Lab 3: Projectile motion under the action of air resistance

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1 Introduction

The aim of this lab was was to perform an investigation into projectile motion, particularly that of motion involving air resistance. To do this we modeled projecting a mass with a certain diameter through certain mediums. The air resistance a particle experiences is proportional to the velocity of the particle at that given moment but is in the opposite direction. Thus the motion of each particle is governed by a differential equation as it's velocity is the rate of change of it's displacement. The first exercise explored how air resistance has both a quadratic and linear component of the diameter. These terms dominate the the contribution to the total air resistance at different values of diameter, these ranges were roughly calculated by investigating plotted graphs. Then for different objects of varying size, such as a baseball and a drop of oil, the ideal form of the air resistance (either quadratic or linear) was investigated.

In Exercise 2 the motion of a dust particle with diameter of 1×10^{-4} m, was investigated. For this both the approximated method using the linear air resistance and the analytical solution were compared. Using this a plot of the time for a particle to hit the ground for different masses was made. In Exercise 3 Plots of the motion of a projectile projected with both an x and y component were made. This then lead to a plot of the optimal launch angle for different masses for linear air resistance.In the final Exercise plots of the trajectory of a particle were compared for linear/quadratic air resistance and its motion in a vacuum.

This Lab was carried out using python 3 and was written in the Spyder application.

2 Methodology

2.1 Exercise 1: Air resistance scaling with the velocity

Air resistance depends heavily on the geometry of the object in motion, so to simplify in these simulation each particle is assumed to be a spherical ball with diameter D. Then the air resistance takes the form:

$$f(V) = bV + cV^2. \tag{1}$$

where,

$$b = BD \quad , \quad c = CD^2. \tag{2}$$

Here B and C are constants that depend on the object and the nature of the medium. For example, a spherical projectile in air has $B = 1.6 \times 10^{-4}\ s/m^2$ and $C = 0.25\ Ns^2/m^4$.

The first task was to create some code that plotted the the air resistance as seen in equation 1. In particular the function was split up so that the linear and the quadratic components were plotted separately on the same graph against $D \times V$. This along with the plot of the total air resistance allowed for the ranges over which the two components were relatively dominant, to be determined. The ranges are hard to see exactly on one singular graph so multiple plots were made, zoomed in to various degrees, to ensure the ranges were found to a higher degree of accuracy. The reason that the ranges over which the two terms became dominant were tested was to find out when approximations of the air resistance to just one term can be made.

Next to test that the air resistance can indeed be approximated to just one term, a simulation was ran for three different real life objects. First a baseball of diameter 7 cm was thrown at 5 m/s, next a drop of oil with a diameter of just 1.5×10^{-6} m at 5×10^{-5} m/s and finally a 1 mm raindrop at 1 m/s. In each case the air resistance was calculated and then compared with the plot of the components in equation 1, to see weather one of them was the "ideal form of $f(V)$" and an approximation could have been made.

2.2 Exercise 2: Vertical motion under the action of air resistance

Here the motion of a dust particle of density 2×10^3 Kg/m^3 and diameter 1×10^{-4} m, was investigated. Using these parameters and the ranges discussed in Exercise 1, which approximation of the air resistance (quadratic or linear) could be used was determined. In this case for the dust particle it turned out that the linear component is the dominant term for these values of the dust particle. Thus the differential equation governing the y-position of the particle over time is:

$$\frac{dV_y}{dt} = -g - \frac{b}{m}V_y. \tag{3}$$

Solving this differential involved an approximation of splitting time up into small increments Δt. Then calculations were made, updating the change in velocity, the velocity and the displacement y of the particle for each increment. This produced a list of values of y at a certain time which could then be plotted for different masses, to compare the curves. The plots for different masses showed just how when the mass gets large it is equivalent to reducing the magnitude of the air resistance. This is explored further in the results.

