

# Lecture 1

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Lectures on QM

For quantization  
of constrained  
systems

I - semester - Free-fields  
II - interacting fields

Books:  
Intro to QFT Peskin & Schoeder → (main book)  
Schweber  
Dirac → Representations  
of Lorentz groups  
"Only look at it for 2 hours"

Finish chapter  
do all Qs  
will only do  
(1/3) of it.

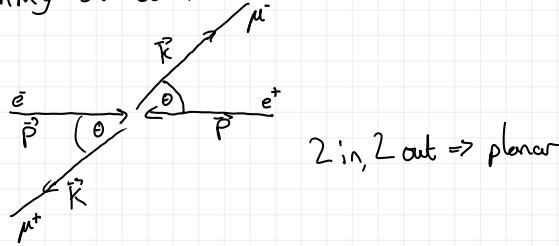
Einstein: SR - 1905  
QFT → QM - 1905 (photo electric effect)  
GR - 1915

"Knock offs"  
Frenchmen & Mathematicians say pioncare  
Planck, Heisenberg, Schrödinger, De Broglie, Mathematicians say Hilbert

Fast → Relativistic

QFT trailer

0 M ↗ Happening at Cern



Notation:  
 $\hbar = 1$  "God given units"  
 $c = 1$

$$\Rightarrow E = \sqrt{p^2 + m^2}$$

$$\frac{M_e}{M_\mu} \sim \frac{1}{200}$$

$$p_0^2 - p^2 = m^2, p_0 \equiv E$$

$$E_0 = \sqrt{p^2 + m_0^2}$$

$$E_\mu = \sqrt{k^2 + M_\mu^2}$$

$$E_{\text{init}} = 2E_e = 2\sqrt{p^2 + m_e^2}$$

$$E_{\text{final}} = 2E_\mu = 2\sqrt{k^2 + M_\mu^2}$$

Conservation  
of energy

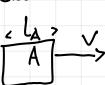
$$E_{\text{init}} = E_{\text{final}}, M_\mu > M_e \Rightarrow |\vec{R}| < |\vec{p}|$$

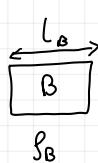
Assume  
(fast collision)

$$E > M_e, M_\mu \quad 2E = 2|\vec{p}| = 2|\vec{R}|$$

$$\text{cross section } \sigma = \frac{\text{Number of events}}{S_A S_B e_A e_B \cdot A}$$

planar process  
with overlap of  
Area A?

Bunches of  
electrons  
  
Packed with  
density  $\rho_A$



Differential cross section

$$\frac{d\sigma}{d\Omega_{(s)}} [\text{length}] = [l]$$

$$[\text{mass}] = [E] \cdot [v] = [p]$$

$$\Delta p \Delta x \sim 1$$

$$\Rightarrow [m] \cdot [l] \sim 1$$

$$\Delta E \cdot \Delta t \sim 1$$

$$\Rightarrow [E] \cdot [t] \sim 1$$

$$\Rightarrow [E] = [l]$$

$$\frac{S_A S_B l_A l_B \cdot A}{[m]^2}$$

$$\Rightarrow [\sigma] = \frac{1}{[m]^2}$$

$$\Rightarrow \left[ \frac{d\sigma}{d\Omega} \right] = \left[ \frac{1}{E^2} \right]$$

$$(E > m_e, m_\mu)$$

don't know proportionality

$$\frac{d\sigma}{dL_3} = \frac{1}{64\pi^2} \frac{1}{E^2} |M|^2, \text{ what does } M \text{ depend on?} \Rightarrow M = M(\theta)$$

for us  $\frac{M}{E} = 0$  | Is cases where  $M\left(\frac{m_e}{E}, \frac{m_\mu}{E}, \theta\right)$  | More complicated  
 $\Rightarrow$  cannot create Dimensionless quantity (cont involve  $E$  either)

Cannot depend on energy

We expect from interacting Hamiltonian

$$M \sim \langle \text{final state} | H_I | \text{initial state} \rangle$$

$e^-$ ,  $e^+$  don't interact with  $\mu^-$ ,  $\mu^+$

$$\Rightarrow \langle \text{final} | H_I | \text{initial} \rangle = 0$$

$M \sim \sum_{\alpha} \langle e^+ e^- | H_I | \gamma \rangle \stackrel{\text{light}}{\sim} \langle \gamma | H_I | \mu^+ \mu^- \rangle$

Light is a vector so need to sum components

- Also need to include spin, can think of this as angular momentum
- easiest to average over all spin

$$\frac{1}{2} \sum_{\substack{\text{init spin} \\ \text{final spin}}} \frac{d\sigma}{dL_3}$$

4-vector

$$\Sigma^\alpha = (0, 1, i, 0)$$

$$\Sigma^\alpha \Sigma_\alpha = 0^2 - 1 + 1 - 0 = 0$$

rotated around z-axis:

$$\Sigma^\alpha = (0, \cos\theta, i, \sin\theta)$$

$$M \sim \sum_{\alpha} \langle e^+ e^- | H_I | \gamma \rangle \stackrel{\text{light}}{\sim} \langle \gamma | H_I | \mu^+ \mu^- \rangle$$

$e \tilde{\epsilon}_r^{\alpha}$

charge is strength and  $\Rightarrow$  length of interaction

$$M(RL \rightarrow RL) \sim e^2(1 - \cos\theta)$$

$$\frac{d\sigma}{dL_3} = \frac{\alpha^2}{4E_{cm}^2} (1 + \cos^2\theta)$$

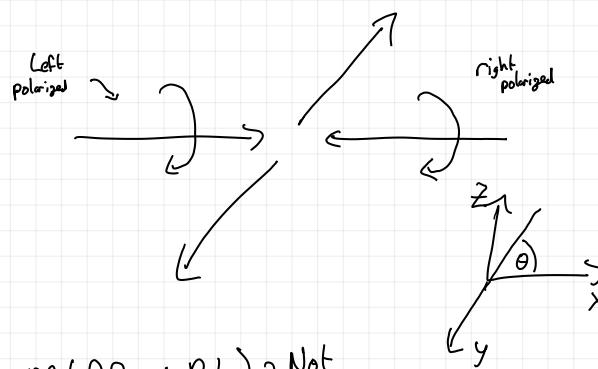
$$\sim \frac{1}{2} \frac{1}{E_{cm}^2} \sum_{\text{spin}} |M|^2$$

$$dL_3 = \sin\theta d\theta d\phi$$

$$= - d(\cos\theta) d\phi$$

$$\theta \in [0, \pi]$$

$$\phi \in [0, 2\pi]$$



$$\begin{aligned} & M(RR \rightarrow RL) \\ & M(LL \rightarrow RL) \end{aligned} \quad \begin{cases} \text{Not} \\ \text{Allowed} \end{cases}$$

pestin says +

$$M(RL \rightarrow RL) \sim e^2(1 - \cos\theta)$$

$$M(RR \rightarrow RL)$$

$$M(RR \rightarrow LL) = 0$$

$$M(RL \rightarrow LR) \sim e^2(1 - \cos\theta)$$

$$M(LR \rightarrow LR) \sim e^2(1 + \cos\theta)$$

proportionality  $\propto = \frac{e^2}{4\pi^2} \sim \frac{1}{137}$

Integrating

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{dL_3} dL_3 = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

Trailer over

What we want to do:

1. Relaxing condition  $E_m > M_\mu$

amplitude

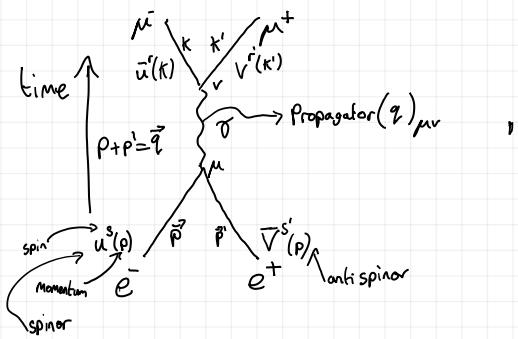
$$\Rightarrow M = M\left(\frac{m}{E}, \theta\right)$$

2. Relax  $E_m > M_e$   $M = M\left(\frac{M_n}{E}, \frac{M_e}{E}, \theta\right)$

3. Relax  $\alpha^2 \ll 1 \rightarrow$  calculate  $\alpha^2, \alpha^6, \dots$

4. Calculate any other process

These are Feynmann rule



$e, \mu, \nu \dots$  leptons

$q, \bar{q} \rightarrow$  hadrons

- We ask ourselves the question  
If we have a particle at  $x$ , what  
is the probability of finding it at  $x'$ ?

This will lead us to thinking  
about waves and fields instead of  
particles  $\Rightarrow$  Relativistic wave equation

Space time:  $(t, x_1, x_2, x_3)$ ,  $s^2 = t^2 - x_1^2 - x_2^2 - x_3^2 = t^2 - \vec{x}^2$

$$\mathbb{R}^{1,3} \text{ "Minkowski Space"} = x^\mu \gamma_{\mu\nu} x^\nu, \quad \gamma_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\langle x | e^{-iH(t-t')} | x' \rangle$$

$$\text{Single free particle of mass } m \\ H = \frac{\vec{p}^2}{2m} \Rightarrow \langle x | e^{-i\frac{\vec{p}^2}{2m}(t-t')} | x' \rangle$$

In classical mechanics  $\{p, x\} = 1, [p, x] = -i$

recall completeness relation

$$\hat{x} \rightarrow \text{multiply by } x \\ \hat{p} \rightarrow \frac{i}{i} \frac{\partial}{\partial x}$$

$$\int \frac{d^3 p}{(2\pi)^3} |\mathbf{p} \times \mathbf{p}| = 1$$

$$\begin{aligned} &= \int \frac{d^3 p}{(2\pi)^3} \langle x | e^{-i\frac{\vec{p}^2}{2m}(t-t')} | \mathbf{p} \times \mathbf{p} | x' \rangle e^{-ipx} \\ &= \int \frac{d^3 p}{(2\pi)^3} \langle x | p \rangle e^{-\frac{p^2(t-t')}{2m}} \langle p | x' \rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} e^{-i\frac{\vec{p}^2}{2m}(t-t')} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \\ &= \left( \frac{m}{2\pi i(t-t')} \right)^{\frac{3}{2}} e^{im \frac{(\vec{x} - \vec{x}')^2}{2(t-t')}} \end{aligned}$$

when  $(t-t')^2 - (\vec{x} - \vec{x}')^2 < 0$

space like

need to use  $\sqrt{p^2 + m^2}$

## Lecture 4

$s^2 > 0$  time-like  
 $s^2 \leq 0$  space-like  
 $s^2 = 0$  light-like

- should not have possible

space like propagation. QM should not break causality!

- Let us try relativistic Hamiltonian

$$H = \sqrt{\vec{p}^2 + m^2}$$

$$\langle x | e^{-i\sqrt{\vec{p}^2 + m^2} t} | x' \rangle$$

$$= \int \langle x | p \rangle e^{-i\sqrt{\vec{p}^2 + m^2} t} \langle p | x' \rangle \frac{d^3 p}{(2\pi)^3}$$

$$= \int \frac{d^3 p}{(2\pi)^3} e^{-i\sqrt{\vec{p}^2 + m^2} t + i\vec{p} \cdot (\vec{x}' - \vec{x})}$$

$$\vec{p} \cdot (\vec{x} - \vec{x}') = |\vec{p}| |\vec{x} - \vec{x}'| \cos\theta$$

can choose these to be the same

$$\Rightarrow \int \frac{d^3 p}{(2\pi)^3} = \int \frac{p^2 dp \sin\theta d\theta d\phi}{(2\pi)^3}$$

$$= \int \frac{p^2 dp}{(2\pi)^2} \int_1^\infty d\gamma e^{-i\sqrt{p^2 + m^2} t + i p |\vec{x} - \vec{x}'| \gamma}$$

$$= \int_0^\infty \frac{p^2 dp}{(2\pi)^2} e^{-i\sqrt{p^2 + m^2} t} \cdot \frac{1}{i p |\vec{x} - \vec{x}'|} \left( e^{i p |\vec{x} - \vec{x}'|} - e^{-i p |\vec{x} - \vec{x}'|} \right)$$

$$= \int_{-\infty}^{\infty} \frac{p^2 dp}{(2\pi)^2} e^{-i\sqrt{p^2 + m^2} t} \frac{2i}{i p |\vec{x} - \vec{x}'|} e^{i p |\vec{x} - \vec{x}'|}$$

$$\text{In our case } f(p) = \sqrt{p^2 + m^2} t + i p |\vec{x} - \vec{x}'|$$

$$f'(p) = \frac{-p^2}{\sqrt{p^2 + m^2}} + i |\vec{x} - \vec{x}'| = 0 \Rightarrow p^2 = \frac{m^2(\vec{x} - \vec{x}')^2}{t^2}$$

$$\Rightarrow U(t) = e^{-\frac{m\sqrt{(\vec{x}-\vec{x}')^2 - t^2}}{t}}$$

We use the following facts:

$$f(\vec{p}) |p\rangle = f(p) |p\rangle$$

$$\int \frac{d^3 p}{(2\pi)^3} |p \times p| = 1$$

$$\langle x | p \rangle = \bar{e}^{-i\vec{p} \cdot \vec{x}}$$

Samson uses  
 $\gamma = \cos\theta$   
 $d\gamma = -\sin\theta d\theta$

An Aside: Laplace steepest descent

expanding around a critical point  $x_0$  with  $f'(x_0) = 0$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{f(x)} dx &= \int_{-\infty}^{\infty} dx e^{f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) + \dots} \\ &\sim e^{f(x_0)} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}(x-x_0)^2 / a} \end{aligned}$$

Need to check this

Representation of Poincaré group,  $p_\mu p^\mu = m^2$

$$p_0^2 - \vec{p}^2 = m^2$$

$$\Rightarrow E = p_0 = \pm \sqrt{p^2 + m^2}$$

what if  $E < 0$ ?

- a) anti-particles  
 b) multi-particle states  
 c) spins & statistics  
 d) tools to calculate anything

need to go from particles  $\rightarrow$  waves

classical

$$S = \int_t^t L(q_i, \dot{q}_i) dt$$

$i=1, \dots, N$

; becomes continuous       $i \rightarrow \vec{x}$   
 $q \rightarrow \varphi(\vec{x}, t)$

$$\Rightarrow S = \int_t^t dt \int d^3x \mathcal{L}(\varphi(\vec{x}, t), \partial_\mu \varphi(\vec{x}, t))$$

$$\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} = 0$$

$$H: L \longrightarrow \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} = p_i \Rightarrow \text{solve } \dot{q}_i = (p, q)$$

$$H = \sum_i p_i \dot{q}_i - L(q, \dot{q}) = H(p, q)$$

The condition for us being able to solve  $\dot{q}$ :

$$\frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}_i \partial \dot{q}_j} = M_{ij} \Leftrightarrow \det(M) \neq 0$$

$$L = \sum_i M_{ij} \dot{q}_i \dot{q}_j, \quad \dot{q}_i = \sum_j M^{ij} p_j$$

$P = \frac{\partial L}{\partial \dot{q}_i} = M_{ii}$ , for this we need  $M$  to have an inverse

This we call a non-degenerate Lagrangian

Back to field theory

$$S = \int d^4x \mathcal{L}(\varphi, \partial_\mu \varphi) = \int L(\dots) dt$$

$$P(x) = \frac{\partial L}{\partial \dot{\varphi}} = \Pi(x) \underbrace{d^3x}_{\text{volume form}}$$

Then what are EoM? Need to vary

$$\delta S = \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \varphi} \delta \varphi - \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \delta \partial_\mu \varphi \right\}$$

Integrating by parts on second term

$$= \int d^4x \left( \frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \right) \right) \delta \varphi = 0$$

Field eqs = 0

Sanson gives the example

$$L = \dot{q} \Rightarrow P = \frac{\partial L}{\partial \dot{q}} = 1$$

$\Rightarrow P = 1$  how to solve for  $\dot{q}$ ? we can't

$$H = \sum p_i \dot{q}_i - L = H(p, q, \dot{q})$$

$$dH = \sum_i \left( \frac{\partial H}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i \right)$$

$$= \sum \left[ (\dots) dq^i + \dot{q}^i dp_i + \left( p_i - \frac{\partial L}{\partial \dot{q}^i} \right) d\dot{q}^i \right]$$

But our construction of  $H$  is

such that it is not a function of  $\dot{q}^i$

$$\Rightarrow \left( p_i - \frac{\partial L}{\partial \dot{q}^i} \right) = 0$$

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}$$

$$H = \int \mathcal{L} d^3x$$

$$= \int (\Pi(x) \dot{\varphi}(x) - \mathcal{L}) d^3x$$

What is the simplest Lagrangian we can think of?

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2$$

$\mu = 0, 1, 2, 3$

$$= \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} m^2 \varphi^2$$

We can lower and raise indices with Minkowski metric  $\eta^{\mu\nu}$

What is Hamiltonian?

$$T(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \dot{\varphi}(x)$$

$$\Rightarrow H = \int d^3x \left[ T(x) \dot{\varphi}(x) - \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right]$$

$$= \int d^3x \left[ \frac{1}{2} T^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{m^2}{2} \varphi^2 \right]$$

EoM?

$$\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} = -m^2 \varphi - \partial_\mu (\partial^\mu \varphi)$$

$$= -(\square + m^2) \varphi = 0 \quad \begin{matrix} \square = \partial_\mu \partial^\mu \\ = \partial_t^2 - \nabla^2 \end{matrix}$$

Klein-Gordan eqn!

