QFT

Lecture 1

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Books: intro to QFT
- Peskin & Schoeder → (main book)
- Schweber → Repesentations of Lorentz groups "Only look at it for 2 hours"
- Dirac → Lectures on QM

Finish chapter do all Q's will only do (1/3) of it.

For quantization of constrained systems "maxwell Theory"

I - semester - Free fields
II - interacting fields

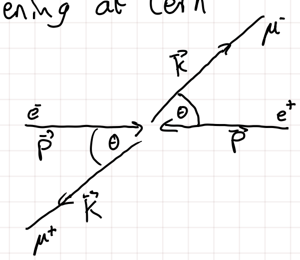
Einstein:
- SR - 1905
- QM - 1905 (photo electric affect
- GR - 1915

QFT

"knock offs"
Frenchmen & Mathematicians say pioncare
Plank, Heizenburg, Schr, De Broglie, Mathematicians say Hilbert

Fast → Reletivistic
QM
Happening at Cern

QFT trailer



2 in, 2 out ⇒ planar

Notation:
$\hbar = 1$
$c = 1$
"God given units"

$\Rightarrow E = \sqrt{p^2 + m^2}$

M_{e^-, e^+}, M_{μ^-, μ^+}

$$\frac{M_e}{M_\mu} \sim \frac{1}{200}$$

$p_0^2 - p^2 = m^2$, $p_0 \equiv E$

$E_e = \sqrt{p^2 + m_e^2}$
$E_\mu = \sqrt{k^2 + M_\mu^2}$

$$E_{init} = 2E_e = 2\sqrt{\vec{p}^2 + m_e^2}$$
$$E_{final} = 2E_\mu = 2\sqrt{\vec{k}^2 + M_\mu^2}$$

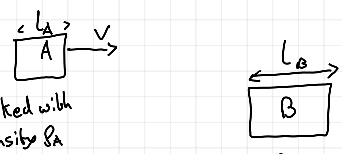
Conservation of energy $E_{init} = E_{final}$, $m_\mu > m_e \Rightarrow |\vec{k}| < |\vec{p}|$

Assume (fast collision) $E > m_e, m_\mu$ $\quad 2E = 2|\vec{p}| = 2|\vec{k}|$

Cross section $\sigma = \dfrac{\text{Number of events}}{\rho_A \rho_B \ell_A \ell_B \cdot A}$

planar process with overlap of Area A?

Buches of electrons



Packed with density ρ_A

ρ_B

Diferential cross section

$[\text{mass}] = [\text{Energy}] = [P]$

$$\frac{d\sigma}{d\Omega_{(3)}}$$

$[\text{Length}] = [L]$

$\Delta p \Delta x \sim 1$
$\Rightarrow [M] \cdot [L] \sim 1$

$\Delta E \cdot \Delta t \sim 1$
$\Rightarrow [E] \cdot [t] \sim 1$
$\Rightarrow [t] = [L]$

$$\underbrace{\rho_A \rho_B \ell_A \ell_B \cdot A}_{[M]^2}$$

$\Rightarrow [\sigma] = \dfrac{1}{[m^2]}$

$(E > m_e, m_\mu)$

$$\Rightarrow \left[\frac{d\sigma}{d\Omega}\right] = \left[\frac{1}{E^2}\right]$$

dont know proportionality

$$\frac{d\sigma}{d\Omega_3} = \frac{1}{64\pi^2}\frac{1}{E^2}|M|^2$$, what does M depend on?.

Cannot depend on energy $\Rightarrow M = M(\theta)$

for us $\frac{M}{E}=0$
$\Rightarrow$ cannot create Dimensionless quantity (cant involve E either)

Is cases where $M(\frac{M_p}{E}, \frac{M_e}{E}, \theta)$

more complicated

we expect from QM

interacting Hamiltonian

$$M \sim \langle \text{final state} | H_I | \text{inital state} \rangle$$

e^-, e^+ dont interact with μ^-, μ^+

$\Rightarrow \langle \text{final} | H_I | \text{inital} \rangle = 0$

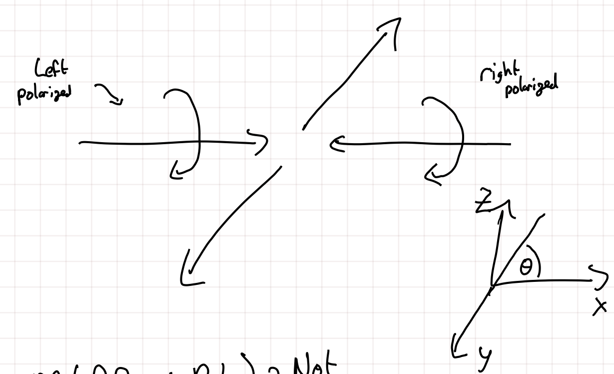
light

$$M \sim \sum_\alpha \langle e^+e^- | H_I | \gamma \rangle^\alpha \langle \gamma |_\alpha H_I | \mu^+\mu^- \rangle$$

Light is a vector so need to sum components

- Also need to include spin, can think of this as angular momentum
- easiest to average over all spin

$$\frac{1}{2}\sum_{\substack{\text{init spin}\\ \text{final spin}}} \frac{d\sigma}{d\Omega}$$



4-vector

$$\varepsilon^\alpha = (0, 1, i, 0)$$

$$\varepsilon^\alpha \varepsilon_\alpha = 0^2 - 1 + 1 - 0 = 0$$

rotated: around z-axis $\tilde{\varepsilon}^\alpha = (0, \cos\theta, i, \sin\theta)$

$$M \sim \sum \underbrace{\langle e^+e^- | H_I | \gamma \rangle^\kappa}_{e\varepsilon^\mu} \underbrace{\langle \gamma |_\alpha H_I | \mu^+\mu^- \rangle}_{e\,\tilde{\varepsilon}_r}$$

charge is strength and $\Rightarrow$ length of interaction

$M(RL \to RL) \sim e^2(1-\cos\theta)$

- $M(RR \to RL)$
$M(LL \to RL)$ } Not Allowed

peskin says +

$$M(RL \to RL) \sim e^2(1-\cos\theta)$$

$M(RR \to RL)$

$M(RR \to LL) = 0$

$M(RL \to LR) \sim e^2(1-\cos\theta)$

$M(LR \to LR) \sim e^2(1+\cos\theta)$

proportionality $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}(1+\cos^2\theta)$$

$$\sim \frac{1}{2}\frac{1}{E_{cm}^2}\sum_{\text{spin}}|M|^2$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$
$$= -d(\cos\theta)\, d\phi$$
$$\theta \in [0, \pi]$$
$$\phi \in [0, 2\pi]$$

Integrating

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega_{(3)}} d\Omega_{(3)} = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

Trailer over

What we want to do:

1. Relaxing condition $E_{cm} \gg M_\mu$

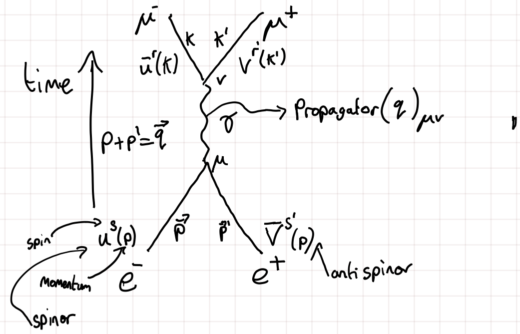
$\Rightarrow \underset{\text{amplitude}}{M} = M\left(\frac{m}{E}, \theta\right)$

2. Relax $E_{cm} \gg m_e$ $\quad M = M\left(\frac{M_\mu}{E}, \frac{M_e}{E}, \theta\right)$

3. Relax $\alpha^2 \ll 1 \Rightarrow$ calculate $\alpha^2, \alpha^6, \ldots$

4. Calculate any other process

These are Feynmann rule



$e, \mu, \nu \ldots$ leptons

$q, \bar{q} \rightarrow$ hadrons

- We ask ourselves the question If we have a particle at x, what is the probability of finding it at x?

This will lead us to thinking about waves and fields instead of particles $\Rightarrow$ Reletivistic wave equation

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Space time: (t, x_1, x_2, x_3), $\quad s^2 = t^2 - x_1^2 - x_2^2 - x_3^2 = t^2 - \vec{x}^2$

$\mathbb{R}^{1,3}$ "Minkovski Space" $= x^\mu \eta_{\mu\nu} x^\nu$, $\quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\langle x | e^{-iH(t-t')} | x' \rangle$

Single free particle of mass m

$\hat{H} = \frac{\hat{p}^2}{2m} \Rightarrow \langle x | e^{-i\frac{\hat{\vec{p}}^2}{2m}(t-t')} | x' \rangle$

In classical mechanics $\{p, x\} = 1$, $[p, x] = -i$

$\hat{x} \rightarrow$ muliply by x

$\hat{p} \rightarrow \frac{1}{i}\frac{\partial}{\partial x}$

recall completness relation

$$\int \frac{d^3p}{(2\pi)^3} |p\rangle\langle p| = \mathbb{1}$$

$$= \int \frac{d^3p}{(2\pi)^3} \langle x | e^{-\frac{\hat{\vec{p}}^2}{2m}(t-t')} | p \rangle \underbrace{\langle p | x' \rangle}_{e^{-ipx'}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\langle x | p \rangle}_{e^{ipx}} e^{-\frac{p^2(t-t')}{2m}} \langle p | x' \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{-\frac{i\hat{\vec{p}}^2}{2m}(t-t')} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

$$= \left(\frac{m}{2\pi i(t-t')}\right)^{\frac{3}{2}} e^{im\frac{(\vec{x}-\vec{x}')^2}{2(t-t')}}$$

when $(t-t')^2 - (\vec{x}-\vec{x}')^2 < 0$

Space like

need to use $\sqrt{p^2+m^2}$

Lecture 4

$s^2 = (t-t')^2 - (\vec{x}-\vec{x}')^2$

$s^2 > 0$ time-like
$s^2 < 0$ space-like
$s^2 = 0$ light-like

- should not have possible space like propogation. QM should not break causality!

- Let us try relativistic Hamiltonian

$$H = \sqrt{\hat{\vec{p}}^2 + m^2}$$

we use the following facts:

$$f(\hat{p})|p\rangle = f(p)|p\rangle \quad , \quad \int \frac{d^3p}{(2\pi)^3}|p\rangle\langle p| = \mathbb{1}$$

$$\langle x|p\rangle = e^{-i\vec{p}\cdot\vec{x}}$$

$$\langle x| e^{-i\sqrt{\hat{\vec{p}}^2+m^2}\, t} |x'\rangle$$

$$= \int \langle x|p\rangle e^{-i\sqrt{\vec{p}^2+m^2}\, t} \langle p|x'\rangle \frac{d^3p}{(2\pi)^3}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{-i\sqrt{\vec{p}^2+m^2}\, t + i\vec{p}(\vec{x}'-\vec{x})}$$

$$\vec{p}\cdot(\vec{x}-\vec{x}') = |\vec{p}||\vec{x}-\vec{x}'|\cos\theta$$

can choose these to be the same

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} = \int \frac{p^2 dp \sin\theta\, d\theta\, d\phi}{(2\pi)^3}$$

Samson uses
$\eta = \cos\theta$
$\Rightarrow d\eta = -\sin\theta d\theta$

$$= \int \frac{p^2 dp}{(2\pi)^2} \int_{-1}^{1} d\eta\, e^{-i\sqrt{p^2+m^2} + ip|\vec{x}-\vec{x}'|\eta}$$

$$= \int_0^\infty \frac{p^2 dp}{(2\pi)^2} e^{-i\sqrt{p^2+m^2}t} \cdot \frac{1}{ip|\vec{x}-\vec{x}'|}\left(e^{ip|\vec{x}-\vec{x}'|} - e^{-ip|\vec{x}-\vec{x}'|}\right)$$

$$= \int_{-\infty}^{\infty} \frac{p^2 dp}{(2\pi)^2} e^{-i\sqrt{p^2+m^2}\, t} \frac{2i}{ip|x-x'|} e^{ip|x-x'|}$$

An Aside: Laplace steepest decent

expanding around a critical point x_0 with $f'(x_0)=0$

$$\int_{-\infty}^{\infty} e^{f(x)} dx = \int_{-\infty}^{\infty} dx\, e^{f(x_0) + (x-x_0)\cancel{f'(x_0)}^{0} + \frac{1}{2}(x-x_0)^2 f''(x_0) + \cdots}$$

$$\simeq e^{f(x_0)} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x-x_0)^2 a}$$

Need to check this

In our case $f(p) = \sqrt{p^2+m^2}\, t + ip|x-x'|$

$$f'(p) = \frac{-p^2}{\sqrt{p^2+m^2}} + |x-x'| = 0 \Rightarrow p^2 = \frac{m(x-x')}{t^2}$$

$$\Rightarrow U(t) = e^{-m\sqrt{(x-x')^2 - t^2}} \neq 0$$

?

Representation of poincare group, $P_\mu P^\mu = m^2$

$$P_0^2 - \vec{P}^2 = m^2$$

$$\Rightarrow E = P_0 = \pm\sqrt{p^2+m^2}$$

what if $E<0$?

a) anti-particals
b) multiparticle states
c) spins & statistics
d) tools to calculate anything

need to go from particles → waves

classical

$$S = \int_t^{t'} L(q_i, \dot{q}_i) dt \qquad i = 1, \dots, N$$

i becomes continuous $\quad i \to \vec{x}$, $\quad q \to \varphi(\vec{x}, t)$

$$\Rightarrow S = \int_t^{t'} dt\, d^3x\, \mathcal{L}\left(\varphi(\vec{x},t), \partial_\mu \varphi(\vec{x},t)\right)$$

Then what are EoM? Need to vary

$$\delta S = \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \varphi} \delta\varphi - \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \delta \partial_\mu \varphi \right\}$$

Integrating by parts on second term

$$= \int d^4x \underbrace{\left(\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \right) \right)}_{\text{Field eqs } = 0} \delta\varphi = 0$$

Lecture 5

$$\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} = 0$$

$$H : L \longrightarrow \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} = p_i \Rightarrow \text{solve} \quad \dot{q}_i = (p, q)$$

↓ Not garenteed

$$H = \sum_i p_i \dot{q}^i - L(q, \dot{q}) = H(p, q)$$

Samson gives the example

$$L = \dot{q} \Rightarrow p = \frac{\partial L}{\partial \dot{q}} = 1$$

$\Rightarrow p = 1$ how to solve for $\dot{q}$? we can't

The condition for us being able to solve $\dot{q}$:

$$\frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^i \partial \dot{q}^j} = M_{ij} \Leftrightarrow \det(M) \neq 0$$

$$L = \sum_j M_{ij} \dot{q}^i \dot{q}^j, \quad \dot{q}^i = \sum_j M^{-1\, ij} p_j$$

$$p = \frac{\partial L}{\partial \dot{q}^i} = M_{ij} \dot{q}^j$$

, for this we need M to have an inverse

This we call a non-degenerate Lagrangian

$$H = \sum p_i \dot{q}^i - L = H(p, q, \dot{q})$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{-i}} = -i\sqrt{\pi}$$

$$dH = \sum_i \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial \dot{q}^i} d\dot{q}^i \right)$$

$$= \sum \left[(\dots) dq^i + \dot{q}^i dp_i + \left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right) d\dot{q}^i \right]$$

But our construction of H is such that it is not a function of $\dot{q}^i$

$$\Rightarrow \left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right) = 0$$

Back to field theory

$$S = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi) = \int L(\dots)\, dt$$

$$p(x) = \frac{\partial L}{\partial \dot{\varphi}} = \Pi(x) \underbrace{d^3x}_{\text{volume form}}$$

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}, \quad H = \int \mathcal{H}\, d^3x$$

$$= \int \left(\Pi(x) \dot{\varphi}(x) - \mathcal{L} \right) d^3x$$

What is the simplest Lagrangian we can think of?

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$$

$\mu = 0,1,2,3$

We can lower and raise indices with minkowski metric $\eta^{\mu\nu}$

$$= \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

What is Hamiltonian?

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \dot{\varphi}(x)$$

$$\Rightarrow H = \int d^3x\left[\Pi(x)\dot{\varphi}(x) - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2\right]$$

$$= \int d^3x\left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{m^2}{2}\varphi^2\right]$$

EOM?

$$\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} = -m^2\varphi - \partial_\mu(\partial^\mu\varphi)$$

$$= -(\Box + m^2)\varphi = 0$$

$$\Box = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$$

Kleine Gordon eqn!

