

# Differential Geometry

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“hokay” -Sergey Frolov

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# 1 Definition of a Manifold

## 1.1 Regions

- A *region* (“open set”) is a set of  $D$  points in  $\mathbb{R}^n$  such that together with each point  $p_0$ ,  $D$  also contains all points sufficiently closer to  $p_0$ , i.e.:

$$\forall p_0 = (x_0^1, \dots, x_0^n) \in D \exists \epsilon > 0, \\ \text{st } p = (x^1, \dots, x^n) \in D, \text{ iff } |x^i - x_0^i| < \epsilon.$$

- A *region with out a boundary* is obtained from a region  $D$  by adjoining all boundary points to  $D$ . The *boundary* of a region is the set of all boundary points.

## 1.2 Differentiable Manifold

- A differentiable  $n$ -dimensional manifold is a set  $M$  together with the following structure on it. The set  $M$  is the union of a finite or countably infinite collection of subsets  $U_q$  with the following properties:
  - Each subset  $U_q$  has defined on it co-ords  $x_q^\alpha, \alpha = 1, \dots, n$  called local co-ords by virtue of which  $U_q$  is identifiable with a region of Euclidean  $n$ -space  $\mathbb{R}^n$  with Euclidean co-ords  $x_q^\alpha$ . The  $U_q$  with their co-ord systems are called *charts* or *local coordinate neighborhoods*.
  - Each non-empty intersection  $U_q \cap U_p$  of a pair of charts thus has defined on it two co-ord systems, the restriction of  $x_p^\alpha$  and  $x_q^\alpha$ . It is required that under each of these coordinatizations the intersection  $U_q \cap U_p$  is identifiable with a region of  $\mathbb{R}^n$  and that each of these co-ordinate systems be expressible in terms of the other in a one to one differentiable manner. Thus, if a *transition* functions from  $x_p^\alpha$  to  $x_q^\alpha$  and back are given by:

$$x_p^\alpha = x_p^\alpha(x_q^1, \dots, x_q^n), \quad \alpha = 1, \dots, n \\ x_q^\alpha = x_q^\alpha(x_p^1, \dots, x_p^n), \quad \alpha = 1, \dots, n$$

Then the *Jacobian*  $J_{pq} = \det(\partial x_p^\alpha / \partial x_q^\alpha)$  is non-zero on  $U_p \cap U_q$ .

## 1.3 Abuse of notation

- Regular partial derivative do not have the same “canceling” that total derivative have ( $dx * dy / dx = dy$ ) But we can restore this property through Einstein summation convention. That is that:

$$\sum_{\gamma=1}^n \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \delta_\beta^\alpha$$

## 2 Elements of Topology

### 2.1 Topological space

- A topological space is a set  $X$  of points of which certain subsets called *open sets* of the topological space, are distinguished, these open sets have to satisfy:
  - The intersection of any two (and hence of any finite collection) open sets should again be an open set.
  - The union of any collection of open sets must again be open.
  - The empty set and the whole set  $X$  must be open.
- The compliment of any open set is called a *closed* set of the topological space.

In Euclidean space  $\mathbb{R}^n$  the “Euclidean topology” is the usual one where the open sets are the open regions.

#### 2.1.1 Induced topology

- Given any subset  $A \in \mathbb{R}^n$ , the *induced topology* on  $A$  is that where the open sets are the intersections  $A \cap U$ , where  $U$  ranges over all open sets of  $\mathbb{R}^n$ .

#### 2.1.2 Continuity

- A map  $f : X \rightarrow Y$  of one topological space to another is called *continuous* if the complete inverse image  $f^{-1}(U)$  of every open set  $U \subset Y$  is open in  $X$ .

#### 2.1.3 Homeomorphic

- Two topological space are *topologically equivalent* or *homeomorphic* if there is a one to one and onto map (bijective) between them, such that it and its inverse are continuous.

#### 2.1.4 Topology on a manifold

- The topology on a manifold  $M$  is given by the following specifications of the open sets. In every local co-ordinate neighborhood  $U_q$  the open regions are to be open in the topology on  $M$ ; the totality of open sets of  $M$  is then obtained by admitting as open, also arbitrary unions countable collections of such regions, i.e. by closing under countable unions.