### Noethers theorem

Theory: "space of fields  $\varphi$ ", action,  $H(0,0)$

$$\varphi \rightarrow \varphi'(x) = \varphi(x) + \alpha \Delta \varphi(x)$$

$\downarrow$  small

$$\text{st: } \mathcal{L}(x) \rightarrow \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x)$$

a symmetry of EoM

$$\alpha \Delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \varphi} \alpha (\Delta \varphi) + \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \partial_\mu (\alpha \Delta \varphi)$$

Then there is a conserved current

$$j^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \Delta \varphi - J^\mu$$

$$SS = \int d^4x \left( \alpha \left[ \frac{\delta \mathcal{L}}{\delta \varphi} \Delta \varphi - \partial^\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \right) \Delta \varphi \right] + \alpha \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \Delta \varphi \right) \right)$$

$\downarrow$   
= 0 with EoM

$$\text{Ex/ } \mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2 \quad X + iY = \varphi$$

$$\varphi \rightarrow e^{i\alpha} \varphi, \quad \varphi \rightarrow \varphi + i\alpha \varphi \quad \Delta \varphi = i$$

$$\varphi^* \rightarrow e^{-i\alpha} \varphi \quad \varphi^* \rightarrow \varphi - i\alpha \varphi$$

$$\Rightarrow j_\mu = i(\partial_\mu \varphi^* \cdot \varphi - \varphi^* \partial_\mu \varphi)$$

- with this current there is a charge

$$\int j_0 d^3x = Q \quad \text{as} \quad \dot{Q} = \int \partial_0 j^0 d^3x = \int \partial_i j^i d^3x = 0$$

boundary conditions

$$\frac{\delta \mathcal{J}(x)}{\delta J^{(i)}} = f(x-i)$$

$$S = \int J(x) A(x) dx$$

$$\frac{\delta S}{\delta J^{(i)}} = \int \left[ \delta J^{(i)}(x) A(x) \right] dx$$

$$= f(x_i, \Delta)$$

### Stress Energy Tensor

(in infinitesimal transformation)

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi + O(a^2)$$

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x+a) = \mathcal{L}(x) + a^\nu \partial_\mu (\delta_\nu^\mu \mathcal{L})$$

$$T_{\nu}^{\mu} = \frac{\delta \mathcal{L}}{\delta \partial_\nu \phi} \partial_\mu \phi - g_{\nu}^{\mu} \mathcal{L}, \quad \partial_\mu T^{\mu\nu} = 0$$

$$\Rightarrow \int T^0_0 d^3x = \int \left( \underbrace{\frac{\delta \mathcal{L}}{\delta \dot{\phi}}}_{\Pi} \dot{\phi} - \mathcal{L} \right) = \int \mathcal{L} d^3x$$

$$\int T^0_i d^3x = \dots = \int P_i d^3x = P_i$$

$$\Pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}}$$

For Klein-Gordon field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \quad (\text{complex?})$$

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

$$\text{with } \partial_\mu T^{\mu\nu} = 0$$

Thm  
can  
Always  
Symmetrize  
Stress  
Tensor  
(in book)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

$$\Rightarrow \Pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}$$

$$\begin{aligned} \delta \mathcal{L} &= \Pi \dot{\phi} - \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \\ &= \frac{\Pi^2}{2} + \frac{(\nabla \phi)^2}{2} + \frac{m^2 \phi^2}{2} > 0 \end{aligned}$$

$$\text{EOM } (\square + m^2) \phi = 0 \quad H = \int \frac{d^3x}{2} (\Pi^2 + (\nabla \phi)^2 + m^2 \phi^2)$$

$$\text{Classically we had: } \{ P_i, q_j \} = \delta_{ij}$$

$$\text{Now we have: } \{ \} \rightarrow [ ]$$

$$P \rightarrow \Pi, \dot{q} \rightarrow \dot{\phi}$$

i is here  
so that  
Hermitian operators  
Have antisymmetric  
commutator

$$\Rightarrow [\Pi(\vec{x}), \phi(\vec{y})] = i \delta^{(3)}(\vec{x} - \vec{y})$$

## Harmonic oscillators

$$H = \frac{p^2 + \omega^2 q^2}{2}$$

What we want!

$$\therefore \frac{\partial A}{\partial t} = [A, H], \quad A(t) = e^{-iHt} A(0) e^{iHt}$$

$$[A, B]^* = (AB)^* - (BA)^*$$

$$= -[A^*, B^*]$$

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^3} \int d^3 p \phi(\vec{p}, t) e^{i\vec{p} \cdot \vec{x}}$$

$$(\square + m^2) \phi = 0 \Rightarrow \frac{1}{(2\pi)^3} \int d^3 p \left( \left( \frac{\partial}{\partial t} \right)^2 + \vec{p}^2 + m^2 \right) \phi(\vec{p}, t) e^{i\vec{p} \cdot \vec{x}} = 0$$

$$\downarrow \partial_t^2 - \vec{p}^2 \Rightarrow = 0$$

$$\text{Notation: } \omega_p = \sqrt{\vec{p}^2 + m^2} \Rightarrow (\partial_t^2 + \omega_p^2) \phi(\vec{p}, t) = 0$$

$$\text{Want } \hat{a} \text{ and } \hat{a}^+ \text{ for fields: } \phi(\vec{p}, t) = \frac{(a + a^*)}{\sqrt{2\omega_p}}$$

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^3} \int d^3 p \left( \hat{a}(\vec{p}) e^{i\vec{p} \cdot \vec{x}} + \hat{a}^*(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} \right) \quad \text{which we write as}$$

"Calculus of QFT"

"For every  $p$  we have harmonic oscillator"

$$\ln \text{QM: } \hat{q} = \frac{1}{i\omega_p} (\hat{a} + \hat{a}^+) \quad [\hat{a}, \hat{a}^+] = \mathbb{I}$$

$$\hat{p} = -i \sqrt{\frac{\hbar}{2}} (\hat{a} - \hat{a}^+) \quad H_{\text{SHO}} = \frac{\hat{p}^2 + \omega_p^2 \hat{q}^2}{2} = \omega_p(a^+ a + \frac{1}{2})$$

$a^+$  → creation  
 $a$  → annihilation

$$[H_{\text{SHO}}, \hat{a}^+] = \omega \hat{a}^+$$

$$[H_{\text{SHO}}, a] = -\omega \hat{a}$$

$$H(a^+ \psi) = (E + \omega) \hat{a}^+ \psi$$

$$\text{States given by: } |n\rangle = \frac{(\hat{a}^+)^n}{n!} |0\rangle$$

$$H = \int d^3 x = \int d^3 p \left( T^2 + (\nabla \phi)^2 + m^2 \phi^2 \right)$$

We should also have that  $[T(x), \phi(y)] = i \delta(x-y) \Leftrightarrow [\hat{a}(p), a^*(p')] = \delta(\vec{p} - \vec{p}')$

$$= \int d^3 p E_p \left( \hat{a}^*(p) \hat{a}(p) + \frac{1}{2} [a(p), a^*(p)] \right) \quad \text{Ignore}$$

$$\text{can also then } [H_{\text{KG}}, \hat{a}(p)] = \omega_p \hat{a}^*(p)$$

check  $[H_{\text{KG}}, \hat{a}(p)] = -\omega_p \hat{a}(p)$

$$|\vec{p}_1, \vec{p}_2\rangle = a^*(p_1) a^*(p_2) |0\rangle = (E_{p_1} + E_{p_2}) |\vec{p}_1, \vec{p}_2\rangle$$

Quantum spectrum of KG  
Bosonic-Fock space

$$a(p)|0\rangle = 0 \sim \text{vacuum} = |0\rangle$$

$$|\vec{p}\rangle = a^*(\vec{p}) |0\rangle, \quad H|\vec{p}\rangle = E|\vec{p}\rangle = \omega_p |\vec{p}\rangle$$

$$\text{but we have } [a(\vec{p}), a(\vec{p}')] = \frac{1}{2} \delta(\vec{p})$$

what matters is difference in energies

⇒ we ignore this constant  
Also we had infinite many oscillators, what did we expect?

$$|\vec{p}\rangle = a^*(\vec{p}) |0\rangle$$

$$H|\vec{p}\rangle = \int d^3 p' E_{p'} a^*(p') a(p)$$

$$\cdot a^*(p') |0\rangle$$

$$+ |a^*(p)|0\rangle = a^*(p) |0\rangle + E_p a^*(p) |0\rangle$$

$$= E_p |\vec{p}\rangle$$

$$\langle p'|p\rangle = ?$$

$$= \langle 0| a(p') a^*(p') |0\rangle \quad \text{This should be Lorentz invariant, but is not}$$

$$= \langle 0| [\hat{a}(\vec{p}), \hat{a}^*(\vec{p}')] |0\rangle$$

$$= \delta(\vec{p} - \vec{p}')$$

$$\text{recall: } \delta(f(x)) \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i) \quad f(x_i) = 0$$

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$d^3 p \delta^{(3)}(\vec{p} - \vec{p}') = d^3 p' \delta^{(3)}(\vec{p}' - \vec{p}')$$

$$\Rightarrow \delta^{(3)}(\vec{p} - \vec{p}') = \delta(\vec{p} - \vec{p}') \frac{E'}{E}$$

$$\Rightarrow |\vec{p}\rangle = \sum E_p a^*(\vec{p}) |0\rangle$$

$$\Rightarrow 2E_p \langle p'|p\rangle \text{ is Lorentz invariant}$$

$$\begin{pmatrix} E'_1 \\ p'_3 \end{pmatrix} = \begin{pmatrix} \gamma(p_3 + \beta E) \\ \gamma(E + \beta p_3) \end{pmatrix}$$

$$\phi(\vec{x}, t) = e^{iHt} \phi(\vec{x}, 0) e^{-iHt}$$

For this we need to know

$$a(\vec{p}, t) = e^{iHt} a(\vec{p}) e^{-iHt}$$

Lemma:  $H^* a(\vec{p}) = a(\vec{p})(H - E_p)^*$  can use  $[H, a(\vec{p})] = -E_p a(\vec{p})$

$$e^{iHt} = \sum_{n=0}^{\infty} \frac{(iHt)^n}{n!} \Rightarrow e^{iHt} a(\vec{p}) = a(\vec{p}) e^{i(H-E)t}$$

$$\Rightarrow a(\vec{p}, t) = a(\vec{p}) e^{-iEt}$$

$$\psi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a(\vec{p}) e^{-iP_m x^m} + a^+(\vec{p}) e^{iP_m x^m} \right)$$

$$\Phi_{\text{KE}}(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a_p e^{-iP_x x} + a_p^+ e^{iP_x x} \right)$$

$$E_p^2 = \vec{p}^2 + m^2$$

$$e^{-i\vec{p} \cdot \vec{x}} a(\vec{p}) e^{i\vec{p} \cdot \vec{x}} = a(\vec{p}) e^{i\vec{p} \cdot \vec{x}}$$

Want to calculate

2 pt correlation function

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x-y)$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$$\frac{1}{ipr} (e^{ipr} - e^{-ipr})$$

$$\langle 0 | a(p) a^+(p') | 0 \rangle$$

$$= \langle 0 | [a_p, a_{p'}^+] | 0 \rangle$$

$$= \delta^{(3)}(\vec{p} - \vec{p}')$$

time like

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} (\dots) = \int \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) (\dots) \quad p_0 > 0$$

$$p^2 - m^2 = (p_0 - \sqrt{p^2 + m^2})(p_0 + \sqrt{p^2 + m^2})$$

$$D(x-y) = \frac{4\pi r}{(2\pi)^3} \int \frac{dp^2}{2\sqrt{p^2 + m^2}} e^{-i\sqrt{p^2 + m^2} t} = \frac{1}{4\pi r} \int \frac{dp}{\sqrt{E^2 - m^2}} e^{-iEt}$$

$$|x-y| = \vec{r}, x^0 - y^0 = 0 \text{ (space like)}$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p} \cdot \vec{r}} = 2\pi \int_0^\infty dp \int_{-1}^1 \frac{dp}{(2\pi)^3 2E_p} e^{ipr}$$

$$= \frac{-i}{2(2\pi)^2} \int_0^\infty \frac{dp}{p} p \frac{e^{ipr}}{\sqrt{p^2 + m^2}}$$

$$\sim \frac{e^{-mr}}{r} \neq 0$$

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip(x-y)} - e^{ip(x-y)} \right)$$

$$= D(x-y) - D(y-x)$$

$$[f](x-y) \in \mathbb{C}^3$$

There is a continuous Lorentz transformation  $(x-y) \rightarrow -(x-y)$

$$\Rightarrow (x-y)^2 < 0$$

$$\Rightarrow \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = 0$$

For complex field  $(\phi, \phi^*)$

$$\langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle = \\ = D(x-y) - D(y-x)$$

If we assign charge  
+1 to  $\phi$  and -1 to  $\phi^*$   
can consider the anti-particles  
going opposite directions

$$e^{ip \cdot x} \quad E_p + p \vec{\cdot} \vec{x}$$

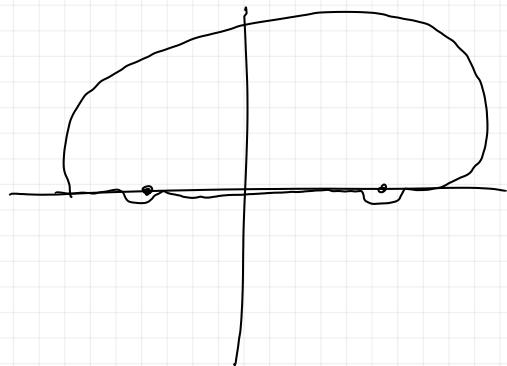
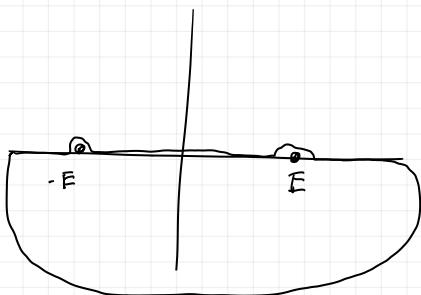
$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip(x-y)} - e^{ip(x-y)} \right) \\ = \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{P_0=E_p} + \frac{1}{-2E_p} e^{-ip(x-y)} \Big|_{P_0=E_p} \right]$$

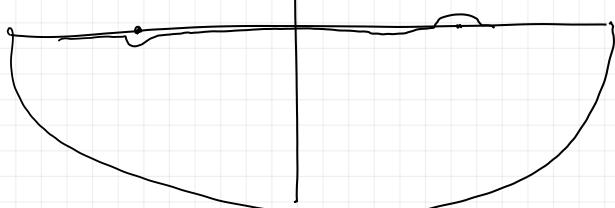
$$= \int \frac{d^3 p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip(x-y)} = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

for 2-point function  
 $\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = [\phi(x), \phi(y)]$

Contours:



Feynmann



$$D(x^0 - y^0) = \Theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$\text{Something} = \begin{cases} D(x-y) & , x^0 > y^0 \\ D(y-x) & , y^0 > x^0 \end{cases}$$

$$\langle 0 | \phi(x) \phi(y) - \phi(y) \phi(x) | 0 \rangle$$

Time ordering

when  $x^0 > y^0$   $\uparrow$  only this term contributes

$$\Rightarrow \Theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle = \langle 0 | T(\phi(x), \phi(y)) | 0 \rangle$$

$$\langle 0 | T(\phi(x), \phi(y)) | 0 \rangle = \lim_{F_{\text{cav}} \rightarrow 0} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{ip(x-y)} = D_F(x-y)$$

gives advanced function  
for negative energy modes  
and retarded for positive energy

$$p_0^2 = \vec{p}^2 + m^2 - i\epsilon$$

$$p_0 = \pm \sqrt{\vec{p}^2 + m^2} \pm \frac{i\epsilon}{2}$$

Classical source

$$(\square + m^2) \phi(x) = j(x)$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + j(x) \phi(x)$$

$$\phi(x) = \phi_0(x) + i \int d^4 y D(x-y) j(y)$$

As this satisfies

$$(\square + m^2) D(x-y) = \delta(x-y) \quad \phi_0(x) = \int \frac{d^3 p}{(2\pi)^3} \left( a_p e^{-ip \cdot x} + a_p^+ e^{ip \cdot x} \right)$$

$$\Rightarrow \phi(x) = \int \frac{d^3 p}{(2\pi)^3} \left( a_p e^{-ip \cdot x} + a_p^+ e^{ip \cdot x} \right) + \int d^4 y \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( e^{-ip(x-y)} - e^{ip(x-y)} \right) j(y)$$

$$= " " + \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} \left[ \left( a_p + \frac{i}{\sqrt{E_p}} j(p) \right) e^{-ipx} + c.c. \right]$$

Can then modify Hamiltonian

$$H = \int \frac{d^3 p}{(2\pi)^3} E_p \left( a_p^+ - \frac{i}{\sqrt{E_p}} j(p) \right) \left( a_p + \frac{i}{\sqrt{E_p}} j(p) \right)$$

$$\langle 0 | H | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} | j(p) |^2$$

# Representations of Poincaré group

$$P = L \times P$$

$$\mathbb{R}^3 ; \quad \gamma^\mu = \text{diag}(1, -1, -1, -1)$$

$L \rightarrow$  Rotations in  $\mathbb{R}^3 \Leftrightarrow$

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

$P \rightarrow$  Translations  $x \rightarrow x + a$

$$x^\mu x_\mu = (x^0)^2 - \vec{x}^2 = \text{invariant}$$

$$x_\mu x^\mu = x'^\mu \gamma_{\mu\nu} x^\nu = x'^\mu \gamma_{\mu\nu} x'^\nu$$

$$= \underbrace{\Lambda^\mu_\alpha}_{x'^\mu} \underbrace{x^\alpha \gamma_{\mu\nu}}_{x'^\nu} \underbrace{\Lambda^\nu_\beta}_{x'^\nu} x^\beta$$

$$\Rightarrow \gamma_{\alpha\beta} = \Lambda^\mu_\alpha \gamma_{\mu\nu} \Lambda^\nu_\beta$$

