# Calculus on Manifolds

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Notes taken in Professor Florian Naef's class, Hilary Term 2024

" $p \in N$ is "

-Florian

# Contents

# 1 Topology on $\mathbb{R}^n$

## 1.1 Metric space

- Let $X$ be a set, A *metric* on a set is a function that measures distances $d : X \times X \to \mathbb{R}$. It has the following properties:

$$\boxed{\begin{aligned} &d(x,y) = d(y,x) \\ &d(x,y) \geq 0 \\ &d(x,y) = 0 \ \text{ iff } \ x = y \\ &d(x,z) \leq d(x,y) + d(y,z) \end{aligned}} \tag{1.1}$$

  $(X,d)$ together make a *metric space*.

- Any subset $Y \subset X$ is itself a metric space with $d(x,y)\Big|_{Y\times Y}$ (restricted to $Y$).

## 1.2 Open/Closed

- Let $(X,d)$ be a metric space $U \subset X$ is *open* if $\forall\ p \in U$, $\exists\ \epsilon > 0$ st. $B_\epsilon(p) := \{x \in X | d(x,y) \leq \epsilon\}$ and *closed* if $X - U$ (the compliment set) is open.

- If we have $U \subset Y \subset X$, $(X,d)$ a metric space, for us in all applications $X = \mathbb{R}^n$. $U$ is open/closed in $(Y, d|_{Y\times Y}) \iff \exists\ V \subset X$ open/closed st. $U = V \cap Y$.

## 1.3 Continuity

- If we have $f : X \to Y$, with $X$ and $Y$ metric spaces, is *continuous* if, $f^{-1}(U)$ is open with $U \subset Y$ is open.

  If $f : X \to Y$ is a bijection, continuous and $f^{-1}$ continuous we call $f$ a homomorphism.

## 1.4 Compact

- $X$ is *compact* if every open cover has a finite subcover , i.e. $\forall\ \{U_\alpha\}_{\alpha\in I}$, $U_\alpha \subset X$ ($U_\alpha$ open) st. $X \subset \bigcup_{\alpha\in I} U_\alpha$ , then $\exists\ \alpha_1, ..., \alpha_k \ \in\ I$ st. $X \subset U_{\alpha_1} \cup ... \cup U_{\alpha_k}$.

## 1.5 Heine Boral theorem

- $X \subset \mathbb{R}^n$ is compact if bounded ($\exists\ R \in \mathbb{R}$ st $X \subset B_R(0)$ ) and closed in $\mathbb{R}^n$.

## 1.6 Differentiation

- $f : U \to V$, $(U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^m)$ is differentiable at $p \in U$ with *derivative* $Df(p) \in Mat(m,n)$ if:

$$\boxed{\lim_{x\to p} \frac{f(x) - f(p) - Df(p)(x-p)}{\|x-p\|} = 0} \tag{1.2}$$

- $f$ is (of class) $C^1$ if it is differentiable at all $p \in U$ and $Df : U \to Mat(m,n) \cong \mathbb{R}^{mn}$ is continuous.
- $f$ is $C^r$ if $Df$ is $C^{r-1}$, $f$ is *smooth* or $C^\infty$ if it is $C^t$ $\forall\, t > 0$.
- If we have $f : U \to \mathbb{R}^m$ , $(U \in \mathbb{R}^n)$. Then $x \mapsto (f_1(x), ..., f_m(x))$ is $C^r$, if:

$$\boxed{\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f_j : U \to \mathbb{R}} \tag{1.3}$$

  Exists, and is continuous for all $k \in \{1, ..., r\}, i_1, ..., i_k \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. In which case the derivative can then be expressed as:

$$\boxed{Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n} \end{pmatrix}} \tag{1.4}$$

## 1.7 Chain rule

- Consider $U \xrightarrow{g} V \xrightarrow{f} W$, where $f$ and $g$ are differentiable (or $C^r$), then so is $f \circ g$ and:

$$\boxed{D(f \circ g)(x) = Df(g(x)) \cdot Dg(x)} \tag{1.5}$$

  This is the chain rule and the $\cdot$ here refers to matrix multiplication.

## 1.8 Diffeomorphism

- If we have $f : U \to V$ , a smooth bijection and $U, V$ open (in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively) st. $f^{-1}V \to U$ exists and is also smooth. Then we call $f$ a diffeomorphism.

## 1.9 Inverse function Theorem

- Let $f : V \to \mathbb{R}^n$ be $C^r$ $(1 \leq r \leq \infty\ )$ and $V \subset \mathbb{R}^n$. For $p \in V$ , suppose $Df(p)$ is non-singular (i.e an invertible $n \times n$ matrix $\iff$ $det(Df) \neq 0$). Then $\exists\ p \in U \subset V$ , $U$ open, st,
  - $f|_U : U \to f(U)$ , is a $C^r$-diffeomorphism. i.e. $f|_U : U \to f(U)$ is a bijection
  - $f(U)$ is open
  - $f^{-1}|_U : U \to f(U)$ is $C^r$.

