Investigation of post Keplerian parameters in General Relitvity and Yukawa gravity

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Contents

1 General Relativity

Here we provide the full derivations of the post Keplerian parameters that determine the timing model for how a pulsar in a binary with a companion star changes in time due to carious effects of gravity. The first section derives these equations using the Einsteins general relativity.

1.1 Periastron advance $\dot{\omega}$:

Einsteins field equations have an exact solution when considering a spherically symmetric star of mass M. This results in the Schwarzschild metric:

$$g = ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.1.1)$$

Then using the fact that the action of a relativistic point particle is:

$$S = -mc^2 \int d\tau. \quad (1.1.2)$$

With the definition of proper time τ as $d\tau^2 = -\frac{1}{c^2} g_{\mu\nu} x^\mu x^\nu$, integrating along the path of this point particle results in:

$$\tau = \frac{1}{c} \int_0^1 d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \quad (1.1.3)$$

Thus the action becomes:

$$S = -mc \int_0^1 d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \quad (1.1.4)$$

Letting $G = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$ and noting that if the Lagrangian is defined as $\mathcal{L} = G^2$ then varying the action results in the same geodesic equation , since an extremum of G must be an extremum of $\mathcal{L}$, due to: $\delta\mathcal{L} = 2G\delta G$. This means for the Schwarzschild metric along with Restriction that this is a Binary system, i.e. we let $\theta = \pi/2$. The Lagrangian is:

$$\mathcal{L} = \left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2. \quad (1.1.5)$$

Here still over-dots denote derivative with respect to λ. Both t and ϕ are cyclic so the Lagrange equations are:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0 \implies E \equiv \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}. \quad (1.1.6)$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0 \implies L \equiv -\frac{1}{2} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \dot{\phi}. \quad (1.1.7)$$

There is also the constraint that the metric $-g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = -u_\mu u^\mu = c^2$, (letting $\lambda \to \tau$), a constant. Thus since $\mathcal{L} = c^2$:

$$-E^2 + \dot{r}^2 + \left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} + c^2\right) = 0. \tag{1.1.8}$$

Then letting $\mathcal{E} = \frac{E^2}{2c^2}$ this takes the form:

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E} \tag{1.1.9}$$

. Where:

$$V(r) \equiv \frac{c^2}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{c^2 r^3} \tag{1.1.10}$$

Using:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{d\phi}{d\tau}\right)^2 \left(\frac{dr}{d\phi}\right)^2 = \frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2. \tag{1.1.11}$$

Then 1.1.9 becomes:

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{c^2 r^4}{L^2} - \frac{2GMr^3}{L^2} + r^2 + \frac{-2GMr}{c^2} = \frac{2\mathcal{E}r^4}{L^2}. \tag{1.1.12}$$

Using the substitutions $u \equiv \frac{L^2}{GMr} \implies \frac{du}{dr} = -\frac{L^2}{GMr^2} \implies \left(\frac{dr}{d\phi}\right)^2 = \left(\frac{du}{d\phi}\right)^2 \left(\frac{GMr^2}{L^2}\right)^2$. Thus:

$$\left(\frac{du}{d\phi}\right)^2 + \frac{c^2 L^2}{(GM)^2} - 2u + u^2 - \frac{2(GM)^2 u^3}{c^2 L^2} = \frac{2\mathcal{E}L^2}{(GM)^2}. \tag{1.1.13}$$

Differentiating with respect to ϕ results in the following after canceling all the factors of $\frac{du}{d\phi}$:

$$\frac{d^2 u}{d\phi^2} - 1 + u = \frac{3(GM)^2 u^2}{c^2 L^2}. \tag{1.1.14}$$

In Newtonian mechanics the term on the RHS is 0, so to solve this problem in GR we consider u to be the Newtonian solution plus a small deviation. Thus $u = u_0 + u_1$. Then:

$$\frac{d^2 u_0}{d\phi^2} - 1 + u_0 = 0. \tag{1.1.15}$$

Then letting $\alpha = \frac{3(GM)^2}{c^2 L^2}$:

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \alpha u_0^2 + \mathcal{O}(\alpha^2). \tag{1.1.16}$$

Where we have dropped the two terms on the RHS with u_1 in them as u_1 is small and so is the factor of α. This assumption that $u_1 \approx \alpha$, is self consistent as can be seen in the equation below where the solution to u_1 is proportional to α. The first ODE's solution is well known: $u_0 = 1 + e\cos\phi$. Subbing this in to the second ODE results in:

$$\frac{d^2 u_1^2}{d\phi^2} + u_1 = \alpha\left[1 + \frac{1}{2}e^2 + 2e\sin\phi + \frac{1}{2}e^2\cos 2\phi\right]. \tag{1.1.17}$$

To which one solution can be obtained courtesy of mathematica:

$$u_1 = \alpha \left[1 + \frac{1}{2}e^2 + e\phi\cos\phi - \frac{1}{6}e^2\cos 2\phi \right]. \tag{1.1.18}$$

The only non-periodic term here is $e\phi\cos\phi$,:

$$u = 1 + e\cos\phi + \alpha e\phi\sin\phi. \tag{1.1.19}$$

Then assuming α is small as it is for pulsars the orbit equation becomes:

$$r = \frac{L^2}{GM(1 + e\cos((1-\alpha)\phi))} + \mathcal{O}(\alpha^2). \tag{1.1.20}$$

As then $\alpha\phi \approx \sin\alpha\phi$ and $1 \approx \cos\alpha\phi$. Thus if we measure from periastron to periastron, r starts at a minimum when $\phi = 0$ and returns to this minimum when $\cos((1-\alpha)\phi) = 1 \implies (1-\alpha)\Delta\phi = 2\pi$. Thus the precession of the angle is:

$$\Delta\phi = \frac{2\pi}{1-\alpha} = 2\pi + 2\pi\alpha + \mathcal{O}(\alpha^2) = 2\pi\alpha + \mathcal{O}(\alpha^2) = \frac{6\pi(GM)^2}{c^2L^2} + \mathcal{O}(\alpha^2). \tag{1.1.21}$$

Since $L^2 \approx GM(1-e^2)a$ and $P_b^2 = \frac{4\pi^2a^3}{GM}$ and in the case that ϕ is the periastron angle ω. Then $\dot{\omega} = \frac{\Delta\omega}{P_b}$ and thus:

$$\dot{\omega} = \frac{3(2\pi)^{5/3}(GM)^{2/3}}{P_b^{5/3}c^2(1-e^2)}. \tag{1.1.22}$$

Here $M = m_p + m_c$ i.e. is the total mass of the system and if we use mass units of the sun ($M_\odot$) and the variable $T_\odot = \frac{GM_\odot}{c^3}$, we get:

$$\dot{\omega} = \frac{3(2\pi)^{5/3}T_\odot^{2/3}(m_p + m_c)^{2/3}}{P_b^{5/3}(1-e^2)}. \qquad \blacksquare \tag{1.1.23}$$

1.2 Amplitude of Einstein delay γ:

Just as above we start with the Schwarzschild metric, choosing again to set $\theta = \pi/2$ and also choosing some constant value for ϕ as the equations we need have no ϕ dependence. Thus $d\phi = 0$ and the metric becomes:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2. \tag{1.2.1}$$

Seeing as we are dealing with the motion of a pulsar the gravitational field effect it feels comes from the companion star thus, $M \to m_c$ and r in the denominator of the potential is the distance between the two pulsars, where as the r in dr is the distance from the pulsar to the barycenter r_p. Also for this system since all the masses are on the order of solar masses we can consider the quantity $\frac{GM}{rc^2} \approx \frac{v_p^2}{c^2} \approx 10^{-6}$ is small, where $v_p = \frac{dr_p}{dt}$, is the velocity of the pulsar in the binary. Then taylor expanding:

$$-c^2 d\tau^2 = -\left(1 - \frac{2Gm_c}{rc^2}\right) c^2 dt^2 + \left(1 + \frac{2Gm_c}{rc^2} + \mathcal{O}(\frac{v_p^4}{c^4})\right) dr_p^2. \tag{1.2.2}$$

Then using $v_p = \frac{dr_p}{dt}$:

$$c^2 d\tau^2 = \left(1 - \frac{2Gm_c}{rc^2}\right) c^2 dt^2 - \left(v_p^2 + \frac{2Gm_c v_p^2}{rc^2}\right) dt^2. \tag{1.2.3}$$

Here we can drop the last term in the last bracket as $\frac{GM}{rc^2} \approx \frac{v_p^2}{c^2} \approx 10^{-6}$. Thus:

$$c^2 d\tau^2 = dt^2 \left(c^2 - \frac{2Gm_c}{r} - v_p^2 + \mathcal{O}(\frac{v_p^4}{c^2})\right). \tag{1.2.4}$$

Then:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}(\frac{2Gm_c}{r} + v_p^2)} = 1 - \frac{Gm_c}{rc^2} - \frac{v_p^2}{2c^2} + \mathcal{O}(\frac{v_p^4}{c^4}). \tag{1.2.5}$$

Then we use the fact that the total energy of a binary system is given by:

$$E = \frac{1}{2}\mu v^2 - \frac{Gm_p m_c}{r}. \tag{1.2.6}$$

Where r and v represent the relative displacement and velocity. There is no GR term in this equation as it is very small compared to the Then using the fact that $E = -\frac{Gm_p m_c}{2a}$, where a is the relative semi-major axis, we can express the velocity as:

$$v^2 = \frac{Gm_p m_c}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right) = G(m_p + m_c)\left(\frac{2}{r} - \frac{1}{a}\right). \tag{1.2.7}$$

Relations like $E = -\frac{Gm_p m_c}{2a}$ remain the same as in the Keplerian case as the contributions from GR are small enough to be ignored. This is equivalent to deriving these relations from equation 2.1.22. Then since $r_p = -\frac{\mu}{m_p} r$, then $v_p = -\frac{\mu}{m_p} v$. So:

$$v_p^2 = \frac{Gm_c^2}{m_p + m_c}\left(\frac{2}{r} - \frac{1}{a}\right). \tag{1.2.8}$$

