1

An introduction to unitary symmetry
(1966)

1 The search for higher symmetries

1.1 *The eight-baryon puzzle*

Let us begin with a very simple observation: there are eight baryons. By this I mean there are eight positive-parity particles with spin one-half and nucleon number one (the nucleons, the Λ, the Σs, and the Ξs), and that the masses of these particles are close together, all lying within 20% of their common mean mass. There are no other particles with the same parity, spin, and nucleon number which lie at all close in mass to these eight. Now, eight is a disquietingly large number of particles; although there is no fundamental reason why we should not have eight independent particles in a field theory, with eight arbitrary masses and eight sets of arbitrary coupling constants, still, it would be very pleasant if there were some way to reduce the number of entities. Two methods immediately suggest themselves:

(i) Perhaps some of the baryons are fundamental and the others are composite meson–baryon states. This was the central idea of the original Sakata model, where p, n, and Λ were taken as fundamental. Then the approximate mass degeneracy would arise because the masses of the mesons are small compared to the masses of the fundamental baryons. This idea has not met with much success.

(ii) Perhaps the strong interactions are approximately symmetric under a group larger than the ordinary isospin–hypercharge group. The eight baryons form a basis for an irreducible representation of this 'higher symmetry group'. If the symmetry were exact, the masses would be degenerate; because the symmetry is only approximate, the masses are split. (The situation envisioned here is something like that which prevails for isospin symmetry; if we turned off the weak interactions and electromagnetism, the world would be isospin symmetric, and all the particles in an isotopic multiplet would be degenerate; in fact, the world is not isospin symmetric, and the masses are slightly different.) This is sometimes phrased by saying that the strong-interaction Hamiltonian is a sum of two terms: one due to 'very strong interactions', which is symmetric under the higher symmetry group, and one due to 'medium-strong interactions', which is not.

The second viewpoint is one which has met with considerable success, and we intend to explore it in more detail here.

Now, what do we know about the structure of such a 'higher symmetry group'? Well, in the first place, there are several assumptions we can make, not because they are forced upon us by the problem, but because they are straightforward generalizations of what we know about the isospin group, and because they simplify the problem considerably. We assume, in the limit of exact symmetry, that:

(i) The group commutes with the Poincaré group.
(ii) The group acts on the Hilbert space of the states of the world as a group of unitary operators. (This is connected with conservation of probability.)
(iii) The one-particle states form an invariant subspace under the action of the group. The many-particle asymptotic states transform like tensor products of one-particle states.
(iv) The group commutes with the S-matrix.

A group which satisfies these conditions is usually called 'an internal symmetry group'. It is worth noting that in a field theory any group whose generators are obtained by integrating, over all space, the time components of a set of conserved Hermitian currents is an internal symmetry group.

In addition to these assumptions, there are several conditions forced on the group by the nature of the problem.

(i) The group turns baryon states at rest, with spin up, into baryon states at rest, with spin up. Therefore, it must have an eight-dimensional representation.
(ii) Since the whole point of the hypothesis is to explain the approximate equality of the baryon masses, this representation must be irreducible.
(iii) (This is a rather fine technical point, but it is necessary in order

to avoid some pathological cases.) The eight-dimensional representation should be closed, in the sense that any matrix which is the limit of a sequence of representation matrices should be a representation matrix itself. Since the S-matrix is continuous, if we are given a representation that is not closed, we can always close it, and thus obtain a representation of a larger group which is still an internal symmetry group.

(iv) The representation must be faithful. All observed particles can, in principle, be constructed out of the eight baryons and their antiparticles. Thus, any group element which acts trivially on the baryons will act trivially on everything known, and might as well be forgotten.

These conditions are sufficient to ensure that the higher symmetry group must be a compact Lie group. This is a class of mathematical objects which has been thoroughly investigated; all the compact Lie groups have been classified, and so have all of their representations. So finding all the compact Lie groups with eight-dimensional representations is merely a question of knowing what mathematics books to look in.

However, there is another condition, which arises because we want our higher symmetry group to contain the isospin–hypercharge group, the group of the old familiar symmetries of strong-interaction physics. This is:

(v) The group should contain a subgroup isomorphic to the isospin–hypercharge group. Furthermore, when we restrict ourselves to this subgroup, the eight-dimensional representation should decompose into an isodoublet of hypercharge one (N), an isosinglet of hypercharge zero (Λ), an isotriplet of hypercharge zero (Σ), and an isodoublet of hypercharge minus one (Ξ).

To determine the groups which satisfy this last condition, as well as all the others, requires some independent effort on the part of the investigator; however, the work is straightforward and can be done in a few hours. At the end of all this one has a long, and unenlightening, list. However, it has a surprising property; every group on it contains either SU(3), the group of all unimodular unitary transformations on a three-dimensional complex vector space, or G_0, a group called 'the connected part of the minimal global symmetry group', whose structure I will explain below.

These two groups are minimal; if their predictions are wrong, the predictions of any other higher symmetry group must be wrong, and the whole idea of higher symmetry must be abandoned. We will begin with G_0, since that turns out to be wrong; SU(3), of course, turns out to be right – otherwise I would not be giving these lectures.

1.2 *The elimination of G_0*

G_0 is the direct product of three factors of SU(2),

$$G_0 \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(2).$$

Thus, every element of G_0 can be written as a triplet of elements of SU(2),

$$(g_1, g_2, g_3),$$

the generators of G_0 are three commuting 'angular-momentum vectors',

$$\mathbf{I}^{(1)}, \mathbf{I}^{(2)}, \mathbf{I}^{(3)},$$

and the irreducible representations of G_0 are labeled by three 'spins',

$$(s_1, s_2, s_3).$$

Isospin and hypercharge are imbedded in G_0 in the following way:

$$\mathbf{I} = \mathbf{I}^{(1)} + \mathbf{I}^{(2)},$$

and $Y = 2I_z^{(3)}$.

The eight baryons transform according to the representation

$$(\tfrac{1}{2}, 0, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, \tfrac{1}{2}, 0).$$

(Thus, we have the desired isospin–hypercharge decomposition $I=\frac{1}{2}$, $Y=\pm 1$, and $I=0$, $Y=0$.) This is a reducible representation of G_0; therefore G_0 does not meet our conditions – though it is contained in many groups which do. The easiest way to display a group which does meet the conditions and which contains G_0 is to add to G_0 a discrete element which has the effect of interchanging $\mathbf{I}^{(2)}$ and $\mathbf{I}^{(3)}$. The representation above then becomes an irreducible representation of the enlarged group.

One particular element of G_0 is

$$R = e^{i\pi I_y^{(3)}}.$$

R is a hypercharge reflection operator,

$$RYR^{-1} = -Y \quad R\mathbf{I}R^{-1} = \mathbf{I}.$$

Simply from R invariance one can deduce an almost endless list of contradictions with experiment. I will give three here.

(i) For every particle or resonance there must be another particle or resonance, of the same spin and nucleon number, with opposite hypercharge, and with approximately the same mass. Thus, there must be a low-lying $\frac{3}{2}^+$ resonance with hypercharge -1. Such a resonance does not exist.

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(ii) It is easy to show from R invariance, isospin invariance, and the assumption that the electric current transforms like the electric charge (i.e. like $I_z + Y/2$), that the electromagnetic self-energy of the Σ^+ should be the same as that of the Σ^-. In fact, they are separated by 13 MeV.

(iii) From the same assumptions, the magnetic moment of the Λ should be zero. In fact, it is of the order of magnitude of the nucleon moments.

Thus, G_0 must be rejected. All that is left, our last hope, is SU(3).

2 SU(3) and its representations

In this section we will develop some of the properties of SU(3) and its representations, and also develop some methods for doing simple SU(3) calculations. We will begin by making some remarks about SU(n).

2.1 *The representations of* SU(n)

SU(n) is defined as the group of all unimodular unitary $n \times n$ complex matrices. This definition immediately tells us one representation of the group. If we let x^i be a complex n-vector, on which the group acts in the following manner

$$U: x^i \to U^i_j x^j,$$

then it is clear that the space of all x^i forms a basis for a representation of U(n). A basis for another representation is formed by a set of vectors y_i, which transform according to

$$U: y^i \to U^j_i y_j,$$

where

$$U^j_i = \bar{U}^j_i.$$

(We use an overbar throughout to indicate complex conjugation.)

The notation we have used, with its upper and lower indices, mimics that of ordinary tensor analysis. This mimicry is not deceptive, for, as a consequence of the unitarity of U,

$$U: x_i y^i \to x_i y^i,$$

and thus, just as in ordinary tensor analysis, the summation of upper and lower indices is an invariant operation.

By the usual method – taking direct products – we can form, from these primitive objects, the spaces of all tensors of rank $n+m$, with n upper and m lower indices. Each of these spaces clearly forms the basis for a representation of U(n).

These representations are, however, not necessarily irreducible. If a_{ij} is a tensor of rank two, we may divide it into the sum of a symmetric and an antisymmetric part

$$a_{ij} = a_{\{ij\}} + a_{[ij]}.$$

This separation is invariant under the action of the group; thus we have divided the original representation space (of dimension n^2) into two invariant subspaces (of dimension $n(n+1)/2$ and $n(n-1)/2$, respectively). Likewise, if a^i_j is a mixed tensor, we may divide it into two parts,

$$a^i_j = \frac{1}{n}\delta^j_i a^k_k + \hat{a}^i_j,$$

where

$$\hat{a}^i_i = 0,$$

and again we have two invariant subspaces, in this case of dimension 1 and $n^2 - 1$. (Note that in this case we cannot make a further reduction by symmetrizing and antisymmetrizing, because the indices transform differently under the action of the group.)

