Differential Geometry

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Notes taken in Professor Sergey Frolov's class, Michaelmas Term 2024 “hokay” -Sergey Frolov

Contents

1 Definition of a Manifold

1.1 Regions

- A *region* ("open set") is a set of D points in $\mathbb{R}^n$ such that together with each point p_0, D also contains all points sufficiently closer to p_0, i.e.:

$$\forall\ p_0 = (x_0^1, \ldots, x_0^n) \in D\ \exists\ \epsilon > 0,$$
$$st : p = (x^1, \ldots, x^n) \in D,\ \text{iff}\ |x^i - x_0^i| < \epsilon.$$

- A *region with out a boundary* is obtained fro ma region D by adjoining all boundary points to D. The *boundary* of a region is the set of all boundary points.

1.2 Differentiable Manifold

- A differentiable n-dimensional manifold is a set M together with the following structure on it. The set M is the union of a finite or countably infinite collection of subsets U_q with the following properties:
  - Each subset U_q has defined on it co-ords $x_q^\alpha, \alpha = 1, \ldots, n$ called local co-ords by virtue of which U_q is identifiable with a region of Euclidean n-space $\mathbb{R}^n$ with Euclidean co-ords x_q^α. The U_q with their co-ord systems are called *charts* or *local coordinate neighborhoods*.
  - Each non-empty intersection $U_q \cap U_p$ of a pair of charts thus has defined on it two co-ord systems, the restriction of x_p^α and x_q^α. It is required that under each of these coordinatizations the intersection $U_q \cap U_p$ is identifiable with a region of $\mathbb{R}^n$ and that each of these co-ordinate systems be expressible in terms of the other in a one to one differentiable manner. Thus, if a the *transition* functions from x_p^α to x_q^α and back are given by:

$$x_p^\alpha = x_p^\alpha(x_q^1, \ldots, x_q^n), \quad \alpha = 1, \ldots, n$$
$$x_q^\alpha = x_q^\alpha(x_p^1, \ldots, x_p^n), \quad \alpha = 1, \ldots, n$$

    Then the *Jacobian* $J_{pq} = det(\partial x_p^\alpha / \partial x_q^\alpha)$ is non-zero on $U_p \cap U_q$.

1.3 Abuse of notation

- Regular partial derivative do not have the same "canceling" that total derivative have ($dx * dy/dx = dy$) But we can restore this property through Einstein summation convention. That is that:

$$\boxed{\sum_{\gamma=1}^{n} \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \delta_\beta^\alpha}$$

2 Elements of Topology

2.1 Topological space

- A topological space is a set X of points of which certain subsets called *open sets* of the topological space, are distinguished, these open sets have to satisfy:
  - The intersection of any two (and hence of any finite collection) open sets should again be an open set.
  - The union of any collection of open sets must again be open.
  - The empty set and the whole set X must be open.
- The compliment of any open set is called a *closed* set of the topological space.

  In Euclidean space $\mathbb{R}^n$ the "Euclidean topology" is the usual one where the open sets are the open regions.

2.1.1 Induced topology

- Given any subset $A \in \mathbb{R}^n$, the *induced topology* on A is that where the open sets are the intersections $A \cap U$, where U ranges over all open sets of $\mathbb{R}^n$.

2.1.2 Continuity

- A map $f : X \to Y$ of one topological space to another is called *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subset Y$ is open in X.

2.1.3 Homeomorphic

- Two topological space are *topologically equivalent* or *homeomorphic* if there is a one to one and onto map (bijective) between them, such that it and its inverse are continuous.

2.1.4 Topology on a manifold

- The topology on a manifold M is given by the following specifications of the open sets. In every local co-ordinate neighborhood U_q the open regions are to be open in the topology on M; the totality of open sets of M is then obtained by admitting as open, also arbitrary unions countable collections of such regions, i.e. by closing under countable unions.

2.2 Metric space

- A *metric space* is a set which comes equipped with a "distance function" i.e. a real-valued function $\rho(x, y)$, defined on pairs x, y of its elements and having the following properties:
  - Symmetry: $\rho(x, y) = \rho(y, x)$.
  - Positivity: $\rho(x, x) = 0, \quad \rho(x, y) > 0$ if $x \neq y$.
  - The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

2.2.1 Hausdorff

- A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centers at c, y do not intersect.

  All topological spaces we consider will be Hausdorff.

2.2.2 Compact

- A topological space X is said to be compact if every countable collection of open sets covering X contains a finite sub-collection already covering X.

  If X is a metric space the compactness is equivalent to the condition that from every sequence of points of X a convergent sub-sequence can be selected.

2.2.3 Connected

- A topological space is connected if any two points can be joined by a continuous path.

2.3 Orientation

- A manifold M is said to be *orientated* of one can choose its atlas (collection of all the charts) so that for every pair U_p, U_q of intersecting co-ordinate neighborhoods the Jacobian of the transition functions is positive.
- We say that the co-ordinate systems x and y define the *same orientation* if $J > 0$ and the *opposite orientation* if $J < 0$.

3 Mappings on Manifolds

3.1 Manifold mappings

- A mapping $f : M \to N$ is said to be smooth of smoothness class k if for all p, q for which f determines functions $y^b_q(x^1_p, \ldots, x^m_p) = f(x^1_p, .., x^m_p)^b_p$, these functions are, where defined, smooth of smoothness class k (i.e. all their partial derivatives up to those of k-th order exist and are continuous).

  the smoothness class of f cannot exceed the maximum class of the manifolds.