This means equation 2.2.4 becomes the following:

$$\begin{aligned} \frac{d\tau}{dt} &= 1 - \frac{Gm_c}{rc^2} - \frac{Gm_c^2}{c^2(m_p + m_c)r} \\ &= 1 - \frac{Gm_c}{c^2 r}\left(1 + \frac{m_c}{m_p + m_c}\right) = 1 - \frac{Gm_c}{c^2 r}\left(\frac{m_p + 2m_c}{m_p + m_c}\right). \end{aligned} \tag{1.2.9}$$

Here we have dropped any constant terms as they end being proportional to t once we integrate and can then be dropped by the re-scaling of t. To integrate this expression we consider how r and t relate to the eccentric anomaly E. $r = a(1 - e\cos E)$, $\omega t = E - e\sin E$ and thus $\omega dt = (1 - e\cos E)dE$. This means the integral becomes:

$$\begin{aligned} \tau &= \frac{1}{\omega}\int\left[1 - \frac{Gm_c}{ac^2(1 - e\cos E)}\frac{m_p + 2m_c}{m_p + m_c}\right](1 - e\cos E)dE, \\ &= \frac{1}{\omega}\int\left[1 - e\cos E - \frac{Gm_c}{ac^2}\frac{m_p + 2m_c}{m_p + m_c}\right]dE \\ \implies \tau &= \frac{1}{\omega}(E - e\sin E - \frac{Gm_c}{ac^2}\frac{m_p + 2m_c}{m_p + m_c}E) \\ \tau &= t - \frac{Gm_c}{\omega ac^2}\frac{m_p + 2m_c}{m_p + m_c}E \to t - \frac{Gm_c}{\omega ac^2}\frac{m_p + 2m_c}{m_p + m_c}e\sin E \end{aligned} \tag{1.2.10}$$

Where we have gotten rid of the last term as the time t can be re-scaled to get rid of the factor that is introduced from subbing in $E = \omega t + e\sin E$. Then seeing as $P_b^2 = \frac{4\pi^2 a^3}{GM}$, $\omega = \frac{2\pi}{P_b}$, $(M = m_p + m_c)$ and once again taking the masses to be in units of solar masses. The amplitude of this change in the Einstein delay is:

$$\gamma = -T_\odot^{\frac{2}{3}}\left(\frac{2\pi}{P_b}\right)^{-\frac{1}{3}}\frac{m_c(m_p + 2m_c)}{(m_p + m_c)^{\frac{4}{3}}}e. \qquad \blacksquare \tag{1.2.11}$$

1.3 Shapiro delay ΔS:

Once again we begin with the Schwarzschild metric. Though this time as we are dealing with the delay in the travel time of light beam, thus the metric becomes light like meaning: $ds^2 = 0$. Then once again fixing ϕ and θ so that the solid angle is 0 the metric becomes:

$$-\left(1-\frac{2GM}{|\mathbf{r}|c^2}\right)c^2dt^2 + \left(1-\frac{2GM}{|\mathbf{r}|c^2}\right)^{-1}d\mathbf{r}^2 = 0, \tag{1.3.1}$$

$$\implies \left|\frac{d\mathbf{r}}{dt}\right| = c\left(1-\frac{2Gm_c}{c^2|\mathbf{r}|}\right) \implies \left|\frac{dt}{d\mathbf{r}}\right| = \frac{1}{c}\left(1+\frac{2Gm_c}{c^2|\mathbf{r}|}\right) + \mathcal{O}\left((\frac{2Gm_c}{c^2r})^2\right) \tag{1.3.2}$$

Where $M \to m_c$ as the gravity well of the pulsar is just a constant and doesn't contribute to the spacing of the time of arrivals. Integrating from the time of emission t_{em} to the time of arrival t_{arr}, results in the total time take for the light beam to travel from the pulsar to earth. Here the RHS of the above equation when divided across has been taylor expanded as $\frac{2Gm_c}{c^2r}$ is small.

$$t_{arr} - t_{em} = \int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})} \frac{1}{c}\left(1+\frac{2Gm_c}{c^2|\mathbf{r}|}\right)|d\mathbf{r}|\,. \tag{1.3.3}$$

Here r_e is the vector pointing from the origin which is the binary barycenter to the earth. Then:

$$t_{arr} - t_{em} = \frac{1}{c}(\mathbf{r}_p(t_{em}) - \mathbf{r}_e(t_{arr})) + \frac{2Gm_c}{c^3}\int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})}\left(\frac{1}{|\mathbf{r}|}\right)|d\mathbf{r}|\,. \tag{1.3.4}$$

The last term of the RHS of this equation corresponds to the Shapiro delay, which when considering $\mathbf{r}$ as a function of time and thus picking up an extra factor of c as $|d\mathbf{r}| = cdt$ becomes:

$$\Delta S = \frac{2Gm_c}{c^2}\int_{t_{em}}^{t_{arr}} \frac{dt}{|\mathbf{x}(t) - \mathbf{r}_c(t_{em})|} + const. \tag{1.3.5}$$

The $|\mathbf{r}|$ here has become the distance between the light beam and the companion mass. Here $x(t)$ is the straight line path that the light travels and is given by:

$$\mathbf{x}(t) = \mathbf{r}_p(t_{em}) + \frac{t - t_{em}}{t_{arr} - t_{em}}(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})). \tag{1.3.6}$$

Using the substitution $\theta = \frac{t-t_{em}}{t_{arr}-t_{em}}$, the bounds of the integral now go from 0 to 1. Then:

$$\Delta S = \frac{2Gm_c}{c^2}\int_0^1 \frac{(t_{arr} - t_{em})d\theta}{|\mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em}) + \theta(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em}))|}. \tag{1.3.7}$$

Then seeing as $|\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})| \approx c(t_{arr} - t_{em})$ and $|\mathbf{r}_e(t_{arr})| \equiv |\mathbf{r}_e| >> |\mathbf{r}_p(t_{em})|$:

$$\Delta S \approx \frac{2Gm_c}{c^3}\int_0^1 \frac{d\theta}{|\frac{\mathbf{r}}{|\mathbf{r}_e|} + \theta\hat{\mathbf{r}}_e|} = \frac{2Gm_c}{c^3}\int_0^1 \frac{d\theta}{\sqrt{\theta^2 + 2\theta\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|} + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2}}. \tag{1.3.8}$$

Where $\mathbf{r} = \mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em})$. Then:

$$\Delta S = \frac{2Gm_c}{c^3} \int_0^1 \frac{d\theta}{\sqrt{(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2 + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2}}. \tag{1.3.9}$$

Then using the substitution $\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} = k \sinh(\psi)$, where $k^2 = (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2$.

$$\frac{2Gm_c}{c^3} \int \frac{k \cosh(\psi) d\psi}{k\sqrt{1 + \sinh^2(\psi)}} = \frac{2Gm_c}{c^3} \sinh^{-1}\left(\frac{1}{k}(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})\right)\Bigg|_0^1. \tag{1.3.10}$$

Then using the fact that $\sinh^{-1}(u) = \ln\left(\sqrt{u^2+1} + u\right)$:

$$\Delta S = \frac{2Gm_c}{c^3}\left(\ln\left|\sqrt{(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2 + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2} + \theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}\right| - \ln|k|\right)\Bigg|_0^1. \tag{1.3.11}$$

Then expanding the inside of the square root and imposing the bounds:

$$\Delta S = \frac{2Gm_c}{c^3} \ln\left|\frac{\sqrt{1 + 2\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2} + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} + 1}{\frac{|\mathbf{r}|}{|\mathbf{r}_e|} + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}}\right|. \tag{1.3.12}$$

Here k is a constant so we have lost the last term in 1.3.11. Then dropping the terms in the square root with $|\mathbf{r}_e|$ in the denominator and imposing once again that $|\mathbf{r}_e| >> |\mathbf{r}|$, results in:

$$\Delta S \approx \frac{2Gm_c}{c^3} \ln\left|\frac{2|\mathbf{r}_e|}{|\mathbf{r}| + \hat{\mathbf{r}}_e \cdot \mathbf{r}}\right| = \frac{2Gm_c}{c^3}\left(\ln\left|\frac{1}{|\mathbf{r}| + |\mathbf{r}| \sin(i)\sin(\omega + \phi)}\right| + \ln 2|\mathbf{r}_e|\right). \tag{1.3.13}$$

Here we can drop the constant term once again. Then using Kepler's ellipse equation to write $|\mathbf{r}|$ and once again dropping the constant factor of $\ln|a|$:

$$\Delta S \approx \frac{2Gm_c}{c^3} \ln\left|\frac{1 + e\cos(\phi)}{(1 - e^2)(1 + \sin(i)\sin(\omega + \phi))}\right| \tag{1.3.14}$$

Here we have used the fact that $\hat{\mathbf{r}}_e \cdot \mathbf{r} = |\mathbf{r}| \sin(i)\sin(\omega + \phi)$, as the dot product is the projection of $\mathbf{r}$ onto $\hat{\mathbf{r}}_e$, scaled by $|\mathbf{r}|$. Then:

$$\Delta S = -\frac{2Gm_c}{c^3} \ln\left|\frac{(1 - e^2)(1 + \sin i(\sin\omega\cos\phi + \cos\omega\sin\phi))}{1 + e\cos\phi}\right|. \tag{1.3.15}$$

