

Differential Geometry

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“hokay” -Sergey Frolov

Contents

1	Definition of a Manifold	5
1.1	Regions	5
1.2	Differentiable Manifold	5
1.3	Abuse of notation	5
2	Elements of Topology	6
2.1	Topological space	6
2.1.1	Induced topology	6
2.1.2	Continuity	6
2.1.3	Homeomorphic	6
2.1.4	Topology on a manifold	6
2.2	Metric space	6
2.2.1	Hausdorff	7
2.2.2	Compact	7
2.2.3	Connected	7
2.3	Orientation	7
3	Mappings on Manifolds	8
3.1	Manifold mappings	8
3.2	Equivalent manifolds	8
3.3	Tangent vector	8
3.4	Push forward	8
3.5	Directional derivative	9
3.6	Riemann metric	9
4	Tensors	10
4.1	Tensor def	10
4.2	Operations on Tensors	10
4.2.1	Permutation of indices	10
4.2.2	Contraction of indicies	10
4.2.3	Product of Tensors	10
4.3	Co-Vectors	11
4.4	Skew-Symmetric Tensor	11
4.5	Volume element	11
4.6	Generalized push forward	12
4.7	Pull back	12
5	Manifolds and surfaces	13
5.1	Immersion	13
5.1.1	Embedding	13
5.2	Manifold with boundary	13
5.2.1	Closed manifold	13
5.3	Surfaces as Manifolds	13
5.4	Orientation of surfaces	13
5.4.1	Orientation class	13
5.4.2	Orientability	14
5.5	Two-sided hyper-surface	14
6	Lie Groups	15

6.1	Group	15
6.2	Lie Group	15
6.3	Example of Lie groups	15
6.3.1	General Linear group	15
6.3.2	Special Linear group	15
6.3.3	Orthogonal group	15
6.3.4	Special Orthogonal group	15
6.3.5	Pseudo Orthogonal group	16
6.3.6	Unitary group	16
6.3.7	Special Unitary group	16
7	Projective spaces	17
7.1	Real protective space	17
7.2	Quaternions	17
7.3	Complex Projective spaces	17
8	Lie Algebras	18
8.1	Neighborhood of identity element	18
8.1.1	Taylor expansion	18
8.2	Commutator	18
8.3	Lie Algebra	19
8.3.1	Structure Constants	19
9	One parameter subgroups	20
9.1	Co-ords of the first kind	20
9.2	Co-ords of the second kind	20
10	Linear Representations	21
10.1	Representation	21
10.1.1	Matrix Invariance	21
10.2	Schur's Lemma	21
10.3	Push Forward Representation	21
10.4	Faithful	21

1 Definition of a Manifold

1.1 Regions

- A *region* (“open set”) is a set of D points in \mathbb{R}^n such that together with each point p_0 , D also contains all points sufficiently closer to p_0 , i.e.:

$$\forall p_0 = (x_0^1, \dots, x_0^n) \in D \exists \epsilon > 0, \\ \text{st } p = (x^1, \dots, x^n) \in D, \text{ iff } |x^i - x_0^i| < \epsilon.$$

- A *region with out a boundary* is obtained from a region D by adjoining all boundary points to D . The *boundary* of a region is the set of all boundary points.

1.2 Differentiable Manifold

- A differentiable n -dimensional manifold is a set M together with the following structure on it. The set M is the union of a finite or countably infinite collection of subsets U_q with the following properties:
 - Each subset U_q has defined on it co-ords $x_q^\alpha, \alpha = 1, \dots, n$ called local co-ords by virtue of which U_q is identifiable with a region of Euclidean n -space \mathbb{R}^n with Euclidean co-ords x_q^α . The U_q with their co-ord systems are called *charts* or *local coordinate neighborhoods*.
 - Each non-empty intersection $U_q \cap U_p$ of a pair of charts thus has defined on it two co-ord systems, the restriction of x_p^α and x_q^α . It is required that under each of these coordinatizations the intersection $U_q \cap U_p$ is identifiable with a region of \mathbb{R}^n and that each of these co-ordinate systems be expressible in terms of the other in a one to one differentiable manner. Thus, if a *transition* functions from x_p^α to x_q^α and back are given by:

$$x_p^\alpha = x_p^\alpha(x_q^1, \dots, x_q^n), \quad \alpha = 1, \dots, n \\ x_q^\alpha = x_q^\alpha(x_p^1, \dots, x_p^n), \quad \alpha = 1, \dots, n$$

Then the *Jacobian* $J_{pq} = \det(\partial x_p^\alpha / \partial x_q^\alpha)$ is non-zero on $U_p \cap U_q$.