# 2 Manifolds

- ”Slogan” (informal definition) $M \subset \mathbb{R}^n$ is a manifold if it is ”smooth” without corners/intersections

## 2.1 Manifolds

- Let $d > 0$, $M \subset \mathbb{R}^n$ is a smooth/$C^r$ manifold of dimension $d$ if $\forall\, p \in M$, $\exists\, p \in V \subset M$ , $U \subset \mathbb{R}^d$ ($V$ and $U$ open) and $\alpha : U \to V$, st:
  - $\alpha$ is smooth/$C^r$
  - $\alpha$ is a bijection with a continuous inverse ( $\iff$ is a homomorphisim)
  - $D\alpha(x)$ has Rank $d$.

  We will see this means $\alpha$ is a diffeomorphism.

### 2.1.1 Parameterised manifold

- Sometimes a only a single function $\alpha : U \to M$ is needed in the definition of a manifold. In this case we call $(M, \alpha)$ a parameterized manifold.

  From now on we will only discuss smooth/$C^\infty$ manifolds

## 2.2 Alternate definitions

- If we have a set $M \subset \mathbb{R}^n$ , $d > 0$, $p \in M$. Then the following are equivalent:
  - $\exists\, p \in V \subset M$, $U \subset \mathbb{R}^d$, ($V$ and $U$ open), $\alpha : U \to V$ a smooth homomorphism, st, $D\alpha(x)$ has rank $d$, $\forall\, x \in U$.
  - $\exists\, p \in V \subset \mathbb{R}^n$ , $U \subset \mathbb{R}^n$ ($V$ and $U$ open), $\beta : U \to V$ a diffeomorphism and $\beta(U \cap (\mathbb{R}^d \times \{0\})) = V \cap M$.
- This second definition is new and the set $U \cap (\mathbb{R}^d \times \{0\})$ is just the intersection of $U \subset \mathbb{R}^n$ and the space $\mathbb{R}^d$ extended into $\mathbb{R}^n$ by adding 0 to the $d$ dimensional tuples $n - d$ times until they become $\mathbb{R}^n$. This is effectively saying we want to be able to straighten out manifold neighbourhoods.

## 2.3 Locally smooth

- We want to say a $d$- manifold looks locally like $\mathbb{R}^d$.
- Let $M \subset \mathbb{R}^n$ , $N \subset \mathbb{R}^m$ be subsets. A function $f : M \to N$ is smooth if $\exists\, M \subset V \subset \mathbb{R}^n$ ($V$ open) and $\tilde{f} : V \to \mathbb{R}^m$ smooth st, $f|_M = f$ and $f : M \to N$ is a diffeomorphism (It is a smooth bijection and has a smooth inverse). Note that we do not require $\tilde{f}$ to have $\tilde{f}^{-1} \circ \tilde{f} = \mathbb{I}$.

  It follows that we can say: $f : M \to N$ a diffeomorphism, $A \subset M \implies f|_A : A \to f(A)$ is a diffeomorphism.
  - Remark These two facts are used in the proof of the following theorem. This theorem looks exactly like the definition of a manifold but note the swapping of $V$ and $U$, which changes the statement to that the condition for a manifold is that there is a smooth mapping from the manifold to $\mathbb{R}^d$.

### 2.3.1 Local smoothness definition of a manifold

- Let $d > 0$, $M \subset \mathbb{R}^n$ is a smooth/$C^r$ manifold of dimension $d$ if $\forall\, p \in M,\ \exists\, p \in V \subset M\ ,\ U \subset \mathbb{R}^d$ ($V$ and $U$ open) and $\alpha : V \to U$, st $\alpha$ is a diffeomorphism.

# 3 Partitions of unity

## 3.1 Main idea

- Given $\{U_i\}_{i\in I}$ a partition of unity is a collection of smooth functions $\{\psi_i\}$, $\psi_i : \mathbb{R}^n \to [0,\infty)$ a diffeomorphism st $\{x|\psi_i(x) \neq 0\} \subset U_i$ st $\sum_{i\in I} \psi_i(x) = 1$.
- We have a local definition of a manifold and we want to extend it so that we have one single function smooth across all of $M$.

## 3.2 Theorem

- Let $\mathbb{R} \supset V = \bigcup_{\alpha\in A} U_\alpha$ , where $U_\alpha$ are open, then there exists $\phi_1, \phi_2, \ldots \quad V \to [0,1]$, st:
  - For each $i \in \mathbb{N} \ \ \exists\ \alpha \in A$ st $S_i := \mathrm{supp}(\phi_i) = \overline{\{x \in V|\psi_i(x) \neq 0\}} \subset U_\alpha$
  - Each $p \in A$ has a neighbourhood intersecting finitely many $S_i$'s.
  - $\sum_{i\in I} \psi_i(x) = 1$, $\forall\ x \in V$.
  - $S_i$'s are compact.
  - $\psi_i$ are smooth.

