Quantum Field Theory I

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"We will work in "God-given" units, where $\hbar = 1 = c$"

-Peskin & Schroeder

Contents

1 QFT trailer

- We wish to see if we can perform a calculation in quantum field theory, just by elementary means, i.e. via dimensional analysis ect.

  Consider the head on collision of an electron e^- and e^+ that results in the production of a muon μ^- anti muon μ^+ pair, shown below:

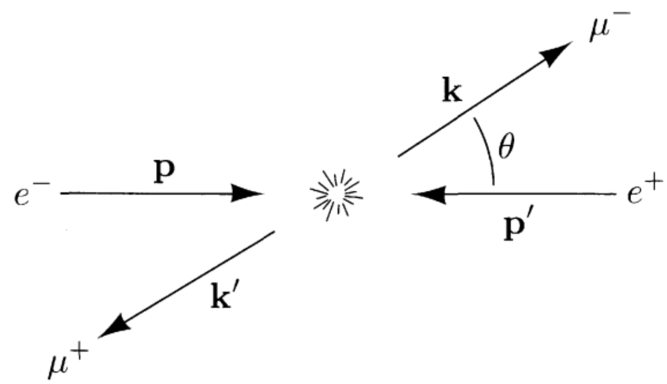

Figure 1: *electron positron annihilation*

- The calculation we would like to perform is the differential cross section, that is the derivative of the cross section σ with respect to the solid angle Ω, $\frac{d\sigma}{d\Omega}$. This is a useful quantity as it is easily experimentally observed. In a particle collider, electrons and positrons are prepared in batches of length l_A and l_B and densities ρ_A and ρ_B respectively. When the two batches collide if the over lapping area of the head on collision is A, then the cross section is given by:

$$\sigma = \frac{\text{Number of events}}{\rho_A \rho_B l_A l_B A}$$

- We now look at the dimensions of our quantities. Conveniently from our use of God given units, i.e. $\hbar = c = 1$ we have that momentum, mass and energy have the same units as the energy mass equivalence becomes $E^2 = p^2 + m^2$. We also have from the Heisenberg's uncertainty principle that $\Delta p \Delta x \sim 1$. Thus the dimensions of mass and length are inversely related.

  We can from this easily see that the dimensions of the quantity $[\rho_A \rho_B l_A l_B A]$ is $[m^2]$, which makes the dimensions of $\left[\frac{d\sigma}{d\Omega}\right] = \left[\frac{1}{m^2}\right]$ as angles are unitless. With this we can say that this quantity is also inversely proportional to the energy squared times some positive quantity that depends on the angle:

$$\boxed{\frac{d\sigma}{d\Omega} \propto \frac{1}{E^2} |\mathcal{M}(\theta)|^2} \tag{1.1}$$

  Here $\mathcal{M}$ is a dimensionless quantity, that is essentially the quantum mechanical amplitude for the process to occur. It does not depend on the energy E, as we are considering the limit where $E >> m_e, m_\mu$. This means we are unable to construct any dimensionless quantity like E/m_e or E/m_μ as we have set $m_e/E = 0 = m_\mu/e$. Note that we could not take this limit and them $\mathcal{M}$ would depend on E, but it is simpler to consider the high energy limit. We will later calculate what the constant of proportionality in this equation is, and it turn out to be $1/64\pi^2$.

- If we recall from Quantum mechanics, in perturbation theory we had that at first order the transition amplitude is related to the initial and final states along with the interaction Hamiltonian H_i, so:

$$\mathcal{M} \sim \langle \text{final state} | H_I | \text{initial state} \rangle$$

But we know physically that the electrons do not interact with the muons, Instead what we know happens is that the electrons annihilate to form photons which in turn form the muons. This then means that $\langle \mu^+\mu^- | H_I | e^+e^- \rangle = 0$ and instead we have that:

$$\boxed{\mathcal{M} \sim \langle \mu^+\mu^- | H_I | \gamma \rangle^\alpha \langle \gamma | H_I | e^+e^- \rangle_\alpha} \tag{1.2}$$

This is a heuristic way of writing this second order contribution, but it makes sense physically as we have the electron positron pair interacting to become a photon ($|\gamma\rangle \langle\gamma|$) and then the photon in turn becoming muons. Note the addition of vector indices (α) as the photon is a vector particle (non-zero spin), so the photon created has 4 intermediate states, 3 for spin as it has spin-1 and one extra that comes from the fact that we are adding angular momenta in the four dimensional Lorentz group and must consider boosts also. (Remember the three spin components are generated by the three angular momentum operators and there is a corresponding generator for boosts).

- Since the photon must conserve angular momentum going to or from either of the two particles, the photon vector must be in the same direction as the axes of the particle pairs. We also know that the strength of the coupling between electrons (or muons) and photons is given by the electric charge e. This means for the case where electron has spin up along x axis, the positron has spin down:

$$\langle \gamma | H_I | e^+e^- \rangle^\alpha \propto e(0, 1, i, 0)$$

And if we have the same for muon and anti-muon:

$$\langle \gamma | H_I | \mu^+\mu^- \rangle^\alpha \propto e(0, \cos\theta, i, -\sin(\theta))$$

The vectors here have the first component as the time component and the last three are part of the the polarization vector of the photon. See Jackson third edition page 299 for these vectors.

- When considering the experimental calculation of $\frac{d\sigma}{d\Omega}$ it also easier to account for all possible initial and final spin states, for which we need to take in to account conservation of angular momentum. This means we cant have two right polarized electron and positrons going to a left and right polarized muon and anti-muon. This condition then leaves 4 possible transitions which we can calculate the contributions. These calculations are done by dotting the two vectors we have above as in 1.2, making sure to take the complex conjugate of the first 4 vector and also remember to properly contract with the $(+---)$ metric. This results in:

$$\boxed{\begin{aligned} M(RL \to RL) &\sim -e^2(1+\cos\theta) \\ M(RL \to LR) &\sim -e^2(1-\cos\theta) \\ M(LR \to RL) &\sim -e^2(1-\cos\theta) \\ M(LR \to LR) &\sim -e^2(1+\cos\theta) \end{aligned}}$$

There are no states that have 0 total angular momentum before and after as it turn out the intermediate photon (despite being a spin-1 boson) cant have spin 0 along an axis. Though in this case we would be requiring that the photon would have 0 total spin which is also not possible.

- We can finally take these 4 probabilities and square and sum them to get $|\mathcal{M}|^2$. Resulting in:

$$|\mathcal{M}|^2 \sim 4e^2(1+\cos^2\theta)$$

So using 1.1 and its proportionality constant $1/64\pi^2$ we can write:

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{32E^2}(1+\cos^2\theta)$$

Defining the constant $\alpha = e^2/4\pi \sim 137^{-1}$, we can then integrate this to get a expression for the total cross section as:

$$\boxed{\sigma = \frac{4\alpha^2\pi}{3E^2}}$$

This is the correct first order approximation!

2 The need for Fields

In this section we will see where regular quantum mechanics fails and what we need to do to fix it.

