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Soft pions

(1967)

The purpose of these lectures is to explain certain techniques, developed in the last few years by Adler, Weisberger, Weinberg, and others, for the analysis of processes involving low-energy pions. I have tried, as far as possible, to make the lectures self-contained; the only background required of the reader is an understanding of field theory on the Feynman diagram level. In particular, no previous knowledge of current commutators or low-energy theorems is assumed.

In Sec. 1 the reduction formula is developed and some of its consequences discussed. Sec. 2 is a brief summary of the relevant parts of weak-interaction theory. Soft pions first appear in Sec. 3, a discussion of the Goldberger–Treiman relation. Sec. 4 is an analysis of the various definitions of PCAC. Sec. 5 is a discussion of Lagrangian models in general, and the gradient-coupling model in particular. In Sec. 6, Adler’s rule for the emission of one soft pion is derived. The current commutation relations are introduced in Sec. 7. In Sec. 8 the formula for the s -wave pion–hadron scattering length is derived. In Sec. 9, the special case of pion–pion scattering is treated. In Sec. 10, a few remarks are made about leptonic decays of kaons.

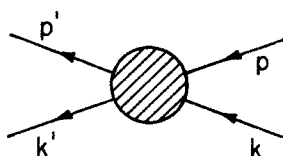
I have done no original work in this field; most of what I know I have learned in conversations with S. Adler, H. Schnitzer, and S. Weinberg. Those who were lucky enough to attend Weinberg’s Loeb lectures at Harvard will know how much these lectures owe to him. I am also indebted to J. Bernstein, who allowed me to read the manuscript of his forthcoming book on currents.

1 The reduction formula

Let us suppose we have a Lagrangian field theory, in which the Lagrange density, \mathcal{L} , depends only on a single scalar field, ϕ , and its

derivatives. (We assume a single scalar field only for simplicity; everything we say will be readily generalizable to a theory of many fields of arbitrary spin.) Given \mathcal{L} , we know, in principle, how to calculate S -matrix elements: we sum all Feynman diagrams contributing to the process of interest.* (Fig. 1 represents such a sum for a two-particle scattering process; the

Fig. 1



arrows distinguish incoming and outgoing particles.) To obtain the S -matrix element, we multiply this sum by appropriate kinematic factors. (For the Feynman conventions used here, see Appendix 1.)

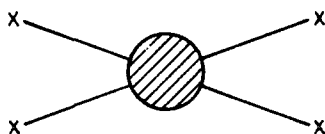
However, we can also calculate the sum of diagrams with the external lines not on the mass shell. The expression thus obtained clearly has nothing to do with any S -matrix element. Does it have any meaning at all?

One way to give it a meaning is to introduce a linear coupling of ϕ to an external c -number source ρ . That is to say, we change the Lagrange density in the following way:

$$\mathcal{L} \rightarrow \mathcal{L} + \rho(x)\phi(x). \quad (1.1)$$

Now, let us consider the matrix element $\langle 0|S|0\rangle$ to fourth order in ρ . This expression is given graphically by Fig. 2, where the crosses represent the

Fig. 2



interaction with the source. But this is the same sum of diagrams that occurs in Fig. 1, except that every external line in Fig. 1 is now multiplied by a Feynman propagator, a factor of $i(p^2 - m^2)^{-1}$, because it has now become an internal line.

* This is not strictly true. There are always terms in the S -matrix where one or more particles do not interact at all; these must be added to the diagrams. Thus in elastic scattering, the diagrams give not S , but $S - 1$. We will avoid the complications introduced by this phenomenon by always assuming that no two momenta are equal.

However, we can calculate Fig. 2 in another way. Remember Dyson's formula for the S -matrix

$$S = T \exp i \int d^4x \mathcal{H}_I(x), \quad (1.2)$$

where \mathcal{H}_I is the interaction Hamiltonian density in the interaction picture, and T is the time-ordering operator. Conventionally, this formula is applied to a field theory by calling all the quadratic terms in the Hamiltonian the free Hamiltonian and calling the remainder the interaction. However, we can just as well call the entire old Hamiltonian the free Hamiltonian; then only the extra term in Eq. (1.1) is the interaction.

Thus we obtain:

$$\langle 0|S|0\rangle = T \left\langle 0 \left| \exp -i \int \rho(x) \phi(x) d^4x \right| 0 \right\rangle, \quad (1.3)$$

where ϕ is, as always, the field operator in the 'interaction picture', which is, in our case, the Heisenberg picture when ρ vanishes. To find the object of interest (Fig. 2), we must differentiate four times with respect to ρ and set ρ equal to zero. This gives

$$T \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle \quad (1.4)$$

We can now forget about diagrams and external sources, for we have established our principal result: S -matrix elements, extrapolated off the mass shell, with every external line multiplied by a free propagator, are simply the Fourier transforms of vacuum expectation values of time-ordered products of Heisenberg fields.

This lengthy sentence may also be expressed by an equation, giving the S -matrix element in terms of the Fourier transform. In our case, this equation is

$$\begin{aligned} \langle k', p' | S | k, p \rangle = & (\text{K.F.}) \int d^4x_1 \dots d^4x_n (i)^4 (\square_1^2 + m^2) \dots (\square_4^2 + m^2) \\ & e^{+ik' \cdot x_1} e^{+ip' \cdot x_2} e^{-ik \cdot x_3} e^{-ip \cdot x_4} \\ & \times T \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle, \end{aligned} \quad (1.5)$$

where (K.F.) stands for kinematic factors (defined in Appendix 1). The i 's and Klein-Gordon operators just serve to cancel the extra propagators. Eq. (1.5) is the famous reduction formula. As we have seen, it is just the Feynman prescription for calculating an S -matrix element, written in coordinate space. It is useful because it enables us to use coordinate-space information (e.g. commutation relations) to place restrictions on the S -matrix.

We need not take all the particles off the mass-shell; for example, by

applying reasoning similar to that above to $\langle p'|S|p\rangle$, we can establish that

$$\langle k', p'|S|k, p\rangle = (\text{K.F.}) \int d^4x_1 d^4x_2 (i)^2 (\Box_1^2 + m^2)(\Box_2^2 + m^2) e^{+ik'\cdot x_1} e^{-ik\cdot x_2} T \langle p'|\phi(x_1)\phi(x_2)|p\rangle, \quad (1.6)$$

as well as thirteen other formulae, in which other combinations of particles are taken off the mass shell.

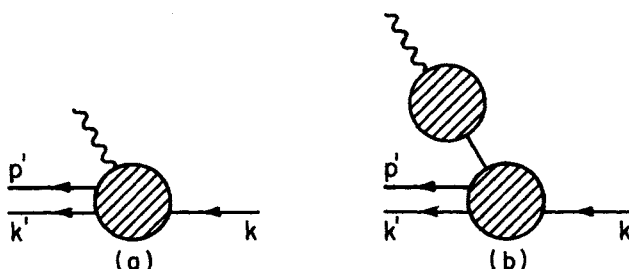
Although we have proven the reduction formula for the case in which ϕ is the canonical field, the field that appears in the Lagrangian, the formula is still true if ϕ is any local scalar field, provided ϕ is properly normalized; that is to say, provided

$$\langle p|\phi(x)|0\rangle = (2\pi)^{-3/2} (2E)^{-1/2} e^{ip\cdot x}. \quad (1.7)$$

(Notice that everything in this equation, except the scale, is determined by Lorentz invariance.)

We will show this for the simple case where only one particle is taken off the mass shell; the generalization is straightforward. Let us return to momentum space. The reduction formula tells us to calculate the matrix element given by Fig. 3(a), multiply by $(p^2 - m^2)$, and go on to the mass

Fig. 3



shell. (In the figure, the wiggly line represents the new field.) Thus, only those diagrams which have a pole at $p^2 = m^2$ can contribute to the final result. The only diagrams with poles are those with one-particle intermediate states (Fig. 3(b)). The residue at the pole is the product of the two shaded blobs. The two-pronged blob is one, by Eq. (1.7). The four-pronged blob is the S -matrix element. Q.E.D.

Thus, if all we know are S -matrix elements, we cannot uniquely assign a field to a particle. If ϕ is a good field for a given particle, so are such peculiar objects as

$$m^{-16} \Box^{16} \phi, \phi + (\Box^2 + m^2)^3 \phi^5,$$

etc. These define different off-the-mass-shell extrapolations, but they all lead to the same S -matrix on the mass shell.

