

Calculus on Manifolds

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” $p \in N$ is ”

-Florian

Contents

1	Topology on \mathbb{R}^n	4
1.1	Metric space	4
1.2	Open/Closed	4
1.3	Continuity	4
1.4	Compact	4
1.5	Heine Boral theorem	4
1.6	Differentiation	4
1.7	Chain rule	5
1.8	Diffeomorphism	5
1.9	Inverse function Theorem	5
2	Manifolds	6
2.1	Manifolds	6
2.1.1	Parameterised manifold	6
2.2	Alternate definitions	6
2.3	Locally smooth	6
2.3.1	Local smoothness definition of a manifold	7
3	Partitions of unity	8
3.1	Main idea	8
3.2	Theorem	8
3.3	Lemma 1	8
3.4	Sub-lemma	8
3.5	Lemma 2	8
3.6	Extension of locally smooth functions	9
4	Boundary of manifolds	10
4.1	Upper half plane	10
4.2	Boundary of a manifold	10
4.2.1	Proposition	10
4.3	Lemma	10
4.4	Change of co-ordinates transformation	10
4.5	Interior and Boundary points	10
4.6	Boundary of manifold is manifold	11
4.7	Lemma	11
4.8	Manifolds from functions	11
5	Tangent spaces	12
5.1	Tangent spaces	12
5.1.1	Lemma	12
5.2	Maps Between tangent spaces	12
5.3	Tangent Bundle	12
5.4	Regular and Critical values	12
5.4.1	Regular value manifold	12
5.5	Sard's Theorem	13
6	Multi-linear Algebra	14
6.1	Basis vectors	14
6.2	Tensor product	14

6.3	Dual transformation	14
7	Alternating Tensors	16
7.1	Symmetric/Alternating tensors	16
7.2	Symmetric group	16
7.2.1	Elementary permutation	16
7.2.2	Lemma	16
7.3	Sign function	16
7.4	Permutation of tensors	17
7.4.1	Lemma	17
7.5	Sgn definition of tensors	17
7.5.1	Lemma	17
7.5.2	Lemma	17
7.6	Alternating Tensor	17
7.7	Basis of Alternating tensors	17
7.8	Alternating Dual	18
7.9	Alternating dual	18
7.9.1	Dual determinant	18
8	The wedge product	19
8.1	The Wedge product	19
8.2	Alternating algebra	19
8.3	Form of the wedge product	19
8.3.1	Averaging operator	19
9	Differential forms	20
9.1	Space of differential forms	20
9.2	Basis k forms	20
9.3	Pullback	21
9.3.1	Smoothness condition	21
9.3.2	Differential form of wedge	21
9.3.3	Conclusion	21
10	Exterior derivative	22
10.0.1	Exterior Derivative of k -forms	22
10.0.2	Alternate definition	22
10.1	Naturality	22
11	Vector Fields	23
11.1	Isomorphisms to differential forms	23
12	Integrating forms	24
12.1	Fubini's Theorem	24
12.2	Change of Variables	24
12.2.1	Claim	24
12.3	Compact Support	24
12.4	Integral of a d -form	24
13	Orientations	25
13.1	Orientation preserving/reversing	25
13.2	Proposition	25

13.3	Overlapping Charts	25
13.4	Oriented manifold	25
13.5	Reversing orientation	25
13.6	Extension of interior orientation	25
13.6.1	Corollary	25
13.7	Induced orientation	26
13.8	Oriented maps	26
13.9	Positive charts wrt d-forms	26
13.10	Volume Forms	26
14	Stokes Theorem	27
14.1	The integral	27
14.2	Stokes Theorem	27
14.2.1	Corollary	27

1 Topology on \mathbb{R}^n

1.1 Metric space

- Let X be a set, A *metric* on a set is a function that measures distances $d : X \times X \rightarrow \mathbb{R}$. It has the following properties:

$$\begin{aligned} d(x, y) &= d(y, x) \\ d(x, y) &\geq 0 \\ d(x, y) &= 0 \text{ iff } x = y \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned} \tag{1.1}$$

(X, d) together make a *metric space*.

- Any subset $Y \subset X$ is itself a metric space with $d(x, y) \Big|_{Y \times Y}$ (restricted to Y).

1.2 Open/Closed

- Let (X, d) be a metric space $U \subset X$ is *open* if $\forall p \in U, \exists \epsilon > 0$ st. $B_\epsilon(p) := \{x \in X | d(x, p) \leq \epsilon\}$ and *closed* if $X - U$ (the compliment set) is open.
- If we have $U \subset Y \subset X$, (X, d) a metric space, for us in all applications $X = \mathbb{R}^n$. U is open/closed in $(Y, d|_{Y \times Y}) \iff \exists V \subset X$ open/closed st. $U = V \cap Y$.

1.3 Continuity

- If we have $f : X \rightarrow Y$, with X and Y metric spaces, is *continuous* if, $f^{-1}(U)$ is open with $U \subset Y$ is open.

If $f : X \rightarrow Y$ is a bijection, continuous and f^{-1} continuous we call f a homomorphism.

1.4 Compact

- X is *compact* if every open cover has a finite subcover, i.e. $\forall \{U_\alpha\}_{\alpha \in I}, U_\alpha \subset X$ (U_α open) st. $X \subset \bigcup_{\alpha \in I} U_\alpha$, then $\exists \alpha_1, \dots, \alpha_k \in I$ st. $X \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$.

1.5 Heine Boral theorem

- $X \subset \mathbb{R}^n$ is compact if bounded ($\exists R \in \mathbb{R}$ st $X \subset B_R(0)$) and closed in \mathbb{R}^n .