Where we have expanded $\sin(\omega + \phi)$ and flipped the fraction. Next we introduce again the eccentric anomaly E. Which is related to the phase angle ϕ by: $\cos E = \frac{e + \cos\phi}{1 + e\cos\phi}$ and $\sin E = \frac{\sqrt{1-e^2}\sin\phi}{1+e\cos\phi}$, Thus:

$$1 - e\cos E = 1 - \frac{e^2 + e\cos\phi}{1 + e\cos\phi} = \frac{1 - e^2}{1 + e\cos\phi}, \tag{1.3.16}$$

$$\frac{(1 - e^2)(\sin\phi)}{1 + e\cos\phi} = \sqrt{1 - e^2}\sin E, \tag{1.3.17}$$

and:

$$\cos E - e = \frac{\cos\phi(1 - e^2)}{1 + \cos\phi} \implies \frac{1 - e^2}{1 + e\cos\phi} = \frac{\cos E - e}{\cos\phi}. \tag{1.3.18}$$

Thus plugging into **??**:

$$\Delta S = -\frac{2Gm_c}{c^3} \ln\left|1 - e\cos E - \sin i(\sin\omega(\cos E - e) + \sqrt{1 - e^2}\sin E\cos\omega)\right| \quad \blacksquare \tag{1.3.19}$$

1.4 Shapiro shape, range and the binary mass function:

The Shapiro delay is parameterised by the Shapiro range r and the Shapiro shape s. Where:

$$r = \frac{Gm_c}{c^3} = T_\odot m_c, \tag{1.4.1}$$

$$s = \sin i. \tag{1.4.2}$$

To find the from of s we derive the binary mass function $f(m_c, m_p)$. We start with the semi-major axis of the relative orbit a, which can be written in terms of the semi-major axis of the pulsar and it's companion by: $a = a_p + a_c$. The total mass of the system is $M = m_p + m_c$ and by definition of center of mass $m_p a_p = m_c a_c$. This leads to:

$$a = a_p(1 + \frac{m_p}{m_c}) = \frac{a_p M}{m_c}. \tag{1.4.3}$$

Inserting this into Kepler's law:

$$GM = \frac{a_p^3 M^3 4\pi^2}{m_c^3 P_b^2}. \tag{1.4.4}$$

Then defining the projected semi-major axis as $x = a_p \sin i$, The binary mass function becomes:

$$f(m_c, m_p) = \frac{(m_c \sin i)^3}{(m_c + m_p)^2} = \frac{4\pi^2 x^3}{T_\odot P_b^2}. \tag{1.4.5}$$

Thus:

$$s = \sin i = T_\odot^{-\frac{1}{3}} \left(\frac{2\pi}{P_b}\right)^{\frac{2}{3}} x \frac{(m_p + m_c)^{\frac{2}{3}}}{m_c} \qquad \blacksquare \tag{1.4.6}$$

2 Yukawa addition to General Relativity

The simplest way to modify GR is to generalize the Einstein - Hilbert action by changing from just the Ricci scalar to an arbitrary function of the Ricci scalar $f(R)$, with the simple condition that in the weak field limit the theory tends towards GR. This is preferred over alternate theories that have to produce ad-hoc mechanisms to explain the lack of evidence in Solar system environments. $f(R)$ gravity results in a Yukawa-like term being added to the gravitational potential $\delta e^{-\frac{r}{\lambda}}$. Here λ is the scale length of the theory and must very large in order to reduce to GR in the weak field regime and δ the strength of this theory. Modifying the Einstein-Hilbert action leads to:

$$S = \int \sqrt{-g} f(R) d^4x + S_m. \tag{2.0.1}$$

This then leads to a slightly changed Schwarzschild metric that takes the following form:

$$g = -\left(1 - \frac{2GM}{rc^2}\frac{(1+\delta e^{-\frac{r}{\lambda}})}{1+\delta}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\frac{(1+\delta e^{-\frac{r}{\lambda}})}{1+\delta}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{2.0.2}$$

From here the new PK parameters can be derived, However since through methods like gravitational wave detection, a lower bound of $\lambda > 1.6 \times 10^{16}$ m has been set [1]. Thus it will often be necessary to taylor expand the exponential term.

2.1 Periastron advance $\dot{\omega}$:

Just as we did in section 1.1 we start with a metric, which in this case is the above Yukawa metric 2.0.2. Then as we did before letting the Lagrangian $\mathcal{L} = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ and setting $\theta = \frac{\pi}{2}$, results in:

$$\mathcal{L} = (1+2\Phi)\, c^2\dot{t}^2 - (1+2\Phi)^{-1}\,\dot{r}^2 - r^2\dot{\phi}^2. \tag{2.1.1}$$

Where:

$$\Phi = -\frac{GM}{rc^2}\frac{(1+\delta e^{-\frac{r}{\lambda}})}{1+\delta}. \tag{2.1.2}$$

Both t and ϕ are cyclic so the Lagrange equations become:

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0 \implies E \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{t}} = (1+2\Phi)\, c^2\dot{t}. \tag{2.1.3}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0 \implies L \equiv -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi}. \tag{2.1.4}$$

Thus, subbing these back into the Lagrangian and letting $\lambda \to \tau$ such that $\mathcal{L} = c^2$, results in:

$$\frac{1}{2}(1+2\Phi)c^2 = \frac{E^2}{2c^2} - \frac{1}{2}\dot{r}^2 + \frac{(1+2\Phi)L^2}{2r^2}. \tag{2.1.5}$$

Then letting $\mathcal{E} = \frac{E^2}{2c^2}$ this takes the form:

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}. \tag{2.1.6}$$

. Where:

$$V(r) \equiv \frac{c^2}{2} - \frac{GM(1+\delta e^{-\frac{r}{\lambda}})}{r(1+\delta)} + \frac{L^2}{2r^2} - \frac{L^2 GM((1+\delta e^{-\frac{r}{\lambda}})}{c^2 r^3(1+\delta)}. \tag{2.1.7}$$

Using:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{d\phi}{d\tau}\right)^2 \left(\frac{dr}{d\phi}\right)^2 = \frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2. \tag{2.1.8}$$

Then 2.1.6 becomes:

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{c^2 r^4}{L^2} - \frac{2GM(1+\delta e^{-\frac{r}{\lambda}})r^3}{(1+\delta)L^2} + r^2 + \frac{-2GM(1+\delta e^{-\frac{r}{\lambda}})r}{c^2(1+\delta)} = \frac{2\mathcal{E}r^4}{L^2}. \tag{2.1.9}$$

Using the substitutions $u \equiv \frac{L^2}{GMr} \implies \frac{du}{dr} = -\frac{L^2}{GMr^2} \implies \left(\frac{dr}{d\phi}\right)^2 = \left(\frac{du}{d\phi}\right)^2 \left(\frac{GMr^2}{L^2}\right)^2$. Thus:

$$\left(\frac{du}{d\phi}\right)^2 + \frac{c^2L^2}{(GM)^2} - \frac{2(1+\delta e^{-\frac{L^2}{GM\lambda u}})u}{(1+\delta)} + u^2 - \frac{2(GM)^2(1+\delta e^{-\frac{L^2}{GM\lambda u}})u^3}{c^2L^2(1+\delta)} = \frac{2\mathcal{E}L^2}{(GM)^2}. \tag{2.1.10}$$

Differentiating and canceling all the factors of $2\left(\frac{du}{d\phi}\right)$:

$$\frac{d^2u}{d\phi^2} = -u + \frac{(1+\delta e^{-\frac{L^2}{GM\lambda u}})}{1+\delta} + \frac{\delta e^{-\frac{L^2}{GM\lambda u}} L^2}{GM\lambda(1+\delta)u}$$

$$+ \frac{3(1+\delta e^{-\frac{L^2}{GM\lambda u}})(GM)^2u^2}{c^2L^2(1+\delta)} + \frac{\delta e^{-\frac{L^2}{GM\lambda u}} GMu}{\lambda c^2(1+\delta)}. \tag{2.1.11}$$

Now to simplify this ODE taking advantage of the lower bound of λ, we can taylor expand the exponential terms. $e^{-\frac{L^2}{GM\lambda u}} \approx 1 - \frac{(1-e^2)}{u}(\frac{a}{\lambda}) + \frac{(1-e^2)^2}{2u^2}(\frac{a}{\lambda})^2$. Here we have also used the low eccentricity approximation that $L^2 \approx aGM(1-e^2)$. Thus the ODE becomes:

$$\frac{d^2u}{d\phi^2} = -u + \frac{(1+\delta(1-\frac{(1-e^2)}{u}(\frac{a}{\lambda}) + \frac{(1-e^2)^2}{2u^2}(\frac{a}{\lambda})^2))}{1+\delta} + \frac{\delta(1-\frac{(1-e^2)}{u}(\frac{a}{\lambda}) + \frac{(1-e^2)^2}{2u^2}(\frac{a}{\lambda})^2)L^2}{GM\lambda(1+\delta)u}$$

$$+ \frac{3(1+\delta(1-\frac{(1-e^2)}{u}(\frac{a}{\lambda}) + \frac{(1-e^2)^2}{2u^2}(\frac{a}{\lambda})^2))(GM)^2u^2}{c^2L^2(1+\delta)}$$

$$+ \frac{\delta(1-\frac{(1-e^2)}{u}(\frac{a}{\lambda}) + \frac{(1-e^2)^2}{2u^2}(\frac{a}{\lambda})^2)GMu}{\lambda c^2(1+\delta)}. \tag{2.1.12}$$

Then rearranging and ignoring any terms of order $(\frac{a}{\lambda})^3$ we get the following ODE:

$$\frac{d^2u}{d\phi^2}+u=1-\frac{\delta}{1+\delta}\frac{(1-e^2)}{u}(\frac{a}{\lambda})+\frac{\delta}{1+\delta}\frac{(1-e^2)}{u}(\frac{a}{\lambda})+\frac{3(GM)u^2}{c^2a(1-e^2)}$$
$$+\frac{\delta}{1+\delta}\frac{(1-e^2)^2}{2u^2}(\frac{a}{\lambda})^2-\frac{\delta}{1+\delta}\frac{(1-e^2)^2}{u^2}(\frac{a}{\lambda})^2+\frac{3\delta GM(1-e^2)}{c^2(1+\delta)}(\frac{a}{\lambda^2})$$
$$-\frac{\delta GMu}{c^2(1+\delta)\lambda}+\frac{\delta GMu}{c^2(1+\delta)\lambda}-\frac{\delta(1-e^2GM)}{c^2(1+\delta)}(\frac{a}{\lambda})^2+\mathcal{O}\left(\frac{a^3}{\lambda^3}\right). \tag{2.1.13}$$