1.3 Abuse of notation

- Regular partial derivative do not have the same “canceling” that total derivative have ($dx * dy / dx = dy$) But we can restore this property through Einstein summation convention. That is that:

$$\sum_{\gamma=1}^n \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \delta_\beta^\alpha$$

2 Elements of Topology

2.1 Topological space

- A topological space is a set X of points of which certain subsets called *open sets* of the topological space, are distinguished, these open sets have to satisfy:
 - The intersection of any two (and hence of any finite collection) open sets should again be an open set.
 - The union of any collection of open sets must again be open.
 - The empty set and the whole set X must be open.
- The complement of any open set is called a *closed* set of the topological space.

In Euclidean space \mathbb{R}^n the “Euclidean topology” is the usual one where the open sets are the open regions.

2.1.1 Induced topology

- Given any subset $A \in \mathbb{R}^n$, the *induced topology* on A is that where the open sets are the intersections $A \cap U$, where U ranges over all open sets of \mathbb{R}^n .

2.1.2 Continuity

- A map $f : X \rightarrow Y$ of one topological space to another is called *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subset Y$ is open in X .

2.1.3 Homeomorphic

- Two topological spaces are *topologically equivalent* or *homeomorphic* if there is a one to one and onto map (bijective) between them, such that it and its inverse are continuous.

2.1.4 Topology on a manifold

- The topology on a manifold M is given by the following specifications of the open sets. In every local co-ordinate neighborhood U_q the open regions are to be open in the topology on M ; the totality of open sets of M is then obtained by admitting as open, also arbitrary unions countable collections of such regions, i.e. by closing under countable unions.

2.2 Metric space

- A *metric space* is a set which comes equipped with a “distance function” i.e. a real-valued function $\rho(x, y)$, defined on pairs x, y of its elements and having the following properties:
 - Symmetry: $\rho(x, y) = \rho(y, x)$.
 - Positivity: $\rho(x, x) = 0$, $\rho(x, y) > 0$ if $x \neq y$.
 - The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

2.2.1 Hausdorff

- A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centers at x, y do not intersect.

All topological spaces we consider will be Hausdorff.

2.2.2 Compact

- A topological space X is said to be compact if every countable collection of open sets covering X contains a finite sub-collection already covering X .

If X is a metric space the compactness is equivalent to the condition that from every sequence of points of X a convergent sub-sequence can be selected.

2.2.3 Connected

- A topological space is connected if any two points can be joined by a continuous path.

2.3 Orientation

- A manifold M is said to be *orientated* if one can choose its atlas (collection of all the charts) so that for every pair U_p, U_q of intersecting co-ordinate neighborhoods the Jacobian of the transition functions is positive.
- We say that the co-ordinate systems x and y define the *same orientation* if $J > 0$ and the *opposite orientation* if $J < 0$.

3 Mappings on Manifolds

3.1 Manifold mappings

- A mapping $f : M \rightarrow N$ is said to be smooth of smoothness class k if for all p, q for which f determines functions $y_q^b(x_p^1, \dots, x_p^m) = f(x_p^1, \dots, x_p^m)_p^b$, these functions are, where defined, smooth of smoothness class k (i.e. all their partial derivatives up to those of k -th order exist and are continuous).

the smoothness class of f cannot exceed the maximum class of the manifolds.

3.2 Equivalent manifolds

- The manifolds M and N are said to be *smoothly equivalent* or *diffeomorphic* if there is a one to one and onto map f such that both $f : M \rightarrow N$ and $f^{-1} : N \rightarrow M$ are smooth of some class $k \geq 1$.

Since f^{-1} exists then the Jacobian $J_{pq} \neq 0$ wherever it is defined.

3.3 Tangent vector

- A *tangent* vector to an m -dim manifold M at an arbitrary point x is represented in terms of local co-ords x_p^α by an m tuple ξ^α of components which are linked to the components in terms of any other system x_q^β of local co-ords by:

$$\xi_p^\alpha = \left(\frac{\partial x_p^\alpha}{\partial x_q^\beta} \right)_x \xi_q^\beta, \quad \forall \alpha \quad (3.1)$$

- The set of all tangent vectors to an m -dim manifold M at a point x forms an m -dim vector space $T_x = T_x M$, the *tangent space* to M at the point x .
- Thus, the velocity at x of any smooth curve M through x is a tangent vector to M at x .