### 2.2 Metric space

- A *metric space* is a set which comes equipped with a “distance function” i.e. a real-valued function  $\rho(x, y)$ , defined on pairs  $x, y$  of its elements and having the following properties:
  - Symmetry:  $\rho(x, y) = \rho(y, x)$ .
  - Positivity:  $\rho(x, x) = 0$ ,  $\rho(x, y) > 0$  if  $x \neq y$ .
  - The triangle inequality:  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ .

**2.2.1 Hausdorff**

- A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius  $\rho(x, y)/3$  with centers at  $x, y$  do not intersect.

All topological spaces we consider will be Hausdorff.

**2.2.2 Compact**

- A topological space  $X$  is said to be compact if every countable collection of open sets covering  $X$  contains a finite sub-collection already covering  $X$ .

If  $X$  is a metric space the compactness is equivalent to the condition that from every sequence of points of  $X$  a convergent sub-sequence can be selected.

**2.2.3 Connected**

- A topological space is connected if any two points can be joined by a continuous path.

**2.3 Orientation**

- A manifold  $M$  is said to be *orientated* if one can choose its atlas (collection of all the charts) so that for every pair  $U_p, U_q$  of intersecting co-ordinate neighborhoods the Jacobian of the transition functions is positive.
- We say that the co-ordinate systems  $x$  and  $y$  define the *same orientation* if  $J > 0$  and the *opposite orientation* if  $J < 0$ .

## 3 Mappings on Manifolds

### 3.1 Manifold mappings

- A mapping  $f : M \rightarrow N$  is said to be smooth of smoothness class  $k$  if for all  $p, q$  for which  $f$  determines functions  $y_q^b(x_p^1, \dots, x_p^m) = f(x_p^1, \dots, x_p^m)_p^b$ , these functions are, where defined, smooth of smoothness class  $k$  (i.e. all their partial derivatives up to those of  $k$ -th order exist and are continuous).

the smoothness class of  $f$  cannot exceed the maximum class of the manifolds.

### 3.2 Equivalent manifolds

- The manifolds  $M$  and  $N$  are said to be *smoothly equivalent* or *diffeomorphic* if there is a one to one and onto map  $f$  such that both  $f : M \rightarrow N$  and  $f^{-1} : N \rightarrow M$  are smooth of some class  $k \geq 1$ .

Since  $f^{-1}$  exists then the Jacobian  $J_{pq} \neq 0$  wherever it is defined.

### 3.3 Tangent vector

- A *tangent* vector to an  $m$ -dim manifold  $M$  at an arbitrary point  $x$  is represented in terms of local co-ords  $x_p^\alpha$  by an  $m$  tuple  $\xi^\alpha$  of components which are linked to the components in terms of any other system  $x_q^\beta$  of local co-ords by:

$$\xi_p^\alpha = \left( \frac{\partial x_p^\alpha}{\partial x_q^\beta} \right)_x \xi_q^\beta, \quad \forall \alpha \quad (3.1)$$

- The set of all tangent vectors to an  $m$ -dim manifold  $M$  at a point  $x$  forms an  $m$ -dim vector space  $T_x = T_x M$ , the *tangent space* to  $M$  at the point  $x$ .
- Thus, the velocity at  $x$  of any smooth curve  $M$  through  $x$  is a tangent vector to  $M$  at  $x$ .

### 3.4 Push forward

- A smooth map  $f$  from  $M$  to  $N$  gives rise for each  $x$  to a *push forward* or an *induced linear* map to tangent spaces:

$$f_* : T_x M \rightarrow T_{f(x)} N$$

defined as sending the velocity at  $x$  of any smooth curve  $x = x(\tau)$  on  $M$  to the velocity vector at  $f(x)$  of the curve  $f(x(\tau))$  on  $N$ . If the map  $f$  is given by:  $y^b = f^b(x^1, \dots, x^m)$  for  $x \in M$  and  $y \in N$ , then the push forward map  $f_*$  is:

$$\xi^\alpha \rightarrow \eta^b = \frac{\partial f^b}{\partial x^\alpha} \xi^\alpha.$$

