Differential Geometry

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Notes taken in Professor Sergey Frolov's class, Michaelmas Term 2024 "hokay" -Sergey Frolov

Contents

1 Definition of a Manifold

1.1 Regions

- A *region* ("open set") is a set of D points in $\mathbb{R}^n$ such that together with each point p_0, D also contains all points sufficiently closer to p_0, i.e.:

$$\forall\ p_0 = (x_0^1, \ldots, x_0^n) \in D\ \exists\ \epsilon > 0,$$
$$st : p = (x^1, \ldots, x^n) \in D,\ \text{iff}\ |x^i - x_0^i| < \epsilon.$$

- A *region with out a boundary* is obtained fro ma region D by adjoining all boundary points to D. The *boundary* of a region is the set of all boundary points.

1.2 Differentiable Manifold

- A differentiable n-dimensional manifold is a set M together with the following structure on it. The set M is the union of a finite or countably infinite collection of subsets U_q with the following properties:
  - Each subset U_q has defined on it co-ords $x_q^\alpha, \alpha = 1, \ldots, n$ called local co-ords by virtue of which U_q is identifiable with a region of Euclidean n-space $\mathbb{R}^n$ with Euclidean co-ords x_q^α. The U_q with their co-ord systems are called *charts* or *local coordinate neighborhoods*.
  - Each non-empty intersection $U_q \cap U_p$ of a pair of charts thus has defined on it two co-ord systems, the restriction of x_p^α and x_q^α. It is required that under each of these coordinatizations the intersection $U_q \cap U_p$ is identifiable with a region of $\mathbb{R}^n$ and that each of these co-ordinate systems be expressible in terms of the other in a one to one differentiable manner. Thus, if a the *transition* functions from x_p^α to x_q^α and back are given by:

$$x_p^\alpha = x_p^\alpha(x_q^1, \ldots, x_q^n), \quad \alpha = 1, \ldots, n$$
$$x_q^\alpha = x_q^\alpha(x_p^1, \ldots, x_p^n), \quad \alpha = 1, \ldots, n$$

    Then the *Jacobian* $J_{pq} = det(\partial x_p^\alpha / \partial x_q^\alpha)$ is non-zero on $U_p \cap U_q$.

1.3 Abuse of notation

- Regular partial derivative do not have the same "canceling" that total derivative have ($dx * dy/dx = dy$) But we can restore this property through Einstein summation convention. That is that:

$$\boxed{\sum_{\gamma=1}^{n} \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \delta_\beta^\alpha}$$

2 Elements of Topology

2.1 Topological space

- A topological space is a set X of points of which certain subsets called *open sets* of the topological space, are distinguished, these open sets have to satisfy:
  - The intersection of any two (and hence of any finite collection) open sets should again be an open set.
  - The union of any collection of open sets must again be open.
  - The empty set and the whole set X must be open.
- The compliment of any open set is called a *closed* set of the topological space.

  In Euclidean space $\mathbb{R}^n$ the "Euclidean topology" is the usual one where the open sets are the open regions.

2.1.1 Induced topology

- Given any subset $A \in \mathbb{R}^n$, the *induced topology* on A is that where the open sets are the intersections $A \cap U$, where U ranges over all open sets of $\mathbb{R}^n$.

2.1.2 Continuity

- A map $f : X \to Y$ of one topological space to another is called *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subset Y$ is open in X.

2.1.3 Homeomorphic

- Two topological space are *topologically equivalent* or *homeomorphic* if there is a one to one and onto map (bijective) between them, such that it and its inverse are continuous.

2.1.4 Topology on a manifold

- The topology on a manifold M is given by the following specifications of the open sets. In every local co-ordinate neighborhood U_q the open regions are to be open in the topology on M; the totality of open sets of M is then obtained by admitting as open, also arbitrary unions countable collections of such regions, i.e. by closing under countable unions.

2.2 Metric space

- A *metric space* is a set which comes equipped with a "distance function" i.e. a real-valued function $\rho(x, y)$, defined on pairs x, y of its elements and having the following properties:
  - Symmetry: $\rho(x, y) = \rho(y, x)$.
  - Positivity: $\rho(x, x) = 0, \quad \rho(x, y) > 0 \text{ if } x \neq y$.
  - The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

2.2.1 Hausdorff

- A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centers at c, y do not intersect.

  All topological spaces we consider will be Hausdorff.

2.2.2 Compact

- A topological space X is said to be compact if every countable collection of open sets covering X contains a finite sub-collection already covering X.

  If X is a metric space the compactness is equivalent to the condition that from every sequence of points of X a convergent sub-sequence can be selected.

2.2.3 Connected

- A topological space is connected if any two points can be joined by a continuous path.

2.3 Orientation

- A manifold M is said to be *orientated* of one can choose its atlas (collection of all the charts) so that for every pair U_p, U_q of intersecting co-ordinate neighborhoods the Jacobian of the transition functions is positive.
- We say that the co-ordinate systems x and y define the *same orientation* if $J > 0$ and the *opposite orientation* if $J < 0$.

