# 8

# 1/N

(1979)

## 1 Introduction

More variables usually means greater complexity, but not always. There exist families of field theories with symmetry group SO($N$) (or SU($N$)) that become simpler as $N$ becomes larger. More precisely, the solutions to these theories possess an expansion in powers of $1/N$. This expansion is the subject of these lectures.

There are two reasons to study the $1/N$ expansion.

(1) It can be used to analyze model field theories. This is important. Most of us have a good intuition for the phenomena of classical mechanics. We were not born with this intuition; we developed it toiling over problems involving rigid spheres that roll without slipping and similar extreme but instructive simplifications of reality. One reason we have such a poor intuition for the phenomena of quantum field theory is that there are so few simple examples; essentially all we have to play with is perturbation theory and a handful of soluble models. The $1/N$ expansion enables us to enlarge this set.

In Section 2 I develop the $1/N$ expansion for $\phi^4$ theory and apply it to two-dimensional models with similar combinatoric structures, the Gross–Neveu model and the $\mathbb{C}P^{N-1}$ model. These models display (in the leading $1/N$ approximation) such interesting phenomena as asymptotic freedom, dynamical symmetry breaking, dimensional transmutation, and nonperturbative confinement; they are worth studying.

(2) It is possible that the $1/N$ expansion, with $N$ the number of colors, might fruitfully be applied to quantum chromodynamics. In the real world, $N$ is 3, so an expansion in powers of $1/N$ may not seem like such a good idea. This objection is without force, as is shown by the following wisecrack by Ed Witten:

$$\frac{e^2}{4\pi}=\frac{1}{137}\Leftrightarrow e=0.30.$$

Of course, this does not show that the $1/N$ expansion in QCD will necessarily be as good an approximation as perturbation theory in QED, but it does show that there is no reason to reject it *a priori*.

Unfortunately, it is not possible to make a decisive test of the approximation, because no one knows how to compute even the first term in the expansion in closed form. However, it is possible to argue that this first term, whatever its detailed form, has many properties that are also shared by the real world, and which are otherwise underived from field theory. These include the saturation of scattering amplitudes by an infinite number of narrow resonances, the essential feature of dual-resonance models. I discuss these matters in Section 3.

Although united here, these two classes of applications have very different standings. The work on model field theories is modest but solid, a permanent part of our knowledge. In contrast, the work on chromodynamics is ambitious but conjectural. It is possible that it will lead to great breakthroughs; it is possible that it will fizzle out, like so many hopeful programs before it.

I should warn you that these lectures are introductory rather than encyclopedic. Much more could be said about every topic I discuss. I have not gone into more depth in part because of lack of time and in part because of lack of competence. I am not an expert in these matters; one reason I decided to lecture on them this summer was to force myself to learn them.

Much of what I do know I have learned from conversations with Roman Jackiw, Hugh Osborn, Howard Schnitzer, Gerard 't Hooft, Ken Wilson, and Edward Witten. My debt to Witten is enormous; most of the second half of Section 3 is plagiarism of his ideas.

(Note to the reader: If you are only interested in chromodynamic applications, I suggest you read just the first five paragraphs of Section 2, and then proceed directly to Section 3.)

## 2 Vector representations, or, soluble models

### 2.1 $\phi^4$ *theory (half-way)*[1]

I will begin the development of the $1/N$ expansion with a theory that is (I hope) familiar to you, the $O(N)$ version of $\phi^4$ theory. I warn you in advance that I will stop the discussion half-way, after I have worked out all the combinatorics but before I have evaluated any Feynman integrals. This is not because the details of the model are not interesting and instructive (they are, especially in the Nambu–Goldstone mode), but because I am using this theory only as a warm-up, and want to get on to the even more interesting and instructive Gross–Neveu and $\mathbb{C}P^{N-1}$ models.

The dynamical variables of the theory are a set of $N$ scalar fields, $\phi^a$, $a=1 \ldots N$, with dynamics defined by the Lagrange density,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \tfrac{1}{2}\mu_0^2 \phi^a \phi^a - \tfrac{1}{8}\lambda_0 (\phi^a \phi^a)^2, \tag{2.1}$$

where the sum on repeated indices is implied. Since I am going to stop the investigation before evaluating any Feynman integrals, I might as well keep the dimension of space-time arbitrary (but less than or equal to four).

To get an idea of what is going on, I have written down in Fig. 1 the first few diagrams (in ordinary perturbation theory) for the scattering of two mesons of type $a$ into two mesons of type $b$ ($a \neq b$). The first diagram displayed, the Born term, is $O(\lambda_0)$. The second diagram is $O(\lambda_0^2 N)$, because there are $N$ possible choices for the internal index, $c$. The third diagram, in contrast, is only $O(\lambda_0^2)$; the internal indices are fixed and there is no sum to do.

The explicit factor of $N$ in the second diagram makes the large-$N$ limit seem nonsensical, but this is easily rectified. All we need do is define

$$g_0 \equiv \lambda_0 N, \tag{2.2}$$

and declare that we wish to study the limit of large-$N$ with fixed $g_0$ (not fixed $\lambda_0$). The first diagram is now $O(g_0/N)$; as we shall see, this is the leading non-trivial order in $1/N$. The second diagram is $O(g_0^2/N)$, the same order in $1/N$. The third diagram is $O(g_0^2/N^2)$, next order in $1/N$ and negligible compared to the two preceding diagrams in the large-$N$ limit.