For this particular differential there is also an analytical solution for V_y that takes the form of the equation below:

$$V_y = \frac{mg}{b}(e^{-\frac{bt}{m}} - 1). \tag{4}$$

Plots for the results of this function were made, comparing it to the approximation of the differential. The error between the two was also plotted as a function of time, to show where the approximation was most accurate.

Finally a plot was made for the time for the function to reach the ground as a function of the mass of the object. This was achieved by dropping the ball from a height of 5 m and using the integrated analytical solution to calculate when the particle had reached the ground.

$$y(t) = \frac{m^2 g}{b^2}(1 - e^{-\frac{bt}{m}} - \frac{bt}{m}) + 5 \tag{5}$$

2.3 Exercise 3: Projectile motion under the action of air resistance - 1

To plot the actual trajectory of a particle when it is projected at an angle to the horizon, both the x and y components of the motion need to be calculated. Since the the particle now has a velocity in the x direction air resistance also acts against this velocity. Thus the x component of the particle velocity also has its own differential equation.. In this case the particle is considered to be a spherical object, small enough so that only linear air resistance (c=0) is relevant, Thus the differential equation for the y component is as shown above in equation 3, and the x component's differential is:

$$\frac{dV_x}{dt} = -\frac{b}{m}V_x. \tag{6}$$

Then the same iteration method that was utilised earlier was applied once again to approximate the these differentials. The trajectories were then plotted until the particle reached the ground, i.e. where y=0 for the second time after launch. This plot was then compared to a plot of a launch with the exact same initial conditions but taken place in a vacuum so that the particle experienced no air resistance.

The second task was to use these trajectories to calculate the optimum launching angle so that the x distance traveled by the particle is at a maximum. Intuitively one may think this will always be 45° (pi/2 rads) to the horizon, as this is the result for all projectiles in a vacuum with simple parabolic motion. However air resistance changes this result, so to calculate the optimal angle, the approximated trajectory for a given mass was calculated for a large sample of angles ranging between 0 to pi/2 radians. Then at the end of this calculation the value of theta that corresponded to the largest x distance when the particle fell back to the ground, was recorded. This was repeated for several masses until a plot of optimal launch angle vs mass could be made.

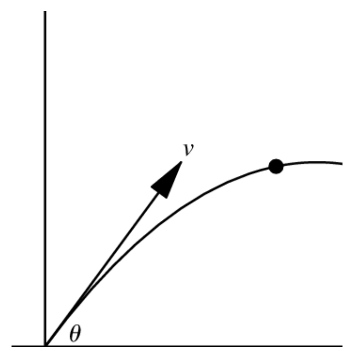

Figure 1: *Launch angle theta*

2.4 Exercise 4: Projectile motion under the action of air resistance - 2

The final exercise was to compare the impact of quadratic air resistance vs linear air resistance. To do this the trajectory of a particle was compered for no air resistance, B air resistance (Linear) and C air resistance (Quadratic). These were all plotted on the same graph in order to visually see their differences. For no air resistance and B air resistance, the equations explored in Exercise 2 and 3 were used, however, for the C air resistance new differential equations were need to model the motion. These equations are as shown below.

$$\frac{dV_x}{dt} = -\frac{c}{m}\sqrt{V_x^2 + V_y^2}V_x, \tag{7}$$

$$\frac{dV_y}{dt} = -g - \frac{c}{m}\sqrt{V_x^2 + V_y^2}V_y. \tag{8}$$

3 Results

3.1 Exercise 1: Air resistance scaling with the velocity

The plots below 2, 13 show both the linear and quadratic components as well as the overall air resistance f(V). The two plots show the exact same functions, the only difference being the range of values over which the plot is shown. The first plot is zoomed out to show where quadratic dominance occurs, where as, the second is zoomed in to show linear dominance.