Noethers thm

Theory: "space of fields φ", action, $H(p,q)$

$$\varphi \to \varphi'(x) = \varphi(x) + \underbrace{\alpha}_{\text{small}}\Delta\varphi(x)$$

$$\text{st: } \mathcal{L}(x) \to \mathcal{L}(x) + \alpha\,\partial_\mu J^\mu(x)$$

St $\Delta\varphi$ is a symmetry of EoM

$$\alpha\Delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi}\alpha(\Delta\varphi) + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\partial_\mu(\alpha\Delta\varphi)$$

Then there is a conserved current

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\Delta\varphi - J^\mu$$

$$\delta S = \int d^4x\left(\alpha\underbrace{\left[\frac{\delta\mathcal{L}}{\delta\varphi}\Delta\varphi - \partial^\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right)\Delta\varphi\right]}_{=0 \text{ with EOM}} + \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\delta\partial_\mu\varphi}\Delta\varphi\right)\right)$$

Ex/ $\mathcal{L} = |\partial_\mu\varphi|^2 - m^2|\varphi|^2$ $\quad X + iY = \varphi$

$\varphi \to e^{i\alpha}\varphi$, $\varphi \to \varphi + i\alpha\varphi$ $\quad \Delta\varphi = i$

$\varphi^* \to e^{-i\alpha}\varphi$ $\quad \varphi^* \to \varphi - i\alpha\varphi$

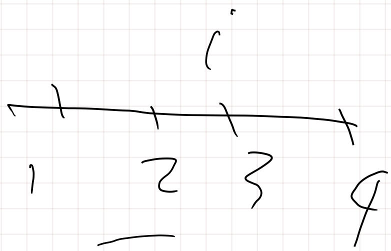
$$\Rightarrow j_\mu = i(\partial_\mu\varphi^*\cdot\varphi - \varphi^*\partial_\mu\varphi)$$

- with this current there is a charge

$$\int j_0\, d^3x = Q \quad \text{as} \quad \dot{Q} = \int \partial_0 j^0 d^3x = \int \partial_i j^i d^3x = 0$$

boundry conditions $=0$

$$\frac{\delta J(x)}{\delta J(y)} = \delta(x-y)$$



$$S = \int J(x) A(x)\, dx$$

J_i

$$\frac{\delta J_i}{\delta J_j} = \delta_{ij}$$

$$\frac{\delta S}{\delta J(y)} = \int \delta(x-y) A(x)\, dx = A(y)$$

Stress Energy Tensor

(infinitesimal transformation)

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi \quad + \mathcal{O}(a^2)$$

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

$$T^\mu{}_\nu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \quad , \quad \partial_\mu T^{\mu\nu} = 0$$

$$\Rightarrow \int T^0{}_0\, d^3x = \int \Big(\underbrace{\frac{\delta \mathcal{L}}{\delta \dot\phi}}_{\Pi} \dot\phi - \mathcal{L} \Big) = \int \mathcal{H}\, d^3x$$

$$\int T^0{}_i\, d^3x = \dots = \int \mathcal{P}_i\, d^3x = P_i$$

$$\Pi = \frac{\delta \mathcal{L}}{\delta \dot\phi}$$

For Klein Gordan field

$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$ (complex?)

$$T^\mu{}_\nu = \partial^\mu \phi\, \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

with $\partial_\mu T^\mu{}_\nu = 0$

Thm can Always Symmetize Stress Tensor (In book)

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \tfrac{m^2}{2} \phi^2$$

$$\Rightarrow \Pi = \frac{\delta \mathcal{L}}{\delta \dot\phi} = \dot\phi$$

$$\Rightarrow \mathcal{H} = \Pi \dot\phi - \tfrac{1}{2} \dot\phi^2 + \tfrac{1}{2} (\nabla \phi)^2 + \tfrac{m^2}{2} \phi^2$$

$$= \frac{\Pi^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{m^2 \phi^2}{2} > 0$$

EoM $(\Box + m^2)\phi = 0$ $\qquad H = \int \mathcal{H}\, d^3x = \int \frac{d^3x}{2} (\Pi^2 + (\nabla\phi)^2 + m^2 \phi^2)$

Classically we had: $\{P_i, q_j\} = \delta_{ij}$

Now we have: $\{\ \} \to [\ ]$

$p \to \Pi$, $q \to \phi$

i is here so that Hermitian operators have anti-symmetric commutator

$$\Rightarrow [\Pi(\vec{x}), \phi(\vec{y})] = i\, \delta^{(3)}(\vec{x} - \vec{y})$$

Harmonic Oscillators

$$H = \frac{p^2 + \omega^2 q^2}{2}$$

What we want!

$$i\frac{\partial A}{\partial t} = [A, H] \quad , \quad A(t) = e^{-iHt} A(0) e^{iHt}$$

$$[A,B]^* = (AB)^* - (BA)^* = -[A^*, B^*]$$

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3p\, \phi(\vec{p},t) e^{i\vec{p}\cdot\vec{x}}$$

$$(\Box + M^2)\phi = 0 \Rightarrow \frac{1}{(2\pi)^3}\int d^3p \underbrace{\left(\left(\frac{\partial}{\partial t}\right)^2 + p^2 + m^2\right)\phi(\vec{p},t)}_{\Rightarrow \; =0} e^{i\vec{p}\cdot\vec{x}} = 0$$

$\Box \to \partial_t^2 - \partial_{\vec{x}}^2$

Notation: $\omega_p = \sqrt{\vec{p}^2 + m^2}$ $\Rightarrow (\partial_t^2 + \omega_p^2)\phi(\vec{p},t) = 0$

In QM: $\hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^+)$, $\hat{p} = -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^+)$

$[\hat{a}, \hat{a}^+] = \mathbb{I}$

$H_{SHO} = \frac{\hat{p}^2 + \omega^2 \hat{q}^2}{2} = \omega a^+ a = \omega(a^+ a + \frac{1}{2})$

$a^+ \to$ creation

$a \to$ anhilation

Want $\hat{a}$ and $\hat{a}^+$ for fields: $\phi(\vec{p},t) = \frac{(a + a^+)}{\sqrt{2\omega_p}}$ which we write as

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3}{\sqrt{2E_p}}\left(\hat{a}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + \hat{a}^+(p) e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\Pi(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3p\,(-i)\sqrt{\frac{E_p}{2}}\left(\hat{a}(p) e^{i\vec{p}\cdot\vec{x}} - \hat{a}^+ e^{-i\vec{p}\cdot\vec{x}}\right)$$

"Calculus of QFT"

"For every p we have harmonic oscilator"

$[H_{SHO}, \hat{a}^+] = \omega \hat{a}^+$

$[H_{SHO}, \hat{a}] = -\omega \hat{a}$

$H(\hat{a}^+\psi) = (E + \omega)\hat{a}^+\psi$

States given by: $|n\rangle = \frac{(\hat{a}^+)^n}{n!}|0\rangle$

$H = \int \mathcal{H} d^3x = \int \frac{d^3x}{2}(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2)$ Hw, check this

We should also have that $[\Pi(\vec{x}), \phi(\vec{y})] = i\delta(\vec{x} - \vec{y}) \Leftrightarrow [\hat{a}(p), a^+(\vec{p}')] = \delta(\vec{p} - \vec{p}')$

$$= \int d^3p\, E_p\left(\hat{a}^+(p)\hat{a}(\vec{p}) + \cancel{\tfrac{1}{2}[a(\vec{p}), a^+(\vec{p})]}\right)$$ Ignored

Can also then check $[H_{KG}, \hat{a}^+(\vec{p})] = \omega_p \hat{a}^+(p)$

$[H_{KG}, \hat{a}(\vec{p})] = -\omega_p \hat{a}(p)$

Quantum Spectrum of KG

Bosonic-Fock space

$a(\vec{p})|0\rangle = 0$ ~ Vacuum $= |0\rangle$

$|\vec{p}\rangle = a^+(\vec{p})|0\rangle$, $H|\vec{p}\rangle = E|p\rangle = \omega_p|p\rangle$, $\omega_p = \sqrt{\vec{p}^2 + m^2}$

but we have

$[a(\vec{p}), a(\vec{p})] = \frac{1}{2}\delta^{(3)}(0)$

What matters is difference in energies

$\Rightarrow$ we ignore this constant

Also we had infinite many oscillators, what did we expect?

$$|\vec{p}_1, \vec{p}_2\rangle = a^+(p_1)a^+(p_2)|0\rangle = (E_{p_1} + E_{p_2})|\vec{p}_1, \vec{p}_2\rangle \; ?$$

$|\vec{p}\rangle = a^+(\vec{p})|0\rangle$

$$H|\vec{p}\rangle = \int d^3p'\, E_{p'} a^+(\vec{p}')a(p') \cdot a^+(\vec{p})|0\rangle$$

$$H a^+(p)|0\rangle = \cancel{a^+(p)H} + E_p a^+(p)|0\rangle = E_p|\vec{p}\rangle$$

$\langle p'|p\rangle = ?$

$= \langle 0|a(\vec{p}')\, a^+(\vec{p})|0\rangle$

$= \langle 0|[a(\vec{p}), a^+(\vec{p})]|0\rangle$

$= \delta(\vec{p} - \vec{p}')$

This should be Lorentz invariant, but isnot

Is invariant under rotation

But Boosts:

$$\begin{pmatrix} E' \\ p_3' \end{pmatrix} = \begin{pmatrix} \gamma(p_3 + \beta E) \\ \gamma(E + \beta p_3) \end{pmatrix}$$

recall: $\delta(f(x)) \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$ $f(x_i) = 0 \; \forall i$

$$\delta(f(x) - f(x_0)) = \frac{1}{f'(x_0)}\delta(x - x_0)$$

$$d^3p\, \delta^{(3)}(\vec{p} - \tilde{\vec{p}}) = d^3p'\, \delta^{(3)}(\vec{p}' - \tilde{\vec{p}}')$$

$$\Rightarrow \delta^{(3)}(\vec{p} - \tilde{\vec{p}}) = \delta(\vec{p}' - \tilde{\vec{p}}')\frac{E'}{E}$$

$$\Rightarrow |\vec{p}\rangle = \sqrt{2E_p}\, a^+(\vec{p})|0\rangle$$

$\Rightarrow 2E_p\langle p'|p\rangle$ Is Lorentz invariant

$$\phi(\vec{x},t) = e^{iHt}\phi(\vec{x},0)e^{-iHt}$$

For this we need to know

$$a(\vec{p},t) = e^{iHt}a(\vec{p})e^{-iHt}$$

Lemma: $H^n a(\vec{p}) = a(\vec{p})(H-E_p)^n$ ✓ can use $[H,a(\vec{p})] = -E_p a(\vec{p})$

$$e^{iHt} = \sum_{n=0}^{\infty}\frac{(iHt)^n}{n!} \Rightarrow e^{iHt}a(\vec{p}) = a(\vec{p})e^{i(H-E)t}$$

$$\Rightarrow a(\vec{p},t) = a(\vec{p})e^{-iEt}$$

$-iEt + i\vec{p}\cdot\vec{x}$

$$\varphi(\vec{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})e^{-ip_\mu x^\mu} + a^\dagger(p)e^{ip_\mu x^\mu}\right)$$

$$\Phi_{KG}(\vec{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

$\leftarrow \mathbb{R}^3!$

$$E_p^2 = \vec{p}^2 + m^2$$

$$e^{-i\mathbb{P}\cdot x}a(\vec{p})e^{i\mathbb{P}\cdot x} = a(p)e^{ip\cdot x}$$

Want to calculate

2pt correlation function

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y)$$

$$D(x-y) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

$$\langle 0|a(p)a^\dagger(p')|0\rangle$$
$$= \langle 0|[a_p, a_{p'}^\dagger]|0\rangle$$
$$= \delta^{(3)}(\vec{p}-\vec{p}')$$

$$p^2 - m^2 = \left(p_0 - \sqrt{|p|^2+m^2}\right)\left(p_0 + \sqrt{|p|^2+m^2}\right)$$

Time Like

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\dots) = \int\frac{d^4p}{(2\pi)^4}2\pi\,\delta(p^2-m^2)(\dots)$$

$p_0 > 0$

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty\frac{dp\,p^2}{2\sqrt{|p|^2+m^2}}e^{-i\sqrt{|p|^2+m^2}\,t} = \frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2-m^2}\,e^{-iEt}$$

$|\vec{x}-\vec{y}| = \vec{r}$, $x^0 - y^0 = 0$ (space like)

$$D(x-y) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}\cdot\vec{r}} = 2\pi\int_0^\infty dp\int_{-1}^{1}\frac{d\eta\,p^2}{(2\pi)^3 2E_p}e^{ipr\eta}$$

$$\frac{1}{ipr}\left(e^{ipr} - e^{-ipr}\right)$$

$$= \frac{-i}{2(2\pi)^2}\frac{1}{r}\int_{-\infty}^{+\infty}dp\,p\frac{e^{ipr}}{\sqrt{p^2+m^2}}$$

$$\sim \frac{e^{-mr}}{r} \neq 0$$

$$\langle 0|[\phi(x),\phi(y)]|0\rangle$$
$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$
$$= D(x-y) - D(y-x)$$

$[f]$ $(x-y)^2 < 0$ in $\mathbb{R}^{1,3}$

there is a continuous Lorentz transformation $(x-y) \to -(x-y)$

$$\Rightarrow (x-y)^2 < 0$$
$$\Rightarrow \langle 0|[\phi(x),\phi(y)]|0\rangle = 0$$

For complex field (ϕ, ϕ^*)

$$\langle 0|[\phi(x), \phi^*(y)]|0\rangle =$$
$$= D(x-y) - D(y-x)$$

If we assign charge $+1$ to ϕ and -1 to ϕ^*

Can consider the anti-particles going opposite directions

$e^{ip.x}$ — $E_p t - \vec{p}.\vec{x}$

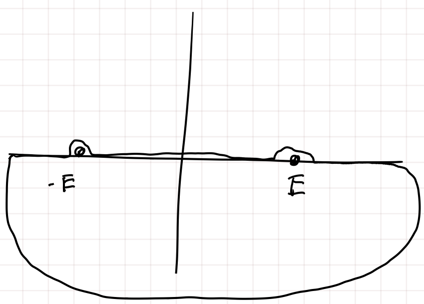
$$\langle 0|[\phi(x), \phi(y)]|0\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$
$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_p} e^{-ip(x-y)}\Big|_{p^2=|p|^2+m^2=E_p} + \frac{1}{-2E_p} e^{-ip(x-y)}\Big|_{p_0=-E_p}\right]$$

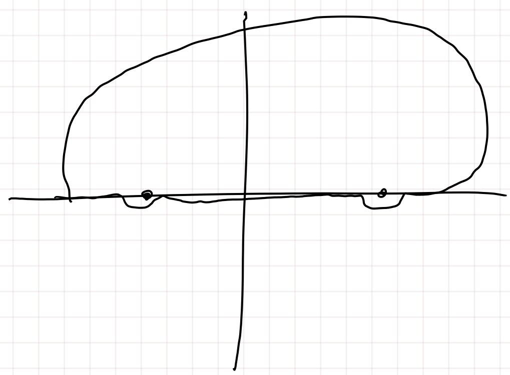
For 2-point function

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = [\phi(x), \phi(y)]$$

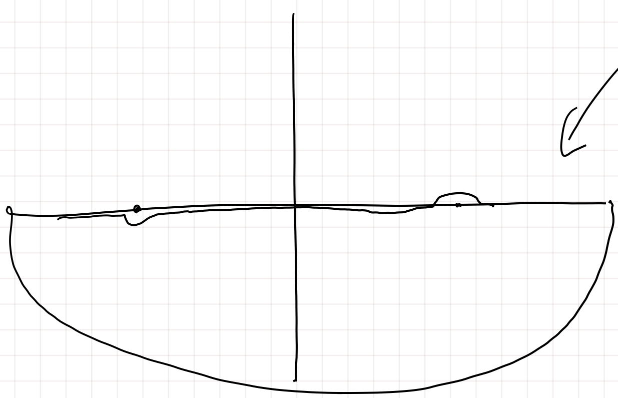
$$\underset{\substack{x^0>y^0 \\ t>t'}}{=} \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip(x-y)} = \langle 0|[\phi(x), \phi(y)]|0\rangle$$

contours:





Feynman



$$D(x^0-y^0) = \Theta(x^0-y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

$$\text{Something} = \begin{cases} D(x-y), & x^0>y^0 \\ D(y-x), & y^0>x^0 \end{cases}$$

$$\langle 0|\,\phi(x)\phi(y) - \phi(y)\phi(x)\,|0\rangle$$

Time ordering

when $x^0 > y^0$ only this term contributes

$$\Rightarrow = \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle = \langle 0|T(\phi(x),\phi(y))|0\rangle$$

gives advanced function for negative energy modes and retarded for positive energy

$$\langle 0|T(\phi(x),\phi(y))|0\rangle = \lim_{\text{Feyn } \epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)} = D_F(x-y)$$

$$p_0^2 = \vec{p}^2+m^2-i\epsilon$$

$$p_0 = \pm\sqrt{\vec{p}^2+m^2} \pm \frac{i\epsilon}{2}$$

Classical source

$$(\Box + m^2)\phi(x) = j(x)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + j(x)\phi(x)$$

$$\phi(x) = \phi_0(x) + i\int d^4y\, D(x-y)\, j(y)$$

As this satisfies

$$(\Box+m^2)D(x-y) = \delta(x-y) \qquad \phi_0(x) = \int\frac{d^3p}{(2\pi)^3}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

$$\Rightarrow \phi(x) = \int\frac{d^3p}{(2\pi)^3}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right) + \int d^4y\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)j(y)$$

$$= \text{''} \qquad\qquad \text{''} + \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[\left(a_p + \frac{i}{\sqrt{E_p}}j(p)\right)e^{-ipx} + cc\right]$$

Can then modify Hamiltonian

$$H = \int\frac{d^3p}{(2\pi)^3}E_p\left(a_p^\dagger - \frac{i}{\sqrt{2E}}j(p)\right)\left(a(p) + \frac{i}{\sqrt{2E}}j(p)\right)$$

$$\langle 0|H|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2}|\tilde{j}(p)|^2$$

Represenlations of Poincaré group

$P = L \rtimes P$ $\qquad \mathbb{R}^{1,3}$; $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$L \to$ Rotations in $\mathbb{R}^{1,3}$ $\iff$ $x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$

$P \to$ Translations $\quad x \to x + a$

$$x^\mu x_\mu = (x^0)^2 - \vec{x}^2 = \text{invariant}$$

$$x_\mu x^\mu = x^\mu \eta_{\mu\nu} x^\nu = x'^\mu \eta_{\mu\nu} x'^\nu$$

$$= \underbrace{\Lambda^\mu_{\ \alpha} x^\alpha}_{x'^\mu} \eta_{\mu\nu} \underbrace{\Lambda^\nu_{\ \beta} x^\beta}_{x'^\nu}$$

$$\Rightarrow \eta_{\alpha\beta} = \Lambda^\mu_{\ \alpha} \eta_{\mu\nu} \Lambda^\nu_{\ \beta}$$

index free

$x' = \Lambda x$; $\eta = \Lambda^T \eta \Lambda$ — Transpose as we had to swap index's

$\eta^{-1} = \eta \Rightarrow \mathbb{1} = \Lambda \Lambda^T$

$m \in M$, group acts on M

Left action

$m \to m' = g \cdot m$

$m'' = h m' = (h \cdot g) \cdot m$

$V(h) V(g) = V(h \cdot g)$

Right action

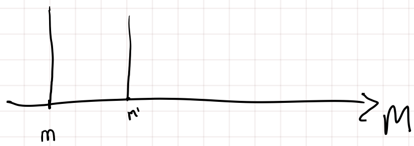
instead if:

$m' = m \cdot g^{-1}$

$m'' = m' h^{-1} = m (g^{-1} h^{-1})$

$V(h^{-1}) V(g^{-1}) = V(g^{-1} \cdot h^{-1})$

$g^{-1} \cdot h^{-1} = (h \cdot g)^{-1} \Rightarrow \quad = V((h \cdot g)^{-1})$



Space of functions on M $\qquad m \xrightarrow{f(m)} \mathbb{R}$

$f(m) \to$ sections

$\psi \to$ section

$\psi^m_{(x)} \to \alpha^m_{\ n} \psi^n(g \cdot x)$

each point m has a tangent space $\mathbb{R}^n$, as we move from point to point these tangent spaces rotate?

$\psi^m \to \psi'^m_{(x)} = \alpha^m_{\ n}(g) \psi(g^{-1} x)$ — so that end order is right

$$\psi''(x)^m = \alpha^m_{\ n}(h) \psi^n(h^{-1} x) = \alpha^m_{\ n}(h) \alpha^n_{\ k}(g) \psi^k(g^{-1}(h^{-1} x))$$

$$= \alpha \cdot \alpha \, \psi((h \cdot g)^{-1} x)$$

So for these maps

Left $\to$ inverse mulitplication

Right $\to$ multi

$\mathbb{R}^{1,3} \ni x$; $x' = \Lambda x$

$$\psi'(x') = L(\Lambda) \psi(x)$$

$$= L(\Lambda) \psi(\Lambda^{-1} x')$$

$$\Rightarrow \psi'(x) = L(\Lambda) \psi(\Lambda^{-1} x)$$

What kind of space is $L(\Lambda)$

$$\eta = \Lambda^T \eta \Lambda$$

$$1 \neq \cancel{\det \eta} = \det \Lambda^T \cdot \cancel{\det \eta} \det \Lambda$$

$$\Rightarrow \det \Lambda = \pm 1$$

$$\eta^{00} = 1 = \Lambda^0_{\ \alpha} \eta^{\alpha\beta} \Lambda^0_{\ \beta}$$

$$= (\Lambda^0_{\ 0})^2 - (\Lambda^0_{\ i})^2$$

$$\Rightarrow (\Lambda^0_{\ 0})^2 = 1 + (\Lambda^0_{\ i})^2 \geq 1$$

$\Lambda^0_{\ 0} \geq 1$, $\Lambda^0_{\ 0} \leq -1$

orthocronas $\qquad$ non-orthocronas

$\det \Lambda = 1$ $\quad$ proper

$\det \Lambda = -1$ $\quad$ improper

4 possibilities

a) $\det\Lambda = 1$, $\Lambda^0{}_0 \geq 1$ | $L_+^\uparrow$
b) $\det\Lambda = 1$, $\Lambda^0{}_0 \leq -1$ | $L_+^\downarrow$
c) $\det\Lambda = -1$, $\Lambda^0{}_0 \geq 1$ | $L_-^\uparrow$
d) $\det\Lambda = -1$ $\Lambda^0{}_0 \leq -1$ | $L_-^\downarrow$

1) Rotations

$$\Lambda = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) \quad A \in SO(3)$$

$$\det\Lambda = \det A = 1$$

$SO(1,3)$ is a double cover of $SO(3)$?

2) Boost

$$\Lambda = \left(\begin{array}{cc|c} \cosh\theta & -\sinh\theta & 0 \\ -\sinh\theta & \cosh\theta & \\ \hline & 0 & \mathbb{1} \end{array}\right)$$

$$\cosh\theta = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma$$

Nov 6^{th}, 13^{th} Hamilton Institute (9-11) am

3) time Reversals / reflections

$$x^0 = t \to -x^0, \quad \vec{x} \to \vec{x}$$

$$\det L = -1, \ \Lambda^0{}_0 = -1 \ | \ L_-^\downarrow$$

4) whole reversals / Reflections

$$\det L = 1, \ \Lambda^0{}_0 = -1 \ | \ L_+^\downarrow$$

6 parameters

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu \Rightarrow \Lambda = I + \varepsilon$$

$$\Lambda^T \eta \Lambda = \eta \quad \Big| \quad I = I + \varepsilon^T + \varepsilon$$

$$\Rightarrow \varepsilon^T + \varepsilon = 0$$

$$\Rightarrow \varepsilon^{\mu\nu} + \varepsilon^{\nu\mu} = 0 \Rightarrow \text{Antisymmetric}$$

upper triangular

$$\frac{n(n+1)}{2} - n \quad \text{diag} = 0$$

$$= \frac{n(n-1)}{2}$$

For symmetric, $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad \frac{1}{i}\frac{\partial}{\partial x^\mu} = P_\mu$$

$$f(x+a) = f(x) + a f'(x)|_a = e^{a\frac{\partial}{\partial x}} f(x)$$

Momentum generates shift $\Rightarrow e^{iaP} f(x)$

Poincare Group $= L \rtimes P$

Lorentz generator

$$M^{\alpha\beta}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta_{\alpha\beta} + (S_{\mu\nu})_{\alpha\beta}$$

$$M = L \cdot I + S$$

$\mathbb{R}^{1,3}$ fibre

recall $\psi'(x) = \Omega(\Lambda)\psi(\Lambda^{-1}x)$

$$\Omega = I - \tfrac{i}{2}\varepsilon \cdot S$$

$$\psi(\Lambda^{-1}x) = -\tfrac{i}{2}\varepsilon^{\mu\nu}$$

So similarly

$$e^{i\varepsilon^{\mu\nu}L_{\mu\nu}} f(x) = f(\overbrace{x^\mu + \varepsilon^\mu{}_\nu x^\nu}^{\Lambda x})$$

$$= f(x) + \varepsilon^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} f(x)$$

$$= f(x) + \frac{i}{2}\varepsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

$$\Rightarrow f(x + \varepsilon x) = e^{\frac{i}{2}\varepsilon^{\mu\nu}L_{\mu\nu}} f(x)$$

$\frac{\partial}{\partial x} S = 0 \qquad [M_{\mu\nu}, M_{\lambda\sigma}] = \underbrace{\cdots}_{\text{determined by } [L,L]} M$ (Linear)

Commutators of Lorentz group

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\nu\rho} L_{\mu\sigma} - i\eta_{\mu\rho} L_{\nu\sigma} + i\eta_{\mu\sigma} L_{\nu\rho} - i\eta_{\nu\sigma} L_{\mu\rho}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\nu\rho} M_{\rho\sigma} - i\eta_{\rho\sigma} M_{\nu\sigma} + i\eta_{\mu\sigma} M_{\nu\rho} - i\eta_{\nu\sigma} M_{\rho\sigma}$$

$$\psi(x) \Rightarrow \psi'(x) = V(\Lambda)\psi(x) = \Omega(\Lambda)\psi(\Lambda^{-1}x)$$

Taking $0, i = 1,2,3$

$[M_{ij}, M_{kl}]$ $\quad \eta \quad M_{\mu\nu} = L_{\mu\nu} I + S_{\mu\nu}$

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk} \qquad K_i = M_{0i}$$

$[J_i, J_j] = i\epsilon_{ijk} J_k \longrightarrow SO(3)$

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

$[J_i, K_j] = i\epsilon_{ijk} K_k$

6 generators
continuous Lorentz transformations are connected.

If we take a combination

$N_i = \frac{1}{2}(J_i + iK_i)$

$\Rightarrow N_i^+ = \frac{1}{2}(J_i - iK_i)$

$\Rightarrow [N_i, N_j] = i\epsilon_{ijk} N_k$

$[N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$ $\quad \leftarrow SL(2,\mathbb{C})$

$[N_i, N_j^+] = 0$

Recall from QM II

$L^2 = L_1^2 + L_2^2 + L_3^2 = l(l+1)$

$l_3 = -l \cdots 0 \cdots l$

L = half integer

$\Rightarrow N^2 \equiv N_1^2 + N_2^2 + N_3^2 = n(n+1)$

choosing $N_3 \Rightarrow -n, \ldots 0, \ldots n \leftarrow 2n+1$

$(N^+)^2 = -m(m+1)$

$N_3^+ = -m, \ldots 0 \ldots, m \leftarrow 2m+1$

Reckoned

$\Rightarrow 2n+1 + 2m+1 = 2(n+m) + 2$

~~Total independant states~~

Should be $(2n+1)(2m+1)$?

$m, n = 0, \frac{1}{2}, \cdots$

Now we can write

$J_i = N_i + N_i^+ = $ Spin $\quad \underline{\underline{n+m}}$

The operation $\dagger$ takes $N \to N^+$

$\Rightarrow$ It takes $(0, \frac{1}{2}) \longleftrightarrow (\frac{1}{2}, 0)$

	(n, m)	No. states	KG Theory
a)	$(0,0)$	1	(spin=0) 1d complex
b)	$(\frac{1}{2}, 0)$	2	2d complex (Spin $= \frac{1}{2}$) (spinors)
c)	$(0, \frac{1}{2})$	2	2d complex (Spin $= \frac{1}{2}$)
d)	$(\frac{1}{2}, \frac{1}{2})$	4	vector (spin(1)) Candidate for maxwell
	$(1, 0)$	3	spin(1)
	$(0, 1)$	3	

Left spinor

$(\frac{1}{2}, 0) = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$ can direct sum $\Rightarrow (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) = \begin{pmatrix} \cdot \\ \cdot \\ * \\ * \end{pmatrix}$ Dirac Spinor

$(0, \frac{1}{2}) \quad \begin{pmatrix} * \\ * \end{pmatrix}$ $\quad \dagger$ takes us from one space to the other.

Right spinor

$(\frac{1}{2}, 0) \times (0, \frac{1}{2}) \in (\frac{1}{2}, \frac{1}{2}) \leftarrow$ Irrep

2D $\qquad$ 4D

Reducible representation

$(\frac{1}{2}, 0) \times (\frac{1}{2}, 0) = (1,0) \oplus (0,0)$

If we have $\psi_\alpha = (\frac{1}{2}, 0) \quad \alpha = 1,2$

Think of 2D matricies

$$\psi_\alpha \cdot \chi_\beta = \underbrace{(\text{Symm})_{\alpha\beta}}_{3 \,\downarrow\, (1,0)} \oplus \underbrace{(\text{anti-sym})_{\alpha\beta}}_{\downarrow\, (0,0)}$$

4d $\mathbb{R}^{1,3}$

$$B_{\mu\nu}\, dx^\mu \wedge dx^\nu \longrightarrow *B = \underbrace{\tfrac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu} B_{\mu\nu}}_{*B_{\alpha\beta}} dx^\alpha \wedge dx^\beta$$

"$B_{\mu\nu}$" | $*^2 = 1$

"$B + *B$" $(1,0)$ self-dual Tensor

"$B - *B$" $(0,1)$ anti-dual Tensor

Quatenions

- can use σ_i

$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma_k$

$x^\mu \to x^0 \mathbb{I} + i x^i \sigma^i = \hat{x}$

& $x^\mu \to x^\mu + a^\mu = x'^\mu$, $x^\mu \to \Lambda^\mu{}_\nu x^\nu = x'^\mu$ } Group

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

Generators:

$$P_\mu = -i\frac{\partial}{\partial x^\mu}, \quad M_{\mu\nu} = \underbrace{L_{\mu\nu}}_{i(x_\mu\partial_\nu - x_\nu\partial_\mu)} \mathbb{I} + \underbrace{S_{\mu\nu}}_{\text{constant}}$$

$[M,M] = 0 = [P,P]$

10 parameter group

$$P_G = LT \rtimes T$$

Then!
$$[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu$$

What are the Casamirs (What commutes with everything)

1) $P_\mu P^\mu = m^2$

2) $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$; $W^\mu W_\mu = W^2$

Pauli-Lubansky vector

$\to$ Can check $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu S_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} S_{\rho\sigma}\frac{\partial}{\partial x^\nu}$

$[W^\mu, P^\nu] = 0$

$[M_{\mu\nu}, W_\rho] = \dots$

$\frac{\partial}{\partial x^\nu} S^{\rho\sigma} = 0$

Types of repre...