One of the problems we set in the introduction was finding the irreducible representations of SU(3). Our general method of attack will be to

(i) construct all tensors with a given number of upper and lower indices;
(ii) divide them invariantly into as many parts as we can;
(iii) discard the parts which we can show lead to representations equivalent to those obtained from tensors of lower rank; and
(iv) identify the remaining parts with (hopefully) new irreducible representations.

This method is, in principle, capable of generating all representations of SU(n), for any n. However, the combinatorics required to keep track of the irreducible tensors becomes formidable for n greater than 3, and other methods seem to be more efficient.

We will first tackle SU(2), to warm up.

2.2 *The representations of* SU(2)

It is a great convenience to introduce the antisymmetric two-index tensor ε^{ij}. Under the action of the group

$$U: \varepsilon^{ij} \to (\det U)\varepsilon^{ij};$$

but since U is unimodular, ε^{ij} is invariant. So is ε_{ij}, the corresponding entity with two lower indices. Given any tensor, we can use the ε tensors to raise and lower indices, just as the metric tensor is used in ordinary tensor analysis. In particular, we may write any tensor invariantly in terms of a tensor with all lower indices, and thus there is no representation of SU(2) induced on a general tensor that is not equivalent to one induced on a tensor with only lower indices. Let

$$a_{i_1 \dots i_n}$$

be a tensor of this kind. We may divide a into two parts, which are respectively symmetric and antisymmetric under interchange of the first two indices. With the aid of ε_{ij} we may write the antisymmetric part in terms of a tensor of lower rank:

$$a_{[i_1, i_2]\dots} = \varepsilon_{i_1 i_2} b_{i_3 \dots}.$$

Thus, we need only consider completely symmetric tensors. Thus we have (hopefully) an inequivalent irreducible representation of SU(2) associated with every space of completely symmetric tensors of a given rank, with all indices lower. (We adhere to an ancient convention, and call this rank $2s$.) We will call the representation $D^{(s)}$, or (s) for short. The dimension of (s) is the number of linearly independent tensors of the proper sort. Since the tensors are completely symmetric, and since the indices are allowed to assume only two values, this is the same as the number of ways $2s$ objects can be divided into two sets. That is to say,

$$\dim (s) = 2s + 1,$$

a result which should be familiar.

2.3 *The representations of* SU(3)

Now let us apply the same techniques to SU(3). The invariant ε tensors now have three indices, and thus cannot be used to raise and lower indices. Thus we have to work with tensors that have both lower and upper indices:

$$a^{i_1 \dots i_n}_{j_1 \dots j_n}.$$

However, we may still use the ε tensors to write the antisymmetric part of any tensor in terms of a tensor of lower rank

$$a^{[i_1, i_2]\dots}_{\dots} = \varepsilon^{i_1 i_2 k}\, b^{\dots}_{k \dots},$$

and therefore we need only consider tensors completely symmetric in both their upper and lower indices. By similar reasoning, we need only consider traceless tensors.

We will define the representation induced on the space of all traceless, completely symmetric tensors with n upper indices and m lower indices as $D^{(n,m)}$, or simply (n, m) for short. Eventually, we will show that this family of representations forms a complete set of inequivalent irreducible

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representations. However, it is more expedient to first extract some properties of these representations that are of practical importance, and only afterwards to prove their completeness, inequivalence, and irreducibility. Therefore, for the time being, we will simply call them IRs. If you want to think of 'IR' as an acronym for 'irreducible representation', you are welcome (indeed encouraged) to do so, but, in fact, we shall not use this property until we have proved it.

Note that a simple consequence of our definitions is that

$$\overline{(n, m)} = (m, n),$$

where the overbar indices complex conjugation.

2.4 *Dimensions of the IRs*

To calculate the dimension of an IR is a straightforward exercise in combinatorics. The space of all completely symmetric tensors with n upper indices and m lower indices can be decomposed into the space of all symmetric tensors and to a space of tensors equivalent to the traces. (By 'the traces' we mean those tensors which are obtained by summing one upper index with one lower index.) The space of the traces is equivalent to the space of all completely symmetric tensors with $(n-1)$ upper indices and $(m-1)$ lower indices. In representation language,

$$(n, 0) \otimes (0, m) = (n, m) \oplus [(n-1, 0) \otimes (0, m-1)].$$

Taking the dimensions of both sides

$$\dim (n, m) = \dim (n, 0) \times \dim (0, m) - [\dim (n-1, 0) \times \dim (0, m-1)].$$

By arguments similar to those we used in the discussion of SU(2),

$$\dim (n, 0) = \tfrac{1}{2}(n+2)(n+1) = \dim (0, \mathrm{n}),$$

and therefore,

$$\dim (n, m) = \tfrac{1}{2}(n+1)(m+1)(n+m+2).$$

There is an alternative method of designating representations, much used in the literature, in which a representation is labeled by its dimension. Since (n, m) and (m, n) have the same dimension, they are distinguished by labeling a representation by its dimension if n is greater than m, and by its dimension with an overbar if m is greater than n. Thus

(1, 0) is often called				**3**,
(0, 1)	"	"	"	$\bar{\mathbf{3}}$,
(1, 1)	"	"	"	**8**,
(3, 0)	"	"	"	**10**,
(2, 2)	"	"	"	**27**, etc

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(This is still not totally unambiguous, since dim $(4, 0) =$ dim $(2, 1) = 15$, but it suffices for practical purposes.)

2.5 *Isospin and hypercharge*

Merely to state that the very strong interactions are invariant under a given higher symmetry group is not sufficient information to construct a physical theory. We also have to know how isospin rotations and hypercharge rotations, the old familiar symmetries of strong-interaction physics, are imbedded in the group. The most convenient way of specifying this, for SU(3), is by giving the isospin and hypercharge transformation properties of the fundamental representation $(1, 0)$.

$(1, 0)$ is a three-dimensional representation. Therefore, when we restrict SU(3) to the isospin group SU(2), it can decompose in only three ways: into the sum of three isosinglets, into an isosinglet and an isodoublet, or into an isotriplet. The first case is mathematically impossible, for it implies that all the elements of SU(2) are inside the identity element of SU(3). The third case is mathematically possible, but physically uninteresting: if the fundamental triplet contains only integral isospin, then all the IRs (which are made from direct products of fundamental triplets) would contain only integral isospins, which is not very satisfactory for explaining nature.

Thus, only the second possibility remains. We will choose our basis in unitary space so that the $(1, 0)$ representation looks like

$$\begin{pmatrix} q^0 \\ q^{+1/2} \\ q^{-1/2} \end{pmatrix},$$

where we have labeled the basis vectors by the appropriate eigenvalue of I_z. All that remains is to assign the hypercharge.

In order that the hypercharge differences between observed particles be integers, it is necessary that the hypercharge difference between the singlet and the doublet be of magnitude one. We choose the hypercharge of the doublet to be the greater. (This is just a matter of convention, although it does not appear to be so at first glance. If we were to choose the opposite assignment, the representation $(0, 1)$ would have our original assignment. Thus, the structure of all calculations would be the same, except that (n, m) would everywhere be replaced by (m, n). This only has to do with our conventions about how we write things on paper, not with what goes on in the world. Formally, this degree of freedom corresponds to the existence of an outer automorphism of SU(3).)

Thus, if the hypercharge of the singlet is y, the hypercharge of the doublet is $1+y$. Now, if hypercharge rotations are to be a subgroup of SU(3), the hypercharge itself must be a generator of SU(3). The generators of unimodular matrices are traceless and, therefore, if Y is the 3×3 hypercharge matrix, then

$$\mathrm{Tr}\, Y = 3y + 2 = 0$$

which means

$$y = -\tfrac{2}{3}.$$

(If we had assigned a different value to y, we could not have put the hypercharge rotations inside SU(3). However, we could still have put them inside U(3), which is just as good a symmetry group for the purposes of physics. Unlike the choice connected with the previous degree of freedom, a different choice for y would make a real difference in the physics of the world. However, it would not effect what we plan to calculate here, the isospin and hypercharge assignments of the observed particles. The reason is that all observed particles may, in principle, be constructed from baryons and antibaryons. The baryons are assigned to the representation (1, 1), and the hypercharge assignments within this representation do not depend on y.)