2.1 Non-relativistic free particle

- We can recall from QM that the probability of a particle at point x at time t propagating to x' at time t' is given by:

$$\langle x| e^{-iH(t-t')} |x'\rangle \tag{2.1}$$

Here H is the Hamiltonian. For a Non-relativistic free particle we have that $H = \hat{\mathbf{P}}^2/2m$. We can then go about solving dor this propagator with this Hamiltonian in the usual way. This involves inserting the identity $\int \frac{d^3p}{(2\pi)^3} |p\rangle \langle p| = \mathbb{I}$ into the above propagator 2.1before the $|x'\rangle$

$$\langle x| e^{-iH(t-t')} |x'\rangle = \int \frac{d^3p}{(2\pi)^3} \langle x| e^{-i\frac{\hat{\mathbf{P}}^2}{2m}(t-t')} |p\rangle \langle p|x'\rangle$$

Then if we recall that $\langle p|x'\rangle = e^{-i\mathbf{p}\cdot\mathbf{x}'}$, $\langle x|p\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}$ and $e^{-i\frac{\hat{\mathbf{P}}^2}{2m}(t-t')} |p\rangle = e^{-i\frac{p^2}{2m}(t-t')} |p\rangle$. We get:

$$\boxed{\begin{aligned}\langle x| e^{-iH(t-t')} |x'\rangle &= \int \frac{d^3p}{(2\pi)^3} e^{-i\frac{p^2}{2m}(t-t')} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\ &= \left(\frac{m}{2\pi i(t-t')}\right)^{3/2} e^{im\frac{(\mathbf{x}-\mathbf{x}')^2}{2(t-t')}}\end{aligned}}$$

- What this is saying, is that for any two points x and x', no matter how far they are separated, have a non-zero probability of propagation from one to another. But this is direct contrast with what we know from special relativity! Two points separated by enough distance so that there space time interval, $\Delta s^2 = (t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2$ is negative, i.e. space like. Which would require a propagation speed faster then light.

2.1.1 Non-trivial Gaussian integral

- In the above calculation we have to evaluate the following integral:

$$\int_{\infty}^{\infty} e^{-(p+ia)^2} dp$$

Now we may think we can just make the substitution $u = p + ia$, but since we are involving complex numbers, we get a non trivial problem with the limits, as they change to $\infty - ia$ and $\infty + ia$. What we can do instead is calculate the integral of a closed contour. Since the integrand above has no poles, we know that any closed contour on the complex plain should evaluate to 0. What we will do is then modify the limits of our integral to be from $-q$ to q, we will take the limit at the end to recover what we got. We can then integrate along the following path: We start at $q + ia$ and move to $-q + ia$, then from $-q + ia$ to $-q$, from $-q$ to q and from q to $q + ia$. This is a closed contour, that we will call C. We then have that:

$$0 = \int_C e^{-z^2} dz = \int_{q+ia}^{-q+ia} e^{-(x)^2} dx + \int_a^0 e^{-(-q+iy)^2} dy + \int_{-q}^{q} e^{-x^2} dx + \int_0^a e^{-(q+iy)^2} dy$$

We can then immediately argue that the second and fourth integrals should vanish when we send q to infinity. Then the first integral we can reverse the limits and pick up a minus sign so:

$$\int_{-q+ia}^{q+ia} e^{-(x)^2}dx = \int_{-q}^{q} e^{-x^2}dx = \sqrt{\pi}$$

And then finally now we can make a u-substitution of the LHS, to get back our original integral! So we can see in this case that since the two ends of this contour vanished in the arge q limit we are essentially able to move our contour up and down to get the regular Gaussian integral.

Relativistic free particle

- But we can chalk this up to a mistake. We were not considering the Hamiltonian of a **relativistic** free particle. That is, with the Hamiltonian $H = \sqrt{\mathbf{P}^2 + m^2}$. The propagator in a similar manner to before then becomes:

$$\begin{aligned}\langle x| e^{-i\sqrt{\mathbf{P}^2+m^2}(t-t')}|x'\rangle &= \int \langle x|p\rangle\, e^{-i\sqrt{\mathbf{P}^2+m^2}(t-t')}\langle p|x'\rangle \frac{d^3p}{(2\pi)^3} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i\sqrt{p^2+m^2}(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\end{aligned}$$

We can then parametrize the angle part of this integral by letting θ be the angle between $\mathbf{p}$ and $(\mathbf{x}-\mathbf{x}')$, we can also further parametrize this with $\eta = \cos\theta$:

$$\begin{aligned}\langle x| e^{-i\sqrt{\hat{\mathbf{P}}^2+m^2}(t-t')}|x'\rangle &= \int_0^\infty \frac{p^2dp}{(2\pi)^2}\int_{-1}^{1} d\eta e^{-i\sqrt{p^2+m^2}(t-t')+ip|\mathbf{x}-\mathbf{x}'|\eta} \\ &= \int_0^\infty \frac{p^2dp}{(2\pi)^2} e^{-i\sqrt{p^2+m^2}(t-t')}\frac{1}{ip|\mathbf{x}-\mathbf{x}'|}\left(e^{ip|\mathbf{x}-\mathbf{x}'|} - e^{-ip|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned}$$

We can then turn this into an integral from $-\infty$ to ∞ with a factor of 2:

$$= \frac{-i}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty pdp\; e^{-i\sqrt{p^2+m^2}(t-t')+ip|\mathbf{x}-\mathbf{x}'|} \tag{2.2}$$

2.1.2 Laplace steepest decent

- We can then use a useful approximation of integrals of this form by expanding around critical points. That is points x_0 where $f'(x_0) = 0$. The approximation is as follows, for a critical point x_0 of $f(x)$:

$$\boxed{\begin{aligned}&\int_{-\infty}^{\infty} h(x)e^{Af(x)}dx = \int_{-\infty}^{\infty} h(x)e^{A\left[f(x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+\mathcal{O}((x-x_0)^3)\right]}dx \\ &\approx h(x_0)e^{Af(x_0)}\int_{-\infty}^{\infty} e^{-A\frac{1}{2}|f''(x_0)|(x-x_0)^2}dx = \sqrt{\frac{2\pi}{A|f''(x_0)|}}h(x_0)e^{Af(x_0)}\end{aligned}}$$

- Where we have assumed x_0 is a global maxima so that $f''(x_0) \leq 0$. This approximation works well as the exponential ensures that any small deviation from x_0 contributes very little to the integral. It of course depends also on $h(x)$ being "well behaved".

- Back to the relevant integral 2.2, we can see that for us $f(p) = -i\sqrt{p^2+m^2}(t-t') + ip|\mathbf{x}-\mathbf{x}'|$ and $h(p) = p$. So we find the stationary point of $f(p)$ via $f'(p_0) = 0$:

$$f'(p_0) = \frac{-p_0(t-t')}{\sqrt{p_0^2+m^2}} + |\mathbf{x}-\mathbf{x}'| = 0 \implies p_0^2 = \frac{m^2|\mathbf{x}-\mathbf{x}'|^2}{(t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2}$$

Then we can evaluate $f(p_0)$:

$$\begin{aligned} f(p_0) &= -i\sqrt{\frac{m^2|\mathbf{x}-\mathbf{x}'|^2}{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}} + m^2(t-t') + i\sqrt{\frac{m^2|\mathbf{x}-\mathbf{x}'|^2}{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}} \\ &= -\frac{m(|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2)}{\sqrt{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2}} = -m\sqrt{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2} \end{aligned}$$

Where we have used the fact that we are considering probabilities outside the light-cone, i.e. where $|\mathbf{x}-\mathbf{x}'| > (t-t')$, so that the square root $\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2} = i\sqrt{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2}$.

- With this we can now use Laplace's steepest decent integral approximation to write:

$$\boxed{\langle x|\, e^{-i\sqrt{\hat{\mathbf{P}}^2+m^2}(t-t')}\,|x'\rangle \propto h(p_0)e^{f(p_0)} \propto e^{-\sqrt{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2}}}$$

This again does not solve our problem, we have that there is a non-zero probability of propagating to outside the light-cone. Some more radical approach is needed to solve this problem.