3.2 Equivalent manifolds

- The manifolds M and N are said to be *smoothly equivilent* or *diffeomorphic* if there is a one to one and onto map f such that both $f : M \to N$ and $f^{-1} : N \to M$ are smooth of some class $k \geq 1$.

  Since f^{-1} exits then the Jacobian $J_{pq} \neq 0$ wherever it is defined.

3.3 Tangent vector

- A *tangent* vector to an m-dim manifold M at an arbitrary point x is represented in terms of local co-ords x^α_p by an m tuple ξ^α of components which are linked to the components in terms of any other system x^β_q of local co-ords by:

$$\xi^\alpha_p = \left(\frac{\partial x^\alpha_p}{\partial x^\beta_q}\right)_x \xi^\beta_q, \quad \forall\ \alpha \tag{3.1}$$

- The set of all tangent vectors to an m-dim manifold M at a point x forms an m-dm vector space $T_x = T_x M$, the *tangent space* to M at the point x.
- Thus, the velocity at x of any smooth curve M through x is a tangent vector to M at x.

3.4 Push forward

- A smooth map f from M to N gives rise for each x to a *push forward* or an *induced linear* map to tangent spaces:

$$f_* : T_x M \to T_{f(x)} N$$

  defined as sending the velocity at x of any smooth curve $x = x(\tau)$ on M to the velocity vector at $f(x)$ of the curve $f(x(\tau))$ on N. If the map f is given by: $y^b = f^b(x^1, \ldots, x^m)$ for $x \in M$ and $y \in N$, then the push forward map f_* is:

$$\xi^\alpha \to \eta^b = \frac{\partial f^b}{\partial x^\alpha}\xi^\alpha.$$

- For a real valued function $f : M- > \mathbb{R}$, the push-forward map f_* corresponding to each $x \in M$ is a real valued linear function on the tangent space to M at x:

$$\xi^a \to \eta = \frac{\partial f}{\partial x^\alpha}\xi^\alpha$$

  and it is represented by the gradiant of f at x, and is a co-vector or one form. Thus f_* can be identified with the differential df, in particular:

$$dx^\alpha_p : \xi^\alpha \to \eta = \xi^\alpha_p$$

3.5 Directional derivative

- We can associate with each vector $\xi = (\xi^i)$ a linear differential operator as follows: Since the gradient $\frac{\partial f}{\partial x^i}$ of a function f is a co-vector, the quantity:

$$\partial_\xi f = \xi^i \frac{\partial f}{\partial x^i}$$

is a scalar called the directional derivative of f in the direction of ξ.

- Thus an arbitrary vector ξ corresponds to the operator:

$$\partial_\xi = \xi^i \frac{\partial}{\partial x^i}$$

So we can identify $\frac{\partial}{\partial x^i} \equiv e_i$ as the *Canonical basis of the tangent space.*

3.6 Riemann metric

- A *Riemann metric* on a manifold M is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local co-ords of the points.

Thus at each point $x = (x_p^1, \dots, x_p^m)$ of each chart U_p, the metric is given by a symmetric metric $g_{\alpha\beta}(x_p^1, ..., x_p^m)$, and determines a symmetric scalar product of pairs of tangent vectors at the point x.

$$\langle \xi, \eta \rangle = g^{(p)}_{\alpha\beta} \xi_p^\alpha \eta_p^\beta = \langle \eta, \xi \rangle \,, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

This scalar product is to be co-ordinate independent:

$$g^{(p)}_{\alpha\beta} \xi_p^\alpha \eta_p^\beta = g^{(q)}_{\alpha\beta} \xi_q^\alpha \eta_q^\beta$$

And therefor the coefficients $g^{(p)}_{\alpha\beta}$ of the quadratic form transform as:

$$g^{(q)}_{\gamma\delta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_p^\beta}{\partial x_q^\delta} g^{(p)}_{\alpha\beta} \tag{3.2}$$

For a *pseudo-Riemann* metric M one just requires the quadratic form to be *nondegenerate.* Note that 3.2 can be re-written as:

$$ds^2 = g^{(p)}_{\alpha\beta} dx_p^\alpha dx_p^\beta = g^{(q)}_{\alpha\beta} dx_q^\alpha dx_q^\beta$$

Where ds is called a line element, and it is chart-independent. ds is used to measure the distance between two infinitesimally close points.

4 Tensors

4.1 Tensor def

- A *tensor of type* (k,l) and rank $k+l$ on an m-dim manifold M is given each local co-ord system (x_p^i) by a family of functions:

$$^{(p)}T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x) \text{ of the point } x.$$

In other local co-ord (x_q^i) the components $^{(p)}T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x)$ of the same tensor are:

$$^{(p)}T^{s_1,\dots,s_k}_{t_1,\dots,t_l}(x) = \frac{\partial x_q^{s_1}}{\partial x_p^{i_1}} \cdots \frac{\partial x_q^{s_k}}{\partial x_p^{i_k}} \frac{\partial x_p^{j_1}}{\partial x_q^{t_1}} \cdots \frac{\partial x_p^{j_l}}{\partial x_q^{t_l}} \cdot \; ^{(p)}T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x)$$

4.2 Operations on Tensors

4.2.1 Permutation of indices

- Let σ be some permutation of $1,2,\dots,l$. σ acrs on the ordered tuple $(j_1,\dots,j_l)$ as $\sigma(j_1,\dots,j_l) = (j_{\sigma_1},\dots,j_{\sigma_l})$. We say that a tensor $\tilde{T}^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x)$ =is obtained from a tensor $T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x)$ by means of a permutation σ of the lower indices if at each point of M:

$$\tilde{T}^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x) = T^{i_1,\dots,i_k}_{\sigma(j_1,\dots,j_l)}(x)$$

Permutation of upper indicies is defined similarly.