3.4 Push forward

- A smooth map f from M to N gives rise for each x to a *push forward* or an *induced linear* map to tangent spaces:

$$f_* : T_x M \rightarrow T_{f(x)} N$$

defined as sending the velocity at x of any smooth curve $x = x(\tau)$ on M to the velocity vector at $f(x)$ of the curve $f(x(\tau))$ on N . If the map f is given by: $y^b = f^b(x^1, \dots, x^m)$ for $x \in M$ and $y \in N$, then the push forward map f_* is:

$$\xi^\alpha \rightarrow \eta^b = \frac{\partial f^b}{\partial x^\alpha} \xi^\alpha.$$

- For a real valued function $f : M \rightarrow \mathbb{R}$, the push-forward map f_* corresponding to each $x \in M$ is a real valued linear function on the tangent space to M at x :

$$\xi^\alpha \rightarrow \eta = \frac{\partial f}{\partial x^\alpha} \xi^\alpha$$

and it is represented by the gradient of f at x , and is a co-vector or one form. Thus f_* can be identified with the differential df , in particular:

$$dx_p^\alpha : \xi^\alpha \rightarrow \eta = \xi_p^\alpha$$

3.5 Directional derivative

- We can associate with each vector $\xi = (\xi^i)$ a linear differential operator as follows: Since the gradient $\frac{\partial f}{\partial x^i}$ of a function f is a co-vector, the quantity:

$$\partial_\xi f = \xi^i \frac{\partial f}{\partial x^i}$$

is a scalar called the directional derivative of f in the direction of ξ .

- Thus an arbitrary vector ξ corresponds to the operator:

$$\partial_\xi = \xi^i \frac{\partial}{\partial x^i}$$

So we can identify $\frac{\partial}{\partial x^i} \equiv e_i$ as the *Canonical basis of the tangent space*.

3.6 Riemann metric

- A *Riemann metric* on a manifold M is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local co-ords of the points.

Thus at each point $x = (x_p^1, \dots, x_p^m)$ of each chart U_p , the metric is given by a symmetric metric $g_{\alpha\beta}(x_p^1, \dots, x_p^m)$, and determines a symmetric scalar product of pairs of tangent vectors at the point x .

$$\langle \xi, \eta \rangle = g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = \langle \eta, \xi \rangle, \quad |\xi|^2 = \langle \xi, \xi \rangle$$

This scalar product is to be co-ordinate independent:

$$g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = g_{\alpha\beta}^{(q)} \xi_q^\alpha \eta_q^\beta$$

And therefor the coefficients $g_{\alpha\beta}^{(p)}$ of the quadratic form transform as:

$$g_{\gamma\delta}^{(q)} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_p^\beta}{\partial x_q^\delta} g_{\alpha\beta}^{(p)} \quad (3.2)$$

For a *pseudo-Riemann* metric M one just requires the quadratic form to be *nondegenerate*. Note that 3.2 can be re-written as:

$$ds^2 = g_{\alpha\beta}^{(p)} dx_p^\alpha dx_p^\beta = g_{\alpha\beta}^{(q)} dx_q^\alpha dx_q^\beta$$

Where ds is called a line element, and it is chart-independent. ds is used to measure the distance between two infinitesimally close points.

4 Tensors

4.1 Tensor def

- A *tensor of type* (k, l) and rank $k + l$ on an m -dim manifold M is given each local co-ord system (x_p^i) by a family of functions:

$${}^{(p)}T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x) \text{ of the point } x.$$

In other local co-ord (x_q^i) the components ${}^{(p)}T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$ of the same tensor are:

$${}^{(p)}T_{t_1, \dots, t_l}^{s_1, \dots, s_k}(x) = \frac{\partial x_q^{s_1}}{\partial x_p^{i_1}} \dots \frac{\partial x_q^{s_k}}{\partial x_p^{i_k}} \frac{\partial x_p^{j_1}}{\partial x_q^{t_1}} \dots \frac{\partial x_p^{j_l}}{\partial x_q^{t_l}} \cdot {}^{(p)}T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$$

4.2 Operations on Tensors

4.2.1 Permutation of indices

- Let σ be some permutation of $1, 2, \dots, l$. σ acts on the ordered tuple (j_1, \dots, j_l) as $\sigma(j_1, \dots, j_l) = (j_{\sigma_1}, \dots, j_{\sigma_l})$. We say that a tensor $\tilde{T}_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$ is obtained from a tensor $T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$ by means of a permutation σ of the lower indices if at each point of M :

$$\tilde{T}_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x) = T_{\sigma(j_1, \dots, j_l)}^{i_1, \dots, i_k}(x)$$

Permutation of upper indicies is defined similarly.