- For a real valued function  $f : M \rightarrow \mathbb{R}$ , the push-forward map  $f_*$  corresponding to each  $x \in M$  is a real valued linear function on the tangent space to  $M$  at  $x$ :

$$\xi^\alpha \rightarrow \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha$$

and it is represented by the gradient of  $f$  at  $x$ , and is a co-vector or one form. Thus  $f_*$  can be identified with the differential  $df$ , in particular:

$$dx_p^\alpha : \xi^\alpha \rightarrow \eta = \xi_p^\alpha$$



### 3.5 Directional derivative

- We can associate with each vector  $\xi = (\xi^i)$  a linear differential operator as follows: Since the gradient  $\frac{\partial f}{\partial x^i}$  of a function  $f$  is a co-vector, the quantity:

$$\partial_\xi f = \xi^i \frac{\partial f}{\partial x^i}$$

is a scalar called the directional derivative of  $f$  in the direction of  $\xi$ .

- Thus an arbitrary vector  $\xi$  corresponds to the operator:

$$\partial_\xi = \xi^i \frac{\partial}{\partial x^i}$$

So we can identify  $\frac{\partial}{\partial x^i} \equiv e_i$  as the *Canonical basis of the tangent space*.

### 3.6 Riemann metric

- A *Riemann metric* on a manifold  $M$  is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local co-ords of the points.

Thus at each point  $x = (x_p^1, \dots, x_p^m)$  of each chart  $U_p$ , the metric is given by a symmetric metric  $g_{\alpha\beta}(x_p^1, \dots, x_p^m)$ , and determines a symmetric scalar product of pairs of tangent vectors at the point  $x$ .

$$\langle \xi, \eta \rangle = g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = \langle \eta, \xi \rangle, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

This scalar product is to be co-ordinate independent:

$$g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = g_{\alpha\beta}^{(q)} \xi_q^\alpha \eta_q^\beta$$

And therefor the coefficients  $g_{\alpha\beta}^{(p)}$  of the quadratic form transform as:

$$g_{\gamma\delta}^{(q)} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_p^\beta}{\partial x_q^\delta} g_{\alpha\beta}^{(p)} \quad (3.2)$$

For a *pseudo-Riemann* metric  $M$  one just requires the quadratic form to be *nondegenerate*. Note that 3.2 can be re-written as:

$$ds^2 = g_{\alpha\beta}^{(p)} dx_p^\alpha dx_p^\beta = g_{\alpha\beta}^{(q)} dx_q^\alpha dx_q^\beta$$

Where  $ds$  is called a line element, and it is chart-independent.  $ds$  is used to measure the distance between two infinitesimally close points.

## 4 Tensors

### 4.1 Tensor def

- A *tensor of type*  $(k, l)$  and rank  $k + l$  on an  $m$ -dim manifold  $M$  is given each local co-ord system  $(x_p^i)$  by a family of functions:

$${}^{(p)}T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x) \text{ of the point } x.$$

In other local co-ord  $(x_q^i)$  the components  ${}^{(p)}T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$  of the same tensor are:

$${}^{(p)}T_{t_1, \dots, t_l}^{s_1, \dots, s_k}(x) = \frac{\partial x_q^{s_1}}{\partial x_p^{i_1}} \dots \frac{\partial x_q^{s_k}}{\partial x_p^{i_k}} \frac{\partial x_p^{j_1}}{\partial x_q^{t_1}} \dots \frac{\partial x_p^{j_l}}{\partial x_q^{t_l}} \cdot {}^{(p)}T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$$

### 4.2 Operations on Tensors

#### 4.2.1 Permutation of indices

- Let  $\sigma$  be some permutation of  $1, 2, \dots, l$ .  $\sigma$  acts on the ordered tuple  $(j_1, \dots, j_l)$  as  $\sigma(j_1, \dots, j_l) = (j_{\sigma_1}, \dots, j_{\sigma_l})$ . We say that a tensor  $\tilde{T}_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$  is obtained from a tensor  $T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$  by means of a permutation  $\sigma$  of the lower indices if at each point of  $M$ :

$$\tilde{T}_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x) = T_{\sigma(j_1, \dots, j_l)}^{i_1, \dots, i_k}(x)$$

Permutation of upper indicies is defined similarly.