Transpose  
as we had  
to swap  
index's

$$x = \Lambda x ; \quad \gamma = \Lambda^T \gamma \Lambda$$

$$\gamma^i \Rightarrow \mathbb{1} = \Lambda \Lambda^T$$

$m \in M$ , group acts on  $M$

Left action

$$m \rightarrow m^l = g \cdot m$$

$$m'' = h m^l = (h \cdot g) \cdot m$$

$$V(h)V(g) = V(h \cdot g)$$

Right action  
instead if:

$$m^r = m \cdot g^{-1}$$

$$m''' = m^r h = m(g \cdot h)$$

$$V(h) V(g) = V(g \cdot h)$$

$$g \cdot h^{-1} = (h \cdot g)^{-1} \Rightarrow -V((h \cdot g)^{-1})$$

$$\psi^m \rightarrow \psi^{(m)}_x = \alpha^m_n(g) \psi(g \cdot x)$$

$$\psi^{(m)}_x = \alpha^m_n(h) \psi^n(h \cdot x) = \alpha^m_n(h) \alpha^k_n(g) \psi^k(g \cdot (h \cdot x))$$

$$= \alpha \cdot \alpha \psi((h \cdot g) \cdot x)$$

So for these maps

Left  $\rightarrow$  inverse multiplication

Right  $\rightarrow$  multi.

$$\Rightarrow (\Lambda^0_0)^2 = 1 + (\Lambda^0_i)^2 \geq 1$$

$$\Lambda^0_0 \geq 1, \quad \Lambda^0_0 \leq -1$$

orthocronous non-orthocronous

$\det \Lambda = 1$  Proper

$\det \Lambda = -1$  Improper

$$\mathbb{R}^3 \ni x : x^i = \Lambda x$$

$$\psi'(x') = L(\Lambda) \psi(x)$$

$$= L(\Lambda) \psi(\Lambda^i x')$$

$$\Rightarrow \psi'(x) = L(\Lambda) \psi(\Lambda^i x)$$

What kind of space is  $L(\Lambda)$

$$\gamma = \Lambda^T \gamma \Lambda$$

$$1 \neq \det \gamma = \det \Lambda^T \cdot \det \gamma \det \Lambda$$

$$\Rightarrow \det \Lambda = \pm 1$$

$$\gamma^{00} = 1 = \Lambda^0_\alpha \gamma^{\alpha\beta} \Lambda^0_\beta$$

$$= (\Lambda^0_0)^2 - (\Lambda^0_i)^2$$

4 Possibilities

- |   |                  |  |
|---|------------------|--|
| a) $\det \Lambda = 1, \Lambda^0 \geq 1$   | $L^{\uparrow}$   | 1) Rotations<br>$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \text{ SO}(3)$ |
| b) $\det \Lambda = 1, \Lambda^0 \leq -1$  | $L^{\downarrow}$ |  |
| c) $\det \Lambda = -1, \Lambda^0 \geq 1$  | $L^{\uparrow}$   |  |
| d) $\det \Lambda = -1, \Lambda^0 \leq -1$ | $L^{\downarrow}$ |  |
- $\det \Lambda = \det A = 1$

$SO(1,3)$  is a double cover of  $SO(3)$ ?

2) Boost

$$\Lambda = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 \\ -\sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cosh \theta = \sqrt{\frac{1}{1 - (\frac{v}{c})^2}} = \gamma$$

Nov 6<sup>th</sup>, 13<sup>th</sup> Hamilton Institute (9-11) am

3) Time Reversals / reflections

$$x^0 = t \rightarrow -x^0, \vec{x} \rightarrow \vec{x}$$

$$\det L = -1, \Lambda^0 = -1 \quad | \quad L^-$$

4) Whole reversals / Reflections

$$\det L = 1, \Lambda^0 = -1 \quad | \quad L^+$$

6 parameters

$$\Lambda^\mu_{\nu} = \delta^\mu_\nu + \epsilon^\mu_\nu \Rightarrow \Lambda = I + \epsilon$$

$$\Lambda^T \Lambda = I \quad | \quad I = I + \epsilon^T + \epsilon$$

$$\Rightarrow \epsilon^T + \epsilon = 0$$

$$\Rightarrow \epsilon^{\mu\nu} + \epsilon^{\nu\mu} = 0 \Rightarrow \text{Anti-Symmetric}$$

Upper triangular

$$\downarrow$$

$$\frac{n(n+1)}{2} - n$$

$\checkmark \text{ diag} = 0$

$$= \frac{n(n-1)}{2}$$

$$\text{For symmetric, } n^2 - \frac{n(n-1)}{2} \approx \frac{n(n+1)}{2}$$

$$L_{\mu\nu} = i(X_\mu \partial_\nu - X_\nu \partial_\mu), \quad i \frac{\partial}{\partial x^\mu} = P_\mu$$

$$f(x+a) = f(x) + a f'(x)|_a = e^{a \frac{\partial}{\partial x}} f(x)$$

$$\text{momentum generates shift} \Rightarrow e^{ia P} f(x)$$

$$\text{Poincare Group} = L \times P$$

Lorentz generator

$$M_{\mu\nu} = i(X_\mu \partial_\nu - X_\nu \partial_\mu) \delta_{ab} + (\delta_{\mu\nu})_{ab}$$

$$M = L \cdot I + S'$$

$$\mathbb{R}^4 \downarrow \text{ fibre}$$

$$\text{recall } \psi(x) = \Omega(\Lambda) \psi(\Lambda' x)$$

$$\Omega = I - \frac{i}{2} \epsilon \cdot S'$$

$$\psi(\Lambda' x) = -\frac{i}{2} \epsilon^{\mu\nu}$$

so similarly

$$e^{i \epsilon^{\mu\nu} L_\mu} f(x) = f(x^\mu + \epsilon^\mu_\nu x^\nu)$$

$$= f(x) + \epsilon^\mu_\nu x^\nu \frac{\partial}{\partial x^\mu} f(x)$$

$$= f(x) + \frac{i}{2} \epsilon^{\mu\nu} (X_\mu \partial_\nu - X_\nu \partial_\mu)$$

$$\Rightarrow f(x + \epsilon x) = e^{\frac{i}{2} \epsilon^{\mu\nu} L_\mu} f(x)$$

$$\frac{\partial}{\partial x} S = 0 \quad [M_{\mu\nu}, M_{\lambda\sigma}] = \underbrace{[L_{\mu\nu}, L_{\lambda\sigma}]}_{\text{determined by } [L, L]} + M$$

Commutators of Loren by  
Gauge

$$[L_{\mu\nu}, L_{\lambda\sigma}] = i\gamma_{\nu\lambda} L_{\mu\sigma} - i\gamma_{\mu\lambda} L_{\nu\sigma} + i\gamma_{\mu\nu} L_{\lambda\sigma} - i\gamma_{\nu\lambda} L_{\mu\sigma}$$

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i\gamma_{\nu\lambda} M_{\mu\sigma} - i\gamma_{\mu\lambda} M_{\nu\sigma} + i\gamma_{\mu\nu} M_{\lambda\sigma} - i\gamma_{\nu\lambda} M_{\mu\sigma}$$

$$\psi(x) \Rightarrow \psi'(x) = V(\lambda)\psi(x) = \Omega(\lambda)\psi(\lambda'x)$$

Taking  $\sigma_i, i=1,2,3$

$$[M_{ij}, M_{kl}] \quad M_{\mu\nu} = L_{\mu\nu}I + S_{\mu\nu}$$

$$J_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \quad K_i = M_{oi}$$

$$[J_i, J_j] = i \epsilon_{ijk} J_k \rightarrow SO(3)$$

$$[K_i, K_j] = -i \epsilon_{ijk} J_k$$

$$[J_i, K_j] = i \epsilon_{ijk} K_k$$

6 generators  
continuous Lorentz  
transformations  
are connected.

If we take a combination

$$N_i = \frac{1}{2}(J_i + iK_i) \Rightarrow [N_i, N_j] = i \epsilon_{ijk} N_k$$

$$\Rightarrow N_i^+ = \frac{1}{2}(J_i - iK_i) \quad [N_i^+, N_j^+] = i \epsilon_{ijk} N_k^+ \quad [N_i, N_j^+] = 0$$

Recall from QM II

$$L^2 = L_1^2 + L_2^2 + L_3^2 = l(l+1)$$

$$l_3 = -l, \dots, 0, \dots, l$$

$l$  = half integer

$$\Rightarrow N^2 = N_1^2 + N_2^2 + N_3^2 = n(n+1)$$

choosing  $N_3 \Rightarrow -n, \dots, 0, \dots, n \leftarrow 2n+1$

$$(N^+)^2 = -m(m+1)$$

$$N_3^+ = -m, \dots, 0, \dots, m \leftarrow 2m+1$$

$$\Rightarrow 2n+1 + 2m+1 = 2(n+m)+2$$

Total independent states  
Should be  $(2n+1)(2m+1)$ ?

$$m, n = 0, \frac{1}{2}, \dots$$

Now we can write

$$J_i = N_i + N_i^+ = \text{Spin} \quad \underline{m+n}$$

The operation + takes  $N \rightarrow N^+$   
 $\Rightarrow$  It takes  $(0, \frac{1}{2}) \leftrightarrow (\frac{1}{2}, 0)$

	$(n, m)$	No. states	KG Theory
a)	$(0, 0)$	1	(spin=0) 12 complex
b)	$(\frac{1}{2}, 0)$	2	2d complex (spin = $\frac{1}{2}$ ) (spinors)
c)	$(0, \frac{1}{2})$	2	2d complex (spin = $\frac{1}{2}$ )
d)	$(\frac{1}{2}, \frac{1}{2})$	4	vector (spin 1) candidate for Maxwell spin(1)
	$(1, 0)$	3	
	$(0, 1)$	3	

Left spinor  
 $(\frac{1}{2}, 0) = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$  can direct sum  $\Rightarrow (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) = \begin{pmatrix} \cdot \\ \cdot \\ * \\ * \end{pmatrix}$  Dirac Spinor

$(0, \frac{1}{2}) \quad (*)$  + takes us from one space  
to the other.

Right spinor  
 $(\frac{1}{2}, 0) \times (0, \frac{1}{2}) \in (\frac{1}{2}, \frac{1}{2})$  Irrep  
 $\downarrow 2D$        $\uparrow 4D$   
 $(\frac{1}{2}, 0) \times (\frac{1}{2}, 0) = (1, 0) \oplus (0, 0)$  Reducible representation

If we have  $\Psi_\alpha = (\frac{1}{2}, 0) \quad \alpha=1,2$

think of  
2D matrices

$$\Psi_\alpha \cdot \mathcal{X}_\beta = (\text{Symm})_{\alpha\beta} \oplus (\text{anti-sym})_{\alpha\beta}$$

$3 \downarrow (1, 0) \quad \downarrow (0, 0)$

4d  $\mathbb{R}^4$

$$B_{\mu\nu} dx^\mu dx^\nu \rightarrow *B = \frac{1}{2} \epsilon_{\alpha\beta}^{\mu\nu} B_{\mu\nu} dx^\alpha dx^\beta$$

$$"B_{\mu\nu}" | *^2 = 1 \quad *B_{\alpha\beta}$$

" $B + *B$ " (1,0) self-dual Tensor  
 " $B - *B$ " (0,1) anti-dual Tensor

quaternions  
 - can use  $\sigma_i$ :  
 $[\sigma_i, \sigma_j] = 2i e^{ik} \sigma_k$

$x^\mu \rightarrow x^\mu \mathbb{I} + i x^i \sigma^i = \hat{x}$

$X^\mu \rightarrow X^\mu + a^\mu = x^\mu$   
 $X^\mu \rightarrow \Lambda_\nu^\mu X^\nu = x^\mu$

Group Generators:  $P_\mu = -i \frac{\partial}{\partial x^\mu}$ ,  $M_{\mu\nu} = L_{\mu\nu} \mathbb{I} + S_{\mu\nu}$

$i(x_\mu \partial_\nu - x_\nu \partial_\mu)$  constant

$[M_{\mu\nu}, P_\lambda] = i \gamma_{\mu\lambda} P_\nu + i \gamma_{\nu\lambda} P_\mu$

(0) parameter group

What are the Casamirs (what commutes with everything)

$P_\mu P^\mu = m^2$

$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} P_\nu M_{\sigma\tau}; W^\mu W_\mu = W^2$

Pauli-Lubansky vector

$\nabla \cdot W = \frac{2}{c} S^{\mu\nu} = 0$

Can check  $W^\mu - \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} P_\nu S_{\sigma\tau} = \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} S_{\sigma\tau} \frac{2}{c}$

$[W^\mu, P^\nu] = 0$

$[M_{\mu\nu}, W_\lambda] = \dots$

Types of representations  
 $S = n+m$

 $P_\mu P^\mu = m^2 > 0$

$W^2 = W_\mu W^\mu = -m^2 s(s+1)$

$s = 0, \frac{1}{2}, 1, \dots$  states  $s_3 = -s, \dots, s$

$P_\mu P^\mu = 0 \Rightarrow m=0$

$b) \text{ or } W^2 = 0, W^\mu P_\mu = 0$

$c) W \sim P; W_\mu = c P_\mu \quad c = \text{helicity}$

$P_\mu P^\mu = m^2 < 0 \quad \text{Tachyon}$

$C = \pm S$

Example  $(S_{\mu\nu})^\beta_\alpha$

$S(\partial_\mu \phi) = [\delta, \partial_\mu] \phi + \underbrace{\partial_\mu (\delta \phi)}_{=0}?$ 
 $[\delta, \partial_\mu] = [\delta_\mu, \partial_\mu] + [\delta^\nu \partial_\nu, \partial_\mu]$ 
 $[\epsilon_\mu^\nu x^\lambda \partial_\nu, \partial_\mu] = \epsilon_\mu^\nu \partial_\nu$

$= \delta(\partial_\mu \phi) = \delta_\mu (\partial_\mu \phi) + \delta x^\nu \partial_\nu (\partial_\mu \phi)$

$= -\frac{i}{2} \epsilon^{\mu\nu} L_{\mu\nu} \partial_\mu \phi - \frac{i}{2} (\epsilon^{\mu\nu} S_{\mu\nu})^\nu_\mu \partial_\nu \phi$

$(S_{\mu\nu})^\nu_\mu = i(\gamma_\mu \gamma^\nu - \gamma_\nu \gamma^\mu) - \text{spin matrix}$

$\Rightarrow \partial_\mu \phi \Leftrightarrow \text{vectors}$

$M_{\mu\nu} = L_{\mu\nu} \mathbb{I} + S_{\mu\nu} \quad \text{derived for vectors}$

$x^1 = x + Sx$ 
 $\delta f = f'(x + Sx) - f(x)$ 
 $= f'(x) - f(x) + \int x^\mu \frac{\partial}{\partial x^\mu} f'(x)$ 
 $= f'(x) - f(x) + \delta x^\mu \frac{\partial}{\partial x^\mu} f(x)$ 
 $= \delta_\mu f + \int x^\mu \partial_\mu f \quad \underline{\delta_\mu f = f'(x) - f(x)}$ 
 $\delta = \delta_\mu + \delta x^\mu \partial_\mu$

$\phi'(x') - \phi(x) = 0$

$0 = \delta \phi = \delta_\mu \phi + \delta x^\mu \partial_\mu \phi$

$\delta_\mu \phi = -\frac{i}{2} \epsilon^{\mu\nu} M_{\mu\nu} \phi$

$\left( -\frac{i}{2} \epsilon^{\mu\nu} M_{\mu\nu} + \int x^\mu \partial_\mu \right) \phi = 0$

Write Poincaré invariant functionals; Lagrangian

$\phi'(x) = \phi(x), \quad \phi^2, \phi^3, \dots, \phi^n - \text{invariants}$

$a \dot{\phi}^2 - b \partial_\mu \phi \partial^\mu \phi - \text{6 this is all}$

$L_{KG} = -\frac{m^2}{2} \dot{\phi}^2 + \frac{1}{2} (\partial_\mu \phi)^2$

$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

$\Leftrightarrow S_{\mu\nu} = 0$

for scalar field

Spinors  $\mathbb{C}^2$

$$\left(\frac{1}{2}, 0\right) \quad \left(0, \frac{1}{2}\right)$$

$$\psi_L(x) \rightarrow \psi'_L(x') = \bigwedge_L \psi_L(x) \Rightarrow \psi'_L(x') = \bigwedge_L \psi_L(x)$$

Fibre?  
vector in  $\mathbb{C}^2$

$2 \times 2$  matrix  $x' = \lambda x$

$M$

fibre  $X$

$M = X \times B$

Base

$\Lambda_L(\Lambda_R) =$  are  $2 \times 2$  complex matrices

$$x^o \rightarrow x^o, x^i = \omega^i_j x^j$$

$$\Rightarrow \Lambda_L = e^{\frac{i}{2} \vec{\sigma} \cdot \vec{\omega}} \quad \omega^i = \epsilon^{ijk} \omega_{jk}$$

$$\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$$

$$\Lambda_L = e^{\frac{i}{2} \vec{\sigma} \cdot (\vec{\omega} - i \vec{v})} \quad \Lambda_R = e^{\frac{i}{2} \vec{\sigma} \cdot (\vec{\omega} + i \vec{v})}$$

$$- \psi_L, \psi_R, \chi_L = \sigma^2 \psi_R^*$$

Properties

- $\Lambda_L^{-1} = \Lambda_R^\dagger$
- $\sigma^2 \sigma^i \sigma^2 = -\sigma^i \sigma^2 \Rightarrow \sigma^2 \Lambda_L \sigma^2 = e^{-i \vec{\sigma}^* (\vec{\omega} - i \vec{v})} = \Lambda_R^*$
- $\Lambda_L^\dagger = \sigma^2 \Lambda_L^{-1} \sigma^2$
- $\psi_L \in \left(\frac{1}{2}, 0\right)$
- $\sigma^2 \psi_L^* \in \left(0, \frac{1}{2}\right)$
- $\sigma^2 \psi_L^* \rightarrow \sigma^2 \Lambda_L^* (\sigma^2 \psi_L^*) \psi_L^* \dots = \Lambda_R \sigma^2 \psi_L^*$