$S = n + m$

1) $P_\mu P^\mu = m^2 > 0$

$W^2 = W_\mu W^\mu = -m^2 s(s+1)$

$s = 0, \frac{1}{2}, 1, \dots$ states $S_3 = -s, \dots, s$

2) $P_\mu P^\mu = 0 \Rightarrow m = 0$

b) or $W^2 = 0$, $W^\mu P_\mu = 0$

c) $W \sim P$; $W_\mu = c P_\mu$ | c = helicity | $c = \pm S$

3) $P_\mu P^\mu = m^2 < 0$ Tachyon

Example $(S_{\mu\nu})_\alpha{}^\beta$

φ. KG $-$ $(0,0)$

$\partial_\mu\phi$ $-$?

$$\delta(\partial_\mu\phi) = [\delta, \partial_\mu]\phi + \underbrace{\partial_\mu(\delta\phi)}_{=0}\ ?$$

$$[\delta, \partial_\mu] = \cancel{[\delta_0, \partial_\mu]} + [\delta x^\nu \partial_\nu, \partial_\mu]$$

$$[\epsilon^\nu{}_\rho x^\rho \partial_\nu, \partial_\mu] = \epsilon^\nu{}_\mu \partial_\nu$$

$$= \delta(\partial_\mu\phi) = \delta_0(\partial_\mu\phi) + \delta x^\nu \partial_\nu(\partial_\mu\phi)$$

$$= -\tfrac{i}{2}\epsilon^{\rho\sigma} L_{\rho\sigma}\partial_\mu\phi - \tfrac{i}{2}(\epsilon^{\rho\sigma}S_{\rho\sigma})_\mu{}^\nu \partial_\nu\phi$$

$$(S_{\rho\sigma})_\mu{}^\nu = i(\eta_{\rho\mu}\eta_\sigma^\nu - \eta_{\sigma\mu}\eta_\rho^\nu) \text{ - spin matrix}$$

$\Rightarrow \partial_\mu\phi \Leftrightarrow$ vectors

$M_{\rho\sigma} = L_{\rho\sigma}\mathbb{I} + S_{\rho\sigma}$ derived for vectors

$x' = x + \delta x$

$$\delta f = f'(x+\delta x) - f(x)$$

$$= f'(x) - f(x) + \delta x^\mu \frac{\partial}{\partial x^\mu} f'(x)$$

$\downarrow$ dropping higher order

$$= f'(x) - f(x) + \delta x^\mu \frac{\partial}{\partial x^\mu} f(x)$$

$$= \delta_0 f + \delta x^\mu \partial_\mu f \qquad \delta_0 = f'(x) - f(x)$$

$$\delta = \delta_0 + \delta x^\mu \partial_\mu$$

$$\phi'(x') - \phi(x) = 0$$

$$0 = \delta\phi = \delta_0\phi + \delta x^\mu \partial_\mu \phi$$

$$\delta_0\phi = -\tfrac{i}{2}\epsilon^{\rho\sigma} M_{\rho\sigma}\phi$$

$$\left(-\tfrac{i}{2}\epsilon^{\rho\sigma} M_{\rho\sigma} + \delta x^\mu \partial_\mu\right)\phi = 0$$

Write Poincare Invariant functionals; Lagrangian

$\phi'(x') = \phi(x)$, $\phi^2, \phi^3, \dots, \phi^n$ - invariants

$a\phi^2$ $\quad b\,\partial_\mu\phi\,\partial^\mu\phi$ - this is all

$$L_{KG} = -\frac{m^2}{2}\phi^2 + \frac{1}{2}(\partial_\mu\phi)^2$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

$\Longleftrightarrow S_{\mu\nu} = 0$ for scalar field

Spinors $\mathbb{C}^2$

$\underbrace{(\tfrac{1}{2},0)\ (0,\tfrac{1}{2})}$

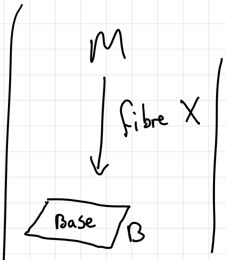
$$\psi_L(x) \to \psi_L'(x') = \Lambda_L \psi_L(x) \Rightarrow \psi_L'(\Lambda x) = \Lambda_L \cdot \psi_L(x)$$

2x2 matrix

$x' = \Lambda x$

$$\psi_R'(x') = \Lambda_R \psi_R(x)$$

Fibre? vector in $\mathbb{C}^2$



$M = X \times B$

$\Lambda_L(\Lambda_R) =$ are 2x2 complex matricies

$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$

$x^0 \to x^0,\ x^i = \omega^i{}_j x^j$

$\Rightarrow \Lambda = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\omega}}$ $\qquad \omega^i = \epsilon^{ijk}\omega_{jk}$

$$\Lambda_L = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega} - i\vec{v})} \qquad \Lambda_R = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega} + i\vec{v})}$$

---

properties

- $\Lambda_L^{-1} = \Lambda_R^\dagger$
- $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*} \Rightarrow \sigma^2\Lambda_L\sigma^2 = e^{-i\vec{\sigma}^*(\omega - iv)} = \Lambda_R^*$
- $\Lambda_L^T = \sigma^2\Lambda_L^{-1}\sigma^2$
- $\psi_L \in (\tfrac{1}{2}, 0)$

$\sigma^2\psi_L^* \in (0, \tfrac{1}{2})$

$\sigma^2\psi_L^* \to \sigma^2\Lambda_2^*(\sigma^2\sigma^2)\psi_L^*$

$\ldots = \Lambda_R\sigma^2\psi_2^*$

$\boxed{\text{Looking for Lorentz invariant object from Spinors}}$

$\cdot)\ \chi_L \in (\tfrac{1}{2}, 0),\ \psi_2 \in (\tfrac{1}{2}, 0)$

$$\chi_L^T\sigma^2\psi_L \to \chi^T \underbrace{\Lambda_L^T}_{\sigma^2\Lambda_L^{-1}\sigma^2}\sigma^2\Lambda_L\psi_L = \chi_L^T\sigma^2\psi_L$$

$\cdot\cdot)\ \chi = \psi$?

$\to \psi_L^T\sigma^2\psi_L = 0$

$-\ \psi_L, \psi_R, \chi_L = \sigma^2\psi_R^*$

$\Rightarrow \chi_L^T\sigma^2\psi_L = \psi_R^\dagger\psi_L$ - invariant $\to$ scalar

$-\ \psi_L^\dagger\sigma^2\psi_L \to \psi_L^\dagger e^{\frac{\vec{\sigma}\cdot\vec{v}}{2}}\sigma^i e^{\frac{\vec{\sigma}\cdot\vec{v}}{2}}\psi_L$

$$= \psi_L^\dagger\left(1 + \frac{\vec{\sigma}\cdot\vec{v}}{2}\right)\sigma^i\left(1 + \frac{\vec{\sigma}\cdot\vec{v}}{2}\right)\psi_L = (\text{Same}) +$$

$$= \psi_L^\dagger\sigma^i\psi_L + \tfrac{1}{2}v^j\psi_L^\dagger(\sigma^j\sigma^i + \sigma^i\sigma^j)\psi_L$$

$$= \psi_L^\dagger\sigma^i\psi_L + v^i\psi_L^\dagger\psi_L$$

$\psi_L^\dagger\psi_L \to e^{\frac{\vec{\sigma}\cdot\vec{v}}{2}}e^{\frac{\vec{\sigma}\cdot\vec{v}}{2}}\psi_L$

$= \psi_L^\dagger\psi_L + \vec{v}\,\psi_L^\dagger\vec{\sigma}\,\psi_L$

$$\delta(\psi_L^\dagger\psi_L) = v^i\psi_L^\dagger\sigma^i\psi_L \cdot \delta(\psi_L^\dagger\sigma^2\psi_2) = v^i\psi_L^\dagger\psi_2$$

$\Rightarrow$ We can define

$$i\psi_2^\dagger\sigma^\mu\psi_L \to \delta V^\mu = \varepsilon^\mu{}_\nu V^\nu$$

$\boxed{\text{all }\text{~~same~~}\text{ part is } V^\mu}$

$\varepsilon^0{}_i = V_i$

$\sigma^\mu = (I, \sigma^i)$

$\bar{\sigma}^\mu = (I, -\sigma^i)$

$$V^\mu = i\psi_L^\dagger\sigma^\mu\psi_L$$

$$U^\mu = i\psi_R^\dagger\bar{\sigma}^\mu\psi_R$$

Conclusion, following objects are invariant

$$\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L, \; \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R$$

$$\partial_\mu \psi_L^\dagger \sigma^\mu \psi_L, \; \partial_\mu \psi_R^\dagger \bar{\sigma}^\mu \psi_R$$

$$\frac{\partial}{\partial x^\mu} \to \Lambda^{-1} \frac{\partial}{\partial x^\mu}$$

$$\frac{1}{2}[\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - (\partial_\mu \psi_L^\dagger) \sigma^\mu \psi_L]$$
$$= \frac{1}{2} \psi_L^\dagger \sigma^\mu \overleftrightarrow{\partial_\mu} \psi_L \quad \Big| \quad \frac{1}{2} \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R$$

$$\psi_L = \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}, \; \psi_R = \begin{pmatrix} \psi_R^1 \\ \psi_R^2 \end{pmatrix}$$

$$\mathcal{L}_{\mathbb{1}} = \frac{i}{2}[\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \partial_\mu \psi_L^\dagger \sigma^\mu \psi_L]$$
$$= \frac{i}{2} \psi_L^\dagger \sigma^\mu \overleftrightarrow{\partial_\mu} \psi_L$$

Parity invariant Dirac 4-comp spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\quad (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$$P\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} O_2 & I_2 \\ I_2 & O_2 \end{pmatrix} \psi = \gamma_0 \psi$$

Projection L & R

$$\frac{1}{2}(1 \pm \gamma_5)\psi = \psi_{L(R)}$$

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow \frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$
$$\frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

$$\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$$ ← Both invariants

$$= \bar{\psi}\psi, \quad \bar{\psi} = \psi^\dagger \gamma_0$$

- Then the Lagrangian which contains all the symmetries (scalars)

Integrating by parts inside action integral:

$$\mathcal{L} = i[\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R] + m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \qquad \gamma^5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3$$

$$= i \bar{\psi} \gamma^\mu \partial_\mu \psi + m \bar{\psi} \psi \qquad \text{where } \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Dirac Lagrangian

$$\psi_L = \sigma^2 \psi_R^*$$

$$\psi^c = \begin{pmatrix} \sigma^2 \psi_R^* \\ -\sigma^2 \psi_L^* \end{pmatrix}$$

Is there a spinor such that $\psi^c = \psi$?

Marjorana spinor:

$$(\psi^M)^c = \psi^M = \begin{pmatrix} \psi_L \\ -\sigma^2 \psi_L^* \end{pmatrix}$$

Properties: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}_{4\times4}$

Euclidian rotation: $\mathbb{R}^{1,3} \to \mathbb{R}^4$

$$t \mapsto \underset{\tau}{it}, \; \vec{x} \mapsto \vec{x}$$

$$ds^2 = dt^2 - d\vec{x}^2 \to -dx_4^2$$

(After rotation in t)

$$\Lambda_L = e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}+\vec{v})} \qquad \Lambda_R = e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}-\vec{v})}$$

$$SL(2,\mathbb{C}) \xrightarrow[\text{rotation}]{\text{After } t} SO(3) \times SO(3) \cong SU(2) \times SU(2)$$

$$A^\mu \Rightarrow A = \begin{pmatrix} A^0 + A^3 & A^1 + iA^2 \\ A^1 - iA^2 & A^0 - A^3 \end{pmatrix}$$
$$= A^\mu \sigma_\mu$$

$\Rightarrow A = A^\dagger$, $\det A =$ invariant

$$\left.\begin{matrix} A_\mu A^\mu, \; \partial_\mu A_\nu \partial^\nu A^\mu \\ \partial_\mu A_\nu \partial^\mu A^\nu, \; \partial_\mu A^\mu, \; (\cdot)^2 \leftarrow ? \end{matrix}\right] \text{ more invariants}$$

Dirac Equation | construct fock space for spinors and find propagator

If we define $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ then $S^{\mu\nu}$ satisfies Lorentz Algebra

$$\psi \to \Lambda \psi(\Lambda^{-1}x) = \psi'(x')$$

$$\Lambda = e^{\frac{i}{2} S^{\mu\nu} \epsilon_{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\begin{pmatrix} m\mathbb{I}_2 & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ -i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & m\mathbb{I}_2 \end{pmatrix} \psi = \begin{pmatrix} m\mathbb{I}_2 \psi_L + i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R \\ -i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L - m\mathbb{I}_2 \psi_L \end{pmatrix} = 0$$

chiral

Plane wave solution:

$$\psi = u(p) e^{-ip\cdot x}$$

$u(p)$: spinor

$$\Rightarrow (\gamma^\mu p_\mu - m) u(p) = 0$$

$$(i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu + m)\psi = 0$$

$$(-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2)\psi = 0$$

$$= (-\gamma^{\mu\nu} \partial_\mu \partial_\nu - m^2)\psi = 0$$

$a\cdot b = \frac{1}{2}\{a,b\} + \frac{1}{2}[a,b]$

does not contribute?

$\vec{p}$ general

$\vec{p} = 0 \Longleftrightarrow$

$$\Lambda_{\frac{1}{2}}(p)\psi(\Lambda x) = \psi'(x')$$

$\Lambda_{\frac{1}{2}}(p)$: boost

$$(\gamma^0 p_0 - m)u(p_0, \vec{p}=0) = 0$$

$$\left[\begin{pmatrix} -mI & 0 \\ 0 & -mI \end{pmatrix} + \begin{pmatrix} 0 & p_0 I \\ p_0 I & 0 \end{pmatrix}\right] u(p) = 0$$

$$m\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0 = m) = 0$$

$$u(p_0 = m) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\xi^\dagger \xi = I \quad \text{not } 1?$$

Boost back

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

infinitesimally

$$= \left[1 + \eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} m \\ 0 \end{pmatrix}$$

need to calculate

$$S^{03} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\Rightarrow$$

$$u(p) = e^{-\frac{1}{2}\eta\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}} \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

$$p_\mu \sigma^\mu = \sqrt{p_0 I + p_3 \sigma^3}$$

Taylor expand use $\sigma^{3\,2} = \mathbb{I}$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \qquad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Helicity operator

$$h = \hat{p}\cdot\vec{S} = \frac{1}{2}\hat{p}_i\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$\lim_{\eta\to 0} u(p) = \begin{pmatrix} \sqrt{E - p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \longrightarrow \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$u^\dagger u = 2E_p \xi^\dagger \xi \quad | \quad \bar{u} = u^\dagger \gamma^0$$

$$\bar{u}u = 2m\xi^\dagger\xi = 2m$$

$$u^{\dagger r} u^s = 2m\,\xi^{s\dagger}\xi^r = 2m\,\delta^{sr}$$

$$\bar{v}^s_p v^r_p = -2m\,\delta^{rs}$$

We could also have

$$\psi = v(p)\, e^{ip\cdot x} \qquad p_0 > 0$$

$$v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(\not{p} + m)v^s(p) = 0$$

$$\bar{u}^r(p)v^s(p) = \bar{v}^r u^s(p) = 0$$

Are these complete

$$\sum_{s=1,2}\left(u^s(p)\bar{u}^s(p) + v^s(p)\bar{v}^s(p)\right)$$

needs to be − for completeness?

$$\begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi^s \\ \sqrt{\bar{\sigma}\cdot p}\,\xi^s \end{pmatrix}\left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^s\sqrt{p\cdot\sigma}\right)$$

$$\sum_{s=1}^{2}\xi^s\xi^{s\dagger} = I$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$\sum_s \bar{u}^s u^s(p) \cdots$

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

relation between co-ords and momenta with out derivative $\longrightarrow$ constrained system

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \psi^\dagger$$

$$\Pi_{\psi^\dagger} = \psi$$

$$H = \int d^3x \left(\Pi_\psi \cdot \dot{\psi} - \mathcal{L}\right) = \int d^3x \left[\bar{\Psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\Psi\right]$$

$$\vec{\alpha} = \gamma^0\vec{\gamma}, \ \beta = \gamma^0 \quad = \int d^3x \left[-i\vec{\alpha}\cdot\vec{\nabla} + m\beta\right]\Psi$$

$$H = \int \left(\Pi_\psi \dot{\psi} - \mathcal{L}\right) d^3x = \int \bar{\Psi}\left(-i\vec{\gamma}\cdot\vec{\partial} + m\right)\Psi d^3x$$

Try $\left[\Psi_\alpha(\vec{x}), \Psi^\dagger_\beta(\vec{y})\right] = \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})$

$$\Psi_\alpha(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\, e^{i\vec{p}\cdot\vec{x}} \sum_{s=1,2}\left[a^s(\vec{p})u^s_\alpha(p) + b^s(-\vec{p})\, v^s_\alpha(-\vec{p})\right]$$

$$\left[a^s(\vec{p}), a^{\dagger s}(\vec{p}')\right] = \left[b^s(\vec{p}), b^{\dagger s}(\vec{p}')\right] = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta^{rs}$$

$$\sum_s u^s(p)\bar{u}^s(p) = p\!\!\!/ + m, \quad \sum_s v^s(p)\bar{v}^s(p) = p\!\!\!/ - m$$

$\Psi_+(\text{Pos } E) = u^s(p)e^{i\vec{p}\cdot\vec{x}}$

$\Psi_-(\text{neg } E) = v^s(p)\,e^{-i\vec{p}\cdot\vec{x}} = v^s(-\vec{p})e^{i\vec{p}\cdot\vec{x}}$

$H\Psi_+ = E_p\Psi_+ \qquad E_p > 0$

$H\Psi_- = -E_p\Psi_- \qquad E > 0$

$$H = \int\frac{d^3p}{(2\pi)^3}\sum_s E_p\left(a^{s\dagger}(\vec{p})\,a^s(\vec{p}) - b^{s\dagger}(\vec{p})b(\vec{p})\right)$$

Not always >0 (Problem #1)

---

what if we change $b^\dagger \to b$ doesn't work...