#### 4.2.2 Contraction of indicies

- By the contraction of a tensor  $T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$  of type  $(k, l)$  with respect to the indicies  $i_a, j_a$  we mean the tensor (summation over  $n$ ):

$$T_{j_1, \dots, j_{l-1}}^{i_1, \dots, i_{k-1}}(x) = T_{j_1, \dots, j_{a-1}, n, j_{a+1}, \dots, j_l}^{i_1, \dots, i_{a-1}, n, i_{a+1}, \dots, i_k}(x)$$

Of type  $(k-1, l-1)$

#### 4.2.3 Product of Tensors

- Given two tensors  $T = (T_{j_1, \dots, j_l}^{i_1, \dots, i_k})$  of type  $(k, l)$  and  $P = (P_{j_1, \dots, j_q}^{i_1, \dots, i_p})$  of type  $(p, q)$ , we define their product to be the tensor product  $S = T \otimes P$  of type  $(k+p, l+q)$  with components:

$$S_{j_1, \dots, j_{l+q}}^{i_1, \dots, i_{k+p}} = T_{j_1, \dots, j_l}^{i_1, \dots, i_k} P_{j_{l+1}, \dots, j_q}^{i_{k+1}, \dots, i_p}$$

This multiplication is *not commutative* but it is associative.

- The result of applying the above three operations to tensors are again tensors.

### 4.3 Co-Vectors

- Recall that the differential of a function  $f$  of  $x^1, \dots, x^n$  corresponding to the increments  $dx^i$  in the  $x^i$  is:

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Since  $dx^i$  is a vector  $df$  has the same value in any co-ord system. In general, given any co-vector  $(T_i)$ , the differential form  $T_i dx^i$  is invariant under change of chart. We can thus identify  $dx^i \equiv e^i$  as the *canonical basis of co-vectors or cotangent space*.

### 4.4 Skew-Symmetric Tensor

- A *skew-symmetric tensor* of type  $(0, k)$  is a tensor  $T_{i_1, \dots, i_k}$  satisfying:

$$T_{\sigma(i_1, \dots, i_k)} = \mathfrak{s}(\sigma) T_{i_1, \dots, i_k}$$

where for all permutations  $\mathfrak{s}(\sigma)$  is the sign function. i.e.  $\mathfrak{s}(\sigma) = +1(-1)$  for even(odd) permutation. If two indices of  $T_{i_1, \dots, i_k}$  are the same then the corresponding component of  $T_{i_1, \dots, i_k}$  is 0. This means if  $k > n$  the tensor is automatically 0.

- The standard basis at a given point is:

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad i_1 < i_2 < \dots < i_k$$

Where:

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathfrak{s}(\sigma) e^{i_{\sigma_1}} \otimes \dots \otimes e^{i_{\sigma_k}}$$

Here  $S_k$  is the symmetric group. i.e. the group of all permutations of  $k$  elements.

- The differential form of the skew-symmetric tensor  $(T_{i_1, \dots, i_k})$  is:

$$\begin{aligned} T_{i_1, \dots, i_k} e^{i_1} \otimes \dots \otimes e^{i_k} &= \sum_{i_1 < i_2 < \dots < i_k} T_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ &= \frac{1}{k!} T_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

Where the last step can be made as both  $dx^{i_1} \wedge \dots \wedge dx^{i_k}$  and  $T_{i_1, \dots, i_k}$  are anti-symmetric.