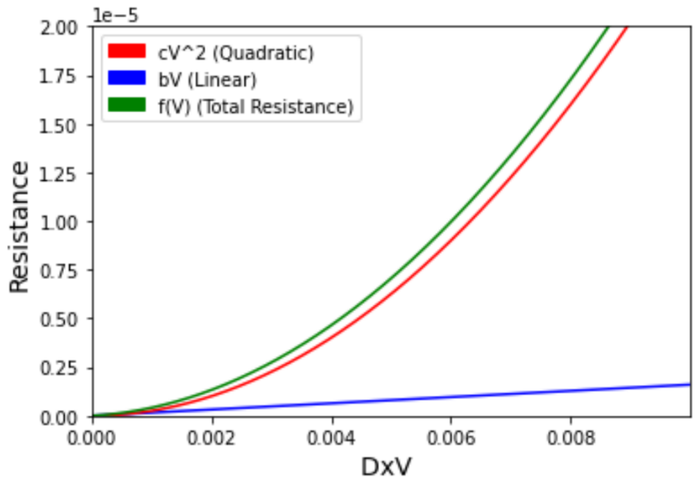

Figure 2: *Plot showing Quadratic dominance for* $D \times V > 0.002$

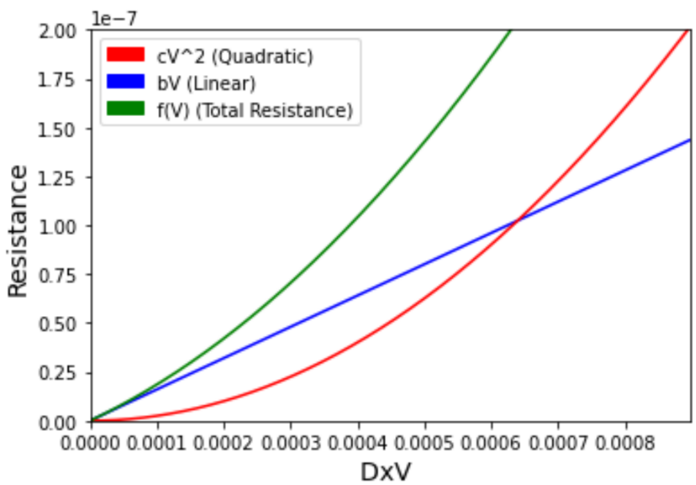

Figure 3: *Plot showing Linear dominance for* $D \times V < 0.0001$

The first plot shows how f(V) has Quadratic dominance for roughly $D \times V > 0.002$. As it can be seen that the red Quadratic curve closely follows the green f(V) curve after that point. Plot two is zoomed in much further and shows how there is linear dominance roughly $D \times V < 0.0001$ as f(V) and the linear term are indistinguishable below this value. This leads to the conclusion that one can ignore the quadratic term, when $D \times V < 0.0001$ and can ignore the linear term when $D \times V > 0.002$.

Next up the ideal form for f(V), i.e. what term linear or quadratic could be used for three different particles was tested. First up was the baseball with diameter 7 cm, thrown at 5 m/s. This resulted in a calculated resistance value of 0.030681. This was then plotted below.

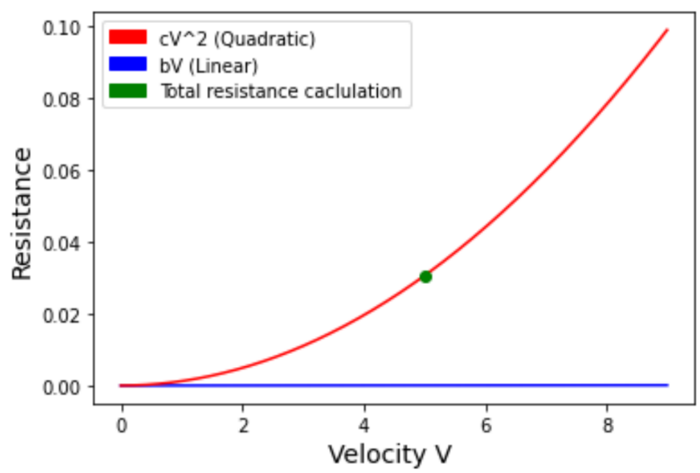

Figure 4: *Plot showing which term the total resistance calculation for the baseball depends on*

As can be clearly seen from the graph above, The total resistance almost entirely depends on the quadratic term. This makes sense in the context of the ranges calculated earlier as for the baseball,$D \times V = (.07)*(5) = 0.35$ which is >> 0.002. Thus the Linear term could be neglected for the baseball.