4.2.2 Contraction of indicies

- By the contraction of a tensor $T^{i_1,\dots,i_k}_{j_1,\dots,j_l}(x)$ of type (k,l) with respect to the indcies i_a, j_a we mean the tensor (summation over n):

$$T^{i_1,\dots,i_{k-1}}_{j_1,\dots,j_{l-1}}(x) = T^{i_1,\dots i_{a-1},n,i_{a+1},\dots,i_k}_{j_1,\dots,j_{a-1},n,j_{a+1},\dots,j_l}(x)$$

Of type $(k-1,l-1)$

4.2.3 Product of Tensors

- Given two tensors $T = \left(T^{i_1,\dots,i_k}_{j_1,\dots,j_l}\right)$ of type (k,l) and $P = \left(P^{i_1,\dots,i_p}_{j_1,\dots,j_q}\right)$ of type (p,q), we define their product to be the tensor product $S = T \otimes P$ of type $(k+p, l+q)$ with components:

$$S^{i_1,\dots,i_{k+p}}_{j_1,\dots,j_{l+q}} = T^{i_1,\dots,i_k}_{j_1,\dots,j_l} P^{i_{k+1},\dots,i_p}_{j_{l+1},\dots,j_q}$$

This multiplication is *not commutative* but it is associative.

- The result of applying the above three operations to tensors are again tensors.

4.3 Co-Vectors

- Recall that the differential of a function f of $x^1, \dots, x^n$ corresponding to the increments dx^i in the x^i is:

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Since dx^i is a vector df has the same value in any co-ord system. In general, given any co-vector (T_i), the differential form $T_i dx^i$ is invariant under change of chart. We can thus identify $dx^i \equiv e^i$ as the *canonical basis of co-vectors or cotangent space.*

4.4 Skew-Symmetric Tensor

- A *skew-symmetric tensor* of type $(0, k)$ is a tensor $T_{i_1,\dots,i_k}$ satisfying:

$$T_{\sigma(i_1,\dots,i_k)} = \mathfrak{s}(\sigma) T_{i_1,\dots,i_k}$$

where for all permutations $\mathfrak{s}(\sigma)$ is the sign function. i.e. $\mathfrak{s}(\sigma) = +1(-1)$ for even(odd) permutation. If two indices of $T_{i_1,\dots,i_k}$ are the same then the corresponding component of $T_{i_1,\dots,i_k}$ is 0. This means if $k > n$ the tensor is automatically 0.

- The standard basis at a given point is:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < i_2 < \cdots < i_k$$

Where:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathfrak{s}(\sigma) e^{i_{\sigma_1}} \otimes \cdots \otimes e^{i_{\sigma_k}}$$

Here S_k is the symmetric group. i.e. the group of all permutations of k elements.

- The differential form of the skew-symmetric tensor $(T_{i_1,\dots,i_k})$ is:

$$\boxed{\begin{aligned} T_{i_1,\dots,i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} &= \sum_{i_1 < i_2 < \cdots < i_k} T_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!} T_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}}$$

Where the last step can be made as both $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $T_{i_1,\dots,i_k}$ are anti-symmetric.

4.5 Volume element

- A metric g_{ij} on a manifold is a tensor of type $(0, 2)$ and on an oriented manifold of $dim(M) = n$ such a metric gives rise to a *volume element*:

$$T_{i_1,\dots,i_n} = \sqrt{|g|} \epsilon i_1, \dots, i_n, \quad g = det(g_{ij})$$

It is convenient to write the volume element in the notation of differential forms:

$$\Omega = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

If g_{ij} is Riemann then the *volume* V of M is:

$$\boxed{V = \int_M \Omega = \int_M \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n}$$

4.6 Generalized push forward

- We can generalize the push froward map we had on vectors earlier to the space of tensors $(k,0)$:

$$\boxed{f_* : \xi^{i_1,\ldots,i_k} \to \eta^{a_1,\ldots,a_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi^{i_1,\ldots,i_k}}$$

4.7 Pull back

- Let $T_x^{(0,k)}M$ denote the space of tensors of type $(0,k)$ at $x \in M$. Let f be a smooth map from M to N. It gives rise to a map:

$$f^* : T_{f(x)}^{(0,k)}N \to T_x^{(0,k)}M$$

  which in terms of $x^i \in U \subset M$, and $y^a \in V \subset N$ is written as:

$$\boxed{f^* : \eta_{a_1,\ldots,a_k} \to \xi_{i_1,\ldots,i_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1,\ldots,a_k}}$$

  The map f^* is called the *pullback*.

