Calculus on Manifolds

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Notes taken in Professor Florian Naef's class, Hilary Term 2024

" $p \in N$ is "

-Florian

Contents

1 Topology on $\mathbb{R}^n$

1.1 Metric space

- Let X be a set, A *metric* on a set is a function that measures distances $d : X \times X \to \mathbb{R}$. It has the following properties:

$$\boxed{\begin{aligned} &d(x, y) = d(y, x) \\ &d(x, y) \geq 0 \\ &d(x, y) = 0 \ \text{ iff } \ x = y \\ &d(x, z) \leq d(x, y) + d(y, z) \end{aligned}} \tag{1.1}$$

  (X, d) together make a *metric space*.

- Any subset $Y \subset X$ is itself a metric space with $d(x, y)\Big|_{Y\times Y}$ (restricted to Y).

1.2 Open/Closed

- Let (X, d) be a metric space $U \subset X$ is *open* if $\forall\ p \in U$, $\exists\ \epsilon > 0$ st. $B_\epsilon(p) := \{x \in X | d(x, y) \leq \epsilon\}$ and *closed* if $X - U$ (the compliment set) is open.

- If we have $U \subset Y \subset X$, (X, d) a metric space, for us in all applications $X = \mathbb{R}^n$. U is open/closed in $(Y, d|_{Y\times Y}) \iff \exists\ V \subset X$ open/closed st. $U = V \ \cap\ Y$.

1.3 Continuity

- If we have $f : X \to Y$, with X and Y metric spaces, is *continuous* if, $f^{-1}(U)$ is open with $U \subset Y$ is open.

  If $f : X \to Y$ is a bijection, continuous and f^{-1} continuous we call f a homomorphism.

1.4 Compact

- X is *compact* if every open cover has a finite subcover , i.e. $\forall\ \{U_\alpha\}_{\alpha\in I}$, $U_\alpha \subset X$ (U_α open) st. $X \subset \bigcup_{\alpha\in I} U_\alpha$, then $\exists\ \alpha_1, ..., \alpha_k \ \in\ I$ st. $X \subset U_{\alpha_1} \cup ... \cup U_{\alpha_k}$.

1.5 Heine Boral theorem

- $X \subset \mathbb{R}^n$ is compact if bounded ($\exists\ R \in \mathbb{R}$ st $X \subset B_R(0)$) and closed in $\mathbb{R}^n$.

1.6 Differentiation

- $f : U \to V$, $\ (U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^m)$ is differentiable at $p \in U$ with *derivative* $Df(p) \in Mat(m, n)$ if:

$$\boxed{\lim_{x\to p} \frac{f(x) - f(p) - Df(p)(x - p)}{\|x - p\|} = 0} \tag{1.2}$$

- f is (of class) C^1 if it is differentiable at all $p \in U$ and $Df : U \to Mat(m,n) \cong \mathbb{R}^{mn}$ is continuous.
- f is C^r if Df is C^{r-1}, f is *smooth* or C^∞ if it is C^t $\forall\, t > 0$.
- If we have $f : U \to \mathbb{R}^m$, $(U \in \mathbb{R}^n)$. Then $x \mapsto (f_1(x), ..., f_m(x))$ is C^r, if:

$$\boxed{\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f_j : U \to \mathbb{R}} \tag{1.3}$$

Exists, and is continuous for all $k \in \{1, ..., r\}, i_1, ..., i_k \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. In which case the derivative can then be expressed as:

$$\boxed{Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n} \end{pmatrix}} \tag{1.4}$$

1.7 Chain rule

- Consider $U \xrightarrow{g} V \xrightarrow{f} W$, where f and g are differentiable (or C^r), then so is $f \circ g$ and:

$$\boxed{D(f \circ g)(x) = Df(g(x)) \cdot Dg(x)} \tag{1.5}$$

This is the chain rule and the $\cdot$ here refers to matrix multiplication.

1.8 Diffeomorphism

- If we have $f : U \to V$, a smooth bijection and U, V open (in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively) st. $f^{-1} V \to U$ exists and is also smooth. Then we call f a diffeomorphism.

1.9 Inverse function Theorem

- Let $f : V \to \mathbb{R}^n$ be C^r $(1 \leq r \leq \infty$) and $V \subset \mathbb{R}^n$. For $p \in V$, suppose $Df(p)$ is non-singular (i.e an invertible $n \times n$ matrix $\iff$ $det(Df) \neq 0$). Then $\exists\ p \in U \subset V$, U open, st,
  - $f|_U : U \to f(U)$, is a C^r-diffeomorphism. i.e. $f|_U : U \to f(U)$ is a bijection
  - $f(U)$ is open
  - $f^{-1}|_U : U \to f(U)$ is C^r.

2 Manifolds

- "Slogan" (informal definition) $M \subset \mathbb{R}^n$ is a manifold if it is "smooth" without corners/intersections

2.1 Manifolds

- Let $d > 0$, $M \subset \mathbb{R}^n$ is a smooth/C^r manifold of dimension d if $\forall\, p \in M$, $\exists\, p \in V \subset M$, $U \subset \mathbb{R}^d$ (V and U open) and $\alpha : U \to V$, st:
  - α is smooth/C^r
  - α is a bijection with a continuous inverse ($\iff$ is a homomorphisim)
  - $D\alpha(x)$ has Rank d.

  We will see this means α is a diffeomorphism.

2.1.1 Parameterised manifold

- Sometimes a only a single function $\alpha : U \to M$ is needed in the definition of a manifold. In this case we call (M, α) a parameterized manifold.