Second up was the tiny drop of oil of diameter 1.5×10^{-6} m travelling at 5×10^{-5} m/s. This yielded a total resistance calculation value of 1.2×10^{-14}. Plotting this again.

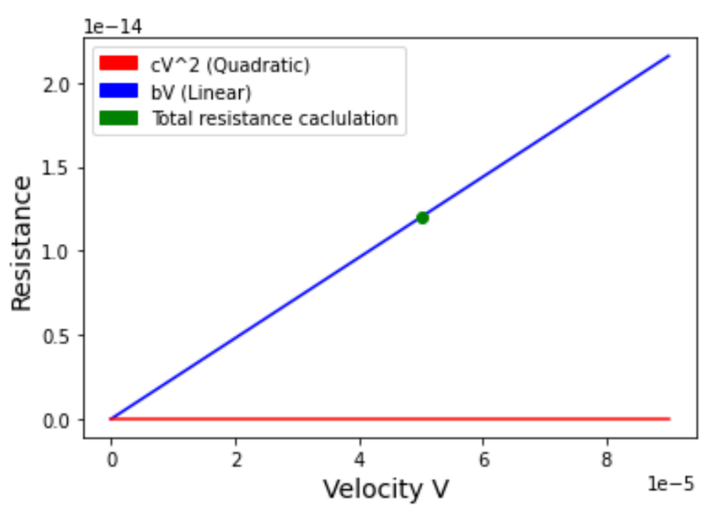

Figure 5: *Plot showing which term the total resistance calculation for the oil drop depends on*

Now it can be seen from the graph that the total resistance almost completely depends on the linear resistance term. This once again makes sense in the context of the parameters as for the oil drop $D \times V = (1.5 \times 10^{-6})*(5 \times 10^{-5}) = 7.5 \times 10^{-11}$ which is << 0.0001. Thus the quadratic term could be neglected for these parameters.

Finally there was the rain droplet of diameter 1 mm traveling at 1 m/s. The total resistance calculation for this resulted in 4.1×10^{-7}. This was once again plotted.

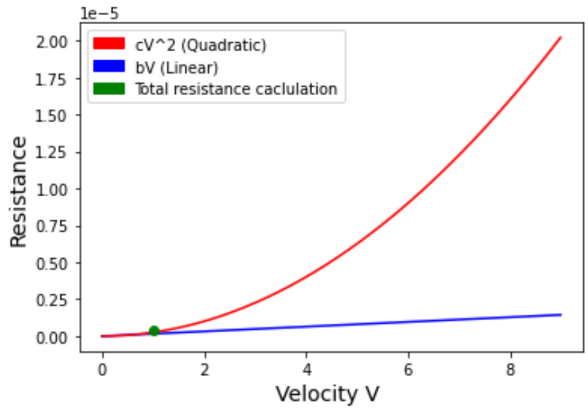

Figure 6: *Plot showing which term the total resistance calculation of the rain drop depends on*

However, now it can be seen from the graph that the total resistance depends both on the linear and quadratic terms. This makes sense in the context of the parameters as for the rain drop $D \times V = (1 \times 10^{-3})*(1) = 1 \times 10^{-3} = 0.001$ which is both > 0.0001 and < 0.002. Thus neither term could be neglected for a particle of this size and speed.