### 4.5 Volume element

- A metric  $g_{ij}$  on a manifold is a tensor of type  $(0, 2)$  and on an oriented manifold of  $\dim(M) = n$  such a metric gives rise to a *volume element*:

$$T_{i_1, \dots, i_n} = \sqrt{|g|} \epsilon_{i_1, \dots, i_n}, \quad g = \det(g_{ij})$$

It is convenient to write the volume element in the notation of differential forms:

$$\Omega = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$$

If  $g_{ij}$  is Riemann then the *volume*  $V$  of  $M$  is:

$$V = \int_M \Omega = \int_M \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$$

#### 4.6 Generalized push forward

- We can generalize the push forward map we had on vectors earlier to the space of tensors  $(k, 0)$ :

$$f_* : \xi^{i_1, \dots, i_k} \rightarrow \eta^{a_1, \dots, a_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi^{i_1, \dots, i_k}$$

#### 4.7 Pull back

- Let  $T_x^{(0,k)}M$  denote the space of tensors of type  $(0, k)$  at  $x \in M$ . Let  $f$  be a smooth map from  $M$  to  $N$ . It gives rise to a map:

$$f^* : T_{f(x)}^{(0,k)}N \rightarrow T_x^{(0,k)}M$$

which in terms of  $x^i \in U \subset M$ , and  $y^a \in V \subset N$  is written as:

$$f^* : \eta_{a_1, \dots, a_k} \rightarrow \xi_{i_1, \dots, i_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1, \dots, a_k}$$

The map  $f^*$  is called the *pullback*.

- We can then note the following relationship between pullbacks and push forwards. Let us denote the action of a vector on another vector as follows:

$$\zeta(\theta) \equiv \zeta_{i_1, \dots, i_k} \theta^{i_1, \dots, i_k}$$

Then we can write that:

$$(f^*\eta)(\xi) = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1, \dots, a_k} \xi^{i_1, \dots, i_k} = \eta(f_*\xi)$$

## 5 Manifolds and surfaces

### 5.1 Immersion

- A manifold  $M$  of dim  $m$  is said to be immersed in a manifold  $N$  of dim  $n \geq m$  if  $\exists$  a smooth map  $f : M \rightarrow N$  such that the push forward map  $f_*$  is at each point a one to one map of the tangent space.

The map  $f$  is called the *immersion* of  $M$  to  $N$ .

Since  $f_*$  is at each a point one to one map of the tangent space, in terms of local co-ords the Jacobian matrix of  $f$  at each point has rank equal to  $m = \dim M$ .

#### 5.1.1 Embedding

- An immersion of  $M$  to  $N$  is called an *embedding* if it one to one. Then  $M$  is called a *sub-manifold* of  $N$ .
- To see the difference between these two definitions note that a Klein bottle is immersed in  $\mathbb{R}^3$  but not embedded as its tangent spaces are distinct (intersecting points can have different tangent spaces) but the map of points is not one- to one as there are cross overs.

### 5.2 Manifold with boundary

- A closed region  $A$  of a manifold  $M$  defined by an inequality:

$$f(x) \leq 0, \quad (\text{or } f(x) \geq 0)$$

where  $f$  is a real-valued function on  $M$ . This region is a *Manifold with boundary*. It is assumed that the boundary  $\partial A$  given by  $f(x) = 0$  is a non-singular sub-manifold of  $M$  i.e.  $\nabla f \neq 0$  on  $\partial A$ .

#### 5.2.1 Closed manifold

- A compact manifold without a boundary is called *closed*.

### 5.3 Surfaces as Manifolds

- A *Non-singular surface*  $M$  of dimension  $k$  in  $n$ -dim Euclidean space is given by a set of  $n - k$  equations:

$$f_i(x^1, \dots, x^n) = 0, \quad i = 1, \dots, n - k$$

where  $\forall x$  the matrix  $\left( \frac{\partial f_i}{\partial x^\alpha} \right)$  has rank  $n - k$ .

### 5.4 Orientation of surfaces

#### 5.4.1 Orientation class

- Consider a frame  $\tau_1 = (e_1^{(1)}, \dots, e_n^{(1)})$  called an ordered basis and another frame  $\tau_1 = (e_1^{(2)}, \dots, e_n^{(2)})$  then we say that they lie in the *same orientation class* if  $\det A > 0$  and the *opposite orientation*

class if  $\det A < 0$ . Where  $A$  is defined as:

$$A : e_k^{(1)} \rightarrow e_k^{(2)}$$

#### 5.4.2 Orientability

- A manifold is said to be *orientable* if it is possible to choose at every point of it a single orientation class depending continuously on the points.