2.2 Field theory EoM

- The Idea will be to go from dealing with particle, to waves. For this we need to generalize our idea of action being the integral of a Lagrangian over time to being the integral of a Lagrange density over time and space as a field is spread out, not localized. This means:

$$S = \int L(q_i, \dot{q}_i, t)dt \to \int d^4x\mathcal{L}(\varphi(\mathbf{x},t), \partial_\mu\varphi(\mathbf{x},t))$$

We then have to vary this action to set $\delta S = 0$:

$$\delta S = \int d^4x \left(\frac{\delta\mathcal{L}}{\delta\varphi}\delta\varphi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\delta\partial_\mu\varphi\right) = \int d^4x \left(\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right)\right)\delta\varphi$$

$$\boxed{\implies \frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} = 0} \tag{2.3}$$

Where we have integrated by parts the second term.

2.3 Non-degenerate Lagrangian

- It is often the case, that when we have constructed our Lagrangian, that we would like to perform a Legendre transform from the variables q and $\dot{q}$, to q and p. This is gives us our Hamiltonian and takes the form $H = \sum_i p_i\dot{q}_i - L(q,\dot{q})$. In doing this we are required to solve for $\dot{q}$ in terms of q and p. But this is not always possible, for example given the Lagrangian $L \propto \dot{q} \implies p = \frac{\partial L}{\partial q} = 1$, so we are unable to solve for $\dot{q}(p,q)$. It turn out the condition for us to always be able to solve for $\dot{q}$ is as follows:

If we denote the matrix $M_{ij} = \frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}$, then the condition becomes, we can solve for $\dot{q}_i \iff \det(M) \neq 0$. The Lagrangian can then be written as:

$$L = \sum_{i,j} M_{ij}\partial\dot{q}^i\partial\dot{q}^j, \quad \text{and} \quad \dot{q}^i = \sum_j M_{ij}^{-1}p_j$$

This is called a Non-degenerate Lagrangian.

2.4 Hamiltonian field theory

- We wish to find a Hamiltonian for our field theory. The best way to go about extending our definition of $H = \sum_i p_i \dot{q}_i - L(q, \dot{q})$, is to think of space as a discretized space, as we had the co-ordinates q indexed by i. We can then calculate:

$$p(\mathbf{x}) = \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \sum_{\mathbf{y}} \mathcal{L}(\phi(\mathbf{y}), \dot{\phi}(\mathbf{y})) d^3 y = \pi(\mathbf{x}) d^3 x.$$

Where:

$$\boxed{\pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x})}} \tag{2.4}$$

This is called the *momentum density* conjugate to $\phi(\mathbf{x})$. We can now write the Hamiltonian as:

$$H = \sum_{\mathbf{x}} p(\mathbf{x}) \dot{\phi}(\mathbf{x}) - L$$

Which in the continuum limit becomes:

$$\boxed{H = \int d^3 x [\pi(\mathbf{x}) \dot{\phi}(\mathbf{x}) - \mathcal{L}] \equiv \int d^3 x \mathcal{H}} \tag{2.5}$$

2.5 Noether's Theorem

- We are used to Noether's Theorems relation between symmetries and conservation in the context of particles. We will now discuss this in the context of fields. If we have an infinitesimal transformation of fields taking the form:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

Here α is small. This transformation is a symmetry if it results in the same EoM, i.e. leaves the action unchanged. This means the Lagrangian must be the same up to the addition of a total derivative, which in this context is the 4-gradient:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x)$$

Where $\mathcal{J}^\mu$ is some 4-vector. We can then find the expected form of $\Delta\mathcal{L}(\phi, \partial_\mu \phi)$:

$$\begin{aligned}
\alpha \Delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \Delta \phi) + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \partial_\mu (\alpha \Delta \phi) \\
&= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta \phi
\end{aligned}$$

We can then recognize that the second term vanished via our EoM 2.3, so we can set the remaining term to $\alpha \partial_\mu \mathcal{J}^\mu(x)$, which is equivalent to:

$$\boxed{\partial_\mu j^\mu(x) = 0, \quad \text{for } j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - \mathcal{J}^\mu} \tag{2.6}$$

- If the symmetry involves more then one field the first term above should be a sum of those terms, for the different fields. This conservation law also implies that there is a charge constant in time namely:

$$\boxed{Q \equiv \int j^0 d^3x}$$

2.6 Stress-Energy Tensor

- We can also consider transformations of space itself, for example, translations or rotations. This takes the for $x^\mu \to x^\mu - a^\mu$, which has the following affect on the fields (when infinitesimal):

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

The Lagrangian must also transform in the same way as it is also a scalar:

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}(x))$$

We can then recognize the $\delta^\mu_\nu \mathcal{L}(x)$ as $\sim \mathcal{J}^\mu$ from 2.6, however since the change in the scalar field became a vector $\Delta\phi \to \partial_\mu \phi(x)$, then we must add a second index to our $\mathcal{J}^\mu \to \mathcal{J}^\mu_{\ \nu}$. This is for the reason for the δ^μ_ν in the above expression and means our conserved vector j^μ becomes a conserved tensor of the form:

$$\boxed{\partial_\mu T^\mu_{\ \nu}(x) = 0, \qquad \text{for } \ T^\mu_{\ \nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}}$$

This is the *Stress-Energy Tensor* also called the *Energy-Momentum Tensor*. We can then notice that the conserved charge associated with the time translations is nothing more then out Hamiltonian:

$$\int T^{00} d^3x = \int \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right) d^3x = \int \mathcal{H} d^3x = H$$

Similarly for space translation the conserved charges are the physical momenta, not to be confused with the canonical momentum.

$$\int T^{0i} d^3x = \int \mathcal{P}_i d^3x = P_i$$

3 The Klein-Gordon Field

- We ease ourselves into the concepts of quantum fields with the discussion of the Klein-Gordon field. This is one of the simplest types of fields and its use becomes obvious when we see the EoM, which is just the Schrödinger equation but made relativistic by replacing $\hat{\mathbf{p}}^2/2m \to \sqrt{\hat{\mathbf{p}}^2 + m^2}$. We will see more of this later.

- The Lagrange density for the Klein-Gordon field takes the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \tag{3.1}$$

From this we can calculate the momentum density via 2.4 to get $\pi = \dot{\phi}$. And thus via 2.5 the Hamiltonian density is:

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \tag{3.2}$$

- The equations of motion can also be calculated easily via 2.3 resulting in the *Klein-Gordon equation*:

$$\boxed{(\partial_\mu \partial^\mu + m^2)\phi = 0} \tag{3.3}$$

This $\partial_\mu \partial^\mu$ is often denoted $\Box$. As an aside, if we "make" the Schrödinger equation relativistic, i.e. $\mathbf{p}^2 + m^2 = E^2$ and use the definition of the operators $i\partial_t \psi = E\psi$ and $\hat{\mathbf{p}} = -i\nabla$, so: $-\nabla^2 + m^2 = -\partial_t^2$, recovering us the above Klein-Gordon equation.