3.2 Exercise 2: Vertical motion under the action of air resistance

In the case of the dust particle from calculation one can get its maximum (terminal) velocity by letting the differential equation 3 = 0. This resulted in a value of 6.414085×10^{-5} m/s which means, $D \times V = (1 \times 10^{-4})*(6.414085 \times 10^{-5}) = 6.414085 \times 10^{-9}$ which is $<< 0.0001$ thus the linear term is dominant and quadratic air resistance can be ignored (c=0).

The first task was to use the differential equation 3 to approximate and plot V_y as a function of time. Different plots were made for different masses as shown below

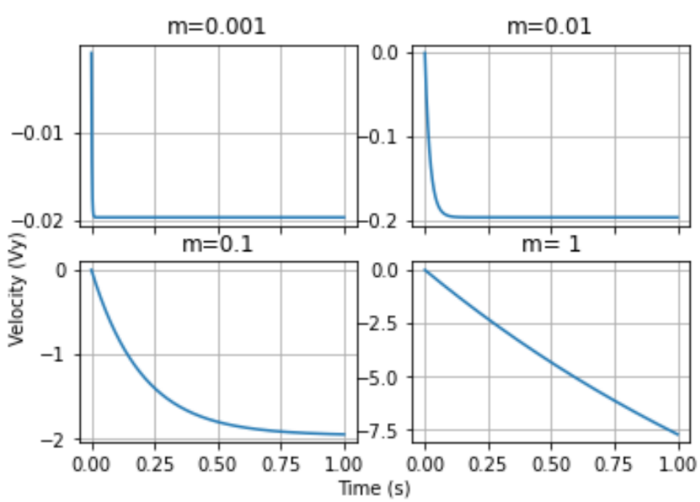

Figure 7: *Plots of V_y vs time for different masses*

What is noticeable from these graphs is that the higher the mass, the lower the affect air resistance seemed to have on the velocity of the particle over time. This is equivalent to saying that particles of higher mass experience less air resistance.

Next up was to compare the analytical solution to the approximate solution. This was done by plotting them both on the same graph of velocity (V_y) vs time. This was done for a projectile of mass 0.1 Kg.

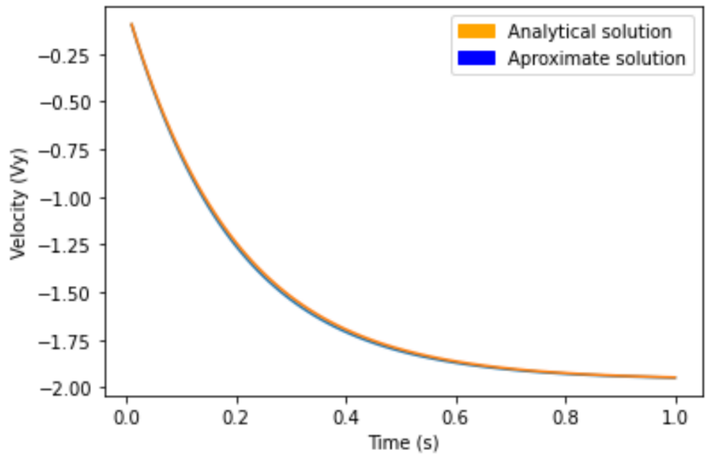

Figure 8: *Plot of analytical and approximate solutions*

Here the difference between the two solutions is barely noticeable. To see the difference between them more clearly a plot of the error over time was made.

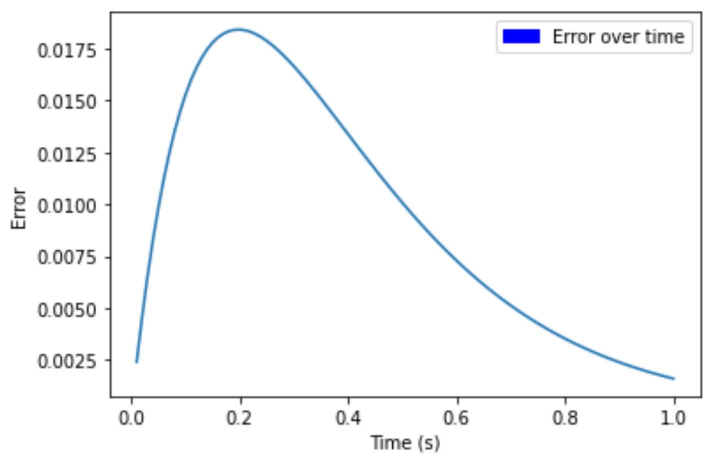

Figure 9: *Plot of the difference between analytical and approximate solutions over time*

