Differential Geometry

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Notes taken in Professor Sergey Frolov's class, Michaelmas Term 2024 “hokay” -Sergey Frolov

Contents

1 Definition of a Manifold

1.1 Regions

- A *region* ("open set") is a set of D points in $\mathbb{R}^n$ such that together with each point p_0, D also contains all points sufficiently closer to p_0, i.e.:

$$\forall\ p_0 = (x_0^1, \dots, x_0^n) \in D\ \exists\ \epsilon > 0,$$
$$st : p = (x^1, \dots, x^n) \in D,\ \text{iff}\ |x^i - x_0^i| < \epsilon.$$

- A *region with out a boundary* is obtained fro ma region D by adjoining all boundary points to D. The *boundary* of a region is the set of all boundary points.

1.2 Differentiable Manifold

- A differentiable n-dimensional manifold is a set M together with the following structure on it. The set M is the union of a finite or countably infinite collection of subsets U_q with the following properties:
  - Each subset U_q has defined on it co-ords $x_q^\alpha, \alpha = 1, \dots, n$ called local co-ords by virtue of which U_q is identifiable with a region of Euclidean n-space $\mathbb{R}^n$ with Euclidean co-ords x_q^α. The U_q with their co-ord systems are called *charts* or *local coordinate neighborhoods*.
  - Each non-empty intersection $U_q \cap U_p$ of a pair of charts thus has defined on it two co-ord systems, the restriction of x_p^α and x_q^α. It is required that under each of these coordinatizations the intersection $U_q \cap U_p$ is identifiable with a region of $\mathbb{R}^n$ and that each of these co-ordinate systems be expressible in terms of the other in a one to one differentiable manner. Thus, if a the *transition* functions from x_p^α to x_q^α and back are given by:

$$x_p^\alpha = x_p^\alpha(x_q^1, \dots, x_q^n), \quad \alpha = 1, \dots, n$$
$$x_q^\alpha = x_q^\alpha(x_p^1, \dots, x_p^n), \quad \alpha = 1, \dots, n$$

    Then the *Jacobian* $J_{pq} = det(\partial x_p^\alpha / \partial x_q^\alpha)$ is non-zero on $U_p \cap U_q$.

1.3 Abuse of notation

- Regular partial derivative do not have the same "canceling" that total derivative have ($dx * dy/dx = dy$) But we can restore this property through Einstein summation convention. That is that:

$$\boxed{\sum_{\gamma=1}^{n} \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \delta_\beta^\alpha}$$

2 Elements of Topology

2.1 Topological space

- A topological space is a set X of points of which certain subsets called *open sets* of the topological space, are distinguished, these open sets have to satisfy:
  - The intersection of any two (and hence of any finite collection) open sets should again be an open set.
  - The union of any collection of open sets must again be open.
  - The empty set and the whole set X must be open.
- The compliment of any open set is called a *closed* set of the topological space.

  In Euclidean space $\mathbb{R}^n$ the "Euclidean topology" is the usual one where the open sets are the open regions.

2.1.1 Induced topology

- Given any subset $A \in \mathbb{R}^n$, the *induced topology* on A is that where the open sets are the intersections $A \cap U$, where U ranges over all open sets of $\mathbb{R}^n$.

2.1.2 Continuity

- A map $f : X \to Y$ of one topological space to another is called *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subset Y$ is open in X.

2.1.3 Homeomorphic

- Two topological space are *topologically equivalent* or *homeomorphic* if there is a one to one and onto map (bijective) between them, such that it and its inverse are continuous.

2.1.4 Topology on a manifold

- The topology on a manifold M is given by the following specifications of the open sets. In every local co-ordinate neighborhood U_q the open regions are to be open in the topology on M; the totality of open sets of M is then obtained by admitting as open, also arbitrary unions countable collections of such regions, i.e. by closing under countable unions.

2.2 Metric space

- A *metric space* is a set which comes equipped with a "distance function" i.e. a real-valued function $\rho(x, y)$, defined on pairs x, y of its elements and having the following properties:
  - Symmetry: $\rho(x, y) = \rho(y, x)$.
  - Positivity: $\rho(x, x) = 0, \quad \rho(x, y) > 0 \text{ if } x \neq y$.
  - The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

2.2.1 Hausdorff

- A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centers at c, y do not intersect.

  All topological spaces we consider will be Hausdorff.

2.2.2 Compact

- A topological space X is said to be compact if every countable collection of open sets covering X contains a finite sub-collection already covering X.

  If X is a metric space the compactness is equivalent to the condition that from every sequence of points of X a convergent sub-sequence can be selected.

2.2.3 Connected

- A topological space is connected if any two points can be joined by a continuous path.