2.6 *Isospin–hypercharge decompositions*

We want to determine how an IR decomposes into irreducible representations of the isospin–hypercharge subgroup when we restrict SU(3) to that subgroup. We will label representations of the subgroup by their isospin and hypercharge, thus

$$(i)^Y.$$

We already know one decomposition from the preceding section:

$$(1, 0) \to \left(\frac{1}{2}\right)^{1/3} \oplus (0)^{-2/3}.$$

From this formula it is trivial to calculate the decomposition of $(n, 0)$ since this is constructed by forming the completely symmetric product of n factors of (1, 0). Thus,

$$(n, 0) \to \left(\frac{n}{2}\right)^{n/3} \oplus \left(\frac{n-1}{2}\right)^{n/3-1} \oplus \ldots (0)^{-2n/3}.$$

Likewise,

$$(0, m) \to \left(\frac{m}{2}\right)^{-m/3} \oplus \left(\frac{m-1}{2}\right)^{-m/3+1} \oplus \ldots (0)^{2m/3}.$$

Table 1. *Graphical representation of the decomposition of* $(n, 0)\otimes(0, m)$

	$\left(\frac{n}{2}\right)^{\frac{n}{3}}$	$\left(\frac{n-1}{2}\right)^{\frac{n}{3}-1}$	$\cdots$
$\left(\frac{m}{2}\right)^{-\frac{m}{3}}$			
$\left(\frac{m-1}{2}\right)^{-\frac{m}{3}+1}$			

From these formulae we may graphically (see Table 1) represent the decomposition of $(n, 0)\otimes(0, m)$. We form an $m+1$ by $n+1$ rectangular array, and label each column by a term in the first decomposition and each row by a term in the second. Inside each box we put the product of the factor associated with the row and the factor associated with the column. (To calculate this requires only the ability to add hypercharges and to multiply representations of the rotation group.) The total content of the array is then the desired isospin–hypercharge decomposition.

The table shows such an array. (The shading will be explained momentarily.)

Likewise, in the same manner, we may decompose $(n-1, 0)\otimes(0, m-1)$. The relevant primary decompositions are

$$(n-1, 0)\rightarrow\left(\frac{n-1}{2}\right)^{n-1/3}\oplus\ldots,$$

and

$$(0, m+1)\rightarrow\left(\frac{m-1}{2}\right)^{-m-1/3}\oplus\ldots.$$

However, as far as calculating the product goes, we could with no error subtract $\frac{2}{3}$ from all the hypercharges occurring in the first series and add the same amount to all the hypercharges in the second series. But then the decomposition of this product is revealed as just the content of the shaded portion of the original array.

But since

$$(n, 0)\otimes(0, m)=(n, m)\oplus[(n-1, 0)\otimes(0, m-1)],$$

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Table 2. *Decomposition of* (2, 2)

	$(1)^{\frac{1}{3}}$	$(1/2)^{-\frac{2}{3}}$	$(0)^{-\frac{3}{3}}$
$(1)^{-\frac{1}{3}}$	$(2)^0 \oplus (1)^0 \oplus (0)^0$	$(3/2)^{-1} \oplus (1/2)^{-1}$	$(1)^{-2}$
$(1/2)^{\frac{2}{3}}$	$(3/2)^1 \oplus (1/2)^1$		
$(0)^{\frac{5}{3}}$	$(1)^2$		

this means that the decomposition of (n, m) is nothing but the content of the unshaded portion of the array, the border. In Table 2, we have calculated the decomposition of (2, 2) by this method.

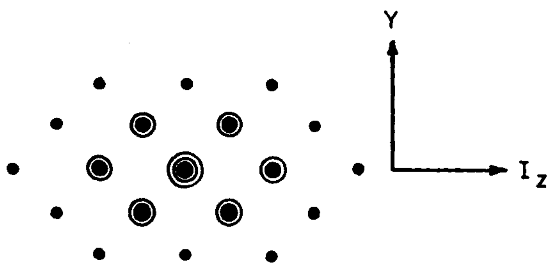
A common (although extremely awkward) method of displaying the isospin–hypercharge decomposition of a representation of SU(3) is to plot a graph, in which the vertical axis is Y and the horizontal axis is I_z. If a state occurs in a representation with a given (Y, I_z) assignment, a dot is placed on the graph in the appropriate location. If two such states occur, the dot is circled. If three, the dot is circled twice, etc. Such a graph is called a weight diagram. Our result may be used to construct the weight diagram of (2, 2) shown in Fig. 1.

It is a good exercise to use this method to construct the weight diagrams for (1, 1), (3, 0) and (4, 1), all of which are of physical interest. ((1, 1) is usually identified with the baryons and mesons: (3, 0) with the $\frac{3}{2}^+$ resonances in baryon–meson scattering. It has recently been proposed that some of the higher resonances may be part of (4, 1).)

2.7 *The Clebsch–Gordan series*

I now want to present a method for decomposing the direct product of two IRs into a direct sum of IRs. The method proceeds in two steps: first we decompose the direct product of two IRs into a direct sum of certain special reducible representations, which will be defined

Fig. 1



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below. Then we decompose the special reducible representations into a direct sum of IRs.

We shall denote the special reducible representations by $D^{(n,n',m,m')}$ – or for brevity, simply by $(n, n'; m, m')$. The representation $(n, n'; m, m')$ is defined as that representation which has for its basis the set of all tensors with $n+n'$ upper indices and $m+m'$ lower indices, that are completely symmetric among the first n upper indices, completely symmetric among the last n' upper indices, completely symmetric among the first m lower indices, completely symmetric among the last m' lower indices, and traceless. Roughly speaking, $(n, n'; m, m')$ may be thought of as the direct product of (n, m) and (n', m') with all traces removed, but without any symmetrization.

It is a simple matter to decompose the direct product of IRs into special reducible representations. We merely separate out all tensors that can be obtained by contracting, in all possible ways, indices from the set of n with indices from the set of m', and indices from the set of n' with indices from the set of m. That is to say,

$$\begin{aligned}(n, m)\otimes(n', m') = (n, n'; m, m') &\oplus (n-1, n'; m, m'-1)\\ &\oplus (n, n'-1; m-1, m')\\ &\oplus (n-1, n'-1; m-1, m'-1)\oplus \dots .\end{aligned}$$

The process terminates whenever we run out of indices to contract; that is, whenever a zero appears in the series on the right. In more compact form,

$$(n, m)\otimes(n', m') = \sum_{i=0}^{\min(n,m')} \sum_{j=0}^{\min(n'm)} (n-i, n'-j; m-j, m'-i),$$

where the summation sign indicates a direct sum.

We now wish to decompose one of our special reducible representations into direct sums of IRs. In the language of tensors, we want to decompose an arbitrary tensor from the basis of $(n, n'; m, m')$ into a sum of linear combinations of completely symmetric traceless tensors. Let us begin with the upper indices. Let

$$T^{i_1 \dots i_n i_{n+1} \dots i_{n+n'}}_{j_1 \dots j_m j_{m+1} \dots j_{m+m'}}$$

be an arbitrary tensor of the type under discussion. Let us choose a pair of upper indices; with no loss of generality they may be i_1 and i_{n+1}. We may write the tensor as the sum of two tensors, one of which is symmetric under interchange of these indices, and the other of which is antisymmetric. Using the ε tensor, we may write the antisymmetric part in terms of a tensor of lower rank,

$$S^{i_2 \dots i_n i_{n+2}}_{k j_1 \dots} = \varepsilon_{k i_1 i_{n+1}} T^{i_1 \dots}_{j_1 \dots}.$$

The surprising fact, which enormously simplifies the whole reduction, is that this tensor is already completely symmetric in its lower indices.

Proof. For example, let us take the indices j_1 and j_{m+1}. We prove the tensor is symmetric under interchange of these indices by showing that their contraction with the ε tensor vanishes,

$$\begin{aligned}\varepsilon^{rj_1j_{m+1}}S^{i_2\cdots}_{k} &= \varepsilon^{rj_1j_{m+1}}\varepsilon_{kl_1l_{n+1}}T^{l_1\cdots}_{j_1} \\ &= (\delta^r_k\delta^{j_1}_{i_1}\delta^{j_{m+1}}_{i_{n+1}} - \delta^{j_1}_k\delta^r_{i_1}\delta^{j_{m+1}}_{i_{n+1}} + \text{cyclic perms.}) \\ &\quad\times T^{i_1\cdots}_{j_1\cdots}.\end{aligned}$$

But, by the tracelessness of T, the right-hand side of this equation is zero. Similar arguments work for any pair of indices.

Thus, the symmetrization is very simple. We may remove pairs of upper indices, adding a lower index whenever we do so; or, alternatively, we may remove pairs of lower indices, adding an upper index whenever we do so – but we can never remove both a pair of upper indices and a pair of lower indices, for once we have removed a pair of upper(lower) indices, the tensor is already completely symmetric in its lower (upper) indices. The process terminates when we run out of indices. Returning from the basis space to the representation, we may write the decomposition in compact form:

$$\begin{aligned}(n, n'; m, m') = (n+n', m+m') &\\ \oplus \sum_{i=1}^{\min(n,n')} &(n+n'-2i, m+m'+i) \\ \oplus \sum_{j=1}^{\min(m,m')} &(n+n'+j, m+m'-2j),\end{aligned}$$

where the summation sign again represents the direct sum.

To demonstrate the efficiency of this method, we conclude with two examples. All arithmetic is shown.