3 Mappings on Manifolds

3.1 Manifold mappings

- A mapping $f : M \to N$ is said to be smooth of smoothness class k if for all p, q for which f determines functions $y^b_q(x^1_p, \ldots, x^m_p) = f(x^1_p, .., x^m_p)^b_p$, these functions are, where defined, smooth of smoothness calss k (i.e. all theire partial derivatives up to those of k-th order exist and are continuous).

  the smoothness class of f cannot exceed the maximum class of the manifolds.

3.2 equivilent manifolds

- The manifolds M and N are said to be *smoothly equivilent* or *diffeomorphic* if there is a one to one and onto map f such that both $f : M \to N$ and $f^{-1} : N \to M$ are smooth of some class $k \geq 1$.

  Since f^{-1} exits then the jacobian $J_{pq} \neq 0$ wherever it is defined.

3.3 Tangent vector

- A *tangent* vector to an m-dim manifold M at an arbitrary point x is represented in terms of local co-ords $x^\alpha_- p$ by an m tuple ξ^α of components which are linked to the components in terms of any oher system x^β_q of local co-ords by:

$$\xi^\alpha_p = \left(\frac{\partial x^\alpha_p}{\partial x^\beta_q}\right)_x \xi^\beta_q, \quad \forall\ \alpha \tag{3.1}$$

- The set of all tangent vectors to an m-dim manifold M at a point x forms an m-dm vector space $T_x = T_x M$, the *tangent space* to M at the point x.
- Thus, the velocity at x of any smooth curve M through x is a tangent vector to M at x.

3.4 Push forward

- A smooth map f from M to N gives rise for each x to a *push forward* or an *induced linear* map to tangent spaces:

$$f_* : T_x M \to T_{f(x)} N$$

  defined as sending the velocity at x of any smooth curve $x = x(\tau)$ on M to the velocity vector at $f(x)$ of the curve $f(x(\tau))$ on N. If the map f is given by: $y^b = f^b(x^1, \ldots, x^m)$ for $x \in M$ and $y \in N$, then the push forward map f_* is:

$$\xi^\alpha \to \eta^b = \frac{\partial f^b}{\partial x^\alpha} \xi^\alpha.$$

- For a real valued function $f : M- > \mathbb{R}$, the push-forward map f_* corresponding to each $x \in M$ is a real valued linear function on the tangent space to M at x:

$$\xi^a \to \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha$$

  and it is represented by the gradiant of f at x, and is a co-vector or one form. Thus f_* can be identified with the differential df, in particular:

$$dx^\alpha_p : \xi^\alpha \to \eta = \xi^\alpha_p$$

3.5 Directional derivative

- We can associate with each vector $\xi = (\xi^i)$ a linear differential operator as follows: Since the gradiant $\frac{\partial f}{\partial x^i}$ of a function f is a co-vector, the quantity:

$$\partial_\xi f = \xi^i \frac{\partial f}{\partial x^i}$$

  is a scalar called the directional derivative of f in the direction of ξ.

- Thus an arbitrary vector ξ corresponds to the operator:

$$\partial_\xi = \xi^i \frac{\partial}{\partial x^i}$$

  So we can identify $\frac{\partial}{\partial x^i} \equiv e_i$ as the *Canonical basis of the tangent space.*

3.6 Riemann metric

- A *Riemann metric* on a manifold M is a point-depedant, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local co-ords of the points.

  Thus at each point $x = (x_p^1, \ldots, x_p^m)$ of each cahrt U_p, the metric is given by a symetric matric $g_{\alpha\beta}(x_p^1, ..., x_p^m)$, and determines a symmetric scalar product of pairs of tangent vectors at the point x.

$$\langle \xi, \eta \rangle = g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = \langle \eta, \xi \rangle \,, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

  This scalar product is to be co-ordiante independant:

$$g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = g_{\alpha\beta}^{(q)} \xi_q^\alpha \eta_q^\beta$$

  And therefor the coefficants $g_{\alpha\beta}^{(p)}$ of the quadratic form transform as:

$$g_{\gamma\delta}^{(q)} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_p^\beta}{\partial x_q^\delta} g_{\alpha\beta}^{(p)} \tag{3.2}$$

  For a *psudo-Reimann* metric M one just requires the quadratic fom to be *nondegenerate*. Note that 3.2 can be re-written as:

$$ds^2 = g_{\alpha\beta}^{(p)} dx_p^\alpha dx_p^\beta = g_{\alpha\beta}^{(q)} dx_q^\alpha dx_q^\beta$$

  Where ds is called a line element, and it is chart-independant. ds is used to measure the distance between two infitesimally close points.