  From now on we will only discuss smooth/C^∞ manifolds

2.2 Alternate definitions

- If we have a set $M \subset \mathbb{R}^n$, $d > 0$, $p \in M$. Then the following are equivalent:
  - $\exists\, p \in V \subset M$, $U \subset \mathbb{R}^d$, (V and U open), $\alpha : U \to V$ a smooth homomorphism, st, $D\alpha(x)$ has rank d, $\forall\, x \in U$.
  - $\exists\, p \in V \subset \mathbb{R}^n$, $U \subset \mathbb{R}^n$ (V and U open), $\beta : U \to V$ a diffeomorphism and $\beta(U \cap (\mathbb{R}^d \times \{0\})) = V \cap M$.
- This second definition is new and the set $U \cap (\mathbb{R}^d \times \{0\})$ is just the intersection of $U \subset \mathbb{R}^n$ and the space $\mathbb{R}^d$ extended into $\mathbb{R}^n$ by adding 0 to the d dimensional tuples $n - d$ times until they become $\mathbb{R}^n$. This is effectively saying we want to be able to straighten out manifold neighbourhoods.

2.3 Locally smooth

- We want to say a d- manifold looks locally like $\mathbb{R}^d$.
- Let $M \subset \mathbb{R}^n$, $N \subset \mathbb{R}^m$ be subsets. A function $f : M \to N$ is smooth if $\exists\, M \subset V \subset \mathbb{R}^n$ (V open) and $\tilde{f} : V \to \mathbb{R}^m$ smooth st, $f|_M = f$ and $f : M \to N$ is a diffeomorphism (It is a smooth bijection and has a smooth inverse). Note that we do not require $\tilde{f}$ to have $\tilde{f}^{-1} \circ \tilde{f} = \mathbb{I}$.

  It follows that we can say: $f : M \to N$ a diffeomorphism, $A \subset M \implies f|_A : A \to f(A)$ is a diffeomorphism.
  - Remark These two facts are used in the proof of the following theorem. This theorem looks exactly like the definition of a manifold but note the swapping of V and U, which changes the statement to that the condition for a manifold is that there is a smooth mapping from the manifold to $\mathbb{R}^d$.

2.3.1 Local smoothness definition of a manifold

- Let $d > 0$, $M \subset \mathbb{R}^n$ is a smooth/C^r manifold of dimension d if $\forall\, p \in M$, $\exists\, p \in V \subset M$, $U \subset \mathbb{R}^d$ (V and U open) and $\alpha : V \to U$, st α is a diffeomorphism.

3 Partitions of unity

3.1 Main idea

- Given $\{U_i\}_{i\in I}$ a partition of unity is a collection of smooth functions $\{\psi_i\}$, $\psi_i : \mathbb{R}^n \to [0,\infty)$ a diffeomorphism st $\{x|\psi_i(x) \neq 0\} \subset U_i$ st $\sum_{i\in I} \psi_i(x) = 1$.
- We have a local definition of a manifold and we want to extend it so that we have one single function smooth across all of M.

3.2 Theorem

- Let $\mathbb{R} \supset V = \bigcup_{\alpha\in A} U_\alpha$, where U_α are open, then there exists $\phi_1, \phi_2, \ldots \quad V \to [0,1]$, st:
  - For each $i \in \mathbb{N} \;\; \exists\; \alpha \in A$ st $S_i := \text{supp}(\phi_i) = \overline{\{x \in V|\psi_i(x) \neq 0\}} \subset U_\alpha$
  - Each $p \in A$ has a neighbourhood intersecting finitely many S_i's.
  - $\sum_{i\in I} \psi_i(x) = 1$, $\forall\; x \in V$.
  - S_i's are compact.
  - ψ_i are smooth.

  $\{\psi_i\}$ is called a partition of unity subordinate to $\{U_\alpha\}$.

3.3 Lemma 1

- $\{U_\alpha\}$ as above, then $\exists\; p_1, p_2, \ldots \in \mathbb{R}^n, \epsilon_1, \epsilon_2, \ldots \in \mathbb{R}_{>0}$ st:
  - $\bigcup_{i=1} B_{\epsilon_i}(p_i) = V$
  - Each $B_{2\epsilon_i}(p_i)$ is contained in a U_α.
  - Each point $p \in V$ has a neighbourhood intersecting finitely many $B_{2\epsilon_i}(p_i)$.

3.4 Sub-lemma

- One can find $k_1 \subset k_2 \subset \ldots \subset V$ st:
  - k_i are compact.
  - $k_i \subset \mathring{k}_{i+1}$
  - $\bigcup_{i=1} k_i = V$

3.5 Lemma 2

- Let $p \in \mathbb{R}^n, \epsilon > 0$ Then $\exists\; \psi : \mathbb{R}^n \to [0,1]$ st:
  - ψ smooth.
  - $\text{supp}(\psi) \subset B_{2\epsilon}(p)$
  - $\psi > 0$, on $B_\epsilon(p)$

3.6 Extension of locally smooth functions

- Let $M \subset \mathbb{R}^n$ a subset , $f : M \to \mathbb{R}^n$. Suppose f is locally smooth , i.e $\forall\ p \in M \exists\ p \in V \subset M$, st $f|_V : V \to \mathbb{R}^m$ is smooth, Then f is smooth on M.

  This theorem is proved using partitions of unity.