$$\left[\Psi_\alpha(x), \bar{\Psi}_\beta(y)\right] \overset{(x-y)^2<0}{=} \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left((p\!\!\!/+m)e^{-ip(x-y)} + (p\!\!\!/-m)e^{ip(x-y)}\right)$$

$$= (i\partial\!\!\!/_x + m)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip(x-y)} - e^{ip\cdot(x-y)}\right) = 0 \quad (x-y)^2<0$$

Suppose $a(\vec{p})|0\rangle = b(\vec{p})|0\rangle = 0$

$$\Rightarrow \left[\Psi(x), \Psi^\dagger(y)\right] = \langle 0|\left[\Psi(x), \bar{\Psi}(y)\right]|0\rangle = \langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle$$

$$- \underbrace{\cancel{\langle 0|\bar{\Psi}(y)\Psi(x)|0\rangle}}_{=0} \ ?$$

Problem 2

$$e^{i\vec{P}\cdot\vec{x}} A\, e^{-i\vec{P}\cdot\vec{x}} = A(\vec{x}) = e^{i\vec{p}\cdot\vec{x}}A(0)$$

$$\longrightarrow = e^{i(\vec{p}-\vec{p}')x}\langle 0|a^s(\vec{p})a^s(p')|0\rangle$$

$$\Rightarrow = \delta^{rs}\delta(\vec{p}-\vec{p}')A(\vec{p})$$

All we know is this is Lorentz invariant

$\Rightarrow A = A(p^2) = A(m^2) = \text{const}$

We start from scratch

Need to change something

Let us calculate $\langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle > \langle 0|\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_r a^r(p)u^r(p)\, e^{-ipx}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum a^{s\dagger}\bar{u}^{s}(p')e^{ip'y}|0\rangle$

postulate $b^\dagger|0\rangle = 0 \qquad \langle 0|a^s(p)a^\dagger(p')|0\rangle = ?$

Consider

$$\langle 0|\, a^r(\vec{p})a^{\dagger s}(\vec{p}')\, \underbrace{e^{i\vec{P}\cdot\vec{x}}}_{=0}|0\rangle$$

$\Rightarrow$

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_r a^r(p)u^r(p)e^{-ipx}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum a^{s\dagger}\bar{u}^{s'}(p')e^{ip'y}|0\rangle$$

$$= \theta(i\partial\!\!\!/ + m)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}A$$

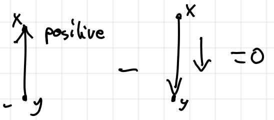
added

similarly

$$\langle 0|\bar{\Psi}(y)\psi(x)|0\rangle = -(i\partial\!\!\!/_x + m)\int\frac{d^3p}{(2\pi)^3}\frac{e^{iP(x-y)}}{2E_p}B$$

where

$$\langle 0|b^\dagger(p)b^s(p')|0\rangle = \delta^{rs}\delta(p-p')B$$



$- \qquad = 0 \quad \Rightarrow A = -B \qquad A>0,\ B>0$

so we change our addition

Fermionic fock space

$a(p)|0\rangle = 0$

$b^\dagger(p)|0\rangle = 0$

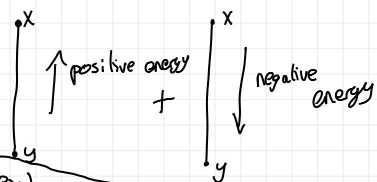
$\langle 0|\{\psi(x)\bar{\psi}(y)\}|0\rangle$

$$\{a^r(p), a^{s\dagger}(p')\} = \{b^r(p), b^{s\dagger}(p')\} = \delta^{rs}\delta(\vec{p}-\vec{p}')$$

rename $b^\dagger \to \tilde{b} \to b$

$$H = \int d^3p\, E_p\left(a^\dagger(p)a(p) + b^\dagger(p)b(p)\right) > 0$$

$$\{\psi^\dagger(\vec{x},t), \psi(\vec{y},t)\} = \delta(\vec{x}-\vec{y})$$



$$\psi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(u^s(p)a^s(p)e^{-ipx} + v^s(p)b^{\dagger s}(p)e^{ipx}\right)$$

Dirac Fermion

Follows Fermi Dirac statistics

Anti commutator is poisson brackets

$\Rightarrow$ Grassman variables

$AB + BA = 0 \Rightarrow A\cdot A = 0$

We now discuss Fermionic Fock space

Plan for notes

- Anti commutation relations
- Pauli exclusion principle
- charges of positron and electron
- Dirac propogator

$a^\dagger$ creates electrons

$b^\dagger$ creates positrons

$|0\rangle$ Vacuum, $H|0\rangle = 0$

$a^s(\vec{p})|0\rangle = b^s(\vec{p})|0\rangle = 0$

$a^{\dagger s}(\vec{p})|0\rangle = |\vec{p}, s\rangle$, $b^{\dagger s}(\vec{p})|0\rangle = |\tilde{\vec{p}}, \tilde{s}\rangle$ (1 particle state)

(2 particle state)

$a^{\dagger s}(\vec{p}_1)a^{\dagger r}(\vec{p}_2)|0\rangle = |p_1, s, p_2, r\rangle$ This will vanish if $p_1 = p_2$, $r = s$

$b^{\dagger s}(p_1)b^{\dagger r}(p_2)|0\rangle = |\tilde{p}_1, \tilde{s}, \tilde{p}_2, \tilde{r}\rangle$ $\Rightarrow$ Pauli exclusion principle!

vanishes for $\tilde{p}_1 = \tilde{p}_2$, $\tilde{r} = \tilde{s}$

$b^\dagger(p_2)a^{\dagger r}(p_1)|0\rangle = |p_1, r, \tilde{p}_2, \tilde{s}\rangle$

$a^{\dagger s}(p_2)b^{\dagger r}(p_1)|0\rangle = |p_2, s, \tilde{p}_1, \tilde{r}\rangle$

$|p_1, s_1, \ldots, p_n, s_n, \tilde{p}_1, \tilde{s}_1, \ldots, \tilde{p}_n, \tilde{s}_n\rangle$

$\hookrightarrow |n, m\rangle$

n particles

n anti-particles

$$\overline{\Psi}(\vec{x},t) = \psi^{+}(\vec{x},t)\gamma^{0} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(\bar{u}^{s}(p)a^{st}(p)e^{ipx} + \bar{v}^{s}(p)b^{s}(p)e^{-ipx}\right)$$

Once again to restore Lorentz invariance

$$\sqrt{2E_p}\, a^{+s}(\vec{p})|0\rangle = |\vec{p},s\rangle \;,\; \sqrt{2E_p}\, b^{+s}(\vec{p})|0\rangle = |\tilde{p},\tilde{s}\rangle$$

- Lets investigate if their is a noether current.

Returning to Dirac Lagrangian

eletric charge

$$\mathcal{L}_D \equiv i\bar{\Psi}\not{\partial}\Psi + m\bar{\Psi}\Psi \;;\; \Psi \to e^{i\alpha}\Psi$$

Symmetry!

$$\psi^{+} \to e^{-i\alpha}\psi^{+}$$

- infitesimly $\psi \to \psi + i\alpha\psi$

$\psi^{+} \to \psi^{+} - i\alpha\psi^{+}$

Noether current: $J^{\mu} = \bar{\psi}\gamma^{\mu}\psi \;;\; \partial_{\mu}J^{\mu} = 0$

$$Q \equiv \int J^{0} d^3x$$

$$J^{0} = \psi^{+}\gamma^{0}\gamma^{0}\psi = \psi^{+}\psi$$

recall electron

$$|p,s\rangle = \sqrt{2E_p}\, a^{+s}(p)|0\rangle$$

Do this!

$$Q = \int \psi^{+}\psi\, d^3x = \cdots = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a^{+s}(p)a^{s}(p) - b^{+s}(p)b^{s}(p)\right)$$

Now lets act on these

$$Q\, a^{+s}(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\sum_r \underbrace{a^{+r}(\vec{p})a^{r}(\vec{p})a^{+s}(0)|0\rangle}$$

$$= [-a^{+s}(0)\cancel{a^{r}(\vec{p})} + \{a^{r}(\vec{p}), a^{+s}(0)\}]|0\rangle$$

$$= \delta^{rs}\delta(\vec{p})|0\rangle$$

$$\Rightarrow Q\, a^{+s}(0)|0\rangle = a^{+s}(0)|0\rangle$$

$$Q\, b^{+s}(0)|0\rangle = -b^{+s}(0)|0\rangle$$

$\Rightarrow$ Can distinguish particle and anti-particle with opposit charges.

Propogator

$$\langle 0|\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s \underbrace{u^{s}\bar{u}^{s}(p)}_{\not{p}+m} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)\int \frac{1}{2E_p} e^{-ip(x-y)}\frac{d^3p}{(2\pi)^3}$$

$$\langle 0|\bar{\psi}_{\alpha}(x)\psi_{\beta}(x)|0\rangle = -(i\not{\partial}_x + m)\int \frac{1}{2E_p} e^{ip(x-y)}\frac{d^3p}{(2\pi)^3}$$

Want to write down Fynmann propagator with time ordering but, when we swap the order we pick up a minus sign due to grassman variables

$$S_F(x-y) = \Theta(x_0-y_0)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle$$

$$-\Theta(y_0-x_0)\langle 0|\bar{\psi}(y)\psi(x)|0\rangle = \langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle$$

recall $D_F(x-y) = \langle 0|T(\varphi(x)\varphi(y))|0\rangle$

$$\Rightarrow S_F(x-y) = (i\not{\partial}_x + m)D_F(x-y)$$

$$= (i\not{\partial}_x + m)\int \frac{d^4p}{(2\pi)^3}\frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$$

Calculating the integral:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{(\not{p}+m)}{p^2-m^2+i\varepsilon}e^{-ip(x-y)}$$

Discrete Symmetries

Quadratic in ψ

$$\gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda]} = \gamma^5\gamma^{\mu}$$

$$\gamma^5 = -\frac{i}{4}\varepsilon^{\mu\nu\lambda\sigma}\gamma_{\mu}\gamma_{\nu}\gamma_{\lambda}\gamma_{\sigma}$$

Γ^{μ} matricies 4x4 $\to$ 16 matricies

$\mathbb{1}$ - 1

γ^{μ} - 4

$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$ - 6

$\gamma^5\gamma^{\mu}$ - 4

γ^5 - 1

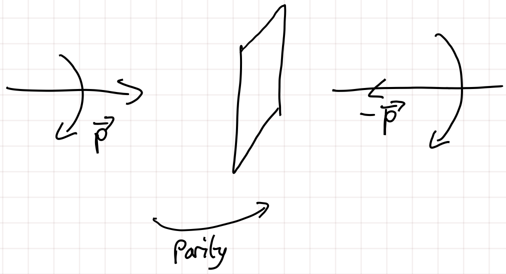
$\Rightarrow$ Most general combonation

$$\bar{\psi}\Gamma^{a}\psi \quad a=1,\ldots,16$$

parity transformation

$$(t,\vec{x}) \to (t,-\vec{x})$$

Time reversal $(t,\vec{x}) \to (-t,\vec{x})$



$$U(\vec{p})\, a^s(\vec{p})\, U^\dagger(\vec{p}) = \eta_a\, a^s(-\vec{p})$$

$$\eta_a = \text{phase} \quad (\eta)^2 = 1$$

Similarly $\quad U b^s(p) U^\dagger = \eta_b\, b^s(p)$

$$U(P)\psi(x)U^\dagger(P) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\eta_a a^s(-\vec{p})u^s(p)e^{-ipx} + \eta_b^* b^{\dagger s}(-\vec{p})v^s(p)e^{ipx}\right)$$

$$\tilde{p} = (p^0, -\vec{p}) \quad = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\eta_a a^s(-\vec{p})u^s(p)e^{-ipx} + \eta_b^* b^{\dagger s}(\vec{p})v^s(p)e^{i\tilde{p}\tilde{x}}\right)$$

$$p\cdot x = \tilde{p}\cdot\tilde{x}$$

$$u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix} = \gamma^0 u(\tilde{p})$$

$$v^s(p) = -\gamma^0 v^s(\tilde{p})$$

Free to choose

$$\eta_a = -\eta_b^* = 1$$

$$U\psi(t,\vec{x})U^\dagger = \gamma^0\psi(t,-\vec{x})$$

$$U\bar{\psi}(t,\vec{x})U^\dagger = \bar{\psi}(t,\vec{x})\gamma^0$$

$$\bar{\psi}(t,\vec{x})\psi(t,\vec{x}) = \text{Scalar} = \bar{\psi}(t,-\vec{x})\psi(t,-\vec{x})$$

$$\bar{\psi}\gamma^\mu\psi = \text{vector} = \underbrace{\pm}_{\mu=0;\ \mu=1,2,3}\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x})$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi = \text{psudo vector} = \underbrace{\mp}_{\mu=0;\ \mu=1,2,3}\bar{\psi}(t,-\vec{x})\gamma^\mu\gamma^5\psi(t,-\vec{x})$$

$$\bar{\psi}\gamma^5\psi = \text{psudo scalar} = -\bar{\psi}(t,-\vec{x})\gamma^5\psi(t,-\vec{x})$$

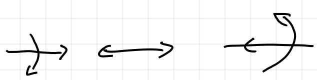
e^{iHT} act on it with time reversal

$$U(T)e^{iHT}U^\dagger(T) = e^{-iHT}$$

$\Rightarrow$ does not commute with Hamiltonian $\quad [U(T), H] \neq 0$

does not commute but still a symmetry

time reversal flips the spin



$$U(T)[A] = [A^*]U(T)$$

postulate $\Rightarrow$ Formular works

$$\xi(\uparrow) = \begin{pmatrix}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix} \qquad \xi(\downarrow) = \begin{pmatrix}-e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

$$\xi^s = (\xi(\uparrow), \xi(\downarrow)) \qquad \text{"natural definition"}$$

$$\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))$$

Final def of time reversal

$$U(T)a^s(\vec{p})U^\dagger(T) = a^{-s}(-\vec{p})$$

$$U(T)b^s(\vec{p})U^\dagger(T) = b^{-s}(-\vec{p})$$

$$u^{s*}(p) = -\gamma^3\gamma^1 u^{-s}(\tilde{p})$$

$$v^{s*}(p) = \gamma^3\gamma^1 v^{-s}(\tilde{p})$$

$$U(T)\psi(x)U^\dagger(T) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a^s(-\vec{p})u^{*s}(p)e^{ipx} + b^{\dagger s}(-\vec{p})v^{*s}(p)e^{-ipx}\right)$$

$$= \int\frac{d^3p}{2\pi}\frac{1}{\sqrt{2E_p}}\sum_s\left(a^s(-\vec{p})u^{*s}(p)e^{ipx} + b^{\dagger s}(-\vec{p})v^{*s}(p)e^{-ipx}\right)$$

$$= -\gamma^1\gamma^3\psi(-t,\vec{x})$$

$\bar{\psi}\psi$ scalar

$\bar{\psi}\gamma^5\psi$ psudo vector

$\bar{\psi}\gamma^\mu\psi$ - vectors

Charge conjugation

$$C^2 = \mathbb{1} \Rightarrow C^{-1} = C$$

$$C a^s(\vec{p}) C = b^s(\vec{p})$$

$$C b^s(\vec{p}) C = a^s(\vec{p})$$

$$C\psi(t,\vec{x})C = C\int \frac{d^3p}{(2\pi)^3}\sum_s \left(u^s(p)a^s(p)e^{-ip\cdot x} + u^s(p) b^{s\dagger}(p) e^{ipx}\right)C$$

$$= -i\gamma^2\psi^*(t,\vec{x}) = -i\gamma^2\left(\psi^\dagger\right)^T$$

$$\boxed{\begin{aligned} u^s(p) &= -i\gamma^2 v^{s*}(p) \\ v^s(p) &= -i\gamma^2 u^{*s}(p)\end{aligned}}$$

CPT Theorem

	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi S^{\mu\nu}\psi$
CPT	1	1	-1	-1	+1

Little book of Dirac:

Quantise Gravity $\ddot{\frown}$

Maxwell:

Degenerate Hamiltonian system

$$L(q,\dot q) \Rightarrow p_i = \frac{\partial L}{\partial \dot q_i}$$ Cannot express $\dot q(p,q)$

$$\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} = M_{ij}, \quad \det M_{ij} \neq 0$$

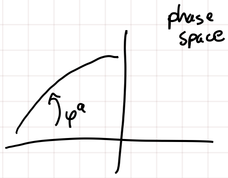
$$H = \sum p_i \dot q_i - \mathcal{L}(q,\dot q) = H(p,q)$$

If M is not invertible means there are some relations between rows

$$\Rightarrow \varphi^a(p,q) = 0 \qquad a = 1,\dots,m$$

There is not one Hamiltonian but m

$$H' = H(p,q) + \sum_{a=1}^m \lambda^a\varphi^a(p,q)$$



phase space reduction

$$\dot\varphi(p,q) = \{\varphi^a, H'\} = 0$$

$$\Rightarrow \{\varphi^a, H\} + \sum_{b=1}^{\lambda}\lambda^b\{\varphi^a,\varphi^b\} = 0$$

$$\{\varphi^a, H\} = f^a(p,q) = 0$$

Now $\tilde\varphi^k(p,q) = 0$; $k = 1,\dots,l < m$ $\quad m+l = k$?