1.6 Differentiation

- $f : U \rightarrow V$, ($U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$) is differentiable at $p \in U$ with *derivative* $Df(p) \in Mat(m, n)$ if:

$$\lim_{x \rightarrow p} \frac{f(x) - f(p) - Df(p)(x - p)}{\|x - p\|} = 0 \tag{1.2}$$

- f is (of class) C^1 if it is differentiable at all $p \in U$ and $Df : U \rightarrow \text{Mat}(m, n) \cong \mathbb{R}^{mn}$ is continuous.
- f is C^r if Df is C^{r-1} , f is *smooth* or C^∞ if it is $C^t \forall t > 0$.
- If we have $f : U \rightarrow \mathbb{R}^m$, ($U \subset \mathbb{R}^n$). Then $x \mapsto (f_1(x), \dots, f_m(x))$ is C^r , if:

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f_j : U \rightarrow \mathbb{R} \quad (1.3)$$

Exists, and is continuous for all $k \in \{1, \dots, r\}$, $i_1, \dots, i_k \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. In which case the derivative can then be expressed as:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (1.4)$$

1.7 Chain rule

- Consider $U \xrightarrow{g} V \xrightarrow{f} W$, where f and g are differentiable (or C^r), then so is $f \circ g$ and:

$$D(f \circ g)(x) = Df(g(x)) \cdot Dg(x) \quad (1.5)$$

This is the chain rule and the \cdot here refers to matrix multiplication.

1.8 Diffeomorphism

- If we have $f : U \rightarrow V$, a smooth bijection and U, V open (in \mathbb{R}^n and \mathbb{R}^m respectively) st. $f^{-1}V \rightarrow U$ exists and is also smooth. Then we call f a diffeomorphism.

1.9 Inverse function Theorem

- Let $f : V \rightarrow \mathbb{R}^n$ be C^r ($1 \leq r \leq \infty$) and $V \subset \mathbb{R}^n$. For $p \in V$, suppose $Df(p)$ is non-singular (i.e. an invertible $n \times n$ matrix $\iff \det(Df) \neq 0$). Then $\exists p \in U \subset V$, U open, st,
 - $f|_U : U \rightarrow f(U)$, is a C^r -diffeomorphism. i.e. $f|_U : U \rightarrow f(U)$ is a bijection
 - $f(U)$ is open
 - $f^{-1}|_U : U \rightarrow f(U)$ is C^r .

2 Manifolds

- "Slogan" (informal definition) $M \subset \mathbb{R}^n$ is a manifold if it is "smooth" without corners/intersections

2.1 Manifolds

- Let $d > 0$, $M \subset \mathbb{R}^n$ is a smooth/ C^r manifold of dimension d if $\forall p \in M$, $\exists p \in V \subset M$, $U \subset \mathbb{R}^d$ (V and U open) and $\alpha : U \rightarrow V$, st:
 - α is smooth/ C^r
 - α is a bijection with a continuous inverse (\iff is a homeomorphism)
 - $D\alpha(x)$ has Rank d .

We will see this means α is a diffeomorphism.

2.1.1 Parameterised manifold

- Sometimes a only a single function $\alpha : U \rightarrow M$ is needed in the definition of a manifold. In this case we call (M, α) a parameterized manifold.

From now on we will only discuss smooth/ C^∞ manifolds

2.2 Alternate definitions

- If we have a set $M \subset \mathbb{R}^n$, $d > 0$, $p \in M$. Then the following are equivalent:
 - $\exists p \in V \subset M$, $U \subset \mathbb{R}^d$, (V and U open), $\alpha : U \rightarrow V$ a smooth homomorphism, st, $D\alpha(x)$ has rank d , $\forall x \in U$.
 - $\exists p \in V \subset \mathbb{R}^n$, $U \subset \mathbb{R}^n$ (V and U open), $\beta : U \rightarrow V$ a diffeomorphism and $\beta(U \cap (\mathbb{R}^d \times \{0\})) = V \cap M$.
- This second definition is new and the set $U \cap (\mathbb{R}^d \times \{0\})$ is just the intersection of $U \subset \mathbb{R}^n$ and the space \mathbb{R}^d extended into \mathbb{R}^n by adding 0 to the d dimensional tuples $n - d$ times until they become \mathbb{R}^n . This is effectively saying we want to be able to straighten out manifold neighbourhoods.

2.3 Locally smooth

- We want to say a d - manifold looks locally like \mathbb{R}^d .
- Let $M \subset \mathbb{R}^n$, $N \subset \mathbb{R}^m$ be subsets. A function $f : M \rightarrow N$ is smooth if $\exists M \subset V \subset \mathbb{R}^n$ (V open) and $\tilde{f} : V \rightarrow \mathbb{R}^m$ smooth st, $\tilde{f}|_M = f$ and $f : M \rightarrow N$ is a diffeomorphism (It is a smooth bijection and has a smooth inverse). Note that we do not require \tilde{f} to have $\tilde{f}^{-1} \circ \tilde{f} = \mathbb{I}$.

It follows that we can say: $f : M \rightarrow N$ a diffeomorphism, $A \subset M \implies f|_A : A \rightarrow f(A)$ is a diffeomorphism.

- Remark These two facts are used in the proof of the following theorem. This theorem looks exactly like the definition of a manifold but note the swapping of V and U , which changes the statement to that the condition for a manifold is that there is a smooth mapping from the manifold to \mathbb{R}^d .

2.3.1 Local smoothness definition of a manifold

- Let $d > 0$, $M \subset \mathbb{R}^n$ is a smooth/ C^r manifold of dimension d if $\forall p \in M$, $\exists p \in V \subset M$, $U \subset \mathbb{R}^d$ (V and U open) and $\alpha : V \rightarrow U$, st α is a diffeomorphism.