4 Boundary of manifolds

4.1 Upper half plane

- We define the upper have plane in $\mathbb{R}^d$ to be: $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$, that is:

$$\boxed{\mathbb{H} = \{(x_1, x_2, ..., x_d)|x_d \geq 0\}} \tag{4.1}$$

- The boundary of this plane is then defined as: $\partial\mathbb{H} := \mathbb{R}^{d-1} \times 0 \subset \mathbb{H}$. We then have that $\mathring{\mathbb{H}} := \mathbb{H} \backslash \partial\mathbb{H}$.

4.2 Boundary of a manifold

- A subset $M \subset \mathbb{R}^n$ is a d-manifold with a boundary if it is basically diffeomorphic to open subsets of $\mathbb{H}^d$. that is $\forall\, p \in M \exists\, p \in V \subset M$, $U \subset \mathbb{H}^d$ (V and U open) and $\alpha : U \to V$ a diffeomorphism.

4.2.1 Proposition

- The condition is equivalent to α being a smooth homomorphism and $D\alpha(x)$ being of rank $d\, \forall\, x \in U$.

4.3 Lemma

- If we have $\mathbb{H}^d \supset U \xrightarrow{\alpha} \mathbb{R}^n$ smooth with extensions $\tilde{\alpha} : \tilde{U} \to \mathbb{R}^n$, (here $U \subset \tilde{U}$). Then $D\tilde{\alpha}(x)\, \forall\, x \in U$ does not depend on the extension.
  - For $x \in \mathring{\mathbb{H}}^d$, $\quad D\tilde{\alpha}(x) = D\alpha(x)$
  - For $x \in \partial\mathbb{H}^d \cap U$, $\quad D\tilde{\alpha}(x) = \left(\frac{\partial\tilde{\alpha}_i(x)}{\partial x_j}\right)_{i,j}$. Where for $j \neq d$ this derivative is defined in the normal way, but for $j = d$, instead of having a two sided limit in the definition we use a one sided limit,from the side that is in the half-plane.

$$\boxed{\frac{\partial\tilde{\alpha}_i(x)}{\partial x_d} = \lim_{\epsilon \to 0^+} \frac{\tilde{\alpha}_i(x + \epsilon e_d) - \tilde{\alpha}_i(x)}{\epsilon}} \tag{4.2}$$

4.4 Change of co-ordinates transformation

- Let M be a manifold with a boundary and $\alpha_i : V_i \to U_i$, $i = 1, 2$, two co-ordinate patches. Then $\alpha_2^{-1} \circ \alpha_i : \alpha_1^{-1}(U_1 \cap U_2) \to \alpha_2^{-1}(U_1 \cap U_2)$ is a diffeomorphism.

  This is essentially saying we should be able to map smoothly between the pre-images of the co-ordinate patches that map to the same part of the manifold.

4.5 Interior and Boundary points

- Let M be a manifold with a boundary.

- We call $p \in M$ an interior point if $\exists\ \alpha : U \to V$ a co-ordinate patch, st, $p = \alpha(x),\ \forall x \in \mathring{\mathbb{H}}^d \cap U$. Then we can define:

$$\boxed{\mathring{M} = \{x \in M | x \text{ is an interior point }\}} \tag{4.3}$$

- We call $p \in M$ an boundary point if $\exists\ \alpha : U \to V$ a co-ordinate patch, st, $p = \alpha(x),\ \forall x \in \partial \mathbb{H}^d \cap U$.

$$\boxed{\partial M = \{x \in M | x \text{ is an boundary point }\}} \tag{4.4}$$

- Warning: These definitions are not the same as in topology. $\mathring{M}$ is not equal to the topological interior of M in $\mathbb{R}^n$ and the same for ∂M.

4.6 Boundary of manifold is manifold

- let M be a d-manifold with a boundary. Then ∂M is a $(d-1)$-manifold with a boundary.

4.7 Lemma

- $M = \mathring{M} \sqcup \partial M$.

  (Disjoint union, a union with the additional information that the sets don't have any elements in common).

4.8 Manifolds from functions

- Let $f : \mathbb{R}^n \supset U \to \mathbb{R}$ be a smooth function (U open), we define:

$$\boxed{M = \{x \in U | f(x) = 0\} = f^{-1}(\{0\}).} \tag{4.5}$$

  And:

$$\boxed{N = \{x \in U | f(x) \geq 0\} = f^{-1}([0, \infty)).} \tag{4.6}$$

  Suppose that $\forall\ x \in M,\quad Df(x)$ has rank 1, i.e. $Df(x) \neq 0$, then N is a manifold with boundary $\partial N = M$.

5 Tangent spaces

5.1 Tangent spaces

- Let $M \in \mathbb{R}^n$ be a manifold with a boundary, $p \in M$, $\alpha : U \to V$ be a chart around p, $x_0 \in U$ be st $\alpha(x_0) = p$. The *tangent space* of M at p is:

$$\boxed{T_pM := \text{Image}(D\alpha(x_0)) \subset \mathbb{R}^n} \tag{5.1}$$

5.1.1 Lemma

- This definition does not depend on α.

5.2 Maps Between tangent spaces

- Let M, N be manifolds with boundaries and $f : M \to N$ a smooth map. Then $Df(p) = D\tilde{f}(p)$, for some extension $\tilde{f}$ of f, defines a linear map $D\tilde{f}(p) : T_pM \to T_{f(p)}N$ for all $p \in M$.