Looking for Lorentz invariant object from spinors

$$\Rightarrow \chi_L^\dagger \sigma^2 \psi_L = \psi_R^\dagger \psi_L - \text{invariant scalar}$$

$$\leftarrow \psi_L^\dagger \sigma^2 \psi_L \rightarrow \psi_L^\dagger e^{\frac{i \vec{\sigma} \cdot \vec{v}}{2}} \sigma^i e^{\frac{i \vec{\sigma} \cdot \vec{v}}{2}} \psi_L$$

$$= \psi_L^\dagger \left(1 + \frac{\vec{\sigma} \cdot \vec{v}}{2}\right) \sigma^i \left(1 + \frac{\vec{\sigma} \cdot \vec{v}}{2}\right) \psi_L = (\text{same}) +$$

$$= \psi_L^\dagger \sigma^i \psi_L + \frac{1}{2} v^i \psi_L^\dagger (\sigma^j \sigma^i + \sigma^i \sigma^j) \psi_L$$

$$= \psi_L^\dagger \sigma^i \psi_L + v^i \psi_L^\dagger \psi_L$$

$$\psi_L^\dagger \psi_L \rightarrow e^{\frac{i \vec{\sigma} \cdot \vec{v}}{2}} e^{\frac{i \vec{\sigma} \cdot \vec{v}}{2}} \psi_L$$

$$= \psi_L^\dagger \psi_L + \vec{v} \psi_L^\dagger \vec{\sigma} \psi_L$$

$$\delta(\psi_L^\dagger \psi_L) = v^i \psi_L^\dagger \sigma^i \psi_L \cdot \delta(\psi_L^\dagger \sigma^2 \psi_L) = v^i \psi_L^\dagger \psi_L$$

$\Rightarrow$  We can define

$$i \psi_L^\dagger \sigma^m \psi_L \rightarrow \delta V^m = \epsilon^m_v V^v$$

all  $\downarrow$   
some part  
is  $v^m$

$$\epsilon^0_i = V_i$$

$$\sigma^m = (J, \sigma^i)$$

$$\bar{\sigma}^m = (I, -\sigma^i)$$

$$V^m = i \psi_L^\dagger \sigma^m \psi_L$$

$$U^m = i \psi_R^\dagger \bar{\sigma}^m \psi_R$$

$\dots \chi = \psi ?$

$$\rightarrow \psi_L^\dagger \sigma^2 \psi_L = 0$$

Conclusion, following objects are invariant

$$\psi_L^+ \sigma^m \partial_\mu \psi_L, \psi_R^+ \bar{\sigma}^m \partial_\mu \psi_R$$

$$\partial_\mu \psi_L^+ \sigma^m \psi_L, \partial_\mu \psi_R^+ \bar{\sigma}^m \psi_R$$

$$\frac{\partial}{\partial x^\mu} \rightarrow \lambda^{-1} \frac{\partial}{\partial x^\mu}$$

$$\begin{aligned} & \frac{1}{2} [\psi_L^+ \sigma^m \partial_\mu \psi_L - (\partial_\mu \psi_L^+) \sigma^m \psi_L] \\ &= \frac{1}{2} \psi_L^+ \sigma^m \partial_\mu \psi_L \quad \Big| \quad \frac{1}{2} \psi_R^+ \bar{\sigma}^m \partial_\mu \psi_R \end{aligned}$$

$$\psi_L = \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}, \psi_R = \begin{pmatrix} \psi_R^1 \\ \psi_R^2 \end{pmatrix}$$

$$\text{Parity invariant Dirac 4-comp. Spinor } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$$

$$\mathcal{L}_{11} = \frac{i}{2} [\psi_L^+ \sigma^m \partial_\mu \psi_L - \partial_\mu \psi_L^+ \sigma^m \psi_L]$$

$$P\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \psi = \gamma_0 \psi$$

$$= \frac{i}{2} \psi_L^+ \sigma^m \partial_\mu \psi_L$$

Projection L & R

$$\frac{1}{2}(\pm \gamma_5)\psi = \psi_{L(R)}$$

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow \frac{1}{2}(1 \mp \gamma_5)\psi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

$$\begin{aligned} & \psi_L^+ \psi_R + \psi_R^+ \psi_L \quad \leftarrow \text{Both invariants} \\ &= \bar{\psi} \psi, \bar{\psi} = \psi^+ \gamma_0 \end{aligned}$$

- Then the Lagrangian which contains all the symmetries (scalars)

Integrating by parts inside action integral:

$$\mathcal{L} = i[\psi_L^+ \sigma^m \partial_\mu \psi_L + \psi_R^+ \bar{\sigma}^m \partial_\mu \psi_R] + m(\psi_L^+ \psi_R + \psi_R^+ \psi_L) \quad \gamma^5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3$$

$$= i\bar{\psi} \gamma^\mu \partial_\mu \psi + m\bar{\psi} \psi \quad \text{where } \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Dirac Lagrangian

$$\psi_L = \sigma^2 \psi_R^*$$

$$\psi^c = \begin{pmatrix} \sigma^2 \psi_R^* \\ -\sigma^2 \psi_L^* \end{pmatrix}$$

Is there a spinor such that  $\psi^c = \psi$ ?

Majorana spinor:

$$(\psi^m)^c = \psi^m = \begin{pmatrix} \psi_L \\ \sigma^2 \psi_L^* \end{pmatrix}$$

$$\text{Properties: } \{\gamma^\mu, \gamma^\nu\} = 2\gamma^{\mu\nu} \mathbb{I}_{4 \times 4}$$

Euclidian rotation:  $\mathbb{R}^{1,3} \rightarrow \mathbb{R}^4$   
 $t \mapsto it, \vec{x} \mapsto \vec{x}'$

$$ds^2 = dt^2 - dx^2 \rightarrow -dx_4^2$$

(After rotation in t)

$$\Lambda_L = e^{\frac{i}{2} \vec{\sigma}^2 (\vec{\omega} + \vec{v})} \quad \Lambda_R = e^{\frac{i}{2} \vec{\sigma}^2 (\vec{\beta} - \vec{v}^*)}$$

$$\begin{aligned} A^\mu &\Rightarrow A = \begin{pmatrix} A^0 + A^3 & A^1 + iA^2 \\ A^1 - iA^2 & A^0 - A^3 \end{pmatrix} \\ &= A^\mu \delta_\mu \end{aligned}$$

$$\Rightarrow A = A^+, \det A = \text{invariant}$$

$$\begin{aligned} & A^\mu A^\nu, \partial_\mu A_\nu \partial^\nu A^\mu \\ & \partial_\mu A_\nu \partial^\nu A^\mu, \partial_\mu A^\mu, (\cdot)^2 \end{aligned} \quad \left[ \text{more invariants} \right]$$

$$SL(2, \mathbb{C}) \xrightarrow{\text{After rotation in } t} SO(3) \times SO(3) \cong SU(2) \times SU(2)$$

Dirac Equation / construct Fock space for spinors and find propagator

If we define  $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$  then  $S^{\mu\nu}$  satisfies Lorentz Algebra

$$\psi \rightarrow \Lambda \psi (\lambda' x) = \psi'(x')$$

$$\Lambda = e^{\frac{i}{2} S^{\mu\nu} \sigma_{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

Plane wave solution:

$$\psi = u(p) e^{-ip \cdot x}$$

↓  
spinor

$$\Rightarrow (\gamma^\mu p_\mu - m) u(p) = 0$$

$\vec{p}$  general

$$\vec{p} = 0 \Leftrightarrow$$

$$\Lambda_{\frac{1}{2}}(p) \psi(\lambda x) = \psi'(x')$$

boost

$$(\gamma^0 p_0 - m) u(p_0, \vec{p} = 0) = 0$$

$$\left[ \begin{pmatrix} -m & 0 \\ 0 & m \end{pmatrix} + \begin{pmatrix} 0 & p_0 I \\ p_0 & 0 \end{pmatrix} \right] u(p) = 0$$

Boost back

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \begin{pmatrix} m \cosh \gamma \\ m \sinh \gamma \end{pmatrix}$$

infinitesimally

$$= \left[ 1 + \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

need to calculate

$$S^3 = \frac{1}{4} [\gamma^\mu, \gamma^\nu]$$

$$\Rightarrow u(p) = e^{-\frac{1}{2} \gamma^3 \sigma^3} \sqrt{m} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \sigma} \bar{\xi} \end{pmatrix}$$

$$\lim_{\gamma \rightarrow 0} u(p) = \begin{pmatrix} \sqrt{E-p^3} & (1) \\ \sqrt{E+p^3} & (1) \end{pmatrix} \rightarrow \sqrt{2E} \begin{pmatrix} (0) \\ (0) \\ (1) \\ (0) \end{pmatrix}$$

$$U^\dagger U = 2E_p \xi^+ \bar{\xi} \quad | \quad \bar{U} = U^\dagger \gamma^0$$

$$\bar{U} U = 2m \xi^+ \bar{\xi} = 2m$$

$$U^{+r} U^s = 2m \xi^+ \bar{\xi}^r = 2m \delta^{sr}$$

$$\bar{V}_p^s V_p^r = -2m \delta^{rs}$$

We could also have  
 $v = v(p) e^{ip \cdot x} \quad p_0 > 0$

$$v(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \gamma^3 \\ -\sqrt{p \cdot \sigma} \gamma^5 \end{pmatrix}$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\xi}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(p + m) v^s(p) = 0$$

$$(i \gamma^\mu \partial_\mu - m) \psi = 0$$

$$\begin{pmatrix} m \mathbb{I}_2 & i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \\ -i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) & m \mathbb{I}_2 \end{pmatrix} \psi = \begin{pmatrix} m \mathbb{I}_2 \psi_L + i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \psi_R \\ -i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) \psi_L - m \mathbb{I}_2 \psi_R \end{pmatrix} = 0$$

chiral

$$(i \gamma^\mu \partial_\mu - m)(i \gamma^\mu \partial_\mu + m) \psi = 0$$

$$(-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2) \psi = 0$$

$$(-\gamma^{\mu\nu} \partial_\mu \partial_\nu - m^2) \psi = 0$$

$$a \cdot b = \frac{1}{2} \{a, b\} + \frac{1}{2} [a, b]$$

does not contribute?

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\xi^+ \xi^- = I \text{ not } 1?$$

$$p_M \alpha^M = \sqrt{p_0 I + p_3 \alpha^3}$$

Taylor expand  
use  $\alpha^2 = 1$

Helicity operator  
 $h = \vec{p} \cdot \vec{\sigma} = \frac{1}{2} p_i \sigma_i$

Are these complete

$$\sum_{s=1,2} (u^s(p) \bar{u}^s(p) + v^s(p) \bar{v}^s(p)) \quad \text{needs to be complete}$$

$$\begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \sigma} \bar{\xi}^s \end{pmatrix} \left( \xi^{s+} \sqrt{p \cdot \sigma}, \bar{\xi}^{s+} \sqrt{p \cdot \sigma} \right)$$

$$= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix}$$

$$\sum_{s=1}^2 \xi^s \bar{\xi}^{s+} = I$$

$$\sum_s \bar{u}^s u^s(p) \dots$$

relation between co-ords  
and momenta with out derivative  
→ constrained system

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \psi^\dagger$$

$$\Pi_{\psi^\dagger} = \psi$$

$$H = \int d^3x (\Pi_\psi \cdot \dot{\psi} - \mathcal{L}) = \int d^3x [\bar{\psi}(-i \vec{\sigma} \cdot \vec{\nabla} + m) \psi]$$

$$\vec{\alpha} = \vec{\sigma}^0 \vec{\sigma}^1, \beta = \sigma^0 \stackrel{\leftarrow}{=} \int d^3x [-i \vec{\alpha} \cdot \vec{\nabla} + m \beta] \psi$$

$$H = \int (\Pi_\psi \dot{\psi} - \mathcal{L}) d^3x = \int \bar{\psi} (-i \vec{\sigma} \cdot \vec{\nabla} + m) \psi d^3x$$

$$[H, [\psi_\alpha(\vec{x}), \psi_\beta^+(\vec{y})]] = \delta_{\alpha\beta} \delta^{(3)}(\vec{x} - \vec{y})$$

$$\psi_\alpha(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} e^{i\vec{p} \cdot \vec{x}} \sum_{s=1,2} [a_s^s(p) u_s^s(p) + b_s^s(p) v_s^s(p)]$$

$$[a_s^s(p), a_t^s(p')] = [b_s^s(p), b_t^s(p')] = (2\pi)^3 \delta(\vec{p} - \vec{p}') \delta^{rs}$$

$$\sum_s u_s^s(p) \bar{u}_s^s(p) = p^2 - m, \sum_s v_s^s(p) \bar{v}_s^s(p) = p^2 - m$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p (a_s^+(p) a_s^s(p) - b_s^+(p) b_s^s(p))$$

Not always  $\geq 0$  (Problem #1)

what if we change  $b^+ \rightarrow b$  doesn't work...

$$[\psi_\alpha(x), \bar{\psi}_\beta(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} ((p+m)e^{-ip(x-y)} + (p-m)e^{ip(x-y)})$$

$$= (i\gamma_x + m) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip(x-y)} - e^{ip(x-y)}) = 0 \quad (x-y \neq 0)$$

$$\text{suppose } a(\vec{p})|0\rangle = b(\vec{p})|0\rangle = 0$$

$$\Rightarrow [\psi(x), \psi^\dagger(y)] = \langle 0 | [\psi(x), \bar{\psi}(y)] | 0 \rangle = \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle - \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle ?$$

problem 2

We start from scratch Need to change something

$$\text{Let us calculate } \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \langle 0 | \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} \sum_r a_r^s(p) u_r^s(p) e^{-ipx} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{E_{p'}}} \sum_r a_r^{s'}(\vec{p}') u_r^{s'}(\vec{p}') e^{ip'y} | 0 \rangle$$

$$\text{postulate } b^\dagger |0\rangle = 0 \quad \langle 0 | a_s^s(p) a_t^t(p') | 0 \rangle = ?$$

$$\text{consider } \langle 0 | a_s^s(p) a_t^{s'}(\vec{p}') e^{i\vec{p} \cdot \vec{x}} | 0 \rangle$$

$$e^{i\vec{p} \cdot \vec{x}} A \bar{e}^{i\vec{p}' \cdot \vec{x}} = A(\vec{x}) = e^{i\vec{p} \cdot \vec{x}} A(0)$$

$$\Rightarrow e^{i(\vec{p} - \vec{p}') \cdot \vec{x}} \langle 0 | a_s^s(p) a_t^{s'}(p') | 0 \rangle$$

$$\Rightarrow \delta^{rs} \delta(\vec{p} - \vec{p}') A(\vec{p})$$

All we know is this is Lorentz invariant  
 $\Rightarrow A = A(p^2) = A(m^2) = \text{const}$

$$\Rightarrow \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r a^r(p) u^r(p) e^{-ipx} \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_{r'} a^{r*}(p') \bar{u}^{r'}(p') e^{ip'y} | 0 \rangle$$

$$= \Theta(i\gamma + m) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} A$$

Similarly added

$$\langle 0 | \bar{\psi}(x) \psi(y) | 0 \rangle = - (i\gamma_x + m) \int \frac{d^3 p}{(2\pi)^3} \frac{e^{ip(x-y)}}{2E_p} B$$

$$\begin{array}{c} \uparrow \\ y \\ \downarrow \\ - \end{array} \quad \begin{array}{c} \uparrow \\ x \\ \downarrow \\ y \end{array} = 0 \Rightarrow A = -B \quad A > 0, B > 0$$

so we change our addition

where

$$\langle 0 | b^+(p) b^s(p') | 0 \rangle = \delta^{rs} \delta(p-p') B$$

### Fermionic Fock space

$$a(p)|0\rangle = 0$$

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle$$

$$b^+(p)|0\rangle = 0$$

$$\{a^r(p), a^{s*}(p)\} = \{b^r(p), b^{s*}(p')\}$$

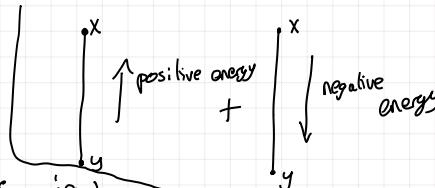
$$= \delta^{rs} \delta(\vec{p} - \vec{p}')$$

rename  $b^+ \rightarrow \tilde{b} \rightarrow b$

$$H = \int d^3 p E_p (a^+(p)a(p) + b^+(p)b(p)) > 0$$

$$\{ \psi_{(x,t)}^+, \psi_{(y,t)}^- \} = \delta(\vec{x} - \vec{y})$$

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} (u^s(p) a^s(p) e^{-ipx} + v^s(p) b^s(p) e^{ipx})$$



### Plan for notes

- Anti commutation relations
- Pauli exclusion principle
- charges of positron and electron
- Dirac propagator

$a^+$  creates electrons  
 $b^+$  creates positrons

↑  
Dirac Fermion  
follows Fermi Dirac statistics

Anti-commutator is poisson brackets  
 $\Rightarrow$  Grassmann variables

$$AB + BA = 0 \Rightarrow A \cdot A = 0$$

We now discuss Fermionic Fock space

$|0\rangle$  Vacuum,  $H|0\rangle = 0$

$$a^s(\vec{p})|0\rangle = b^s(\vec{p})|0\rangle = 0$$

$$a^{+s}(\vec{p})|0\rangle = |\vec{p}, s\rangle, b^{+s}(\vec{p})|0\rangle = |\vec{p}, \tilde{s}\rangle \quad (1 \text{ particle state})$$

(2 particle state)

$$a^{+s}(\vec{p}_1) a^{+r}(\vec{p}_2) |0\rangle = |\vec{p}_1, s, \vec{p}_2, r\rangle \quad \text{This will vanish if } p_1 = p_2 \quad r = s$$

$$b^{+s}(\vec{p}_1) b^{+r}(\vec{p}_2) |0\rangle = |\vec{p}_1, \tilde{s}, \vec{p}_2, \tilde{r}\rangle \quad \Rightarrow \text{Pauli exclusion principle!}$$

↑  
vanishes for  
 $\vec{p}_1 = \vec{p}_2$   
 $\tilde{s} = \tilde{r}$

$$b^+(\vec{p}_1) a^+(\vec{p}_2) |0\rangle = |\vec{p}_1, \tilde{s}, \vec{p}_2, \tilde{r}\rangle$$

$$a^{+s}(\vec{p}_1) b^+(\vec{p}_2) |0\rangle = |\vec{p}_2, s, \vec{p}_1, \tilde{r}\rangle$$

$$|\vec{p}_1, s_1, \dots, \vec{p}_n, s_n, \vec{p}_1, \tilde{s}_1, \dots, \vec{p}_n, \tilde{s}_n\rangle$$

$|n, m\rangle$

$n$  particles  $\hookrightarrow n$  anti-particles

$$\bar{\Psi}(x, t) = \psi^\dagger(x, t) \gamma^0 = \int \frac{d^3 p}{(2\pi)^3} \sum_{S=1,2} \left( \bar{U}^S(p) a^S(p) e^{ipx} + \bar{V}^S(p) b^S(p) e^{-ipx} \right)$$

Once again to restore Lorentz invariance

$$\sum E_p a^{+S}(\vec{p}) |0\rangle = |\vec{p}, S\rangle, \sum E_p b^{+S}(\vec{p}) |0\rangle = |\vec{p}, \tilde{S}\rangle$$

- Let's investigate if there is a Noether current.