Here we can see that a few terms cancel. We will deal with this ODE in a similar manner to how we dealt with the ODE in the GR case. First writing the ODE in terms of single constants:

$$\frac{d^2u}{d\phi^2}+u-1=\frac{b}{u^2}-au^2+c. \tag{2.1.14}$$

Here:

$$a=\frac{3(GM)}{c^2a(1-e^2)},$$
$$b=-\frac{\delta(1-e^2)^2}{2(1+\delta)}(\frac{a}{\lambda})^2,$$
$$c=\frac{3\delta GM(1-e^2)}{c^2(1+\delta)}(\frac{a}{\lambda^2})-\frac{\delta(1-e^2GM)}{c^2(1+\delta)}(\frac{a}{\lambda})^2. \tag{2.1.15}$$

We consider u to be the Newtonian solution plus a small deviation. Thus $u=u_0+u_1$. Then:

$$\frac{d^2u_0}{d\phi^2}-1+u_0=0, \tag{2.1.16}$$
$$\frac{d^2u_1}{d\phi^2}+u_1=au_0^2+\frac{b}{u_0^2}+c+\mathcal{O}(a^2). \tag{2.1.17}$$

Here we impose u_1 is small and so is the factor of a. This assumption that $u_1 \approx a$, is self consistent as can be seen in the equation below where the solution to u_1 is proportional to a. The factor b is also small so to keep consistency we also drop terms in the denominator of the b term. The solution to the first ODE here is well known: : $u_0 = 1 + e\cos\phi$. Subbing this in to the second ODE results in:

$$\frac{d^2u_1}{d\phi^2}+u_1=a(1+e\cos\phi)^2+\frac{b}{(1+e\cos\phi)^2}+c. \tag{2.1.18}$$

To which one solution from mathematica is:

$$u_1=\frac{2be\sin(\phi)\tan^{-1}\left(\frac{(1-e)\tan\left(\frac{\phi}{2}\right)}{\sqrt{1-e^2}}\right)}{(1-e^2)^{3/2}}$$

$$-\frac{e\left[3ae^4+3ae^2+a\left(1-e^2\right)e^2\cos(2\phi)-6\left(1-e^2\right)\sin(\phi)(ae\phi+c_2)-6a-6b+6ce^2-6c\right]}{6e\left(1-e^2\right)}$$

$$-\frac{3\cos(\phi)\left[2b-e\left(1-e^2\right)(ae+2c_1)\right]}{6e\left(1-e^2\right)}. \tag{2.1.19}$$

From this we extract the only term which contributes to the precession, which is the only non-periodic term, thus:

$$u_1 = ae\phi\sin\phi. \tag{2.1.20}$$

Thus:

$$u = 1 + e\cos\phi + ae\phi\sin\phi. \tag{2.1.21}$$

Then by similar methods as section 1.1:

$$r = \frac{L^2}{GM(1+e\cos((1-a)\phi))} + \mathcal{O}(a^2). \tag{2.1.22}$$

Noticeably since a is the same as α in section 1.1, this results in Yukawa gravity predicting the same equation for the periastron advance as GR.

$$\dot{\omega}_{Yk} = \dot{\omega}_{GR} = \frac{3(2\pi)^{5/3}T_{\odot}^{2/3}(m_p+m_c)^{2/3}}{P_b^{5/3}(1-e^2)}. \qquad \blacksquare \tag{2.1.23}$$

2.2 Amplitude of Einstein delay γ:

Again following a similar method to section 1.2 we start with our modified Schwarzschild metric, once again choosing constant values for ϕ and θ so that $d\Omega = 0$. Thus:

$$-c^2 d\tau^2 = -\left(1 + 2\Phi\right) c^2 dt^2 + \left(1 + 2\Phi\right)^{-1} dr^2. \tag{2.2.1}$$

Where Φ is the same as 2.1.2. Then taylor expanding the right most term as once again for pulsars: $\frac{GM}{rc^2} \approx \frac{v_p^2}{c^2} \approx 10^{-6}$, results in:

$$c^2 d\tau^2 = \left(1 - 2\Phi\right) c^2 dt^2 - \left(1 - 2\Phi + \mathcal{O}(\frac{v_p^4}{c^4})\right) dr^2. \tag{2.2.2}$$

The r in dr here is the distance from the pulsar to the barycenter r_p, as to measure the time dilation we need to see how the pulsar is rotating around its center of mass. Then using $v_p = \frac{dr_p}{dt}$:

$$\begin{aligned} c^2 d\tau^2 &= \left(1 - 2\Phi\right) c^2 dt^2 - \left(v_p^2 + 2\Phi v_p^2\right) dt^2, \\ c^2 d\tau^2 &= dt^2 \left(c^2 + 2\Phi c^2 - v_p^2 + \mathcal{O}(\frac{v_p^4}{c^2})\right). \end{aligned} \tag{2.2.3}$$

Then:

$$\frac{d\tau}{dt} = \sqrt{1 + \frac{1}{c^2}(2\Phi c^2 - v_p^2)} = 1 + \Phi - \frac{v_p^2}{2c^2} + \mathcal{O}(\frac{v_p^4}{c^4}). \tag{2.2.4}$$

For the pulsar, the r in the Φ term is the distance from the pulsar to its companion star and the M is m_c as this field is due to the presence of the companion star. Thus:

$$\frac{d\tau}{dt} = 1 - \frac{Gm_c}{rc^2}\frac{(1 + \delta e^{-\frac{r}{\lambda}})}{1 + \delta} - \frac{v_p^2}{2c^2}. \tag{2.2.5}$$

Now we wish to obtain an expression for v_p as we did before in section 1.2. To do this we consider once again the the orbital energy of the binary system:

$$E = \frac{1}{2}\mu v^2 - \frac{Gm_p m_c}{r}\frac{(1 + \delta e^{-\frac{r}{\lambda}})}{(1 + \delta)}. \tag{2.2.6}$$

Where the relation between a and E is:

$$E = -\frac{Gm_p m_c}{2a}\frac{(1 + \delta e^{-\frac{2a}{\lambda}})}{(1 + \delta)}. \tag{2.2.7}$$

This relation can be thought to come from the fact that there are many different possible elliptical orbits with the same a and different E, but as the limit of **??** as $v \to 0$ r must go to $2a$, the length of the ellipses major axis. Using this relation results in the following expression for the orbital velocity:

$$v^2 = \frac{Gm_p m_c}{\mu(1 + \delta)}\left[\frac{2(1 + e^{-\frac{r}{\lambda}})}{r} - \frac{(1 + \delta e^{-\frac{2a}{\lambda}})}{a}\right]. \tag{2.2.8}$$

Then with $v_p = -\frac{\mu}{m_p} v$:

$$v_p^2 = \frac{Gm_c^2}{(m_p + m_c)(1+\delta)} \left[\frac{2(1+e^{-\frac{r}{\lambda}})}{r} - \frac{(1+\delta e^{-\frac{2a}{\lambda}})}{a} \right]. \tag{2.2.9}$$

Then subbing into 2.2.5:

$$\frac{d\tau}{dt} = 1 - \frac{Gm_c}{rc^2}\frac{(1+\delta e^{-\frac{r}{\lambda}})}{1+\delta} - \frac{Gm_c^2}{c^2(m_p+m_c)(1+\delta)} \left[\frac{(1+e^{-\frac{r}{\lambda}})}{r} - \frac{(1+\delta e^{-\frac{2a}{\lambda}})}{2a} \right].$$

$$= 1 - \frac{Gm_c}{c^2(m_p+m_c)(1+\delta)} \left[\frac{(m_p+2m_c)(1+e^{-\frac{r}{\lambda}})}{r} - \frac{m_c(1+\delta e^{-\frac{2a}{\lambda}})}{2a} \right]. \tag{2.2.10}$$

Here we can ignore any constant terms as they just integrate to be proportional to t and can then be ignored by scaling t at the end. Then letting $\sigma = \frac{Gm_c(m_p+2m_c)}{c^2(m_p+m_c)}$ and taylor expanding $e^{-\frac{r}{\lambda}} = 1 - \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} + \mathcal{O}(\frac{r^3}{\lambda^3})$:

$$\frac{d\tau}{dt} = 1 - \frac{\sigma}{r} + \frac{\sigma\delta}{\lambda(1+\delta)} - \frac{\sigma\delta r}{2\lambda^2(1+\delta)}. \tag{2.2.11}$$

Just as in section 1.2 we use the eccentric anomaly E to integrate this expression, as $r = a(1 - e\cos E)$, $\omega t = E - e\sin E$ and thus $\omega dt = (1 - e\cos E)dE$. Thus:

$$\tau = \frac{1}{\omega}\int \left[1 - \frac{\sigma}{a(1-e\cos E)} + \frac{\sigma\delta}{\lambda(1+\delta)} - \frac{\sigma\delta(1-e\cos E)}{2\lambda^2(1+\delta)} \right](1-e\cos E)dE,$$

$$= \frac{1}{\omega}\int \left[(1-e\cos E)(1+\frac{\sigma\delta}{\lambda(1+\delta)}) - \frac{\sigma}{a} - \frac{\sigma\delta a(1-e\cos E)^2}{2\lambda^2(1+\delta)} \right] dE \tag{2.2.12}$$

Which becomes:

$$\implies \tau = \frac{1}{\omega}(E - e\sin E)(1 + \frac{\sigma\delta}{\lambda(1+\delta)}) - \frac{\sigma}{\omega a}E$$
$$- \frac{\sigma\delta a}{2\omega\lambda^2(1+\delta)}(E - 2e\sin E + \frac{e^2}{2}(E + \frac{1}{2}\sin 2E)) \tag{2.2.13}$$