Example 1: $(1, 1)\otimes(1, 1)$

$$(1, 1)\otimes(1, 1) = (1, 1; 1, 1)\oplus(1, 0; 0, 1)\oplus(0, 1; 1, 0)\oplus(0, 0; 0, 0).$$

$$\begin{aligned}(1, 1; 1, 1) &= (2, 2)\oplus(0, 3)\oplus(3, 0),\\ (1, 0; 0, 1) &= (1, 1),\\ (0, 1; 1, 0) &= (1, 1),\\ (0, 0; 0, 0) &= (0, 0).\end{aligned}$$

The desired decomposition is the sum of all the terms on the right. If we write this in terms of the notation in which representations are labelled

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by their dimensions, we find

$$\mathbf{8}\otimes\mathbf{8}=\mathbf{27}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}.$$

This is a familiar (and useful) result. It tells us the number of independent amplitudes for the scattering of two octets into two octets (eight, if time reversal imposes no further restrictions), the number of independent Yukawa couplings for antibaryon–baryon–pseudoscalar-meson (two), the number of amplitudes for the decay of a $\frac{3}{2}^+$ resonance into baryon and pseudoscalar meson (one), etc.

Example 2: $(2,2)\otimes(3,0)$

$$(2,2)\otimes(3,0)=(2,3;2,0)\oplus(2,2;1,0)\oplus(2,1;0,0).$$

$$(2,3;2,0)=(5,2)\oplus(3,3)\oplus(1,4),$$

$$(2,2;1,0)=(4,1)\oplus(2,2)\oplus(0,3),$$

$$(2,1;0,0)=(3,0)\oplus(1,1).$$

In the alternative notation,

$$\mathbf{27}\otimes\mathbf{10}=\mathbf{81}\oplus\mathbf{64}\oplus\mathbf{35}\oplus\overline{\mathbf{35}}\oplus\mathbf{27}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{8}.$$

2.8 *Some theorems*

I now wish to show that the IRs we have been discussing do indeed form a complete set of inequivalent irreducible representations of SU(3). In order to do this, we need a simple theorem from group theory. Let G be any compact group, and let the irreducible representations of G be arranged in a series, $D^{(0)}$, $D^{(1)}, \ldots,$ with $D^{(0)}$ the trivial representation. Then any representation of G may be decomposed into a direct sum of irreducible representations. We will write this in the following way

$$D=\oplus\sum n_i D^{(i)},$$

where n_i is the number of times $D^{(i)}$ occurs in the sum. The theorem we need is this:

Theorem. In the decomposition of $\bar{D}^{(i)}\otimes D^{(j)}$, $n_0=\delta_{ij}$.

Corollary. If, in the decomposition of $\bar{D}\otimes D$, $n_0=1$, D is irreducible.

Now, we know how to decompose the direct product of any two IRs into a sum of IRs, from the algorithms of the preceding section. Thus, if we know which IRs contain $D^{(0)}$, we can use the corollary to check if the IRs are irreducible, and the theorem to check if they are inequivalent.

We will now prove the following:

Lemma. Of all the IRs, only (0, 0) contains $D^{(0)}$ in its decomposition, and it contains it once.

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Proof. Let us assume an IR contains $D^{(0)}$. Then it must contain a hypercharge-zero isosinglet. It is a trivial consequence* of our algorithm for the hypercharge–isospin decomposition of an IR that only (m, m) contains a hypercharge-zero isosinglet, and further, that it only contains one such state. Thus, any tensor component of such an IR which transforms like a hypercharge-zero isosinglet must be pure $D^{(0)}$. Such a component is

$$a^{11\ldots1}_{11\ldots1}.$$

But this cannot be pure $D^{(0)}$, for there are evidently transformations of the group (for example, those which interchange the first and second axes and leave the third invariant) under which it is not invariant. The only exception to this argument occurs for (0, 0).

Thus, to find how many times $D^{(0)}$ occurs in the decomposition of $\overline{(n, m)}\otimes(n, m)=(m, n)\otimes(n, m)$, we need only find how many times (0, 0) occurs. But it is a trivial consequence* of our algorithm for the Clebsch–Gordan series that this only occurs once. Thus we have the following.

Theorem. (n, m) is an irreducible representation of SU(3).

By the same token, to prove the inequivalence of the IRs we need only see how many times (0, 0) occurs in the decomposition of $\overline{(n, m)}\otimes(n', m')$ $=(m, n)\otimes(n', m')$. But it follows trivially* from the Clebsch–Gordan algorithm that (0, 0) only occurs in this series if $m=m'$ and $n=n'$. Thus we have the following.

Theorem. (n, m) and (n', m') are equivalent only if $n=n'$ and $m=m'$.

To prove the completeness of the IRs, we need some more information from group theory. For any compact group it is possible to define an integral over group space in such a way that

$$\int \bar{D}^{(i)}_{\alpha\beta}(g)D^{(j)}_{\gamma\delta}(g)\mathrm{d}g=0 \quad (i\neq j),$$

where the Greek subscripts indicate matrix elements. Now, let us assume that there exists an irreducible representation of SU(3) which is not equivalent to any (n, m). Let us call this representation $D^{(?)}$. Then, by assumption

$$\int \bar{D}^{(?)}_{\alpha\beta}(g)D^{(n,m)}_{\gamma\delta}(g)\mathrm{d}g=0.$$

The representation (1, 0) has eight independent matrix elements. Let me call these z_i $(i=1\cdots8)$. They form a set of coordinates in group space. The matrix elements of (0, 1) are the $\bar{z}_i$. The equation above says that the matrix elements of $D^{(?)}$ are orthogonal to z_i and $\bar{z}_i$, that is to say, to all

* Prove it.

linear functions of z_i and $\bar{z}_i$. It also says that the matrix elements of $D^{(?)}$ are orthogonal to all the matrix elements of $(1,0)\otimes(1,0)$, $(1,0)\otimes(0,1)$, and $(0,1)\otimes(0,1)$, that is to say, to $z_i z_j$, $z_i\bar{z}_j$, and $\bar{z}_i\bar{z}_j$. In other words, these matrix elements are orthogonal to all polynomials of order two in the coordinates. In fact, by similar arguments, we can show that they are orthogonal to all polynomials in the coordinates and, therefore,

$$D^{(?)}_{\alpha\beta}(g)=0 \quad \text{for all } g.$$

But this is a contradiction, because for any representation $D_{\alpha\beta}(1)=\delta_{\alpha\beta}$. Thus, we have our final

Theorem. The IRs form a complete set of inequivalent irreducible representations of SU(3).

2.9 *Invariant couplings*

The last general subject I am going to talk about is the construction of invariant couplings. (For trilinear interactions, this is equivalent to the problem of constructing the Clebsch–Gordan coefficients.) I know of no methods here as simple and powerful as the ones discussed above. However, there are some special tricks based on tensorial methods which are useful for a very restricted class of problems. (Which, through the grace of God, turns out to include many cases of physical interest.) My main aim here will be to discuss these; however, I would like to begin with a general discussion, to place these methods within a wider framework.

2.10 *The problem of Cartesian components*

Let us suppose we wish to couple three octets of fields. Just to be definite, let us make them baryon (denoted by ψ), antibaryon ($\bar{\psi}$) and meson (ϕ). The Clebsch–Gordan algorithm tells us that there are only two invariant couplings, and a moment's thought shows what they must be;

$$\bar{\psi}^i_k \psi^j_i \phi^k_j \qquad \text{and} \qquad \bar{\psi}^i_k \psi^k_j \phi^j_i.$$

(Of course, if these are really baryon and meson fields, ψ is an octet of Dirac bispinors, and there should be a γ_5 in the above expressions to conserve parity. However, we will ignore degrees of freedom associated with space-time transformation properties here, and treat all fields as if they were scalars, for the sake of simplicity.) In a sense, this expression solves the problem completely, for it explicitly gives the invariant coupling in terms of the fields. Unfortunately, it gives the coupling in terms of the Cartesian components of the fields (ϕ^1_1, ϕ^1_2, etc.), and for practical applications, we need the coupling in terms of isospin–hypercharge eigenstates (π^+, K^0, etc.).

Thus, in this formulation, the whole problem of constructing invariant couplings reduces to the problem of expressing the Cartesian components of SU(3) tensors in terms of hypercharge–isospin eigenstates. If I could present a simple algorithm for constructing such expressions, the problem would be completely solved. Regrettably, I know of no such algorithm. However, we can always construct such expressions by what are essentially cut-and-try methods; the main portion of the subsequent exposition will be devoted to an explanation of these techniques. They are definitely not of the back of the envelope class; to do calculations, you need tables, which is unfortunate. However, with this way of looking at the problem you only need one table per representation, while with the more usual technique, you need one table per triplet of representations. So there is some gain.

As before, we will begin by discussing SU(2), where things are simple.

2.11 SU(2) *again*

Let us begin with a fundamental doublet of fields, which we will call p and n, and which we will write as a column vector,

$$\mathrm{N}=\begin{pmatrix}\mathrm{p}\\ \mathrm{n}\end{pmatrix}.$$

The conjugate doublet of antiparticle fields transforms like a row vector

$$\bar{\mathrm{N}}=(\bar{\mathrm{p}}\bar{\mathrm{n}}).$$

But, of course, with the aid of the ε tensor, we may write this as a column vector also:

$$\begin{pmatrix}\bar{\mathrm{n}}\\ -\bar{\mathrm{p}}\end{pmatrix}.$$

(The minus sign appears because of the antisymmetry of the ε tensor.)