Returning to Dirac Lagrangian

electric charge

$$\mathcal{L}_0 \equiv i \bar{\psi} \not{D} \psi + m \bar{\psi} \psi; \quad \psi \rightarrow e^{i\alpha} \psi \\ \psi^\dagger \rightarrow e^{-i\alpha} \psi^\dagger$$

↙ Symmetry!

$$- infinitesimally \quad \psi \rightarrow \psi + i\alpha \psi \\ \psi^\dagger \rightarrow \psi^\dagger - i\alpha \psi^\dagger$$

$$\text{Noether current: } J^\mu = \bar{\psi} \not{\partial}^\mu \psi; \quad \partial_\mu J^\mu = 0$$

recall electron ↴

$$|\vec{p}, s\rangle = \sum E_p a^{+S}(\vec{p}) |0\rangle$$

$$Q = \int \psi^\dagger \not{\partial}^\mu \psi d^3 x \stackrel{\text{Do this!}}{=} \int \frac{d^3 p}{(2\pi)^3} \sum_S (a^{+S}(\vec{p}) \bar{U}^S(\vec{p}) - b^{+S}(\vec{p}) \bar{V}^S(\vec{p}))$$

$$Q \equiv \int \mathcal{J}^\mu d^3 x$$

$$\mathcal{J}^\mu = \bar{\psi}^\dagger \not{\partial}^\mu \psi = \bar{\psi} \not{\partial}^\mu \psi$$

Now let's act on these

$$Q a^{+S}(0) |0\rangle = \int \frac{d^3 p}{(2\pi)^3} \sum_r \bar{U}^r(\vec{p}) a^r(0) \bar{U}^S(\vec{p}) |0\rangle \\ = [a^{+S}(0) \bar{U}^1(\vec{p}) + \bar{U}^2(\vec{p}) a^{+S}(0)] |0\rangle \\ = g^{rs} \delta(\vec{p}) |0\rangle$$

$$\Rightarrow Q a^{+S}(0) |0\rangle = a^{+S}(0) |0\rangle$$

$$Q b^{+S}(0) |0\rangle = - b^{+S}(0) |0\rangle$$

⇒ Can distinguish particle and anti-particle with opposite charges.

Propagator

$$\langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_S U^S(\vec{p}) e^{-ip(x-y)}$$

$\vec{p} + m$

$$= (i \not{\partial}_x + m) \int \frac{1}{2E_p} e^{-ip(x-y)} \frac{d^3 p}{(2\pi)^3}$$

Want to write down Feynmann propagator with time ordering but, when we swap the order we pick up a minus sign due to grassman variables

$$S_F(x-y) = \Theta(x_0 - y_0) \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle$$

$$- \Theta(y_0 - x_0) \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle = \langle 0 | T(\psi(x) \bar{\psi}(y)) | 0 \rangle$$

$$\text{recall } D_F(x-y) = \langle 0 | T(\psi(x) \psi(y)) | 0 \rangle$$

$$\Rightarrow S_F(x-y) = (i \not{\partial}_x + m) D_F(x-y)$$

$$= (i \not{\partial}_x + m) \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}$$

Calculating the integral:

$$S_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{(p+m)}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$

Discrete Symmetries

Quadratic in  $\gamma^\mu$

$$[\gamma^\mu, \gamma^\nu, \gamma^\lambda] = \gamma^\sigma \gamma^\mu \gamma^\nu \gamma^\lambda$$

$$\gamma^\mu = -\frac{i}{4} \epsilon^{\mu\nu\lambda\sigma} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma$$

$\Gamma^\mu$  matrices  $4 \times 4 \rightarrow 16$  matrices

$$\begin{array}{cc} 1 & -1 \\ \gamma^\mu & -4 \end{array}$$

⇒ Most general combination

$$\bar{\psi} \Gamma^\mu \psi \quad \mu = 1, \dots, 16$$

$$S^{\mu\nu} = \frac{1}{4} [\gamma^\mu, \gamma^\nu] - 6$$

$$\gamma^\mu \gamma^\nu - 4$$

$$\gamma^\sigma - 1$$

parity transformation

$$(t, \vec{x}) \rightarrow (t, -\vec{x})$$

Time reversal

$$(t, \vec{x}) \rightarrow (-t, \vec{x})$$

$$U(p) a^s(p) U^\dagger(p) = \gamma_a a^s(-p)$$

$$\gamma_a = \text{phase} \left( \gamma \right)^2 = 1$$

$$\text{Similarly } U b^s(p) U^\dagger(p) = \gamma_b b^s(-p)$$

$$U(p) \psi(x) U^\dagger(p) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (\gamma_a a^s(p) U^s(p) e^{-ipx} + \gamma_b^* b^{+s}(p) V^s(p) e^{ipx})$$

$$\tilde{p} = (p^0, -\vec{p}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (\gamma_a a^s(p) U^s(p) e^{-ipx} + \gamma_b^* b^{+s}(p) V^s(p) e^{ipx})$$

$$U(p) = \begin{pmatrix} \sqrt{p^0} & \xi \\ \sqrt{p^0} & \bar{\xi} \end{pmatrix} = \gamma^0 U(\hat{p})$$

$$V^s(p) = -\gamma^0 V^s(\hat{p})$$

Free to choose

$$\gamma_a = -\gamma_b^* = 1$$

$$p \cdot x = \tilde{p} \cdot \tilde{x}$$

$$U \psi(t, \vec{x}) U^\dagger = \gamma^0 \psi(t, -\vec{x})$$

$$U \bar{\psi}(t, \vec{x}) U^\dagger = \bar{\psi}(t, \vec{x}) \gamma^0$$

$$\bar{\psi}(t, \vec{x}) \psi(t, \vec{x}) = \text{scalar} = \bar{\psi}(t, -\vec{x}) \psi(t, -\vec{x})$$

$$\bar{\psi} \gamma^\mu \psi = \text{vector} = \pm \sum_{\mu=0}^3 \bar{\psi}(t, -\vec{x}) \gamma^\mu \psi(t, \vec{x})$$

$$\bar{\psi} \gamma^\mu \gamma^5 \psi = \text{pseudo vector} = \mp \sum_{\mu=0}^3 \bar{\psi}(t, -\vec{x}) \gamma^\mu \gamma^5 \psi(t, \vec{x})$$

$$\bar{\psi} \gamma^5 \psi = \text{pseudo scalar} = -\bar{\psi}(t, -\vec{x}) \gamma^5 \psi(t, -\vec{x})$$

time reversal flips the spin

$$\rightarrow \leftarrow \leftarrow \rightarrow$$

$$U(T) [A] = [A^*] U(T)$$

postulate  $\uparrow \rightarrow \text{Formular works} \rightarrow \downarrow$

Final def of time reversal

$$U(T) a^s(p) U^\dagger(T) = \bar{a}^s(-\vec{p})$$

$$U(T) b^s(p) U^\dagger(T) = \bar{b}^s(-\vec{p})$$

$$U(T) \psi(x) U^\dagger(T) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^s(-p) U^s(p) e^{ipx} + b^{+s}(-p) V^s(p) e^{ipx})$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^s(-p) U^s(p) e^{ipx} + b^{+s}(-p) V^s(p) e^{ipx})$$

$$= -\gamma^1 \gamma^3 \psi(-t, \vec{x})$$

$$\xi(\uparrow) = \begin{pmatrix} \cos \frac{\Omega}{2} \\ e^{i\phi} \sin \frac{\Omega}{2} \end{pmatrix} \quad \xi(\downarrow) = \begin{pmatrix} -e^{i\phi} \sin \frac{\Omega}{2} \\ \cos \frac{\Omega}{2} \end{pmatrix}$$

$$(\uparrow) \qquad \qquad \qquad (0)$$

$$\xi^S = (\xi(\uparrow), \xi(\downarrow)) \quad \text{"natural definition"}$$

$$\xi^{-S} = (-\xi(\downarrow), -\xi(\uparrow))$$

$$U(T) a^s(p) U^\dagger(T) = \bar{a}^s(-\vec{p})$$

$$U(T) b^s(p) U^\dagger(T) = \bar{b}^s(-\vec{p})$$

$$U(T) \psi(x) U^\dagger(T) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^s(-p) U^s(p) e^{ipx} + b^{+s}(-p) V^s(p) e^{ipx})$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^s(-p) U^s(p) e^{ipx} + b^{+s}(-p) V^s(p) e^{ipx})$$

$$= -\gamma^1 \gamma^3 \psi(-t, \vec{x})$$

$$\bar{\psi} \psi \text{ scalar}$$

$$\bar{\psi} \gamma^5 \psi \rightarrow \text{pseudo vector}$$

$$\bar{\psi} \gamma^\mu \psi \rightarrow \text{vectors}$$

## Charge conjugation

$$C^2 = \mathbb{1} \Rightarrow C^{-1} = C$$

$$C \alpha^s(\vec{p}) C = b^s(\vec{p})$$

$$C b^s(\vec{p}) C = \alpha^s(\vec{p})$$

$$C \Psi(t, \vec{x}) C = C \int \frac{d^3 p}{(2\pi)^3} \sum_s \left( \alpha^s(p) a^s(p) e^{ipx} + \bar{\alpha}^s(p) b^s(p) e^{ipx} \right) C$$

$$= -i \gamma^2 \psi^*(t, \vec{x}) = -i \gamma^2 (\psi^+)^T$$

Little book of Dirac:

Quantise Gravity

Maxwell:

Degenerate Hamiltonian system

$$L(q_i, \dot{q}_i) \Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{cannot express } \dot{q}_i(p, q)$$

$$\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} = M_{ij}, \quad \det M_{ij} \neq 0$$

$$H = \sum p_i \dot{q}_i - \mathcal{L}(q, \dot{q}) = H(p, q)$$

If  $M$  is not invertible means  
there are some relations between rows

$$\Rightarrow \varphi^a(p, q) = 0 \quad a = 1, \dots, m$$

There is not one Hamiltonian but  $m$

$$H' = H(p, q) + \sum_{a=1}^m \lambda^a \varphi^a(p, q)$$

$$H'' = H' + \sum_{a=1}^n \lambda^a \varphi^a(p, q) + \sum_{a=1}^m \lambda^a \varphi^a(p, q)$$

$$\Rightarrow H'' = H + \sum_{a=1}^m \lambda^a \varphi^a(p, q)$$

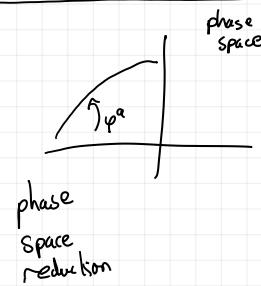
Now repeat first step?

$$\Phi^\alpha = \{\varphi^\alpha, H\} = 0$$

$$\begin{cases} U^s(p) = -i \gamma^2 \bar{\nu}^s(p) \\ V^s(p) = -i \gamma^2 \bar{u}^s(p) \end{cases}$$

CPT Theorem

CPT	$\bar{\psi} \psi$	$\bar{\psi} \gamma^5 \psi$	$\bar{\psi} \gamma^\mu \psi$	$\bar{\psi} \gamma^\mu \gamma^5 \psi$	$\bar{\psi} \gamma^\mu \gamma^5 \psi$
	1	1	-1	-1	+1



$$\{\varphi(p, q)\} = \{\varphi^a, H\} = 0$$

$$\Rightarrow \{\varphi^a, H\} + \sum_{b=1}^k \lambda^b \{\varphi^a, \varphi^b\} = 0$$

$$\{\varphi^a, H\} = f^a(p, q) = 0$$

$$\text{Now } \varphi^k(p, q) = 0; \quad k = 1, \dots, l < m \quad m+l=k?$$

Secondary + Primary

$$\text{This equation: } \{\varphi^a, H\} + \sum_{b=1}^k \lambda^b \{\varphi^a, \varphi^b\} = 0 \Rightarrow \lambda^a = (N^{-1})^{ab} \{\varphi^b, H\}$$

$$\text{If } \{\varphi^a, \varphi^b\} = N^{ab}$$

$\varphi^a$  is invertible ( $\det N \neq 0$ )

there are a few cases

①  $N$ -invertible (All good)

②  $N$ -not-invertible, No solutions

③ New equations?

$$\begin{cases} H(p, q) & \text{Constraint} \\ \{\varphi^\alpha(p, q), \varphi^\beta(p, q)\} = 0 & \text{Hamiltonian system} \\ \alpha = 1, \dots, m & \{\varphi^\alpha, H\} = 0 \\ \text{but others} & \end{cases}$$

First class constraints

$$\{\varphi^\alpha, \varphi^\beta\} = 0$$

Second class

all others

$$H|\psi\rangle = E|\psi\rangle$$

$$\varphi^\alpha |\psi\rangle$$

Weak sense

Strong sense

2nd class

$$\phi^\alpha(p, q) = 0 \quad \text{2nd class}$$

$$\{f(p, q), g(p, q)\}_D = \{f, g\} - \sum_{\alpha, \beta} C_{\alpha\beta} \{g^\beta, g\}$$

$$C_{\alpha\beta} = (C^{\alpha\beta})^{-1}, \quad C^{\alpha\beta} = \{f^\alpha, f^\beta\}$$

$$\{f^\alpha = 0, g\}_D = \{f^\alpha, g\}$$

$\Rightarrow$  constraints are 0  
in the strong sense

$$\alpha = 1, \dots, k \quad k = \text{Even} = 2n \quad 2N \text{ variables}$$

and  $2n$  equations

$$\phi^\alpha = 0 \quad \Rightarrow \# \text{ independent variables } 2N - 2n = 2(N-n)$$

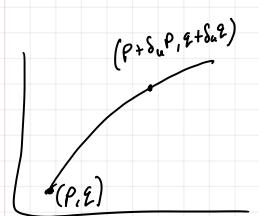
$p_a^*, q^{*\alpha}$ ,  $\alpha = 1, \dots, N-n \Rightarrow$  Reduced phase space

$$\begin{aligned} \{f^\alpha, g\}_D &= \{f^\alpha, g\}_P \\ &- \{f^\alpha, f^\beta\}_P C_{\alpha\beta} \{f^\beta, g\}_P \\ &= \{f^\alpha, g\} - C^{\alpha\beta} C_{\alpha\beta} \{f^\beta, g\} \end{aligned}$$

$\underbrace{\delta^\alpha_\beta}_{\text{Repeated indices mean sum}}$

$$= 0$$

$$\begin{aligned} \text{generating function } \phi(u) &= \sum_\alpha u^\alpha \phi^\alpha \\ \phi(v) &= \sum_\alpha v^\alpha \phi^\alpha \end{aligned}$$



$$\begin{aligned} \text{gauge transformation} \\ p_u &= p + \delta_u p \\ q_u &= q + \delta_u q \end{aligned}$$

$$\begin{aligned} \delta_u P_i &= \{f(u), P_i\}_P \\ \delta_u q_i &= \{f(u), q_i\}_P \end{aligned}$$

$$\begin{aligned} \delta_t P &= \{H, P\} \\ \delta_t q &= \{H, q\} \\ u &= k\text{-dim vector} \end{aligned}$$

2nd class constraints

Wed morning class

$$\{f^\alpha, f^\beta\}_P = \{f, f\}$$

$$C_{ab} C^{bc} = \delta_a^c$$

$$\text{poisson}$$

$$\{f(p, q), g(p, q)\}_D = \{f, g\}_P$$

$$- \{f, \phi^\alpha\}_P C_{ab} \{f^\beta, \phi^\gamma\}_P$$

With this we can put  $\phi^\alpha = 0$   
in a strong sense  $\Leftrightarrow$  solving equations  $(p_i, q_i)_{i=1, \dots, n}$

$$H^{\text{Final}} = H + \sum_\alpha \lambda^\alpha \phi^\alpha = H^F(p, q)$$

Done

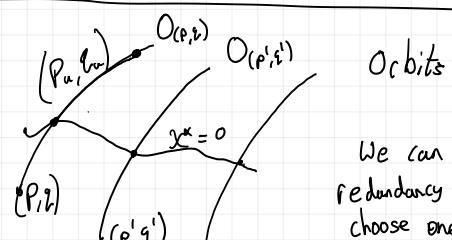
evaluation with  
 $H$  &  $\phi^\alpha = 0$

1st class

Dirac proved

$$\delta_u f = \{\phi(u), f\}_P$$

like poisson bracket with  $H$   
but for different generators



We can eliminate redundancy we choose one point in the  $k$ -dim space

$$\chi^\alpha(p, q) \quad \alpha = 1, \dots, k$$

constraints

These are gauge fixing conditions.

$$p + \delta_u p = p + \{\phi(u), p\}$$

$$q + \delta_u q = q + \{\phi(u), q\}$$

If we find a  $u \neq 0$ :

$$\chi(p, q) = 0 \text{ and } \chi(p + \delta_u p, q + \delta_u q)$$

$\Rightarrow$  We cross same orbit twice for  $u \neq 0$

$$\chi(p + \delta_u p, q + \delta_u q) = \chi + \{\phi(u), \chi\} = \sum_{\beta} u^{\beta} \phi^{\beta}, \chi^{\alpha} = 0$$

only one sol  $u=0$

In this sense  $\phi(u)$  are generators.