3.1 Second Quantization

- We will now see if we can "quantize" this field. What we can do to help figure out how to do this is to take inspiration from the quantization of the single particle Harmonic oscillator in QM. This works nicely as we can see the Lagrangian density for the Klein-Gordon field 3.1 resembles that of the harmonic oscillator. By quantize here we mean to reinterpret the dynamical variables as operators that obey canonical commutation relations, like $[p_i, q_j] = i\delta_{i,j},\ [q_i, q_j] = 0 = [p_i, p_j]$. Instead however ϕ and π will become our operators and we will require that $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}),\ [\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})]$. This procedure is often called *Second Quantization* to distinguish it from the old one particle version.

- Our goal is to now find the spectrum of the Hamiltonian. We start by taking our field $\phi(\mathbf{x})$ and writing it in terms of its Fourier transform to momentum (p)-space

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\phi(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}$$

Here we also require that $\phi(\mathbf{x}, t)$ is real so $\phi^*(\mathbf{p}) = \phi(-\mathbf{p})$ Then using the Klein-Gordon equation 3.3 we get the condition:

$$\left[\partial_t^2 + p^2 + m^2\right]\phi(\mathbf{p}) = 0$$

Where we have acted the $-\partial_i^2$ from $\Box$ on the exponential to pull out the p^2 This equation of motion is the same as that of a harmonic oscillator with frequency $\omega_{\mathbf{p}} = \sqrt{p^2 + m^2}$.

- We now recall from QM that in order to solve the harmonic oscillator we introduced the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators, so that we could write $\hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger)$. This then meant the Hamiltonian took the form $H = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ and also that the commutators were: $[H, \hat{a}^\dagger] = \omega\hat{a}^\dagger$, $[H, \hat{a}] = -\omega\hat{a}$ and $[a, a^\dagger] = \mathcal{I}$.

- This can be done for our field ϕ, except we have to consider that $\hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger)$ is a function of momentum as for us we have $\omega_{\mathbf{p}} = \sqrt{p^2 + m^2}$. We can then recall that since we expect ϕ to be real we require that $\phi^\dagger(\mathbf{p}) = \phi(-\mathbf{p})$, this means we should have $\phi(\mathbf{p}) \propto a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}$ as then $\phi(\mathbf{p})^\dagger \propto a^\dagger_{\mathbf{p}} + a_{-\mathbf{p}} \propto \phi(-\mathbf{p})$. This means our corresponding expression for ϕ should take the form:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{3.4}$$

$$\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{3.5}$$

We can also think of this expression as each Fourier mode of the field being treated as an independent oscillator with its own a and $a^\dagger$ (which will also be a function of the momentum). We can then isolate the second term in this integral and change $\mathbf{p}$ to $-\mathbf{p}$, since we are integrating over all momentum space, this just results in the signs of the p's in the term changing so we can write:

$$\boxed{\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right)} \tag{3.6}$$

In a similar manner the momentum density can be written as:

$$\boxed{\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right)} \tag{3.7}$$

3.1.1 Commutators

- We can then go about checking the commutator $[\phi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{y})$ which should be equivalent to the the new creation and annihilation operators satisfying $[a, a^\dagger] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')$. Using the above definitions 3.4 and 3.5 for this calculation is easier, resulting in:

$$\begin{aligned}
[\phi(\mathbf{x}), \pi(\mathbf{x}')] &= -\frac{i}{2} \int \frac{d^3p d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \left[-a_{\mathbf{p}} a^\dagger_{\text{-}\mathbf{p}'} + a^\dagger_{\text{-}\mathbf{p}} a_{\mathbf{p}'} - (-a^\dagger_{\mathbf{p}'} a_{\text{-}\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\text{-}\mathbf{p}'})\right] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} \\
&= -\frac{i}{2} \int \frac{d^3p d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \left[[a^\dagger_{\text{-}\mathbf{p}}, a_{\text{-}\mathbf{p}'}] - [a_{\mathbf{p}}, a^\dagger_{\text{-}\mathbf{p}'}]\right] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} \\
&= -\frac{i}{2} \int \frac{d^3p d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} (2\pi)^3 \left[-\delta(\mathbf{p}' + \mathbf{p}) - \delta(\mathbf{p} + \mathbf{p}')\right] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} \\
&= i \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}')} = i\delta(\mathbf{x} - \mathbf{x}')
\end{aligned} \tag{3.8}$$

Where we have used the fact that $\omega_{\mathbf{p}} = \omega_{\text{-}\mathbf{p}}$ by definition. We can then also calculate the Hamiltonian via 3.2:

$$\begin{aligned}
H = \frac{1}{2} \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} \Bigg[& \frac{-\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}}{2} \left(a_{\mathbf{p}} - a^\dagger_{\text{-}\mathbf{p}}\right) \left(a_{\mathbf{p}'} - a^\dagger_{\text{-}\mathbf{p}'}\right) \\
&+ \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}} \left(a_{\mathbf{p}} + a^\dagger_{\text{-}\mathbf{p}}\right) \left(a_{\mathbf{p}'} + a^\dagger_{\text{-}\mathbf{p}'}\right) \Bigg] e^{i\mathbf{x}\cdot(\mathbf{p} + \mathbf{p}')}
\end{aligned}$$

We can then notice that the $\mathbf{x}$ part of this integral creates a delta function $\int d^3x e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{p}')} = (2\pi)^3\delta(\mathbf{p}+\mathbf{p}')$, so $\mathbf{p} = -\mathbf{p}'$. This allows us to write:

$$\begin{aligned} H &= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[-\left(a_{\mathbf{p}} - a^\dagger_{\mathbf{-p}}\right)\left(a_{\mathbf{-p}} - a^\dagger_{\mathbf{p}}\right) + \left(a_{\mathbf{p}} + a^\dagger_{\mathbf{-p}}\right)\left(a_{\mathbf{-p}} + a^\dagger_{\mathbf{p}}\right)\right] \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{-p}}a_{-\mathbf{p}}\right] \end{aligned}$$

We can then for this second term do the same trick of changing $\mathbf{p} \to -\mathbf{p}$, which does not affect the integral's value as $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$. Then also using the fact that $a^\dagger_{\mathbf{p}}a_{\mathbf{p}} = a_{\mathbf{p}}a^\dagger_{\mathbf{p}} - [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}]$ we get the final result that:

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[a_{\mathbf{p}}a^\dagger_{\mathbf{p}} - \frac{1}{2}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}]\right]} \tag{3.9}$$

If we stop to think about the second term in this expression we realize something strange, $[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] = (2\pi)^3\delta(0)$, which is and infinite constant (constant as in it wont affect any state we act H on). But since this is constant we can never measure it experimentally as we only ever measure the energy shift between two different energies. We can also think of this as the zero point energy for each point in space, except that we made this space continuous resulting in infinite many oscillator ground states, so what did we expect really? This means we can ignore the second term in 3.9.

To make sure everything is consistent we also check the commutators, $[H, a^\dagger_{\mathbf{p}}] = \omega a^\dagger_{\mathbf{p}}$, $[H, a_{\mathbf{p}}] = -\omega a_{\mathbf{p}}$:

$$\begin{aligned} [H, a_{\mathbf{p}}] &= \int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}\left[a_{\mathbf{p}'}a^\dagger_{\mathbf{p}'}a_{\mathbf{p}} - a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}a_{\mathbf{p}'}\right] \\ &= \int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}\left[a_{\mathbf{p}'}a_{\mathbf{p}}a^\dagger_{\mathbf{p}'} - a_{\mathbf{p}'}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] - a_{\mathbf{p}}a_{\mathbf{p}'}a^\dagger_{\mathbf{p}'}\right] \\ &= -\omega_{\mathbf{p}}a_{\mathbf{p}} \end{aligned}$$

Here we have used the fact that $[a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0 = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]$ to make the first and last term cancel. The commutator $[H, a^\dagger_{\mathbf{p}}] = \omega_{\mathbf{p}}a^\dagger_{\mathbf{p}}$, can be calculated in the same manner.