A particular choice of such an orientation class for each point is called an orientation of the manifold, and a manifold equipped with a particular orientation is said to be *oriented*.

If no orientation exists the manifold is said to be *non-orientable*

#### 5.5 Two-sided hyper-surface

- A connected  $(n - 1)$ -dim sub-manifold of  $\mathbb{R}^n$  is called two sided if a single valued continuous field of unit normals can be defined on it.

such a sub-manifold is called a *two-sided hyper-surface*.

## 6 Lie Groups

### 6.1 Group

- A *group* is a non-empty set  $G$  on which there is defined a binary operation  $(a, b) \rightarrow ab$  satisfying the following properties:
  - Closure: If  $a$  and  $b$  belong to  $G$ , then  $ab \in G$ .
  - Associativity:  $\forall a, b, c \in G, \quad a(bc) = (ab)c$ .
  - Identity:  $\exists$  an element  $1 \in G$  st:  $a1 = 1a = a, \quad \forall a \in G$
  - Inverse: If  $a \in G$  then  $\exists a^{-1} \in G$  st:  $aa^{-1} = a^{-1}a = 1$ .

### 6.2 Lie Group

- A manifold  $G$  is called a *Lie Group* if it has given on it a group operation with the properties that the maps  $\varphi : G \rightarrow G$ , defined by  $\varphi(g) = g^{-1}$  and  $\psi : G \times G \rightarrow G$  defined by  $\psi(g, h) = gh$ , are smooth maps.

### 6.3 Example of Lie groups

#### 6.3.1 General Linear group

- This is  $GL(n, \mathbb{R})$  consisting of all  $n \times n$  real matrices with non zero determinant in a region  $\mathbb{R}^{n^2}$ .  
 $\dim GL(n, \mathbb{R}) = n^2$ .

#### 6.3.2 Special Linear group

- This is  $SL(n, \mathbb{R})$  consisting of all  $n \times n$  real matrices with determinant equal to 1. It is a hyper-surface in  $\mathbb{R}^{n^2}$ .

$$\det A = 1, \quad A \in Mat(n, \mathbb{R})$$

$$\dim SL(n, \mathbb{R}) = n^2 - 1.$$

#### 6.3.3 Orthogonal group

- This is  $O(n, \mathbb{R})$  consisting of all  $n \times n$  real matrices Satisfying:

$$A^T \cdot A = \mathbb{I}, \quad A \in Mat(n, \mathbb{R})$$

$$\dim O(n, \mathbb{R}) = \frac{1}{2}n(n-1).$$

#### 6.3.4 Special Orthogonal group

- This is  $SO(n, \mathbb{R})$  consisting of all  $n \times n$  real matrices Satisfying:

$$A^T \cdot A = \mathbb{I}, \quad \det(A) = 1, \quad A \in Mat(n, \mathbb{R})$$

$$\dim SO(n, \mathbb{R}) = \frac{1}{2}n(n-1).$$

### 6.3.5 Pseudo Orthogonal group

- This is  $O(p, q, n)$  consisting of all  $n \times n$  real matrices Satisfying:

$$A^T \cdot \eta \cdot A = \eta, \quad \det(A) = 1, \quad \eta = \text{diag}\{\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q\}$$

$$\dim O(p, q, n) = \frac{1}{2}n(n-1).$$

### 6.3.6 Unitary group

- This is  $U(n)$  consisting of all  $n \times n$  complex matrices Satisfying:

$$A^\dagger \cdot A = \mathbb{I}, \quad A \in \text{Mat}(n, \mathbb{C})$$

$$\dim U(n) = n^2.$$

### 6.3.7 Special Unitary group

- This is  $SU(n)$  consisting of all  $n \times n$  complex matrices Satisfying:

$$A^\dagger \cdot A = \mathbb{I}, \quad \det(A) = 1, \quad A \in \text{Mat}(n, \mathbb{C})$$

$$\dim U(n) = n^2 - 1.$$



## 7 Projective spaces

### 7.1 Real projective space

- The *real Projective space*  $\mathbb{R}P^n$  is the set of all straight lines in  $\mathbb{R}^{n+1}$  passing through the origin. Equivalently it is the set of equivalence classes of non-zero vectors in  $\mathbb{R}^{n+1}$  where two non-zero vectors are equivalent if they are scalar multiples of one another.
- We may think of  $\mathbb{R}P^n$  as obtained from  $S^n$  by gluing, that is identifying diametrically opposite points. This means we have the isomorphism  $\mathbb{R}P^n \simeq S^n/Z_2$ .