Clearly the analytical solution is all ready quite close to the analytical solution, but, one method of increasing the accuracy would be to decrease the value of Δt, the increment time mention in section 2.2.

Finally a plot was made showing time taken to hit the ground for different masses.

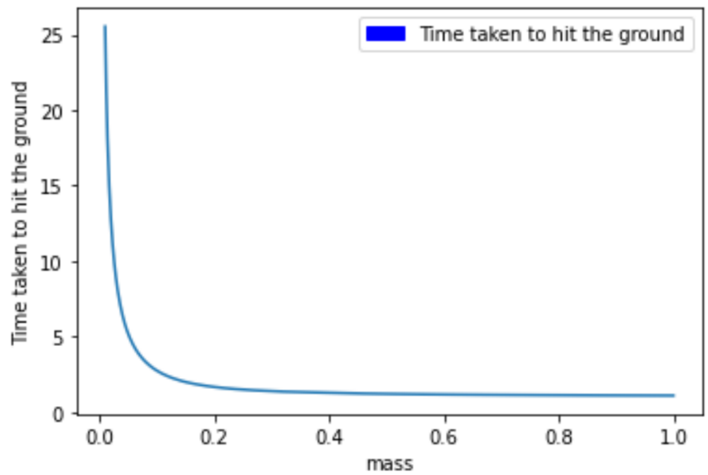

Figure 10: *Plot showing time taken to hit the ground for different masses*

Once again it can be seen that lighter mass are more affected by air resistance, where as heavier masses appear to experience less and less of an affect the heavier the mass.

3.3 Exercise 3: Projectile motion under the action of air resistance - 1

Now for the plots of the actual trajectory of the motion, plots of motion with no air resistance were compared to those with air resistance. In this first case there was only linear air resistance and the medium and dimensions of the ball were such that the constant b=0.5. The same particle was used for each trajectory with the same initial velocity.

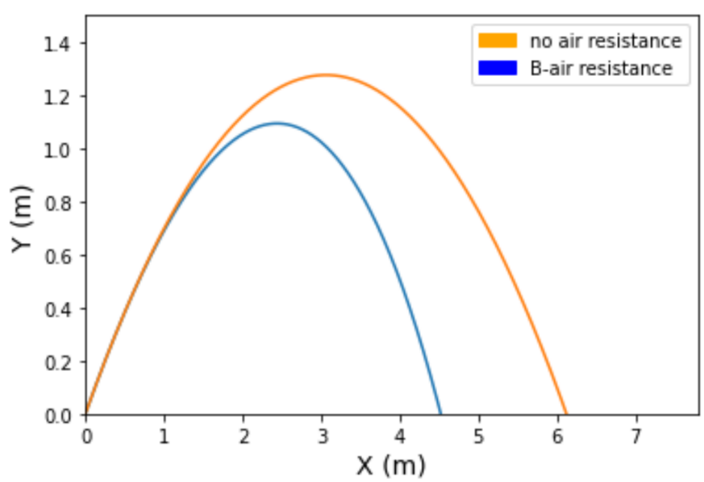

Figure 11: *Plot showing motion with air resistance as well as motion in a vacuum*

As expected the particle when projected in a vacuum travels higher and farther than the particle experiencing air resistance.

The next task was to plot the optimal launching angle as a function of the mass of the particle being fired.