- We can now start to think about the creation and annihilation operators as their name suggests. The ground state is $|0\rangle$ st. $a_{\mathbf{p}}|0\rangle = 0$ and has $E = 0$, where as acting with $a^\dagger_{\mathbf{p}}$ increases the energy. These operators clearly depend on momentum $\mathbf{p}$ so we can think of them as creating (and annihilating) momentum eigenstates.

3.2 Particle creation

- The next step is to see if we can write out all the possible momentum states as powers of the creation operator $a^\dagger_{\mathbf{p}}$. The question is then what is the choice of proportionality? We should of course have that the states are normalized, that is $\langle\mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}')$. This however, leads to a problem, namely that this quantity $\delta(\mathbf{p}-\mathbf{p}')$ is not Lorentz invariant. To see why, consider a boost along the x_3 direction. Using the property that $\delta(f(x) - f(x_0)) = \delta(x - x_0)/|f'(x_0)|$ we have that if we boost from momenta $\mathbf{p}, \mathbf{p}' \to \tilde{\mathbf{p}}, \tilde{\mathbf{p}}'$,so that $\tilde{p}_3 = \gamma(p_3 + \beta E)$ and $\tilde{E} = \gamma(E + \beta p_3)$. Then we have that:

$$\begin{aligned}
\delta(\mathbf{p} - \mathbf{p}') &= \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')\frac{d\tilde{p}_3}{dp_3} = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')\frac{d\gamma(p_3 + \beta E)}{dp_3} \\
&= \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')\gamma\left(1 + \frac{dE}{dp_3}\right) = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')\frac{\gamma}{E}(E + \beta p_3) \\
&= \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')\frac{\tilde{E}}{E}
\end{aligned}$$

Where we have used $E^2 = |\mathbf{p}|^2 + m^2$ to do the second last step. We can see from this that if we choose the normalization $|\mathbf{p}\rangle = \sqrt{2E_\mathbf{p}}(a^\dagger_\mathbf{p} |0\rangle$ then $\langle \mathbf{p}|\mathbf{p}'\rangle$ is Lorentz invariant, as needed.

- With this we can consider the action of ϕ on the ground state $|0\rangle$. Via 3.6 this becomes:

$$\phi(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle$$

Where from here and now on we replace $\omega_\mathbf{p}$ with $E_\mathbf{p}$. This is similar to the Fourier expansion we have of $|\mathbf{x}\rangle$ in regular QM, except for the factor of $1/E_\mathbf{p}$, which is almost constant in the non-relativistic, so we can put forward the interpretation that the operator $\phi(\mathbf{x})$ acting on the vacuum $|0\rangle$, creates a particle at position $|\mathbf{x}\rangle$.

3.3 Time evolution

- We would now like to turn our heads to adding time evolution to our fields. So far we have been working in the Schrödinger picture and interpreted the resulting theory in terms of particles. We now switch to the Heisenberg picture where it is simpler to add time dependence.

We know the time evolution operator is a unitary operator $U(t)$, st. $\phi(x) = \phi(\mathbf{x}, t) = U^\dagger\phi(\mathbf{x})U$. For a time independent Hamiltonian we just have the usual $U = e^{-iHt}$. If we want to calculate $\phi(x)$ we will need to know what $a(\mathbf{p}, t) = e^{iHt}a_\mathbf{p}e^{-iHt}$ is. We know that $[H, a_\mathbf{p}] = -E_\mathbf{p}a_\mathbf{p}$. So we can write:

$$\begin{aligned}
Ha_\mathbf{p} &= [H, a_\mathbf{p}] + a_\mathbf{p}H = a_\mathbf{p}(H - E_\mathbf{p}) \\
\implies H^n a_\mathbf{p} &= a_\mathbf{p}(H - E_\mathbf{p})^n
\end{aligned}$$

Where we can do the last step as H commutes with both itself and $E_\mathbf{p}$, so acting with further H's from the left only yields more of the same factor. Similarly this means $H^n a^\dagger_\mathbf{p} = a^\dagger_\mathbf{p}(H + E_\mathbf{p})^n$. This then means that:

$$e^{iHt}a_\mathbf{p}e^{-iHt} = a_\mathbf{p}e^{-iE_\mathbf{p}t}, \quad e^{iHt}a^\dagger_\mathbf{p}e^{-iHt} = a^\dagger_\mathbf{p}e^{-iE_\mathbf{p}t}$$

This means using 3.6 we can write $\phi(x)$ as:

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip^\mu x_\mu} + a^\dagger_\mathbf{p}e^{ip^\mu x_\mu}\right)} \tag{3.10}$$

Where in the 4-vector (which we often just write as p or x) $p_0 = E_\mathbf{p}$.

- We can also use the Heisenberg equation of motion, which states that for an operator $\mathcal{O}$, $i\frac{\partial\mathcal{O}}{\partial t} =$

$[\mathcal{O}, H]$. This allows us to calculate $i\frac{\partial\phi(x)}{\partial t}$ as it must equal $[\phi(x), H]$, with H given by 3.2:

$$\begin{aligned} i\frac{\partial\phi(x)}{\partial t} &= [\phi(x), \frac{1}{2}\int d^3x' \left(\pi^2(x') + (\nabla\phi(x'))^2 + m^2\phi^2(x')\right)] \\ &= \frac{1}{2}\int d^3x \left(\pi(x')[\phi(x), \pi(x')] + [\phi(x), \pi(x')]\pi(x')\right) \\ &= i\pi(x) \end{aligned}$$

3.4 2-point correlation function

- With all this formalism we now see if we can solve the problem of causality. That is, does $\langle \mathbf{x}, \mathbf{y}\rangle$ vanish for $\mathbf{y}$ outside the light cone. In the above formalism, $\langle \mathbf{x}, \mathbf{y}\rangle$ takes the form $\langle 0|\,\phi(x)\phi(y)\,|0\rangle$. This is our propagator and will be denoted $D(x-y)$. We can use 3.10 to write this as:

$$\begin{aligned} D(x-y) &= \int \frac{d^3p d^3p'}{(2\pi)^6}\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}\langle 0|\left(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\left(a_{\mathbf{p}'}e^{-ip'\cdot y} + a^\dagger_{\mathbf{p}'}e^{ip'\cdot y}\right)|0\rangle \\ &= \int \frac{d^3p d^3p'}{(2\pi)^6}\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}\langle 0|\,a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}\,|0\rangle\, e^{i(p'\cdot y - p\cdot x)} \end{aligned}$$

$$\boxed{\implies D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}} \tag{3.11}$$

Where in the first line we have used the fact that $\langle 0|\,a_{\mathbf{p}} = 0 = a^\dagger_{\mathbf{p}}\,|0\rangle$ and in the last line we have used a corollary that $\langle 0|\,a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}\,|0\rangle = \langle 0|\,[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]\,|0\rangle$

3.4.1 Time-like separation

- If we consider a time like separation, that is $x^0 - y^0 > |\mathbf{x}-\mathbf{y}|$. Then we can always boost to a frame where $x^0 - y^0 = t$ and $|\mathbf{x}-\mathbf{y}| = 0$. This is just boosting to the frame of the world-line. The above expression for $D(x-y)$, in 3.11 becomes:

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\frac{p^2}{2\sqrt{p^2+m^2}}e^{-i\sqrt{p^2+m^2}t}$$

If we look at this integral in the limit as $t \to \infty$, then we can see that this is a stationary phase problem, i.e. the exponential oscillates fast around around the unit circle averaging to 0, and only becoming non-zero where the function $\sqrt{p^2+m^2}$ is stationary. This occurs when $\frac{d}{dp}\sqrt{p^2+m^2} = 0 \implies \frac{p}{\sqrt{p^2+m^2}} = 0 \implies p = 0$. So we can Taylor expand the terms in this integral around $p = 0$. Using the fact that $\sqrt{p^2+m^2} \simeq m + \frac{p^2}{2m}$ and $\frac{p^2}{\sqrt{p^2+m^2}} \simeq \frac{p^2}{m}(1-\frac{p^2}{2m}) \simeq \frac{p^2}{m}$. We can write this integral as:

$$\begin{aligned} D(x-y) &\simeq \frac{4\pi}{2(2\pi)^3}\int_0^\infty dp\frac{p^2}{m}e^{-i(m+\frac{p^2}{2m})t} \\ &= \frac{4\pi}{2(2\pi)^3}\left(\int_0^\infty dp\frac{p^2}{m}e^{-i\frac{p^2}{2m}t}\right)e^{-imt} \sim e^{-imt} \end{aligned}$$

The integral in brackets is just a Generalized Fresnel integral, the main behavior as $t \to \infty$ is given by e^{-imt}.

3.4.2 Space-like separation

- If we consider a space like separation, that is $x^0 - y^0 < |\mathbf{x} - \mathbf{y}|$. Then we can always boost to a frame where $x^0 - y^0 = 0$ and $\mathbf{x} - \mathbf{y} = \mathbf{r}$. Then we can write 3.11 as:

$$\begin{aligned} D(x-y) &= \frac{1}{(2\pi)^3} \int d^3p \frac{1}{2\sqrt{p^2+m^2}} e^{-i\mathbf{p}\cdot\mathbf{r}} \\ &= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} \frac{e^{ipr} - e^{-ipr}}{ipr} \\ &= -\frac{i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{p}{\sqrt{p^2+m^2}} e^{ipr} \end{aligned}$$

  Where in the first step we choose our angular integration variable θ to be the angle between $\mathbf{p}$ and $\mathbf{r}$. The last step is then a result of taking the second term and changing $p \to -p$. This integral is non-trivial and the manner in which Peskin explains it is not quite correct. For a discussion of this and a solution of the behavior as $r \to \infty$ see the answer to this stack-exchange post.

  Anyway the behavior as $r \to \infty$ is $D(x-y) \sim e^{-mr}$, i.e. we still have the problem of non-zero propagation outside the light-cone.

3.5 Measurement

- Perhaps we have been thinking too classically about the situation. If we think about quantum mechanics, what matters is measurement. We should be asking if I perform a measurement, can it possibly affect another measurement that occurs outside my light-cone and visa versa. This means we should instead be calculating the *commutator* of $\phi(x)$ and $\phi(y)$. i.e. does it matter the order in which we perform on the field at two points that are outside the light cones of each other. It is easy to see that by definition $\langle 0| [\phi(x), \phi(y)] |0\rangle = D(x-y) - D(y-x)$. So we can easily write this down as:

$$\boxed{\langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right)} \tag{3.12}$$

- There is a nice argument for why this vanishes for points that are space-like separated. If we pick two events x and y, that have space like separation, so $(x-y)^2 < 0$. Then there exits a continuous Lorentz transform from $(x-y)$ to $(y-x)$. It is continuous as it at no point has to cross over the light-cone boundaries. (Easiest to picture this with SO(1,2)). Then since $D(x-y)$ and $D(x-y)$ must be Lorentz invariant, we must have that $D(x-y) = D(y-x)$, so the commutator is 0 for points that are space like separated! Causality is maintained!

- David Tong has a nice comment on this: "There are words you can drape around this calculation. When $(x-y)^2 < 0$, there is no Lorentz invariant way to order events. If a particle can travel in a spacelike direction from $x \to y$, it can just as easily travel from $y \to x$. In any measurement, the amplitudes for these two events cancel."

  What he means by this statement that due to spacelike separation there is no agreed upon temporal ordering as these two points are not causally connected. Thus the amplitudes for the propagation from $x \to y$ is the same magnitude as the propagation $y \to x$, but they cancel due to the commutator.

3.6 Klein-Gordon Propagator

- We can actually find a better expression for the propagator $D(x-y)$ given by 3.11. Let us first consider manipulating the following integral:

$$\int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip\cdot(x-y)} = \int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int_{-\infty}^{\infty}\frac{dp^0}{2\pi i}\frac{-1}{(p^0)^2-\mathbf{p}^2-m^2}e^{-ip^0\cdot(x^0-y^0)} \tag{3.13}$$

This second integral on the RHS is an integral along the real line and has two poles at $p^0 = \pm E_{\mathbf{p}}$, where $E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$. We can see they are simple poles by expanding with partial fractions i.e.:

$$\frac{1}{(p^0)^2 - E_{\mathbf{p}}^2} = -\frac{1}{2E_{\mathbf{p}}}\left[\frac{1}{p^0+E_{\mathbf{p}}} - \frac{1}{p^0-E_{\mathbf{p}}}\right]$$

- To solve this integral we have to shift the poles by a small amount ϵ below the real line so that we can apply the Residue theorem. We will then take the limit as $\epsilon \to 0$ to obtain the true result. Our contour for this integral will be a semi-circle that goes along the real line and then loops background either in the upper half plane C_1 or the lower half plane C_2. We will show then that the integral on the circular part does not contribute as we send the radius of the semi-circle $R \to \infty$.

  To decide which path to use we have to look at the form of the $e^{-ip^0(x^0-y^0)}$ part of the integral, since we are considering the complex plane k_0 is complex we can write this term as:

$$e^{-ip^0(x^0-y^0)} = e^{-i\mathrm{Re}(p^0)(x^0-y^0)}e^{\mathrm{Im}(p^0)(x^0-y^0)}$$

  In the complex plane along the curve the imaginary part of p^0 gets sent to infinity so in order for this term to not contribute, we have to use the path C_2 when $x^0 - y^0 > 0$ and C_1 when $x^0 - y^0 < 0$. We can then go ahead and apply the Residue theorem. For now we will assume $x^0 - y^0 > 0$, so we use the path C_2. This means the following integral is a closed path around a simple pole, so we can apply the Residue theorem:

$$I_{\pm} = \lim_{\epsilon\to 0}\oint_{C_2}\frac{dp^0}{2\pi i}e^{-ip^0(x^0-y^0)}\frac{1}{p^0 \pm (E_{\mathbf{p}}+i\epsilon)} = \lim_{\epsilon\to 0} e^{\pm i(E_{\mathbf{p}}+i\epsilon)(x^0-y^0)} = e^{\pm iE_{\mathbf{p}}(x^0-y^0)}$$

  This means we can write our integral 3.13 as:

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip\cdot(x-y)} &= \int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{(I_+ - I_-)}{2E_{\mathbf{p}}}\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)\end{aligned}$$

  Which is exactly what we had in 3.12! so we can say our proposition integral 3.13 is the same as $\langle 0|\,[\phi(x),\phi(y)]\,|0\rangle$. We should note that in the last step we have relabeled the p integral multiplying the I_+ integral, from integrating over $\mathbf{p}$ to $-\mathbf{p}$. We also have set p^0 to $E_{\mathbf{p}}$.