- We can then note the following relationship between pullbacks and push forwards. Let us denote the action of a vector on another vector as follows:

$$\zeta(\theta) \equiv \zeta_{i_1,..,i_k}\theta^{i_1,\ldots,i_k}$$

  Then we can write that:

$$(f^*\eta)(\xi) = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1,\ldots,a_k}\xi^{i_1,\ldots,i_k} = \eta(f_*\xi)$$

5 Manifolds and surfaces

5.1 Immersion

- A manifold M of dim m is said to be immersed in a manifold N of dim $n \geq m$ if $\exists$ a smooth map $f : M \to N$ such that the push forward map f_* is at each point a one to one map of the tangent space.

  The map f is called the *immersion* of M to N.

  Since f_* is at each a point one to one map of the tangent space, in terms of local co-ords the Jacobian matrix of f at each point has rank equal to $m = \dim M$.

5.1.1 Embedding

- An immersion of M to N is called an *embedding* if it one to one. Then M is called a *sub-manifold* of N.
- To see the difference between these two definitions note that a Klein bottle is immersed in $\mathbb{R}^3$ but not embedded as its tangent spaces are distinct (intersecting points can have different tangent spaces) but the map of points is not one- to one as there are cross overs.

5.2 Manifold with boundary

- A closed region A of a manifold M defined by an inequality:

$$f(x) \leq 0, \quad (\text{or} f(x) \geq 0)$$

  where f is a real-valued function on M. This region is a *Manifold with boundary*. It is assumed that the boundary ∂A given by $f(x) = 0$ is a non-singular sub-manifold of M i.e. $\nabla f \neq 0$ on ∂A.

5.2.1 Closed manifold

- A compact manifold without a boundary is called *closed*.

5.3 Surfaces as Manifolds

- A *Non-singular surface* M of dimension k in n-dim Euclidean space is given by a set of $n - k$ equations:

$$f_i(x^1, .., x^n) = 0, \quad i = 1, \dots, n - k$$

  where $\forall x$ the matrix $\left(\frac{\partial f_i}{\partial x^\alpha}\right)$ has rank $n - k$.

5.4 Orientation of surfaces

5.4.1 Orientation class

- Consider a frame $\tau_1 = (e_1^{(1)}, \dots, e_n^{(1)})$ called an ordered basis and another frame $\tau_1 = (e_1^{(2)}, \dots, e_n^{(2)})$ then we say that they lie in the *same orientation class* if $detA > 0$ and the *opposite orientation* *class* if $detA < 0$. Where A is defined as:

$$A : e_k^{(1)} :\to e_k^{(2)}$$

5.4.2 Orientability

- A manifold is said to be *orientable* if it is possible to choose at every point of it a single orientation class depending continuously on the points.

  A particular choice of such an orientation class for each point is called an orientation of the manifold, and a manifold equipped with a particular orientation is said to be *oriented*.

  If no orientation exists the manifold is said to be *non-orientable*

5.5 Two-sided hyper-surface

- A connected $(n-1)$-dim sub-manifold of $\mathbb{R}^n$ is called two sided if a single valued continuous field of unit normals can be defined on it.

  such a sub-manifold is called a *two-sided hyper-surface*.

6 Lie Groups

6.1 Group

- A *group* is a non-empty set G on which there is defined a binary operation $(a, b) \to ab$ satisfying the following properties:
  - Closure: If a and b belong to G, then $ab \in G$.
  - Associativity: $\forall a, b, c \in G, \quad a(bc) = (ab)c$.
  - Identity: $\exists$ an element $1 \in G$ st: $a1 = 1a = a, \quad \forall a \in G$
  - Inverse: If $a \in G$ then $\exists\ a^{-1} \in G$ st: $aa^{-1} = a^{-1}a = 1$.

6.2 Lie Group

- A manifold G is called a *Lie Group* if it has given on it a group operation with the properties that the maps $\varphi : G \to G$, defined by $\varphi(g) = g^{-1}$ and $\psi : G \times G \to G$ defined by $\psi(g, h) = gh$, are smooth maps.

6.3 Example of Lie groups

6.3.1 General Linear group

- This is $GL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with non zero determinant in a region $\mathbb{R}^{n^2}$. dim $GL(n, \mathbb{R}) = n^2$.

6.3.2 Special Linear group

- This is $SL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with determinant equal to 1. It is a hyper-surface in $\mathbb{R}^{n^2}$.

$$detA = 1, \quad A \in Mat(n, \mathbb{R})$$

dim $SL(n, \mathbb{R}) = n^2 - 1$.

6.3.3 Orthogonal group

- This is $O(n, \mathbb{R})$ consisting of all $n \times n$ real matrices Satisfying:

$$A^T \cdot A = \mathbb{I}, \quad A \in Mat(n, \mathbb{R})$$

dim $O(n, \mathbb{R}) = \frac{1}{2}n(n-1)$.

6.3.4 Special Orthogonal group

- This is $SO(n, \mathbb{R})$ consisting of all $n \times n$ real matrices Satisfying:

$$A^T \cdot A = \mathbb{I}, \quad det(A) = 1, \quad A \in Mat(n, \mathbb{R})$$

dim $SO(n, \mathbb{R}) = \frac{1}{2}n(n-1)$.