Then converting back to t using $t = \frac{1}{\omega}(E - e\sin E)$ and for any singular E terms using $E = \omega t + e\sin E$:

$$\implies \tau = t\left(1 + \frac{\sigma\delta}{\lambda(1+\delta)} - \frac{\sigma}{a} - \frac{\sigma\delta a}{2\lambda^2(1+\delta)}(1+\frac{e^2}{2})\right)$$
$$-\left(\frac{\sigma}{a} - \frac{\sigma\delta a}{2\lambda^2(1+\delta)}(1-\frac{e^2}{2})\right)\frac{e}{\omega}\sin E + \frac{\sigma\delta a e^2}{8\lambda^2\omega(1+\delta)}\sin 2E \tag{2.2.14}$$

Re-scaling t to absorb the constants results in:

$$\tau = t - \left(\frac{\sigma}{a} - \frac{\sigma\delta a}{2\lambda^2(1+\delta)}(1-\frac{e^2}{2})\right)\frac{e}{\omega}\sin E + \frac{\sigma\delta a e^2}{8\lambda^2\omega(1+\delta)}\sin 2E \tag{2.2.15}$$

Notably we can see here that Yukawa like gravity not only predicts an extra term added to the amplitude of the GR Einstein delay, but also predicts there to be an extra delay proportional to $\sin 2E$. Thus the over all modification to the arrival times caused by gravitational red shift and time dilatation is:

$$\tau = t + \gamma_1 \sin E + \gamma_2 \sin 2E \tag{2.2.16}$$

Where:

$$\gamma_1 = \gamma_{GR} + \gamma_0 = -\frac{Gm_c(m_p + 2m_c)e}{c^2(m_p + m_c)a\omega} + \frac{Gm_c(m_p + 2m_c)e\delta a}{2c^2\lambda^2\omega(1+\delta)(m_p + m_c)}(1 - \frac{e^2}{2})$$

$$\gamma_2 = \frac{Gm_c(m_p + 2m_c)\delta a e^2}{8c^2\lambda^2\omega(1+\delta)(m_p + m_c)} \tag{2.2.17}$$

Thus:

$$\gamma_1 = -T_\odot^{\frac{2}{3}} \left(\frac{2\pi}{P_b}\right)^{-\frac{1}{3}} \frac{m_c(m_p + 2m_c)}{(m_p + m_c)^{\frac{4}{3}}} e + T_\odot^{\frac{4}{3}} c^2 \left(\frac{2\pi}{P_b}\right)^{-\frac{5}{3}} \frac{m_c(m_p + 2m_c)e\delta(1 - \frac{e^2}{2})}{2\lambda^2(1+\delta)(m_p + m_c)^{\frac{2}{3}}}, \tag{2.2.18}$$

$$\gamma_2 = T_\odot^{\frac{4}{3}} c^2 \left(\frac{2\pi}{P_b}\right)^{-\frac{5}{3}} \frac{m_c(m_p + 2m_c)\delta e^2}{8\lambda^2(1+\delta)(m_p + m_c)^{\frac{2}{3}}}. \qquad \blacksquare \tag{2.2.19}$$

2.3 Shapiro delay ΔS:

Once again we begin with the modified Schwarzschild metric. Though this time as we are dealing with the delay in the travel time of light beam, thus the metric becomes light like meaning: $ds^2 = 0$. Then once again fixing ϕ and θ so that the solid angle is 0 the metric becomes:

$$
\begin{aligned}
&-(1+2\Phi)\,c^2dt^2 + (1+2\Phi)^{-1}\,dr^2 = 0, \\
\implies &\left|\frac{d\mathbf{r}}{dt}\right| = c\,(1+2\Phi) = c\left(1 - \frac{2Gm_c(1+\delta e^{-\frac{r}{\lambda}})}{c^2|\mathbf{r}|(1+\delta)}\right).
\end{aligned} \tag{2.3.1}
$$

Where $M \to m_c$ as the gravity well of the pulsar is just a constant and doesn't contribute to the spacing of the time of arrivals. Dividing across and taylor expanding as $\frac{2Gm_c}{c^2r}$ is small, results in:

$$
\begin{aligned}
&\left|\frac{dt}{d\mathbf{r}}\right| = \frac{1}{c}\left(1 + \frac{2Gm_c(1+\delta e^{-\frac{r}{\lambda}})}{c^2|\mathbf{r}|(1+\delta)}\right), \\
\implies t_{arr} - t_{em} &= \int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})} \frac{1}{c}\left(1 + \frac{2Gm_c(1+\delta e^{-\frac{r}{\lambda}})}{c^2|\mathbf{r}|(1+\delta)}\right)|d\mathbf{r}|, \\
\implies t_{arr} - t_{em} &= \frac{1}{c}(\mathbf{r}_p(t_{em}) - \mathbf{r}_e(t_{arr})) + \frac{2Gm_c}{c^3(1+\delta)}\int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})} \frac{(1+\delta e^{-\frac{r}{\lambda}})}{|\mathbf{r}|}|d\mathbf{r}|.
\end{aligned} \tag{2.3.2}
$$

Here $\mathbf{r}_e$ is the vector pointing from the origin which is the binary barycenter to the earth, t_{em} is the time of emission and t_{arr} is the time of arrival. Then once again taylor expanding $e^{-\frac{r}{\lambda}} = 1 - \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} + \mathcal{O}(\frac{r^3}{\lambda^3})$, The integral on the right hand side, which corresponds to the total Shapiro delay, becomes:

$$
\Delta S = \frac{2Gm_c}{c^3(1+\delta)}\int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})}\left(\frac{(1+\delta)}{|\mathbf{r}|} - \frac{\delta}{\lambda} + \frac{\delta|\mathbf{r}|}{2\lambda^2}\right)|d\mathbf{r}| \tag{2.3.3}
$$

From here we can again drop the constant term in the integral. Then changing from $|\mathbf{r}|$ to a function of time, as $|\mathbf{r}| = |\mathbf{x}(t) - \mathbf{r}_c(t_{em})|$, i.e. $|\mathbf{r}|$ is the distance from the path of the light to the companion star. Here the path of the light $\mathbf{x}(t)$ is given by:

$$
\mathbf{x}(t) = \mathbf{r}_p(t_{em}) + \frac{t - t_{em}}{t_{arr} - t_{em}}(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})). \tag{2.3.4}
$$

Changing to a function of time picks up an extra factor of c as $|d\mathbf{r}| = cdt$. Thus the full Shapiro delay becomes:

$$
\Delta S = \Delta S_{GR} + \frac{\delta Gm_c}{c^2\lambda^2(1+\delta)}\int_{t_{em}}^{t_{arr}} |\mathbf{x}(t) - \mathbf{r}_c(t_{em})|dt. \tag{2.3.5}
$$

The calculation of ΔS_{GR} is omitted here but can be found in section 1.3. So in Yukawa gravity the Shapiro delay is given by:

$$
\Delta S = \Delta S_{GR} + \Delta S_{YK}. \tag{2.3.6}
$$

Then again using the substitution $\theta = \frac{t - t_{em}}{t_{arr} - t_{em}}$ the the bounds of the integral now go from 0 to 1, resulting in:

$$\Delta S_{YK} = \frac{\delta G m_c (t_{arr} - t_{em})}{c^2 \lambda^2 (1+\delta)} \int_0^1 |\mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em}) + \theta(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em}))| d\theta. \tag{2.3.7}$$

Then seeing as $|\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})| \approx c(t_{arr} - t_{em})$ and $|\mathbf{r}_e(t_{arr})| \equiv |\mathbf{r}_e| >> |\mathbf{r}_p(t_{em})|$:

$$\Delta S_{YK} = \frac{\delta G m_c |\mathbf{r}_e|}{c^3 \lambda^2 (1+\delta)} \int_0^1 |\mathbf{r} + \theta \mathbf{r}_e| d\theta = \frac{\delta G m_c |\mathbf{r}_e|^2}{c^3 \lambda^2 (1+\delta)} \int_0^1 |\frac{\mathbf{r}}{|\mathbf{r}_e|} + \theta \hat{\mathbf{r}}_e| d\theta \tag{2.3.8}$$

Where $\mathbf{r} = \mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em})$. Then:

$$\begin{aligned} \Delta S_{YK} &= \frac{\delta G m_c |\mathbf{r}_e|^2}{c^3 \lambda^2 (1+\delta)} \int_0^1 \sqrt{\theta^2 + 2\theta \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|} + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2} d\theta. \\ &= \frac{\delta G m_c |\mathbf{r}_e|^2}{c^3 \lambda^2 (1+\delta)} \int_0^1 \sqrt{(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2 + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2} d\theta. \end{aligned} \tag{2.3.9}$$

Then letting $ku = \theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}$, where $k^2 = (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})^2$. The integral then becomes:

$$\Delta S_{YK} = \frac{\delta G m_c |\mathbf{r}_e|^2}{c^3 \lambda^2 (1+\delta)} \int k^2 \sqrt{u^2+1} du. \tag{2.3.10}$$

Then letting $u = \sinh\psi$, the integral can be solved by:

$$\begin{aligned} I = \int \sqrt{u^2+1} du &= \int cosh^2 \psi d\psi = \int \frac{1}{2}(1 + \cosh 2\psi) d\psi \\ &= \frac{1}{2}(\psi + \frac{1}{2}\sinh 2\psi) = \frac{1}{2}(\psi + \sinh\psi \cosh\psi). \end{aligned} \tag{2.3.11}$$

Then using the fact that $\cosh(\sinh^{-1} u) = \sqrt{u^2+1}$ and $\sinh^{-1} = \ln|u + \sqrt{u^2+1}|$. Then:

$$I = \frac{1}{2} u \sqrt{u^2+1} + \frac{1}{2} \ln|u + \sqrt{u^2+1}|. \tag{2.3.12}$$