3.6.1 Advanced/Retarded Propagators

- The above calculated propagator 3.13 which we showed was the same as $D(x-y)$ is actually one of many propagators. The one we calculated is actually called the *retarded Greens function* as we assumed $x^0 - y^0 > 0$ so we only considered measurements propagating from the past to the future. If we instead had $x^0 - y^0 < 0$ then we would have had to have chosen the contour C_1 in the upper half plane, but then since our poles were shifted into the lower half plane now our contour does not enclose the poles, so our integral must be 0. With this we can conclude that the Retarded Greens function can be written as:

$$\boxed{D_R(x-y) = \theta(x^0 - y^0) \langle 0| [\phi(x), \phi(y)] |0\rangle} \tag{3.14}$$

Where θ is Heaviside's step function. We could have also shifted the poles in the upper half plane and then required that $x^0 - y^0 < 0$ for the integral to be non-zero, thus we can similarly define the *Advanced Greens function*, as:

$$\boxed{D_A(x-y) = \theta(y^0 - x^0) \langle 0| [\phi(x), \phi(y)] |0\rangle} \tag{3.15}$$

This function is usually not used though as it corresponds to measurements propagating from future to past.

3.6.2 Feynman Propagator

- There is one last propagator that we can construct by a combination of the others. What we can do is shift one pole down and the other up. This way both of the paths C_1 and C_2 only enclose a single pole. We usually choose the convention of shifting the $-E_{\mathbf{p}}$ pole down and the other up. This means that we have to use the C_2 contour when $x^0 - y^0 > 0$ and the C_1 contour when $x^0 - y^0 < 0$. This means the *Feynman Propagator* is:

$$\boxed{\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & x^0 < y^0 \end{cases} \\ &= \theta(x^0 - y^0) \langle 0| \phi(x), \phi(y) |0\rangle + \theta(y^0 - x^0) \langle 0| [\phi(y), \phi(x) |0\rangle \\ &= \langle 0| T\phi(x), \phi(y) |0\rangle \end{aligned}} \tag{3.16}$$

Where in the last step we have defined the *Time ordering symbol* T.

4 Representations of the Poincaré Group

- We are quite familiar with the Lorentz group $SO(3,1)$, that generates rotations and boosts. In general we can find Lorentz invariant quantities by finding anything that is a scalar, i.e. all its indices contract such that it can't change under transformations. This is an excellent method for finding invariants, but it does not capture everything. For all Isometries (distance preserving maps) we need to extend the Lorentz group to the *Poincaré* group. This is nothing more then the semi-Direct product of Rotations and translations in $\mathbb{R}^{1,3}$. This is usually written as $P = \mathbf{R}^{1,3} \rtimes O(1,3)$.

  A general transformation of the Poincaré group takes the form $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$. So the elements of the group take the form:

$$\begin{pmatrix} \Lambda_0^0 & \Lambda_1^0 & \Lambda_2^0 & \Lambda_3^0 & a^0 \\ \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 & a^1 \\ \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 & a^2 \\ \Lambda_0^3 & \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4.1 Group actions

- We would like to be able to classify how fields, as well as other quantities transform under Lorentz Transformations. To aid with this we can use the fact that each of the transformation matrices $\Lambda^\mu{}_\nu$ is an element of the Lorentz group $SO(1,3)$. Then if we take the perspective of *active* transformations we can think of the action of a Lorentz transformation as not affecting a single particle, but as acting on the manifold of spacetime itself. This is then a group action.

- This Group G acting on the manifold must then be a Lie group and must act on the elements of the manifold M such that:

  - There is, associated with each of the elements $g \in G$ a diffeomorphism from M to itself, $x \mapsto \mathcal{T}_g(x),\ x \in M$. Such that $\mathcal{T}_g\mathcal{T}_h = \mathcal{T}_{gh} \quad \forall\ g,h \in G$
  - $\mathcal{T}_g$ depends smoothly on the arguments x and g.

  This first condition is actually quite constringent as we will see later.

4.1.1 Left actions

- For elements of a manifold $x \in M$, we can define the following *Left* action of a group G as:

$$\begin{aligned} x \to x' &= g \cdot x \\ \implies x'' &= h \cdot g \cdot x = (h \cdot g) \cdot x \end{aligned}$$

  As can be seen in the last line, this action satisfies $\mathcal{T}_h\mathcal{T}_g = \mathcal{T}_{hg}$.

4.1.2 Right actions

- Similarly we can define the *Right* action as:

$$\begin{aligned} x \to x' &= x \cdot g^{-1} \\ \implies x'' &= x \cdot g^{-1} \cdot h^{-1} = x \cdot (h \cdot g)^{-1} \end{aligned}$$

  As can be seen in the last line, this action satisfies $\mathcal{T}_h\mathcal{T}_g = \mathcal{T}_{hg}$. Note this could not work if we didn't change to the action by inverses.

4.2 Transformation of fields

- Equipped with the above tools, we can now go about considering how the fields transform under active transformations. If we have a field $\phi(x)$ for $x \in M$ our manifold of spacetime. Then under active transformation these co-ordinates x transform via $x \to x' = \Lambda x$. For now if we denote this matrix Λ by g as it is an element of a group, then we can consider the transformation of the field ϕ as:

$$\begin{aligned} \phi(x) \to \phi'(x) &= \phi(g \cdot x) \\ \implies \phi''(x) &= \phi'(h \cdot x) = \phi(g \cdot (h \cdot x)) \end{aligned}$$

  But this does not respect the order we needed from a Lie group, namely here we have $\mathcal{T}_h \mathcal{T}_g = \mathcal{T}_{gh}$. To fix this we can take inspiration from the definition of the Right action above and say that if we replace g with g^{-1} then we have that $\phi''(x) = \phi'(h^{-1} \cdot x) = \phi(g^{-1} \cdot (h^{-1} \cdot x)) = \phi((h \cdot g)^{-1} \cdot x)$ as needed. Thus we can say that the fields must transform via:

$$\boxed{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} \cdot x)} \tag{4.1}$$

4.3 Lorentz Group

- Let us remind ourselves what we know about the Lorentz group. It can be described as the space of operations that leave the quantity $x_\mu x^\mu$ unchanged. That is if $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, and the raising and lowering of indices is governed by the Minkowski metric such that $x^\mu_\mu = x^\mu \eta_{\mu\nu} x^\nu$, then we can write that:

$$\begin{aligned} x^\mu \eta_{\mu\nu} x^\nu = x_\mu x^\mu = x'_\mu x'^\mu &= x'^\mu \eta_{\mu\nu} x'^\nu \\ &= \Lambda^\mu{}_\alpha x^\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta x^\beta \\ \implies \eta_{\alpha\beta} &= \Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta \end{aligned} \tag{4.2}$$

  Where in the last step we have relabeled some indices to get an equality. This last statement is often used as the defining definition of the Lorentz group.

- We should also note that swapping which order we have the indices, transposes our matrix as that is the definition of transposing. So if we want to write The above equation 4.2 in index free form, we have to swap the indies on the first Lambda resulting in $\eta = \Lambda^T \eta \Lambda$. We can then take the determinant of both sides to see that $det(\Lambda^T \Lambda) = 1 \implies det(\Lambda) = \pm 1$ as $det(\Lambda^T) = det(\Lambda)$. A Lorentz transform is called *proper* if $det(\Lambda) = 1$ and *improper* if $det(\Lambda) = -1$.