4.2.2 Contraction of indicies

- By the contraction of a tensor $T_{j_1, \dots, j_l}^{i_1, \dots, i_k}(x)$ of type (k, l) with respect to the indicies i_a, j_a we mean the tensor (summation over n):

$$T_{j_1, \dots, j_{l-1}}^{i_1, \dots, i_{k-1}}(x) = T_{j_1, \dots, j_{a-1}, n, j_{a+1}, \dots, j_l}^{i_1, \dots, i_{a-1}, n, i_{a+1}, \dots, i_k}(x)$$

Of type $(k-1, l-1)$

4.2.3 Product of Tensors

- Given two tensors $T = (T_{j_1, \dots, j_l}^{i_1, \dots, i_k})$ of type (k, l) and $P = (P_{j_1, \dots, j_q}^{i_1, \dots, i_p})$ of type (p, q) , we define their product to be the tensor product $S = T \otimes P$ of type $(k+p, l+q)$ with components:

$$S_{j_1, \dots, j_{l+q}}^{i_1, \dots, i_{k+p}} = T_{j_1, \dots, j_l}^{i_1, \dots, i_k} P_{j_{l+1}, \dots, j_q}^{i_{k+1}, \dots, i_p}$$

This multiplication is *not commutative* but it is associative.

- The result of applying the above three operations to tensors are again tensors.

4.3 Co-Vectors

- Recall that the differential of a function f of x^1, \dots, x^n corresponding to the increments dx^i in the x^i is:

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Since dx^i is a vector df has the same value in any co-ord system. In general, given any co-vector (T_i) , the differential form $T_i dx^i$ is invariant under change of chart. We can thus identify $dx^i \equiv e^i$ as the *canonical basis of co-vectors or cotangent space*.

4.4 Skew-Symmetric Tensor

- A *skew-symmetric tensor* of type $(0, k)$ is a tensor T_{i_1, \dots, i_k} satisfying:

$$T_{\sigma(i_1, \dots, i_k)} = \mathfrak{s}(\sigma) T_{i_1, \dots, i_k}$$

where for all permutations $\mathfrak{s}(\sigma)$ is the sign function. i.e. $\mathfrak{s}(\sigma) = +1(-1)$ for even(odd) permutation. If two indices of T_{i_1, \dots, i_k} are the same then the corresponding component of T_{i_1, \dots, i_k} is 0. This means if $k > n$ the tensor is automatically 0.

- The standard basis at a given point is:

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad i_1 < i_2 < \dots < i_k$$

Where:

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathfrak{s}(\sigma) e^{i_{\sigma_1}} \otimes \dots \otimes e^{i_{\sigma_k}}$$

Here S_k is the symmetric group. i.e. the group of all permutations of k elements.

- The differential form of the skew-symmetric tensor (T_{i_1, \dots, i_k}) is:

$$\begin{aligned} T_{i_1, \dots, i_k} e^{i_1} \otimes \dots \otimes e^{i_k} &= \sum_{i_1 < i_2 < \dots < i_k} T_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ &= \frac{1}{k!} T_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

Where the last step can be made as both $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and T_{i_1, \dots, i_k} are anti-symmetric.

4.5 Volume element

- A metric g_{ij} on a manifold is a tensor of type $(0, 2)$ and on an oriented manifold of $\dim(M) = n$ such a metric gives rise to a *volume element*:

$$T_{i_1, \dots, i_n} = \sqrt{|g|} \epsilon_{i_1, \dots, i_n}, \quad g = \det(g_{ij})$$

It is convenient to write the volume element in the notation of differential forms:

$$\Omega = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$$

If g_{ij} is Riemann then the *volume* V of M is:

$$V = \int_M \Omega = \int_M \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$$

4.6 Generalized push forward

- We can generalize the push forward map we had on vectors earlier to the space of tensors $(k, 0)$:

$$f_* : \xi^{i_1, \dots, i_k} \rightarrow \eta^{a_1, \dots, a_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \xi^{i_1, \dots, i_k}$$