2 The weak interactions: first principles

The weak interaction Hamiltonian is the space integral of the Hamiltonian density, \mathcal{H}_W . We assume that \mathcal{H}_W has the usual current-current form

$$\mathcal{H}_W = \frac{G}{2} \mathcal{J}_\mu \mathcal{J}^{\mu\dagger}. \quad (2.1)$$

\mathcal{J}_μ is the sum of a leptonic part and a hadronic part,

$$\mathcal{J}_\mu = \mathcal{J}_\mu^H + \mathcal{J}_\mu^L. \quad (2.2)$$

Although we know very little about the form of \mathcal{J}_μ^H , that of \mathcal{J}_μ^L has been well-established:

$$\mathcal{J}_\mu^L = \bar{\nu}_e \gamma_\mu (1 + i\gamma_5) e + (e \rightarrow \mu). \quad (2.3)$$

(We use a notation in which $\gamma_5^2 = -1$.) G can then be determined from muon decay experiments. It is approximately $10^{-5} M_p^{-2}$. \mathcal{J}_μ^H can be written as the sum of a vector current and an axial-vector current:

$$\mathcal{J}_\mu^H = V_\mu + A_\mu. \quad (2.4)$$

Each of these, in turn, can be written as the sum of a set of currents with definite strangeness-changing properties. For a major part of these lectures we will concentrate on strangeness-conserving processes, and thus we will need to consider only the strangeness-conserving parts of the current.

Semi-leptonic decays are processes of the form

$$i \rightarrow f + \text{leptons},$$

where i and f are hadrons. If we neglect electromagnetic effects and higher-order weak corrections, the matrix elements for such a decay factors into the product of two terms:

$$\langle f, l | \mathcal{J}_\mu^H \mathcal{J}_\mu^{L\dagger} | i \rangle = \langle f | \mathcal{J}_\mu^H | i \rangle \langle l | \mathcal{J}_\mu^{L\dagger} | 0 \rangle. \quad (2.5)$$

The second term is known exactly by virtue of Eq. (2.3). Unfortunately, the first term is not so easy to calculate. We can use symmetry principles – Lorentz invariance, CP conservation, assumed isospin or $SU(3)$ transformation properties of the current – to express this matrix element in terms of a few unknown functions of four-momentum-transfer; however, to calculate these functions we need strong-interaction dynamics.

Let me give two examples of semi-leptonic processes. The first is neutron β -decay. The relevant matrix element is

$$\langle p | \mathcal{J}_\mu^H(x) | n \rangle = (\text{K.F.}) e^{-ik \cdot x} \bar{u}_p [\gamma_\mu g_V(k^2) + i\gamma_\mu \gamma_5 g_A(k^2) + \cdots] u_n, \quad (2.6)$$

where k is the four-momentum transfer, the u s are the appropriate Dirac bispinors, and the three dots represents other terms that are of order k^2 , and hence do not make a significant contribution to this process. (Although they are important in muon capture by nuclei, and also in high-energy neutrino reactions.) The A and V terms clearly come from the axial-vector and vector currents, respectively. Experimentally,

$$g_V(0) \approx 1, g_A(0)/g_V(0) \approx 1.25.$$

We shall often denote these quantities simply by g_V and g_A .

The second example is pion decay. Here, by parity, only the axial current can contribute. The relevant matrix element is

$$\langle 0 | A_\mu(x) | \pi^- \rangle = (K.F.) e^{-ip \cdot x} (ip_\mu F_\pi / \sqrt{2}). \quad (2.7)$$

where p is the pion four-momentum. Unlike the previous example, here the quantity F_π has no p^2 dependence, since p^2 is fixed at m_π^2 . From the observed pion lifetime F_π is readily calculated to be $0.19 M_p$.

If we take the divergence of Eq. (2.7), we find that

$$\langle 0 | \partial^\mu A_\mu(x) | \pi^- \rangle = (K.F.) m_\pi^2 F_\pi e^{-ip \cdot x} / \sqrt{2}. \quad (2.8)$$

The most important property of this equation is that the right-hand side is not zero! For, in view of the arguments of the proceeding section, this means that

$$\phi_{\pi^-} = \sqrt{2} \partial_\mu A^\mu / F_\pi m_\pi^2, \quad (2.9)$$

is a perfectly good pion field, suitable for use in the reduction formula. We shall begin to exploit this fact in the next section.

3 The Goldberger–Treiman relation and a first glance at PCAC

Let us define $g(k^2)$ by

$$\langle p | \phi_{\pi^-}(x) | n \rangle = (K.F.) e^{-ik \cdot x} (k^2 - m_\pi^2)^{-1} \bar{u}_p \gamma_5 u_n g(k^2) \sqrt{2}. \quad (3.1)$$

The only thing we know about $g(k^2)$ experimentally is its value at m_π^2 . For, by the arguments of Sec. 1, this must be the renormalized pion–nucleon coupling constant, the quantity we measure by extrapolating to the pion pole in nucleon–nucleon scattering, or by extrapolating to the nucleon pole in pion photoproduction. (The $\sqrt{2}$ in the formula is just an isospin factor.) Thus,

$$g(m_\pi^2) = g = 13.5. \quad (3.2)$$

Now let us return to the axial-vector contribution to neutron beta-decay. This time I will write out all the invariants.

$$\begin{aligned} \langle p | A_\mu(x) | n \rangle = & i(K.F.) e^{-ik \cdot x} \bar{u}_p \\ & \times [\gamma_\mu \gamma_5 g_A(k^2) + k_\mu \gamma_5 g_p(k^2) + \sigma_{\mu\nu} \gamma_5 k^\nu g_M(k^2)] u_n. \end{aligned} \quad (3.3)$$

Taking the divergence, we find

$$\langle p | \partial^\mu A_\mu(x) | n \rangle = (K.F.) e^{-ik \cdot x} \bar{u}_p \gamma_5 u_n [-2Mg_A(k^2) + k^2 g_p(k^2)], \quad (3.4)$$

where M is the nucleon mass. Comparing this with Eqs. (2.9) and (3.1), we obtain

$$\sqrt{2} g(k^2)/(k^2 - m_\pi^2) = [-2Mg_A(k^2) + k^2 g_p(k^2)] \sqrt{2}/F_\pi m_\pi^2. \quad (3.5)$$

This equation is simply a consequence of our definition of ϕ_π , and is without predictive power unless supplemented by further assumptions. We now make such an assumption. We assume that $g(k^2)$ is ‘slowly varying’ over a distance of the order of m_π^2 , that is to say, that

$$g(m_\pi^2) \approx g(0). \quad (3.6)$$

This is a special case of the PCAC hypothesis which we will discuss in more detail shortly. (PCAC is an acronym for ‘partially conserved axial current’. We will explain this peculiar phrase in the next section.)

I stress that Eq. (3.6) is pure assumption. It would certainly be false, for example, if ϕ_π were like the peculiar fields involving Klein–Gordon operators which we constructed at the end of Sec. 1.

However, if we accept Eq. (3.6), then, evaluating Eq. (3.5) at $k^2 = 0$, we instantly obtain

$$F_\pi g \approx 2Mg_A. \quad (3.7)$$

This is the famous Goldberger–Treiman relation. It is in excellent agreement with experiment:

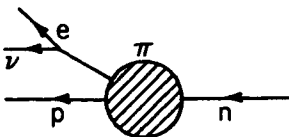
$$2.56M \approx 2.50M.$$

4 A hard look at PCAC

Many statements about the meaning of PCAC and the explanation of the Goldberger–Treiman relation exist in the literature and in the folklore of physics. Here are some of them:

1. ‘The Goldberger–Treiman relation is just polology. It is simply the statement that neutron beta-decay is dominated by the one-pion pole.’ This is *wrong*. If the one-pion pole diagram (Fig. 4) made the only contribu-

Fig. 4



tion to neutron beta-decay, we would predict

$$g_p = g F_\pi (k^2 - m_\pi^2)^{-1}, \quad (4.1)$$

and

$$g_A = 0. \quad (4.2)$$

This is (1) not the Goldberger–Treiman relation and (2) in flat contradiction with experiment.