3 Partitions of unity

3.1 Main idea

- Given $\{U_i\}_{i \in I}$ a partition of unity is a collection of smooth functions $\{\psi_i\}$, $\psi_i : \mathbb{R}^n \rightarrow [0, \infty)$ a diffeomorphism st $\{x | \psi_i(x) \neq 0\} \subset U_i$ st $\sum_{i \in I} \psi_i(x) = 1$.
- We have a local definition of a manifold and we want to extend it so that we have one single function smooth across all of M .

3.2 Theorem

- Let $\mathbb{R} \supset V = \bigcup_{\alpha \in A} U_\alpha$, where U_α are open, then there exists $\phi_1, \phi_2, \dots : V \rightarrow [0, 1]$, st:
 - For each $i \in \mathbb{N} \exists \alpha \in A$ st $S_i := \text{supp}(\phi_i) = \overline{\{x \in V | \psi_i(x) \neq 0\}} \subset U_\alpha$
 - Each $p \in A$ has a neighbourhood intersecting finitely many S_i 's.
 - $\sum_{i \in I} \psi_i(x) = 1, \forall x \in V$.
 - S_i 's are compact.
 - ψ_i are smooth.
- $\{\psi_i\}$ is called a partition of unity subordinate to $\{U_\alpha\}$.

3.3 Lemma 1

- $\{U_\alpha\}$ as above, then $\exists p_1, p_2, \dots \in \mathbb{R}^n, \epsilon_1, \epsilon_2, \dots \in \mathbb{R}_{>0}$ st:
 - $\bigcup_{i=1} B_{\epsilon_i}(p_i) = V$
 - Each $B_{2\epsilon_i}(p_i)$ is contained in a U_α .
 - Each point $p \in V$ has a neighbourhood intersecting finitely many $B_{2\epsilon_i}(p_i)$.

3.4 Sub-lemma

- One can find $k_1 \subset k_2 \subset \dots \subset V$ st:
 - k_i are compact.
 - $k_i \subset \overset{\circ}{k}_{i+1}$
 - $\bigcup_{i=1} k_i = V$

3.5 Lemma 2

- Let $p \in \mathbb{R}^n, \epsilon > 0$ Then $\exists \psi : \mathbb{R}^n \rightarrow [0, 1]$ st:
 - ψ smooth.
 - $\text{supp}(\psi) \subset B_{2\epsilon}(p)$
 - $\psi > 0$, on $B_\epsilon(p)$

3.6 Extension of locally smooth functions

- Let $M \subset \mathbb{R}^n$ a subset, $f : M \rightarrow \mathbb{R}^m$. Suppose f is locally smooth, i.e. $\forall p \in M \exists V \subset M$, st $f|_V : V \rightarrow \mathbb{R}^m$ is smooth, Then f is smooth on M .

This theorem is proved using partitions of unity.

4 Boundary of manifolds

4.1 Upper half plane

- We define the upper half plane in \mathbb{R}^d to be: $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_{\geq 0}$, that is:

$$\mathbb{H} = \{(x_1, x_2, \dots, x_d) | x_d \geq 0\} \quad (4.1)$$

- The boundary of this plane is then defined as: $\partial\mathbb{H} := \mathbb{R}^{d-1} \times 0 \subset \mathbb{H}$. We then have that $\mathring{\mathbb{H}} := \mathbb{H} \setminus \partial\mathbb{H}$.

4.2 Boundary of a manifold

- A subset $M \subset \mathbb{R}^n$ is a d -manifold with a boundary if it is basically diffeomorphic to open subsets of \mathbb{H}^d . that is $\forall p \in M \exists p \in V \subset M$, $U \subset \mathbb{H}^d$ (V and U open) and $\alpha : U \rightarrow V$ a diffeomorphism.

4.2.1 Proposition

- The condition is equivalent to α being a smooth homomorphism and $D\alpha(x)$ being of rank $d \forall x \in U$.

4.3 Lemma

- If we have $\mathbb{H}^d \supset U \xrightarrow{\alpha} \mathbb{R}^n$ smooth with extensions $\tilde{\alpha} : \tilde{U} \rightarrow \mathbb{R}^n$, (here $U \subset \tilde{U}$). Then $D\tilde{\alpha}(x) \forall x \in U$ does not depend on the extension.
 - For $x \in \mathring{\mathbb{H}}^d$, $D\tilde{\alpha}(x) = D\alpha(x)$
 - For $x \in \partial\mathbb{H}^d \cap U$, $D\tilde{\alpha}(x) = \left(\frac{\partial \tilde{\alpha}_i(x)}{\partial x_j} \right)_{i,j}$. Where for $j \neq d$ this derivative is defined in the normal way, but for $j = d$, instead of having a two sided limit in the definition we use a one sided limit, from the side that is in the half-plane.

$$\frac{\partial \tilde{\alpha}_i(x)}{\partial x_d} = \lim_{\epsilon \rightarrow 0^+} \frac{\tilde{\alpha}_i(x + \epsilon e_d) - \tilde{\alpha}_i(x)}{\epsilon} \quad (4.2)$$

4.4 Change of co-ordinates transformation

- Let M be a manifold with a boundary and $\alpha_i : V_i \rightarrow U_i$, $i = 1, 2$, two co-ordinate patches. Then $\alpha_2^{-1} \circ \alpha_1 : \alpha_1^{-1}(U_1 \cap U_2) \rightarrow \alpha_2^{-1}(U_1 \cap U_2)$ is a diffeomorphism.

This is essentially saying we should be able to map smoothly between the pre-images of the co-ordinate patches that map to the same part of the manifold.

4.5 Interior and Boundary points

- Let M be a manifold with a boundary.

