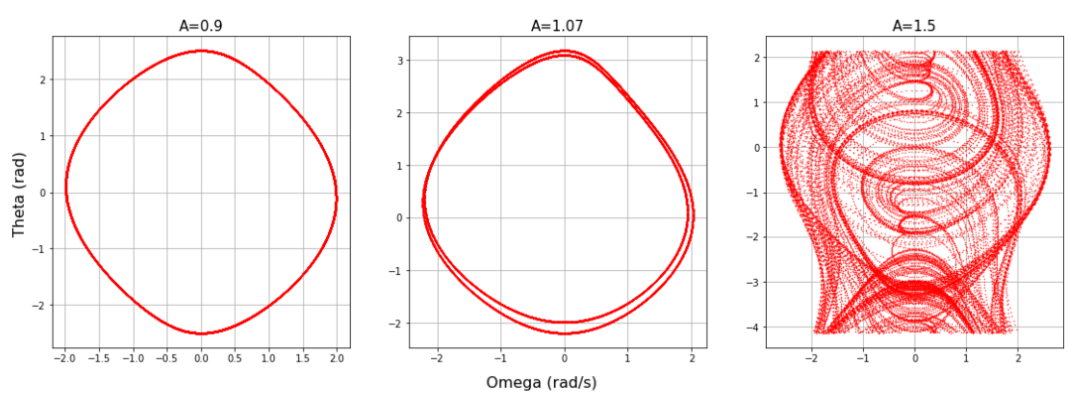

Figure 7: *First three phase portraits, for driven, damped non-linear motion*

For the first phase portrait $A = 0.9$, there is clearly a normal pendulum behaviour of a one-period closed phase trajectory

For the second phase portrait $A = 1.07$, it is evident that period doubling has occurred as can be seen from the two distinct loops. The motion however, remains closed.

For the third portrait $A = 1.5$, It is clear that the system has devolved into chaotic motion, and no closed loop can be formed.

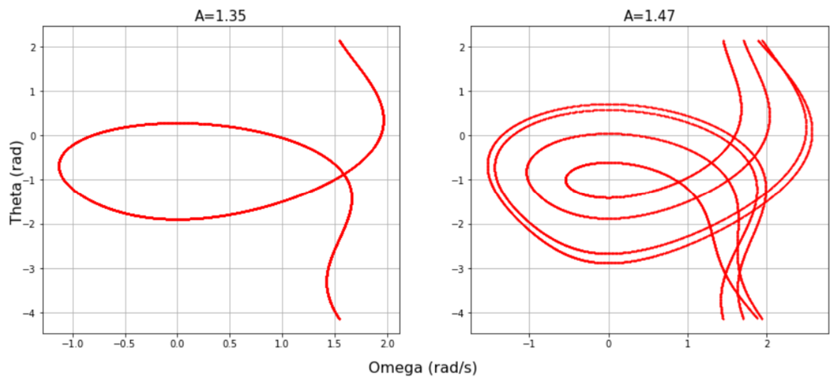

Figure 8: *Final two phase portraits*

For the fourth portrait $A = 1.35$, There is clearly a more complicated motion happening due to full rotations of the pendulum, though the loop is still closed.

For the fifth portrait $A = 1.47$, Here again there is clearly full rotations of the pendulum but also period doubling has occurred several times, and the period is approximately 4 times that of the driving force.

Overall these diagrams show that depending on the value of A, even small changes result in pendulum motions that are drastically different from one another. With some being closed , some closed with full rotations and some taking on complete chaotic motion all together.

4 Conclusion

The aim of this lab was to preform an investigation into dynamical systems and using python, solve coupled ordinary differential equations (ODE's). The first two exercises examined the linear and non-linear pendulum, showing how the linear pendulum broke down in describing motion for large initial angles of theta. Next the two methods of approximations were compared. The Trapezoidal and Runge-Kutta methods. As shown in figure 5, The Runge-Kutta method was the more accurate method, due to it being of order Δt^2 instead of just Δt. Thus the Runge-Kutta was used as the approximation method of choice for the rest of the experiments. Analysis of the damped non-linear pendulum, showed how adding in friction forces slowed the pendulum down over time, a description that is consistent with real world pendulums. Finally for the Driven, damped non-linear pendulum, it was shown that even small changes in the amplitude of the driving force (A) resulted in drastically different motions of the pendulum, with some even becoming chaotic.