2. ‘PCAC is the statement that the matrix elements of $\partial_\mu A^\mu$ are slowly varying.’ This is almost right. (That is to say, it is very close to the viewpoint I will espouse in these lectures.) However, I would prefer to say rather that the matrix elements are *normally varying*: that, just as for S -matrix elements and electromagnetic form factors, about which we do have empirical information, the rate of variation is determined by the distance to the nearest singularity. Thus, in the particular case of $g(k^2)$, the nearest singularity is the beginning of the three-pion cut. This should induce a derivative on the order of $(9m_\pi^2)^{-1}$ at the origin, just as the ρ pole induces a derivative in the electromagnetic form factor of the order of m_ρ^{-2} . Thus the error made in extrapolating from zero to m_π^2 should be of the order of $(m_\pi^2/9m_\pi^2)$ or approximately 10%. The advantage of stating things this way is twofold. (1) It gives a clear idea of how to use PCAC in processes where the kinematics are more complicated than in neutron decay, and where the invariant functions may depend on several variables. (2) It emphasizes that $\partial^\mu A_\mu$ is in no way especially ‘smooth’; the derivatives of its matrix elements are no smaller than those of many familiar operators. What is special is that the pion mass is small, compared to the characteristic masses of strong interaction physics; thus extrapolation over a distance of m_π^2 introduces only small errors.

3. ‘PCAC is the statement that if the pion mass were zero the axial vector current would be exactly conserved.’ This does not look at all like the preceding version of PCAC; nevertheless, it also leads to the Goldberger–Treiman relation. Let us see how this works.

Let us imagine ourselves in a world in which the pion mass is zero and the axial current is conserved. We can still define F_π , G_A , etc. by Eqs. (2.8) and (3.3). (Although their physical meanings are, of course, quite different – F_π is now not the pion decay constant, but the electron decay constant.) Because the axial current is conserved, Eq. (3.4) becomes

$$-2Mg_A(k^2) + k^2g_p(k^2) = 0. \quad (4.3)$$

However, one-pion exchange (Fig. 4 again!) now produces a pole in g_p :

$$g_p(k^2) = g F_\pi / k^2 + \text{non-singular terms}. \quad (4.4)$$

Thus, when we evaluate (4.3) at $k^2=0$, we obtain

$$-2Mg_A + gF_\pi = 0, \quad (4.5)$$

which is the Goldberger–Treiman relation!

Of course, in the real world the pion mass is not zero and the axial current is not conserved, so Eq. (4.5) is not exact. However, the pion mass is small; thus if it is the only thing that keeps the axial current from being conserved, we might expect Eq. (4.5) to be accurate up to terms of the order of m_π^2 . One might express this viewpoint by saying that the axial current is ‘almost conserved’. Actually, the phrase used is ‘partially conserved’ – hence the term PCAC.

Note that although this formulation of PCAC is quite different from the preceding one, it leads to the same result and the same rough estimate of the error. Although in these lectures I will try and stick with the preceding formulation (principally because I believe it offers a clearer idea of the sources of possible errors in the more complicated applications which we shall do shortly), this way of looking at things is also a good one. Sometimes we will refer back to it, and it will give us new insights.

4. ‘PCAC is the statement that the pion field is the divergence of the axial-vector current.’ This statement occurs frequently in the literature. Unfortunately, it is completely free of content, since, as we have seen, it is true by definition. Of course, if we are working within the framework of a Lagrangian field theory with a specific strong-interaction Lagrangian, it is a well-defined statement to say that $\partial^\mu A_\mu$ is proportional to the canonical pion field that occurs in the Lagrangian. However, it is a statement without predictive power, since we know no more about the variation of the matrix elements of canonical fields than we do about those of any other local operator.

Nevertheless, we will sometimes have occasion to write down Lagrangian field theories which possess PCAC in this sense. Our motivation is this: when we make assumptions about the properties of the weak interaction currents, we do not have absolute freedom; we must always be sure that our assumptions do not contradict each other or the general principles of relativistic local field theory. The easiest way to check consistency is to construct a Lagrangian field theory that embodies all of our assumptions. However, since we are *only* using this theory to check consistency, there is no reason why the strong interaction coupling constants must be large. Indeed, we will usually take them to be small. In this case, we can calculate the matrix elements of the canonical pion field in lowest-order perturba-

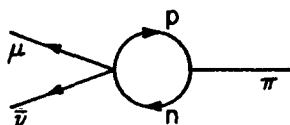
tion theory, and verify that the definition of PCAC given here implies the correct version (statement 2 above).

In the next section we will construct such a Lagrangian model to verify the consistency of the assumptions that lie behind the Goldberger–Treiman relation.

5 The gradient-coupling model

In the early days of pion physics, a popular model of pion decay was the one expressed in Fig. 5. In this model, F_π is clearly proportional to

Fig. 5



g . The Goldberger–Treiman relation, however, makes precisely the opposite assertion:

$$F_\pi = 2Mg_A/g. \quad (5.1)$$

This strongly suggests that the Goldberger–Treiman relation is a characteristically strong-interaction result – that it would not be true if the strong interactions were weak. We will show that this suggestion is misleading by displaying a Lagrangian model in which the Goldberger–Treiman relation holds even if the strong interactions are weak. (Indeed, because of the remarks at the end of the last section, this is the only domain in which, for a Lagrangian field theory, we can be sure it holds.)

However, first I will review the familiar method for obtaining currents from a Lagrangian. A local Lagrangian field theory is determined by a Lagrange density, \mathcal{L} , which is a function of a set of fields ϕ^α and their first derivatives $\partial_\mu \phi^\alpha$. The dynamics of the theory is given by Hamilton's principle,

$$\delta \int \mathcal{L} d^4x = 0, \quad (5.2)$$

for variations of the fields which vanish at infinity.

Let us consider an infinitesimal transformation of the fields of the form

$$\delta \phi^\alpha = F^\alpha \delta \lambda, \quad (5.3)$$

where the F s are arbitrary functions of the ϕ s at the point x , and $\delta \lambda$ is an infinitesimal constant. Let us assume that under this transformation

\mathcal{L} is invariant. That is to say,

$$\delta \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi^a} F^a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \partial_\mu F^a \right) d\lambda = 0, \quad (5.4)$$

where the sum on repeated indices is implied. Now let us consider a transformation of the same form as (5.3), but with $\delta\lambda$ an arbitrary function of space and time. Then

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} F^a \partial_\mu \delta\lambda, \quad (5.5)$$

no longer vanishes. However, by (5.2), its integral must still vanish;

$$\int d^4x \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} F^a \right] \partial_\mu \delta\lambda = 0, \quad (5.6)$$

which implies, since $\delta\lambda$ is arbitrary,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} F^a \right] = 0. \quad (5.7)$$

Thus, for every invariance of \mathcal{L} we have a conserved current

$$\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} F^a. \quad (5.8)$$

If \mathcal{L} is the sum of two terms,

$$\mathcal{L} = \mathcal{L}_s + \mathcal{L}', \quad (5.9)$$

where the first term is invariant under (5.3) and the second term does not depend on the $\partial_\mu \phi$ s, then, by the same reasoning, Eq. (5.7) is replaced by

$$\partial_\mu \mathcal{J}^\mu = \frac{\partial \mathcal{L}'}{\partial \phi^a} F^a, \quad (5.10)$$

with \mathcal{J}^μ defined as above.

We are now in a position to analyze our model. It involves an isodoublet of nucleon fields, denoted by ψ , and an isotriplet of pion fields, denoted by ϕ . The Lagrange density is

$$\mathcal{L} = \mathcal{L}_0 - i \frac{g}{2M} \bar{\psi} \gamma_\mu \gamma_5 \tau \psi \cdot \partial^\mu \phi, \quad (5.11)$$

where \mathcal{L}_0 is the usual free Lagrange density and τ are the standard 2×2 isospin matrices. This is called the gradient-coupling model. The coupling constant has been chosen such that $g(k^2)$, defined by Eq. (3.1), is (in lowest order) a constant equal to g . Now let us consider the infinitesimal transformations

$$\delta \phi = \delta \lambda, \quad \delta \psi = 0. \quad (5.12)$$

Associated with these transformations is an isotriplet of axial-vector currents:

$$\mathcal{A}_\mu = \partial_\mu \phi - i \frac{g}{2M} \bar{\psi} \gamma_\mu \gamma_5 \tau \psi. \quad (5.13)$$

The only part of \mathcal{L} not invariant under (5.12) is the pion mass term; therefore,

$$\partial^\mu \mathcal{A}_\mu = -m_\pi^2 \phi. \quad (5.14)$$

This current, as it stands, is not a suitable candidate for A_μ ; its one-nucleon matrix elements do not have the right value. Therefore, we define

$$A_\mu = \frac{2Mg_A}{g} \mathcal{A}_\mu. \quad (5.15)$$

and take A_μ to be the positively-charged component of this triplet.

