Standard Model

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"If you can't explain it simply enough you don't understand it well enough"

- Albert Einstein

Contents

1 The Quark Model

- Long ago it was realized that the proton and the neutron have very similar masses and thus in regions where the Electromagnetic is weak compared to the strong force, there is an approximate symmetry between the neutron. With this, it was postulated that these two particles were two states of the same particle, *the nucleon*. With this in analogous to spin states, we can write the proton as $|p\rangle = (1,0)^T$ and the neutron as $|n\rangle = (0,1)^T$. This lead to the idea of *Isospin*, as the proton and the neutron can be considered to form an isospin doublet, with total Isospin 1/2 and a third component of $I_3 = \pm 1/2$. Just like spin, our Lagrangian (if we ignore the electromagnetic terms) should be invariant under unitary transformations of these states (which will be $\in U(2)$), meaning there is a conserved charge associate with this transformation. We can do this exact same procedure for the up and down quarks. The strong force treats all the quarks equally, and seeing as the up and down quark have approximately the same masses, we can treat them as spin states just like the nucleon. In this case the conserved quantity associated with this symmetry is known as *flavor*.

1.1 Isospin

- $U(2)$ has 4 degrees of freedom, and thus 4 generators, one of these can be chosen to be a scaling by a phase factor of the identity, $e^{i\theta}\mathbb{1}$, since overall phase factors of $U(1)$ do not change our states, this can be ignored, leaving us with the 3 Pauli matrices σ^i, which are the generators of $SU(2)$, the main symmetry group here. From this we can proceed in the exact same manner as we do with spin, recognizing that these matrices form a non-Abelian Lie algebra based on their commutators, where we can define raising and lowering operators, to enable us to write down states analogous to $|lm\rangle$ for spin. This quantity that is like spin is called *Isospin*. This is 3-vector and is defined as:

$$\mathbf{T} = \frac{1}{2}\boldsymbol{\sigma}$$

- The components obey the following commutations relations:

$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

Where we have sum of the index k. The measurable quantity from this system is the total isospin, $\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2$. We will label states by their total isospin and the third component of isospin I_3, i.e. $\phi(I, I_3)$. The up quark is then $|u\rangle = \phi(\frac{1}{2}, \frac{1}{2})$ and the down quark is $|d\rangle = \phi(\frac{1}{2}, -\frac{1}{2})$.

1.2 Anti-quark Doublet

- The above treatment of up and down quarks is called a *quark doublet*, which we write as:

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

We would like to have the same treatment of anti-quarks. We know that the complex conjugate of any quark will give us the anti-quark (eg. $u^* = \bar{u}$), but we don't want to write down something like $\bar{q} = (\bar{u}, \bar{d})^T$ as then this will transform via U^* instead of U and will not follow the same symmetries. Instead we should find some combination that does transform via U. We write this as $\bar{q} = S(\bar{u}, \bar{d})^T$ and then impose that $SU^* = US$. Solving this equation results in the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Meaning the quark anti-state can be written as:

$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

1.3 Mesons

- Mesons are bound states of a quark and anti-quark pair, since quarks are spin half, this makes mesons bosons. Since Mesons are comprised of two quarks, we can think of adding their Isospin in exactly the same way we add the spin of two particles together. Since the quarks have isospin $\frac{1}{2}$, this will create the familiar $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$. Meaning there will be four possible states, a triplet with $I_3 = 1$ and a singlet with $I_3 = 0$. These state correspond to meson particles! Each state will correspond to more then one particle as we are not considering different combinations of spins (which affect the mass!). These particles are:

$$
\begin{aligned}
\phi(1,1) &= -|u\bar{d}\rangle & &= |\pi^+\rangle, |\rho^+\rangle \\
\phi(1,0) &= \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right) & &= |\pi^0\rangle, |\rho^0\rangle \\
\phi(1,-1) &= |d\bar{u}\rangle & &= |\pi^-\rangle, |\rho^-\rangle \\
\phi(0,0) &= \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right) & &= |\eta\rangle, |\omega\rangle
\end{aligned}
$$