And thus equation 2.3.10 becomes:

$$\frac{\delta G m_c |\mathbf{r}_e|^2 k^2}{c^3 \lambda^2 (1+\delta)} \Bigg[\frac{1}{2}(\frac{1}{k}(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})) \sqrt{(\frac{1}{k}(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}))^2 + 1}$$
$$+ \frac{1}{2} \ln \left| (\frac{1}{k}(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})) + \sqrt{(\frac{1}{k}(\theta + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}))^2 + 1} \right| \Bigg] \Bigg|_0^1$$

$$= \frac{\delta G m_c |\mathbf{r}_e|^2 k^2}{c^3 \lambda^2 (1+\delta)} \Bigg[\frac{1}{2}(\frac{1}{k}(1 + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})) \sqrt{(\frac{1}{k}(1 + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}))^2 + 1}$$
$$+ \frac{1}{2} \ln \left| (\frac{1}{k}(1 + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})) + \sqrt{(\frac{1}{k}(1 + \hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}))^2 + 1} \right|$$
$$- \frac{1}{2}(\frac{1}{k}(\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})) \sqrt{(\frac{1}{k}(\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}))^2 + 1}$$
$$- \frac{1}{2} \ln \left| (\frac{1}{k}(\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|})) + \sqrt{(\frac{1}{k}(\hat{\mathbf{r}}_e \cdot \frac{\mathbf{r}}{|\mathbf{r}_e|}))^2 + 1} \right| \Bigg].$$

Then pulling out factors of k:

$$\Delta S_{YK} = \frac{\delta G m_c |\mathbf{r}_e|^2}{c^3\lambda^2(1+\delta)}\left[\frac{1}{2}(1+\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})\sqrt{(1+\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2+k^2}\right.$$
$$+\frac{k^2}{2}\ln\left|(1+\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})+\sqrt{(1+\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2+k^2}\right|$$
$$-\frac{1}{2}((\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|}))\sqrt{(\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2+k^2}$$
$$\left.-\frac{k^2}{2}\ln\left|(\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})+\sqrt{(\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2+k^2}\right|+\ln|k|-\ln|k|\right].$$

Then subbing in for k and imposing $\frac{|\mathbf{r}|}{|\mathbf{r}_e|} << 1$, leaves the only remaining terms to be the logarithm terms:

$$\Delta S_{YK} = \frac{\delta G m_c |\mathbf{r}_e|^2 k^2}{2c^3\lambda^2(1+\delta)}\left[\ln|2|-\ln\left|\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|}+\frac{|\mathbf{r}|}{|\mathbf{r}_e|}\right|\right]$$
$$= -\frac{\delta G m_c\big((\mathbf{r})^2-(\hat{\mathbf{r}}_e\cdot\mathbf{r})^2\big)}{2c^3\lambda^2(1+\delta)}\ln|\hat{\mathbf{r}}_e\cdot\mathbf{r}+|\mathbf{r}||\,. \tag{2.3.13}$$

Where we have dropped any constant terms to get to the last line. Then seeing as $\hat{\mathbf{r}}_e\cdot\mathbf{r} = |\mathbf{r}|\sin(i)\sin(\omega+\phi)$ and writing the logarithm term as we did in section 1.3 results in:

$$\Delta S_{YK} = -\frac{\delta G m_c a^2(1-e^2)^2(1-\sin^2(\omega+\phi)\sin^2(i))}{2c^3\lambda^2(1+\delta)(1+e\cos\phi)}$$
$$\times\ln\left|1-e\cos E-\sin i(\sin\omega(\cos E-e)+\sqrt{1-e^2}\sin E\cos\omega)\right|. \tag{2.3.14}$$

Then by equations 1.3.16 - 1.3.18 and Kepler's ellipse equation:

$$\Delta S_{YK} = -\frac{\delta G m_c a^2[(1-e\cos E)^2-\sin^2 i(\sin\omega(\cos E-e)+\sqrt{1-e^2}\sin E\cos\omega)^2]}{2c^3\lambda^2(1+\delta)}$$
$$\times\ln\left|1-e\cos E-\sin i(\sin\omega(\cos E-e)+\sqrt{1-e^2}\sin E\cos\omega)\right|. \quad\blacksquare \tag{2.3.15}$$

3 Examining higher order corrections in GR

Due to the upper bound on λ [1] the corrections to the GR PK parameters are exceedingly small, smaller in fact then the terms we drop in the derivations of the GR parameters above. Thus if we wish to tell these deviations apart from those that come from ignoring terms of similar magnitude in GR, we need to find expressions for the GR PK parameters that only ignore terms smaller then those of Yukawa gravity corrections.

3.1 Periastron advance $\dot{\omega}$:

We begin by looking at the periastron advance, however there is an immediate problem that in equation 1.1.16 we the terms of order $\alpha u_1 u_0$ or higher from the ODE as we make the assumption that u_1 is a small deviation, which is self consistent as it becomes proportional to α. However attempting to restore just the $\alpha u_1 u_0$ term to the ODE results in an ODE that is not solvable by mathematica, preventing any consistent higher order corrections from being made to the periastron advance. This is not a huge deal as the Yukawa gravity predicts the same periastron advance formula as GR so it can be left at its current order.

3.2 Einstein delayγ:

For the Einstein delay we can repeat the same method of calculation instead now ignoring terms any higher than the smallest terms in the equations derived in section 2.2, i.e. ignoring any terms smaller than $\gamma_0 \approx \gamma_2 \approx 10^{-16}$. We can note that this is ridiculously small, but a full breakdown of terms to see if deviations of the theories could be seen at the leading order in Yukawa gravity corrections, is needed.

Through out this derivation we use the fact that $\frac{GM}{rc^2} \approx \frac{v_p^2}{c^2} \approx 10^{-6}$. We begin a step on from the Schwarzschild metric, equation 1.2.3:

$$c^2 d\tau^2 = \left(1 - \frac{2Gm_c}{rc^2}\right) c^2 dt^2 - \left(v_p^2 + \frac{2Gm_c v_p^2}{rc^2}\right) dt^2.$$

$$= dt^2 \left(c^2 - \frac{2Gm_c}{r} - v_p^2 - \frac{2Gm_c v_p^2}{rc^2}\right). \tag{3.2.1}$$

Where $v_p = \frac{dr_p}{dt}$, Thus:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\left(\frac{2Gm_c}{r} + v_p^2 + \frac{2Gm_c v_p^2}{rc^2}\right)}$$

$$= 1 - \frac{1}{2c^2}\left[\frac{2Gm_c}{r} + v_p^2(1 + \frac{2Gm_c}{rc^2})\right]$$

$$-\frac{1}{8c^4}\left[\frac{4(Gm_c)^2}{r^2} + \frac{4Gm_c}{r}(v_p^2 + \frac{2Gm_c v_p^2}{rc^2}) + v_p^4 + \frac{4Gm_c v_p^4}{rc^2} + \frac{4(Gm_c)^2 v_p^4}{r^2c^4}\right] + \mathcal{O}(\frac{v_p^6}{c^6}). \tag{3.2.2}$$

Where we have taylor expanded in the last step. Then dropping more terms:

$$\frac{d\tau}{dt} = 1 - \frac{Gm_c}{rc^2} - \frac{v_p^2}{2c^2} - \frac{Gm_c v_p^2}{rc^4} - \frac{(Gm_c)^2}{2r^2c^4} - \frac{Gm_c v_p^2}{2rc^4} - \frac{v_p^4}{8c^4} + \mathcal{O}(\frac{v_p^6}{c^6}). \tag{3.2.3}$$

Then using equation 1.2.8, which says:

$$v_p^2 = \frac{Gm_c^2}{m_p + m_c}\left(\frac{2}{r} - \frac{1}{a}\right). \tag{3.2.4}$$

Thus:

$$v_p^4 = \frac{(Gm_c^2)^2}{(m_p + m_c)^2}\left(\frac{4}{r^2} - \frac{2}{ra} + \frac{1}{a^2}\right). \tag{3.2.5}$$

Then ignoring constant terms as we have done before, due to the fact that after we integrate we can re scale t to include any constant terms in the RHS of equation 3.2.3. Then 3.2.3 becomes:

$$\begin{aligned}\frac{d\tau}{dt} = 1 &- \frac{Gm_c}{rc^2} - \frac{Gm_c^2}{c^2(m_p+m_c)r} - \frac{3G^2m_c^3}{c^4(m_p+m_c)r^2} - \frac{3G^2m_c^3}{2c^4(m_p+m_c)ra} \\ &- \frac{(Gm_c)^2}{2r^2c^4} - \frac{G^2m_c^4}{2c^4(m_p+m_c)^2r^2} + \frac{G^2m_c^4}{4c^4(m_p+m_c)^2ra}.\end{aligned} \tag{3.2.6}$$

$$\begin{aligned}= 1 &- \frac{1}{r}\left[\frac{Gm_c}{c^2} + \frac{Gm_c^2}{c^2(m_p+m_c)} - \frac{G^2m_c^4}{4c^4(m_p+m_c)^2a} + \frac{3G^2m_c^3}{2c^4(m_p+m_c)a}\right] \\ &- \frac{1}{r^2}\left[\frac{3G^2m_c^3}{c^4(m_p+m_c)} + \frac{(Gm_c)^2}{2c^4} + \frac{G^2m_c^4}{2c^4(m_p+m_c)^2}\right].\end{aligned} \tag{3.2.7}$$

We wish to re-arrange to return the original GR parameter plus extra contributions. Thus:

$$\begin{aligned}= 1 &- \frac{1}{r}\left[\frac{Gm_c(m_p+2m_c)}{c^2(m_p+m_c)} + \frac{G^2m_c^3(6m_p+5m_c)}{4c^4(m_p+m_c)^2a}\right] \\ &- \frac{1}{r^2}\left[\frac{G^2m_c^2(9m_c^2+10m_cm_p+m_p^2)}{c^4(m_p+m_c)^2}\right].\end{aligned} \tag{3.2.8}$$

