Quantum Mechanics II

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Contents

1 3D Schrodinger equation

- In QM the Hamiltonian is written in terms of operators. Momenta becomes $\mathbf{p} \to -i\hbar\nabla$ and the potential $V \to \hat{V}$ also an operator. So the Hamiltonian becomes:

$$\boxed{\frac{-\hbar^2}{2m}\nabla^2 + \hat{V}} \tag{1.1}$$

- The (time independent) Schrodinger equation is:

$$\boxed{H\psi(x, y, z) = E\psi} \tag{1.2}$$

- If we then have a central potential, that is $V = V(r)$, so rotation symmetry generated by a rotational group, we find it best to change co-ords to spherical co-ords. Here the Laplacian is:

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi}{\partial\phi^2} = 0} \tag{1.3}$$

We then make the typical assumption of separation of variables, that is:

$$\boxed{\psi = R(r)Y(\theta, \phi)} \tag{1.4}$$

The Schrodinger equation then becomes:

$$\boxed{\begin{aligned} -\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta}) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] \\ + V(r)R(r)Y(\theta, \phi) = ER(r)Y(\theta, \phi) \end{aligned}} \tag{1.5}$$

Dividing by $R(r)Y(\theta, \phi)$ and multiplying by r^2, we get two terms, one a function of r only and one a function of θ and ϕ, adding to a constant E. This means that both these terms must themselves be constant. This gives us two separate equations.

$$\boxed{\begin{aligned} -\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}[V(r) - E] = l(l+1) \\ \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = -l(l+1) \end{aligned}} \tag{1.6}$$

The choice $l(l+1)$ as a constant, may seem strange at first but makes sense later. On the later equation we again perform separation of variables, $Y = \Theta(\theta)\Phi(\phi)$, so we get two more equations:

$$\boxed{\begin{aligned} \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} &= -m^2 \\ \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(l(l+1)\sin^2\theta - m^2\right)\Theta &= 0 \end{aligned}} \tag{1.7}$$