- If we look at just the $\eta^{00} = 1$ term we know $\eta^{00} = \Lambda^0{}_\alpha \eta^{\alpha\beta} \Lambda^0{}_\beta = (\Lambda^0{}_0)^2 - (\Lambda^0{}_i)^2 = 1$. So we then have two cases, $\Lambda^0{}_0 \geq 1$ which is called *orthochronous* or $\Lambda^0{}_0 \leq -1$ which is called *non-orthochronous*. The Lorentz Group contains Rotations, Boosts, time reversals/space reversals and space-time reversals. All of which fall into one of the 4 categories of being orthochronous/non-orthochronous and proper/improper.

4.3.1 Generators

- We can find the generators of these transformations by considering an infinitesimal transformation, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}$. For this to then satisfy 4.2 we then have that:

$$\begin{aligned} \eta_{\alpha\beta} &= (\delta^{\mu}{}_{\alpha} + \varepsilon^{\mu}{}_{\alpha})\eta_{\mu\nu}(\delta^{\nu}{}_{\beta} + \varepsilon^{\nu}{}_{\beta}) \\ &= \eta_{\alpha\beta} + \varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} + \mathcal{O}(\varepsilon^2) \\ &\implies \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \end{aligned}$$

  So the generators must be anti-symmetric.

- If we then have some transformation that is parametrized by some small parameter λ then we can find out what the generator for that transformation is via:

$$M^{\mu\nu} = \frac{d}{d\lambda}\Lambda(\lambda;\mu,\nu)\bigg|_{\lambda=0}$$

  Where by saying Λ is a function of μ and ν we mean that Λ has a different matrix structure for each of the different types of transformations. i.e. one for each possible rotation direction (which requires two indices μ and ν describing the plane on which the rotation is happening). Recall also that boosts are just rotations with one of the indices being time. This means that each $M^{\mu\nu}$ is an anti-symmetric 4×4 matrix. We can write these explicitly as:

$$(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu} \tag{4.3}$$

- To recover the full transformation generated by these generators we have an infinite number of infinitely small steps so:

$$\Lambda = \lim_{N\to\infty}\left(\mathbb{I} + \frac{1}{N}\omega_{\mu\nu}M^{\mu\nu}\right)^N = e^{\omega_{\mu\nu}M^{\mu\nu}}$$

  Where here $\omega_{\mu\nu}$ is an anti-symmetric tensor containing the shift corresponding to each transformation as we define the generators $M^{\mu\nu}$ to be anti-symmetric. It can then be shown that these generators follow the following commutation relations:

$$\boxed{[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}} \tag{4.4}$$

  This can be shown from the matrices that one gets from 4.3. We can recognize that the boost generators are just the $K_i = M_{0i}, \quad i = 1,2,3$ and the rotation generators are $L_i = \frac{1}{2}\epsilon_{ijk}M_{jk}\ i,j = 1,2,3$. Where the rotation generator, L_i generates a rotation in the $j-k$ plane. It can then be shown that the above commutators 4.4 lead to the regular commutators we are used to. Namely:

$$\begin{aligned} [L_i, L_j] &= \epsilon_{ijk}L_k \\ [K_i, K_j] &= -\epsilon_{ijk}L_k \\ [L_i, K_j] &= \epsilon_{ijk}K_k \end{aligned} \tag{4.5}$$

- For our purposes of QFT we can obtain the form of these generators by recognizing that the rotations will be the standard angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p} = \mathbf{x} \times (i\nabla) \implies J^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$. To fit in with these operators being hermitian we must add and i to the RHS of 4.4 as here $J^{\mu\nu}$ is the same as the $M^{\mu\nu}$ we had before. This then means that the commutators take the form:

$$\boxed{[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}\right)} \tag{4.6}$$

This can also just be checked from the definition of $J^{\mu\nu}$ as $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$.

4.3.2 $\mathfrak{sl}(2, \mathbb{C})$ representation

- We can define the following operators:

$$N_i = \frac{1}{2}(L_i + iK_i)$$
$$N_i^\dagger = \frac{1}{2}(L_i - iK_i)$$

With these definitions, it can be shown using the above commutation relations 4.5, that:

$$[N_i, N_j] = i\epsilon_{ijk}N_k$$
$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger$$
$$[N_I, N_J^\dagger] = 0$$

We can see that these have the structure of two sets separate angular momentum generators. They also have the same structure as the sigma generators of $\mathfrak{sl}(2, \mathbb{C})$. In the same manner we deal with regular angular momentum we can pick N_3 and $N_3^\dagger$ as the z direction which will then have eigenvalues $-n, \ldots, 0, \ldots, n$, where n is the "spin". Similarly $N^\dagger$ has eigenvalues $-m, \ldots, 0, \ldots, m$. That means the total number of possible states of this system is $(2m+1)(2n+1)$. Here $m, n = 0, \frac{1}{2}, 1, \ldots$. This result means that all finite representations of the Lorentz group correspond to pairs of integers or half integers corresponding to pairs of representations of the rotation group.

- As it turns out the $m, n = 0$ corresponds to the Klein-Gordon theory for a spin 0 scalar particle we developed earlier. The two cases $m = \frac{1}{2}, n = 0$ and $m = 0, n = \frac{1}{2}$ correspond to spin half particles which we will see more off later.

4.4 Generators of the Poincaré group

- The Poincaré group has the same generators as the $J^{\mu\nu}$ above 4.6, for the $SO(3,1)$ part, as well as the translation generators $P_\mu = -i\frac{\partial}{\partial x^\mu}$. It can be shown then that they have the following commutation relations:

$$\boxed{\begin{gathered}[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}) \\ [J_{\mu\nu}, P_\sigma] = i(\eta_{\nu\sigma}P_\mu - \eta_{\mu\sigma}P_\nu) \\ [P_\mu, P_\nu] = 0\end{gathered}} \tag{4.7}$$

If we have any sort of generators that act on certain objects to generate transformations of the Poincaré group the only condition on them is that they must obey these commutation relations which are known as the *Commutation relations of the Poincarè Algebra.*

4.4.1 Casimir Invariants

- What we would then like to do is find out what are the Casimir's of these commutators, i.e. operators that commute with everything. Recall that the Casimir operator in quantum mechanics was L^2. It can be shown that the following quantities commute with both $J^{\mu\nu}$ and P^μ:

$$\boxed{\begin{aligned} P_\mu P^\mu &= m^2 \\ W^\mu W_\mu &= W^2 \end{aligned}}$$

Here $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}$ is the *Pauli-Lubański vector*.

5 The Dirac Field

- We have so far only discussed scalar fields that transform via 4.1 and seen that quantization of these fields leads to spin-0 particles that obey the Klein Gordon equation. To find out what describes particles with non-zero spin it is natural to think that we will need some fields that are vector fields of some kind. This means we need to find alternate representation of the Lorentz/Poincaré algebra. Remember that the only condition on the generators of these algebras is that they must satisfy the commutation relations 4.6. We have already found some matrices 4.3 that do this. But what if there is another representation.

5.1 Gamma matrices

- We can find another of these representations due to a trick from Dirac. This turn out to be nothing other then the *Clifford algebra*. Suppose we have a set of $n \times n$ matrices γ^μ satisfying:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_n} \tag{5.1}$$

This is also known as a Dirac algebra. Then using this, it turns out that if we define:

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]} \tag{5.2}$$

Then $S^{\mu\nu}$ satisfy the commutation relations 4.6.