$H, \Phi^{\alpha}$  is a new system of second class constraints

$\det \{\phi^{\alpha}, \chi^{\beta}\} \neq 0$   
 Condition from fixing gauge

Maxwell theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$A_{\mu}$  = vector potential

$$F = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}$$

$$H = \int d^3x, \Pi_{\mu} = \frac{\partial \mathcal{L}}{\partial A_{\mu}}$$

$$\begin{aligned} \Pi_0 = 0 & \quad \Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{oi} = +F_{oi} \\ & = E; \quad \text{electric field} \end{aligned}$$

Constraint!  
 Maxwell theory has a primary constraint

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{oi} F^{oi} - \frac{1}{4} F_{io} F^{io} - \frac{1}{4} F_{ij} F^{ij} \\ &= \frac{1}{2} E_i E_i - \frac{1}{4} F_{ij} F^{ij} \quad -\frac{1}{2} B_i B_i \\ B_i &= \frac{1}{2} E_{ijk} F_{jk} \end{aligned}$$

$$\begin{aligned} H &= \int d^3x \left[ \Pi_0 \dot{A}_0 + \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + E_i \partial_i A_0 \right] \\ &= \int d^3x \left[ \Pi_0 \dot{A}_0 + \frac{1}{2} (\vec{E}^2 + \vec{B}^2) - A_0 \partial_i E_i \right] \quad \text{can integrate by parts} \end{aligned}$$

+ constraint of  $\Pi_0 = 0$

$$\dot{\Pi}_0 = \Pi_0(\vec{x}) = \{\Pi_0(\vec{x}), H\} = \{\Pi_0, \int d^3y A_0(\vec{y}) \partial_i E_i\}$$

$$\{\Pi_0, A_0(\vec{y})\} = \delta(\vec{x} - \vec{y})$$

$$\{\dot{E}_i, A_j(\vec{y})\} = \delta(\vec{x} - \vec{y}) \delta_{ij}$$

$$\Rightarrow \{\vec{E}_i(\vec{x}), \vec{B}_j(\vec{y})\} = \epsilon_{ijk} \partial_k \delta(\vec{x} - \vec{y})$$

$$\Rightarrow \int \{\dot{E}_i(\vec{x}), B_j(\vec{y})\} d^3y = \int \epsilon_{ijk} \partial_k \delta(\vec{x} - \vec{y}) B_j(y) d^3y$$

$$= \partial_i (\epsilon_{ijk} \partial_k B_j(\vec{y})) = 0$$

$\Rightarrow \partial_i E_i = 0$  is only constraint

and is first class

as  $\{\phi(\vec{x}), \phi(\vec{y})\} = 0$

$$\Rightarrow \sum_{\beta} u^{\beta} \{\phi^{\beta}, \chi^{\alpha}\} = V^{\alpha} \quad u = ( )^V$$

only if  $\det \{\phi^{\beta}, \chi^{\alpha}\} \neq 0$

$$+ \{ \phi^{\alpha} = 0, \chi^{\alpha} = 0 \}$$

gauge fixed

equivalent description

$$\prod_{\alpha=1}^{2k} \phi^{\alpha}, \alpha=1, \dots, 2k \quad \det \left( \{\phi^{\alpha}, \phi^{\beta}\}_{\alpha, \beta} \right) \neq 0$$

$$\Phi = \begin{pmatrix} \phi^{\alpha} \\ \chi^{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} \{\phi^{\alpha}, \phi^{\beta}\} \\ \{\phi^{\alpha}, \chi^{\beta}\} \\ \{\chi^{\alpha}, \phi^{\beta}\} \\ \{\chi^{\alpha}, \chi^{\beta}\} \end{pmatrix}$$

One condition  
 $C^{\alpha\beta} = \{\phi^{\alpha}, \chi^{\beta}\}$   
 invertible

$$\begin{aligned} \mathcal{H} &= \Pi_0 \dot{A}_0 + \Pi_i \dot{A}_i - \mathcal{L} \\ &= \Pi_0 \dot{A}_0 + E_i E_i + F_i \partial_i A_0 \\ &\quad - \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \end{aligned}$$

(Hamiltonian density)

$$H' = \int d^3x \left[ \frac{1}{2} (\vec{E}^2 + \vec{B}^2) - A_0 \partial_i E_i + \dot{A}_0 \Pi_0 \right]$$

could add  $\lambda_1 \partial_i E_i, \lambda_2 \Pi_0$  but this is the same as shifting  $A_0, \dot{A}_0$  so we can just consider these our constraints

phase Space  $(E_i, A_i)$  with constraints

now we can plug in  $\Pi_0 = 0$  as that is the same as solving that equation

left with the

constraint:  $\phi(x) = \partial_i E_i, \phi(x) = \{\phi(x), H'\}$

$$= \{\partial_i E_i, H'\} = \{\partial_i E_i(x), \int d^3y \frac{1}{2} \vec{B}^2\}$$

$$\{\dot{E}_i(\vec{x}), B_j(\vec{y})\} = \{\dot{E}_i(\vec{x}), \frac{1}{3} \epsilon_{ijk} F_{kl}(\vec{y})\}$$

$$= \frac{1}{2} \epsilon_{jkl} \{\dot{E}_i(\vec{x}), \partial_k A_l(\vec{y}) - \partial_l A_k(\vec{y})\}$$

$$= \frac{1}{2} \epsilon_{jkl} (\partial_k \partial_l \delta(\vec{x} - \vec{y}) - \delta_{ik} \partial_l \delta(\vec{x} - \vec{y}))$$

$$= \frac{1}{2} \epsilon_{jki} \partial_k \delta(\vec{x} - \vec{y}) + \frac{1}{2} \epsilon_{jki} \partial_k \delta(\vec{x} - \vec{y})$$

$A_i(\vec{x}), E_i(\vec{x})$  are phase space co-ords  
infinite dim.  $(i, \vec{x})$ -index  
 $2N = 2(3 \times \mathbb{R}^3) = \dim \text{phase space}$

- dim of reduced space

$\phi(\vec{x}) \rightarrow \text{gauge fixing conditions } \mathbb{R}^3 \Rightarrow \dim \text{of reduced phase space} = 2(2 \times \mathbb{R}^2)$

$= 4 \times \text{space-co-ords}$   
 $\begin{cases} \text{went from 8 to 4} \\ \text{2 coord 2-momenta} \\ \hookrightarrow \dots \rightarrow \text{photon has 2 polarizations} \end{cases}$

$$\left\{ \begin{array}{l} \phi = \partial_i; E_i = 0 \text{ Gauss} \\ x = \partial_i; A_i = 0 \text{ Coulomb Gauge} \end{array} \right.$$

$$\{ \phi(\vec{x}), \chi(\vec{y}) \} = \partial_i \partial_j \delta(\vec{x} - \vec{y}) \delta_{ij} = -\Delta \delta(\vec{x} - \vec{y})$$

Invertible. (only crosses orbit once)

$$A_i + S_u A_i$$

$$S_u(A_i) = \{ \phi(u), A_i \} = \left\{ \int u(\vec{z}) \partial_j E_j d^3 z, A_i \right\}_{\substack{\text{sum IBP} \\ -\partial_i u}}$$

$$\Rightarrow A_i \rightarrow A_i - \partial_i u$$

$$\begin{aligned} &\Rightarrow \partial_i A_i - \partial_i \partial_i u = 0 \\ &\quad \text{or} \\ &\quad -\Delta u = 0 \\ &\Rightarrow u = 0 \end{aligned}$$

If  $u = \text{const} + \text{linear}$   
cannot happen as boundary  
must be 0.

Notes plan

- Dirac propagator
- Dirac book (skip discrete)
- Maxwell

$${}^{ab} \tilde{C} = -\Delta \delta(\vec{x} - \vec{y})$$

$$\Rightarrow {}^{ab} C = -\frac{1}{\Delta} (\vec{x}, \vec{y})$$

$$A_i \rightarrow A_i - \partial_i u; E_i \rightarrow E_i$$

$$\{ f, g \}_D = \{ f, g \}_P - \sum_{a,b} \{ f, \phi^a \}_P C_{ab} \{ \phi^b, g \}_P$$

$$\begin{aligned} &\xrightarrow{\text{field theory}} \{ f, g \}_P + \int d^3 x d^3 y \{ f, \partial_j E_j(\vec{x}) \}_P \frac{1}{\Delta}(\vec{x}, \vec{y}) \{ \partial_i A_i(\vec{y}), g \}_P \end{aligned}$$

$$\text{old case: } \{ E_i(\vec{x}), A_j(\vec{y}) \} = \delta_{ij} \delta(\vec{x} - \vec{y})$$

$$\begin{aligned} \{ E_i(\vec{x}), A_j(\vec{y}) \}_D &= \delta_{ij} \delta(\vec{x} - \vec{y}) + \int d\vec{z}_1 d\vec{z}_2 \underbrace{\{ E_i(\vec{x}), \partial_k A_k(\vec{z}_1) \}_P}_{S_{ik} \delta(\vec{z}_1)} \underbrace{\frac{1}{\Delta}(\vec{z}_1, \vec{z}_2) \{ \partial_i E_i(\vec{z}_2), A_j(\vec{z}_2) \}}_{\partial_j \delta(\vec{z}_2 - \vec{y})} \end{aligned}$$

these "hit"  $\frac{1}{\Delta}$  giving us two minus signs (From IBP)

$$= \delta_{ij} \delta(\vec{x} - \vec{y}) + \partial_i \partial_j \frac{1}{\Delta}(\vec{x}, \vec{y})$$

$$= \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \left( \delta_{ij} - \frac{p_i p_j}{p^2} \right)$$

$$\Delta \cdot \frac{1}{\Delta} = \delta(\vec{x} - \vec{y})$$

$$\Delta G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$$

$$= \Delta \int G(p) e^{i\vec{p} \cdot (\vec{x} - \vec{y})} = \int e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \Rightarrow -\vec{p} G = 1$$

$$\{f(p_i), g(p_j)\}_D = \{f_i, g\}_D - \sum_{a,b} \{f_a, g_b\}_{ab} \delta^a_i \delta^b_j$$

If all constraint 2nd class

$$\{\phi^a, \phi^b\} = C^{ab}$$

is invertible

$$A_i(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \sum a(p, \lambda) E_i(p, \lambda) e^{ip \cdot \vec{x}} + c.c.$$

- How do  $a(p, \lambda)$  and  $a^\dagger(p, \lambda)$  depend on  $t$

$$\begin{cases} \vec{C}^2 = 1, \vec{E}^2(p, i) \cdot \vec{E}^2(p, 2) = 0 \\ \vec{E}^2(p, \lambda) \cdot \vec{p} = 0 \end{cases}$$

$$\begin{aligned} C a(p, \lambda) e^{iHt} &= a(p, \lambda, t) \\ &= e^{iP_0 t} a(p, \lambda) \quad \square A_i = 0 \\ &\Rightarrow p^2 = 0 \end{aligned}$$

what if we did not read little book of Dirac!

- can write  $\vec{E}$  as 4-vector

$$\begin{aligned} E^m(p, \lambda) &= (\vec{0}, \vec{E}^2(p, \lambda)) \Rightarrow E^m(p, \lambda) E_\mu(p, \nu) = \delta_{\mu\nu}, \\ E_\mu p^\mu &= 0 \quad \text{particular to Coulomb gauge} \end{aligned}$$

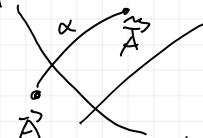
Returning to Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\mathcal{L}$  invariant under  $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$

- In the Hamiltonian gauge transformation generated approach: by Gauss Law  $\int \partial_\mu E_i(x) d^3 x$

$\phi(A_\mu)$



$A_i \rightarrow A_i + \partial_i \alpha$   
polarization  
transformation

- In Lagrangian approach

$$\phi(A_\mu + \partial_\mu \alpha) = 0 = \phi(A_\mu) - \int d^3 x \frac{\partial \phi}{\partial A_\mu} d^4 x \cdot \alpha$$

Lorenz gauge  $\partial_\mu A_\mu = 0$

- Lorentz invariant gauge

$$\partial_\mu (A_\mu + \partial_\mu \alpha) = 0 = \partial_\mu A_\mu + \partial_\mu \partial_\mu \alpha$$

$$\Rightarrow \square \alpha = 0 \quad \left| \begin{array}{l} \text{This has no solutions} \\ \text{only } \alpha = 0? \\ \text{even though before} \\ A \text{ had infinitely many} \\ \text{solutions} \end{array} \right.$$

$$A_\mu(x, t) = \int \frac{d^3 p}{(2\pi)^3} \left[ \vec{C} \cdot \vec{p} \cdot \vec{x} \sum a(p, \lambda) E_\mu(p, \lambda) + c.c. \right] \rightarrow \partial_\mu A_\mu = 0 \Rightarrow p^\mu \epsilon_\mu = 0$$

Note:  $A_0 \propto$  as  $E_0 = 0$

$$F_{\mu\nu} F^{\mu\nu} = \partial_\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) = \square A_\nu - \partial_\nu (\partial_\mu A_\mu) = 0 \Rightarrow \square A_\nu = 0$$

- Lorentz gauge is better for propagator

Bosonic Fock space

$$[a(p, \lambda), a^\dagger(p', \mu)] = \delta_{\mu\lambda} \delta(\vec{p} - \vec{p}')$$

Want

$$\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle$$

Propagator in Coulomb gauge

We have a vacuum  $a(p, \lambda)|0\rangle = 0$

- To have Lorentz invariant particles

$$\sqrt{E_p} a^\dagger(p, \lambda) |0\rangle = |p, \lambda\rangle$$

$$2\text{-particle: } \sqrt{E_p} \sqrt{E_{p'}} a^\dagger(p, \lambda) a^\dagger(p', \lambda') |0\rangle$$

Now we do this for Lorenz gauge

$$\langle 0 | T (A_\mu(x) A_\nu(y)) | 0 \rangle = D_{F, \mu\nu}(x-y) =$$

$$= \int \frac{d^3 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i\epsilon} (\gamma_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})$$

- Now we should have  $\partial_\mu \langle 0 | T (A_\mu(x) A_\nu(y)) | 0 \rangle = 0$

$$\text{Let us call } \Delta_{\mu\nu} = (\gamma_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})$$

$p_\mu \Delta_{\mu\nu} = p_\nu \Delta_{\mu\nu} = 0 \Rightarrow$  is "perpendicular" to  $p^\mu$

- Since we are in Coulomb gauge we must have:

$$\sum_i \frac{\partial}{\partial x_i} \langle 0 | A_i(x) A_\nu(y) | 0 \rangle = 0$$

$$\{E_i(\vec{x}), A_j(\vec{y})\} = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \left( \delta_{ij} - \frac{p_i p_j}{p^2} \right)$$

Coulomb gauge

after gauge fixing

$\partial_i E_i = 0$

$\partial_i A_i = 0$

$$\vec{H} = \int d^3 x \frac{1}{2} (\vec{E}^2 + \vec{B}^2)$$



$$\vec{E}^2(p, \lambda) \vec{E}^2(p, \mu) = \delta_{\mu\lambda}$$

$$A_i(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} (a_i(p) e^{ip \cdot \vec{x}} + c.c.)$$

$$\partial_i A_i = 0 \Rightarrow p_i a_i(p) = 0$$

$$\Rightarrow a_i(p) = \sum \lambda E_i(p, \lambda) \quad \left| \begin{array}{l} \text{Not a gauge} \\ \text{solution to EoM} \end{array} \right. \quad \left| \begin{array}{l} \text{Not a co-ord} \\ \text{in phase space} \end{array} \right. \quad \text{Polarisation}$$

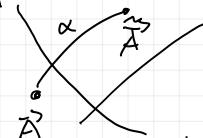
Returning to Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\mathcal{L}$  invariant under  $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$

- In the Hamiltonian gauge transformation generated approach: by Gauss Law  $\int \partial_\mu E_i(x) d^3 x$

$\phi(A_\mu)$



$A_i \rightarrow A_i + \partial_i \alpha$   
polarization  
transformation

- In Lagrangian approach

$$\phi(A_\mu + \partial_\mu \alpha) = 0 = \phi(A_\mu) - \int d^3 x \frac{\partial \phi}{\partial A_\mu} d^4 x \cdot \alpha$$