Next, let us consider a triplet of fields (the pions, for example). We may represent this triplet in three equivalent ways, as tensors of the three forms

$$\phi_{ij},\ \phi^{ij},\ \phi^i_j.$$

We will choose the third way, and write the components in the form of a 2×2 matrix:

$$\begin{pmatrix}\phi^1_1 & \phi^1_2\\ \phi^2_1 & \phi^2_2\end{pmatrix}.$$

Antisymmetry for one form implies tracelessness for the other:

$$0=\varepsilon^{ij}\phi_{ij}=\phi^i_i.$$

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Thus

$$\phi_1^1 = -\phi_2^2.$$

It is clearly an invariant condition to demand that the matrix ϕ be Hermitian, and we shall impose such a condition here. (There is nothing wrong with non-Hermitian matrices, but they would describe sets of six real fields – perfectly suitable for the Σ hyperons, but not for the pions.) The component ϕ_2^1 carries charge $+1$, and we will normalize the matrix by demanding that it be the π^+ field. Then Hermiticity and tracelessness determine everything to within one free real parameter α:

$$\phi = \begin{pmatrix} \alpha\pi^0 & \pi^+ \\ \pi^- & -\alpha\pi^0 \end{pmatrix}.$$

This is determined by looking at the invariant-mass term in the Lagrangian. This is $\phi_j^i\phi_i^j$, and we want to normalize our fields so that it will be the sum of three squares with coefficients one*. Thus,

$$\frac{1}{2}\phi_j^i\phi_i^j = \pi^+\pi^- + \alpha^2\pi^0\pi^0 = \pi^+\pi^- + \frac{1}{2}\pi^0\pi^0,$$

which means

$$\alpha = \pm\sqrt{\left(\frac{1}{2}\right)}.$$

The sign ambiguity is just a matter of phase conventions. We choose the plus sign.

One advantage of writing these fields as row vectors, column vectors, and matrices, is that the ordinary operations of matrix multiplication (which just involve, with these conventions, the summation of upper and lower indices) are invariant under the action of the group. Thus, the usual pion–nucleon interaction may be written as

$$\mathscr{L} = \frac{1}{2}\mathrm{Tr}[\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2] + \bar{\mathrm{N}}(\mathrm{i}\partial_\mu\gamma^\mu - m)\mathrm{N} + g\bar{\mathrm{N}}\gamma_5\phi\mathrm{N} + \lambda\mathrm{Tr}\phi^4;$$

where Tr represents the ordinary matrix trace.

2.12 SU(3) *octets: trilinear couplings*

By exactly the same arguments as we used above, we can write the (1, 1) representation of SU(3) as a 3×3 matrix. To identify this matrix with the physical fields, all we need to do is study its transformation

* We do not have to do this of course; we could choose α to be any non-zero real number, and it would all cancel out of any calculation when we were finished with the renormalizations. This is, however, the traditional and convenient choice; it makes the symmetry between the three pions manifest.

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properties under the SU(2) subgroup. (I remind you that this is the set of all transformations that leave the first index invariant.) This determines everything to within normalization factors, which are determined by the same arguments we used in the SU(2) case. Thus, we find for the octet of baryon fields,

$$\psi=\begin{pmatrix} \sqrt{(\frac{2}{3})}\Lambda & \Xi^- & -\Xi^0 \\ p & -\sqrt{(\frac{1}{6})}\Lambda+\sqrt{(\frac{1}{2})}\Sigma^0 & \Sigma^+ \\ n & \Sigma^- & -\sqrt{(\frac{1}{6})}\Lambda-\sqrt{(\frac{1}{2})}\Sigma^0 \end{pmatrix}$$

The minus sign for the Ξ doublet arises because we are writing it as a row vector. (See the discussion for the nucleon doublet above.) Likewise, we may write a matrix for the pseudoscalar meson octet

$$\phi=\begin{pmatrix} \sqrt{(\frac{2}{3})}\eta & K^- & \bar{K}^0 \\ K^+ & -\sqrt{(\frac{1}{6})}\eta+\sqrt{(\frac{1}{2})}\pi^0 & \pi^+ \\ K^0 & \pi^- & \sqrt{(\frac{1}{6})}\eta-\sqrt{(\frac{1}{2})}\pi^0 \end{pmatrix}$$

Note that ϕ is Hermitian. The corresponding minus sign does not arise here. This is because the usual convention is to define $(K^-, \bar{K}^0)$ as the anti-particles of (K^+, K^0); the particles which form an isotopic doublet with proper phase relations are $(\bar{K}^0, -K^-)$. The two minus signs cancel.

Just as before, the ordinary operations of matrix algebra are invariant under the action of the group. Thus, the two invariant couplings discussed in 2.10 may be written as

$$\mathrm{Tr}\,\bar{\psi}\psi\phi \quad \text{and} \quad \mathrm{Tr}\,\bar{\psi}\phi\psi.$$

Actually, it is conventional to consider not these two couplings, but the two linear combinations

$$\mathrm{Tr}\,\bar{\psi}\{\psi, \phi\} \quad \text{and} \quad \mathrm{Tr}\,\bar{\psi}[\phi, \psi].$$

These are called d-type and f-type couplings, respectively.

2.13 SU(3) *octets: quadrilinear couplings*

We may also use these matrices to study the invariant coupling of four octets. This is useful in the analysis of meson–baryon scattering, and also in the analysis of non-leptonic hyperon decays, using octet spurions. Let us represent the four octets by four 3×3 traceless matrices, A_i $(i=1\ldots 4)$.

The possible invariants fall into two classes. There are those of the form

$$\mathrm{Tr}\,A_1A_2A_3A_4.$$

There are twenty-four permutations of the four As, but due to the invariance of the trace under cyclic permutations, only six of these lead to distinct invariants. Secondly, there are also invariants of the form

$$\mathrm{Tr}\, A_1 A_2\, \mathrm{Tr}\, A_3 A_4.$$

Here, only three permutations lead to distinct invariants.

Thus we have nine invariants in all. Unfortunately, we know that there are only eight independent couplings of four octets. (This was one of the consequences of the first example given of the Clebsch–Gordan algorithm.) This is an apparent contradiction; we can only escape if there is a linear relation among our nine matrix invariants.

Indeed there is such a relation; it is

$$\sum_{\text{six distinct perms}} \mathrm{Tr}\, A_1 A_2 A_3 A_4 = \sum_{\text{three distinct perms}} \mathrm{Tr}\, A_1 A_2\, \mathrm{Tr}\, A_3\, A_4,$$

for any four 3×3 traceless matrices. This identity was no doubt known to Cayley; however, it was first shown to me by Burgoyne, and I will embarrass him by calling it Burgoyne's identity.

Proof. Let A be any 3×3 matrix, with eigenvalues a_1, a_2, a_3; A satisfies its own characteristic equation

$$(A-a_1)(A-a_2)(A-a_3)=0.$$

Let us write the coefficients of the powers of A in terms of invariants. We then obtain

$$A^3-(\mathrm{Tr}\, A)A^2-\tfrac{1}{2}[\mathrm{Tr}\, A^2-(\mathrm{Tr}\, A)^2]A-(\det A)=0.$$

Multiplying this equation by A and taking the trace, we find

$$\mathrm{Tr}\, A^4=\tfrac{1}{2}(\mathrm{Tr}\, A^2)^2,$$

if A is traceless. Now let

$$A=\sum_i \lambda_i A_i,$$

and let us extract the coefficient of $\lambda_1\lambda_2\lambda_3\lambda_4$ from the above equation. We then obtain Burgoyne's identity.

2.14 *A mixed notation*

Actually, for many purposes, it is unnecessary to reduce an SU(3)-invariant interaction to an expression which explicitly involves isospin–hypercharge eigenstates; it suffices to reduce the coupling to a sum of SU(2)-invariant interactions. This can easily be done within our framework. For example, instead of explicitly writing down the matrix for the baryon fields, we can display the decomposition in the following way:

$$\psi_1^1=\sqrt{(\tfrac{2}{3})}\,\Lambda,$$
$$\psi_a^1=\Xi_a,$$
$$\psi_1^a=N^a,$$

and

$$\psi_b^a=\Sigma_b^a-\sqrt{(\tfrac{1}{6})}\,\delta_b^a\Lambda,$$

where a, $b=2$, 3, and the terms on the right are the appropriate SU(2) tensors. If we insert this (and the corresponding expression for the mesons) in the invariant coupling, we will obtain the desired result.

This procedure is evidently insensitive to the convention one adapts for phases within an SU(2) multiplet. This is an advantage because there are two such conventions which are widely used in the literature. The convention we have used here is the one that is common in elementary particle physics and the one which is used in most of the earlier SU(3) literature. There is another convention, which is designed to agree with the standard angular-momentum convention, and which is used in de Swart's tables and in some of the later literature. In this convention, the field conjugate to π^+ is minus π^-.

Using the methods we have explained here, we can construct similar decompositions for any representation. (But the process is tedious.) For example, the decomposition for the (3, 0) representation is

$$\psi^{111}=\Omega,$$
$$\psi^{11a}=\sqrt{(\tfrac{1}{3})}\,\Xi^a,$$
$$\psi^{1ab}=\sqrt{(\tfrac{1}{3})}\,\Sigma^{ab},$$

and

$$\psi^{abc}=\Delta^{abc},$$

where Ξ and Σ label the states with the same isospin–hypercharge assignments as the corresponding baryons, Ω is a $(0)^{-2}$ state, and Δ a $(\tfrac{3}{2})^1$ multiplet, and where we have omitted components obtainable from the ones we have listed by trivial permutations of indices.