- We call $p \in M$ an interior point if $\exists \alpha : U \rightarrow V$ a co-ordinate patch, st, $p = \alpha(x)$, $\forall x \in \mathring{\mathbb{H}}^d \cap U$. Then we can define:

$$\mathring{M} = \{x \in M | x \text{ is an interior point} \} \quad (4.3)$$

- We call $p \in M$ an boundary point if $\exists \alpha : U \rightarrow V$ a co-ordinate patch, st, $p = \alpha(x)$, $\forall x \in \partial \mathbb{H}^d \cap U$.

$$\partial M = \{x \in M | x \text{ is an boundary point} \} \quad (4.4)$$

- Warning: These definitions are not the same as in topology. \mathring{M} is not equal to the topological interior of M in \mathbb{R}^n and the same for ∂M .

4.6 Boundary of manifold is manifold

- let M be a d -manifold with a boundary. Then ∂M is a $(d - 1)$ -manifold with a boundary.

4.7 Lemma

- $M = \mathring{M} \sqcup \partial M$.

(Disjoint union, a union with the additional information that the sets don't have any elements in common).

4.8 Manifolds from functions

- Let $f : \mathbb{R}^n \supset U \rightarrow \mathbb{R}$ be a smooth function (U open), we define:

$$M = \{x \in U | f(x) = 0\} = f^{-1}(\{0\}). \quad (4.5)$$

And:

$$N = \{x \in U | f(x) \geq 0\} = f^{-1}([0, \infty)). \quad (4.6)$$

Suppose that $\forall x \in M$, $Df(x)$ has rank 1, i.e. $Df(x) \neq 0$, then N is a manifold with boundary $\partial N = M$.

5 Tangent spaces

5.1 Tangent spaces

- Let $M \in \mathbb{R}^n$ be a manifold with a boundary, $p \in M$, $\alpha : U \rightarrow V$ be a chart around p , $x_0 \in U$ be st $\alpha(x_0) = p$. The *tangent space* of M at p is:

$$T_p M := \text{Image}(D\alpha(x_0)) \subset \mathbb{R}^n \quad (5.1)$$

5.1.1 Lemma

- This definition does not depend on α .

5.2 Maps Between tangent spaces

- Let M, N be manifolds with boundaries and $f : M \rightarrow N$ a smooth map. Then $Df(p) = D\tilde{f}(p)$, for some extension \tilde{f} of f , defines a linear map $D\tilde{f}(p) : T_p M \rightarrow T_{f(p)} N$ for all $p \in M$.

5.3 Tangent Bundle

- Let $m \subset \mathbb{R}^n$ be a manifold with a boundary. then the *Tangent Bundle* of M is defined as the disjoint union of all the tangent spaces: $TM = \bigsqcup_{p \in M} T_p M$. i.e.:

$$TM = \{(x, v) \in M \times \mathbb{R}^n | v \in T_x M\} \quad (5.2)$$

We then have that:

- TM is a $2d$ -manifold with a boundary.
- $f : M \rightarrow N$ smooth $\implies \tilde{f}(p) : TM \rightarrow TN$ i.e. $(p, v) \rightarrow (f(p), Df(p)v)$ is smooth.
- if we have $M \xrightarrow{f} N \xrightarrow{g} L$ smooth $\implies D(g \circ f) = Dg \circ Df$, (chain rule).

5.4 Regular and Critical values

- let $f : M \rightarrow N$ be smooth, we say $p \in N$ is a *regular value* if $Df(x) : T_x M \rightarrow T_p N$ is *onto* (surjective) $\forall x \in f^{-1}(\{p\})$, otherwise we call p a *critical point*.

5.4.1 Regular value manifold

- If we have $f : M \rightarrow N$ be smooth, $\partial M = \emptyset = \partial N$ and $p \in N$ a regular value. Then $L = f^{-1}(\{p\})$ is a manifold. Moreover, $T_x L = \ker(Df(x) : T_x M \rightarrow T_p N)$.
 - Remark: We can find cases where this doesn't work. For example for $f(x, y, z) = z - xy \geq 0$, 0 is a regular point ($Df = (-y, -x, 1)$) but the corresponding $L = f^{-1}(\{p\})$ is not a manifold with a boundary as ∂L does not have 0 as a regular point for $\partial L = (z - xy, z) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. To fix this we just have to restrict the boundary of L to $\partial L = f^{-1}(\{0\}) \cap \partial M$.

5.5 Sard's Theorem

- Let $f : M \rightarrow N$ be smooth. Then the set of critical values $\text{crit}(f) \subset N$ has "*measure zero*". In particular $\{p \in N \mid p \text{ regular value of } f\}$ is dense in N .

6 Multi-linear Algebra

- let V be a vector space. A function $T : V^k \rightarrow \mathbb{R}$ is called multi-linear, (or a k tensor), if for $v_1, \dots, v_i, \dots, v_k \in T$, the function $v \rightarrow T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear. This just means it is linear in each variable.
- The space of all such functions is denoted $\mathcal{L}^k(V)$, so:

$$\mathcal{L}^k(V) := \{T : V^k \rightarrow \mathbb{R} \mid T \text{ multilinear}\} \quad (6.1)$$

Usually we denote $\mathcal{L}^1(V) = V^*$ and $\mathcal{L}^0(V) = \{0\}$. We can then show that $\mathcal{L}^k(V)$ is a vector space with $(\lambda f + g)(v_1, \dots, v_k) = \lambda f(v_1, \dots, v_k) + g(v_1, \dots, v_k)$, $\lambda \in \mathbb{R}$.

6.1 Basis vectors

- Let e_i be a basis of V , we define $e^j \in V^*$, via $e^j v_i = e^j \sum_i a_i e_i = a_j$. These form what More generally for $I = (i_1, \dots, i_k)$, we defined $e^I(v_1, \dots, v_k) = e^{i_1}(v_1) \cdot \dots \cdot e^{i_k}(v_k)$.
- The set $\{e^I\}, I \in \{1, \dots, d\}^k$ forms a basis of $\mathcal{L}^k(V)$. In the particular $\dim \mathcal{L}^k(V) = (\dim V)^k$.