5.3 Tangent Bundle

- Let $m \subset \mathbb{R}^n$ be a manifold with a boundary. then the *Tangent Bundle* of M is defined as the disjoint union of all the tangent spaces: $TM = \bigsqcup_{p\in M} T_pM$. i.e.:

$$\boxed{TM = \{(x, v) \in M \times \mathbb{R}^n | v \in T_xM\}} \tag{5.2}$$

  We then have that:

  - TM is a $2d$-manifold with a boundary.
  - $f : M \to N$ smooth $\implies$ $\tilde{f}(p) : TM \to TN$ i.e. $(p.v) \to (f(p), Df(p)v)$ is smooth.
  - if we have $M \xrightarrow{f} N \xrightarrow{g} L$ smooth $\implies$ $D(g \circ f) = Dg \circ Df$, (chain rule).

5.4 Regular and Critical values

- let $f : M \to N$ be smooth, we say $p \in N$ is a *regular value* if $Df(x) : T_xM \to T_pN$ is *onto* (surjective) $\forall\ x \in f^{-1}(\{p\})$, otherwise we call p a *critical point.*

5.4.1 Regular value manifold

- If we have $f : M \to N$ be smooth, $\partial M = \varnothing = \partial N$ and $p \in N$ a regular value. Then $L = f^{-1}(\{p\})$ is a manifold. Moreover, $T_xL = \ker(Df(x) : T_xM \to T_pN)$.
  - Remark: We can find cases where this doesn't work. For example for $f(x, y, z) = z - xy \geq 0$, 0 is a regular point ($Df = (-y, -x, 1)$) but the corresponding $L = f^{-1}(\{p\})$ is not a manifold with a boundary as ∂L does not have 0 as a regular point for $\partial L = (z - xy, z) : \mathbb{R}^3 \to \mathbb{R}^2$. To fix this we just have to restrict the boundary of L to $\partial L = f^{-1}(\{0\}) \cap \partial M$.

5.5 Sard's Theorem

- Let $f : M \to N$ be smooth. Then the set of critical values $\mathrm{crit}(f) \subset N$ has "*measure zero*". In particular $\{p \in N | p \text{ regular value of f}\}$ is dense in N.

6 Multi-linear Algebra

- let V be a vector space. A function $T : V^k \to \mathbb{R}$ is called multi-linear, (or a k tensor), if for $v_1, ..., v_i, ..., v_k \in T$, the function $v \to T(v_1, ..., v_{i-1}, v, v_{i+1}, ..., v_k)$ is linear. This just means it is leaner in each variable.
- The space of all such functions is denoted $\mathcal{L}^k(V)$, so:

$$\boxed{\mathcal{L}^k(V) := \{T : V^k \to \mathbb{R} |\ T \text{ multilinear}\}} \tag{6.1}$$

Usually we denote $\mathcal{L}^1(V) = V^*$ and $\mathcal{L}^0(V) = \{0\}$. We can then show that $\mathcal{L}^k(V)$ is a vector space with $(\lambda f + g)(v_1, ..., v_k) = \lambda f(v_1, ..., v_k) = g(v_1, ..., v_k)$, $\lambda \in \mathbb{R}$.

6.1 Basis vectors

- Let e_i be a basis of V, we define $e^j \in V^*$, via $e^j v_i = e^j \sum_i a_i e_i = a_j$. These form what More generally for $I = (i_1, ..., i_k)$, we defined $e^I(v_1, ..., v_k) = e^{i_1}(v_1) \cdot \ \cdots \ \cdot e^{i_k}(v_k)$.
- The set $\{e^I\}, I \in \{1, ..., d\}^k$ forms a basis of $\mathcal{L}^k(V)$. In the particular $\mathrm{dim}\mathcal{L}^{\mathrm{k}}(\mathrm{V}) = (\mathrm{dimV})^{\mathrm{k}}$.

6.2 Tensor product

- let $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^l(V)$, we define the following operation $f \otimes g \in \mathcal{L}^{k+l}(V)$, by:

$$\boxed{f \otimes g(v_1, ..., v_{k+l}) = f(v_1, ..., v_k) \cdot g(v_{k+1}, ..., v_{k+l})} \tag{6.2}$$

This is the *tensor product* and has the following properties. let f, g and h be tensors, then:

$$\boxed{\begin{gathered} f \otimes (g \otimes h) = (f \otimes g) \otimes h \\ (\lambda f) \otimes g = \lambda (f \otimes g) = f \otimes (\lambda g) \\ (f + g) \otimes h = f \otimes h + g \otimes h, \quad h \otimes (f + g) = h \otimes f + h \otimes g \\ e^I = e^{i_1} \otimes \cdots \otimes e^{i_k} \end{gathered}} \tag{6.3}$$

6.3 Dual transformation

- Let $A : V \to W$ be a linear map. We define the dual transformation, $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ by :

$$\boxed{(A^* f)(v_1, ..., v_k) = f(Av_1, ..., Av_k)} \tag{6.4}$$

- It can then be shown that the dual transformation has the following properties:

$$\boxed{\begin{aligned}&A^* \text{ is linear}\\&A^*(f \otimes g) = A^*f \otimes A^*g\\&(A \cdot B)^* = B^* \cdot A^*\end{aligned}} \tag{6.5}$$

7 Alternating Tensors

7.1 Symmetric/Alternating tensors

- A tensor $f \in \mathcal{L}^k(V)$ is called:
  - *symmetric* if $f(v_1, ..., v_i, v_{i+1}, ..., v_k) = f(v_1, ..., v_{i+1}, v_i, ..., v_k)$.
  - *alternating* if $f(v_1, ..., v_i, v_{i+1}, ..., v_k) = -f(v_1, ..., v_{i+1}, v_i, ..., v_k)$ We let S^kV and $\mathcal{A}^kV$ denote the vector space of symmetric/alternating respectively.