I leave it to you to verify that in this model (1) all three statements of PCAC given in the preceding section are true, and (2) the Goldberger–Treiman relation is valid. (Always, of course, working only to lowest order in g .)

Problem. Suppose, instead of choosing (5.15) for the axial current, we choose

$$A_\mu + \alpha \partial_\mu \phi,$$

with α some undetermined constant. Is the Goldberger–Treiman relation still valid? If so, why? If not, why not? Answer the question using all three versions of PCAC, if possible.

6 Adler's rule for the emission of one soft pion

We will now develop a formalism, due to S. Adler, for calculating the matrix element of any hadronic process of the form

$$i \rightarrow f + \pi, \quad (6.1)$$

where i and f are any hadronic states, in terms of that for the process

$$i \rightarrow f. \quad (6.2)$$

For example, we will be able to relate pion production,

$$N + N \rightarrow N + N + \pi, \quad (6.3)$$

to nucleon–nucleon scattering

$$N + N \rightarrow N + N. \quad (6.4)$$

Our method will be to obtain an *exact* formula for the process (6.1), with the pion off the mass shell and with the pion four-momentum, k , close to zero. We will then extrapolate this expression to the mass shell. It is clear

that this extrapolation can be trusted only if all invariants of the form $p \cdot k$ are small, where p is any momentum in the initial or final state. Thus, in the example, the formalism will give an expression for (6.4) valid in the neighborhood of threshold. This kinematic situation is sometimes described by saying that the pion is 'soft'. The terminology comes from the concept of a soft photon – defined in precisely the same way – which arises in the theory of infrared corrections.

Before proceeding to the detailed analysis, it will be convenient to rewrite some of our fundamental formulae in an isospin-symmetric form. We will assume that the axial current is part of an isotriplet of currents A_μ^a ($a=1, 2, 3$), normalized such that A_μ is $(A_\mu^1 + iA_\mu^2)/2$. Thus for, example, Eq. (2.7) becomes

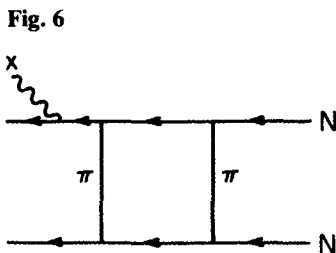
$$\langle \pi^b | A_\mu^a | 0 \rangle = (\text{K.F.}) e^{ip \cdot x} i \delta_{ab} p_\mu F_\pi, \quad (6.5)$$

and our other formulae are altered correspondingly.

Now for the analysis. The S -matrix element for (6.1) is clearly related by the reduction formula to

$$\langle f | \partial^\mu A_\mu^a | i \rangle = i k^\mu \langle f | A_\mu^a | i \rangle. \quad (6.6)$$

We want to investigate this object near $k=0$. At first glance, it might seem that (6.6) vanishes at this point, because of the multiplicative factor of k . However, this is not necessarily the case. Because of energy-momentum conservation, we must alter the momenta of the initial and final states as we send k to zero. Thus there is a possibility that the matrix element of A_μ^a blows up – develops a pole – as k goes to zero, and the product has a finite limit. Fig. 6 shows this happening for a typical Feynman diagram



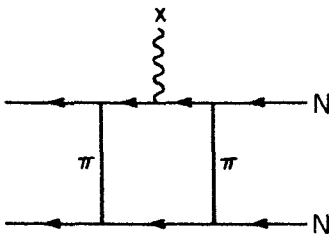
contributing to the process (6.3). (The wiggly line terminating in a cross represents the current.) As k goes to zero, the internal nucleon line is forced on to the mass shell, and the diagram becomes infinite. It requires only an elementary Feynman calculation to show that (6.6) then has a non-zero limit.

This example can readily be generalized. Let us divide all Feynman diagrams that contribute to the matrix element of A_μ^a into two classes: (1) 'pole diagrams' in which the current is attached to an external line, as in Fig. 6; (2) 'guts diagrams' in which the current is attached to an internal line, as in Fig. 7. The guts diagrams cannot develop poles as k goes to zero. (Proof: the locations of the singularities of a diagram are determined by the Landau rules. The Landau rules are purely kinematic; they involve only the possible values of the internal momenta. For a guts diagram, when k is zero, these are the same as for the diagram without the current. The diagram without the current is a diagram for a scattering process in the physical region. Scattering processes do not have poles in the physical region.)* Thus, their contribution to (6.6) vanishes at zero. By PCAC, they may therefore be neglected on the mass shell. Therefore, to calculate the scattering amplitude, we need only compute the pole diagrams.

(A side remark: Suppose, in Fig. 6, the pion is charged, and further suppose that we give the proton and neutron slightly different masses, as is indeed the case in nature. Then the pole diagram would vanish at zero. However, then its contribution to (6.6) would be rapidly varying near zero – since it would have a very-near-by pole – and we would not be able to extrapolate it. Thus we would again be led to the same conclusion.)

But it is trivial to calculate the pole diagrams. For these simply consist of the diagrams which contribute to (6.2), with a current hooked on to an external line. Using the Goldberger–Treiman relation, it is easy to see that, when we take the divergence of this current, we simply reproduce the pion–nucleon coupling of the gradient-coupling theory. Thus we obtain

Fig. 7



* This is not strictly true. For processes with three or more particles in both the initial and final states, trivial macrocausality poles may enter the physical region. In the following, we assume we are working in a range of energies and momenta such that these poles may be neglected.

Adler's rule. In any strong-interaction process, to calculate the matrix element for the emission of one soft pion, take the matrix element for the process without the pion, and sum all the terms obtained by attaching the pion to each of the external lines, using gradient coupling.

In other words, lowest-order perturbation in the gradient-coupling theory is exact for the emission of one soft pion.

7 Current commutators

Suppose we attempt to extend the formalism of the previous section to processes involving two soft pions. Clearly, in this case we would have to analyze objects like

$$T\langle f|\partial^\mu A_\mu^a(x)\partial^\nu A_\nu^b(y)|i\rangle.$$

But now when we attempt to pull out the differential operators we obtain not only a double divergence of a matrix element of two currents, but also an equal time commutator from the derivative of the time-ordering operator. Thus, to go further, we need some information about the equal-time commutator.

7.1 Vector-vector commutators

Let me review the CVC hypothesis of Feynman and Gell-Mann. Recall our observation that, experimentally, g_V is very close to one. Why should this be so? Feynman and Gell-Mann offered the following explanation; suppose that the weak current for nucleons is of exactly the same structure as that for leptons. Then, the bare value of g_V , the number that occurs in the interaction Lagrangian, would be one, as would the bare value of g_A . Further suppose that the vector current is conserved. (CVC = conserved vector current). Then in this case it can be shown that the g_V is not renormalized by the strong interactions. (For a proof, see Appendix 2). Now, currents conserved by the strong interactions are not that easy to find; the only one known with the right quantum numbers is the isospin current. Therefore, Feynman and Gell-Mann further postulated that V_μ is part of an isotriplet of currents, V_μ^a , and that V_μ^a is proportional to the isospin current I_μ^a ; that is to say

$$V_\mu^a = \alpha I_\mu^a, \quad (7.1)$$

with α a constant. This is a stronger hypothesis than conservation, but it has been checked in the famous weak-magnetism experiment, and is now generally accepted.

Originally, α was taken to be two; that is to say, the coupling of the vector current to nucleons was assumed to be exactly equal to the coupling

to leptons. The observed small deviation of g_V from one was then ascribed to electromagnetic corrections. However, ever since the famous SU(3) analysis of the weak currents by Cabibbo, it has been realized that this assumption is based on a too-naïve formulation of universality; Cabibbo theory predicts

$$\alpha = 2g_V, \quad (7.2)$$

where g_V is identified with the cosine of the Cabibbo angle.

Now, in most simple models of the strong interactions (e.g. Yukawa-type Lagrangians), the equal-time commutators of the fourth components of the isospin currents have the same algebraic structure as those of the associated charges; i.e.

$$[I_0^a(\mathbf{x}, 0), I_0^b(\mathbf{y}, 0)] = i\epsilon_{abc}I_0^c(\mathbf{x}, 0)\delta^3(\mathbf{x} - \mathbf{y}). \quad (7.3)$$

In more complicated models, there may be additional terms on the right, proportional to gradients of delta-functions, which vanish when we integrate the currents to make the charges. For simplicity, we will ignore the possible occurrence of these terms. However, as you may readily check, none of the calculations we will do will depend on the assumption of their absence.