6.2 Tensor product

- let $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^l(V)$, we define the following operation $f \otimes g \in \mathcal{L}^{k+l}(V)$, by:

$$f \otimes g(v_1, \dots, v_{k+l}) = f(v_1, \dots, v_k) \cdot g(v_{k+1}, \dots, v_{k+l}) \quad (6.2)$$

This is the *tensor product* and has the following properties. let f, g and h be tensors, then:

$$\begin{aligned} f \otimes (g \otimes h) &= (f \otimes g) \otimes h \\ (\lambda f) \otimes g &= \lambda(f \otimes g) = f \otimes (\lambda g) \\ (f + g) \otimes h &= f \otimes h + g \otimes h, \quad h \otimes (f + g) = h \otimes f + h \otimes g \\ e^I &= e^{i_1} \otimes \dots \otimes e^{i_k} \end{aligned} \quad (6.3)$$

6.3 Dual transformation

- Let $A : V \rightarrow W$ be a linear map. We define the dual transformation, $A^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$ by :

$$(A^* f)(v_1, \dots, v_k) = f(Av_1, \dots, Av_k) \quad (6.4)$$

- It can then be shown that the dual transformation has the following properties:

A^* is linear

$$A^*(f \otimes g) = A^*f \otimes A^*g$$

$$(A \cdot B)^* = B^* \cdot A^*$$

(6.5)

7 Alternating Tensors

7.1 Symmetric/Alternating tensors

- A tensor $f \in \mathcal{L}^k(V)$ is called:
 - *symmetric* if $f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = f(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$.
 - *alternating* if $f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -f(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$. We let S^kV and \mathcal{A}^kV denote the vector space of symmetric/alternating respectively.

7.2 Symmetric group

- The permutation or symmetric group is defined as:

$$S_k = \{\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\} | \sigma \text{ a bijection}\} \quad (7.1)$$

Since it is a group it also follows that for $\sigma, \tau \in S_k$, then $\sigma \circ \tau, \sigma^{-1} \in S_k$

7.2.1 Elementary permutation

- An elementary permutation is defined as:

$$e_i(l) = \begin{cases} i+1, & l = i \\ i, & l = i+1 \\ l, & \text{otherwise} \end{cases} \quad (7.2)$$

7.2.2 Lemma

- Every σ is a composite of the elementary permutations e_i .

7.3 Sign function

- There exists a function, $\text{sgn} : S_n \rightarrow \{\pm 1\}$ st:
 - $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$
 - $\text{sgn}(e_i) = -1$
 - $\text{sgn}(\sigma) = (-1)^m$, if σ is made of m elementary permutations.
 - $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$
 - $\text{sgn}(\sigma) = -1$, if σ keeps $p \neq q$ fixed and keeps everything else fixed.

Moreover, the first and second property here uniquely determine sgn .

7.4 Permutation of tensors

- If we have $f \in \mathcal{L}^k(V)$, $\sigma \in S_k$, then we can define the following:

$$f^\sigma(v_1, \dots, v_k) := f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \quad (7.3)$$

7.4.1 Lemma

- $\mathcal{L}^k(V)$ is a linear S_k -representation if for $f \in \mathcal{L}^k(V)$, $f^{\sigma\tau} = (f^\tau)^\sigma$, $f^\sigma = f$ (i.e. it is symmetric) and $f \mapsto f^\sigma$ is a linear map from $\mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$.

7.5 Sgn definition of tensors

- For $f \in \mathcal{L}^k(V)$, $\sigma \in S_k$
 - f is symmetric iff $f^\sigma = f$, $\forall \sigma \in S_k$
 - f is alternating iff $f^\sigma = \text{sgn}(\sigma)f$, $\forall \sigma \in S_k$

7.5.1 Lemma

- $f \in \mathcal{L}^k(V)$ is alternating iff $f(v_1, \dots, v_k) = 0$, whenever $v_i = v_j$, for some $i \neq j$.

7.5.2 Lemma

- Let $f \in \mathcal{A}^k(V)$ and suppose $f(e_{i_1}, \dots, e_{i_k}) = 0$, $\forall (i_1 \leq \dots \leq i_k)$, then $f = 0$.

7.6 Alternating Tensor

- Let $I = (i_1 \leq \dots \leq i_k)$, we define a unique k-tensor ψ_I as:

$$\psi_I = \sum_{\sigma} \text{sgn}(\sigma) (e^I)^\sigma \quad (7.4)$$

This acts on a set of basis vectors e_{j_1}, \dots, e_{j_k} as follows:

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & \text{if } (j_1, \dots, j_k) = (i_1, \dots, i_k) \\ 0, & \text{otherwise} \end{cases} \quad (7.5)$$

This is because $(e^I)(e_{j_1}, \dots, e_{j_k})$ is defined to act by: $(e^{i_1} e_{j_1})(e^{i_2} e_{j_2}) \cdots (e^{i_k} e_{j_k})$.

7.7 Basis of Alternating tensors

- $\{\psi_I\}$, with I ascending, form a basis for $\mathcal{A}^k(V)$. In particular $\dim \mathcal{A}^k(V) = \binom{n}{k}$, where $n = \dim V$.

- This is because we can write any alternating tensor $f \in \mathcal{A}^k(V)$ in terms of these tensors. Consider $g = \sum_J d_J \psi_J$, where $d_J = f(e_{j_1}, \dots, e_{j_k})$ (output of a tensor so just a scalar) and J is all ascending indices of order k . Then the action of this new function g on the basis vectors $g(e_{i_1}, \dots, e_{i_k}) = d_I \cdot (1) = f(e_{i_1}, \dots, e_{i_k})$, so we can say that $g = f$ and thus any alternating tensor f can be expanded over ψ_I .
- It can also be noted that if $k = \dim V \implies \dim \mathcal{A}^k(V) = 1$.