6.3.5 Pseudo Orthogonal group

- This is $O(p,q,n)$ consisting of all $n \times n$ real matrices Satisfying:

$$A^T \cdot \eta \cdot A = \eta, \quad \det(A) = 1, \quad \eta = \mathrm{diag}\{\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q}\}$$

dim $O(p,q,n) = \frac{1}{2}n(n-1)$.

6.3.6 Unitary group

- This is $U(n)$ consisting of all $n \times n$ complex matrices Satisfying:

$$A^\dagger \cdot A = \mathbb{I}, \quad A \in Mat(n, \mathbb{C})$$

dim $U(n) = n^2$.

6.3.7 Special Unitary group

- This is $SU(n)$ consisting of all $n \times n$ complex matrices Satisfying:

$$A^\dagger \cdot A = \mathbb{I}, \quad \det(A) = 1, \quad A \in Mat(n, \mathbb{C})$$

dim $U(n) = n^2 - 1$.

7 Projective spaces

7.1 Real protective space

- The *real Projective space* $\mathbb{R}P^n$ is the set of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin. Equivalently it is the set of equivalence classes of non-zero vectors in $\mathbb{R}^{n+1}$ where two non-zero vectors are equivalent if they are scalar multiples of one another.
- We may think of $\mathbb{R}P^n$ as obtained from S^n by gluing, that is identifying diametrically opposite points. This means we have the isomorphism $\mathbb{R}P^n \simeq S^n/Z_2$.

7.2 Quaternions

- The set $\mathbb{H}$ of *Quaternions* consists of all linear combinations:

$$q \in \mathbb{H}, \quad q = a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}, \quad a, b, c, d \in \mathbb{R}$$

  Where $\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are linearly independent. Where these bases satisfy the following multiplications:

$$\begin{array}{llll}
\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{k} = -\boldsymbol{j}\cdot\boldsymbol{i}, & \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{i} = -\boldsymbol{k}\cdot\boldsymbol{j}, & \boldsymbol{k}\cdot\boldsymbol{i} = \boldsymbol{j} = -\boldsymbol{i}\cdot\boldsymbol{k}, & \\
\boldsymbol{i}\cdot\boldsymbol{i} \equiv \boldsymbol{i}^2 = -1, & \boldsymbol{j}\cdot\boldsymbol{j} \equiv \boldsymbol{j}^2 = -1, & \boldsymbol{k}\cdot\boldsymbol{k} \equiv \boldsymbol{k}^2 = -1, & \\
\boldsymbol{i}\cdot\mathbf{1} = \boldsymbol{i} = \mathbf{1}\cdot\boldsymbol{i}, & \boldsymbol{j}\cdot\mathbf{1} = \boldsymbol{j} = \mathbf{1}\cdot\boldsymbol{j}, & \boldsymbol{k}\cdot\mathbf{1} = \boldsymbol{k} = \mathbf{1}\cdot\boldsymbol{k}, & \mathbf{1}\cdot\mathbf{1} = \mathbf{1}.
\end{array}$$

  This makes $\mathbb{H}$ an associative algebra over the field of real numbers.

7.3 Complex Projective spaces

- The *complex projective space* $\mathbb{CP}^{\ltimes}$ is the set of equivalence classes of non-zero vectors in $\mathbb{C}^{\ltimes+\nVdash}$ where two nonzero vectors are equivalent if they are scalar multiples of one another.
- In a similar manner to the real projective space we can identify the isomorphism: $\mathbb{C} \simeq S^{2n+1}/U(1)$.

8 Lie Algebras

8.1 Neighborhood of identity element

- Let G be a Lie group. let the point $g_0 \equiv 1 \in G$ be the identity element of G, and let $T = T_{(1)}$ be the tangent space at the identity element. We can now express the group operations on G in a chart U_0 containing g_0 in terms of local co-ords. We choose co-ords in U_0 so that the identity element is the origin. $g_0 \equiv 1 = (0, \ldots, 0)$. then if we let:

$$g_1 = (x^1, \ldots, x^n), \quad g_2 = (y^1, \ldots, y^n), \quad g_3 = (z^1, \ldots z^n)$$

Which allows us to define the product of two elements:

$$g_1 g_2 = (\psi^1(x,y), \ldots, \psi^n(x,y)) = (\psi^i(x,y)) \in U_0$$

An inverse as:

$$g_1^{-1} = (\varphi^1(x), .., \varphi^n(x)) = (\varphi^i(x)) \in U_0$$

These functions $\varphi(x), \psi(x)$ satisfy:

$$\begin{aligned}
&\psi^i(x,0) = \psi^i(0,x) = x^i \\
&\psi^i(x, \varphi(x)) \\
&\psi^i(x, \psi(y,z)) = \psi^i(\psi(x,y), z)
\end{aligned}$$

8.1.1 Taylor expansion

- Let $\psi^i(x,y)$ be sufficiently smooth and for $x, y, z \sim \epsilon$:

$$\begin{aligned}
\psi^i(x,y) =& x^i + y^i + b^i_{jk} x^j y^k + \mathcal{O}(\epsilon^3) \\
b^i_{jk} =& \left. \frac{\partial^2 \psi^i}{\partial x^j \partial y^k} \right|_{x=y=0}
\end{aligned}$$