From the preceding equations, we find

$$[V_0^a(\mathbf{x}, 0), V_0^b(\mathbf{y}, 0)] = 2ig_V\epsilon_{abc}V_0^c(\mathbf{x}, 0)\delta^3(\mathbf{x} - \mathbf{y}). \quad (7.4)$$

7.2 Vector-axial commutators

These follow directly from the identification of V_μ^a with the isospin current, and the statement that A_μ^a is an isotriplet:

$$[V_0^a(\mathbf{x}, 0), A_0^b(\mathbf{y}, 0)] = 2ig_V\epsilon_{abc}A_0^c(\mathbf{x}, 0)\delta^3(\mathbf{x} - \mathbf{y}). \quad (7.5)$$

7.3 Axial-axial commutators

There is no direct experimental check on these objects (other than the applications we are going to discuss). However, there are some general theoretical arguments that make the commutators we are going to assume particularly attractive. I will sketch two lines of argument.

(1) Chirality principle. Suppose we follow the original suggestion of Feynman and Gell-Mann and introduce parity-violation into the weak interactions only through the use of the projection matrix $1 + i\gamma_5$ in the definition of the currents. This means that the parity-transformed currents, $V_\mu - A_\mu$, only involve the matrix $1 - i\gamma_5$. These are orthogonal projection matrices; therefore

$$[(V_0^a(\mathbf{x}, 0) + A_0^a(\mathbf{x}, 0)), (V_0^b(\mathbf{y}, 0) - A_0^b(\mathbf{y}, 0))] = 0. \quad (7.6)$$

Now let us abstract Eq. (7.6) from the Feynman–Gell-Mann theory and adopt it as a general rule. (I will call this ‘the chirality principle’.) Then we deduce that

$$[A_0^a(\mathbf{x}, 0), A_0^b(\mathbf{y}, 0)] = 2ig_V \varepsilon_{abc} V_0^c(\mathbf{x}, 0) \delta^3(\mathbf{x} - \mathbf{y}). \quad (7.7)$$

(2) Universality principle. Roughly, universality is the statement that all the weak interactions have the same strength. Originally, when only nuclear beta-decay was known, this was formulated as the requirement that the baryon–lepton coupling have the same strength as the lepton–lepton coupling. However, with the current plethora of hadrons, it is difficult to unambiguously generalize this statement. Should one require that all hadronic coupling constants be equal? Or should the sum of the squares be equal to the lepton constant squared? Anyway, how does one compare coupling constant for particles of different spin? To avoid these difficulties, Gell-Mann suggested the following definition of universality: ‘The algebra generated by repeated equal-time commutation of the fourth components of the total weak-current with its adjoint must be the same as the corresponding algebra generated from the lepton currents alone.’

This definition has several obvious advantages: (1) if the only hadrons were nucleons, it would lead to the old definition of universality; (2) it is independent of the details of the hadron spectrum and the structure of the strong interactions; (3) within any particular model of the strong interactions, it fixes the relative scale of the hadron and lepton currents, as well as the relative scale of the baryon and boson parts of the hadron current.

It can be shown that this definition of universality, together with the Cabibbo theory, leads to the commutators we have written down. However, since the argument involves considerable use of SU(3), I will not give it here.

I should complete this section by giving some examples of Lagrangian models in which these commutators hold and which also obey PCAC. (Note that the gradient-coupling model will not do; in it, the fourth components of axial currents commute.)* Fortunately, I am spared this labor; any of the Lagrangians discussed by Professor Zumino in his lectures will do the job.†

8 The Weinberg–Tomozawa formula and the Adler–Weisberger relation

In this section we will derive a formula for the *s*-wave threshold scattering length in the elastic scattering of a pion off any hadron target.

* They are canonical momentum densities.

† See *Hadrons and Their Interactions* (Academic Press, New York and London, 1968).

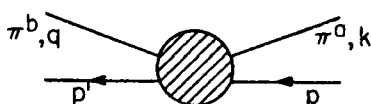
Our general method will be the same as in the derivation of Adler's rule; we will expand the matrix element in a power series in the pion momenta (plus pole terms) near zero momentum, and then extrapolate to the mass shell. In this case, however, we will take two pions off the mass shell; therefore, the current commutators will play an important role.

We consider a process of the type

$$\pi^a + i \rightarrow \pi^b + f, \quad (8.1)$$

where the superscripts indicate the isospin of the pions, and the momenta are as indicated in Fig. 8. The reduction formula tells us that the object

Fig. 8



we need to study is

$$\begin{aligned} I &= \int d^4x d^4y e^{iq \cdot x} e^{-ik \cdot y} T \langle f | \partial^\mu A_\mu^a(x) \partial^\nu A_\nu^b(y) | i \rangle \\ &= - \frac{(2\pi)^4 \delta^4(p + k - p' - q) F_\pi^2 m_\pi^4 \mathcal{M}}{(q^2 - m_\pi^2)(k^2 - m_\pi^2)(2\pi)^3 (4EE')^{1/2}}, \end{aligned} \quad (8.2)$$

where \mathcal{M} is the invariant matrix element for the scattering process, the object to which we wish to apply PCAC. It is related to the S -matrix by

$$\langle f | S - 1 | i \rangle = (\text{K.F.}) \mathcal{M} (2\pi)^4 \delta^4(p + k - p' - q). \quad (8.3)$$

We now wish to expand I , and hence \mathcal{M} , near $q = k = 0$. We will lump together, and eventually neglect, all terms of second order and higher. Note that the invariants k^2 , q^2 , and $k \cdot q$ are of second order, while the invariants $p \cdot k$, $p' \cdot q$, $p \cdot q$, and $p' \cdot k$ are all of first order, and, as a consequence of energy–momentum conservation, all equal, aside from terms of second order.

Pulling the differential operators through the time ordering symbol, we may write I as the sum of three terms,

$$I = I_1 + I_2 + I_3, \quad (8.4)$$

where

$$I_1 = - \int d^4x d^4y e^{iq \cdot x} e^{-ik \cdot y} \delta(x_0 - y_0) \langle f | [A_0^b(x), \partial^\nu A_\nu^a(y)] | i \rangle, \quad (8.5)$$

$$I_2 = \int d^4x d^4y e^{iq \cdot x} e^{-ik \cdot y} \partial_x^\mu \partial_y^\nu T \langle f | A_\mu^a(x) A_\nu^b(y) | i \rangle, \quad (8.6)$$

and

$$\begin{aligned} I_3 &= - \int d^4x d^4y e^{iq \cdot x} e^{-ik \cdot y} \partial_x^\mu \delta(x_0 - y_0) \langle f | [A_0^a(x), A_0^b(y)] | i \rangle \\ &= +i \int d^4x d^4y e^{iq \cdot x} e^{-ik \cdot y} q^\mu \delta(x_0 - y_0) \langle f | [A_0^a(x), A_0^b(y)] | i \rangle, \end{aligned} \quad (8.7)$$

where we have integrated by parts in the last equation. Each of these terms leads to a corresponding term in \mathcal{M} ; we will call these terms \mathcal{M}_1 , \mathcal{M}_2 , and \mathcal{M}_3 .