A problem. Before we leave these matters and go on to some applications, I would like to give you a problem to think about. You may find it amusing.

Consider a field theory in which there are eight pseudoscalar fields, of the same mass, forming an SU(3) octet, coupled together by an SU(3)-invariant interaction Lagrangian, involving no derivatives, and of fourth order in the fields. There are no other fields.

(i) Show that the interaction Lagrangian involves only one arbitrary coupling constant;

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(ii) Assume that for sufficiently strong coupling two pseudoscalar mesons bind to make a scalar bound state. These states must, of course, fall into SU(3) representations. Show that the only representations that can occur are **1**, **8** and **27**.

(iii) Show that **8** and **27** are necessarily degenerate in mass.

3 Applications

The applications of SU(3) fall into two classes: those that deal with the limit of exact symmetry, and those which attempt to treat the three kinds of symmetry-breaking interactions. All the applications I will consider here will be of the second class.

There are three types of symmetry-breaking interactions: the mysterious medium-strong interactions, the electromagnetic interactions, and the weak interactions*. The general procedure one adopts for these problems is as follows: one treats the symmetry-breaking interactions as perturbations to an exactly symmetric world. One calculates the effect of interest to lowest non-vanishing order in a perturbative expansion, and then uses the known (or conjectured) SU(3) transformation properties of the symmetry-breaking interaction to connect the coefficients in the expansion. In this way one obtains sum rules for mass splittings, magnetic moments, leptonic-decay amplitudes, etc.

The computational reason for this reliance on lowest-order perturbation theory is clear: as one goes to higher orders in perturbation theory, more free parameters appear, until, eventually, one has no predictions left. The physical justification is somewhat more obscure. Lowest-order perturbation theory is certainly very plausible for the weak and electromagnetic interactions, but it is much less so for the medium-strong interactions. We will use it, nevertheless, and keep our fingers crossed.

3.1 *Electromagnetism*

The electromagnetic interaction Lagrange density is

$$ej_\mu A^\mu,$$

where j_μ is the electric current and A_μ the electromagnetic field. A_μ does not involve hadronic fields; therefore it commutes with the generators of SU(3), and the transformation properties of the interaction are given only in terms of those of the current.

* Only medium-strong and electromagnetic interactions are discussed here. The lectures as given contained a brief discussion of the Cabibbo theory for semi-leptonic decay, but since this was discussed in more detail by other lecturers, I have not included it in the notes. See *Strong and Weak Interactions – Present Problems* (Academic Press, New York and London, 1966).

Now, we do not know the transformation properties of j_μ; however, we do know those of the electric charge

$$Q=\int j_0(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x},$$

because this is a sum of SU(3) generators,

$$Q=\frac{Y}{2}+I_z.$$

Thus, Q transforms like a group generator, that is to say, like a member of a (1, 1) representation. Which member? This can be most easily determined by inspecting the (1, 0) representation. The three members of this representation have hypercharge $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, and I_z $(0, \frac{1}{2}, -\frac{1}{2})$. Thus, they have charge $(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$, and Q therefore transforms like the diagonal matrix

$$E=\frac{1}{3}\begin{pmatrix}-1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

We now assume that j_μ has the same transformation properties as Q. This is a very plausible assumption; it would be true, for example, if we began with an SU(3)-invariant field theory of baryons and mesons and constructed the minimal electromagnetic coupling.

3.2 *Magnetic moments: baryons*

The electromagnetic form factors for the baryon octet are defined by

$$\langle B|j_\mu(x)|B'\rangle=\mathrm{e}^{\mathrm{i}k\cdot x}\bar{u}(F_1^{BB'}(k^2)\gamma_\mu+F_2^{BB'}\sigma_{\mu\nu}k^\nu)u',$$

where B and B' are one-baryon states with corresponding spinors u and u', and k is the four-momentum transfer. We wish to find the constraints placed on these quantities, calculated to first order in electromagnetism and to zeroth order in all other symmetry-breaking interactions.

The baryon state on the right transforms like (1, 1), as does that on the left. The electric current transforms like (1, 1). $(1, 1)\otimes(1, 1)$ contains (1, 1) twice. Thus there are two invariant couplings. Thus, we can write the electric (magnetic) form factors of all eight baryons – as well as the electric (magnetic) form factor for the Σ^0–Λ transition – in terms of two unknown functions of k^2. It is easy to write down the explicit expression in terms of the matrix methods of Part 2:

$$F_{1,2}^{BB'}(k^2)=F_{1,2}^{(1)}(k^2)\,\mathrm{Tr}\,\bar{B}EB'+F_{1,2}^{(2)}(k^2)\,\mathrm{Tr}\,\bar{B}B'E,$$

where B and B' are the 3×3 matrices associated with the baryon states and $F_{1,2}^{(1)}$ and $F_{1,2}^{(2)}$ are unknown functions. In particular, $F_2^{BB'}(0)$, which tells us the eight baryon magnetic moments and the Σ^0–Λ transition moment, is given in terms of two unknown constants. It is trivial to do the traces and eliminate the two constants; we then find equations for seven moments in terms of $\mu(\mathrm{p})$ and $\mu(\mathrm{n})$. These are:

$$\begin{aligned} \mu(\Lambda) &= \tfrac{1}{2}\mu(\mathrm{n}), \\ \mu(\Sigma^+) &= \mu(\mathrm{p}), \\ \mu(\Xi^0) &= \mu(\mathrm{n}), \\ \mu(\Xi^-) = \mu(\Sigma^-) &= -[\mu(\mathrm{p}) + \mu(\mathrm{n})], \\ \mu(\Sigma^0) &= -\tfrac{1}{2}\mu(\mathrm{n}), \end{aligned}$$

and

$$\mu(\Sigma^0 \to \Lambda) = \tfrac{1}{2}\sqrt{3}\,\mu(\mathrm{n}).$$

Only the first of these formulae has been checked experimentally. The formula predicts $\mu(\Lambda) = -0.95$ nuclear magnetons; the experimental value is -0.73 ± 0.17.

3.3 *Electromagnetic mass splittings*

Electromagnetic mass splittings are second-order effects; the electromagnetic Hamiltonian must act twice. At first glance this would seem to give nine independent parameters:

$$\begin{tikzcd}[column sep=tiny]
(0,0) \arrow[d] & \oplus & (1,1) \arrow[d] \arrow[drr] & \oplus & (1,1) \arrow[d] \arrow[dll] & \oplus & (3,0) \arrow[d] & \oplus & (0,3) \arrow[d] & \oplus & (2,2) \arrow[d] \\
(0,0) & \oplus & (1,1) & \oplus & (1,1) & \oplus & (3,0) & \oplus & (0,3) & \oplus & (2,2),
\end{tikzcd}$$

where the first line is the decomposition of the two Hamiltonians, the second that of the two baryon states, and the arrows the independent couplings. However, this neglects the fact that the mass splitting is symmetric under interchange of the two Hamiltonians. If we analyze the symmetry of the terms in $(1, 1)\otimes(1, 1)$, we find that $(0, 0)$, $(2, 2)$ and one of the $(1, 1)$s are symmetric, while $(0, 3)$, $(3, 0)$, and the other $(1, 1)$ are antisymmetric. So the figure above should read:

$$\begin{tikzcd}[column sep=tiny]
(1,1)\otimes(1,1) = & (0,0) \arrow[d] & \oplus & (1,1) \arrow[d] \arrow[drr] & \oplus & (2,2) \arrow[drr] & \oplus\,\text{antisymmetric terms} & & \\
(1,1)\otimes(1,1) = & (0,0) & \oplus & (1,1) & \oplus & (1,1) & \oplus & (2,2) & \oplus(0,3)\oplus(3,0).
\end{tikzcd}$$

Furthermore, the first of these (the singlet term) contributes equally to all baryon masses. Thus, there are only three significant terms for the mass splittings. Since there are four observable splittings, this means we can find one relation among them.

The three invariants are

$$\mathrm{Tr}\,\bar{B}E^2B,\ \mathrm{Tr}\,\bar{B}BE^2, \quad \text{and} \quad \mathrm{Tr}\,\bar{B}EBE.$$

Again, it is trivial to evaluate the traces. The resulting formula is

$$\Xi^- - \Xi^0 = \Sigma^- - \Sigma^+ + \mathrm{p} - \mathrm{n},$$

where the names of the particles stand for their masses. Experimentally, the right-hand side of this equation is 6.6 ± 0.1 MeV, while the left-hand side is 6.5 ± 1.0 MeV.

3.4 *Electromagnetic properties of the decuplet*

The analysis of the form factors of the $\frac{3}{2}^+$ decuplet proceeds along exactly the same lines as that for the $\frac{1}{2}^+$ octet (except that there are four form factors instead of two). The decuplet state on the right transforms like (0, 3), that on the left like $\overline{(0, 3)} = (3, 0)$:

$$(3, 0) \otimes (0, 3) = (3, 3) \oplus (2, 2) \oplus (1, 1) \oplus (0, 0).$$

Thus, we have to find only one SU(3) invariant. But we already know one – the electric charge! Thus, for the decuplet, *all moments are proportional to the charge.*

Electromagnetic mass splittings are just as simple. Here there are only two possible invariants. Once again, we know of two objects with the right transformation properties, Q and Q^2. Thus

$$\delta M_{\mathrm{em}} = \alpha Q + \beta Q^2,$$

with α and β unknown constants.