Secondary + Primary

This equation: $\{\varphi^a, H\} + \sum_{b=1}^{k}\lambda^b\{\varphi^a,\varphi^b\} = 0$

$$\Rightarrow \lambda^a = (N^{-1})^{ab}\{\varphi^b, H\}$$

If $\{\varphi^a,\varphi^b\}\big|_{\varphi=0} = N^{ab}$ is invertable ($\det N \neq 0$)

There are a few cases
1. N-invertable (All good)
2. N-not-invertable, No solutions
3. New equations?

$$\begin{cases} H(p,q) \\ \phi^\alpha(p,q) = 0 \end{cases}$$ $\alpha = 1,\dots,\tilde{m} \neq m$

Constraint Hamiltonian system

$\{\phi^\alpha, H\} = 0$ tautology

$$H'' = H' + \sum_{a=1}^{h}\tilde\lambda^a\tilde\varphi^a(p,q) + \sum_{a=1}^{m}\lambda^a\varphi^a(p,q)$$

Arbitrary

$$\Rightarrow H'' = H + \sum_{\alpha=1}^{\tilde m}\lambda^\alpha\phi^\alpha(p,q)$$

Now repeat first step?

$$\Phi^\alpha = \{\phi^\alpha, H\} = 0$$

First class constraints

$$\{\phi^\alpha, \phi^\beta\} = 0$$

Second class

all others

$$H|\psi\rangle = E|\psi\rangle$$

$$\phi^\alpha|\psi\rangle$$

Weak sense	Strong sense

2nd class

$\phi^\alpha(p,q)=0$ 2nd class

$\{f(p,q),g(p,q)\}_D = \{f,g\}_p - \sum_{\alpha\beta} \{f,\phi^\alpha\} C_{\alpha\beta} \{\phi^\beta, g\}$

$C_{\alpha\beta} = (C^{\alpha\beta})^{-1}$, $C^{\alpha\beta} = \{\phi^\alpha, \phi^\beta\}$

$\{\phi^\alpha = 0, g\}_D = \{\phi^\alpha, g\}$

=> constraints are 0 in the strong sense

2nd class constraints

Wed morning class

$\{\phi^a, \phi^b\} = C^{ab}$

$C_{ab} C^{bc} = \delta_a^{\ c}$

$\{f(p,q), g(p,q)\}_D = \{f,g\}_p - \{f, \phi^a\}_p C_{ab} \{\phi^b, g\}_p$

poisson

With this we can put $\Phi^a = 0$ in a strong sense $\Longleftrightarrow$ solving equations (p_i, q^i) in 2N $i=1,\dots,N$

$\alpha = 1,\dots,K$ $K = \text{Even} = 2n$ 2N variables and 2n equations

$\Phi^\alpha = 0$

=> # Independent variables $2N - 2n = 2(N-n)$

p^*_a, q^{*a} , $a = 1,\dots,N-n$ => Reduced phase space

$\{\Phi^\gamma, g\}_D = \{\Phi^\gamma, g\}_p - \{\phi^\gamma, \phi^\alpha\}_p C_{\alpha\beta} \{\phi^\beta, g\}_p$

$= \{\Phi^\alpha, g\} - \underbrace{C^{\gamma\alpha} C_{\alpha\beta}}_{\delta^\gamma_\beta} \{\phi^\beta, g\}$

$= 0$

=> No need to solve other eqs?

Repeated indices mean sum

$H^{Final} = H + \sum_\alpha \lambda^\alpha \phi^\alpha = H^F(p,q)$

Done

evaluation with H & $\phi^\alpha = 0$

1st class

---

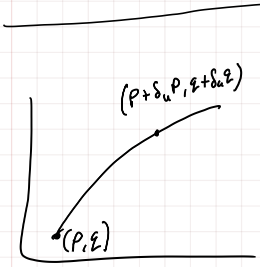
generating function $\phi(u) = \sum_\alpha u^\alpha \phi^\alpha$

$\phi(v) = \sum_\alpha v^\alpha \phi^\alpha$

Dirac proved

$\delta_u f = \{\phi(u), f\}_p$

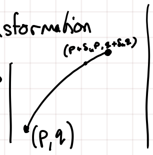
Like poisson bracket with H but for different generators



guage transformation

$p_u = p + \delta_u p$

$q_u = q + \delta_u p$



$\delta_u p_i = \{\phi(u), p_i\}_p$

$\delta_u q_i = \{\phi(u), q^i\}$

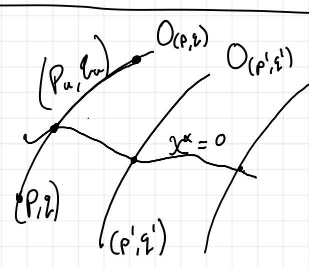
This transformation is a symmetry of the theory.

$\partial_t p = \{H, p\}$

$\partial_t q = \{H, q\}$

u = K-dim vector

$\{\phi(u), H\}$



Orbits

We can eliminate redundancy we choose one point in the K-dim space

$\chi^\alpha(p,q)$ $\alpha = 1,\dots,K$ constraints

These are guage fixing conditions.

$p + \delta_u p = p + \{\phi(u), p\}$

$q + \delta_u q = q + \{\phi(u), q\}$

If we find a u st:

$\chi(p,q)=0$ and $\chi(p+\delta_u p,\, q+\delta_u q)$

$\Rightarrow$ We cross same orbit twice for $u\neq 0$

$\chi^\alpha(p+\delta_u p,\, q+\delta_u q) = \chi^\alpha + \{\phi(u), \chi^\alpha\} = \{\sum_\beta u^\beta \phi^\beta, \chi^\alpha\} = 0$

only one sol $u=0$

In this sense $\phi(u)$ are generators.

$\Rightarrow \sum_\beta u^\beta \{\phi^\beta, \chi^\alpha\} = V^\alpha \qquad u = (\ )^{-1} V = 0$

only if $\det\{\phi^\beta, \chi^\alpha\} \neq 0$

$H,\ \Phi = 0$ is a new system of second class constraints

$\det\{\phi^\alpha, \chi^\beta\}\neq 0$

$\hookrightarrow$ Condition from fixing guage

$$\underbrace{H,\ \phi^\alpha = 0\ ,\ \chi^\alpha = 0}_{\text{guage fixed}}$$

equivelent description

$\Phi^\alpha;\quad \alpha = 1,\dots,2k \qquad \det\left(\{\phi^a, \phi^b\}_P\big|_{\phi=0}\right)\neq 0$

$\Phi = \begin{pmatrix}\phi^\alpha \\ \chi^\alpha\end{pmatrix}$

$\begin{pmatrix}\{\phi^a,\phi^b\} & \{\phi^a,\chi^b\} \\ \{\chi^a,\phi^b\} & \{\chi^a,\chi^b\}\end{pmatrix}$

One condition

$\tilde{C}^{\alpha\beta} = \{\phi^\alpha, \chi^\beta\}$ invertable

Maxwell theory

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

A_μ = vector potential

$F = \partial_\nu A_\mu - \partial_\mu A_\nu$

$H = \int d^3x \quad , \quad \Pi_\mu = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$

$\underline{\Pi_0 = 0} \qquad \Pi_i = \dfrac{\partial\mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = +F_{0i} = E_i$ electric field

Constraint! Maxwell theory has a primary constraint

$\mathcal{L} = -\frac{1}{4}F_{0i}F^{0i} - \frac{1}{4}F_{i0}F^{i0} - \frac{1}{4}F_{ij}F^{ij}$

$= \frac{1}{2}E_iE_i - \frac{1}{4}\underbrace{F_{ij}F^{ij}}_{-\frac{1}{2}B_iB_i}$

$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$

$\mathcal{H} = \Pi_0\dot{A}_0 + \Pi_i\dot{A}_i - \mathcal{L}$

$= \Pi_0\dot{A}_0 + E_iE_i + E_i\partial_iA_0 - \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$

(Hamiltonian density)

$H = \int \mathcal{H} d^3x = \int d^3x\left[\Pi_0\dot{A}_0 + \frac{1}{2}(\vec{E}^2+\vec{B}^2) + \underbrace{E_i\partial_iA_0}_{\text{can integrate by parts}}\right]$

$= \int d^3x\left[\Pi_0\dot{A}_0 + \frac{1}{2}(\vec{E}^2+\vec{B}^2) - A_0\partial_iE_i\right]$

+ constraint of $\Pi_0 = 0$

$H' = \int d^3x\left[\frac{1}{2}(\vec{E}^2+\vec{B}^2) - A_0\partial_iE_i + \dot{A}_0\Pi_0\right]$

could add $\lambda_1\partial_iE_i\ ,\ \lambda_2\Pi_0$

but this is the same as shifting $A_0, \dot{A}_0$ so we we can just consider these our constraints

$\dot{\Pi}_0 = \Pi_0(\vec{x}) = \{\Pi_0(\vec{x}), H\} = \{\Pi_0, \int d^3y\, A_0(\vec{y})\partial_iE_i\}$

$= \underline{-\partial_i\Pi_i = 0}$

$\{\Pi_0, A_0(\vec{y})\} = \delta(\vec{x}-\vec{y})$

$\{E_i, A_j(\vec{y})\} = \delta(\vec{x}-\vec{y})\delta_{ij}$

phase space (E_i, A_i) with constraints

now we can plug in $\Pi_0 = 0$ as that is the same as soliving that equation

left with the constraint: $\phi(x) = \partial_iE_i\ ,\quad \dot{\phi}(x) = \{\phi(x), H'\}$

$\Rightarrow \{\vec{E}_i(\vec{x}), \vec{B}_j(\vec{y})\} = \epsilon_{ijk}\partial^y_k\delta(\vec{x}-\vec{y})$

$\Rightarrow \int\{E_i(\vec{x}), B_j(\vec{y})B_j(\vec{y})\}d^3y = \int \epsilon_{ijk}\partial^y_k\delta(\vec{x}-\vec{y})B_j(y)\,d^3y$

some IBP ?

$= \partial^x_i(\epsilon_{ijk}\partial_kB_j(\vec{y})) = 0$

$\Rightarrow \partial_iE_i = 0$ is only constraint and is first class

as $\{\phi(\vec{x}), \phi(\vec{y})\} = 0$

$= \{\partial_iE_i, H'\} = \{\partial_iE_i(x), \int d^3y\, \frac{1}{2}\vec{B}^2\}$

$\{E_i(\vec{x}), B_j(\vec{y})\} = \{E_i(\vec{x}), \frac{1}{2}\epsilon_{jkl}F_{kl}(\vec{y})\}$

$= \frac{1}{2}\epsilon_{jkl}\{E_i(\vec{x}), \partial_kA_l(\vec{y}) - \partial_lA_k(\vec{y})\}$

$= \frac{1}{2}\epsilon_{jkl}\left(\delta_{il}\partial^y_k\delta(\vec{x}-\vec{y}) - \delta_{ik}\partial^y_l\delta(\vec{x}-\vec{y})\right)$

$= \frac{1}{2}\epsilon_{jki}\partial_k\delta(\vec{x}-\vec{y}) + \frac{1}{2}\epsilon_{jki}\partial_k\delta(\vec{x}-\vec{y})$

$A_i(\vec{x}), E_i(\vec{x})$ are phase space co-ords
infinite dim. $(i, \vec{x})$ - index

$$2N = 2(3 \times \mathbb{R}^3) = \text{dim phase space}$$

- dim of reduced space

$\phi(\vec{x}) \to$ guage fixing conditions $\mathbb{R}^3$ $\Rightarrow$ dim of reduced phase space $= 2(2 \times \mathbb{R}^3)$

$= 4 \times$ space-co-ords

2 coord 2-momenta

went from 8 (4-x^μ, 4 p^μ)
↳ to 4
↳ ... → photon has 2 polarizations

$$\begin{cases} \phi = \partial_i E_i = 0 \text{ Guass} \\ \chi = \partial_i A_i = 0 \text{ Coulomb Guage} \end{cases}$$

$$\{\phi(\vec{x}), \chi(\vec{y})\} = \partial_i^x \partial_j^y \delta(\vec{x}-\vec{y})\delta_{ij} = -\Delta \delta(\vec{x}-\vec{y})$$

Invertable. (only crosses orbit once)

$A_i + \delta_u A_i$

$$\delta_u(A_i) = \{\phi(u), A_i\} = \left\{\int u(\vec{y})\partial_j E_j d^3\vec{y}, A_i\right\}$$

↓ some IBP

$-\partial_i u$

$\Rightarrow A_i \to A_i - \partial_i u$

$\Rightarrow \underbrace{\partial_i A_i}_{0} - \partial_i \partial_i u = 0$

$-\Delta u = 0$

$\Rightarrow u = 0$

If u = const + linear cannot happen as boundry must be 0.

Notes plan

- Dirac propagator
- Dirac book (skip descrete)
- Maxwell

$$\text{"}C^{ab}\text{"} = -\Delta\,\delta(\vec{x}-\vec{y})$$

$$\to \text{"}C_{ab}\text{"} = -\frac{1}{\Delta}(\vec{x},\vec{y})$$

$$A_i \to A_i - \partial_i u_i \quad E_i \to E_i$$

$$\{f,g\}_D = \{f,g\}_P - \sum_{a,b}\{f,\phi^a\}_P C_{ab}\{\phi^b, g\}_P$$

field theory

$$= \{f,g\}_P + \int d^3x\, d^3y \{f, \partial_j E_j(\vec{x})\}_P \frac{1}{\Delta}(\vec{x},\vec{y})\{\partial_i A_i(\vec{y}), g\}_P$$

old case: $\{E_i(\vec{x}), A_j(\vec{y})\} = \delta_{ij}\delta(\vec{x}-\vec{y})$

$$\{E_i(\vec{x}), A_j(\vec{y})\}_D = \delta_{ij}\delta(\vec{x}-\vec{y}) + \int d^3z_1 d^3z_2 \underbrace{\{E_i(\vec{x}), \partial_k A_k(\vec{z}_1)\}}_{\delta_{ik}\partial_k^{z_1}\delta(x-z_1)} \frac{1}{\Delta}(z_1,z_2) \underbrace{\{\partial_l E_l(\vec{z}_2), A_j(\vec{y})\}}_{\partial_l^{z_2}\delta(\vec{z}_2-\vec{y})}$$

these "hit" $\frac{1}{\Delta}$ giving us two minus signs (from IBP)

$$= \delta_{ij}\delta(\vec{x}-\vec{y}) + \partial_i\partial_j \frac{1}{\Delta}(\vec{x},\vec{y})$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)$$

$$\Delta \cdot \frac{1}{\Delta} = \delta(\vec{x}-\vec{y})$$

$$\Delta G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})$$

$$= \Delta \int G(p) e^{i\vec{p}(\vec{x}-\vec{y})} = \int e^{i\vec{p}(\vec{x}-\vec{y})} \Rightarrow -\vec{p}^2 G = 1$$

$$\{f(p,q),g(p,q)\}_D = \{f,g\}_p - \sum_{a,b}\{f,\phi^a\}C_{ab}\{\phi^b,g\}_p$$

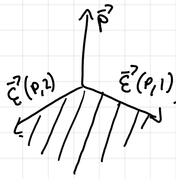
If all constraint 2nd class
$\{\phi^a,\phi^b\} = C^{ab}$
is invertable

$$\{E_i(\vec{x}),A_j(\vec{y})\} = \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left(\delta_{ij}-\frac{p_ip_j}{p^2}\right)$$

Coulomb guage

after guage fixing
$\partial_iE_i=0$
$\partial_iA_i=0$

$\vec{H} = \int d^3x\,\frac{1}{2}(\vec{E}^2+\vec{B}^2)$



$\vec{\varepsilon}(\vec{p},\lambda)\vec{\varepsilon}(\vec{p},\mu) = \delta_{\lambda\mu}$

$a_i(p)$ must live in this plane as $\underline{p_ia_i(p)=0}$.

$$A_i(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}\left(a_i(p)e^{i\vec{p}\cdot\vec{x}}+cc\right)$$

$\partial_iA_i=0 \Rightarrow \underline{p_ia_i(p)=0}$

$\Rightarrow a_i(p) = \sum_\lambda a(p,\lambda)\vec{\varepsilon}(p,\lambda) \leftarrow$ Polarisation

$$A_i(\vec{x},\overset{\downarrow t}{0}) = \int\frac{d^3p}{(2\pi)^3}\sum_\lambda a(p,\lambda)\,\varepsilon_i(p,\lambda)e^{i\vec{p}\cdot\vec{x}} + cc)$$

$$\boxed{\vec{\varepsilon}^2=1,\ \vec{\varepsilon}(p,1)\cdot\vec{\varepsilon}(p,2)=0 \quad \vec{\varepsilon}(p,\lambda)\cdot\vec{p}=0}$$

- How do $a(p,\lambda)$ and $a^\dagger(p,\lambda)$ depend on t

- Going back to maxwell equation, $\partial_\mu F^{\mu\nu}=0$
- In the coulomb guage

$$e^{iHt}a(p,\lambda)e^{-iHt} = a(p,\lambda,t) = e^{-ip_0t}a(p,\lambda)$$

$\Box A_i = 0$, In coulomb guage $A_0=0 \leftarrow$ Not a guage solution to EoM

$\Rightarrow p^2=0$

Not a co-ord in phase space

what if we did not read little book of Dirac!