Lorenz gauge  $\partial_\mu A_\mu = 0$

- Lorentz invariant gauge

$$\partial_\mu (A_\mu + \partial_\mu \alpha) = 0 = \partial_\mu A_\mu + \partial_\mu \partial_\mu \alpha$$

$$\Rightarrow \square \alpha = 0 \quad \left| \begin{array}{l} \text{This has no solutions} \\ \text{only } \alpha = 0? \\ \text{even though before} \\ A \text{ had infinitely many} \\ \text{solutions} \end{array} \right.$$

If we Wick rotate

$$t = i\tau \Rightarrow (-\vec{k}^2 - \Delta) \alpha = 0$$

Now only has one solution:

$\alpha = 0$  Now a "good" gauge

$$\langle 0 | T (A_\mu(x) A_\nu(y)) | 0 \rangle = D_{F, \mu\nu}(x-y) = \gamma_{\mu\nu} D_F(x-y)$$

$$= \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \frac{k^2 \gamma_\mu \gamma_\nu - (k \cdot \gamma)^2 (K_\mu K_\nu + K_\nu K_\mu) + K_\mu K_\nu}{(k \cdot \gamma)^2 - k^2}$$

Now we do this for Lorenz gauge

$$\langle 0 | T (A_\mu(x) A_\nu(y)) | 0 \rangle = D_{F, \mu\nu}(x-y) =$$

$$= \int \frac{d^3 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i\epsilon} (\gamma_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})$$

- Now we should have  $\partial_\mu \langle 0 | T (A_\mu(x) A_\nu(y)) | 0 \rangle = 0$

$$\text{Let us call } \Delta_{\mu\nu} = (\gamma_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})$$

$p_\mu \Delta_{\mu\nu} = p_\nu \Delta_{\mu\nu} = 0 \Rightarrow$  is "perpendicular" to  $p^\mu$

Peskin drops the  $\frac{p_\mu p_\nu}{p^2}$   
and ends up with right answers later due to magic

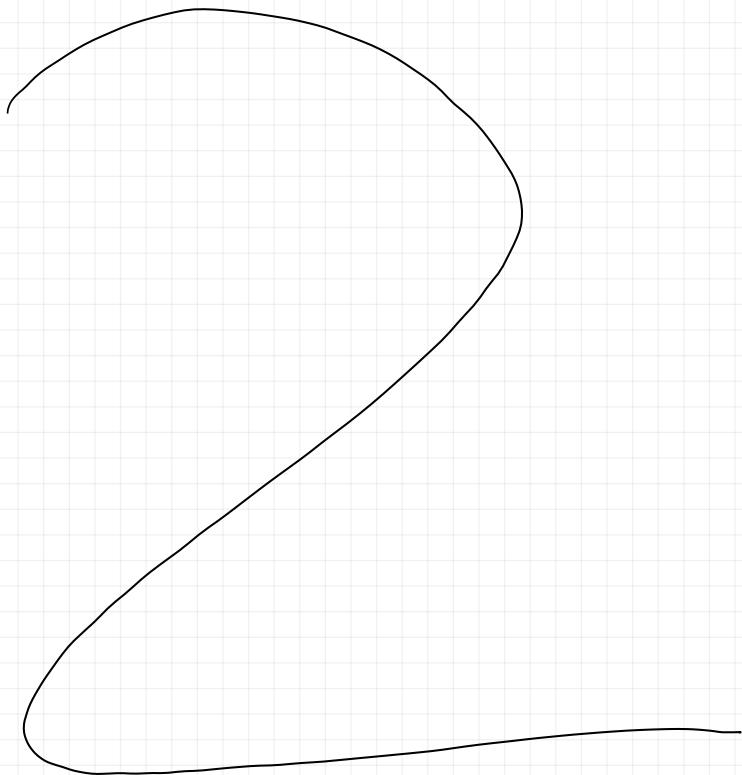
Why does Coulomb gauge which is not Lorentz invariant give right answer

$$\begin{aligned} \partial_i A_i &= 0 & x^\mu &\rightarrow \Lambda_\nu x^\nu \\ A_\mu &\rightarrow \Lambda_\mu^\nu A_\nu = \Lambda_0^\nu A_0 + \Lambda_i^\nu A_i & A_0 &= \Lambda_0^\nu A_\nu \\ && \partial_i^\nu &\rightarrow \Lambda_i^\mu \partial_\mu^\nu \end{aligned}$$

- Lorentz transformation will change Coulomb gauge

$A \rightarrow A' + \partial \alpha$  can choose  $\alpha$  to make  $(A' + \partial \alpha)$  satisfy Coulomb gauge

S F M E S T E R



Interacting fields We studied:  $(\underset{s=0}{KG}, \underset{s=\frac{1}{2}}{\text{Dirac}}, \underset{s=1}{\text{Maxwell}})$

for each we calculated propagators

(a) Start with  $\phi^4$

A  $\xrightarrow{\quad}$  B

- (b) QED  $\equiv$  Dirac + Maxwell
- (c) Gauge Symmetry
- (d) Yukawa Theory ( $KG + \text{Maxwell}$ )
- (e) Renormalization

Dimensions of fields  
 $[\phi] = 1$       Recall  $S = \int d^4x \mathcal{L}(\dots)$   
 $[\psi] = \frac{3}{2}$       for  $t=1$ ,  $S = \text{dimensionless}$   
 $[A_\mu] = 1$        $\Rightarrow S \sim m^4 \sim L^4$   
 $\mathcal{L} = \frac{m^2 \phi^2}{2}$

Propagators

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

call  $|-\lambda\rangle$  the ground state in interaction, may not be the same as  $|0\rangle$   
 $\phi$  not the same       $\phi$ - not the same

$$\text{Goal: } \langle -\lambda | \phi(x) \phi(y) | -\lambda \rangle$$

Strategy,  $D_F(x-y)[\lambda]$  in terms of  $D_F^\circ(x-y) + i\epsilon + \lambda^4(\dots)$   
 $\downarrow$  Free propagator

-power series in  $\lambda$ , where  $\lambda \phi^4 \in \mathcal{L}$

$$H = H_0 + H_{\text{int}} = H_0 + \int d^3x \frac{\lambda}{4!} \phi^4$$

$$\phi(x, t) = e^{iH_0(t-t_0)} \phi(x, t_0) e^{-iH_0(t-t_0)}$$

$\frac{i}{\partial t} \frac{\partial \phi}{\partial t} = [H, \phi]$

$$\phi(x, t) = e^{iH_0(t-t_0)} \phi(x, t_0) e^{-iH_0(t-t_0)} \equiv \phi_I(x, t)$$

$\lambda=0$

free field in interaction picture

$$i \partial_t U(t, t_0) = e^{iH_0(t-t_0)} \left( i H_0 - i t \right) e^{-iH_0(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)}$$

with this we can consider this

$$H_I(\phi_I) \quad (\text{think of power series})$$

$$= i \partial_t U(t, t_0)$$

$$\phi_I(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( a(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x} \right)$$

$$\phi(x, t) = \underbrace{U^\dagger(t, t_0)}_{U^\dagger(t, t_0)} \phi_I(x, t) \underbrace{U(t, t_0)}_{U(t, t_0)}$$

$$\phi(x, t) = U^\dagger(t, t_0) \phi_I(x, t) U(t, t_0)$$

$$U(t, t_0) = I + (-i) \lambda \int_{t_0}^t H_I(\phi_I(t_1)) dt_1$$

Assuming

$$\phi_I(x, t_0) = \phi(x, t_0)$$

$$\Rightarrow U = U(\phi_I)$$

$$\Rightarrow \text{we have } \phi(\phi_I)$$

For this we have a BC  $U(t, t) = 1$

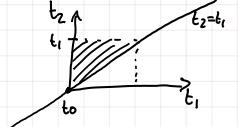
$$+ (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2)$$

$$+ (-i) \lambda H_I(t) \cdot I + (-i)^2 H_I \int_{t_0}^t H(t_2) dt_2$$

$$= \lambda H_I \left( I + (-i) \lambda \int_{t_0}^t H_I(t_2) dt_2 \right) = \lambda H_I U(t, t_0)$$

$$\frac{\partial}{\partial x} \int_0^x f(x') dx' = f(x)$$

$$\begin{aligned}
 & + \dots + \frac{1}{n!} \int_{t_0}^t dt_n \int_{t_0}^{t_n} dt_{n-1} \dots \int_{t_0}^{t_2} dt_1 H(t_n) \dots H(t_1) \\
 \Rightarrow & \int_{t_0}^t dt_1 \int_{t_0}^{t_1} H_I(t_1) + H_I(t_2) = \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 T(H_I(t_1) H_I(t_2)) \\
 \Rightarrow & U(t, t_0) = I + (-i) \int_{t_0}^t H_I(\phi_I(t_1)) dt_1 + \frac{(-i)^2}{2!} \int_{t_0}^t \int_{t_0}^{t_1} dt_1 dt_2 T(H_I(t_1) H_I(t_2))
 \end{aligned}$$


  
 - half this half of the integral over the square as the integral is symmetric

This is because

$$\begin{aligned}
 & T\left(\frac{1}{n!} \int_{t_0}^t dt_1 \dots dt_n (H_I(t_1) \dots H_I(t_n))\right) \\
 & = \frac{1}{n!} \left( -i \int_{t_0}^t H_I(t') dt' \right)^n
 \end{aligned}$$

Unique solution  $\Rightarrow H_I = \frac{\lambda}{4!} \phi^4$

$$U(t, t_0) = T \exp \left\{ -i \int_{t_0}^t H_I(t') dt' \right\}$$

- Not exactly exponent,  $T$  goes

inside integral when we expand?

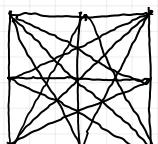
Properties

$$\begin{aligned}
 1. \quad U(t_1, t_2) U(t_2, t_3) &= U(t_1, t_3) \quad \text{— Causality} \quad \rightarrow e^{-i \int_{t_1}^{t_2} H(t) dt} \cdot e^{\int_{t_2}^{t_3} H(t) dt} = e^{-i \int_{t_1}^{t_3} H(t) dt} \\
 2. \quad U(t_1, t_3) U^\dagger(t_2, t_3) &= U(t_1, t_2)
 \end{aligned}$$

$|L\rangle$ ?

$$\langle L | T(\phi(x) \phi(y)) | L \rangle$$

We assume we can write  $|L\rangle$  in terms of  $|0\rangle$



$$e^{-iHT}|0\rangle$$

If interacting theory has good Hilbert Space

$$\Rightarrow |L \times L\rangle + \sum_n |n \times n\rangle = |L\rangle$$

excited space

$$= e^{-iHT}|0\rangle = e^{-iE_0 T}|L \times L\rangle + \sum_n e^{-iE_n T}|n \times n\rangle \quad E_n > E_0$$

$$\lim_{T \rightarrow \infty (1-i\varepsilon)} \left( \frac{e^{-iHT}|0\rangle}{e^{-iE_0 T}|L\rangle} \right) = |L\rangle$$

we can add to here

$$\lim_{T \rightarrow \infty (1-i\varepsilon)} \frac{e^{-H(t_0 - (-T))}}{e^{-iE_0(t_0 - (-T))}} \frac{e^{-iH_0(T-t_0)}}{|L\rangle} |0\rangle = \lim_{T \rightarrow \infty} \frac{U(t_0, -(-T))|0\rangle}{e^{-iE_0(t_0 - (-T))}}$$

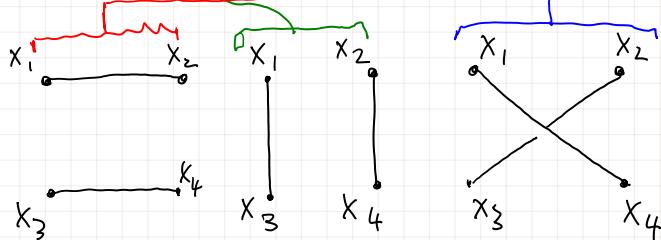
Missing some lectures here that got deleted :)

Monday 27<sup>th</sup> Jan

Wick's Thm:

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) | 0 \rangle = D_F(x_1 - x_2) D_F(x_3 - x_4)$$

$$+ D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3)$$



Recall we want to calculate

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

$$= \langle 0 | T \left\{ \phi(x) \phi(y) e^{-i \int_{x_0}^{y_0} H_I(\phi) dt} \right\} | 0 \rangle$$

↑ interacting  
Free fields

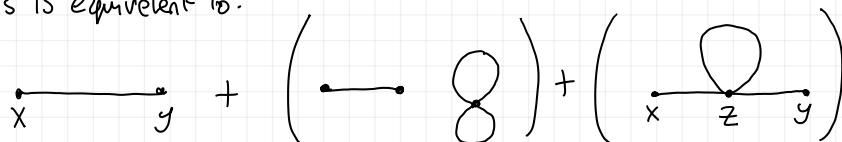
↓ expanding to first order

$$= \langle 0 | T \left\{ \underbrace{\phi(x) \phi(y)}_{D_F(x-y)} - i \phi(x) \phi(y) \int H_I(\phi) dt \right\} | 0 \rangle$$

$$H_I = \int d^3x \frac{\lambda}{4!} \phi^4$$

$$\begin{aligned} & 3 \left( -\frac{i\lambda}{4!} \right) D_F(x-y) \int d^4z D_F(z-z) D_F(z-z) \\ & \text{number of contractions?} + 12 \left( -\frac{i\lambda}{4!} \right) \int d^4z D_F(x-z) D_F(y-z) D_F(z-z) \end{aligned}$$

- This is equivalent to:



Third order in  $\lambda$

$$+ \frac{1}{3!} \left( \frac{-i\lambda}{4!} \right) \int H_I \int H_I \int H_I \dots$$

From baryon expansion

$$\langle 0 | T \{ \phi(x) \phi(y) \frac{1}{3!} \left( \frac{-i\lambda}{4!} \right)^3 \int d^4z \phi \phi \phi \phi(z) \int d^4w \phi \phi \phi \phi(w) \int d^4u \phi \phi \phi \phi(u) \} | 0 \rangle$$

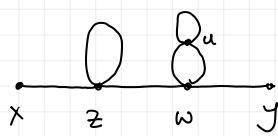
One possible contraction:

$$\langle 0 | T \{ \phi(x) \phi(y) \frac{1}{3!} \left( \frac{-i\lambda}{4!} \right)^3 \int d^4z \phi \phi \phi \phi(z) \int d^4w \phi \phi \phi \phi(w) \int d^4u \phi \phi \phi \phi(u) \} | 0 \rangle$$

$$= \frac{1}{3!} \left( \frac{-i\lambda}{4!} \right)^3 D_F(x-z) D_F(z-z) D_F(z-w) D_F(w-u) D_F^2(w-u) D_F(u-u)$$

$$(3!) \text{ in interchange of vertices} \times \left( \begin{array}{l} \text{placement of} \\ \text{contraction in vertex} \end{array} \right) \times \left( \begin{array}{l} \text{placement of} \\ \text{contraction in} \\ w \text{ vertex} \end{array} \right) \times \left( \begin{array}{l} \text{4x3 placement} \\ \text{of contraction} \\ \text{in u} \end{array} \right) \times \frac{1}{2}$$

over counting correction  
( $w \leftrightarrow u$ ) due to  $D_F^2(w-u)$  term



The  $\frac{1}{n!}$  factorial from the baryon expansion

will always cancel the  $n!$  from the number of vertex arrangements

Also each vertex  $\phi \phi \phi \phi$

$$\cdots \frac{1}{4!}$$

$$\Leftrightarrow 4! \cdot \frac{1}{4!}$$

- with this we are just left with a "Symmetry" factor

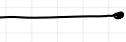
$$\underline{\textcircled{1}} \rightarrow S' = 2$$

$$\underline{\infty} \rightarrow S' = 8$$

$$\underline{\textcircled{1}} \rightarrow S' = 3! = 6$$

$$\underline{\textcircled{1}} \rightarrow S' = ?$$

## Feynmann Rules

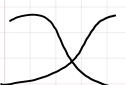
1: propagator  =  $D_F(x-y)$

2:  =  $(-i\lambda) \int d^4z = \int \frac{d^4p}{(2\pi)^4} \frac{-i e^{ip(x-y)}}{p^2 - m^2 + i\epsilon}$

3: External point  = 1

4: Divide by symmetry factor  $S$

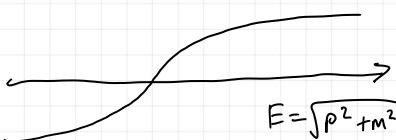
$D_F(z-z) \leftarrow$  corresponds to a loop.

  $\int d^4z e^{-(\sum p) \cdot z}$

recall we have a limit

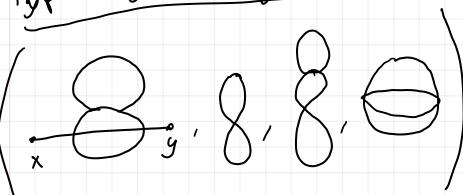
$$\lim_{T \rightarrow \infty} \int_0^T dt \int d^3z e^{-i(\sum p)z} = \lim_{T \rightarrow \infty} (1-i\epsilon) \int_{-T}^T dz^0 \int d^3z e^{-i(\sum p) \cdot z}$$

- This evaluated with  $\sum \vec{p} = 0$  blows up in the limit



- Every disconnected piece will have  $2T \cdot V$

## Typical Feynmann diagram



⇒ Just left with connected diagrams:

$$= \left( \text{---} + \text{---} + \text{---} + \dots \right)$$

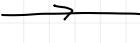
$$\exp\left(\sum V_i\right) \sim \exp[-i E_0(2T)]$$

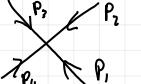
$$\sum V_i \sim 2T V$$

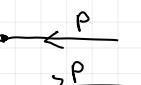
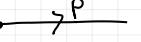
$$\frac{E_0}{V} = i \left[ 8 + 8 + \text{---} + \dots \right]$$

$\underbrace{\delta^{(4)}(0)}_{\hookrightarrow V \cdot 2T} (2\pi)^4$

## Momentum Space

1: For each propagator   $\frac{i}{p^2 - m^2 + i\epsilon}$

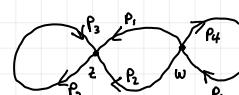
2: Vertex  =  $-i\lambda$

3: External point  =  $e^{-ipx}$   
 =  $e^{ipx}$

4: Each vertex has momentum conservation  $\delta^4(p)$

5: Integrate over all momentum

6: Divide by Symmetry factor.