  $\{\psi_i\}$ is called a partition of unity subordinate to $\{U_\alpha\}$.

## 3.3 Lemma 1

- $\{U_\alpha\}$ as above, then $\exists\ p_1, p_2, \ldots \in \mathbb{R}^n, \epsilon_1, \epsilon_2, \ldots \in \mathbb{R}_{>0}$ st:
  - $\bigcup_{i=1} B_{\epsilon_i}(p_i) = V$
  - Each $B_{2\epsilon_i}(p_i)$ is contained in a $U_\alpha$.
  - Each point $p \in V$ has a neighbourhood intersecting finitely many $B_{2\epsilon_i}(p_i)$.

## 3.4 Sub-lemma

- One can find $k_1 \subset k_2 \subset \ldots \subset V$ st:
  - $k_i$ are compact.
  - $k_i \subset \mathring{k}_{i+1}$
  - $\bigcup_{i=1} k_i = V$

## 3.5 Lemma 2

- Let $p \in \mathbb{R}^n, \epsilon > 0$ Then $\exists\ \psi : \mathbb{R}^n \to [0,1]$ st:
  - $\psi$ smooth.
  - $\mathrm{supp}(\psi) \subset B_{2\epsilon}(p)$
  - $\psi > 0$, on $B_\epsilon(p)$

## 3.6 Extension of locally smooth functions

- Let $M \subset \mathbb{R}^n$ a subset , $f : M \to \mathbb{R}^n$. Suppose $f$ is locally smooth , i.e $\forall\ p \in M \exists\ p \in V \subset M$, st $f|_V : V \to \mathbb{R}^m$ is smooth, Then $f$ is smooth on $M$.

  This theorem is proved using partitions of unity.

# 4 Boundary of manifolds

## 4.1 Upper half plane

- We define the upper have plane in $\mathbb{R}^d$ to be: $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$, that is:

$$\boxed{\mathbb{H} = \{(x_1, x_2, ..., x_d) | x_d \geq 0\}} \tag{4.1}$$

- The boundary of this plane is then defined as: $\partial\mathbb{H} := \mathbb{R}^{d-1} \times 0 \subset \mathbb{H}$. We then have that $\mathring{\mathbb{H}} := \mathbb{H} \backslash \partial\mathbb{H}$.

## 4.2 Boundary of a manifold

- A subset $M \subset \mathbb{R}^n$ is a $d$-manifold with a boundary if it is basically diffeomorphic to open subsets of $\mathbb{H}^d$ . that is $\forall\ p \in M \exists\ p \in V \subset M$ , $U \subset \mathbb{H}^d$ ($V$ and $U$ open) and $\alpha : U \to V$ a diffeomorphism.

### 4.2.1 Proposition

- The condition is equivalent to $\alpha$ being a smooth homomorphism and $D\alpha(x)$ being of rank $d\ \forall\ x \in U$.

## 4.3 Lemma

- If we have $\mathbb{H}^d \supset U \xrightarrow{\alpha} \mathbb{R}^n$ smooth with extensions $\tilde{\alpha} : \tilde{U} \to \mathbb{R}^n$ , (here $U \subset \tilde{U}$). Then $D\tilde{\alpha}(x)\ \forall\ x \in U$ does not depend on the extension.
  - For $x \in \mathring{\mathbb{H}}^d$, $\quad D\tilde{\alpha}(x) = D\alpha(x)$
  - For $x \in \partial\mathbb{H}^d \cap U$, $\quad D\tilde{\alpha}(x) = \left(\frac{\partial\tilde{\alpha}_i(x)}{\partial x_j}\right)_{i,j}$. Where for $j \neq d$ this derivative is defined in the normal way, but for $j = d$, instead of having a two sided limit in the definition we use a one sided limit,from the side that is in the half-plane.

$$\boxed{\frac{\partial\tilde{\alpha}_i(x)}{\partial x_d} = \lim_{\epsilon \to 0^+} \frac{\tilde{\alpha}_i(x + \epsilon e_d) - \tilde{\alpha}_i(x)}{\epsilon}} \tag{4.2}$$

## 4.4 Change of co-ordinates transformation

- Let $M$ be a manifold with a boundary and $\alpha_i : V_i \to U_i$, $i = 1, 2$, two co-ordinate patches. Then $\alpha_2^{-1} \circ \alpha_i : \alpha_1^{-1}(U_1 \cap U_2) \to \alpha_2^{-1}(U_1 \cap U_2)$ is a diffeomorphism.

  This is essentially saying we should be able to map smoothly between the pre-images of the co-ordinate patches that map to the same part of the manifold.