- This allows us to write:

$$\mathcal{A}^k(V) = \{\lambda \psi^{(1,2,\dots,n)} \mid \lambda \in \mathbb{R}\} \quad (7.6)$$

7.8 Alternating Dual

- Let $B : V \rightarrow W$ be a linear transformation, If f is an alternating tensor, then B^*f is also an alternating tensor.

7.9 Alternating dual

- Let $B : V \rightarrow W$ be a linear map, then B^* restricted to $B^* : \mathcal{A}^k(W) \rightarrow \mathcal{A}^k(V)$, that is B^*f is alternating if f is.

7.9.1 Dual determinant

- For $B : V \rightarrow V$ and $k = \dim V = n$, then we have that:

$$B^*f = \det(B)f, \quad f \in \mathcal{A}^k(V) \quad (7.7)$$

8 The wedge product

- The motivation behind this is that we would like to be able to combine alternating tensors in such a way so that the result is also an alternating tensor!

8.1 The Wedge product

- \exists an operation $\mathcal{A}^k(V) \times \mathcal{A}^l(V) \rightarrow \mathcal{A}^{k+l}(V)$ $((f, g) \mapsto f \wedge g)$ satisfying the following:
 - $(f \wedge g) \wedge h = f \wedge (g \wedge h)$
 - $(f, g) \mapsto f \wedge g$ is bilinear, $(f + \lambda g) \wedge h = f \wedge h + \lambda(g \wedge h)$
 - $f \wedge g = (-1)^{kl} g \wedge f$
 - $\psi^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$ for a basis e_i of V , $I = (i_1 \leq \dots \leq i_k)$.

The wedge product is uniquely defined by these for properties, furthermore let $T : V \rightarrow W$ be a linear map, then:

$$T^*(f \wedge g) = T^*f \wedge T^*g \in \mathcal{A}^{k+l}(W) \quad (8.1)$$

8.2 Alternating algebra

- The direct sum $\bigoplus_{k=0}^{\infty} \mathcal{A}^k(V)$ form an associative, graded (anti-symmetric) commutative algebra, module in V st: $e^{i_1} \wedge \cdots \wedge e^{i_k} = \psi^I$.

8.3 Form of the wedge product

- So far we have just said there exists a wedge product but what does it actually look like? To do this we have to define a specific operator:

8.3.1 Averaging operator

- This is $A : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ and acts by:

$$Af := \sum_{\sigma \in S_k} \text{sgn}(\sigma) f^\sigma \quad (8.2)$$

This operator satisfies that:

- A is linear
- $Af \in \mathcal{A}^k(V)$
- If $f \in \mathcal{A}^k(V)$, then $Af = k!f$
- This then allows us to define the wedge product for $l \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^l(V)$:

$$f \wedge g := \frac{1}{k!l!} A(f \otimes g) \quad (8.3)$$

9 Differential forms

- Let $M \subset \mathbb{R}^n$ be a manifold with a boundary. A differential form of order/degree k is a smooth function:

$$\omega : \{(p, v_1, \dots, v_k) \in M \times \mathbb{R}^n \times \dots \times \mathbb{R}^n | v_i \in T_p M\} \quad (9.1)$$

st, $\forall p \in M$, $\omega_p = \omega(p) = \omega(p, v_1, \dots, v_k) : T_p M \rightarrow \mathbb{R}$, is an alternating tensor. Thus we can also say that $\omega : M \rightarrow \bigsqcup_{q \in M} \mathcal{A}^k(T_q M)$, st: $\omega_p \in \mathcal{A}^k(T_p M)$.

9.1 Space of differential forms

- Let $\Omega^k(M)$ be defined as follows:

$$\Omega^k(M) = \{\omega | \omega \text{ a smooth differential form of degree } k\} \quad (9.2)$$

We can also define the similar $\Omega_\delta^K(M)$ as:

$$\Omega_\delta^k(M) = \{\omega | \omega \text{ same as above but not necessarily smooth}\} \quad (9.3)$$

We then have that: $\Omega^0(M) = C^\infty(M) = \{f : M \rightarrow \mathbb{R} | f \text{ smooth}\}$.

- $\Omega^K(M)$ is a vector space under point-wise addition/ multiplication with scalars.

9.2 Basis k forms

- Recalling that $e^j(x_1, x_2, \dots, x_n) = x_j \in \mathbb{R}$, for $x \in \mathbb{R}^n$. If we look at the form of $\psi_I(x)$ in 7.4, and the definition of the wedge product in 8.3 then we can see that we can re-write ψ_I as:

$$\psi^I = e^{i_1} \wedge e^{i_2} \dots \wedge e^{i_k} \quad (9.4)$$

And we end up denoting this:

$$\psi^I = dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_k} = dx^I \quad (9.5)$$

These ψ_I are called elementary k forms (since each e^{i_1} is a 1-form). It is also worth noticing that:

$$dx^I(v_1, v_2, \dots, v_k) = \det([v_1, v_2 \dots v_k]) \quad (9.6)$$

- We then have that each k tensor $\omega(p)$ can be written uniquely in the form:

$$\omega(p) = \sum_{[I]} b_I(p) dx^I(p) \quad (9.7)$$

Where $[I]$ denotes any increasing sequence, that means no repeating index's, but it does not have to be of length k . If each $b_I(p)$ is smooth, then so is ω .