We will begin by analyzing I_3 ; this will turn out to be the most important term. Let us choose q and k such that their space components are zero. (Note that we lose no information by this choice, since, as we have argued above, there is only one first-order invariant.) Then the commutator becomes one we know (Eq. (7.7)) and I_3 becomes

$$I_3 = -2g_V \varepsilon_{abc} \int d^4x d^4y e^{-ik_0 y_0} e^{-iq_0 y_0} \delta^4(x - y) q^0 \langle f | V_0^c(x) | i \rangle. \quad (8.8)$$

The δ -function may be trivially integrated away. Likewise the space integration is trivial since

$$\int d^3x \langle f | V_0^c(x) | i \rangle = 2g_V \langle f | I^c | i \rangle = 2g_V I_i^c \delta^3(\mathbf{p} - \mathbf{p}'), \quad (8.9)$$

where \mathbf{I} is the total isospin and I_i^c the isospin of the target, is the pure isospin part of the matrix element, with the space part of the wave-function neglected. The time integration simply gives a δ -function. Thus we obtain

$$I_3 = -4g_V^2 \varepsilon_{abc} I_i^c 2\pi \delta^4(p + k - p' - q') q_0, \quad (8.10)$$

which leads to

$$\mathcal{M}_3 = + \frac{8g_V^2}{F_\pi^2} \varepsilon_{abc} I_i^c p \cdot q. \quad (8.11)$$

This may be further simplified by noting that, for isotriplet states,

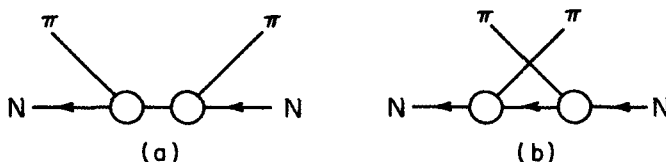
$$\langle b | I^c | a \rangle = i \varepsilon_{abc}. \quad (8.12)$$

This transforms \mathcal{M}_3 into

$$\mathcal{M}_3 = -i \frac{8g_V^2}{F_\pi^2} (\mathbf{I}_\pi \cdot \mathbf{I}_i) p \cdot q. \quad (8.13)$$

Next we turn to \mathcal{M}_2 . In I_2 , all the derivatives are outside the time-ordering symbol. Thus, \mathcal{M}_2 is of the same form as the matrix elements we discussed in Sec. 6; just as we did there, we may divide \mathcal{M}_2 into pole terms (Fig. 9) and a remainder. The remainder is of second order, and we will neglect it. The pole terms, however must be calculated explicitly.

Fig. 9



When this is done, though, a surprising thing happens: *at threshold*, they are of the order of m_π^2 , and therefore can be neglected.

I will show this explicitly for the case of a $\frac{1}{2}^+$ target (e.g. the nucleon). The first diagram in Fig. 9 gives a matrix element proportional to

$$\bar{u}' \gamma_5 k \frac{1}{\not{p} + \not{q} - m_t} \not{q} \gamma_5 u. \quad (8.14)$$

(I have suppressed isospin factors.) This is equal to

$$\bar{u}' k \frac{1}{\not{p} + \not{q} + m_t} \not{q} u. \quad (8.15)$$

Now, *at threshold*,

$$k_\mu = q_\mu = \frac{m_\pi}{m_t} p_\mu. \quad (8.16)$$

Thus, \not{p} , \not{q} , and \not{k} all commute, and, using the Dirac equation, we find that (8.16) is equal to

$$\bar{u}' m_\pi \frac{1}{2m_t + m_\pi} m_\pi u, \quad (8.17)$$

which is indeed of order m_π^2 . The second diagram may be treated in the same way.

The only term left to study is \mathcal{M}_1 . I will give two arguments to show that it is of order m_π^2 , and can be neglected. (1) I_1 explicitly involves $\partial^\mu A_\mu$. According to one version of PCAC, this is of order m_π^2 . (2) By straightforward manipulation of I_1 , \mathcal{M}_1 can be shown to be equal to a constant term plus terms of second order. Only the constant term need concern us. By Adler's Rule, if we send the incoming pion to zero, keeping the outgoing pion on the mass shell, the matrix element should be given exclusively by the pole terms. On the other hand, our preceding analysis shows that, in this limit, the matrix element consists of the sum of the pole terms, the constant term in \mathcal{M}_1 , and terms manifestly of order m_π^2 . Therefore the constant term is of order m_π^2 . (The constant term is sometimes called the σ term, because in the σ model it is related to the matrix element of the σ field.)

The upshot of all this is that, at threshold, only \mathcal{M}_3 is important:

$$\mathcal{M} = -i \frac{8g_V^2}{F_\pi^2} (\mathbf{I}_\pi \cdot \mathbf{I}_t) p \cdot q + O\left(\frac{m_\pi^2}{m_t^2}\right). \quad (8.18)$$

Near threshold, the elastic scattering matrix element is simply related to the *s*-wave scattering lengths. Doing the kinematics (for details see Appendix 3), we find for the scattering lengths

$$a = -L \left(1 + \frac{m_\pi}{m_t}\right)^{-1} 2\mathbf{I}_\pi \cdot \mathbf{I}_t = -L \left(1 + \frac{m_\pi}{m_t}\right)^{-1} \times [I(I+1) - I_t(I_t+1) - 2], \quad (8.19)$$

where *I* is the total isotopic spin, and *L* is a constant, called ‘Weinberg’s universal length’, defined by

$$L = \frac{g_V^2 m_\pi}{2\pi F_\pi^2}. \quad (8.20)$$

This can be rewritten using the Goldberger–Treiman relation;

$$L = \frac{g^2 m_\pi}{8\pi M^2} \left(\frac{g_V}{g_A}\right)^2 = 0.11 m_\pi^{-1}. \quad (8.21)$$

Eq. (8.19) is called the Weinberg–Tomozawa formula. It was derived in its full generality, using the method I have explained here, by Weinberg, and was found for certain special cases, using a different method, by Tomozawa. It is valid for the scattering of pions off any target, subject to the following two restrictions. (1) The pole terms must be negligible at threshold. If they are not, they must be added by hand, altering the formula. Notice that this makes our approach impracticable for, say, the scattering of pions off carbon, since each of the excited states of carbon up to a few hundred MeV contributes a rapidly-varying pole term. (2) The mass of the target must be much heavier than the mass of the pion, in order that the terms we have neglected will be small compared to the term we have calculated. The only case this excludes is that of pion–pion scattering, which we will discuss in some detail in the next section.

Now let us apply (8.19) to pion–nucleon scattering. We find

$$\begin{aligned} a_{1/2} &= 0.20 m_\pi^{-1}, \\ a_{3/2} &= -0.10 m_\pi^{-1}. \end{aligned} \quad (8.22)$$

where the subscripts indicate total isospin. This is in excellent agreement with the experimental numbers, 0.17 and -0.09 .

It is instructive to rewrite Eq. (8.22) in the following way:

$$\begin{aligned} a_{1/2} + 2a_{3/2} &= 0, \\ a_{1/2} - a_{3/2} &= 0.30 m_\pi^{-1}. \end{aligned} \quad (8.23)$$

The first of these equations depends only on the $I \cdot I$ form of the matrix element, not upon the coefficient in front. It was obtained long ago by Sakurai, from a model in which pion-nucleon scattering was dominated by ρ exchange. The derivation we have given is much more general; it is independent of the very existence of the ρ , let alone the role it plays in pion-nucleon scattering.

The second equation can be rewritten, with the aid of dispersion relations, as an integral over total cross sections. Furthermore, we can use Eq. (8.21) to turn this into a formula that expresses g_V/g_A in terms of this integral and hadronic masses and coupling constants. This formula is the famous Adler-Weisberger relation. It is curious that this relation was first derived directly as an integral formula, and only later seen to be a disguised low-energy theorem.

9 Pion-pion scattering à la Weinberg

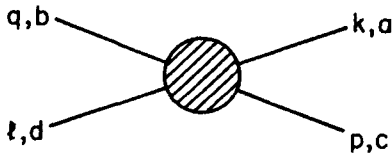
In the last section we derived a formula for the scattering of a pion off an arbitrary target. We remarked, however, that if the target were itself a pion, the formula would break down, because the terms we neglected would then be of the same magnitude as the terms we retained. Thus, pion-pion scattering would appear to be a hopelessly intractable problem. However, Weinberg showed that, by exploiting Bose statistics and crossing symmetry, one could obtain a definite prediction for the threshold scattering lengths, provided only that one made an assumption about the isospin transformation properties of the so-called σ -term. $\pi\pi$ scattering is unique in that all known symmetries (including crossing) connect it to itself. (Compare, e.g., πN scattering.) The calculation is both very simple and very sophisticated: simple because it involves only elementary algebra, sophisticated because it used the very 'modern' concepts of analyticity, crossing, and current commutation relations.

The calculation yields scattering lengths much smaller than those predicted by other techniques. This is important for, if the scattering lengths were large, it would be very difficult to justify our extrapolation procedures for processes involving two or more soft pions. Of course, since we will use these very extrapolation techniques in the calculation, it by no means proves that the scattering lengths are small. It is, however, an important consistency check.

We now turn to the actual calculation. We label the pion momenta and isospin as in Fig. 10. If all four pions are off the mass shell, there are six independent scalar invariants. For convenience, we will use the over-complete set of seven invariants formed by

$$s = (k + p)^2, \quad t = (k - q)^2, \quad u = (p - q)^2,$$

Fig. 10



and the four masses k^2 , p^2 , q^2 and l^2 . Energy-momentum conservation relates these:

$$s + t + u = p^2 + k^2 + q^2 + l^2. \quad (9.1)$$

We will now write down the expansion of the invariant amplitude \mathcal{M}_{abcd} , in a power series in the momenta, discarding terms of the fourth order or higher.