## 4.5 Interior and Boundary points

- Let $M$ be a manifold with a boundary.

- We call $p \in M$ an interior point if $\exists\ \alpha : U \to V$ a co-ordinate patch, st, $p = \alpha(x),\ \forall x \in \mathring{\mathbb{H}}^d \cap U$. Then we can define:

$$\boxed{\mathring{M} = \{x \in M | x \text{ is an interior point }\}} \tag{4.3}$$

- We call $p \in M$ an boundary point if $\exists\ \alpha : U \to V$ a co-ordinate patch, st, $p = \alpha(x),\ \forall x \in \partial\mathbb{H}^d \cap U$.

$$\boxed{\partial M = \{x \in M | x \text{ is an boundary point }\}} \tag{4.4}$$

- Warning: These definitions are not the same as in topology. $\mathring{M}$ is not equal to the topological interior of $M$ in $\mathbb{R}^n$ and the same for $\partial M$.

## 4.6 Boundary of manifold is manifold

- let $M$ be a $d$-manifold with a boundary. Then $\partial M$ is a $(d-1)$-manifold with a boundary.

## 4.7 Lemma

- $M = \mathring{M} \sqcup \partial M$.

  (Disjoint union, a union with the additional information that the sets don't have any elements in common).

## 4.8 Manifolds from functions

- Let $f : \mathbb{R}^n \supset U \to \mathbb{R}$ be a smooth function ($U$ open), we define:

$$\boxed{M = \{x \in U | f(x) = 0\} = f^{-1}(\{0\}).} \tag{4.5}$$

  And:

$$\boxed{N = \{x \in U | f(x) \geq 0\} = f^{-1}([0, \infty)).} \tag{4.6}$$

  Suppose that $\forall\ x \in M,\ \ Df(x)$ has rank 1, i.e. $Df(x) \neq 0$, then $N$ is a manifold with boundary $\partial N = M$.

# 5 Tangent spaces

## 5.1 Tangent spaces

- Let $M \in \mathbb{R}^n$ be a manifold with a boundary, $p \in M$, $\alpha : U \to V$ be a chart around $p$, $x_0 \in U$ be st $\alpha(x_0) = p$. The *tangent space* of $M$ at $p$ is:

$$\boxed{T_pM := \text{Image}(D\alpha(x_0)) \subset \mathbb{R}^n} \tag{5.1}$$

### 5.1.1 Lemma

- This definition does not depend on $\alpha$.

## 5.2 Maps Between tangent spaces

- Let $M, N$ be manifolds with boundaries and $f : M \to N$ a smooth map. Then $Df(p) = D\tilde{f}(p)$, for some extension $\tilde{f}$ of $f$, defines a linear map $D\tilde{f}(p) : T_pM \to T_{f(p)}N$ for all $p \in M$.

## 5.3 Tangent Bundle

- Let $m \subset \mathbb{R}^n$ be a manifold with a boundary. then the *Tangent Bundle* of $M$ is defined as the disjoint union of all the tangent spaces: $TM = \bigsqcup_{p\in M} T_pM$. i.e.:

$$\boxed{TM = \{(x, v) \in M \times \mathbb{R}^n | v \in T_xM\}} \tag{5.2}$$

  We then have that:

  - $TM$ is a $2d$-manifold with a boundary.
  - $f : M \to N$ smooth $\implies$ $\tilde{f}(p) : TM \to TN$ i.e. $(p.v) \to (f(p), Df(p)v)$ is smooth.
  - if we have $M \xrightarrow{f} N \xrightarrow{g} L$ smooth $\implies$ $D(g \circ f) = Dg \circ Df$ , (chain rule).

## 5.4 Regular and Critical values

- let $f : M \to N$ be smooth, we say $p \in N$ is a *regular value* if $Df(p) : T_pM \to T_{f(p)}N$ is *onto* (surjective) $\forall\ x \in f^{-1}(\{p\})$, otherwise we call $p$ a *critical point.*

### 5.4.1 Regular value manifold

- If we have $f : M \to N$ be smooth, $\partial M = \varnothing = \partial N$ and $p \in N$ a regular value. Then $L = f^{-1}(\{p\})$ is a manifold. Moreover, $T_xL = \ker(Df(x) : T_xM \to T_pN)$.
  - Remark: We can find cases where this doesn't work. For example for $f(x, y, z) = z - xy \geq 0$, 0 is a regular point ($Df = (-y, -x, 1)$) but the corresponding $L = f^{-1}(\{p\})$ is not a manifold with a boundary as $\partial L$ does not have 0 as a regular point for $\partial L = (z - xy, z) : \mathbb{R}^3 \to \mathbb{R}^2$ . To fix this we just have to restrict the boundary of $L$ to $\partial L = f^{-1}(\{0\}) \cap \partial M$.