### 7.2 Quaternions

- The set  $\mathbb{H}$  of *Quaternions* consists of all linear combinations:

$$q \in \mathbb{H}, \quad q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{R}$$

Where  $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$  are linearly independent. Where these bases satisfy the following multiplications:

$$\begin{aligned} \mathbf{i} \cdot \mathbf{j} &= \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}, & \mathbf{j} \cdot \mathbf{k} &= \mathbf{i} = -\mathbf{k} \cdot \mathbf{j}, & \mathbf{k} \cdot \mathbf{i} &= \mathbf{j} = -\mathbf{i} \cdot \mathbf{k}, \\ \mathbf{i} \cdot \mathbf{i} &\equiv \mathbf{i}^2 = -1, & \mathbf{j} \cdot \mathbf{j} &\equiv \mathbf{j}^2 = -1, & \mathbf{k} \cdot \mathbf{k} &\equiv \mathbf{k}^2 = -1, \\ \mathbf{i} \cdot \mathbf{1} &= \mathbf{i} = \mathbf{1} \cdot \mathbf{i}, & \mathbf{j} \cdot \mathbf{1} &= \mathbf{j} = \mathbf{1} \cdot \mathbf{j}, & \mathbf{k} \cdot \mathbf{1} &= \mathbf{k} = \mathbf{1} \cdot \mathbf{k}, & \mathbf{1} \cdot \mathbf{1} &= \mathbf{1}. \end{aligned}$$

This makes  $\mathbb{H}$  an associative algebra over the field of real numbers.

### 7.3 Complex Projective spaces

- The *complex projective space*  $\mathbb{C}P^{\kappa}$  is the set of equivalence classes of non-zero vectors in  $\mathbb{C}^{\kappa+\mathbb{K}}$  where two nonzero vectors are equivalent if they are scalar multiples of one another.
- In a similar manner to the real projective space we can identify the isomorphism:  $\mathbb{C} \simeq S^{2n+1}/U(1)$ .

## 8 Lie Algebras

### 8.1 Neighborhood of identity element

- Let  $G$  be a Lie group. let the point  $g_0 \equiv 1 \in G$  be the identity element of  $G$ , and let  $T = T_{(1)}$  be the tangent space at the identity element. We can now express the group operations on  $G$  in a chart  $U_0$  containing  $g_0$  in terms of local co-ords. We choose co-ords in  $U_0$  so that the identity element is the origin.  $g_0 \equiv 1 = (0, \dots, 0)$ . then if we let:

$$g_1 = (x^1, \dots, x^n), \quad g_2 = (y^1, \dots, y^n), \quad g_3 = (z^1, \dots, z^n)$$

Which allows us to define the product of two elements:

$$g_1 g_2 = (\psi^1(x, y), \dots, \psi^n(x, y)) = (\psi^i(x, y)) \in U_0$$

An inverse as:

$$g_1^{-1} = (\varphi^1(x), \dots, \varphi^n(x)) = (\varphi^i(x)) \in U_0$$

These functions  $\varphi(x), \psi(x)$  satisfy:

$$\begin{aligned} \psi^i(x, 0) &= \psi^i(0, x) = x^i \\ \psi^i(x, \varphi(x)) &= 0 \\ \psi^i(x, \psi(y, z)) &= \psi^i(\psi(x, y), z) \end{aligned}$$

#### 8.1.1 Taylor expansion

- Let  $\psi^i(x, y)$  be sufficiently smooth and for  $x, y, z \sim \epsilon$ :

$$\begin{aligned} \psi^i(x, y) &= x^i + y^i + b_{jk}^i x^j y^k + \mathcal{O}(\epsilon^3) \\ b_{jk}^i &= \left. \frac{\partial^2 \psi^i}{\partial x^j \partial y^k} \right|_{x=y=0} \end{aligned}$$