8.2 Commutator

- Let $\xi, \eta \in T$, and their components in terms of x^i are ξ^i and η^i. Then we can define the *commutator* $[\xi, \eta] \in T$ is defined by:

$$\boxed{[\xi, \eta]^i = c^i_{jk} \xi^j \eta^k, \quad c^i_{jk} \equiv b^i_{jk} - b^i_{kj}}$$

- It has three basic quantities:
  - It is *bi-linear* operation on the n-dim vector space T.
  - Skew-symmetry: $[\xi, \eta] = -[\eta, \xi]$.
  - Jacoby identity: $[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi]$

8.3 Lie Algebra

- A *Lie algebra* is a vector space $\mathcal{G}$ over a field F with a bi-linear operation $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ which is called a commutator or a lie bracket, such that the three axioms above are satisfied.
- This means we can identify the tangent space of a Lie Group at the identity is with respect to the commutator operation of a Lie algebra called the *Lie algebra of the Lie group G*.
- If we choose $\xi = e_j, \eta = e_k$, then combined with the fact that $(e_m) = \delta^n_m$, then we have:

$$\boxed{[e_j, e_k]^i = c^i_{jk} e_i}$$

8.3.1 Structure Constants

- The constants c^i_{jk} which determine the commutation operation on a Lie algebra, and which are skew-symmetric in j, k are called the *structure constants* of the Lie algebra.

9 One parameter subgroups

- A *One parameter subgroup* of a lie group G is defined to be a parametric curve $F(t)$ on the manifold G such that:

$$F(0) = 1, \quad F(t_1 + t_2) = F(t_1)F(t_2), \quad F(-t) = F^{-1}(t)$$

The velocity vector at $F(t)$ is:

$$\frac{dF}{dt} = \frac{dF(t+\epsilon)}{dt}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}(F(t)F(\epsilon))\bigg|_{\epsilon=0} = F(t)\frac{dF(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

Hence:

$$\dot{F}(t) = F(t)\dot{F}(0) \quad \text{or} \quad F^{-1}(t)\dot{F}(t) = \dot{F}(0)$$

i.e. the induced action of left multiplication by $F^{-1}(t)$ sends $\dot{F}(t)$ to $\dot{F}(0) = const \in T$.

- conversely, $\forall\ A \in T$ the equation $F^{-1}(t)\dot{F}(t) = A$ is satisfied by a unique one-parameter subgroup $F(t)$ of G. If G is a matrix group then $F(t) - exp(At)$.

9.1 Co-ords of the first kind

- One parameter subgroups can be used to define so called *canonical* in a neighborhood of the identity of a Lie group G.
- Let $A_1, \ldots, A_n$ form a basis for the Lie algebra T. Then $\forall\ A = \sum_i A_i x^i \ \in\ T\ \exists$ a one parameter group $F(t) = exp(At)$. To the point $F(1) = exp(A)$ we assign as co-ords co-officiants $x^1, \ldots, x^n$, which gives us a system of co-ords in a sufficiently small neighborhood of $g_0 = 1 \in G$. These are called the *canonical co-ords of the first kind.*

9.2 Co-ords of the second kind

- Another system of co-ords is obtained by introducing $F_i(t) = exp(At)$ and representing a point g sufficiently close to g_0 as:

$$g = F_1(t_1)F_2(t_2)\cdots F_n(t_n)$$

for small $t_1, ..., t_n$. Assigning co-ords $x^1 = t_1, ..., x^n = t_n$ to the point g, we get the *canonical co-ords of the second kind.*

10 Linear Representations

10.1 Representations

- A *Linear representation* of a group G of $dimG = n$ is a homomorphism:

$$\rho : G \to GL(r, \mathbb{R}), \quad \text{or} \quad \rho : G \to GL(r, \mathbb{C})$$

- Given a representation ρ of G the map:

$$\chi_\rho : G \to \mathbb{R}, \quad \text{or} \quad G \to \mathbb{C}$$

defined by:

$$\chi_\rho(g) = \mathrm{tr}(\rho(g))$$

is called the *character* of the representation ρ.

- A representation ρ of G is said to be *irreducible* if the vector space $\mathbb{R}^r$ contains no proper subspace invariant under the matrix group $\rho(G)$.

10.1.1 Matrix Invariance

- A subspace W of the representation space $\mathbb{R}^r$ is called *invariant under the matrix group* $\rho(G)$ (or simply G invariant) if:

$$\rho(G)W \subset W, \quad \forall\, g \in G$$

Then we can restrict ρ to W and get a *subrepresentation*.

10.2 Schur's Lemma

- Let $\rho_i : G \to GL(r_i, \mathbb{R}), \quad i = 1, 2$ be two irreducible representations (irreps) of a group G. If $A : R^{r_1} \to \mathbb{R}^{r_2}$ is a linear transformation changing ρ_1 to ρ_2, i.e. stratifying:

$$A\rho_1(g) = \rho_2 A, \quad \forall\, g \in G$$

Then either A is the zero transformation or else a bijection, in which case $r_1 = r - 2$.

10.3 Push Forward Representation

- If G is a Lie group and a representation $\rho : G \to GL(r, \mathbb{R})$ is a smooth map, then the push-froward map ρ_* is a linear map from the Lie algebra $\mathfrak{g} = T_{(1)}$ to the space of all $r \times r$ matrices:

$$\rho_* : \mathfrak{g} \to Mat(r, \mathbb{R})$$

It can then be shown that this means ρ_* is a *representation* of the Lie algebra $\mathfrak{g}$, i.e. that it is a Lie algebra homomorphism. Meaning it is linear and preserves the commutators $\rho_*[\xi, \eta] = [\rho_*\xi, \rho_*\eta]$.