### 5.5 Sard's Theorem

- Let $f : M \to N$ be smooth. Then the set of critical values $\mathrm{crit}(f) \subset N$ has "*measure zero*". In particular $\{p \in N | p \text{ regular value of f}\}$ is dense in $N$.

# 6 Multi-linear Algebra

- let $V$ be a vector space. A function $T : V^k \to \mathbb{R}$ is called multi-linear, (or a $k$ tensor), if for $v_1, ..., v_i, ..., v_k \in T$ , the function $v \to T(v_1, ..., v_{i-1}, v, v_{i+1}, ..., v_k)$ is linear. This just means it is leaner in each variable.
- The space of all such functions is denoted $\mathcal{L}^k(V)$, so:

$$\boxed{\mathcal{L}^k(V) := \{T : V^k \to \mathbb{R} | \; T \text{ multilinear}\}} \tag{6.1}$$

Usually we denote $\mathcal{L}^1(V) = V^*$ and $\mathcal{L}^0(V) = \{0\}$. We can then show that $\mathcal{L}^k(V)$ is a vector space with $(\lambda f + g)(v_1, ..., v_k) = \lambda f(v_1, ..., v_k) = g(v_1, ..., v_k)$, $\lambda \in \mathbb{R}$.

## 6.1 Basis vectors

- Let $e_i$ be a basis of $V$, we define $e^j \in V^*$, via $e^j v_i = e^j \sum_i a_i e_i = a_j$. These form what More generally for $I = (i_1, ..., i_k)$, we defined $e^I(v_1, ..., v_k) = e^{i_1}(v_1) \cdot \; \cdots \; \cdot e^{i_k}(v_k)$.
- The set $\{e^I\}, I \in \{1, ..., d\}^k$ forms a basis of $\mathcal{L}^k(V)$. In the particular $\mathrm{dim}\mathcal{L}^\mathrm{k}(\mathrm{V}) = (\mathrm{dimV})^\mathrm{k}$.

## 6.2 Tensor product

- let $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^l(V)$, we define the following operation $f \otimes g \in \mathcal{L}^{k+l}(V)$, by:

$$\boxed{f \otimes g(v_1, ..., v_{k+l}) = f(v_1, ..., v_k) \cdot g(v_{k+1}, ..., v_{k+l})} \tag{6.2}$$

This is the *tensor product* and has the following properties. let $f, g$ and $h$ be tensors, then:

$$\boxed{\begin{gathered} f \otimes (g \otimes h) = (f \otimes g) \otimes h \\ (\lambda f) \otimes g = \lambda (f \otimes g) = f \otimes (\lambda g) \\ (f + g) \otimes h = f \otimes h + g \otimes h, \quad h \otimes (f + g) = h \otimes f + h \otimes g \\ e^I = e^{i_1} \otimes \cdots \otimes e^{i_k} \end{gathered}} \tag{6.3}$$

## 6.3 Dual transformation

- Let $A : V \to W$ be a linear map. We define the dual transformation, $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ by :

$$\boxed{(A^* f)(v_1, ..., v_k) = f(Av_1, ..., Av_k)} \tag{6.4}$$

- It can then be shown that the dual transformation has the following properties:

$$\boxed{\begin{aligned} &A^* \text{ is linear} \\ &A^*(f \otimes g) = A^* f \otimes A^* g \\ &(A \cdot B)^* = B^* \cdot A^* \end{aligned}} \tag{6.5}$$

# 7 Alternating Tensors

## 7.1 Symmetric/Alternating tensors

- A tensor $f \in \mathcal{L}^k(V)$ is called:
  - *symmetric* if $f(v_1, ..., v_i, v_{i+1}, ..., v_k) = f(v_1, ..., v_{i+1}, v_i, ..., v_k)$.
  - *alternating* if $f(v_1, ..., v_i, v_{i+1}, ..., v_k) = -f(v_1, ..., v_{i+1}, v_i, ..., v_k)$ We let $S^kV$ and $\mathcal{A}^kV$ denote the vector space of symmetric/alternating respectively.

## 7.2 Symmetric group

- The permutation or symmetric group is defined as:

$$\boxed{S_k = \{\sigma : \{1, ..., k\} \to \{1, ..., k\} | \sigma \quad \text{a bijection}\}} \tag{7.1}$$

Since it is a group it also follows that for $\sigma, \tau \in S_k$, then $\sigma \circ \tau, \sigma^{-1} \in S_k$

### 7.2.1 Elementary permutation

- An elementary permutation is defined as:

$$\boxed{e_i(l) = \begin{cases} i+1, & l = i \\ i, & l = i+1 \\ l, & \text{otherwise} \end{cases}} \tag{7.2}$$

### 7.2.2 Lemma

- Every $\sigma$ is a composite of the elementary permutations $e_i$.