4.7 Pull back

- Let $T_x^{(0,k)}M$ denote the space of tensors of type $(0, k)$ at $x \in M$. Let f be a smooth map from M to N . It gives rise to a map:

$$f^* : T_{f(x)}^{(0,k)}N \rightarrow T_x^{(0,k)}M$$

which in terms of $x^i \in U \subset M$, and $y^a \in V \subset N$ is written as:

$$f^* : \eta_{a_1, \dots, a_k} \rightarrow \xi_{i_1, \dots, i_k} = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1, \dots, a_k}$$

The map f^* is called the *pullback*.

- We can then note the following relationship between pullbacks and push forwards. Let us denote the action of a vector on another vector as follows:

$$\zeta(\theta) \equiv \zeta_{i_1, \dots, i_k} \theta^{i_1, \dots, i_k}$$

Then we can write that:

$$(f^*\eta)(\xi) = \frac{\partial f^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{a_k}}{\partial x^{i_k}} \eta_{a_1, \dots, a_k} \xi^{i_1, \dots, i_k} = \eta(f_*\xi)$$

5 Manifolds and surfaces

5.1 Immersion

- A manifold M of dim m is said to be immersed in a manifold N of dim $n \geq m$ if \exists a smooth map $f : M \rightarrow N$ such that the push forward map f_* is at each point a one to one map of the tangent space.

The map f is called the *immersion* of M to N .

Since f_* is at each a point one to one map of the tangent space, in terms of local co-ords the Jacobian matrix of f at each point has rank equal to $m = \dim M$.

5.1.1 Embedding

- An immersion of M to N is called an *embedding* if it one to one. Then M is called a *sub-manifold* of N .
- To see the difference between these two definitions note that a Klein bottle is immersed in \mathbb{R}^3 but not embedded as its tangent spaces are distinct (intersecting points can have different tangent spaces) but the map of points is not one- to one as there are cross overs.

5.2 Manifold with boundary

- A closed region A of a manifold M defined by an inequality:

$$f(x) \leq 0, \quad (\text{or } f(x) \geq 0)$$

where f is a real-valued function on M . This region is a *Manifold with boundary*. It is assumed that the boundary ∂A given by $f(x) = 0$ is a non-singular sub-manifold of M i.e. $\nabla f \neq 0$ on ∂A .

5.2.1 Closed manifold

- A compact manifold without a boundary is called *closed*.

5.3 Surfaces as Manifolds

- A *Non-singular surface* M of dimension k in n -dim Euclidean space is given by a set of $n - k$ equations:

$$f_i(x^1, \dots, x^n) = 0, \quad i = 1, \dots, n - k$$

where $\forall x$ the matrix $\left(\frac{\partial f_i}{\partial x^\alpha} \right)$ has rank $n - k$.

5.4 Orientation of surfaces

5.4.1 Orientation class

- Consider a frame $\tau_1 = (e_1^{(1)}, \dots, e_n^{(1)})$ called an ordered basis and another frame $\tau_1 = (e_1^{(2)}, \dots, e_n^{(2)})$ then we say that they lie in the *same orientation class* if $\det A > 0$ and the *opposite orientation*

class if $\det A < 0$. Where A is defined as:

$$A : e_k^{(1)} \rightarrow e_k^{(2)}$$

5.4.2 Orientability

- A manifold is said to be *orientable* if it is possible to choose at every point of it a single orientation class depending continuously on the points.

A particular choice of such an orientation class for each point is called an orientation of the manifold, and a manifold equipped with a particular orientation is said to be *oriented*.

If no orientation exists the manifold is said to be *non-orientable*

5.5 Two-sided hyper-surface

- A connected $(n - 1)$ -dim sub-manifold of \mathbb{R}^n is called two sided if a single valued continuous field of unit normals can be defined on it.

such a sub-manifold is called a *two-sided hyper-surface*.

6 Lie Groups

6.1 Group

- A *group* is a non-empty set G on which there is defined a binary operation $(a, b) \rightarrow ab$ satisfying the following properties:
 - Closure: If a and b belong to G , then $ab \in G$.
 - Associativity: $\forall a, b, c \in G, \quad a(bc) = (ab)c$.
 - Identity: \exists an element $1 \in G$ st: $a1 = 1a = a, \quad \forall a \in G$
 - Inverse: If $a \in G$ then $\exists a^{-1} \in G$ st: $aa^{-1} = a^{-1}a = 1$.