$E_i(\vec{x},t) = \dot{A}_i(\vec{x},t)$

- Can write $\vec{\varepsilon}$ as 4-vector

$\varepsilon^\mu(p,\lambda) = (0,\vec{\varepsilon}(p,\lambda)) \Rightarrow \varepsilon^\mu(p,\lambda)\varepsilon_\mu(p,\mu) = \delta_{\mu\lambda}$,

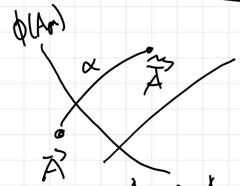
$\varepsilon_\mu p^\mu = 0$ Particular to Coulomb guage

Returning to Maxwell Lagrangian

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\mathcal{L}$ invariant under $A_\mu \to A_\mu + \partial_\mu\alpha$

- In the Hamiltonian approach: guage transformation generated by Guass Law $\int\partial_iE_i\,u(x)\,d^3x$



$A_i \to A_i + \partial_i u$

paramatrises translation

- In Lagrangian approach

$\phi(A_\mu+\partial_\mu\alpha) = 0 = \cancel{\phi(A_\mu)} - \int\partial_\mu\alpha\frac{\partial\phi}{\partial A_\mu}d^4x = 0$

Lorenz gauge $\partial_\mu A_\mu = 0$

- Lorentz invariant guage

$$A_\mu(x,t) = \int\frac{d^3p}{(2\pi)^3}\left[e^{-ip\cdot x}\sum_\lambda a(p,\lambda)\,\varepsilon_\mu(p,\lambda) + cc\right] \to \partial_\mu A_\mu = 0 \Rightarrow p^\mu\varepsilon_\mu = 0$$

justification for this is that in coulomb guage there are only 2 polarisations and the two guages must be equivalent

Note: $A_0=0$ as $\varepsilon_0=0$

$F_\mu F_{\mu\nu} = \partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\nu - \partial_\nu(\cancel{\partial_\mu A_\mu}) = 0 \Rightarrow \Box A_\nu = 0$

- Lorentz guage is better for propogator

$\partial_\mu(A_\mu+\partial_\mu\alpha) = 0 = \partial_\mu A_\mu + \partial_\mu\partial_\mu\alpha$

$\Rightarrow \Box\alpha = 0$
$(\partial_t^2-\Delta)\alpha = 0$

This has no solutions only $\alpha=0$? even though before A had infinitly many solutions

If we wick rotate
$t = i\tau \Rightarrow (-\partial_\tau^2-\Delta)\alpha = 0$
Now only has one solution!
$\underline{\alpha=0}$ Now a "good" guage

Propogator in Coulomb gauge

We have a vacuum $a(p,\lambda)|0\rangle = 0$
- To have lorentz invariant particles

$\sqrt{2E_p}\,a^\dagger(p,\lambda)|0\rangle = |p,\lambda\rangle$

2-particle: $\sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\,a^\dagger(p_1,\lambda_1)\,a^\dagger(p_2,\lambda_2)|0\rangle$

Bosonic fock space

$[a(p,\lambda),a^\dagger(p,\mu)] = \delta_{\mu\lambda}\,\delta(\vec{p}-\vec{p}')$

Want
$\langle 0|TA_\mu(x)A(y)|0\rangle$

$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = D^{tr}_{F,\mu\nu}(x-y) = \eta_{\mu\nu}D_F(x-y)$ $\leftarrow$ trace/transpos???

$$= \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+i\varepsilon}\,\frac{k^2\eta_\mu\eta_\nu-(k\cdot\eta)^2(k_\mu k_\nu+k_\nu k_\mu)+k_\mu k_\nu}{(k\eta)^2-k^2}$$

$\eta^\mu = (1,0,0,0)$

- Since we are in Coulomb gauge we must have:

$$\sum_i\frac{\partial}{\partial x_i}\langle 0|A_i(x)A_\nu(y)|0\rangle = 0$$

Now we do this for Lorenz gauge

$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = D_{F,\mu\nu}(x-y) =$

$$= \int\frac{d^3p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2+i\varepsilon}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)$$

- Now we should have $\partial_\mu\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = 0$

Let us call $\Delta_{\mu\nu} = \left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)$

$p_\mu\Delta_{\mu\nu} = p_\nu\Delta_{\mu\nu} = 0 \Rightarrow$ is "perpendicular" to p^μ

peskin drops the $\frac{p_\mu p_\nu}{p^2}$
and ends up with right answers later due to magic

Why does Coulomb guage which is not Lorentz invariant give right answer

$\partial_i A_i = 0$

$x^\mu \to \Lambda^\mu_{\ \nu} x^\nu$

$A_\mu \to \Lambda_\mu^{\ \nu} A_\nu = \Lambda^0_{\ i} A_0 + \Lambda_i^{\ j} A_j$

$A_0 = \Lambda_0^{\ i} A_i \qquad \partial_i^x \to \Lambda_i^{\ \mu} \partial_\mu^{x'}$

- Lorentz transformation will chang Coulom guage

$A \to A' + \partial\alpha$ can choose α to make $(A' + \partial\alpha)$ satisfy coulomb guage

SEMESTER 2

Interacting fields

We studied: $\left(\underset{s=0}{KG}, \underset{s=\frac{1}{2}}{\text{Dirac}}, \underset{s=1}{\text{Maxwell}}\right)$

For each we calculated propogators

A —— B

(a) Start with ϕ^4 $\longrightarrow$ KG

(b) $\mathcal{L}_{QED} \equiv$ Dirac + Maxwell

(c) Gauge symmetry

(d) Yakawa Theory (KG + maxwell)

(e) Renormalization

Dimensions of fields

$$[\phi] = 1$$
$$[\psi] = \tfrac{3}{2}$$
$$[A_\mu] = 1$$

powers of $\frac{1}{m}$?

Recall $S = \int d^4x\, \mathcal{L}(\dots)$

for $\hbar = 1$, S = dimensionless

$\Rightarrow \mathcal{L} \sim m^{-4} \sim l^{-4}$

$\mathcal{L} = \frac{m^2\phi^2}{2} \Rightarrow$

Propagators

$$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$$

call $|\Omega\rangle$ the ground sate in interaction, may not be the sam as $|0\rangle$

- ϕ not the same

Φ - not the same

Goal: $\langle \Omega | \phi(x) \phi(y) | \Omega \rangle$

Strategy, $D_F(x-y)[\lambda]$ in terms of $D_F^0(x-y) + \lambda(\dots) + \lambda^4(\dots)$

Free propogator

- power series in λ, wher $\lambda\phi^4 \in \mathcal{L}$

$$H = H_0 + H_{int} = H_0^{KG} + \underbrace{\int d^3x \frac{\lambda}{4!}\phi^4}_{H_{int}}$$

$$\phi(x,t) = e^{iH(t-t_0)}\phi(\vec{x},t_0)e^{-iH(t-t_0)}$$

Full Hamiltonian

$$i\frac{\partial \phi}{\partial t} = [H, \phi]$$

$$\underset{(\lambda=0)}{\phi(\vec{x},t)} = e^{iH_0(t-t_0)}\phi(\vec{x},t_0)e^{-iH_0(t-t_0)} \equiv \phi_I(\vec{x},t)$$

free field in interaction picture

$$\phi_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(p)e^{ipx}\right)$$

$$\phi(\vec{x},t) = \underbrace{e^{iH(t-t_0)}e^{-iH_0(t-t_0)}}_{U^\dagger(t,t_0)}\phi_I(\vec{x},t)\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{U(t,t_0)}$$

$$\phi(\vec{x},t) = U^\dagger(t,t_0)\phi_I(\vec{x},t)U(t,t_0)$$

$$i\partial_t U(t,t_0) = e^{iH_0(t-t_0)}(iH_0 - iH)e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)}H_{int}e^{-iH(t-t_0)}$$

$H_{int} = H_{int}(\phi)$

with this we can consider this $H_I(\phi_I)$ (Think of power series)

$$\overset{\lambda?}{H_I}(\phi_I)U(t,t_0)$$
$$= i\partial_t U(t,t_0)$$

Assuming $\phi_I(\vec{x},t_0) = \phi(\vec{x},t_0)$

$U = U(\phi_I)$

$\Rightarrow$ we have $\phi(\phi_I)$

For this we have a BC $U(t,t) = \mathbb{1}$

$$U(t,t_0) = I + (-i)\lambda\int_{t_0}^{t} H_I(\phi_I(t_1))\,dt_1$$

$$+ (-i)^2\lambda^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\, H_I(t_1)H_I(t_2)$$

$$\frac{\partial}{\partial x}\int_0^x f(x')dx' = f(x)$$

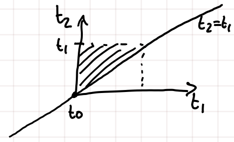
- Taking deriv

$$i(-i)\lambda H_I(t)\cdot I + i(-i)^2 H_I\int_{t_0}^{t}H(t_2)dt_2$$

$$= \lambda H_I\left(I + (-i)\lambda\int_{t_0}^{t}H_I(t_2)dt_2\right) = \lambda H_I\, U(t,t_0)$$

$$+ \cdots + (-i)^n \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \cdots \int_{t_0}^{t_2} dt_1\, H(t_n) \cdots H(t_1)$$

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} H_I(t_1) H_I(t_2)$$



- but this half of the integral over the square as the integral is symmetric

$$\Rightarrow \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\big(H_I(t_1) H_I(t_2)\big)$$

time ordering

$$\Rightarrow U(t, t_0) = I + (-i)\lambda \int_{t_0}^{t} H_I(\phi_I(t_1))\, dt_1 + \frac{(-i\lambda)^2}{2} \int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2\, T\big(H_I(t_1) H_I(t_2)\big)$$

This is because

$$T\left(\frac{-i}{n!} \int_{t_0}^{t} dt_1 \cdots dt_n \big(H_I(t_1) \cdots H_I(t_n)\big)\right) = \frac{1}{n!}\left(-i \int_{t_0}^{t} H_I(t')\, dt'\right)^n$$

Unique solution $\rightarrow H_I = \frac{\lambda}{4!}\phi^4$

$$U(t, t_0) = T \exp\left\{-i \int_{t_0}^{t} H_I(t')\, dt'\right\}$$

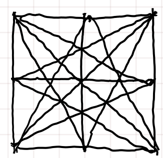
- Not exactly exponent, T goes inside integral when we expand?

Properties

1. $U(t_1, t_2)\, U(t_2, t_3) = U(t_1, t_3)$ — Causality $\rightarrow e^{-i\int_{t_1}^{t_2} H(t)\,dt} \cdot e^{\int_{t_2}^{t_3} H(t)\,dt} = e^{-i\int_{t_1}^{t_3} H(t)\,dt}$
2. $U(t_1, t_3)\, U^{\dagger}(t_2, t_3) = U(t_1, t_2)$

$|\Omega\rangle$? $\langle \Omega | T(\phi(x)\phi(y)) | \Omega \rangle$

We assume we can write $|\Omega\rangle$ in terms of $|0\rangle$



$e^{-iHT}|0\rangle$ If interacting theory has good hilbert space

$$\Rightarrow |\Omega\rangle\langle\Omega| + \underbrace{\sum_n |n\rangle\langle n|}_{\text{excited space}} = \mathbb{1}$$

$$= e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \qquad E_n > E_0$$

$$\lim_{T \to \infty(1-i\varepsilon)} \left(e^{-iHT}|0\rangle \Big/ e^{-iE_0T}\langle\Omega|0\rangle \right) = |\Omega\rangle$$

we can add t_0 here

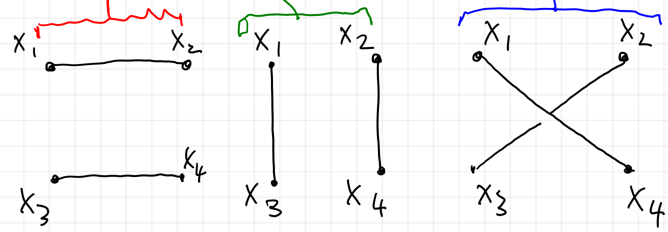
$$\lim_{T \to \infty(1-i\varepsilon)} \frac{e^{-H(t_0-(-T))}\, e^{-iH_0(-T-t_0)}|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle} = \lim_{T \to \infty} \frac{U(t_0, -T)|0\rangle}{e^{-iE_0(t_0-(-T))}}$$

Missing some lectures here that got deleted :'(

Monday 27th Jan

Wick's Thm:

$$\langle 0|\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)|0\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)$$



Recall we want to calculate

$$\langle\Omega|T\{\underbrace{\Phi(x)\Phi(y)}_{\text{interacting}}\}|\Omega\rangle$$

$$= \langle 0|T\{\underbrace{\phi(x)\phi(y)}_{\text{Free fields}}\, e^{-i\int_{x_0}^{y_0} H_I(\phi)\,dt}\}|0\rangle$$

↓ expanding to first order

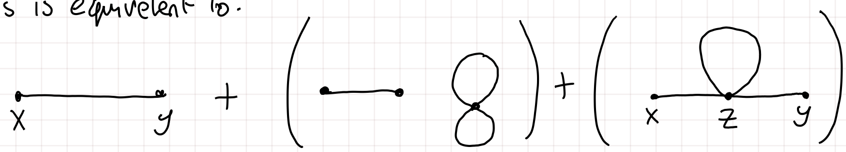
$$= \langle 0|T\{\underbrace{\phi(x)\phi(y)}_{D_F(x-y)} - \underbrace{i\,\phi(x)\phi(y)\int H_I(\phi)\,dt}\}|0\rangle$$

$$H_I = \int d^3x\, \frac{\lambda}{4!}\phi^4$$

$$\to \underbrace{3}_{\text{number of contractions?}}\left(\frac{-i\lambda}{4!}\right) D_F(x-y)\int d^4z\, D_F(z-z)D_F(z-z)$$

$$+\ 12\,\frac{(-i\lambda)}{4!}\int d^4z\, D_F(x-z)D_F(y-z)D_F(z-z)$$

- This is equivalent to:



- **Third order in λ**

$$+\underbrace{\frac{1}{3!}}_{\text{From taylor expansion}}\left(\frac{-i\lambda}{4!}\right)\int H\int H\int H \cdots$$

$$\langle 0|T\{\phi(x)\phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,\phi\phi\phi\phi(z)\int d^4w\,\phi\phi\phi\phi(w)\int d^4u\,\phi\phi\phi\phi(u)|0\rangle$$

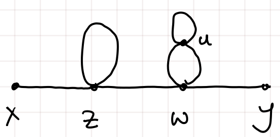
One possible contraction:

$$\langle 0|T\{\phi(x)\phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,\phi\phi\phi\phi(z)\int d^4w\,\phi\phi\phi\phi(w)\int d^4u\,\phi\phi\phi\phi(u)|0\rangle$$

$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int D_F(x-z)D_F(z-z)D_F(z-w)D_F(w-y)D_F^2(w-u)D_F(u-u)$$

$$(3!\text{ interchange of verticies})\times\left(\begin{matrix}4\times3\\ \text{placment of}\\ \text{contraction in vertex}\end{matrix}\right)\times\left(\begin{matrix}4\times3\times2\\ \text{placment of}\\ \text{contraction in}\\ w\text{ vertex}\end{matrix}\right)\times\left(\begin{matrix}4\times3\text{ placment}\\ \text{of contraction}\\ \text{in }u\end{matrix}\right)\times\frac{1}{2}$$

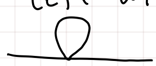
← overcounting correction ($w \leftrightarrow u$) due to $D_F^2(w-u)$ term

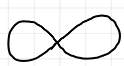


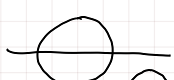
The $\frac{1}{n!}$ factorial from the taylor expansion | will always cancle the $n!$ from the number of vertex arrangments

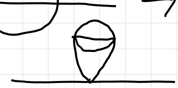
Also each vertex $\phi\phi\phi\phi$ .... $4!$ $\iff 4!\cdot\frac{\lambda}{\cancel{4!}}$

- With this we are just left with a "symmetry" factor


$\to S = 2$

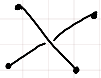

$\to S = 8$


$\to S = 3! = 6$


$\to S = ?$

Fynmann Rules

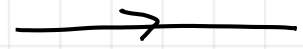
1: propogator $= D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i\, e^{ip(x-y)}}{p^2-m^2+i\epsilon}$

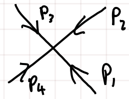
2:  $= (-i\lambda)\int d^4 z$

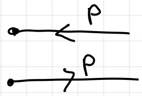
3: External point $= 1$

4: Divide by symmetry factor S

Momentum Space

1: For each propogator  $\dfrac{i}{p^2-m^2+i\epsilon}$

2: Vertex  $= -i\lambda$

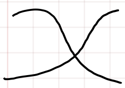
3: External point  $= e^{-ipx}$, $= e^{ipx}$

4: Each vertex has momentum conservation $\delta^4(p)$

5: Integrate over all momentum

6: Divide by symmetry factor.