7.2 Symmetric group

- The permutation or symmetric group is defined as:

$$\boxed{S_k = \{\sigma : \{1, ..., k\} \to \{1, ..., k\} | \sigma \text{ a bijection}\}} \tag{7.1}$$

Since it is a group it also follows that for $\sigma, \tau \in S_k$, then $\sigma \circ \tau, \sigma^{-1} \in S_k$

7.2.1 Elementary permutation

- An elementary permutation is defined as:

$$\boxed{e_i(l) = \begin{cases} i+1, & l = i \\ i, & l = i+1 \\ l, & \text{otherwise} \end{cases}} \tag{7.2}$$

7.2.2 Lemma

- Every σ is a composite of the elementary permutations e_i.

7.3 Sign function

- There exists a function, $\mathrm{sgn} : \mathrm{S_n} \to \{\pm 1\}$ st:
  - $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)$
  - $\mathrm{sgn}(\mathrm{e_i}) = -1$
  - $\mathrm{sgn}(\sigma) = (-1)^{\mathrm{m}}$, if σ is made of m elementary permutations.
  - $\mathrm{sgn}(\sigma^{-1}) = \mathrm{sgn}(\sigma)$
  - $\mathrm{sgn}(\sigma) = -1$, if σ keeps $p \neq q$ fixed and keeps everything else fixed.

  Moreover, the first and second property here uniquely determine sgn.

7.4 Permutation of tensors

- If we have $f \in \mathcal{L}^k(V), \quad \sigma \in S_k$, then we can define the following:

$$\boxed{f^\sigma(v_1, ..., v_k) := f(v_{\sigma(1)}, ..., v_{\sigma(k)})} \tag{7.3}$$

7.4.1 Lemma

- $\mathcal{L}^k(V)$ is a linear S_k-representation if for $f \in \mathcal{L}^k(V)$, $f^{\sigma\tau} = (f^\tau)^\sigma$, $f^\sigma = f$ (i.e. it is symmetric) and $f \mapsto f^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

7.5 Sgn definition of tensors

- For $f \in \mathcal{L}^k(V)$, $\sigma \in S_k$
  - f is symmetric iff $f^\sigma = f, \forall\ \sigma \in S_k$
  - f is alternating iff $f^\sigma = \mathrm{sgn}(\sigma)\mathrm{f}, \forall\ \sigma \in S_k$

7.5.1 Lemma

- $f \in \mathcal{L}^k(V)$ is alternating iff $f(v_1, ..., v_k) = 0$, whenever $v_i = v_j$, for some $i \neq j$.

7.5.2 Lemma

- Let $f \in \mathcal{A}^k(V)$ and suppose $f(e_{i_1}, ..., e_{i_k}) = 0$, $\forall\ (i_1 \leq \cdots \leq i_k)$, then $f = 0$.

7.6 Alternating Tensor

- Let $I = (i_1 \leq ... \leq i_k)$, we define a unique k-tensor ψ_I as:

$$\boxed{\psi_I = \sum_\sigma \mathrm{sgn}(\sigma)(\mathrm{e^I})^\sigma} \tag{7.4}$$

This acts on a set of basis vectors $e_{j_1}, ..e_{j_k}$ as follows:

$$\boxed{\psi_I(e_{j_1}, ..e_{j_k}) = \begin{cases} 1, & \text{if } (\mathrm{j_1, ..., j_k}) = (\mathrm{i_1, ..., i_k}) \\ 0, & \text{otherwise} \end{cases}} \tag{7.5}$$

This is because $(e^I)(e_{j_1}, ..e_{j_k})$ is defined to act by: $(e^{i_1}e_{j_1})(e^{i_2}e_{j_2})\cdots(e^{i_1}e_{j_1})$.

7.7 Basis of Alternating tensors

- $\{\psi_I\}$, with I ascending, form a basis for $\mathcal{A}^k(V)$. In particular $\dim\mathcal{A}^{\mathrm{k}}(\mathrm{V}) = \binom{n}{k}$, where $n = \dim\mathrm{V}$.

- This is because we can write any alternating tensor $f \in \mathcal{A}^k(V)$ in terms of these tensors. Consider $g = \sum_J d_J \psi_J$, where $d_J = f(e_{j_1}, ..., e_{j_k})$ (output of a tensor so just a scalar) and J is all ascending indices of order k. Then the action of this new function g on the basis vectors $g(e_{i_1}, ..., e_{i_k}) = d_I \cdot (1) = f(e_{i_1}, ..., e_{i_k})$, so we can say that $g = f$ and thus any alternating tensor f can be expanded over ψ_I.
- It can also be noted that if $k = \mathrm{dimV} \implies \mathrm{dim}\mathcal{A}^{\mathrm{k}}(\mathrm{V}) = 1$.

- This allows us to write:

$$\boxed{\mathcal{A}^k(V) = \{\lambda\psi^{(1,2,...,n)} | \lambda \in \mathbb{R}\}} \tag{7.6}$$

7.8 Alternating Dual

- Let $B : V \to W$ be a linear transformation, If f is an alternating tensor, then $B^* f$ is also an alternating tensor.

7.9 Alternating dual

- Let $B : V \to W$ be a linear map, then B^* restricted to $B^* : \mathcal{A}^k(W) \to \mathcal{A}^k(V)$, that is $B^* f$ is alternating if f is.

7.9.1 Dual determinant

- For $B : V \to V$ and $k = \mathrm{dimV} = \mathrm{n}$, then we have that:

$$\boxed{B^* f = \mathrm{det(B)f}, \quad \mathrm{f} \in \mathcal{A}^{\mathrm{k}}(\mathrm{V})} \tag{7.7}$$

8 The wedge product

- The motivation behind this is that we would like to be able to combine alternating tensors in such a way so that the result is also an alternating tensor!