## 7.3 Sign function

- There exists a function, $\mathrm{sgn} : \mathrm{S_n} \to \{\pm 1\}$ st:
  - $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)$
  - $\mathrm{sgn}(\mathrm{e_i}) = -1$
  - $\mathrm{sgn}(\sigma) = (-1)^{\mathrm{m}}$, if $\sigma$ is made of $m$ elementary permutations.
  - $\mathrm{sgn}(\sigma^{-1}) = \mathrm{sgn}(\sigma)$
  - $\mathrm{sgn}(\sigma) = -1$, if $\sigma$ keeps $p \neq q$ fixed and keeps everything else fixed.

Moreover, the first and second property here uniquely determine sgn.

## 7.4 Permutation of tensors

- If we have $f \in \mathcal{L}^k(V)$, $\sigma \in S_k$, then we can define the following:

$$\boxed{f^\sigma(v_1, ..., v_k) := f(v_{\sigma(1)}, ..., v_{\sigma(k)})} \tag{7.3}$$

### 7.4.1 Lemma

- $\mathcal{L}^k(V)$ is a linear $S_k$-representation if for $f \in \mathcal{L}^k(V)$, $f^{\sigma\tau} = (f^\tau)^\sigma$, $f^\sigma = f$ (i.e. it is symmetric) and $f \mapsto f^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

## 7.5 Sgn definition of tensors

- For $f \in \mathcal{L}^k(V)$ , $\sigma \in S_k$
  - $f$ is symmetric iff $f^\sigma = f$, $\forall\ \sigma \in S_k$
  - $f$ is alternating iff $f^\sigma = \mathrm{sgn}(\sigma)\mathrm{f}$, $\forall\ \sigma \in S_k$

### 7.5.1 Lemma

- $f \in \mathcal{L}^k(V)$ is alternating iff $f(v_1, ..., v_k) = 0$, whenever $v_i = v_j$, for some $i \neq j$.

### 7.5.2 Lemma

- Let $f \in \mathcal{A}^k(V)$ and suppose $f(e_{i_1}, ..., e_{i_k}) = 0$, $\forall\ (i_1 \leq \cdots \leq i_k)$, then $f = 0$.

## 7.6 Alternating Tensor

- Let $I = (i_1 \leq ... \leq i_k)$, we define a unique k-tensor $\psi_I$ as:

$$\boxed{\psi_I = \sum_\sigma \mathrm{sgn}(\sigma)(\mathrm{e^I})^\sigma} \tag{7.4}$$

  This acts on a set of basis vectors $e_{j_1}, ..e_{j_k}$ as follows:

$$\boxed{\psi_I(e_{j_1}, ..e_{j_k}) = \begin{cases} 1, & \text{if } (\mathrm{j_1}, ..., \mathrm{j_k}) = (\mathrm{i_1}, ..., \mathrm{i_k}) \\ 0, & \text{otherwise} \end{cases}} \tag{7.5}$$

  This is because $(e^I)(e_{j_1}, ..e_{j_k})$ is defined to act by: $(e^{i_1}e_{j_1})(e^{i_2}e_{j_2}) \cdots (e^{i_1}e_{j_1})$.

## 7.7 Basis of Alternating tensors

- $\{\psi_I\}$, with $I$ ascending, form a basis for $\mathcal{A}^k(V)$. In particular $\dim\mathcal{A}^\mathrm{k}(\mathrm{V}) = \binom{n}{k}$, where $n = \dim\mathrm{V}$.

- This is because we can write any alternating tensor $f \in \mathcal{A}^k(V)$ in terms of these tensors. Consider $g = \sum_J d_J \psi_J$, where $d_J = f(e_{j_1}, ..., e_{j_k})$ (output of a tensor so just a scalar) and $J$ is all ascending indices of order $k$. Then the action of this new function $g$ on the basis vectors $g(e_{i_1}, ..., e_{i_k}) = d_I \cdot (1) = f(e_{i_1}, ..., e_{i_k})$, so we can say that $g = f$ and thus any alternating tensor $f$ can be expanded over $\psi_I$.
- It can also be noted that if $k = \mathrm{dimV} \implies \mathrm{dim}\mathcal{A}^{\mathrm{k}}(\mathrm{V}) = 1$.

- This allows us to write:

$$\boxed{\mathcal{A}^k(V) = \{\lambda\psi^{(1,2,...,n)} | \lambda \in \mathbb{R}\}} \tag{7.6}$$

## 7.8 Alternating Dual

- Let $B : V \to W$ be a linear transformation, If $f$ is an alternating tensor, then $B^*f$ is also an alternating tensor.

## 7.9 Alternating dual

- Let $B : V \to W$ be a linear map, then $B^*$ restricted to $B^* : \mathcal{A}^k(W) \to \mathcal{A}^k(V)$ , that is $B^*f$ is alternating if $f$ is.

