Calculus on Manifolds

Thomas Brosnan

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1 Topology on $\mathbb{R}^n$

1.1 Metric space

- Let X be a set, A *metric* on a set is a function that measures distances $d : X \times X \to \mathbb{R}$. It has the following properties:

$$\boxed{\begin{aligned} &d(x,y) = d(y,x) \\ &d(x,y) \geq 0 \\ &d(x,y) = 0 \ \text{ iff } \ x = y \\ &d(x,z) \leq d(x,y) + d(y,z) \end{aligned}} \tag{1.1}$$

  (X, d) together make a *metric space*.

- Any subset $Y \subset X$ is itself a metric space with $d(x,y)\Big|_{Y\times Y}$ (restricted to Y).

1.2 Open/Closed

- Let (X, d) be a metric space $U \subset X$ is *open* if $\forall\ p \in U$, $\exists\ \epsilon > 0$ st. $B_\epsilon(p) := \{x \in X | d(x,y) \leq \epsilon\}$ and *closed* if $X - U$ (the compliment set) is open.
- If we have $U \subset Y \subset X$, (X, d) a metric space, for us in all applications $X = \mathbb{R}^n$. U is open/closed in $(Y, d|_{Y\times Y}) \iff \exists\ V \subset X$ open/closed st. $U = V \cap Y$.

1.3 Continuity

- If we have $f : X \to Y$, with X and Y metric spaces, is *continuous* if, $f^{-1}(U)$ is open with $U \subset Y$ is open.

  If $f : X \to Y$ is a bijection, continuous and f^{-1} continuous we call f a homomorphism.

1.4 Compact

- X is *compact* if every open cover has a finite subcover , i.e. $\forall\ \{U_\alpha\}_{\alpha\in I}$, $U_\alpha \subset X$ (U_α open) st. $X \subset \bigcup_{\alpha\in I} U_\alpha$, then $\exists\ \alpha_1, ..., \alpha_k \in I$ st. $X \subset U_{\alpha_1} \cup ... \cup U_{\alpha_k}$.

1.5 Heine Boral theorem

- $X \subset \mathbb{R}^n$ is compact if bounded ($\exists\ R \in \mathbb{R}$ st $X \subset B_R(0)$) and closed in $\mathbb{R}^n$.

1.6 Differentiation

- $f : U \to V$, $(U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^m)$ is differentiable at $p \in U$ with *derivative* $Df(p) \in Mat(m, n)$ if:

$$\boxed{\lim_{x\to p} \frac{f(x) - f(p) - Df(p)(x-p)}{\|x-p\|} = 0} \tag{1.2}$$

- f is (of class) C^1 if it is differentiable at all $p \in U$ and $Df : U \to Mat(m,n) \cong \mathbb{R}^{mn}$ is continuous.
- f is C^r if Df is C^{r-1}, f is *smooth* or C^∞ if it is C^t $\forall\ t > 0$.
- If we have $f : U \to \mathbb{R}^m$, $(U \in \mathbb{R}^n)$. Then $x \mapsto (f_1(x), ..., f_n(x))$ is C^r, if:

$$\boxed{\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f_j : U \to \mathbb{R}} \tag{1.3}$$

  Exists, and is continuous for all $k \in \{1, ..., m\}, i_1, ..., i_k \in [1, ..., n]$ and $j \in [1, ..., m]$. In which case the derivative can then be expressed as:

$$\boxed{Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n} \end{pmatrix}} \tag{1.4}$$

1.7 Chain rule

- Consider $U \xrightarrow{g} V \xrightarrow{f} W$, where f and g are differentiable (or C^r), then so is $f \circ g$ and:

$$\boxed{D(f \circ g)(x) = Df(g(x)) \cdot Dg(x)} \tag{1.5}$$

  This is the chain rule and the $\cdot$ here refers to matrix multiplication.

1.8 Diffeomorphism

- If we have $f : U \to V$, smooth and U, V open (in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively) st. $f^{-1}V \to U$ exists and is also smooth. Then we call f a diffeomorphism.

1.9 Inverse function Theorem

- Let $f : V \to \mathbb{R}^n$ be C^r $(1 \leq r \leq \infty$) and $V \subset \mathbb{R}^m$. For $p \in V$, suppose $Df(p)$ is non-singular (i.e an invertible $m \times n$ matrix $\iff det(Df) \neq 0)$. Then $\exists\ p \in U \subset V$, U open, st,
  - $f\Big|_U : U \to f(U)$, is a C^r-diffeomorphism. i.e. $f\Big|_U : U \to f(U)$ is a bijection
  - $f(U)$ is open
  - $f^{-1}\Big|_U : U \to f(U)$ is C^r.

2 Manifolds

- "Slogan" (informal definition) $M \subset \mathbb{R}^n$ is a manifold if it is "smooth" without corners/intersections

2.1 Manifolds

- Let $d > 0$ $M \subset \mathbb{R}^n$ is a smooth/C^r manifold of dimension d if $\forall\, p \in M$, $\exists\, p \in V \subset M$, $U \subset \mathbb{R}^d$ (V and U open) and $\alpha : U \to V$, st:
  - α is smooth/C^r
  - α is a bijection with a continuous inverse ($\iff$ is a homomorphisim)
  - $D\alpha(x)$ has Rank d.

  We will see this means α is a diffeomorphism.

2.1.1 Parameterised manifold

- Sometimes a only a single function $\alpha : U \to M$ is needed in the definition of a manifold. In this case we call (M, α) a parameterized manifold.

  From now on we will only discuss smooth/C^∞ manifolds

2.2 Alternate definitions

- If we have a set $M \subset \mathbb{R}^n$, $d > 0$, $p \in M$. Then the following are equivalent:
  - $\exists\, p \in V \subset M$, $U \subset \mathbb{R}^d$, (V and U open), $\alpha : U \to V$ a smooth homomorphism, st, $D\alpha(x)$ has rank d, $\forall\, x \in U$.
  - $\exists\, p \in V \subset \mathbb{R}^n$, $U \subset \mathbb{R}^n$ (V and U open), $\beta : U \to V$ a diffeomorphism and $\beta(U \cap (\mathbb{R}^d \times \{0\})) = V \cap M$.
- This second definition is new and the set $U \cap (\mathbb{R}^d \times \{0\})$ is just the intersection of $U \subset \mathbb{R}^n$ and the space $\mathbb{R}^d$ extended into $\mathbb{R}^n$ by adding 0 to the d dimensional tuples $n - d$ times until they become $\mathbb{R}^n$. This is effectively saying we want to be able to straighten out manifold neighbourhoods.