### 8.2 Commutator

- Let  $\xi, \eta \in T$ , and their components in terms of  $x^i$  are  $\xi^i$  and  $\eta^i$ . Then we can define the *commutator*  $[\xi, \eta] \in T$  is defined by:

$$[\xi, \eta]^i = c_{jk}^i \xi^j \eta^k, \quad c_{jk}^i \equiv b_{jk}^i - b_{kj}^i$$

- It has three basic quantities:
  - It is *bi-linear* operation on the  $n$ -dim vector space  $T$ .
  - Skew-symmetry:  $[\xi, \eta] = -[\eta, \xi]$ .
  - Jacoby identity:  $[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0$

### 8.3 Lie Algebra

- A *Lie algebra* is a vector space  $\mathcal{G}$  over a field  $F$  with a bi-linear operation  $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  which is called a commutator or a lie bracket, such that the three axioms above are satisfied.
- This means we can identify the tangent space of a Lie Group at the identity is with respect to the commutator operation of a Lie algebra called the *Lie algebra of the Lie group*  $G$ .
- If we choose  $\xi = e_j, \eta = e_k$ , then combined with the fact that  $(e_m) = \delta_m^n$ , then we have:

$$[e_j, e_k]^i = c_{jk}^i e_i$$

#### 8.3.1 Structure Constants

- The constants  $c_{jk}^i$  which determine the commutation operation on a Lie algebra, and which are skew-symmetric in  $j, k$  are called the *structure constants* of the Lie algebra.

## 9 One parameter subgroups

- A *One parameter subgroup* of a lie group  $G$  is defined to be a parametric curve  $F(t)$  on the manifold  $G$  such that:

$$F(0) = 1, \quad F(t_1 + t_2) = F(t_1)F(t_2), \quad F(-t) = F^{-1}(t)$$

The velocity vector at  $F(t)$  is:

$$\frac{dF}{dt} = \frac{dF(t + \epsilon)}{dt} \Big|_{\epsilon=0} = \frac{d}{d\epsilon}(F(t)F(\epsilon)) \Big|_{\epsilon=0} = F(t) \frac{dF(\epsilon)}{d\epsilon} \Big|_{\epsilon=0}$$

Hence:

$$\dot{F}(t) = F(t)\dot{F}(0) \quad \text{or} \quad F^{-1}(t)\dot{F}(t) = \dot{F}(0)$$

i.e. the induced action of left multiplication by  $F^{-1}(t)$  sends  $\dot{F}(t)$  to  $\dot{F}(0) = \text{const} \in T$ .

- conversely,  $\forall A \in T$  the equation  $F^{-1}(t)\dot{F}(t) = A$  is satisfied by a unique one-parameter subgroup  $F(t)$  of  $G$ . If  $G$  is a matrix group then  $F(t) = \exp(At)$ .

### 9.1 Co-ords of the first kind

- One parameter subgroups can be used to define so called *canonical* in a neighborhood of the identity of a Lie group  $G$ .
- Let  $A_1, \dots, A_n$  form a basis for the Lie algebra  $T$ . Then  $\forall A = \sum_i A_i x^i \in T \exists$  a one parameter group  $F(t) = \exp(At)$ . To the point  $F(1) = \exp(A)$  we assign as co-ords co-officiants  $x^1, \dots, x^n$ , which gives us a system of co-ords in a sufficiently small neighborhood of  $g_0 = 1 \in G$ . These are called the *canonical co-ords of the first kind*.

### 9.2 Co-ords of the second kind

- Another system of co-ords is obtained by introducing  $F_i(t) = \exp(A_i t)$  and representing a point  $g$  sufficiently close to  $g_0$  as:

$$g = F_1(t_1)F_2(t_2) \cdots F_n(t_n)$$

for small  $t_1, \dots, t_n$ . Assigning co-ords  $x^1 = t_1, \dots, x^n = t_n$  to the point  $g$ , we get the *canonical co-ords of the second kind*.