### 7.9.1 Dual determinant

- For $B : V \to V$ and $k = \mathrm{dimV} = \mathrm{n}$, then we have that:

$$\boxed{B^*f = \mathrm{det(B)f}, \quad \mathrm{f} \in \mathcal{A}^{\mathrm{k}}(\mathrm{V})} \tag{7.7}$$

# 8 The wedge product

- The motivation behind this is that we would like to be able to combine alternating tensors in such a way so that the result is also an alternating tensor!

## 8.1 The Wedge product

- $\exists$ an operation $\mathcal{A}^k(V) \times \mathcal{A}^l(V) \to \mathcal{A}^{k+l}(V)$ $((f, g) \mapsto f \wedge g)$ satisfying the following:
  - $(f \wedge g) \wedge h = f \wedge (g \wedge h)$
  - $(f, g) \to f \wedge g$ is bilinear, $(f + \lambda g) \wedge h = f \wedge + \lambda(g \wedge h)$
  - $f \wedge g = (-1)^k lg \wedge f$
  - $\psi^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$ for a basis $e_i$ of $V$, $I = (i_1 \leq ... \leq i_k)$.

  The wedge product is uniquely defined by these for properties, furthermore let $T : V \to W$ be a linear map, then:

$$\boxed{T^*(f \wedge g) = T^* f \wedge T^* g \quad \in \mathcal{A}^{k+l}(V)} \tag{8.1}$$

## 8.2 Alternating algebra

- The direct sum $\bigoplus_{k=0}^{\infty} \mathcal{A}^k(V)$ form and associative, graded (anti-symmetric ) commutative algebra, module in $V$ st: $e^{i_1} \wedge \cdots \wedge e^{i_k} = \psi^I$.

## 8.3 Form of the wedge product

- So far we have just said there exists a wedge product but what does it actually look like? To do this we have to define a specific operator:

### 8.3.1 Averaging operator

- This is $A : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ and acts by:

$$\boxed{Af := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \mathrm{f}^{\sigma}} \tag{8.2}$$

  This operator satisfies that:
  - $A$ is linear
  - $Af \in \mathcal{A}^k(V)$
  - If $f \in \mathcal{A}^k(V)$, then $Af = k!f$
- This then allows us to define the wedge product for $l \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^l(V)$:

$$\boxed{f \wedge g := \frac{1}{k!l!} A(f \otimes g)} \tag{8.3}$$

# 9 Differential forms

- Let $M \subset \mathbb{R}^n$ be a manifold with a boundary. A differential form of order/degree $k$ is a smooth function:

$$\boxed{\omega : \{(p, v_1, ..., v_k) \in M \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n | v_i \in T_pM\}} \tag{9.1}$$

st, $\forall\ p \in M,\ \ \omega_p = \omega(p) = \omega(p, v_1, ..., v_k) : T_pM \to \mathbb{R}$, is an alternating tensor. Thus we can also say that $\omega : M \to \bigsqcup_{q\in M} \mathcal{A}^k(T_qM)$, st: $\omega_p \in \mathcal{A}^k(T_pM)$.

## 9.1 Space of differential forms

- Let $\Omega^k(M)$ be defined as follows:

$$\boxed{\Omega^k(M) = \{\omega|\ \omega \text{ a smooth differential form of degree } k\}} \tag{9.2}$$

We can also define the similar $\Omega^K_\delta(M)$ as:

$$\boxed{\Omega^k_\delta(M) = \{\omega|\ \omega\ \text{ same as above but not necessarily smooth}\}} \tag{9.3}$$

We then have that: $\Omega^0(M) = C^\infty(M) = \{f : M \to N|\ f \text{ smooth}\}$.

- $\Omega^K(M)$ is a vector space under point-wise addition/ multiplication with scalars.

## 9.2 Basis $k$ forms

- Recalling that $e^j(x_1, x_2, ..., x_n) = x_j \in \mathbb{R}$, for $x \in \mathbb{R}^n$. If we look at the form of $\psi_I(x)$ in 7.4, and the definition of the wedge product in 8.3 then we can see that we can re-write $\psi_I$ as:

$$\boxed{\psi^I = e^{i_1} \wedge e^{i_2} \cdots \wedge e^{i_k}} \tag{9.4}$$

And we end up denoting this:

$$\boxed{\psi^I = dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k} = dx^I} \tag{9.5}$$

These $\psi_I$ are called elementary $k$ forms (since each $e^{i_1}$ is a 1-form). It is also worth noticing that:

$$\boxed{dx^I(v_1, v_2, ..., v_k) = \det([\mathrm{v}_1, \mathrm{v}_2 \cdots \mathrm{v}_\mathrm{k}])} \tag{9.6}$$

- We then have that each $k$ tensor $\omega(p)$ can be written uniquely in the form:

$$\boxed{\omega(p) = \sum_{[I]} b_I(p) dx^I(p)} \tag{9.7}$$

Where $[I]$ denotes any increasing sequence, that means no repeating index's, but it does not have to be of length $k$. If each $b_I(p)$ is smooth, then so is $\omega$.