10.4 Faithful

- A representation $\rho : G \to GL(r, \mathbb{R})$ is called *faithful* if it is one to one i.e. if its Kernel is trivial. So $\rho(g) \neq \mathbb{I}$ unless $g = g_0$.
- If a Lie group has a faithful representation then it can be realized as a matrix Lie group.

10.5 Inner automorphism

- For each $h \in G$ the transformation $G \to G$ defined by $g \to hgh^{-1}$ is called the *inner automorphism.* of G determined by h.
- Any inner automorphism does not move the identity element. i.e. $g_0 = hg_0h^{-1}$ and therefor the push forward (induced linear) map of the tangent space T to G at g_0 is a linear transformation of T denoted by:

$$Ad_h : T \to T$$

  it satisfies the following:

  - $Ad_{g_0} = id$, where id is the identity transformation of T.
  - $Ad_{h_1}Ad_{h_2} = Ad_{h_1h_2}$ for all $h_1, h_2 \in G$. because $h_1h_2gh_2^{-1}h_1^{-1} = (h_1h_2)g(h_1h_2)^{-1}$.
  - Choosing $h_1 = h, h_2 = h^{-1}$, we get that $Ad_{h^{-1}} = Ad_h^{-1}$
- This means that the map $h \to Ad_h$ is a *linear representation* of the group G. i.e. a homomorphism to a group of linear transformations, $Ad : G \to GL(n, \mathbb{R}), h \to Ad_h = Ad(h)$. This representation of G is called *Adjoint*.

10.6 One Parameter Adjoint

- Let $F(t) = e^{At}$ be a one parameter subgroup of a Lie group G. Then $Ad_{F(t)}$ is a one parameter subgroup of $GL(n, \mathbb{R})$.

  The vector $\frac{d}{dt}Ad_{F(t)}\Big|_{t=0}$ lies in the Lie algebra $\mathfrak{g} \sim Mat(n, \mathbb{R})$ of the Group $GL(n, \mathbb{R})$ and can be regarded as a linear operator.
- This operator is denoted ad_A and is given by:

$$\boxed{ad_A : \mathbb{R} \to \mathbb{R}, \quad B \to [A, B], \quad B \in T \simeq \mathbb{R}^n}$$

11 Simple Lie Algebras and Forms

11.1 Simple & Semi-Simple

- A Lie algebra $\mathfrak{g} = \{\mathbb{R}^n, c^i_{jk}\}$ is said to be *simple* if it is *non-commutative* and has *no proper ideals*, i.e. subspaces $\mathcal{I} \neq \mathfrak{g}, 0$ for which $[\mathcal{I}, \mathfrak{g}] \subset \mathcal{I}$.
- It is instead called *semi-simple* if we can write $\mathfrak{g} = \mathcal{I}_1 \otimes \mathcal{I}_2 \otimes \cdots \otimes \mathcal{I}_k$ Where the $\mathcal{I}_j$ are ideals which are simple as Lie algebras. These ideals are pairwise commuting $[\mathcal{I}_i, \mathcal{I}_j] = 0, \quad i \neq j$.

  A Lie group is defined to be simple or semi-simple according to its Lie algebra.
- A theorem that can be proven is that if the Lie algebra $\mathfrak{g}$ of a Lie group G is simple, then the linear representation $Ad : G \to GL(n, \mathbb{R})$ is *irreducible*, i.e. $\mathfrak{g}$ has no proper invariant sub-spaces under the group of inner automorphisms Ad_G.

11.2 Killing Form

- The *Killing form* on an arbitrary Lie algebra $\mathfrak{g}$ is defined (up to a sign) by:

$$\langle A, B \rangle = -\mathrm{tr}(ad_A ad_B)$$

- If the Killing form of a Lie algebra is positive definite then the Lie algebra is semi-simple.
- We also have that a Lie algebra is semi-simple if and only if its Killing form is non-degenerate.

12 Group Actions

12.1 Left and Right actions

- We say that a Lie group G is represented as a *group of transformations* of a manifold M, or has a *left action* on M if:
  - There is associated with each of its elements g a diffeomorphism from M to itself. $x \mapsto \mathcal{T}_g(x), \quad x \in M$. Such that $\mathcal{T}_g \mathcal{T}_h = \mathcal{T}_{gh}, \ \forall \ g, h \in G$.
  - $\mathcal{T}_g(x)$ depends smoothly on the arguments g, x i.e. the map $(g, x) \mapsto \mathcal{T}_g(x)$ is a smooth map from $G \times M \to M$.
- The Lie group is said to have *Right action* on M if the above definition is valid with $\mathcal{T}_g \mathcal{T}_h = \mathcal{T}_{hg}$.

12.2 Transitivity

- The action of a group G on M is said to be *transitive* if for every two points $x, y \in M$ there exists an element of G such that $\mathcal{T}_g(x) = y$.

  To show that an action of a group on a manifold is transitive it is sufficient to choose any point of M as a reference point x_0, and to prove that for any point $y \in M$ there exists an element $g \in G$ such that $y = \mathcal{T}_g(x_0)$.