6.2 Lie Group

- A manifold G is called a *Lie Group* if it has given on it a group operation with the properties that the maps $\varphi : G \rightarrow G$, defined by $\varphi(g) = g^{-1}$ and $\psi : G \times G \rightarrow G$ defined by $\psi(g, h) = gh$, are smooth maps.

6.3 Example of Lie groups

6.3.1 General Linear group

- This is $GL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with non zero determinant in a region \mathbb{R}^{n^2} .
 $\dim GL(n, \mathbb{R}) = n^2$.

6.3.2 Special Linear group

- This is $SL(n, \mathbb{R})$ consisting of all $n \times n$ real matrices with determinant equal to 1. It is a hyper-surface in \mathbb{R}^{n^2} .

$$\det A = 1, \quad A \in Mat(n, \mathbb{R})$$

$$\dim SL(n, \mathbb{R}) = n^2 - 1.$$

6.3.3 Orthogonal group

- This is $O(n, \mathbb{R})$ consisting of all $n \times n$ real matrices Satisfying:

$$A^T \cdot A = \mathbb{I}, \quad A \in Mat(n, \mathbb{R})$$

$$\dim O(n, \mathbb{R}) = \frac{1}{2}n(n-1).$$

6.3.4 Special Orthogonal group

- This is $SO(n, \mathbb{R})$ consisting of all $n \times n$ real matrices Satisfying:

$$A^T \cdot A = \mathbb{I}, \quad \det(A) = 1, \quad A \in Mat(n, \mathbb{R})$$

$$\dim SO(n, \mathbb{R}) = \frac{1}{2}n(n-1).$$

6.3.5 Pseudo Orthogonal group

- This is $O(p, q, n)$ consisting of all $n \times n$ real matrices Satisfying:

$$A^T \cdot \eta \cdot A = \eta, \quad \det(A) = 1, \quad \eta = \text{diag}\{\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q\}$$

$$\dim O(p, q, n) = \frac{1}{2}n(n-1).$$

6.3.6 Unitary group

- This is $U(n)$ consisting of all $n \times n$ complex matrices Satisfying:

$$A^\dagger \cdot A = \mathbb{I}, \quad A \in \text{Mat}(n, \mathbb{C})$$

$$\dim U(n) = n^2.$$

6.3.7 Special Unitary group

- This is $SU(n)$ consisting of all $n \times n$ complex matrices Satisfying:

$$A^\dagger \cdot A = \mathbb{I}, \quad \det(A) = 1, \quad A \in \text{Mat}(n, \mathbb{C})$$

$$\dim U(n) = n^2 - 1.$$

7 Projective spaces

7.1 Real projective space

- The *real Projective space* $\mathbb{R}P^n$ is the set of all straight lines in \mathbb{R}^{n+1} passing through the origin. Equivalently it is the set of equivalence classes of non-zero vectors in \mathbb{R}^{n+1} where two non-zero vectors are equivalent if they are scalar multiples of one another.
- We may think of $\mathbb{R}P^n$ as obtained from S^n by gluing, that is identifying diametrically opposite points. This means we have the isomorphism $\mathbb{R}P^n \simeq S^n/Z_2$.

7.2 Quaternions

- The set \mathbb{H} of *Quaternions* consists of all linear combinations:

$$q \in \mathbb{H}, \quad q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{R}$$

Where $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are linearly independent. Where these bases satisfy the following multiplications:

$$\begin{aligned} \mathbf{i} \cdot \mathbf{j} &= \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}, & \mathbf{j} \cdot \mathbf{k} &= \mathbf{i} = -\mathbf{k} \cdot \mathbf{j}, & \mathbf{k} \cdot \mathbf{i} &= \mathbf{j} = -\mathbf{i} \cdot \mathbf{k}, \\ \mathbf{i} \cdot \mathbf{i} &\equiv \mathbf{i}^2 = -1, & \mathbf{j} \cdot \mathbf{j} &\equiv \mathbf{j}^2 = -1, & \mathbf{k} \cdot \mathbf{k} &\equiv \mathbf{k}^2 = -1, \\ \mathbf{i} \cdot \mathbf{1} &= \mathbf{i} = \mathbf{1} \cdot \mathbf{i}, & \mathbf{j} \cdot \mathbf{1} &= \mathbf{j} = \mathbf{1} \cdot \mathbf{j}, & \mathbf{k} \cdot \mathbf{1} &= \mathbf{k} = \mathbf{1} \cdot \mathbf{k}, & \mathbf{1} \cdot \mathbf{1} &= \mathbf{1}. \end{aligned}$$

This makes \mathbb{H} an associative algebra over the field of real numbers.