9.3 Pullback

- Let $f : M \rightarrow N$ be smooth, we define $f^* : \Omega^k(N) \rightarrow \Omega^k(M)$ as:

$$(f^*\omega)(p, v_1, \dots, v_k) := \omega(f(p), Df_p v_1, Df_p v_2, \dots, Df_p v_k) \quad (9.8)$$

f^* is a well defined linear map. We may also write $(f^*\omega)_p \in A^k(T_p M)$. With these definitions it can be proven that:

- $\text{id}_M^* = \text{id}_{\Omega^*(M)}$
- $(f \circ g)^* = g^* \circ f^*$ For $L \xleftarrow{f} N \xleftarrow{g} M$ smooth. This map is called a *pullback*.

9.3.1 Smoothness condition

- An element $\omega \in \Omega_\delta^k(M)$ is smooth if $\forall p \in M, \exists \alpha : U \rightarrow V \subset M$ a co-ord patch around p st, $\alpha^*\omega \in \Omega_\delta^k(U)$ is smooth.

9.3.2 Differential form of wedge

- Let $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, we define $\omega \wedge \eta \in \Omega^{k+l}(M)$ by:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p \quad (9.9)$$

- It can also be shown that if we have $f : M \rightarrow N$ smooth, then $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

9.3.3 Conclusion

- $\Omega^\bullet(M) := \bigoplus_{k=0}^\infty \Omega^k(M)$ is a *graded-commutative (anti-commutative) associative* algebra structure, with $\Omega^0(M) = C^\infty(M)$ (the set of all smooth 1-d functions on M).
- It also has that if $f : M \rightarrow N$, then $f^* : \Omega^\bullet(N) \rightarrow \Omega^\bullet(M)$, preserves the above structure.

It is also worth noting that $\Omega^k(M) = 0$ for $k > \dim V$ as then there are more elements in the sequence i_1, \dots, i_k , then there are dimensions, so $dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k}$, must have a repeating index, making it 0 and thus each ω is also 0.

10 Exterior derivative

- Let M be a manifold with a boundary, \exists a unique linear map $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ defined for all $k \geq 0$ st:
 - If $f \in C^\infty(M) = \Omega^0(M)$ then $df_p(v) = Df(p)v$
 - If $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
 - $d(d\omega) = 0$, more over if $F : M \rightarrow N$ is smooth then $d(F^*\omega) = F^*d\omega$

10.0.1 Exterior Derivative of k-forms

- We know how this derivative acts on 0-forms based on the first property, but how does it act on a k -form ω ? To find this we can just look at our expression of k -forms in terms of the elementary k -forms in 9.7. With this expression $d\omega$ is defined as:

$$\begin{aligned} d\omega &= d \sum_{[I]} b_I dx^I = \sum_{[I]} db_I \wedge dx^I + \sum_{[I]} b_I d^2 x^I \\ &= \sum_{[I]} db_I \wedge dx^I \end{aligned} \tag{10.1}$$

Where we have used the property that $d^2 = 0$.

- One can also use this expression to show that $d^2\omega = 0$ as since b_I is a 0 form, $d^2 b_I = \sum_{i,j=1}^n D_i D_j b_I$, i.e. all second order partial derivatives, but since b_I , must be a smooth function $D_i D_j b_I = D_j D_i b_I$, so since $d^2\omega = \sum_I \sum_{i,j=1}^n D_i D_j b_I dx^I = \sum_I \sum_{i>j} (D_i D_j b_I - D_j D_i b_I) dx^I = 0$, as swapping dx^{i_i} and dx^{i_j} , picks up a minus sign.

10.0.2 Alternate definition

- Alternatively if we have $\alpha : U \rightarrow M$, be a patch around p . we can define $d\omega$ as:

$$(d\omega)_p := ((\alpha^{-1})^* d_U(\alpha^*\omega))_p \tag{10.2}$$

This is well defined and independent of the choice of α .

10.1 Naturality

- Let $F : U \rightarrow V$ be smooth, $\omega \in \Omega^k(V)$. Then:

$$F^*d\omega = d(F^*\omega) \tag{10.3}$$

11 Vector Fields

- A vector field is a smooth function $X : M \rightarrow TM$, st. $X(p) \in T_p M$. We then define the set of all vector fields on our manifold M :

$$\mathfrak{X}(M) := \{X : M \rightarrow TM \mid X \text{ is a vector field}\} \quad (11.1)$$

11.1 Isomorphisms to differential forms

- We can define the following isomorphisms for $M \subset \mathbb{R}^d$:

$$\begin{aligned} h_1 : \mathfrak{X}(M) &\rightarrow \Omega^1(M) \\ \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{pmatrix} &\mapsto \sum_{i=1}^d x^i dx^i \\ h_{n-1} : \mathfrak{X}(M) &\rightarrow \Omega^{d-1}(M) \\ \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{pmatrix} &\mapsto \sum_{i=1}^d x^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^d \\ h_n : \mathfrak{X}(M) &\rightarrow \Omega^d(M) \\ u &\mapsto u dx^1 \wedge \cdots \wedge dx^d \end{aligned} \quad (11.2)$$

Note that these isomorphisms may not be natural, i.e. $h_1(F^*x) = F^*h_1(x)$, is in general not true.

12 Integrating forms

- In this section we will define what it means to integrate a k -form over a manifold M . To do this we link our integrals back to \mathbb{R}^n , where we have well defined integration.

12.1 Fubini's Theorem

$$\int f(x^1, \dots, x^n) dx^1 \cdots dx^n = \int_{\mathbb{R}^{n-l}} \left[\int_{\mathbb{R}^l} f(x^1, \dots, x^n) dx^1 \cdots dx^l \right] dx^{l+1} \cdots dx^n \quad (12.1)$$

12.2 Change of Variables

- Let $F : U_1 \rightarrow U_2$, ($U_1 \subset \mathbb{H}^d, U_2 \subset \mathbb{H}^n$) a diffeomorphism, then:

$$\int_{\mathbb{R}^n} f(F(x)) |\det DF| dx^1 \cdots dx^n = \int_{\mathbb{R}^n} f(x) dx^1 \cdots dx^n \quad (12.2)$$

12.2.1 Claim

- Change of Variables $\iff \int F^* \omega = \int \omega$, if $|\det DF| = \det DF$.