$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \delta^{(4)}(p_4 - p_4 - p_2 - p_1) \\ = \delta^{(4)}(p_1 + p_2) \delta^{(4)}(0)$$

$$\delta^{(4)}(0) = \int d^4w \cdot e^0 = \int d^3w \int dt \cdot 1 = \text{Volume of } 2T \equiv T \cdot 2T$$

Blows up in limit

## Vacuum diagrams

$$V_i \in \{8, 8, \dots\}$$

- Basis of diagram with no external legs.

- Suppose our FD has  $N_i$  pieces of type  $V_i$

$$(\text{Value of connected part}) \times \prod_i \frac{1}{n_i!} V_i^{n_i}$$

- 2-point function

$$= \sum_{\text{all possible connect}} \sum_{\text{all legs}} (\text{Value of conn}) \times \prod_i \frac{1}{n_i!} V_i^{n_i}$$

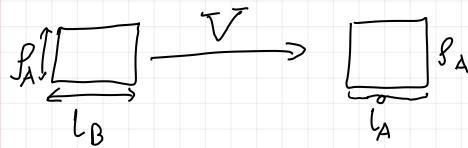
$$= \sum_{\text{all conn}} (\text{value of}) e^{\sum V_i}$$

- Denominator is

$$= \exp[\text{Vacuum}]$$

$$\langle \mathcal{L} | T \{ \phi(x_1) \dots \phi(x_n) \} | \mathcal{L} \rangle = \left( \sum_{\text{connected Feynmann diagrams}} \right)$$

# - Cross Section & S-Matrix



$$\sigma = \text{cross section}$$

$$= \frac{\text{total # events}}{\sigma_A l_A \sigma_B l_B \cdot A}$$

Area where "things" happen

$$\text{Number of } \# = \sigma \int_A \int_B \int_X \sigma_A(x) \rho_B(x) d^3x \left| \frac{d\sigma}{d\Omega^{(n)}} \right|$$

$$\text{For const. } \sigma \quad \#N = \frac{\sigma N_A N_B}{A}$$

- There is also a decay rate

$$\Gamma = \frac{\text{Number of decays/unit time}}{\text{Nume of A particle present}} = \text{const}$$

Wave packet:

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k) |k\rangle$$

- Free particle  $\rightarrow |k\rangle = \sqrt{2E_k} e^{ikx} |0\rangle$

$$\langle \phi | \phi \rangle = 1 \Rightarrow \int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 = 1$$

- Want to calculate probability of an out state, given an in state

$$P = \left| \underbrace{\langle \phi_1 \phi_2 \dots}_{\text{future}} \underbrace{|\phi_A \phi_B \rangle}_{\text{past}} \right|^2$$

$$|\phi_A \phi_B\rangle = \int \frac{d^3 k_A}{(2\pi)^3} \frac{d^3 k_B}{(2\pi)^3} \frac{\phi_A(k_A) \phi_B(k_B)}{\sqrt{2E_A 2E_B}} |k_A k_B\rangle$$

$$|\phi_A \phi_B\rangle_{in} \quad |p \dots p\rangle_{out}$$

- What we need to calculate is:

$T \rightarrow +\infty \quad T \rightarrow -\infty$  - Evolution operator governs the transition in time

(- Some Hilbert spaces)

$$= \lim_{T \rightarrow \infty} \langle_{in} p_1 \dots p_n | e^{-iH(2T)} |_{in} k_A k_B \rangle$$

$$= \langle p_1 \dots p_n | S | k_A k_B \rangle$$

We define the S matrix in this way.

- Needs to include the case when nothing happens

$$\Rightarrow S = 1 + iT$$

- Definition of T matrix

Monday 10/2/25 !

$$\langle p_1 \dots p_n | iT | k_A k_B \rangle = (2\pi)^4 S^{(4)}(k_A + k_B + \sum_i p_i) \times M(k_A, k_B \rightarrow p_f)$$

Amplitude

On shell:

$$p^0 = E_p = \sqrt{p^2 + m^2}$$

off shell anything else

out state

$$\langle \phi_1 \dots \phi_f | = \prod_i \int \frac{d^3 p_i}{(2\pi)^3} \frac{\phi_f(p_i)}{\sqrt{2E_f}} \langle p_1 \dots p_f |$$

$\Rightarrow$  Need to calculate  $\langle p_1 \dots p_f | k_A k_B \rangle$

$$= \langle p_1 \dots p_f | S | k_A k_B \rangle$$

If  $S = 1$  nothing happens

$$\Rightarrow S = 1 + iT$$

$\Rightarrow$

$$P(AB \rightarrow 1, 2, \dots, n) = \left( \prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f} \left| \langle p_1 \dots p_n | \phi_A \phi_B \rangle \right|^2 \right)$$

$$\tilde{N} = \sum P = \int d^3 b n_B P(b)$$

$n_B = \# \text{ of particles/unit area}$

$$\sigma = \frac{\tilde{N} A}{N_A N_B}$$

$$d\sigma = \frac{1}{2E_A^2 E_B} \frac{1}{|v_A - v_B|} \left( \int_{\text{phase space}} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| M(k_A k_B \rightarrow \sum p_f) \right|^2 (2\pi)^4 \delta(k_A + k_B - \sum p_f)$$

We can perform a calculation in the relativistic case where  $m_A = m_B = m_1 = m_2 = 0$

$$\frac{dp}{dx} (2 \rightarrow 2) = \text{diff cross section}$$

$$d^3 p = p^2 d\Omega$$

S Matrix

$$\langle p_1 \dots p_n | S | k_A k_B \rangle$$

$$= \lim_{T \rightarrow \infty} \langle p_1 \dots p_n | e^{-iH(2T)} | k_A k_B \rangle$$

$$H = H_0 + H_{\text{int}}$$

↑ small number

$$= \lim_{T \rightarrow \infty} \langle p_1 \dots p_n | e^{-i \int_{-T}^T H(t) dt} | k_A k_B \rangle$$

$$\langle p_1 \dots p_n | iT | k_A k_B \rangle$$

$$= \lim_{T \rightarrow \infty} \left( \langle p_1 \dots p_n | e^{-i \int_{-T}^T H(t) dt} | k_A k_B \rangle_0 \right)$$

connected amplitude

For this we know the vacuum free

$$|\Omega\rangle = \lim_{T \rightarrow \infty} \frac{e^{iHt} |0\rangle}{e^{iE_f T}} \langle 0 | 0 \rangle$$

- We can write eigenstate  $|k_A k_B\rangle$  in terms of this state in the vacuum

$$|k_A k_B\rangle \sim \lim_{T \rightarrow \infty} e^{-iHt} |\Omega\rangle$$

Somson takes a minute to ask  
Dan if he is recording him

$$= (\delta^{(4)}(k_A + p_1) \delta^{(4)}(k_B - p_2) + \delta^{(4)}(k_B - p_1) \delta^{(4)}(k_A - p_2)) \sqrt{2E_A 2E_B 2E_C 2E_D}$$

- This can be written as!

$$\sigma = \frac{|M|^2}{64\pi^2 E_{\text{cm}}^2}$$

Lowest order

$$\langle p_1 p_2 | k_A k_B \rangle = \sqrt{2E_A 2E_B 2E_C 2E_D} |a(k_1) a(k_2) a^\dagger(k_3) a^\dagger(k_4)| 0 \rangle$$

- We can then compute the creation/annihilation operators

$$\begin{array}{c} | \\ A \end{array} \quad \begin{array}{c} | \\ B \end{array} \quad + \quad \begin{array}{c} | \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array}$$

If we have interaction:

$$\langle p_1 p_2 | \left( -i \int d^4 x \frac{\lambda}{4!} \phi(x) \right) | k_A k_B \rangle$$

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} (a_p e^{-ipx} + a^\dagger_p e^{ipx})$$

$$\Rightarrow \phi(x) \sqrt{E_k} a_k |0\rangle = \dots = e^{-ikx} |0\rangle$$

contraction  $\rightarrow = \phi(x) |k\rangle$

$$\langle k | \phi(x) = e^{ikx}$$

$$\langle p_1 p_2 | \phi \phi \phi \phi | k_A k_B \rangle$$

$$\begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} + \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} \quad \text{Case ①}$$

$$= \frac{1}{4!} \cdot 4! (-i\lambda) \int d^4 x e^{-i(k_A + k_B - p_1 - p_2)} = -i\lambda (2\pi)^4 \delta^{(4)}(k_A + k_B - p_1 - p_2)$$

$$\text{But recall } \langle i | T | \rangle = i M(\rightarrow) \delta^{(4)}(\text{total mom})$$

$$\Rightarrow \frac{d\sigma}{dx} = \frac{x^2}{64\pi^2 E_{\text{cm}}}$$

- We want to do all possible contractions Possibilities

- ① No  $\phi\phi$  contractions  $\phi\phi\phi\phi \rightarrow$  all  $\phi$ 's contracted on all states
- ② 1  $\phi\phi$  contraction  $\phi\phi\phi\phi \rightarrow$
- ③ All  $\phi\phi$  contractions  $\phi\phi\phi\phi \rightarrow$  states contracted themselves

$$\langle p_1 p_2 | \phi \phi \phi \phi | k_A k_B \rangle$$

$$\begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} + \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} \quad \text{Case ②}$$

$$+ \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} + \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array}$$

3 8 (Zeroeth order)

$$\begin{array}{c} | \\ A \end{array} \quad \begin{array}{c} | \\ B \end{array} \quad + \quad \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array}$$

$$\begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} = \frac{1}{4!} \cdot 4! (-i\lambda) \int d^4 x e^{-i(k_A + k_B - p_1 - p_2)} = -i\lambda (2\pi)^4 \delta^{(4)}(k_A + k_B - p_1 - p_2)$$

$$\langle p_1 p_2 | iT | k_A k_B \rangle$$

$$= \begin{array}{c} \times \\ A \end{array} + \begin{array}{c} \times \\ B \end{array} + \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} + \begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} + \dots$$

- Only connected diagrams contribute

We also get rid of "amputated diagrams" i.e. one with



$$\begin{array}{c} \times \\ A \end{array} \quad \begin{array}{c} \times \\ B \end{array} = \frac{\lambda^2}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{p'^2 - m^2 + i\epsilon} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} (-i\lambda) \delta^{(4)}(p_1 + p_2 - k_A - k_B) \times (-i\lambda) \delta^{(4)}(k_B - p' + * + *)$$

$$= \frac{1}{(2\pi)^4} \frac{1}{k_B^2 - m^2 + i\epsilon}$$

This is not good as  $B$  is onshell  $\Rightarrow k_B^2 = m^2 \Rightarrow = \frac{1}{0}$

$$i M S^{(4)}(k_A + k_B - \sum p_f)$$

= (sum of all connected, amputated FD with  $k_A k_B$  incoming &  $\sum p_f$  outgoing)

Rules

$$1) \quad x \rightarrow y = D_F(x-y)$$

$$2) \quad \times = -i\lambda \int d^4 x$$

$$3) \quad \rightarrow = e^{-ipx}$$

4 divide by symmetry factor

5 integrate over all momentum

## Fermions

$$\textcircled{1} \quad T\{\psi(x)\bar{\psi}\} = \begin{cases} \psi(x)\bar{\psi}(y) & x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & x^0 < y^0 \end{cases}$$

$$\textcircled{2} \quad S_F(x-y) = \langle 0 | T\{\psi(x)\bar{\psi}(y)\} | 0 \rangle = \boxed{\int \frac{d^4 p}{(2\pi)^4} \frac{i(\sigma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}}$$

$$T\{\psi_1 \dots \psi_n\} = (-1)^{k-1} \psi_k \psi_1 \dots \psi_{n-1}$$

- Normal ordering:

$$N(a_p a_q a_k^\dagger) = (-1)^3 a_k^\dagger a_q a_p$$

$$T\{\psi(x)\bar{\psi}(y)\} = N(\psi(x)\bar{\psi}(y) + \bar{\psi}(x)\psi(y))$$

$$N(\bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_3 \bar{\psi}_4) = - \bar{\psi}_1 \bar{\psi}_3 N(\bar{\psi}_2 \bar{\psi}_4)$$

## Yukawa theory

$$H = H_{\text{Dirac}} + H_{\text{KG}} + g \int d^3 x \psi(x) \phi(x) \bar{\psi}(x)$$

$$f_p + f_K \rightarrow f_{p'} + f_{K'}$$

- We care only about even powers of  $g$ ? can only contract even terms

$\Rightarrow g^2 = \text{first order}$

$$= \frac{(-ig)^2}{2!} \underbrace{\zeta_{P'K'}}_0 \left| T \left\{ \int d^4 x \bar{\psi}(x) \phi(x) \psi(x) \int d^4 y \bar{\psi}(y) \phi(y) \psi(y) \right\} | P, K \right\rangle \right. \text{ fermions} \quad \text{fermions} \\ \text{only possible} \quad \text{Boson contraction}$$

Most contract  $P$ 's and  $K$ 's with fermion fields  $\psi$  otherwise the terms vanish why?

$$= \frac{(-ig)^2}{2!} \underbrace{\zeta_{P'K'}}_0 \left| T \left\{ \int d^4 x \bar{\psi}(x) \phi(x) \psi(x) \int d^4 y \bar{\psi}(y) \phi(y) \psi(y) \right\} | P, K \right\rangle \right. \text{ spin indices} \\ = (-ig)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(p' - p - q) \times (2\pi)^4 \delta^{(4)}(k' - k - q) \bar{U}^s(p) U^q(q) \bar{U}^r(k') U^t(k) \\ \text{case ①} \quad \text{case ②} \quad = (-ig)^2 \delta(p' - p - k' + k) \frac{i}{(p' - p)^2 - m_\phi^2 + i\epsilon} \bar{U}^s(p) U^q(p) \bar{U}^r(k') U^t(k) \\ \text{case ①} \quad \text{case ②} \quad \text{same but } k' \leftrightarrow p' \quad \text{This swapping gives us a minus sign}$$

$$\psi(x)|P, S\rangle$$

$$= \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_s a^s(q) u^s(q) \times \sqrt{2E_p} a^{s\dagger}(p) | 0 \rangle \\ = e^{i p \cdot q} u^s(p) \equiv \bar{\psi}|P, S\rangle$$

$$\bar{\psi}|P, S\rangle = e^{ipx} \bar{v}^s$$

## Rules in momentum space

$$1. \quad \overline{\phi} \phi = \dots \rightarrow = \frac{i}{q^2 - m_\phi^2 + i\epsilon}$$

$$2. \quad \bar{\psi}_{(s)} \psi_{(s)} = \dots = \frac{i(\sigma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}$$

$$3. \quad \text{Vertex} \quad \langle \dots \dots \dots \dots \rangle = -ig$$

$$4. \quad \text{Extended leg: } \overline{\phi}|q\rangle \quad \langle \dots \dots \dots \dots \rangle = 1$$

$$\bar{\psi}|p, q\rangle = \dots \quad U^s(p) \sim \bar{\psi} \rightarrow \bar{v}^s(p)$$

$$5. \quad \text{Momentum conservation}$$

$$6. \quad \text{Integrate over undefined momentum}$$

$$7. \quad \text{divide by symmetry factor} \quad 8. \quad \text{Don't forget } (-)$$

$$\bar{\psi} \psi_x \bar{\psi} \psi_y \bar{\psi} \psi_z \bar{\psi} \psi_w = \dots = -\text{tr} [S(y-x)S(x-w)S(w-z)S(z-y)]$$

- we get a minus for each swap we have to make to get terms beside each other.

- Yukawa potential - Non relativistic limit

$$p = (m, \vec{p}) \quad k = (m, \vec{k})$$

$$p' = (m, \vec{p}') \quad k' = (m, \vec{k}')$$

$$(p' - p)^2 = -|\vec{p} - \vec{p}'|^2 + O(p^4)$$

$$U^s(p) \Rightarrow \sqrt{m} \left( \frac{\xi^s}{3} \right), \quad \xi^s \xi^s = \delta^{ss}$$

$$\bar{U}^s(p) U^s(p) = 2m \xi^s \xi^s = 2m \delta^{ss}$$

$$iM = \frac{ig^2}{(p^2 - p'^2 + m_\phi^2)}$$

Born Oppenheimer approximation

$$V(x) = \int \frac{dk_p}{(2\pi)^4} e^{ipx} \tilde{V}(p)$$

$$\langle p' | T | p \rangle = -\tilde{V}(p) 2\pi \delta(E_{p'} - E_p) \quad q = p' - p$$

- For Yukawa:

$$\tilde{V}(q) = \frac{-g^2}{|q^2 + m_\phi^2|} \Rightarrow \text{Fourier}$$

$$V(x) = \frac{g^2}{4\pi r} \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q^2 + M_\phi^2} \quad \text{2 poles } q = \pm im_\phi$$

$$\text{angular something} \quad \dots = \frac{1}{q^2 + m_\phi^2} = \frac{1}{2im} \left( \frac{1}{q-im_\phi} - \frac{1}{q+im_\phi} \right)$$

$$\dots = \left[ -\frac{g^2}{4\pi} \frac{e^{-M_\phi r}}{r} \right]$$