7.3 Complex Projective spaces

- The *complex projective space* $\mathbb{C}P^{\kappa}$ is the set of equivalence classes of non-zero vectors in $\mathbb{C}^{\kappa+\mathbb{K}}$ where two nonzero vectors are equivalent if they are scalar multiples of one another.
- In a similar manner to the real projective space we can identify the isomorphism: $\mathbb{C} \simeq S^{2n+1}/U(1)$.

8 Lie Algebras

8.1 Neighborhood of identity element

- Let G be a Lie group. let the point $g_0 \equiv 1 \in G$ be the identity element of G , and let $T = T_{(1)}$ be the tangent space at the identity element. We can now express the group operations on G in a chart U_0 containing g_0 in terms of local co-ords. We choose co-ords in U_0 so that the identity element is the origin. $g_0 \equiv 1 = (0, \dots, 0)$. then if we let:

$$g_1 = (x^1, \dots, x^n), \quad g_2 = (y^1, \dots, y^n), \quad g_3 = (z^1, \dots, z^n)$$

Which allows us to define the product of two elements:

$$g_1 g_2 = (\psi^1(x, y), \dots, \psi^n(x, y)) = (\psi^i(x, y)) \in U_0$$

An inverse as:

$$g_1^{-1} = (\varphi^1(x), \dots, \varphi^n(x)) = (\varphi^i(x)) \in U_0$$

These functions $\varphi(x), \psi(x)$ satisfy:

$$\begin{aligned} \psi^i(x, 0) &= \psi^i(0, x) = x^i \\ \psi^i(x, \varphi(x)) &= 0 \\ \psi^i(x, \psi(y, z)) &= \psi^i(\psi(x, y), z) \end{aligned}$$

8.1.1 Taylor expansion

- Let $\psi^i(x, y)$ be sufficiently smooth and for $x, y, z \sim \epsilon$:

$$\begin{aligned} \psi^i(x, y) &= x^i + y^i + b_{jk}^i x^j y^k + \mathcal{O}(\epsilon^3) \\ b_{jk}^i &= \left. \frac{\partial^2 \psi^i}{\partial x^j \partial y^k} \right|_{x=y=0} \end{aligned}$$

8.2 Commutator

- Let $\xi, \eta \in T$, and their components in terms of x^i are ξ^i and η^i . Then we can define the *commutator* $[\xi, \eta] \in T$ is defined by:

$$[\xi, \eta]^i = c_{jk}^i \xi^j \eta^k, \quad c_{jk}^i \equiv b_{jk}^i - b_{kj}^i$$

- It has three basic quantities:
 - It is *bi-linear* operation on the n -dim vector space T .
 - Skew-symmetry: $[\xi, \eta] = -[\eta, \xi]$.
 - Jacoby identity: $[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0$

8.3 Lie Algebra

- A *Lie algebra* is a vector space \mathcal{G} over a field F with a bi-linear operation $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ which is called a commutator or a lie bracket, such that the three axioms above are satisfied.
- This means we can identify the tangent space of a Lie Group at the identity is with respect to the commutator operation of a Lie algebra called the *Lie algebra of the Lie group* G .
- If we choose $\xi = e_j, \eta = e_k$, then combined with the fact that $(e_m) = \delta_m^n$, then we have:

$$[e_j, e_k]^i = c_{jk}^i e_i$$

8.3.1 Structure Constants

- The constants c_{jk}^i which determine the commutation operation on a Lie algebra, and which are skew-symmetric in j, k are called the *structure constants* of the Lie algebra.