3.5 *The medium-strong interactions*

In his classic first paper on SU(3), Gell-Mann proposed that the medium-strong interaction Hamiltonian should transform like a component of an octet. This assumption is, of course, completely independent from the assumption of SU(3) invariance of the very strong interactions. Nevertheless, the remarkable success of the mass formula that can be derived from the two assumptions offers evidence for both of them.

If the medium-strong interaction transforms like a component of an octet, it must be the Λ-like component, for no other carries zero hypercharge and isospin. It follows from our Clebsch–Gordan algorithm that there are at most two ways of coupling an octet to the product of an IR and its conjugate; therefore, for any hadron multiplet there are, at most, three independent coefficients in the mass formula – one for the common mass, and two for the splitting.

It turns out to be possible to explicitly calculate the two splitting terms, once and for all, for every unitary multiplet. (This is a curious fact; the electromagnetic mass formulae do not have this property.) This was first done by Okubo, using a lengthy method. I shall give here a short proof due to Smorodinski.

For any representation of SU(3), the eight matrices which form the generators of the representation transform, under the action of the group, like an octet. Therefore, we can arrange them in a 3×3 matrix (a matrix of matrices!):

$$\mathbf{G}=\begin{pmatrix} -2\alpha Y & - & - \\ - & \alpha Y+I_z & \sqrt{2}\,I_+ \\ - & \sqrt{2}\,I_- & \alpha Y-I_z \end{pmatrix},$$

where the Is are the usual isospin generators, Y is the hypercharge, and α is an unknown real constant. The blank spaces are occupied by strangeness-changing generators in which we are not interested. To determine α, we observe that those transformations which mix the first and third axes, and leave the second invariant, do not change the electric charge. Electric charge is a generator of the group, and it is evident that the only member of the matrix of generators which is left invariant by these transformations is the 22 entry. Therefore, this must be proportional to the electric charge, which means

$$\alpha=\tfrac{1}{2}.$$

We want to find objects which transform like the 11 components of octets. One such object is clearly

$$G_1^1=-Y.$$

Since matrix operations are invariant under SU(3), any matrix function of G will also transform like a matrix, that is to say, like a mixture of octet and singlet. In particular, let us consider the co-factor matrix. (The co-factor matrix is the matrix composed of the determinants of the minors of a matrix; it occurs in the standard expression for the inverse of a matrix):

$$\begin{aligned}(\text{cof } G)_1^1 &= G_2^2G_3^3-G_2^3G_3^2 \\ &= \frac{Y^2}{4}-\mathbf{I}^2.\end{aligned}$$

This is a second object with the desired transformation properties. (It is not a pure octet, but the singlet part will not affect a formula for mass differences.) Thus, for any unitary multiplet the three terms in the mass formula are

$$m=a+bY+c\left(\frac{Y^2}{4}-I(I+1)\right).$$

This is the famous Gell-Mann–Okubo formula.

I have written this formula in terms of mass; however, it turns out that one obtains a better fit with experiment if one uses masses for fermions and squares of masses for bosons. Nobody knows why this is so. (Covariant perturbation theory certainly suggests that masses are the 'natural' parameters for fermions, and squares of masses for bosons, but this is not much of an argument.)

The experimental tests of the formula are discussed in Professor Barbaro-Galtieri's lectures.

4 Ideas of octet enhancement

Up to now, we have been considering the three kinds of symmetry-breaking interactions as independent effects. I would now like to turn to the attempts that have been made to connect them and, in particular, to the remarkable relations discovered a few years ago between the medium-strong and electromagnetic mass splittings.

'One measures a circle beginning anywhere.' Let us begin with the mass spectrum of the Σ hyperons. These particles have isospin one; therefore the mass operator is the sum of three parts, with isospins zero, one, and two, respectively. The last two are electromagnetic in origin. Let us attempt to estimate their relative magnitude, in the most naive way. Naively, we would expect the major portion of the electromagnetic self-mass of a particle to be given by diagrams of the form shown in Fig. 2 where the blobs represent electromagnetic form factors. If we estimate the magnitude of these diagrams for the nucleon system, using experimental form factors, we find that the magnetic form factors fall off too rapidly to make any significant contribution to the integral giving the self-mass. Therefore the self-mass is, to a good approximation, an integral over a quadratic form in the electric form factors only. Now, for the nucleons. The electric form factors are proportional to the charge, to a very good approximation; therefore, by unitary symmetry, the same will be true for the Σs. Thus, we would expect the two charged Σs to have the same self-mass, that is to say, we would expect the mass splitting to be mostly isospin two.

However, when we examine the experimental situation, we find that

$$\Delta m^{(1)} = (\Sigma^- - \Sigma^+)/\sqrt{2} \approx 5.6 \text{ MeV},$$

Fig. 2

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and

$$\Delta m^{(2)} = (\Sigma^+ + \Sigma^- - 2\Sigma^0)/\sqrt{6} \approx 0.7 \text{ MeV}.$$

Precisely the reverse of our naive prediction!

Clearly, then, there is something in the structure of the strong interactions that tends to enhance the isospin one part of the mass splitting (and perhaps, also, to suppress the isospin two part). The strong interactions are the sum of the very strong interactions and the medium-strong interactions. In the last section we derived sum rules for electromagnetic mass splittings by neglecting the effects of the medium-strong interactions, and obtained good agreement with experiment. So, we would be surprised if the medium-strong interactions had an important effect on the mass splittings. Thus, whatever it is that enhances the isospin-one mass splitting would be present even if we turned off the medium-strong interactions.

However, the very strong interactions do not know isotopic spin; they only know SU(3). They do not have the option of enhancing $I=1$ perturbations. They have only the option of enhancing the octet or the 27-plet part of the perturbation. If they enhanced the 27-plet, we would see enhancement of both isospins, for the 27-plet contains both neutral isotriplets and neutral isoquintuplets. The octet, however, contains only isotriplets.

Thus, we are forced to conclude that there is something in the structure of the very strong interactions that causes them to enhance the octet part of the electromagnetic mass splittings. This is called octet enhancement.

Notice that octet enhancement is a property of the very strong interactions, not of electromagnetism. The observed octet pattern of electromagnetic mass splittings has nothing at all to do with the transformation properties of the electromagnetic interaction. Once we have come to this discovery, it is very tempting to speculate that the same thing is true for the medium-strong interactions. Perhaps the observed octet pattern of medium-strong mass splittings is due, not to the transformation properties of these interactions, but to an octet enhancement mechanism. In fact, perhaps it is the same mechanism for both interactions – perhaps there is a mechanism for *universal octet enhancement.*

Is this a plausible idea or just a wild speculation? Well, one way of checking its plausibility is to construct dynamical models that embody universal octet enhancement. I will discuss three such models here.

(i) First, there is the tadpole model, proposed by Glashow and myself. This model assumes the existence of an octet of scalar mesons. (These need not be stable particles, of course; they would be resonances or anti-bound states. The last is a more likely situation for *s*-wave channels.) These are labeled with the names of the corresponding pseudoscalar mesons, with a prime. Thus, K′, $\bar{\mathrm{K}}'$, π', η'. In a field theory involving scalar mesons there arises the possibility of a class of Feynman diagrams that do not otherwise occur. These are diagrams with only one external line. Fig. 3 shows such a diagram. For obvious reasons, these are called tadpole diagrams. In the absence of the symmetry-breaking interactions these diagrams vanish, for they represent a transition between a unitary octet (the scalar mesons) and a unitary singlet (the vacuum). However, if we turn on the medium-strong interactions we can make an η' tadpole, and if we turn on electromagnetism we can make a $\pi^{0\prime}$ tadpole. These diagrams, by themselves, are of very little interest, but they are important as internal parts of other diagrams. Fig. 4 shows two tadpoles occurring as internal parts of baryon self-energy diagrams. The tadpole on the left makes a contribution to the medium-strong self-mass of the Λ; the one on the right makes a contribution to the electromagnetic self-energy of the proton. Our fundamental (and completely unjustified) assumption is that these diagrams dominate symmetry breaking. Because the scalar mesons form an octet, we get universal octet dominance. If they formed a 27-plet, we would get 27-plet dominance.

(ii) Next, there is the mixing model, proposed by Sakurai, Radicati, Zanello and Picasso. These authors posit that the medium-strong mass splittings are dominated by diagrams that involve the mixing of a singlet vector meson and an octet vector meson. The electromagnetic splittings are, correspondingly, dominated by ρ^0–ω_1 mixing. These mixing matrix elements transform like the components of an octet; thus we obtain universal octet dominance.