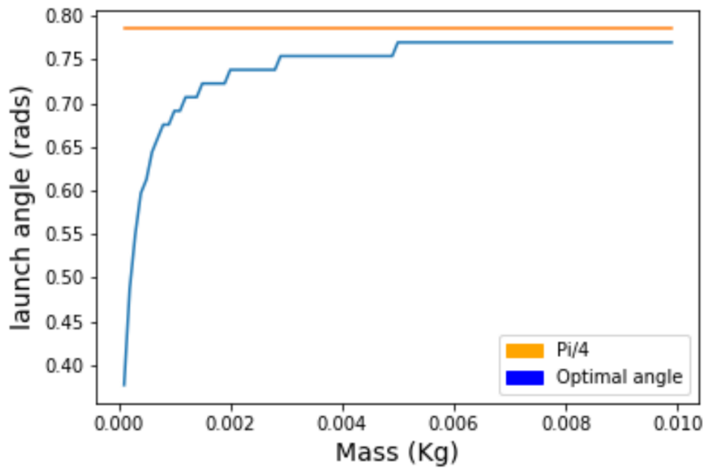

Figure 12: *Plot showing motion with air resistance as well as motion in a vacuum*

As can clearly be seen from the graph the launching angle starts low but as the mass gets bigger, the optimal angle converges towards $\pi/4$ radians. This makes sense in the context of what was discussed earlier, that a higher mass is equivilent to lowering the air resistance acting on the particle, thus one would expect as the mass is increased the optimal angle tends towards what the optimal angle would be in a vacuum, which is known to be $\pi/4$ radians.

3.4 Exercise 4: Projectile motion under the action of air resistance - 2

Lastly was to add back in and compare quadratic air resistances affect to both linear resistance and motion in a vacuum. This was done for various parameters as shown below but all with b=0.5 and c=0.5.

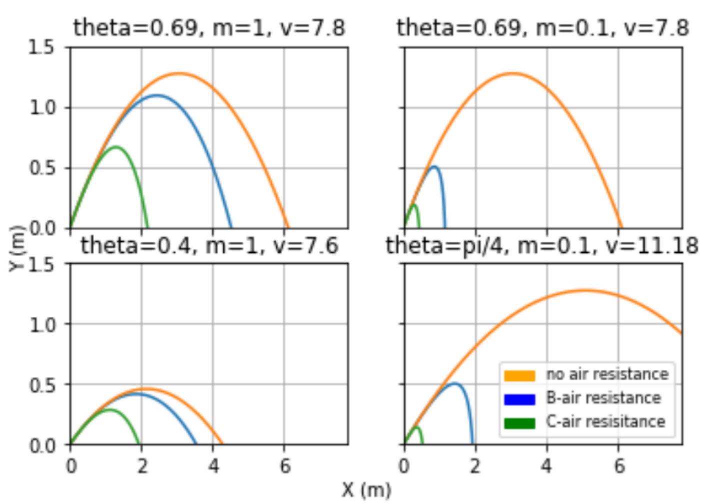

Figure 13: *Plots showing motion with linear and quadratic air resistance as well as motion in a vacuum*

These plots clearly illustrate how for the values chosen the quadratic term is clearly the more dominating factor of the air resistance. They also show once again that smaller masses are more affected by air resistance and that launch angle heavily determines the range of the particle.

4 Conclusion

In conclusion a thorough investigation was performed into projectile motion and how air resistance of different kinds impacts particles of different parameters. Ranges were found for when f(V) (the air resistance) could be reduced to only its linear term ($D \times V < 0.0001$) and when it only depended on its quadratic term ($D \times V > 0.002$). It was also found that how much a particle was affected by air resistance was heavily influenced by the mass of the particle. Higher mass objects are less affected where as lighter masses are more affected. This was further backed up by the fact that the optimal launching angle tended towards $\pi/4$ for heavier mass particles, showing how they are less affected by air resistance and tend towards motion in a vacuum. Lastly plots of the trajectory's of particles with different parameters were made, which fitted out previous observational results and emulated real world expected results.