9 One parameter subgroups

- A *One parameter subgroup* of a lie group G is defined to be a parametric curve $F(t)$ on the manifold G such that:

$$F(0) = 1, \quad F(t_1 + t_2) = F(t_1)F(t_2), \quad F(-t) = F^{-1}(t)$$

The velocity vector at $F(t)$ is:

$$\frac{dF}{dt} = \frac{dF(t + \epsilon)}{dt} \Big|_{\epsilon=0} = \frac{d}{d\epsilon}(F(t)F(\epsilon)) \Big|_{\epsilon=0} = F(t) \frac{dF(\epsilon)}{d\epsilon} \Big|_{\epsilon=0}$$

Hence:

$$\dot{F}(t) = F(t)\dot{F}(0) \quad \text{or} \quad F^{-1}(t)\dot{F}(t) = \dot{F}(0)$$

i.e. the induced action of left multiplication by $F^{-1}(t)$ sends $\dot{F}(t)$ to $\dot{F}(0) = \text{const} \in T$.

- conversely, $\forall A \in T$ the equation $F^{-1}(t)\dot{F}(t) = A$ is satisfied by a unique one-parameter subgroup $F(t)$ of G . If G is a matrix group then $F(t) = \exp(At)$.

9.1 Co-ords of the first kind

- One parameter subgroups can be used to define so called *canonical* in a neighborhood of the identity of a Lie group G .
- Let A_1, \dots, A_n form a basis for the Lie algebra T . Then $\forall A = \sum_i A_i x^i \in T \exists$ a one parameter group $F(t) = \exp(At)$. To the point $F(1) = \exp(A)$ we assign as co-ords co-officiants x^1, \dots, x^n , which gives us a system of co-ords in a sufficiently small neighborhood of $g_0 = 1 \in G$. These are called the *canonical co-ords of the first kind*.

9.2 Co-ords of the second kind

- Another system of co-ords is obtained by introducing $F_i(t) = \exp(A_i t)$ and representing a point g sufficiently close to g_0 as:

$$g = F_1(t_1)F_2(t_2) \cdots F_n(t_n)$$

for small t_1, \dots, t_n . Assigning co-ords $x^1 = t_1, \dots, x^n = t_n$ to the point g , we get the *canonical co-ords of the second kind*.

10 Linear Representations

10.1 Representations

- A *Linear representation* of a group G of $\dim G = n$ is a homomorphism:

$$\rho : G \rightarrow GL(r, \mathbb{R}), \quad \text{or} \quad \rho : G \rightarrow GL(r, \mathbb{C})$$

- Given a representation ρ of G the map:

$$\chi_\rho : G \rightarrow \mathbb{R}, \quad \text{or} \quad G \rightarrow \mathbb{C}$$

defined by:

$$\chi_\rho(g) = \text{tr}(\rho(g))$$

is called the *character* of the representation ρ .

- A representation ρ of G is said to be *irreducible* if the vector space \mathbb{R}^r contains no proper subspace invariant under the matrix group $\rho(G)$.

10.1.1 Matrix Invariance

- A subspace W of the representation space \mathbb{R}^r is called *invariant under the matrix group* $\rho(G)$ (or simply G invariant) if:

$$\rho(G)W \subset W, \quad \forall g \in G$$

Then we can restrict ρ to W and get a *subrepresentation*.

10.2 Schur's Lemma

- Let $\rho_i : G \rightarrow GL(r_i, \mathbb{R})$, $i = 1, 2$ be two irreducible representations (irreps) of a group G . If $A : \mathbb{R}^{r_1} \rightarrow \mathbb{R}^{r_2}$ is a linear transformation changing ρ_1 to ρ_2 , i.e. stratifying:

$$A\rho_1(g) = \rho_2(g)A, \quad \forall g \in G$$

Then either A is the zero transformation or else a bijection, in which case $r_1 = r_2$.

10.3 Push Forward Representation

- If G is a Lie group and a representation $\rho : G \rightarrow GL(r, \mathbb{R})$ is a smooth map, then the push-forward map ρ_* is a linear map from the Lie algebra $\mathfrak{g} = T_{(1)}$ to the space of all $r \times r$ matrices:

$$\rho_* : \mathfrak{g} \rightarrow \text{Mat}(r, \mathbb{R})$$

It can then be shown that this means ρ_* is a *representation* of the Lie algebra \mathfrak{g} , i.e. that it is a Lie algebra homomorphism. Meaning it is linear and preserves the commutators $\rho_*[\xi, \eta] = [\rho_*\xi, \rho_*\eta]$.

10.4 Faithful

- A representation $\rho : G \rightarrow GL(r, \mathbb{R})$ is called *faithful* if it is one to one i.e. if its Kernel is trivial. So $\rho(g) \neq \mathbb{I}$ unless $g = g_0$.
- If a Lie group has a faithful representation then it can be realized as a matrix Lie group.