Fig. 3

Fig. 4

(iii) Finally, there is the bootstrap model, introduced by Cutkosky and Tarjanne, and investigated in more detail by Dashen and Frautschi. Let us suppose we have a system of particles, interacting through some interaction with a high degree of symmetry, under which the particles form degenerate bound states. Let us introduce a new interaction which breaks the symmetry. Then the energies of the bound states will be changed by two mechanisms: the masses of the components will be changed, and the forces will also be changed. In lowest-order perturbation theory this can be written as

$$\delta m_B = A\,\delta m_c + f,$$

where m_B is a vector composed of the masses of the bound states, m_c is a vector composed of the masses of the components, f is a vector composed of the contribution of the change in the forces, and A is a matrix. Now let us suppose the problem is, in fact, a bootstrap. (The example studied by Dashen and Frautschi is the static baryon-octet resonance-decuplet bootstrap.) Then m_B and m_c are identical, and the equation becomes

$$(1-A)\,\delta m = f.$$

If A has an eigenvalue close to one, then there will be enhancement, independent of f. If this eigenvalue corresponds to a set of eigenvectors that transform like an octet, then it will be octet enhancement. Dashen and Frautschi have found, in the static bootstrap, that there *is* an octet of eigenvectors with eigenvalue close to one. This is a strong point in favour of the bootstrap model. In the other theories, octet enhancement is fed in at the beginning; in the bootstrap model it is calculated. The calculation could lead to 27-plet enhancement; it does not.

If universal octet enhancement was the only consequence of these models, they would not really be worth much; there would be hardly more conclusions than assumptions. However, all the models I have discussed possess a second feature which is most easily demonstrated for the tadpole model. Let us return to Fig. 4. The diagram on the left gives the principal part of the medium-strong mass splittings of the hadrons in terms of the strength of their coupling to η' and the magnitude of the η' tadpole. Conversely, we can deduce from the observed splittings the η' hadron–hadron coupling constants, to within an unknown multiplicative factor. Since η' and $\pi^{0\prime}$ are part of the same octet, we can then calculate the $\pi^{0\prime}$ hadron–hadron coupling constants, just by doing a unitary rotation. But if we know these, we know the values of the diagrams on the right, to within an unknown multiplicative constant.

Thus, we can calculate the principal part of the electromagnetic mass

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splittings in terms of the principal part of the medium-strong mass splittings, and one free parameter. (I emphasize, *not* one free parameter for every unitary multiplet, but one free parameter for all the hadrons.) This is a new phenomenon, not a consequence of universal octet enhancement. For example, if we had two octets of scalar mesons, we would have two sets of unknown coupling constants, which would contribute in different proportions to the medium-strong and electromagnetic splittings. Thus, we would not be able to deduce the electromagnetic splittings from the medium-strong ones.

We call this second desirable property of a dynamical model *non-degeneracy*. Non-degeneracy occurs in mixing models as long as there is a unique mixing mechanism. Thus, as long as we have either dominant vector mixing or dominant pseudoscalar mixing, for example, we can play the same game as above. However, if both play an important role, we are out of luck. In the bootstrap approach the symmetry-violating solutions are found by solving an eigenvalue equation. If one of the eigenvalues is close to one, near spontaneous symmetry breakdown occurs. If *only* one is close to one, we have non-degeneracy. (This is the origin of the term.)

Let us try to get a rough estimate of the accuracy of the algorithm sketched out above. We know that for the baryons the medium-strong mass splitting is mainly F type, i.e., proportional to the hypercharge. Non-degeneracy tells us that the electromagnetic splitting must also be mainly F type, i.e., proportional to the electric charge. We know

$$\mathrm{n} > \mathrm{p},$$

so we expect

$$\Sigma^- > \Sigma^0 > \Sigma^+,$$

and

$$\Xi^- > \Xi^0,$$

both of which are in agreement with experiment. If we do the detailed calculation, we find that the general features of this estimate are preserved; we get the right signs, but the magnitudes are not too reliable. This is encouraging; it makes us believe that we indeed have a reliable way of estimating the principal part of the electromagnetic splittings. However, in order to go further to get a real comparison between theory and experiment, we need a theory of the terms beyond the principal term. We need *a theory of the next corrections.*

This is the third desirable feature of a dynamical model. Such a theory has been constructed for the tadpole model. It has not yet been constructed for the other models. This is not a matter of principle: it is certainly possible; it just has not been done yet. Let me explain how things work with tadpoles.

It is well known that the electromagnetic mass of any particle can be written in terms of the forward scattering of unphysical photons off that particle, summed over all polarizations and integrated over all photon four-momenta. Any approximation to this unphysical amplitude generates an approximation to the self-mass. One reasonable approximation to the amplitude is to include all poles in all three Mandelstam variables. Fig. 5 shows all the Feynman diagrams that have poles. All the particles are on the mass-shell except the photons. The first diagram shows the pole in s, the second the pole in u. When we sum and integrate these terms, we just obtain the conventional expression for the self-mass in terms of electromagnetic form factors, first suggested by Feynman and Speisman, and first derived from dispersion relations by Cini, Ferrari, and Gatto. The third diagram shows the pole in t. There are many such diagrams; however, the only ones which will survive the summation and integration to make a contribution to the self-mass are those in which the particle exchanged is a scalar meson. These yield the tadpoles.

Thus, at least in the tadpole model, the prescription for finding the next corrections is simple. We just calculate the conventional contributions to the self-mass, which are given unambiguously in terms of experimentally measurable form factors, and add them to the tadpole terms. Of course, to actually perform the calculation we need some estimates for the form factors. The nucleon form factors are known from experiment; from these we can find those for the hyperons, using unitary symmetry. For the pion, there are reasons to believe that the ρ pole dominates the form factor; the kaon form factor is then determined by unitary symmetry.

Fig. 5

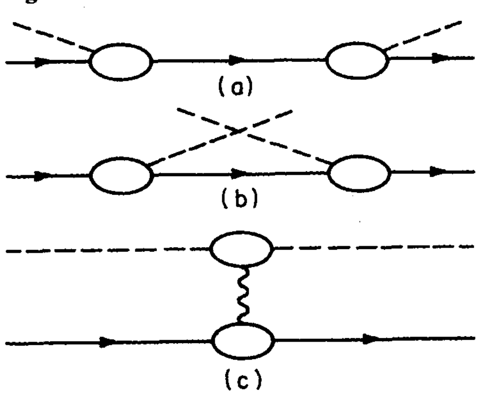

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Table 3 shows the results of such a calculation, done by Schnitzer, Socolow and myself.

As you can see, except for the kaon mass splitting, which is badly off, the agreement with experiment is excellent.

Table 3 *Contributions to electromagnetic mass splittings*

	Non-tadpole	Tadpole	Total	Experiment
n–p	−1.1	2.4	1.3	1.3
Σ^0–Σ^+	−0.7	3.6	2.9	3.1 ± 0.1
Σ^-–Σ^0	1.4	3.6	5.0	4.8 ± 0.1
Ξ^-–Ξ^0	1.2	4.8	6.0	6.2 ± 0.7
$\pi^+ - \pi^0$	4.9	0.0	4.9	4.6
K^+–K^0	2.8	−4.2	−1.4	-4.0 ± 0.2

Finally, I would like to make a few remarks about non-leptonic weak interactions. The most natural interaction for these processes is $J_\mu J_\mu^+$, where J_μ is the Cabibbo current. However, this interaction does not guarantee the most striking experimental regularity of non-leptonic decays, the $\Delta I = \frac{1}{2}$ rule. (As you probably know, to get the $\Delta I = \frac{1}{2}$ rule, one needs at least two pairs of conjugate currents.) This smells very much like another case of octet dominance, and it is very natural to ask whether the mechanisms we have been discussing can be extended to non-leptonic decays. At first glance this seems very easy to do; in the tadpole model, for example, dominance of diagrams involving $K^{0\prime}$ tadpoles would give the $\Delta I = \frac{1}{2}$ rule for parity-conserving decays. To explain parity-violating decays, we would need to hypothesize an octet of pseudoscalar mesons to make pseudoscalar tadpoles, that is to say, we would need to hypothesize them if they were not already there.

This is a very nice suggestion; it is a great pity it is completely wrong. It breaks down for separate reasons in the parity-violating and parity-conserving cases.

First, the parity-violating case. The Cabibbo current transforms like $\cos\theta\,\pi^+ + \sin\theta\,K^+$. It is easy to see that there is an SU(3) transformation that turns this into pure π^+. Furthermore, this transformation commutes with charge and with charge conjugation. In the transformed frame, the only pseudoscalar tadpole the interaction Hamiltonian can make is that associated with π^0. However, the interaction is *CP* even, and π^0 is *CP* odd. Therefore, there is no tadpole.

Second, the parity-conserving case. Let us write the vacuum expectation values of the eight scalar fields as a 3×3 Hermitian matrix. We can find an SU(3) transformation that diagonalizes this matrix. Let us change our definition of hypercharge to mean hypercharge rotations in this new coordinate system. Then, there are no hypercharge-changing tadpoles. That is to say, the only effect of the $K^{0\prime}$ tadpoles is to mix states in such a way as to change the natural definition of strangeness. They do not cause decays.

I have given these arguments in terms of the tadpole model. Corresponding optimistic assumptions can be framed and destroyed within the bootstrap model. I leave this task as an exercise for you.

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Part 1

The arguments referred to in Part 1 can be found in my thesis (Cal. Inst. of Tech., 1962). They can also be constructed, with a little labour, from the results of D. R. Speiser and J. Tarski, *J. Math. Phys.* **4**, 588 (1963).

Part 2

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Part 3

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Part 4

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