12.3 Compact Support

- We say $\omega \in \Omega^k(M)$ has *compact support* if $\text{supp } \omega := \overline{\{p \in M | \omega_p \neq 0\}}$ is compact. We can then denote $\Omega_c^k(M)$ denote all the k -forms with compact supports.

12.4 Integral of a d-form

- Let $M \in \mathbb{H}^d$ (M open), $\omega \in \Omega_c^d(M)$. We define the integral of this d -form as follows:

$$\int_M \omega = \int_{\mathbb{R}^d} u(x^1, \dots, x^d) dx^1 \cdots dx^d \quad (12.3)$$

Here u is defined by $\omega = u dx^1 \wedge \cdots \wedge dx^d$ and is extended by 0 outside of M .

13 Orientations

- In order to determine whether the result of integrating a k -form, has a \pm in front of it, we have to define an orientation on the manifold we are integrating over.

13.1 Orientation preserving/reversing

- Let $F : M \rightarrow N$ ($N, M \subset \mathbb{H}^d$), be a diffeomorphism. We call F *orientation preserving* if $\det DF(p) > 0$, $\forall p \in M$ and *orientation reversing* if $\det DF(p) < 0$, $\forall p \in M$. Note that $\det DF \neq 0$, as F is a diffeomorphism.

13.2 Proposition

- Let F be as above and orientation preserving. Then:

$$\int_M F^* \omega = \int_N \omega, \quad \forall \omega \in \Omega_c^d(N) \quad (13.1)$$

13.3 Overlapping Charts

- Let M be a manifold with a boundary. Let $\alpha_i : U_i \rightarrow V_i \subset M$ be charts. We say two charts α_1 and α_2 *overlap positively* if $\alpha_2^{-1} \circ \alpha_1 : \alpha_1^{-1}(V_1 \cap V_2) \rightarrow \alpha_2^{-1}(V_1 \cap V_2)$ is orientation preserving.

13.4 Oriented manifold

- An orientation on M is the choice of collection of charts that pairwise overlap positively and cover M .

We denote an oriented manifold by $(M, \{\alpha_i\})$.

We call a chart $\beta : U \rightarrow V$, positive if it overlaps positively with all $\alpha_i \in \{\alpha_i\}$. It is easy to see then that. $\{\alpha_i\} \subset \{\beta\} \iff$ they define the same collection of positive charts.

13.5 Reversing orientation

- Set $\tau : \mathbb{H}^d \rightarrow \mathbb{H}^d$, $(x^1, \dots, x^d) \mapsto (-x^1, \dots, x^d)$. Given a patch $\alpha : U \rightarrow M$, then $\alpha \circ \tau$ is also a patch with opposite orientation. Usually we denote this by: $(M, \{\alpha_i \circ \tau\}) = -(M, \{\alpha_i\}) = -M$.

It's clear to see that if we have M orientated then either α , or $\alpha_i \circ \tau$, is positive.

13.6 Extension of interior orientation

- Let M be a manifold with a boundary. Suppose $\overset{\circ}{M} = M \setminus \partial M$ is orientable, then so is M . Moreover, if $A = \alpha_i$, is an orientation on $\overset{\circ}{M}$, then \exists , $B = \beta_i$ an orientation on M , st: $A \subset B$.

13.6.1 Corollary

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth and 0 a regular value. Then $f^{-1}([0, \infty]) = M$, carries a natural orientation.

13.7 Induced orientation

- Let $(M, \{\alpha_i\})$, be a oriented manifold with a boundary. Then, $(\partial M, \alpha_i \Big|_{\partial \mathbb{H}^d \cap U_i})$, is an oriented manifold with what we call a *restricted orientation*.
- The *Induced orientation* on ∂M is $(-1)^d$ times the restricted one. This is so that stokes theorem always holds!

13.8 Oriented maps

- Let $f : M \rightarrow N$ be a diffeomorphism and $(M, \{\alpha_i\}), (N, \{\beta_i\})$, oriented manifolds. We say f is orientation preserving if $\{f \circ \alpha\}$ are positive charts with respect to $\{\beta_i\}$.

13.9 Positive charts wrt d-forms

- Given $\omega \in \Omega^d(M)$ on M a d -manifold with a boundary. We declare $\alpha : U \rightarrow M$, to be positive iff $\alpha^*\omega \in \Omega^d(U)$, $U \in \mathbb{R}^d$, st: $\alpha^*\omega = u dx^1 \wedge \cdots \wedge dx^d$, with $u(x) > 0$, $\forall x \in U$.
- This defines an orientation $\iff \omega_p \neq 0$, $\forall p \in M$.

13.10 Volume Forms

- $\omega \in \Omega^d(M)$ is called an *volume form* if $\omega_p \neq 0$, $\forall p \in M$.

14 Stokes Theorem

14.1 The integral

- The integral is linear:

$$\int_M \lambda\omega + \eta = \lambda \int_M \omega + \int_M \eta \quad (14.1)$$

If $-M$, denotes M , with the opposite orientation then:

$$\int_{-M} \omega = - \int_M \omega \quad (14.2)$$

14.2 Stokes Theorem

- Let M be an oriented manifold with a boundary, then for $\omega \in \Omega_c^{d-1}(M)$:

$$\int_M d\omega = \int_{\partial M} \omega \quad (14.3)$$

14.2.1 Corollary

- If M has no boundary ($\partial M = \emptyset$), then $\int_M d\omega = 0$.