## 9.3 Pullback

- Let $f : M \to N$ be smooth, we define $f^* : \Omega^k(N) \to \Omega^k(M)$ as:

$$\boxed{(f^*\omega)(p, v_1, ..., v_k) := \omega(f(p), Df_p v_1, Df_p v_2, ..., Df_p v_k)} \tag{9.8}$$

$f^*$ is a well defined linear map. We may also write $(f^*\omega)_p \in A^k(T_pM)$. With these definitions it can be proven that:

  - $\mathrm{id}^*_{\mathrm{m}} = \mathrm{id}_{\Omega^*(\mathrm{M})}$
  - $(f \circ g)^* = g^* \circ f^*$ For $L \xleftarrow{f} N \xleftarrow{g} M$ smooth. This map is called a *pullback*.

### 9.3.1 Smoothness condition

- An element $\omega \in \Omega^k_\delta(M)$ is smooth if $\forall\ p \in M,\ \ \exists\ \alpha : U \to V \subset M$ a co-ord patch around $p$ st, $\alpha^*\omega \in \Omega^k_\delta(U)$ is smooth.

### 9.3.2 Differential form of wedge

- Let $\omega \in \Omega^k(M),\ \eta \in \Omega^l(M)$, we define $\omega \wedge \eta \in \Omega^{k+l}(M)$ by:

$$\boxed{(\omega \wedge \eta)_p = \omega_p \wedge \eta_p} \tag{9.9}$$

- It can also be shown that if we have $f : M \to N$ smooth, then $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

### 9.3.3 Conclusion

- $\Omega^\bullet(M) := \bigoplus_{k=0}^\infty \Omega^k(M)$ is a *graded-commutative* (*anti-commutative*) *associative* algebra structure, with $\Omega^0(M) = C^\infty(M)$ (the set of a all smooth 1-d functions on $M$).
- It also has that if $f : M \to N$, then $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$, preserves the above structure.

  It is also worth noting that $\Omega^k(M) = 0$ for $k > \mathrm{dimV}$ as then there are more elements in the sequence $i_1, ..., i_k$, then there are dimensions, so $dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k}$, must have a repeating index, making it 0 and thus each $\omega$ is also 0.

# 10 Exterior derivative

- Let $M$ be a manifold with a boundary, $\exists$ a unique linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined for all $k \geq 0$ st:
  - If $f \in C^\infty(M) = \Omega^0(M)$ then $df_p(v) = Df(p)v$
  - If $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
  - $d(d\omega) = 0$, more over if $F : M \to N$ is smooth then $d(F^*\omega) = F^* d\omega$

### 10.0.1 Exterior Derivative of k-forms

- We know how this derivative acts on 0-forms based on the first property, but how does it act on a $k$-form $\omega$? To find this we can just look at our expression of $k$-forms in terms of the elementary $k$-forms in 9.7. With this expression $d\omega$ is defined as:

$$\boxed{\begin{aligned} d\omega &= d\sum_{[I]} b_I dx^I = \sum_{[I]} db_I \wedge dx^I + \sum_{[I]} b_I d^2 x^I \\ &= \sum_{[I]} db_I \wedge dx^I \end{aligned}} \tag{10.1}$$

  Where we have used the property that $d^2 = 0$.

- One can also use this expression to show that $d^2\omega = 0$ as since $b_I$ is a 0 form, $d^2 b_I = \sum_{i,j=1}^n D_i D_j b_I$, i.e. all second order partial derivatives, but since $b_I$, must be a smooth function $D_i D_j b_I = D_j D_i b_I$, so since $d^\omega = \sum_I \sum_{i,j=1}^n D_i D_j b_I dx^I = \sum_I \sum_{i>j} (D_i D_j b_I - D_j D_i b_I) dx^I = 0$ , as swapping $dx^{i_i}$ and $dx^{i_j}$, picks up a minus sign.

### 10.0.2 Alternate definition

- Alternatively if we have $\alpha : U \to M$, be a patch around $p$. we can define $d\omega$ as:

$$\boxed{(d\omega)_p := ((\alpha^{-1})^* d_U(\alpha^*\omega))_p} \tag{10.2}$$

  This is well defined and independent of the choice of $\alpha$.

## 10.1 Naturality

- Let $F : U \to V$ be smooth, $\omega \in \Omega^k(V)$ . Then:

$$\boxed{F^* d\omega = d(F^*\omega)} \tag{10.3}$$

# 11 Vector Fields

- A vector field is a smooth function $X : M \to TM$, st. $X(p) \in T_pM$. We then define the set of all vector fields on our manifold $M$:

$$\boxed{\mathfrak{X}(M) := \{X : M \to TM | X \text{ is a vector field}\}} \tag{11.1}$$