12.2.1 Homogeneity

- A manifold on which a Lie group acts transitively is called a *homogeneous space* of the Lie group.
- In particular, G is a homogeneous space for itself, e.g. as $h \to \mathcal{T}_g(h) = gh, \ h \in G$. G is called the *principle* homogeneous space.

12.2.2 Isotropy group

- Let x be any point of a homogeneous space M of a Lie group G. The *isotropy* group (or *stationary* group) H_x of the point x is the stabilizer of x under the action of G:

$$H_x = \{h | \mathcal{T}_h(x) = x\}$$

- All isotropy groups H_x of points x of a homogeneous space are isomorphic.
- There is a one to one correspondence between the points of a homogeneous space M of a group G, and the left cosets gH of H in G, where H is the isotropy group and G acts on the left. Thus we can write $M \simeq G/H$, i.e. M is a diffeomorphic to the quotient space G/H.

12.3 Examples of Homogeneous spaces

12.3.1 Stiefel manifolds

- For each n, k the Stiefel manifold $V_{n,k}$ has as its points *all orthonormal* k-frames $x = (e_1, \ldots, e_k)$ of k vectors e_a in $\mathbb{R}^n$.
- The dimension of $V_{n,k}$ is $nk - \frac{1}{2}k(k+1)$ and $V_{n,k} \simeq O(n)/O(n-k) \simeq SO(n)/SO(n-k)$.

12.3.2 Real Grassmanian manifolds

- The points of $G_{n,k}$ are the k dimensional planes passing through the origin of $\mathbb{R}^n$.
- It can be shown that $G_{n,k} \simeq O(n)/(O(k) \times O(n-k)) \simeq G_{n,n-k}$. The dimension of $G_{n,k}$ is $(n-k)k$

13 Vector Bundles

13.1 Tangent Bundle

- The *tangent bundle* $T(M)$ of an n dimensional manifold M is a $2n$ dimensional manifold defined as follows:
  - The points of $T(M)$ are the pairs $(x,\xi), x \in M,\ \xi \in T_x M$.
  - Given a chart U_q of M with the local co-ords (x_q^i), the corresponding chart U_q^T of $T(M)$ is the set of all pairs (x,ξ) where:

    $$x = (x_1^1, \dots, x_q^n) \in U_q, \quad \xi = \xi_q^i \frac{\partial}{\partial x_q^i} \in T_x M$$

    with local co-ords $(y_q^1, \dots, y_q^{2n}) = (x_q^1, \dots, x_q^n, \xi_q^1, \dots, \xi_q^n) = (x_q^i, \xi_q^i)$.
- This tangent bundle is a smooth oriented manifold.

13.2 Cotangent Bundle

- The *cotangent bundle* $T^*(M)$ of an n dimensional manifold M is a $2n$ dim manifold defined as follows:
  - The points $T^*(M)$ are the pairs $(x,p), x \in M$ and p a co-vector at the point x, so $p \in T_x^* M$.
  - Given a chart U_q of M with the local co-ords (x_q^i), the corresponding chart $U_x^{T^*}$ of T^*M is the set of all pairs (x,p), where:

    $$x = (x_1^1, \dots, x_q^n) \in U_q, \quad p = p_{qi} dx_q^i \in T_x^* M$$

    with local co-ords $(y_q^1, \dots, y_q^{2n}) = (x_q^1, \dots, x_q^n, p_{q1}, \dots, p_{qn}) = (x_q^i, p_{qi})$.
- This cotangent bundle is a smooth oriented manifold.

13.3 Symplectic Manifold

- The existence of a metric on M gives rise to a map:

  $$T(M) \to T^*(M) : (x^i, \xi^i) \mapsto (x^i, g_{ij}\xi^i)$$

- Since $\omega = p_i dx^i$, a differential one-form on M, is invariant under a change of co-ords of $T^*(M)$, it is a differential form on $T^*(M)$.

  Its differential $\Omega = d\omega = dp_i \wedge dx^i$ is a *non-degenerate closed*,$(d\Omega = 0)$, 2-form on $T^*(M)$.
- Thus $T^*(M)$ is a *symplectic* manifold, i.e. it is equipped with a closed non-degenerate 2-form.

14 Vector and Tensor Fields

14.1 Vector Field

- A *vector field* is a map that specifies a unique vector at each point x of the manifold M:

$$\xi : M \to T(M), \quad x \mapsto \xi_x \in T_x M$$

  A vector field intersects each tangent space of $T(M)$ at one and only one point, i.e. a vector field is a curve which is no-where parallel to a tangent space. It is a *cross section* of $T(M)$.

- A vector field can be understood as a differential operator that maps a scalar function to a scalar function on M:

$$\xi(f) = \xi^i \frac{\partial f}{\partial x^i}.$$

- These maps are linear and satisfy the Leibniz rule. This means they are *derivations*.

14.2 Tensor Field

- A *Tensor field* of type (r, s) assigns a unique tensor of type (r, s) to each point x of the manifold M:

$$^{(r,s)}\xi : M \to T^{(r,s)}(M), \quad x \mapsto^{(r,s)} \xi_x \in T_x^{(r,s)} M$$

  . It is a cross section of $T_x^{(r,s)} M$.