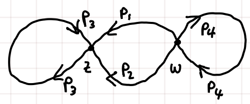
$D_F(z-z) \leftarrow$ corresponds to a loop.

 $\int d^4z\, e^{-(\Sigma p)\cdot z}$

recall we have a limit

$$\lim_{T\to\infty} \int_{-T}^{T} dt \int d^3 z\, e^{-i(\Sigma p)z} = \lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0\, e^{(\Sigma p)z_0} \int d^3 z\, e^{-i(\Sigma \vec{p})\cdot \vec{z}}$$

- This evaluated with $\Sigma \vec{p} = 0$ blows up in the limit



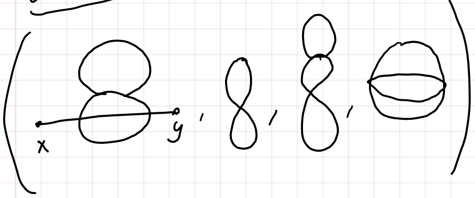
$$\delta^{(4)}(p_1+p_2-p_3+p_3)\,\delta^{(4)}(p_4-p_4-p_2-p_1)$$
$$= \delta^{(4)}(p_1+p_2)\,\delta^{(4)}(0)$$

$$\delta^{(4)}(0) = \int d^4 w \cdot e^0 = \int d^3 w \int dt \cdot 1 = \text{Volume of space} \cdot 2T \equiv V \cdot 2T$$

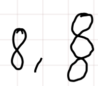
Blows up in limit

- Every disconnected piece will have $2T \cdot V$

Typical Fynmann diagram



Vacuum diagrams

$V_i \in \{$  $, \cdots \}$

- Basis of diagram with no external legs.
- Suppose our FD has n_i pieces of type V_i

$(\text{Value of connected part}) \times \prod_i \frac{1}{n_i!}(V_i)^{n_i}$

- 2-point function

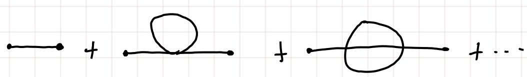
$$= \sum_{\substack{\text{all possible}\\ \text{connect}}} \sum_{\text{all}\{n_i\}} \begin{pmatrix}\text{Value}\\ \text{of Conn}\end{pmatrix} \times \prod_i \frac{1}{n_i!} V_i^{n_i}$$

$$= \sum_{\substack{\text{all}\\ \text{conn}}} \begin{pmatrix}\text{value of}\\ \text{Conn}\end{pmatrix} e^{\sum_i V_i}$$

- Denominator is

$= \exp[\text{Vacuum}]$

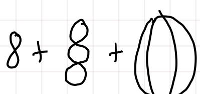
$\Rightarrow$ Just left with connected diagrams:

$= ($  $+ \cdots$

$$\exp\left(\sum_i V_i\right) \sim \exp[-i E_0 (2T)]$$

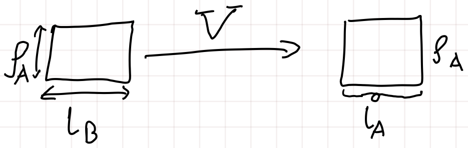
$$\sum_i V_i \sim 2TV$$

$$\frac{E_0}{V} = \frac{i[\ \text{8} + \text{8} + \text{⊖} + \cdots]}{\underbrace{\delta^{(4)}(0)}_{\to V\cdot 2T}\ (2\pi)^4}$$



$$\langle \Omega | T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \begin{pmatrix}\text{Sum of all}\\ \text{connected feynmann}\\ \text{diagrams}\end{pmatrix}$$

- Cross Section & S-Matrix



σ = cross section

$$= \frac{\text{total \# events}}{\rho_A L_A \rho_B L_A \cdot A}$$

Area where "things" happen

$$\text{Number of \#} = \sigma\, l_A l_B \int \rho_A(x)\rho_B(x)\, d^2x \Big| \frac{d\sigma}{d\Omega^{(2)}}$$

For const ρ $\quad \#N = \frac{\sigma N_A N_B}{A}$

- There is also a decay rate

$$\Gamma = \frac{\text{Number of decays/unit time}}{\text{Num of A particle present}} = \text{const}$$

Wave packet:

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(\vec{k}) |\vec{k}\rangle$$

Not $k \in$ field

- Free particle $\to |k\rangle = \sqrt{2E_k}\, a^\dagger(k)|0\rangle$

$$\langle\phi|\phi\rangle = \mathbb{1} \Rightarrow \int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 = \mathbb{1}$$

- Want to calculate probability of an out state, given an in state

$$P = |\underbrace{\langle\phi_1\phi_2\cdots|}_{\text{future}}\underbrace{\phi_A\phi_B\rangle}_{\text{past}}|^2$$

$$|\varphi_A\varphi_B\rangle = \int \frac{d^3k_A}{(2\pi)^3}\frac{d^3k_B}{(2\pi)^3} \frac{\varphi_A(k_A)\varphi_B(k_B)}{\sqrt{2E_A 2E_B}} |k_A, k_B\rangle$$

$|\varphi_A\varphi_B\rangle_{in}$ $\qquad |\varphi\cdots\varphi\rangle_{out}$

- What we need to calculate is:

$$\underset{T\to+\infty}{\text{out}}\langle p_1 p_2 \cdots | \underset{T\to-\infty}{k_A k_B}\rangle_{in}$$

- Evolution operator governs the transition in time

(- Same hilbert spaces)

$$= \lim_{T\to\infty} {}_{in}\langle p_1\ldots p_n| e^{-iH(2T)} |k_A k_B\rangle_{in}$$

$$= \langle p_1\cdots p_n| S |k_A k_B\rangle$$

We define the S matrix in this way.

- Needs to include the case when nothing happens

$$\Rightarrow S = \mathbb{1} + iT$$

- Definition of T matrix

Monday 10/2/25 !

$$\langle p_1\cdots p_N| iT |k_A k_B\rangle = (2\pi)^4 \delta^{(4)}\Big(k_A + k_B + \sum_i p_i\Big) \times \mathcal{M}(k_A, k_B \to p_f)$$

Amplitude

On shell: $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$

off shell anything else

---

out state

$$\langle\phi_1\cdots\phi_f| = \prod_i \int \frac{d^3p_i}{(2\pi)^3} \frac{\phi_f(p_i)}{\sqrt{2E_p}} \langle p_1\cdots p_f|$$

$\Rightarrow$ need to calculate $\langle p_1\cdots p_f | k_A k_B\rangle$

$$= {}_{in}\langle p_1\cdots p_f| S |k_A k_B\rangle$$

If $S = \mathbb{1}$ nothing happens

$\Rightarrow S = \mathbb{1} + iT$

$\Rightarrow$

$$P(AB \to 1,2,\ldots,N) = \Big(\prod_f \frac{d^3p}{(2\pi)^3}\frac{1}{2E_f} |\langle p_1\cdots p_N|\phi_A\phi_B\rangle|\Big)$$

$$\tilde{N} = \sum P = \int d^2b\, n_B\, P(b)$$

n_b = # of particles/unit area

$$\sigma = \frac{\tilde{N} A}{N_A N_B}$$

$$d\sigma = \frac{1}{2E_A 2E_B}\frac{1}{|v_A - v_B|}\underbrace{\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)}_{\text{phase space}}\left|\mathcal{M}(k_A k_B \to \{p_f\})\right|^2 (2\pi)^4 \delta(k_A + k_B - \sum p_f)$$

We can perform a calculation in the relelivistic case where $m_A = m_B = m_1 = m_2 = 0$

$\frac{d\sigma}{d\Omega}(2\to 2)$ = diffcross section

$d^3p = p^2 d\Omega$

: Samson does none of the calculation

$\sigma = \frac{|M|^2}{64\pi^2 E_{cm}^2}$ tadaa

S Matrix

$\langle p_1 \cdots p_N | S | k_A k_B\rangle$

$= \lim_{T\to\infty} \langle p_1 \cdots p_N | e^{-iH(2T)} | k_A k_B\rangle$

$H = H_0 + H_{int}$ (H_{int}: small number)

For this we know the vacuum

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHt}|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle}$$ ($|0\rangle$: free)

- We can write eigenstate $|k_A k_B\rangle$ in terms of this state in the vacuum

$|k_A k_B\rangle \sim \lim_{T\to\infty(1-i\epsilon)} e^{-iHt}|k_A k_B\rangle_0$

$= \lim_{T\to\infty(1-i\epsilon)} {}_0\langle p_1 \cdots p_N | e^{-i\int_{-T}^{T} H_I^{(0)} dt} | k_A k_B\rangle_0$

Samson takes a minuit to ask Dom If he is recording him

$\langle p_1 \cdots p_N | iT | k_A k_B\rangle$

$= \lim_{T\to\infty(1+i\epsilon)} \left({}_0\langle p_1 \cdots p_N | e^{-i\int_{-T}^{T} H_I(t) dt} | k_A k_B\rangle_0\right)\Big/$ conneted ampulated

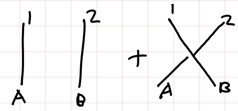
Lowest order

${}_0\langle p_1 p_2 | k_A k_B\rangle_0 = \sqrt{2E_1 2E_2 2E_A 2E_B}\,\langle 0 | a(k_1) a(k_2) a^\dagger(k_A) a^\dagger(k_B) | 0\rangle$

- We can then commute the creation/annihilation operators

$= \left(\delta^{(3)}(k_A - p_1)\delta^{(3)}(k_B - p_2) + \delta^{(3)}(k_B - p_1)\delta^{(3)}(k_A - p_2)\right)\sqrt{2E_1 2E_2 2E_A 2E_B}$

- This can be written as:



If we have interaction:

$\langle p_1 p_2 | \left(-i\int_{-T}^{T} d^4x \frac{\lambda}{4!}\phi^4(x)\right) | k_A k_B\rangle$

$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ipx} + a^\dagger e^{ipx}\right)$

$\Rightarrow \phi(x)\sqrt{2E_k}\, a_k^\dagger |0\rangle = \cdots = e^{-ikx}|0\rangle$

contraction $\longrightarrow \equiv \phi(x)|k\rangle$

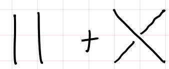
$\langle k | \phi(x) = e^{ikx}$

- We want to do all possible contractions

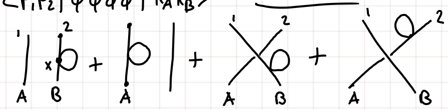
Possibilities

1. no $\phi\phi$ Contractions $\phi\phi\phi\phi \to$ all ϕ's contracted on all states
2. 1 ϕ-ϕ contraction $\phi\phi\phi\phi$
3. All $\phi\phi$ contractions $\phi\phi\phi\phi \to$ states contracted themselves

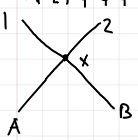
3 8 (Zeroeth order)



$\langle p_1 p_2 | \phi\phi\phi\phi | k_A k_B\rangle$ Case ②



$\langle p_1 p_2 | \phi\phi\phi\phi | k_A k_B\rangle$ Case ①



$= \frac{1}{4!}\cdot 4!(-i\lambda)\int d^4x\, e^{-i(k_A + k_B - p_1 - p_2)}$

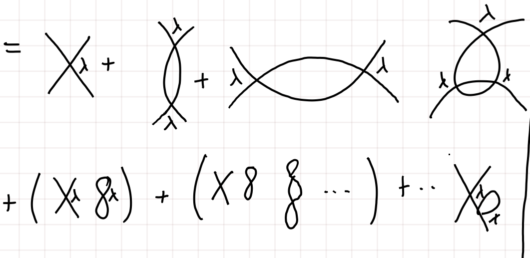
$= -i\lambda(2\pi)^4\delta^{(4)}(k_A + k_B - p_1 - p_2)$

- But recall

$\langle \; | iT | \; \rangle = i\mathcal{M}(\to)\,\delta^{(4)}$(total mom)

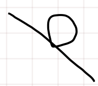
$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 E_{cm}}$

$\langle p_1 p_2 | iT | k_A k_B\rangle$

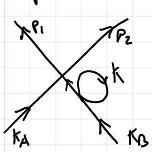


- Only connected diagrams contribute

We also get rid of "ampulated diagrams" i.e. one with



"amputation"



$$= \frac{\lambda^2}{2}\underbrace{\int\frac{d^4p'}{(2\pi)^4}\frac{1}{p'^2 - m^2 + i\epsilon}}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon}(-i\lambda)\delta^{(4)}(p_1 + p_2 - k_A - k_B)\times(-i\lambda)\delta^{(4)}(k_B - p' + * + *)$$

$= \frac{1}{(2\pi)^4}\frac{1}{k_B^2 - m^2 + i\epsilon}$

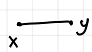
- This is not good as B is on shell

$\Rightarrow k_B^2 = m^2 \Rightarrow = \frac{1}{0}$

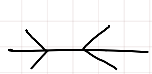
$i\mathcal{M}\,\delta^{(4)}(k_A + k_B - \sum_f p_f)$

$= \big($ sum of all connected, ampulated FD with $k_A k_B$ incoming & $\{p_f\}$ outgoing

Rules

1)  $= D_F(x-y)$

2)  $= -i\lambda\int d^4x$

3  $= e^{-ipx}$

4 divide by symmety factor

5 integrale over all momentum

Fermions

① $T\{\psi(x)\bar{\psi}\} \equiv \begin{cases} \psi(x)\bar{\psi}(y) & x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & x^0 < y^0 \end{cases}$

② $S_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = \sqcap =$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

$$T\{\psi_1 \cdots \psi_n\} = (-1)^{k-1} \psi_k \psi_1 \cdots \psi_n$$

- Normal ordering:

$$N(a_p a_q a_k^\dagger) = (-1)^3 a_k^\dagger a_q a_p$$

$$T\{\psi(x)\bar{\psi}(y)\} = N(\psi(x)\bar{\psi}(y) + \overbrace{\psi(x)\bar{\psi}(y)})$$

$$N(\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4) = -\, \overbrace{\psi_1 \bar{\psi}_3} N(\psi_2 \bar{\psi}_4)$$

Yukawa theory

$$H = H_{Dirac} + H_{KG} + g\overbrace{\int d^3x\, \bar{\psi}(x)\phi(x)\psi(x)}^{\text{interaction}}$$

$f_p + f_k \to f_{p'} + f_{k'}$

- We care only about even powers of g?

$\Rightarrow g^2 = $ first order

$$= \frac{(-ig)^2}{2!} \langle p'k'|T\{\int d^4x\, \bar{\psi}(x)\underbrace{\phi(x)\psi(x)\int d^4y\, \bar{\psi}(y)\phi(y)}_{\text{only possible Boson contraction}}\psi(y)\}|p,k\rangle_0$$

fermions

$\psi(x)|p,s\rangle$ ← spin

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{s'} a^{s'}(q) u^{s'}(q) \times \sqrt{2E_p}\, a^{s\dagger}(p)|0\rangle$$

$$= e^{-ip\cdot x} u^s(p) \equiv \overbrace{\psi|p,s\rangle}$$

$$\bar{\psi}|p,s\rangle = e^{ipx}\bar{v}^s$$