8.1 The Wedge product

- $\exists$ an operation $\mathcal{A}^k(V) \times \mathcal{A}^l(V) \to \mathcal{A}^{k+l}(V)$ $((f,g) \mapsto f \wedge g)$ satisfying the following:
  - $(f \wedge g) \wedge h = f \wedge (g \wedge h)$
  - $(f,g) \to f \wedge g$ is bilinear, $(f + \lambda g) \wedge h = f \wedge + \lambda(g \wedge h)$
  - $f \wedge g = (-1)^k lg \wedge f$
  - $\psi^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$ for a basis e_i of V, $I = (i_1 \leq ... \leq i_k)$.

  The wedge product is uniquely defined by these for properties, furthermore let $T : V \to W$ be a linear map, then:

$$\boxed{T^*(f \wedge g) = T^* f \wedge T^* g \quad \in \mathcal{A}^{k+l}(V)} \tag{8.1}$$

8.2 Alternating algebra

- The direct sum $\bigoplus_{k=0}^{\infty} \mathcal{A}^k(V)$ form and associative, graded (anti-symmetric) commutative algebra, module in V st: $e^{i_1} \wedge \cdots \wedge e^{i_k} = \psi^I$.

8.3 Form of the wedge product

- So far we have just said there exists a wedge product but what does it actually look like? To do this we have to define a specific operator:

8.3.1 Averaging operator

- This is $A : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ and acts by:

$$\boxed{Af := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \mathrm{f}^{\sigma}} \tag{8.2}$$

  This operator satisfies that:
  - A is linear
  - $Af \in \mathcal{A}^k(V)$
  - If $f \in \mathcal{A}^k(V)$, then $Af = k! f$
- This then allows us to define the wedge product for $l \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^l(V)$:

$$\boxed{f \wedge g := \frac{1}{k!l!} A(f \otimes g)} \tag{8.3}$$

9 Differential forms

- Let $M \subset \mathbb{R}^n$ be a manifold with a boundary. A differential form of order/degree k is a smooth function:

$$\boxed{\omega : \{(p, v_1, ..., v_k) \in M \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n | v_i \in T_pM\}} \tag{9.1}$$

st, $\forall\ p \in M,\quad \omega_p = \omega(p) = \omega(p, v_1, ..., v_k) : T_pM \to \mathbb{R}$, is an alternating tensor. Thus we can also say that $\omega : M \to \bigsqcup_{q \in M} \mathcal{A}^k(T_qM)$, st: $\omega_p \in \mathcal{A}^k(T_pM)$.

9.1 Space of differential forms

- Let $\Omega^k(M)$ be defined as follows:

$$\boxed{\Omega^k(M) = \{\omega |\ \omega \text{ a smooth differential form of degree } k\}} \tag{9.2}$$

We can also define the similar $\Omega^K_\delta(M)$ as:

$$\boxed{\Omega^k_\delta(M) = \{\omega |\ \omega\ \text{ same as above but not necessarily smooth}\}} \tag{9.3}$$

We then have that: $\Omega^0(M) = C^\infty(M) = \{f : M \to N |\ f \text{ smooth}\}$.

- $\Omega^K(M)$ is a vector space under point-wise addition/ multiplication with scalars.

9.2 Basis k forms

- Recalling that $e^j(x_1, x_2, ..., x_n) = x_j \in \mathbb{R}$, for $x \in \mathbb{R}^n$. If we look at the form of $\psi_I(x)$ in 7.4, and the definition of the wedge product in 8.3 then we can see that we can re-write ψ_I as:

$$\boxed{\psi^I = e^{i_1} \wedge e^{i_2} \cdots \wedge e^{i_k}} \tag{9.4}$$

And we end up denoting this:

$$\boxed{\psi^I = dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k} = dx^I} \tag{9.5}$$

These ψ_I are called elementary k forms (since each e^{i_1} is a 1-form). It is also worth noticing that:

$$\boxed{dx^I(v_1, v_2, ..., v_k) = \det([\mathrm{v}_1, \mathrm{v}_2 \cdots \mathrm{v}_\mathrm{k}])} \tag{9.6}$$

- We then have that each k tensor $\omega(p)$ can be written uniquely in the form:

$$\boxed{\omega(p) = \sum_{[I]} b_I(p) dx^I(p)} \tag{9.7}$$

Where $[I]$ denotes any increasing sequence, that means no repeating index's, but it does not have to be of length k. If each $b_I(p)$ is smooth, then so is ω.

9.3 Pullback

- Let $f : M \to N$ be smooth, we define $f^* : \Omega^k(N) \to \Omega^k(M)$ as:

$$\boxed{(f^*\omega)(p, v_1, ..., v_k) := \omega(f(p), Df_p v_1, Df_p v_2, ..., Df_p v_k)} \tag{9.8}$$

f^* is a well defined linear map. We may also write $(f^*\omega)_p \in A^k(T_pM)$. With these definitions it can be proven that:

  - $\mathrm{id}_{\mathrm{m}}^* = \mathrm{id}_{\Omega^*(\mathrm{M})}$
  - $(f \circ g)^* = g^* \circ f^*$ For $L \overset{f}{\leftarrow} N \overset{g}{\leftarrow} M$ smooth. This map is called a *pullback*.

9.3.1 Smoothness condition

- An element $\omega \in \Omega^k_\delta(M)$ is smooth if $\forall\ p \in M,\ \ \exists\ \alpha : U \to V \subset M$ a co-ord patch around p st, $\alpha^*\omega \in \Omega^k_\delta(U)$ is smooth.