We can see that the first term will return the delay calculated in section 1.2, we will however, temporarily combine this term for the sake of brevity. Thus:

$$\frac{d\tau}{dt} = 1 - \frac{\alpha}{r} - \frac{\beta}{r^2}$$

Once again to carry out the integral we employ the use of the eccentric anomaly E. $r = a(1 - e\cos E)$, $\omega t = E - e\sin E$ and thus $\omega dt = (1 - e\cos E)dE$. This means the integral becomes:

$$\begin{aligned}\tau &= \frac{1}{\omega}\int\left[1 - \frac{\alpha}{a(1-e\cos E)} - \frac{\beta}{a^2(1-e\cos E)^2}\right](1-e\cos E)dE, \\ &= \frac{1}{\omega}\int\left[1 - e\cos E - \frac{\alpha}{a} - \frac{\beta}{a^2(1-e\cos E)}\right]dE, \\ &= \frac{1}{\omega}(E - e\sin E - \frac{\alpha}{a}E) - \frac{\beta}{\omega a^2}\int\frac{dE}{(1-e\cos E)}.\end{aligned} \tag{3.2.9}$$

To evaluate this integral we use Weierstrass substitution, that is writing the $\cos E$ in terms of $\tan \frac{E}{2}$.

$$I = \int \frac{dE}{(1 - e\cos E)} = \int \frac{dE}{(1 - e\frac{1-\tan^2 \frac{E}{2}}{\tan^2 \frac{E}{2}+1})} = \int \frac{\sec^2 \frac{E}{2} dE}{(1+e)\tan^2 \frac{E}{2} + (1-e)} \tag{3.2.10}$$

Then letting $u = \frac{\sqrt{1+e}}{\sqrt{1-e}} \tan \frac{E}{2}$, thus $dE = \frac{\sqrt{1-e}}{\sqrt{1+e}} \frac{2du}{\sec^2 \frac{E}{2}}$. Which leads to:

$$= 2\frac{\sqrt{1-e}}{\sqrt{1+e}} \int \frac{du}{(1-e)(u^2+1)} = \frac{2}{\sqrt{1-e^2}} \int \frac{du}{u^2+1} \tag{3.2.11}$$

This final integral is easily recognised as $\arctan u$. Thus the final solution becomes:

$$I = \frac{2}{\sqrt{1-e^2}} \arctan\left(\frac{\sqrt{1+e}}{\sqrt{1-e}} \tan \frac{E}{2}\right) = \frac{2}{\sqrt{1-e^2}} \arctan\left(\frac{1+e}{\sqrt{1-e^2}} \tan \frac{E}{2}\right) \tag{3.2.12}$$

To put this in a form more akin to that of the Yukawa gravity contributions, we consider the taylor expansion of the above solution to the integral around $\frac{1+e}{\sqrt{1-e^2}} = 1$, as we are usually dealing with small e, thus $\frac{1+e}{\sqrt{1-e^2}} \approx 1$ The resulting expansion, letting $x = \frac{1+e}{\sqrt{1-e^2}}$ is:

$$\begin{aligned}
\arctan\left(x \tan \frac{E}{2}\right) &= \arctan\left(\tan\left(\frac{E}{2}\right)\right) + \frac{(x-1)\tan(\frac{E}{2})}{\tan^2(\frac{E}{2})+1} - \frac{(x-1)^2 \tan^3(\frac{E}{2})}{\left(\tan^2(\frac{E}{2})+1\right)^2} \\
&+ \frac{(x-1)^3 \tan^3(\frac{E}{2})\left(3\tan^2(\frac{E}{2})-1\right)}{3\left(\tan^2(\frac{E}{2})+1\right)^3} - \frac{(x-1)^4 \tan^5\left(\frac{E}{2}\right)\left(\tan^2\left(\frac{E}{2}\right)-1\right)}{\left(1+\tan^2\left(\frac{E}{2}\right)\right)^4} \\
&+ \frac{(x-1)^5 \tan\left(\frac{e}{2}\right)\left(\frac{16\tan^8(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^4} - \frac{12\tan^6(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^3} + \frac{\tan^4(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^2}\right)}{5\left(1+\tan^2\left(\frac{e}{2}\right)\right)} \\
&+ \frac{(x-1)^7 \tan\left(\frac{e}{2}\right)\left(\frac{64\tan^{12}(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^6} - \frac{80\tan^{10}(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^5} + \frac{24\tan^8(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^4} - \frac{\tan^6(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^3}\right)}{7\left(1+\tan^2\left(\frac{e}{2}\right)\right)} \\
&+ \frac{(x-1)^6 \tan\left(\frac{e}{2}\right)\left(-\frac{32\tan^{10}(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^5} + \frac{32\tan^8(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^4} - \frac{6\tan^6(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^3}\right)}{6\left(1+\tan^2\left(\frac{e}{2}\right)\right)} \\
&+ \frac{(x-1)^8 \tan\left(\frac{e}{2}\right)\left(-\frac{128\tan^{14}(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^7} + \frac{192\tan^{12}(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^6} - \frac{80\tan^{10}(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^5} + \frac{8\tan^8(\frac{e}{2})}{\left(1+\tan^2(\frac{e}{2})\right)^4}\right)}{8\left(1+\tan^2\left(\frac{e}{2}\right)\right)} \\
&+ O\left((x-1)^9\right)
\end{aligned} \tag{3.2.13}$$

Here if we consider $e \approx 10^{-1}$, then $(x-1) \approx 10^{-1}$. Then if we consider $\frac{\beta}{a^2} \approx \frac{v_p^4}{c^4} \approx 10^{-12}$ and the orbital period P_b to be on the order of roughly 0.1 days, then $\omega \approx 10^{-4}$ and thus $\frac{\beta}{\omega a^2} \approx 10^{-8}$. So in order to obtain the same order as that predicted in section 2.2 for Yukawa gravity we keep only terms larger than $O\left((x-1)^9\right) \approx 10^{-9}$. Using mathematica this expansion can be simplified to:

$$\frac{E}{2}+\frac{1}{2}(x-1)\sin(E)-\frac{1}{4}(x-1)^2\sin(E)+\frac{1}{8}(x-1)^3\sin(E)-\frac{1}{16}(x-1)^4\sin(E)+\frac{1}{32}(x-1)^5\sin(E)$$
$$-\frac{1}{64}(x-1)^6\sin(E)+\frac{1}{128}(x-1)^7\sin(E)-\frac{1}{256}(x-1)^8\sin(E)+\frac{1}{8}(x-1)^2\sin(2E)-\frac{1}{8}(x-1)^3\sin(2E)$$
$$+\frac{3}{32}(x-1)^4\sin(2E)-\frac{1}{16}(x-1)^5\sin(2E)+\frac{5}{128}(x-1)^6\sin(2E)-\frac{3}{128}(x-1)^7\sin(2E)+\frac{7}{512}(x-1)^8\sin(2E)$$
$$+\frac{1}{24}(x-1)^3\sin(3E)-\frac{1}{16}(x-1)^4\sin(3E)+\frac{1}{16}(x-1)^5\sin(3E)-\frac{5}{96}(x-1)^6\sin(3E)+\frac{5}{128}(x-1)^7\sin(3E)$$
$$-\frac{7}{256}(x-1)^8\sin(3E)+\frac{1}{64}(x-1)^4\sin(4E)-\frac{1}{32}(x-1)^5\sin(4E)+\frac{5}{128}(x-1)^6\sin(4E)$$
$$-\frac{5}{128}(x-1)^7\sin(4E)+\frac{35(x-1)^8\sin(4E)}{1024}+\frac{1}{160}(x-1)^5\sin(5E)-\frac{1}{64}(x-1)^6\sin(5E)$$
$$+\frac{3}{128}(x-1)^7\sin(5E)-\frac{7}{256}(x-1)^8\sin(5E)+\frac{1}{384}(x-1)^6\sin(6E)-\frac{1}{128}(x-1)^7\sin(6E)$$
$$+\frac{7}{512}(x-1)^8\sin(6E)+\frac{1}{896}(x-1)^7\sin(7E)-\frac{1}{256}(x-1)^8\sin(7E)+\frac{(x-1)^8\sin(8E)}{2048} \quad (3.2.14)$$

Then dropping a few terms that have factors that make them smaller then the limit results in:

$$I=\frac{2}{\sqrt{1-e^2}}\left[\frac{E}{2}+\frac{1}{2}(x-1)\sin(E)-\frac{1}{4}(x-1)^2\sin(E)+\frac{1}{8}(x-1)^3\sin(E)-\frac{1}{16}(x-1)^4\sin(E)+\right.$$
$$\frac{1}{32}(x-1)^5\sin(E)-\frac{1}{64}(x-1)^6\sin(E)+\frac{1}{8}(x-1)^2\sin(2E)-\frac{1}{8}(x-1)^3\sin(2E)$$
$$+\frac{3}{32}(x-1)^4\sin(2E)-\frac{1}{16}(x-1)^5\sin(2E)+\frac{5}{128}(x-1)^6\sin(2E)+\frac{1}{24}(x-1)^3\sin(3E)$$
$$-\frac{1}{16}(x-1)^4\sin(3E)+\frac{1}{16}(x-1)^5\sin(3E)-\frac{5}{96}(x-1)^6\sin(3E)+\frac{1}{64}(x-1)^4\sin(4E)$$
$$\left.-\frac{1}{32}(x-1)^5\sin(4E)+\frac{5}{128}(x-1)^6\sin(4E)+\frac{1}{160}(x-1)^5\sin(5E)-\frac{1}{64}(x-1)^6\sin(5E)\right] \quad (3.2.15)$$