1.4 $SU(3)$ Flavour

- The above described $SU(2)$ symmetry is almost exact as the up and down quarks have almost them same mass. What we can also do is consider extending this symmetry to the strange quark. This makes it a $SU(3)$ symmetry, as the quark doublet now becomes a quark triplet $q = (u, d, s)^T$, for which we will need a new basis of generators. The standard choice of generators are the *Gell Mann matrices*:

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad u \leftrightarrow d
$$

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad u \leftrightarrow s
$$

$$
\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad d \leftrightarrow s
$$

$$
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \text{equal treatment of u,d}
$$

- This particular choice is normalized such that $\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{ij}$. With this we can define the $SU(3)$ Isospin in a similar manner to $SU(2)$ by identifying:

$$
\hat{T}_i = \frac{1}{2}\lambda_i
$$

The total Isospin is then $\sum_i T_i^2 = \frac{1}{4}\lambda_i^2 = \frac{4}{3}\mathbb{1}$. $SU(3)$ is different to $SU(2)$ in the fact that it two mutually commuting generators instead of 1. We can call once again the component along the direction of the third generator T_3 the third component of the isospin I_3 and identify states by this number, but we then also need to consider the component along the direction of $\hat{T}_8$ as $\hat{T}_8$ commutes with $\hat{T}_3$. The component along this direction is known as the *Hyper-charge* and is denoted with a Y. Strictly we define the hyper charge to be $Y = \frac{1}{\sqrt{3}}\lambda_8$.

- From the definitions of the Gell Mann matrices we can check the values of the I_3 and Y, for the 3 quarks:

$$\hat{T}_3 u = +\frac{1}{2}u \qquad \hat{Y}u = +\frac{1}{3}u$$
$$\hat{T}_3 d = -\frac{1}{2}d \qquad \hat{Y}d = +\frac{1}{3}d$$
$$\hat{T}_3 s = 0 \qquad \hat{Y}s = -\frac{2}{3}s$$

We can plot Y and I_3 for the three quarks and their anti-particles:

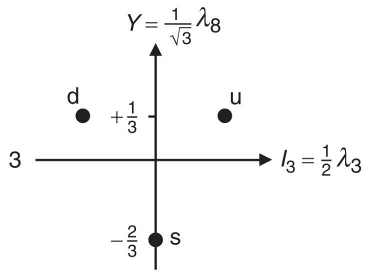

Figure 1: *Quark Isospin and Hyper-charge*

1.5 Light Mesons

- Since we have used λ_3 and λ_8 to form what is know as the Cartan sub-algebra, we can take the remaining λ_i and form raising and lowering operators. There will be three pairs, that which step respectively between the $d \leftrightarrow u, s \leftrightarrow u$ and $d \leftrightarrow s$:

$$\hat{T}_{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \tag{1.1}$$
$$\hat{V}_{\pm} = \frac{1}{2}(\lambda_4 \pm i\lambda_5) \tag{1.2}$$
$$\hat{U}_{\pm} = \frac{1}{2}(\lambda_6 \pm i\lambda_7) \tag{1.3}$$

We can use these to find all possible Mesons made out of these 3 quarks. This is done by identifying the extreme states (stats with maximal I_3 or Y), then apply the raising and lowering operators to exhaust all other possible states. Since we are combining 3 possible quarks with 3 possible anti-quarks [1] then this is a case of $3 \otimes \bar{3} = 8 \oplus 1$ [2]. This decomposition is into a octet and a singlet and takes the below visual form:

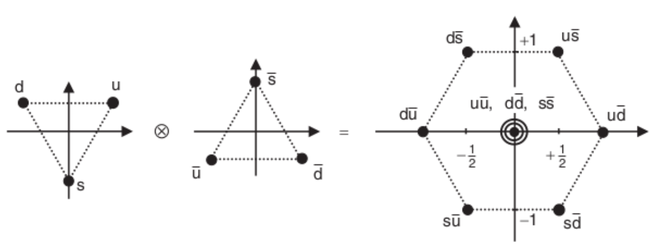

Figure 2: *Decomposition of quark-anti-quark combinations*

[1] Here we are only looking at 2 quark combinations here, i.e. Mesons
[2] The bar on the 3 just indicates that this is the anti-quark triplet

Where the singlet is plotted along with the two octet elements that have $I_3 = Y = 0$. These quark combinations are as we mentioned before Mesons! We now have generated more of these having considered the strange quark as well. We are not how-ever considering spin. Quarks are spin 1/2 particles meaning it is possible to form spin 0 or spin 1 combinations of them. It turns out that these different combinations affect the mass of the resulting bound system, meaning these are different particles. We thus have two sets of 9 particles:

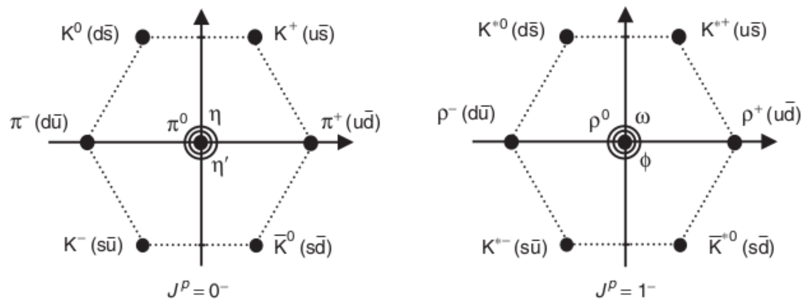

Figure 3: *All Mesons, graphed by Isospin and Hyper-charge*

- In terms of the quarks these are for spin-0:

Pions: $$|\pi^+\rangle = |u\bar{d}\rangle, \quad |\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \quad |\pi^-\rangle = |d\bar{u}\rangle$$

Kaons: $$|K^+\rangle = |u\bar{s}\rangle, \quad |K^0\rangle = |d\bar{s}\rangle, \quad |\bar{K}^0\rangle = |s\bar{d}\rangle, \quad |K^-\rangle = |s\bar{u}\rangle$$

Eta and Eta Prime: $$|\eta\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle), \quad |\eta'\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$$

- And for the spin-1 mesons:

Rho Mesons: $$|\rho^+\rangle = |u\bar{d}\rangle, \quad |\rho^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \quad |\rho^-\rangle = |d\bar{u}\rangle$$

Kaon* Mesons: $$|K^{*+}\rangle = |u\bar{s}\rangle, \quad |K^{*0}\rangle = |d\bar{s}\rangle, \quad |\bar{K}^{*0}\rangle = |s\bar{d}\rangle, \quad |K^{*-}\rangle = |s\bar{u}\rangle$$

Omega and Phi Mesons: $$|\omega\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle), \quad |\phi\rangle = |s\bar{s}\rangle$$

- The Heavy Mesons are constructed from the bottom and charm quarks in a similar way.

1.6 Baryons

- Baryons are combinations of 3 quarks/ anti-quarks. This makes them fermions of spin $\frac{1}{2}$ or 3/2. This means we need to calculate $3 \otimes 3 \otimes 3$. It turns out that the calculation of $3 \otimes 3$ is a little different to that of $3 \otimes \bar{3}$. Since we dont have anti-quarks we can't properly form a state that has $I_3 = Y_3 = 0$. This means the decomposition is $3 \otimes 3 = 6 \oplus 3$. This can be prooved by the standard ladder operator calculations. We are then left with $3 \otimes (6 \oplus 3)$ which breaks down into the standard $3 \otimes \bar{3} = 8 \oplus 1$ and the new $3 \otimes 6 = 10 \oplus 8$. Overall this means the full decomposition is:

$$\boxed{3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1}$$

- We can visulze this nicely with the followong:

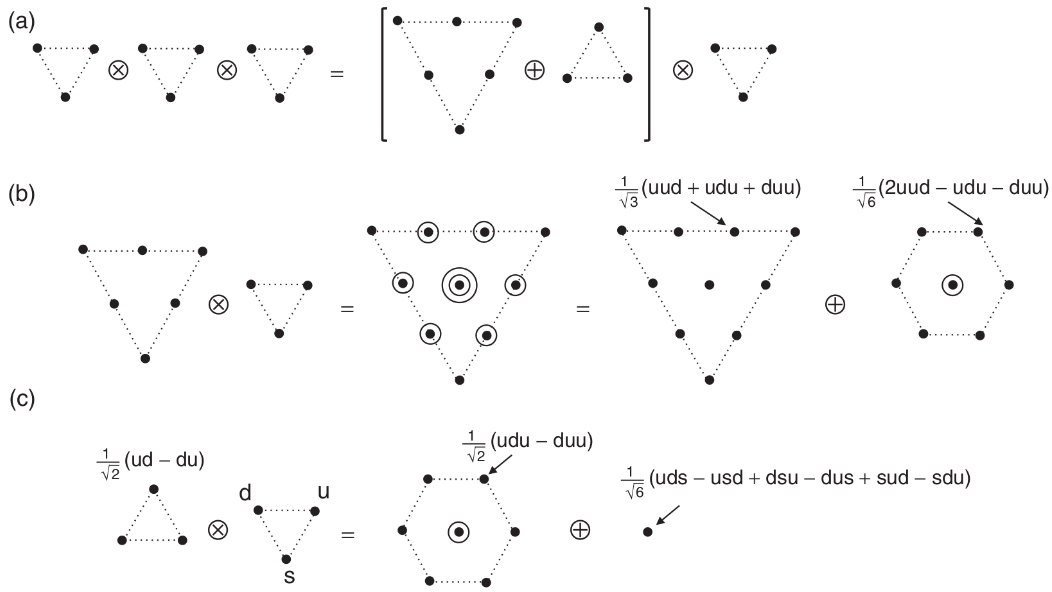

Figure 4: *All Mesons, graphed by Isospin and Hyper-charge*

- The quark composition of the individual Baryons is then:

Nucleons:	$\lvert p\rangle = \lvert uud\rangle, \quad \lvert n\rangle = \lvert udd\rangle$
Delta Baryons:	$\lvert\Delta^{++}\rangle = \lvert uuu\rangle, \quad \lvert\Delta^{+}\rangle = \lvert uud\rangle, \quad \lvert\Delta^{0}\rangle = \lvert udd\rangle, \quad \lvert\Delta^{-}\rangle = \lvert ddd\rangle$
Sigma Baryons:	$\lvert\Sigma^{+}\rangle = \lvert uus\rangle, \quad \lvert\Sigma^{0}\rangle = \lvert uds\rangle, \quad \lvert\Sigma^{-}\rangle = \lvert dds\rangle$
Xi Baryons:	$\lvert\Xi^{0}\rangle = \lvert uss\rangle, \quad \lvert\Xi^{-}\rangle = \lvert dss\rangle$
Omega Baryon:	$\lvert\Omega^{-}\rangle = \lvert sss\rangle$

1.7 Total Wavefunction

- There are many possible values of flavour, spin and colour (which we will encounter later) that a Baryon can have:

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$$

However, not all of these states are valid as since baryons are fermions, there total wavefunctions needs to be anti-symmetric under exchange of any two quarks. We will see later that the colour eavefunction ξ_{colour} is totally anto-symmetric. We are discussing the quarkes with $l = 0$, so zero spatial anular momentum , and since the spatial wavfunction transforms by $(-1)^l$ under parity, then η_{space} is symmetric. This means that $\phi_{\text{flavour}}\chi_{\text{spin}}$ must be symmetric. This allows us to determine the wave-function super positions of the quarks in terms of their flavour and spin.