9.3.2 Differential form of wedge

- Let $\omega \in \Omega^k(M),\ \eta \in \Omega^l(M)$, we define $\omega \wedge \eta \in \Omega^{k+l}(M)$ by:

$$\boxed{(\omega \wedge \eta)_p = \omega_p \wedge \eta_p} \tag{9.9}$$

- It can also be shown that if we have $f : M \to N$ smooth, then $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

9.3.3 Conclusion

- $\Omega^\bullet(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M)$ is a *graded-commutative* (*anti-commutative*) *associative* algebra structure, with $\Omega^0(M) = C^\infty(M)$ (the set of a all smooth 1-d functions on M).
- It also has that if $f : M \to N$, then $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$, preserves the above structure.

  It is also worth noting that $\Omega^k(M) = 0$ for $k > \mathrm{dimV}$ as then there are more elements in the sequence $i_1, ..., i_k$, then there are dimensions, so $dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k}$, must have a repeating index, making it 0 and thus each ω is also 0.

10 Exterior derivative

- Let M be a manifold with a boundary, $\exists$ a unique linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined for all $k \geq 0$ st:
  - If $f \in C^\infty(M) = \Omega^0(M)$ then $df_p(v) = Df(p)v$
  - If $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
  - $d(d\omega) = 0$, more over if $F : M \to N$ is smooth then $d(F^*\omega) = F^*d\omega$

10.0.1 Exterior Derivative of k-forms

- We know how this derivative acts on 0-forms based on the first property, but how does it act on a k-form ω? To find this we can just look at our expression of k-forms in terms of the elementary k-forms in 9.7. With this expression $d\omega$ is defined as:

$$\boxed{\begin{aligned} d\omega &= d\sum_{[I]} b_I dx^I = \sum_{[I]} db_I \wedge dx^I + \sum_{[I]} b_I d^2 x^I \\ &= \sum_{[I]} db_I \wedge dx^I \end{aligned}} \tag{10.1}$$

  Where we have used the property that $d^2 = 0$.

- One can also use this expression to show that $d^2\omega = 0$ as since b_I is a 0 form, $d^2 b_I = \sum_{i,j=1}^n D_i D_j b_I$, i.e. all second order partial derivatives, but since b_I, must be a smooth function $D_i D_j b_I = D_j D_i b_I$, so since $d^\omega = \sum_I \sum_{i,j=1}^n D_i D_j b_I dx^I = \sum_I \sum_{i>j} (D_i D_j b_I - D_j D_i b_I) dx^I = 0$, as swapping dx^{i_i} and dx^{i_j}, picks up a minus sign.

10.0.2 Alternate definition

- Alternatively if we have $\alpha : U \to M$, be a patch around p. we can define $d\omega$ as:

$$\boxed{(d\omega)_p := ((\alpha^{-1})^* d_U(\alpha^*\omega))_p} \tag{10.2}$$

  This is well defined and independent of the choice of α.

10.1 Naturality

- Let $F : U \to V$ be smooth, $\omega \in \Omega^k(V)$. Then:

$$\boxed{F^*d\omega = d(F^*\omega)} \tag{10.3}$$

11 Vector Fields

- A vector field is a smooth function $X : M \to TM$, st. $X(p) \in T_pM$. We then define the set of all vector fields on our manifold M:

$$\boxed{\mathfrak{X}(M) := \{X : M \to TM | X \text{ is a vector field}\}} \tag{11.1}$$

11.1 Isomorphisims to differential forms

- We can define the following isomorphisms for $M \subset \mathbb{R}$:

$$\boxed{\begin{aligned}
&h_1 : \mathfrak{X}(M) \to \Omega^1(M)\\
&\begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{pmatrix} \mapsto \sum_{i=1}^{d} x^i dx^i\\
&h_{n-1} : \mathfrak{X}(M) \to \Omega^d - 1(M)\\
&\begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{pmatrix} \mapsto \sum_{i=1}^{d} x^i dx^1 \wedge \cdots dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^d\\
&h_n : \mathfrak{X}(M) \to \Omega^d(M)\\
&u \mapsto u dx^1 \wedge \cdots \wedge dx^d
\end{aligned}} \tag{11.2}$$

Note that these isomorphisms may not be natural, i.e. $h_1(F^*x) = F^*h_1(x)$, is in general not true.

12 Integrating forms

- In this section we will define what it means to integrate a k-form over a manifold M. To do this we link our integrals back to $\mathbb{R}^n$, where we have well defined integration.

12.1 Fubinis Theorem

$$\int f(x^1,...,x^n)dx^1\cdots dx^n = \int_{\mathbb{R}^{n-l}}\left[\int_{\mathbb{R}^l} f(x^1,...,x^n)dx^1\cdots dx^l\right]dx^{l+1}\cdots dx^n \tag{12.1}$$

12.2 Change of Variables

- Let $F : U_1 \to U_2$, $(U_1 \subset \mathbb{H}^d, U_2 \subset \mathbb{H}^n)$ a diffeomorphism, then:

$$\int_{\mathbb{R}^n} f(F(x))|\mathrm{detDF}|\mathrm{dx}^1\cdots\mathrm{dx}^n = \int_{\mathbb{R}^n}\mathrm{f(x)dx}^1\cdots\mathrm{dx}^n \tag{12.2}$$

12.2.1 Claim

- Change of Variables $\iff$ $\int F^*\omega = \int \omega$, if $|\mathrm{detDF}| = \mathrm{detDF}$.

