

Differential Geometry

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“hokay” -Sergey Frolov

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1 Definition of a Manifold

1.1 Regions

- A *region* (“open set”) is a set of D points in \mathbb{R}^n such that together with each point p_0 , D also contains all points sufficiently closer to p_0 , i.e.:

$$\forall p_0 = (x_0^1, \dots, x_0^n) \in D \exists \epsilon > 0, \\ \text{st } p = (x^1, \dots, x^n) \in D, \text{ iff } |x^i - x_0^i| < \epsilon.$$

- A *region with out a boundary* is obtained from a region D by adjoining all boundary points to D . The *boundary* of a region is the set of all boundary points.

1.2 Differentiable Manifold

- A differentiable n -dimensional manifold is a set M together with the following structure on it. The set M is the union of a finite or countably infinite collection of subsets U_q with the following properties:
 - Each subset U_q has defined on it co-ords $x_q^\alpha, \alpha = 1, \dots, n$ called local co-ords by virtue of which U_q is identifiable with a region of Euclidean n -space \mathbb{R}^n with Euclidean co-ords x_q^α . The U_q with their co-ord systems are called *charts* or *local coordinate neighborhoods*.
 - Each non-empty intersection $U_q \cap U_p$ of a pair of charts thus has defined on it two co-ord systems, the restriction of x_p^α and x_q^α . It is required that under each of these coordinatizations the intersection $U_q \cap U_p$ is identifiable with a region of \mathbb{R}^n and that each of these co-ordinate systems be expressible in terms of the other in a one to one differentiable manner. Thus, if a *transition* functions from x_p^α to x_q^α and back are given by:

$$x_p^\alpha = x_p^\alpha(x_q^1, \dots, x_q^n), \quad \alpha = 1, \dots, n \\ x_q^\alpha = x_q^\alpha(x_p^1, \dots, x_p^n), \quad \alpha = 1, \dots, n$$

Then the *Jacobian* $J_{pq} = \det(\partial x_p^\alpha / \partial x_q^\alpha)$ is non-zero on $U_p \cap U_q$.

1.3 Abuse of notation

- Regular partial derivative do not have the same “canceling” that total derivative have ($dx * dy / dx = dy$) But we can restore this property through Einstein summation convention. That is that:

$$\sum_{\gamma=1}^n \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \frac{\partial x_p^\alpha}{\partial x_q^\gamma} \frac{\partial x_q^\gamma}{\partial x_q^\beta} = \delta_\beta^\alpha$$

2 Elements of Topology

2.1 Topological space

- A topological space is a set X of points of which certain subsets called *open sets* of the topological space, are distinguished, these open sets have to satisfy:
 - The intersection of any two (and hence of any finite collection) open sets should again be an open set.
 - The union of any collection of open sets must again be open.
 - The empty set and the whole set X must be open.
- The complement of any open set is called a *closed* set of the topological space.

In Euclidean space \mathbb{R}^n the “Euclidean topology” is the usual one where the open sets are the open regions.

2.1.1 Induced topology

- Given any subset $A \in \mathbb{R}^n$, the *induced topology* on A is that where the open sets are the intersections $A \cap U$, where U ranges over all open sets of \mathbb{R}^n .

2.1.2 Continuity

- A map $f : X \rightarrow Y$ of one topological space to another is called *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subset Y$ is open in X .

2.1.3 Homeomorphic

- Two topological spaces are *topologically equivalent* or *homeomorphic* if there is a one to one and onto map (bijective) between them, such that it and its inverse are continuous.

2.1.4 Topology on a manifold

- The topology on a manifold M is given by the following specifications of the open sets. In every local co-ordinate neighborhood U_q the open regions are to be open in the topology on M ; the totality of open sets of M is then obtained by admitting as open, also arbitrary unions countable collections of such regions, i.e. by closing under countable unions.

2.2 Metric space

- A *metric space* is a set which comes equipped with a “distance function” i.e. a real-valued function $\rho(x, y)$, defined on pairs x, y of its elements and having the following properties:
 - Symmetry: $\rho(x, y) = \rho(y, x)$.
 - Positivity: $\rho(x, x) = 0$, $\rho(x, y) > 0$ if $x \neq y$.
 - The triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

2.2.1 Hausdorff

- A topological space is called *Hausdorff* if any two points are contained in disjoint open sets. Any metric space is Hausdorff because the open balls of radius $\rho(x, y)/3$ with centers at x, y do not intersect.

All topological spaces we consider will be Hausdorff.

2.2.2 Compact

- A topological space X is said to be compact if every countable collection of open sets covering X contains a finite sub-collection already covering X .

If X is a metric space the compactness is equivalent to the condition that from every sequence of points of X a convergent sub-sequence can be selected.

2.2.3 Connected

- A topological space is connected if any two points can be joined by a continuous path.

2.3 Orientation

- A manifold M is said to be *orientated* if one can choose its atlas (collection of all the charts) so that for every pair U_p, U_q of intersecting co-ordinate neighborhoods the Jacobian of the transition functions is positive.
- We say that the co-ordinate systems x and y define the *same orientation* if $J > 0$ and the *opposite orientation* if $J < 0$.