Subbing back in, with $x=\frac{1+e}{\sqrt{1-e^2}}$. Results in:

$$I=\left[\frac{E}{\sqrt{1-e^2}}+\sin E\left(\frac{(e-2)\left(\sqrt{1-e^2}-e-1\right)\left(-9e^2+7\sqrt{1-e^2}e-14\sqrt{1-e^2}+e+10\right)}{8(e-1)^3(e+1)\sqrt{1-e^2}}\right)\right.$$
$$-\sin 2E\left(\frac{\left(\sqrt{1-e^2}-1\right)\left(e^2-14\sqrt{1-e^2}e+28\sqrt{1-e^2}+26e-32\right)}{8(e-1)^3\sqrt{1-e^2}}\right)$$
$$+\sin 3E\left(\frac{\left(\sqrt{1-e^2}-e-1\right)^3\left(14e^2+21\sqrt{1-e^2}-5e-19\right)}{24(e-1)^3(e+1)^2\sqrt{1-e^2}}\right)$$

$$+\sin 4E\left(\frac{\left(-\sqrt{1-e^2}+e+1\right)^4\left(7\sqrt{1-e^2}+3e-8\right)}{32(e-1)^3(e+1)^2\sqrt{1-e^2}}\right)$$

$$+\sin 5E\left(\frac{\left(7-\frac{5(e+1)}{\sqrt{1-e^2}}\right)\left(\frac{e+1}{\sqrt{1-e^2}}-1\right)^5}{160\sqrt{1-e^2}}\right)\Bigg] \tag{3.2.16}$$

For the sake of brevity we label these functions of e as $f_1(e), f_2(e)$... etc. Now we wish to put this all together, so equation 3.2.9 after subbing in $\omega t = E - e\sin E$ once again becomes:

$$\tau = \left(1-\frac{\alpha}{a}-\frac{\beta}{a^2\sqrt{1-e^2}}\right)t+\left(\frac{\alpha}{\omega a}+\frac{\beta}{\omega a^2}(\frac{1}{\sqrt{1-e^2}}-f_1(e))\right)\sin E$$
$$-\frac{\beta}{\omega a^2}\left[-f_2(e)\sin 2E+f_3(e)\sin 3+f_4(e)\sin 4E+f_5(e)\sin 5E\right]. \tag{3.2.17}$$

Which re-scaling t becomes:

$$\tau = t+\left(\frac{\alpha}{\omega a}+\frac{\beta}{\omega a^2}(\frac{1}{\sqrt{1-e^2}}-f_1(e))\right)\sin E$$

$$+\frac{\beta}{\omega a^2}\left[f_2(e)\sin 2E-f_3(e)\sin 3-f_4(e)\sin 4E-f_5(e)\sin 5E\right]. \tag{3.2.18}$$

We can write this then as:

$$\tau = t+\gamma_1\sin E+\gamma_2\sin 2E+\gamma_3\sin 3E+\gamma_4\sin 4E+\gamma_5\sin 5E \tag{3.2.19}$$

Where if we were to separate the original GR term, we would see that $\gamma_1 = \gamma_{GR} + \gamma_0$, where γ_0 is made of various other small terms.

What this analysis shows is that at the same order we begin to see terms from Yukawa gravity, GR predicts a whole host of higher frequency terms. It can also be noted that the factors of the $\sin E$ and $\sin 2E$, terms predicted by Yukawa gravity have terms from GR that are larger and thus would be far more noticeable. the Yukawa terms are of order 10^{-16}, where as extra GR corrections are as large as 10^{-8} and as small as 10^{-16}, compared to the original GR terms of order 10^{-6}. This is further evidence that Yukawa gravity does not deviate from GR enough in the regime of Solar system size systems, to be noticed in the measurements of post Keplerian parameters.

3.3 Shapiro delay ΔS:

Finally we look at adding higher order corrections to the Shapiro delay. To account for corrections of similar magnitude to the corrections derived under Yukawa gravity in section 2.3, we look at the next leading term in the taylor expansion carried out in equation 1.3.2. Which now becomes:

$$\left|\frac{dt}{d\mathbf{r}}\right| = \frac{1}{c}\left(1 + \frac{2Gm_c}{c^2|\mathbf{r}|} + \frac{4(Gm_c)^2}{c^4|\mathbf{r}|^2}\right) + \mathcal{O}\left((\frac{2Gm_c}{c^2|\mathbf{r}|})^3\right) \tag{3.3.1}$$

Integrating the first two terms returns the Shapiro delay calculated in section **??**, so we will just focus on integrating the last term.

$$t_{arr} - t_{em} = \frac{1}{c}(\mathbf{r}_p(t_{em}) - \mathbf{r}_e(t_{arr})) + \Delta S_{GR} + \int_{\mathbf{r}_p(t_{em})}^{\mathbf{r}_e(t_{arr})} \frac{4(Gm_c)^2}{c^5|\mathbf{r}|^2}\,|d\mathbf{r}|\,. \tag{3.3.2}$$

We follow the same procedure as section 1.3. Writing $|\mathbf{r}|$ as $|\mathbf{x}(t) - \mathbf{r}_c(t_{em})|$, where $\mathbf{x}(t)$ is the same as equation 1.3.6 and switching to integrating over time ($|d\mathbf{r}| = cdt$). Once again we use the substitution $\theta = \frac{t-t_{em}}{t_{arr}-t_{em}}$, making the bounds of the integral go from 0 to 1. Thus:

$$\Delta S = \Delta S_{GR} + \frac{4(Gm_c)^2}{c^4}\int_0^1 \frac{(t_{arr} - t_{em})d\theta}{|\mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em}) + \theta(\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em}))|^2}. \tag{3.3.3}$$

Then seeing as $|\mathbf{r}_e(t_{arr}) - \mathbf{r}_p(t_{em})| \approx c(t_{arr} - t_{em})$ and $|\mathbf{r}_e(t_{arr})| \equiv |\mathbf{r}_e| >> |\mathbf{r}_p(t_{em})|$:

$$\Delta S \approx \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5|\mathbf{r}_e|}\int_0^1 \frac{d\theta}{|\frac{\mathbf{r}}{|\mathbf{r}_e|} + \theta\hat{\mathbf{r}}_e|^2} = \frac{4(Gm_c)^2}{c^5|\mathbf{r}_e|}\int_0^1 \frac{d\theta}{\theta^2 + 2\theta\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|} + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2}. \tag{3.3.4}$$

Where $\mathbf{r} = \mathbf{r}_p(t_{em}) - \mathbf{r}_c(t_{em})$. Then:

$$\Delta S = \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5|\mathbf{r}_e|}\int_0^1 \frac{d\theta}{(\theta + \hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 + (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2}. \tag{3.3.5}$$

Then using the substitution $\theta + \hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|} = ku$, where $k^2 = (\frac{\mathbf{r}}{|\mathbf{r}_e|})^2 - (\hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})^2$. Leads to:

$$\begin{aligned}
\Delta S &= \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5|\mathbf{r}_e|}\int_0^1 \frac{kdu}{k^2(u^2+1)} = \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5|\mathbf{r}_e|k}\arctan(u) \\
&= \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5\sqrt{\mathbf{r}^2 - (\hat{\mathbf{r}}_e\cdot\mathbf{r})^2}}\arctan\left(\frac{1}{k}(\theta + \hat{\mathbf{r}}_e\cdot\frac{\mathbf{r}}{|\mathbf{r}_e|})\right)\Bigg|_0^1 \\
&= \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5\sqrt{\mathbf{r}^2 - (\hat{\mathbf{r}}_e\cdot\mathbf{r})^2}}\left[\arctan\left(\frac{|\mathbf{r}_e| + \hat{\mathbf{r}}_e\cdot\mathbf{r}}{\sqrt{\mathbf{r}^2 - (\hat{\mathbf{r}}_e\cdot\mathbf{r})^2}}\right) - \arctan\left(\frac{\hat{\mathbf{r}}_e\cdot\mathbf{r}}{\sqrt{\mathbf{r}^2 - (\hat{\mathbf{r}}_e\cdot\mathbf{r})^2}}\right)\right] \\
&\approx \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5\sqrt{\mathbf{r}^2 - \hat{\mathbf{r}}_e\cdot\mathbf{r}^2}}\left[\arctan\left(\frac{|\mathbf{r}_e|}{\sqrt{\mathbf{r}^2 - \hat{\mathbf{r}}_e\cdot\mathbf{r}^2}}\right) - \arctan\left(\frac{\hat{\mathbf{r}}_e\cdot\mathbf{r}}{\sqrt{\mathbf{r}^2 - \hat{\mathbf{r}}_e\cdot\mathbf{r}^2}}\right)\right]
\end{aligned} \tag{3.3.6}$$

and since $|\mathbf{r}_e| >> \sqrt{\mathbf{r}^2 - \hat{\mathbf{r}}_e\cdot\mathbf{r}^2}$:

$$\Delta S \approx \Delta S_{GR} + \frac{4(Gm_c)^2}{c^5\sqrt{\mathbf{r}^2 - \hat{\mathbf{r}}_e\cdot\mathbf{r}^2}}\left[\frac{\pi}{2} - \arctan\left(\frac{\hat{\mathbf{r}}_e\cdot\mathbf{r}}{\sqrt{\mathbf{r}^2 - \hat{\mathbf{r}}_e\cdot\mathbf{r}^2}}\right)\right]$$

$$\Delta S = \Delta S_{GR} + \frac{4(Gm_c)^2(1 + e\cos\phi)}{ac^5(1-e^2)\sqrt{(1 + \sin^2(i)\sin^2(\omega+\phi))}}\left[\frac{\pi}{2} - \arctan\left(\frac{\sin(i)\sin(\omega+\phi)}{\sqrt{1 - \sin^2(i)\sin^2(\omega+\phi)}}\right)\right]$$

This is as far as we can reduce this equation. What can be noticed is that this addition is of order $\approx 10^{-12}$ compared to the extra addition from Yukawa gravity of 10^{-16}. So would be more noticeable, though still small compared to the original GR term, $\Delta S_{GR} \approx 10^{-5}$.

References

[1] Clifford M Will. "Solar system versus gravitational-wave bounds on the graviton mass". In: *Classical and Quantum Gravity* 35.17 (2018).