2.3 Orientation

- A manifold M is said to be *orientated* of one can choose its atlas (collection of all the charts) so that for every pair U_p, U_q of intersecting co-ordinate neighborhoods the Jacobian of the transition functions is positive.
- We say that the co-ordinate systems x and y define the *same orientation* if $J > 0$ and the *opposite orientation* if $J < 0$.

3 Mappings on Manifolds

3.1 Manifold mappings

- A mapping $f : M \to N$ is said to be smooth of smoothness class k if for all p, q for which f determines functions $y^b_q(x^1_p, \ldots, x^m_p) = f(x^1_p, .., x^m_p)^b_p$, these functions are, where defined, smooth of smoothness calss k (i.e. all theire partial derivatives up to those of k-th order exist and are continuous).

  the smoothness class of f cannot exceed the maximum class of the manifolds.

3.2 equivilent manifolds

- The manifolds M and N are said to be *smoothly equivilent* or *diffeomorphic* if there is a one to one and onto map f such that both $f : M \to N$ and $f^{-1} : N \to M$ are smooth of some class $k \geq 1$.

  Since f^{-1} exits then the jacobian $J_{pq} \neq 0$ wherever it is defined.

3.3 Tangent vector

- A *tangent* vector to an m-dim manifold M at an arbitrary point x is represented in terms of local co-ords $x^\alpha_- p$ by an m tuple ξ^α of components which are linked to the components in terms of any oher system x^β_q of local co-ords by:

$$\xi^\alpha_p = \left(\frac{\partial x^\alpha_p}{\partial x^\beta_q} \right)_x \xi^\beta_q, \quad \forall\ \alpha \tag{3.1}$$

- The set of all tangent vectors to an m-dim manifold M at a point x forms an m-dm vector space $T_x = T_x M$, the *tangent space* to M at the point x.
- Thus, the velocity at x of any smooth curve M through x is a tangent vector to M at x.

3.4 Push forward

- A smooth map f from M to N gives rise for each x to a *push forward* or an *induced linear* map to tangent spaces:

$$f_* : T_x M \to T_{f(x)} N$$

  defined as sending the velocity at x of any smooth curve $x = x(\tau)$ on M to the velocity vector at $f(x)$ of the curve $f(x(\tau))$ on N. If the map f is given by: $y^b = f^b(x^1, \ldots, x^m)$ for $x \in M$ and $y \in N$, then the push forward map f_* is:

$$\xi^\alpha \to \eta^b = \frac{\partial f^b}{\partial x^\alpha} \xi^\alpha.$$

- For a real valued function $f : M- > \mathbb{R}$, the push-forward map f_* corresponding to each $x \in M$ is a real valued linear function on the tangent space to M at x:

$$\xi^a \to \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha$$

  and it is represented by the gradiant of f at x, and is a co-vector or one form. Thus f_* can be identified with the differential df, in particular:

$$dx^\alpha_p : \xi^\alpha \to \eta = \xi^\alpha_p$$

3.5 Riemann metric

- A *Riemann metric* on a manifold M is a point-depedant, positive- definitr quadratic form on the tangent vectors at each point, depending smoothly on the local co-ords of the points.

  Thus at each point $x = (x_p^1, \ldots, x_p^m)$ of each cahrt U_p, the metric is given by a symetric matric $g_{\alpha\beta}(x_p^1, ..., x_p^m)$, and determines a symmetric scalar product of pairs of tangent vectors at the point x.

  $$\langle \xi, \eta \rangle = g^{(p)}_{\alpha\beta} \xi_p^\alpha \eta_p^\beta = \langle \eta, \xi \rangle \, , \quad |\xi|^2 = \langle \xi, \xi \rangle$$

  This scalar product is to be co-ordiante independant:

  $$g^{(p)}_{\alpha\beta} \xi_p^\alpha \eta_p^\beta = g^{(q)}_{\alpha\beta} \xi_q^\alpha \eta_q^\beta$$

  And therefor the coefficants $g^{(p)}_{\alpha\beta}$ of the quadratic form transform as:

  $$g^{(q)}_{\gamma\delta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_p^\beta}{\partial x_q^\delta} g^{(p)}_{\alpha\beta} \tag{3.2}$$

  For a *psudo-Reimann* metric M one just requires the quadratic fom to be *nondegenerate*. Note that 3.2 can be re-written as:

  $$ds^2 = g^{(p)}_{\alpha\beta} dx_p^\alpha dx_p^\beta = g^{(q)}_{\alpha\beta} dx_q^\alpha dx_q^\beta$$

  Where ds is called a line element, and it is chart-independant. ds is used to measure the distance between two infitesimally close points.