4 Tensors

4.1 Tensor def

- A *tensor of type* (k,l) and rank $k+l$ on an m-dim manifold M is given each local co-ord system (x_p^i) by a family of functions:

$$^{(p)}T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x) \text{ of the point } x.$$

In other local co-ord (x_q^i) the components $^{(p)}T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x)$ of the same tensor are:

$$^{(p)}T^{s_1,\ldots,s_k}_{t_1,\ldots,t_l}(x) = \frac{\partial x_q^{s_1}}{\partial x_p^{i_1}} \cdots \frac{\partial x_q^{s_k}}{\partial x_p^{i_k}} \frac{\partial x_p^{j_1}}{\partial x_q^{t_1}} \cdots \frac{\partial x_p^{j_l}}{\partial x_q^{t_l}} \cdot \; ^{(p)}T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x)$$

4.2 Operations on Tensors

4.2.1 Permutation of indices

- Let σ be some permutation of $1,2,\ldots,l$. σ acrs on the ordered tuple $(j_1,\ldots,j_l)$ as $\sigma(j_1,\ldots,j_l) = (j_{\sigma_1},\ldots,j_{\sigma_l})$. We say that a tensor $\tilde{T}^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x)$ =is obtained from a tensor $T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x)$ by means of a permutation σ of the lower indices if at each point of M:

$$\tilde{T}^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x) = T^{i_1,\ldots,i_k}_{\sigma(j_1,\ldots,j_l)}(x)$$

Permutation of upper indicies is defined similarly.

4.2.2 Contraction of indicies

- By the contraction of a tensor $T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}(x)$ of type (k,l) with respect to the indcies i_a, j_a we mean the tensor (summation over n):

$$T^{i_1,\ldots,i_{k-1}}_{j_1,\ldots,j_{l-1}}(x) = T^{i_1,\ldots i_{a-1},n,i_{a+1},\ldots,i_k}_{j_1,\ldots,j_{a-1},n,j_{a+1},\ldots,j_l}(x)$$

Of type $(k-1,l-1)$

4.2.3 Product of Tensors

- Given two tensors $T = \left(T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}\right)$ of type (k,l) and $P = \left(P^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}\right)$ of type (p,q), we define tehir product to be the tensor product $S = T \otimes P$ of type $(k+p, l+q)$ with components:

$$S^{i_1,\ldots,i_{k+p}}_{j_1,\ldots,j_{l+q}} = T^{i_1,\ldots,i_k}_{j_1,\ldots,j_l} P^{i_{k+1},\ldots,i_p}_{j_{l+1},\ldots,j_q}$$

This multiplication is *not commutative* but it is associative.

- The result of applying the above three operations to tensors are again tensors.

4.3 Co-Vectors

- Recall that the differnetial of a function f of $x^1, \ldots, x^n$ coresponding to the incriments dx^i in the x^i is:

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Since dx^i is a vector df has the same value in any co-ord system. In general, given any co-vector (T_i), the differnetial form $T_i dx^i$ is invariant under change of chart. We can thus identify $dx^i \equiv e^i$ as the *canonical basis of co-vectors or cotangent space.*

4.4 Skew-Symmetric Tesnsor

- A *skew-symmetric tensor* of type $(0, k)$ is a tensor $T_{i_1,\ldots,i_k}$ satisfying:

$$T_{\sigma(i_1,\ldots,i_k)} = \mathfrak{s}(\sigma) T_{i_1,\ldots,i_k}$$

where for all permutations $\mathfrak{s}(\sigma)$ is the sign function. i.e. $\mathfrak{s}(\sigma) = +1(-1)$ for even(odd) permutation. If two indicies of $T_{i_1,\ldots,i_k}$ are the same then the corresponding component of $T_{i_1,\ldots,i_k}$ is 0. This means if $k > n$ the tensor is automatically 0.

- The standard basis at a given point is:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < i_2 < \cdots < i_k$$

Where:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathfrak{s}(\sigma) e^{i_{\sigma_1}} \otimes \cdots \otimes e^{i_{\sigma_k}}$$

Here S_k is the symmetric group. i.e. the group of all permutations of k elements.

- The differential form of the skew-symmetric tensor $(T_{i_1,\ldots,i_k})$ is:

$$\boxed{\begin{aligned} T_{i_1,\ldots,i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} &= \sum_{i_1 < i_2 < \cdots < i_k} T_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!} T_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}}$$

Where the last step can be made as both $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $T_{i_1,\ldots,i_k}$ are anti-symmetric.

4.5 Volume element

- A metric g_{ij} on a manifold is a tensor of type $(0, 2)$ and on an oriented manifold of $dim(M) = n$ such a metric gives rise to a *volume element*:

$$T_{i_1,\ldots,i_n} = \sqrt{|g|} \epsilon i_1, \ldots, i_n, \quad g = det(g_{ij})$$

It is covenient to write te volume element in the notation of differential forms:

$$\Omega = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

If g_{ij} is Riemann then the *volume* V of M is:

$$V = \int_M \Omega = \int_M \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$