There are three independent isospin invariants, which we will choose to be

$$\delta_{ac}\delta_{bd}, \quad \delta_{ab}\delta_{cd}, \quad \text{and} \quad \delta_{ad}\delta_{bc}.$$

Let us consider the coefficient of the first of these. Suppose it contains a term linear in k^2 . Then, by Bose statistics it must contain a term linear in p^2 with the same coefficient. Time reversal says the same must be true of q^2 and l^2 . Thus, the masses can enter only in the combination

$$k^2 + p^2 + q^2 + l^2.$$

But, by (9.1), this is $s + t + u$. Thus, this term (and, *mutatis mutandis*, the other two also) can be written in terms of s , t , and u alone.

It is now easy to write down the most general expression for \mathcal{M} allowed by Bose statistics and crossing:

$$\begin{aligned} i\mathcal{M} = & \delta_{ac}\delta_{bd}[Am_\pi^2 + B(u+t) + Cs] + \delta_{ab}\delta_{cd}[Am_\pi^2 + B(u+s) + Ct] \\ & + \delta_{ad}\delta_{bc}[Am_\pi^2 + B(s+t) + Cu], \end{aligned} \quad (9.2)$$

where A , B , and C are unknown constants. (The i is inserted in the definition to keep the constants real.)

By Adler's rule, if we send k to zero, and keep all the other pions on the mass shell, \mathcal{M} must vanish. (There are no pole terms; parity forbids a three-pion vertex.) At this point,

$$s = t = u = m_\pi^2. \quad (9.3)$$

Therefore,

$$A + 2B + C = 0. \quad (9.4)$$

By the analysis of the preceding section, if we send both k and q to zero, the only term which survives is the σ term. We will assume this is pure $I=0$, i.e. proportional to δ_{ab} . This is the extra assumption mentioned

earlier; without it, the σ term could be any combination of $I=0$ and $I=2$. The only motivation for this assumption is that this is indeed what happens in the σ -model.

At this point ($k=q=0$),

$$s=u=m_\pi^2, \quad t=0. \quad (9.5)$$

Therefore,

$$A+B+C=0. \quad (9.6)$$

From Eq. (9.4) and (9.6) we can already deduce that

$$B=0$$

and

$$A=-C. \quad (9.7)$$

Finally, we observe that if both q and k are close to zero, Eq. (8.11) tells us that the linear part of the matrix element is given by

$$\begin{aligned} i\mathcal{M} &= \frac{8g_V^2}{F_\pi^2} i\varepsilon_{abe}(I^e)_{dc} p \cdot q = \frac{8g_V^2}{F_\pi^2} (i\varepsilon_{abe})(i\varepsilon_{dce}) p \cdot q \\ &= \frac{8g_V^2}{F_\pi^2} (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) p \cdot q. \end{aligned} \quad (9.8)$$

In this region

$$s=m_\pi^2+2p \cdot q, \quad u=m_\pi^2-2p \cdot q, \quad t=0. \quad (9.9)$$

Therefore

$$C = \frac{4g_V^2}{F_\pi^2} = 8\pi L/m_\pi. \quad (9.10)$$

We have now determined all the unknown constants; thus we can evaluate \mathcal{M} at threshold. Here,

$$s=4m_\pi^2, \quad t=u=0. \quad (9.11)$$

Thus, at threshold

$$i\mathcal{M} = m_\pi^2 C [3\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}]. \quad (9.12)$$

We wish to express this result in terms of isospin states. We can think of \mathcal{M}_{bdac} as a 9×9 matrix acting on two-index isospin wave functions, associated with two-particle isospin states by the rule

$$|\psi\rangle = \psi_{ac}|a\rangle|c\rangle$$

(The sum on repeated indices is, as always, implied.) \mathcal{M} has one eigenvalue, $\mathcal{M}^{(0)}$, for isospin zero states, five equal eigenvalues $\mathcal{M}^{(2)}$, for isospin two states, and three eigenvalues, evidently zero, for isospin one states. We wish to find $\mathcal{M}^{(0)}$ and $\mathcal{M}^{(2)}$.

An evident isospin zero state is

$$\psi_{ac} = \delta_{ac}. \quad (9.13)$$

Applying \mathcal{M} to this, we find

$$i\mathcal{M}_{bdac}\delta_{ac} = m_\pi^2 C(9 - 1 - 1)\delta_{bd}. \quad (9.14)$$

Thus,

$$i\mathcal{M}^{(0)} = 7m_\pi^2 C. \quad (9.15)$$

To find $\mathcal{M}^{(2)}$, we take the trace:

$$i(5\mathcal{M}^{(2)} + \mathcal{M}^{(0)}) = \mathcal{M}_{acac} = m_\pi^2 C(9 - 9 - 3), \quad (9.16)$$

whence,

$$i\mathcal{M}^{(2)} = -2m_\pi^2 C. \quad (9.17)$$

Doing the kinematics (see Appendix 3 for the relevant formulae) we find

$$a_0 = \frac{7}{4}L = 0.20m_\pi^{-1},$$

and

$$a_2 = \frac{1}{2}L = -0.06m_\pi^{-1}. \quad (9.18)$$

These scattering lengths are quite small – much smaller than those predicted by any previous calculation. This should not be surprising; previous calculations treated pion dynamics as essentially an autonomous system; the characteristic range of variation; therefore, was on the order of m_π , and the scattering lengths of order m_π^{-1} . (They had to be – there was no other length in the problem.) Our extrapolation procedure, on the other hand is valid only if the characteristic range of variation is several times greater than m_π (for example, on the order of m_ρ). Eventually, the scattering lengths will be measured (from the phases in K_{l4} decay, if in no other way.) It will be interesting to see which approach is correct.

10 Kaon decays

We now turn to a class of processes which, although they only involve one soft pion, nevertheless require the evaluation of commutators. These are weak decays with a soft pion in the final state. These decays may be either leptonic

$$i \rightarrow f + \pi^a + \text{leptons}, \quad (10.1)$$

or non-leptonic

$$i \rightarrow f + \pi^a, \quad (10.2)$$

where i and f are hadronic states. In either case, we have to evaluate a matrix element of the form

$$\langle f | \mathcal{J} | i \rangle, \quad (10.3)$$

where \mathcal{J} is either a weak current or the weak non-leptonic Hamiltonian density.

To analyse this matrix element, we use the same methods as before. We write it in terms of

$$T\langle f|\partial^\mu A_\mu^a(x)\mathcal{J}(y)|i\rangle. \quad (10.4)$$

and pull the divergence through the time-ordering operator. This gives us, as usual, pole terms, plus the equal-time commutator

$$\langle f|[A_0^a(\mathbf{x}, 0), \mathcal{J}(\mathbf{y}, 0)]|i\rangle. \quad (10.5)$$

Unfortunately, in all the cases of interest (kaon decays, hyperon decays, etc.), \mathcal{J} is a strangeness-changing operator, and the commutators we have used so far are of no help in evaluating this expression. However, we can appeal to the chirality principle of Sec. 7 and assume, whatever the detailed structure of \mathcal{J} , that it is made up only of $V+A$ currents. We may then write

$$[A_0^a(\mathbf{x}, 0), \mathcal{J}(\mathbf{y}, 0)] = [V_0^a(\mathbf{x}, 0), \mathcal{J}(\mathbf{y}, 0)]. \quad (10.6)$$

This is an object about which we have some empirical information, for we usually know the isospin transformation properties of \mathcal{J} . I will briefly sketch the application of these methods to leptonic kaon decays. Let us begin with K_{l2} and K_{l3} decays – that is to say, with the processes

$$K^+ \rightarrow \text{leptons}$$

and

$$K^+ \rightarrow \pi^0 + \text{leptons}.$$

If we define \mathcal{J}_μ to be the strangeness-changing weak current, the relevant matrix elements are

$$\langle 0|\mathcal{J}_\mu(0)|K^+\rangle = -ip_\mu(K.F.)F_K, \quad (10.7)$$

and

$$\langle \pi^0|\mathcal{J}_\mu(0)|K^+\rangle = i(K.F.)[f_+(p_K + p_\pi)_\mu + f_-(p_K - p_K - p_\pi)_\mu]. \quad (10.8)$$

We know the commutator (10.6) from the $\Delta I = 1/2$ rule, the statement \mathcal{J}_μ is the charged component of an isospinor. There are no pole terms, since parity forbids a three-pseudo-scalar vertex. Thus we deduce

$$f_+ + f_- = g_V F_K / F_\pi. \quad (10.9)$$

This relation was first found by Callan and Treiman. It is in good agreement with experiment.