12.3 Compact Support

- We say $\omega \in \Omega^k(M)$ has *compact support* if supp $\omega := \overline{\{\mathrm{p \in M}|\omega_\mathrm{p} \neq 0\}}$ is compact. We can then denote $\Omega^k_c(M)$ denote all the k-forms with compact supports.

12.4 Integral of a d-form

- Let $M \in \mathbb{H}^d$ (M open), $\omega \in \Omega^d_c(M)$. We define the integral of this d-form as follows:

$$\int_M \omega = \int_{\mathbb{R}^d} u(x^1,...,x^d)dx^1\cdots dx^d \tag{12.3}$$

Here u is defined by $\omega = udx^1\wedge\cdots\wedge dx^d$ and is extended by 0 outside of M.

13 Orientations

- In order to determine weather the result of integrating a k-form, has a $\pm$ in front of it, we have to define an orientation on the manifold we are integrating over.

13.1 Orientation preserving/reversing

- Let $F : M \to N$ $(N, M \subset \mathbb{H}^d)$, be a diffeomorphism. We call F *orientation preserving* if $\mathrm{detDF(p)} > 0,\ \forall\ \mathrm{p} \in \mathrm{M}$ and *orientation reversing* if $\mathrm{detDF(p)} < 0,\ \forall\ \mathrm{p} \in \mathrm{M}$. Note that $\mathrm{detDF} \neq 0$, as F is a diffeomorphism.

13.2 Proposition

- Let F be as above and orientation preserving. Then:

$$\boxed{\int_M F^*\omega = \int_N \omega, \quad \forall\ \omega \in \Omega^d_c(N)} \tag{13.1}$$

13.3 Overlapping Charts

- Let M be a manifold with a boundary. Let $\alpha_i : U_i \to V_i \subset M$ be charts. We say two charts α_1 and α_2 *overlap positively* if $\alpha_2^{-1} \circ \alpha_1 : \alpha_1^{-1}(V_1 \cap V_2) \to \alpha_2^{-1}(V_1 \cap V_2)$ is orientation preserving.

13.4 Oriented manifold

- An orientation on M is the choice of collection of charts that pairwise overlap positively and cover M.

  We denote an oriented manifold by $(M, \{\alpha_i\})$.

  We call a chart $\beta : U \to V$, positive if it overlaps positively with all $\alpha_i \in \{\alpha_i\}$. It is easy to see then that. $\{\alpha_i\} \subset \{\beta\} \iff$ they define the same collection of positive charts.

13.5 Reversing orientation

- Set $\tau : \mathbb{H}^d \to \mathbb{H}^d$, $(x^1, ..., x^d) \mapsto (-x^1, ..., x^d)$. Given a patch $\alpha : U \to M$, then $\alpha \circ \tau$ is also a patch with opposite orientation. Usually we denote this by: $(M, \{\alpha_i \circ \tau\}) = -(M, \{\alpha_i\}) = -M$.

  Its clear to see that if we have M orientated then either α, or $\alpha_i \circ \tau$, is positive .

13.6 Extension of interior orientation

- Let M be a manifold with a boundary. Suppose $\mathring{M} = M \backslash \partial M$ is orientable, then so is M. More over, if $A = \alpha_i$, is an orientation on $\mathring{M}$, then $\exists$, $B = \beta_i$ an orientation on M, st: $A \subset B$.

13.6.1 Corollary

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and 0 a regular value. Then $f^{-1}([0, \infty]) = M$, carries a natural orientation.

13.7 Induced orientation

- Let $(M, \{\alpha_i\})$, be a oriented manifold with a boundary. Then, $(\partial M, \alpha_i\Big|_{\partial\mathbb{H}^d \cap U_i})$, is an oriented manifold with what we call a *restricted orientation*.
- The *Induced orientation* on ∂M is $(-1)^d$ times the restricted one. This is so that stokes theorem always holds!

13.8 Oriented maps

- Let $f : M \to N$ be a diffeomorphism and $(M, \{\alpha_i\}), (N, \{\beta_i\})$, oriented manifolds. We say f is orientation preserving if $\{f \circ \alpha\}$ are positive charts with respect to $\{\beta_i\}$.

13.9 Positive charts wrt d-forms

- Given $\omega \in \Omega^d(M)$ on M a d-manifold with a boundary. We declare $\alpha : U \to M$, to be positive iff $\alpha^*\omega \in \Omega^d(U)$, $U \in \mathbb{R}^d$, st: $\alpha^*\omega = u dx^1 \wedge \cdots \wedge dx^d$, with $u(x) > 0, \ \ \forall\, x \in U$.
- This defines an orientation $\iff$ $\omega_p \neq 0, \ \ \forall\, p \in M$.

13.10 Volume Forms

- $\omega \in \Omega^d(M)$ is called an *volume form* if $\omega_p \neq 0, \ \ \forall\, p \in M$.

14 Stokes Theorem

14.1 The integral

- The integral is linear:

$$\boxed{\int_M \lambda\omega + \eta = \lambda \int_M \omega + \int_M \eta} \tag{14.1}$$

If $-M$, denotes M, with the opposite orientation then:

$$\boxed{\int_{-M} \omega = -\int_M \omega} \tag{14.2}$$

14.2 Stokes Theorem

- Let M be an oriented manifold with a boundary, then for $\omega \in \Omega_c^{d-1}(M)$:

$$\boxed{\int_M d\omega = \int_{\partial M} \omega} \tag{14.3}$$

14.2.1 Corollary

- If M has no boundary ($\partial M = \varnothing$), then $\int_M d\omega = 0$.