Callan and Treiman attempted to apply the same techniques to K_{l4} decay

$$K^+ \rightarrow \pi^+ + \pi^- + \text{leptons}.$$

The relevant form factors are

$$\langle \pi^+ \pi^- | \mathcal{J}_\mu(0) | K^+ \rangle = (\text{K.F.}) \times [f_1(q_+ + q_-)_\mu + f_2(q_+ - q_-)_\mu + f_3(k - q_+ - q_-)_\mu + f_4 \varepsilon_{\mu\nu\lambda\sigma} k^\nu q_+^\lambda q_-^\sigma], \quad (10.10)$$

where q_\pm and k are the momenta of the pions and the kaon, respectively.

Here we have the option of reducing either the π^+ or the π^- . If we reduce the π^- , the commutator is zero, and we obtain, in the limit $p_- \rightarrow 0$,

$$f_3 = 0, \quad f_1 = f_2. \quad (10.11)$$

If we reduce the π^+ , the commutator is not zero, and we obtain in the limit $p_+ \rightarrow 0$,

$$f_3 = \sqrt{2} g_V (f_+ + f_-) / F_\pi,$$

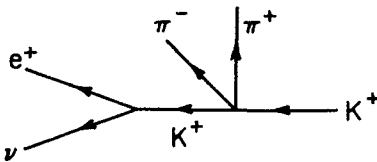
and

$$f_1 + f_2 = \sqrt{2} 2 g_V f_+ / F_\pi. \quad (10.12)$$

Callan and Treiman were puzzled by the discrepancy between the two predictions for f_3 . They ascribed the large variation of f_3 to a strong $\pi\pi$ s -wave interaction. However, this is unacceptable for two reasons. (1) If there is such an interaction, all soft pion calculations involving two or more pions are invalid. (2) All the assumptions of the calculation would be true if the strong interactions were weak, as we pointed out in Sec. 4. In such a case, Eqs. (10.11) and (10.12) would still hold, but there could be no strong final-state interaction.

This apparent paradox was resolved by Weinberg, who observed that there was a neglected pole diagram (Fig. 11). This makes a contribution

Fig. 11



only to f_3 , proportional to

$$\frac{(p_+ - p_-) \cdot (2k - p_+ - p_-)}{(p_+ + p_-)^2 - 2k \cdot (p_+ + p_-)}. \quad (10.13)$$

(The form of the four-boson interaction is taken from Sec. 8). Note that (10.13) is one when p_+ is zero, and minus one when p_- is zero. This diagram thus supplies the desired rapid variation of f_3 ; there is no need to introduce a strong final-state interaction. (For the details of the analysis, see Weinberg's paper, cited in the bibliography.)

Appendix 1. Notational conventions

- 1 We use the metric $g_{\mu\nu}$ for which

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1.$$

- 2 The differential operator ∂_μ is defined as $\partial/\partial x^\mu$; likewise

$$\square^2 = \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2,$$

and the Klein–Gordon equation is

$$(\square^2 + m^2)\phi = 0.$$

- 2 The γ -matrices are defined by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}.$$

Also, for any vector a ,

$$\not{a} = a_\mu \gamma^\mu.$$

The Dirac equation is

$$(i\not{\partial} - m)\psi = 0$$

- 4 The propagators for spin 0 and spin 1/2 fields are

$$i(p^2 - m^2)^{-1} \quad \text{and} \quad i(\not{p} - m)^{-1},$$

respectively.

- 5 In a matrix element, every particle carries with it a kinematic factor of $(2\pi)^{-3/2}(2E)^{-1/2}$. To shorten equations, the product of these factors is frequently denoted simply by (K.F.).

Appendix 2. No-renormalization theorem.

In this appendix we wish to show that if the vector current is proportional to the isospin current, the value of g_V does not depend on the strong interactions. In other words, g_V is not renormalized by the strong interactions.

Let us define the total isospin, \mathbf{I} , by

$$I^a = \int d^3x I_0^a(\mathbf{x}, 0).$$

If we neglect electromagnetism, \mathbf{I} is conserved; it commutes with the Hamiltonian and turns states of a given energy into states of the same energy. Likewise, from its definition, it commutes with the momentum, and turns states of a given momentum into states of the same momentum. Therefore, \mathbf{I} must turn one-nucleon states at rest into one-nucleon states at rest. Now the components of \mathbf{I} are a set of operators closed under commutation; in mathematical language they form a Lie algebra, one that is, in fact, the same as the angular-momentum Lie algebra. Therefore,

the one-nucleon states at rest must form the basis for a two-dimensional representation of this algebra.

However, we know from ordinary angular-momentum theory that there is only one such representation, the spin- $\frac{1}{2}$ representation, unique up to a similarity transformation. We fix the similarity transformation by defining the neutron and proton states to be eigenstates of I_z , and choosing the relative phase such that the matrix elements of the raising and lowering operators are real. Thus, the matrix elements of \mathbf{I} between one-nucleon states at rest are uniquely determined by the isospin algebra; they can not depend on any strong-interaction parameters.

But now we are almost home. For, by Eq. (7.1)

$$I^a = \alpha \int d^3x V_0^a(\mathbf{x}, 0),$$

and the right-hand side of this equation, evaluated between one-nucleon states at rest, is clearly linearly related to g_v , since the momentum transfer, k , is zero. Thus, g_v can not depend on the strong interactions, and we have proved the theorem.

Appendix 3. Threshold S -matrix and threshold scattering lengths

In this appendix, we will consider a two-particle scattering process with initial momenta (p, q) and final momenta (p', q') , and relate the value of the S -matrix near threshold to the s -wave scattering length.

Near threshold

$$\langle p', q' | (S - 1) | p, q \rangle = A \delta^4(p' + q' - p - q).$$

with A some constant. Let us introduce a new basis, labeled by the total and relative momenta, P and k .

$$\langle P', k' | (S - 1) | P, k \rangle = A \delta(E - E') \delta^3(\mathbf{P} - \mathbf{P}').$$

If we go to the center-of-mass frame (i.e. restrict ourselves to states with $P = 0$) we may write the center of mass S -matrix as

$$\langle \mathbf{k}' | (S - 1) | \mathbf{k} \rangle = A \delta(E - E').$$

Note that for small \mathbf{k} ,

$$E = \frac{\mathbf{k}^2}{2m_1} + \frac{\mathbf{k}^2}{2m_2} = \frac{\mathbf{k}^2}{2\mu},$$

where m_1 and m_2 are the masses of the two particles, and μ is the reduced mass. The s -wave state, $|k\rangle$, is defined by

$$|k\rangle = \frac{1}{4\pi k} \int d^3\mathbf{k} \delta(k - |\mathbf{k}|) |\mathbf{k}\rangle.$$

We have normalized $|k\rangle$ such that

$$\langle k'|k\rangle = \delta(k - k').$$

It follows that, near $k=0$,

$$\begin{aligned}\langle k'|S-1|k\rangle &= 4\pi k^2 A \delta(E' - E) \\ &= 4\pi k_\mu A \delta(k' - k).\end{aligned}$$

The s -wave phase shift is defined by

$$S|k\rangle = e^{2i\delta}|k\rangle.$$

Near threshold

$$\delta = ak,$$

where a is the scattering length. Therefore

$$(S-1)|k\rangle = 2iak|k\rangle.$$

Comparison with the earlier formulae shows that

$$a = -2\pi i A_\mu,$$

the desired result.

If we use invariant Feynman amplitudes, defined as in the lectures, then

$$A = (2\pi)^4 (\text{K.F.}) \mathcal{M}.$$

At threshold in the center of mass frame,

$$(\text{K.F.}) = \frac{1}{(2\pi)^6} \frac{1}{4m_1 m_2}.$$

Thus

$$a = -i \frac{\mu \mathcal{M}}{8\pi m_1 m_2} = -i \frac{\mathcal{M}}{8\pi(m_1 + m_2)}.$$

If the two particles are identical, there is an extra factor of $1/\sqrt{2}$ in the definition of $|k\rangle$, because $|k\rangle$ and $|-k\rangle$ are identical states. This introduces an extra factor of $1/2$ into our final formulae.

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