This is the first step in constructing the $1/N$ expansion. We must decide what parameters to hold fixed as $N$ becomes large. If we make the wrong choice, we can obtain either a trivial theory (only the Born term survives) or one without a $1/N$ expansion (there are graphs proportional to positive powers of $N$). Of course, we have not yet shown that the second possibility does not occur in the theory at hand. However, there are clearly an infinite

**Fig. 1**

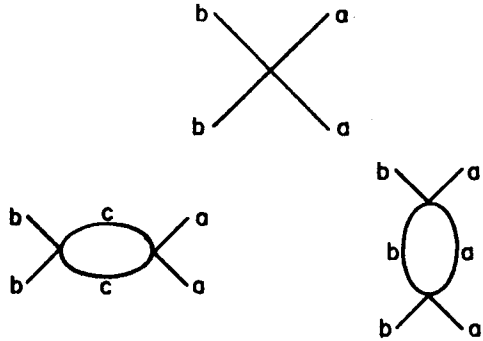


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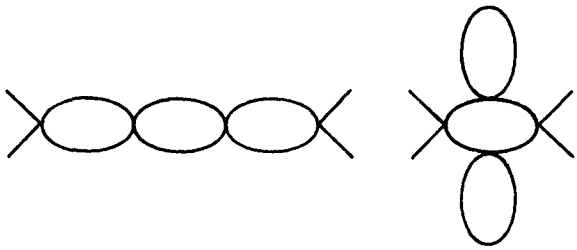
**Fig. 2**



number of graphs proportional to $1/N$, times various powers of $g_0$; two of them are shown in Fig. 2. (To keep the graphs from being hopeless jumbles, I have left out the index labels; I hope you can figure out where they go.)

To keep all these diagrams straight, and to show that there are no diagrams proportional to positive powers of $N$, is a combinatoric challenge. We can simplify life considerably by introducing an auxiliary field, $\sigma$, and altering the Lagrange density:

$$\mathscr{L} \to \mathscr{L} + \frac{1}{2}\frac{N}{g_0}\left(\sigma - \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a\right)^2. \tag{2.3}$$

This added term has no effects on the dynamics of the theory. This is easy to see from the viewpoint of functional integration. The functional integral over $\sigma$ is a trivial Gaussian integral; its only effect is to multiply the generating functional of the theory by an irrelevant constant. It is also easy to see from the viewpoint of canonical quantization. The Euler–Lagrange equation for $\sigma$ is

$$\sigma = \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a. \tag{2.4}$$

This involves no time derivatives; it is not a true equation of motion, but an equation of constraint, like the Euler–Lagrange equation for the fourth component of a massive vector field. When we construct the Hamiltonian, $\sigma$ must be eliminated from the Lagrangian, using Eq. (2.4); this cancels the added term.

However, although the dynamics defined by our new Lagrangian are the same as those defined by the old one, the Feynman rules are different. By elementary algebra

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_0^2\phi^a\phi^a + \frac{1}{2}\frac{N}{g_0}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a. \tag{2.5}$$

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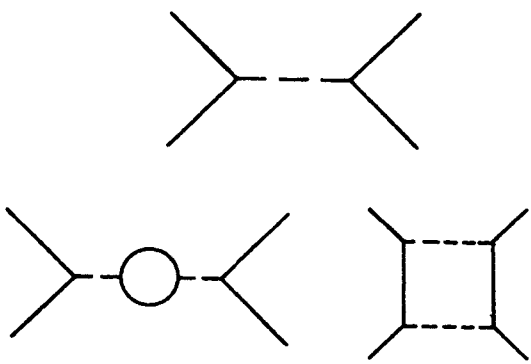
Thus, in the new formalism, the only non-trivial interaction is the $\phi\phi\sigma$ coupling. All factors of $1/N$ come from the $\sigma$ propagator ($ig_0/N$). Every line on a closed $\phi$ loop must always carry the same index, and this index must always be summed over; thus, we need not write explicit indices on $\phi$ loops, and every closed $\phi$ loop always gives a factor of $N$.

Fig. 3 shows the graphs of Fig. 1 in our new formalism. (The dashed line is the $\sigma$ propagator.) Counting powers of $1/N$ is now much easier than before, but things can be made easier yet. Let us imagine analyzing a general Feynman graph as follows: First, let us strip away all the $\phi$ lines that end on external lines, that is to say, that are not part of closed loops. This yields a graph that has only external $\sigma$ lines. Second, let us do all the momentum integrals over the closed $\phi$ loops. Every $\phi$ loop thus becomes a (non-local) interaction between the $\sigma$ fields that terminate on that loop. We thus generate a graph with only $\sigma$ lines; it can be thought of as a graph in an effective field theory whose Feynman rules are derived from an effective action, $S_{\text{eff}}(\sigma)$.

There are two ways of describing $S_{\text{eff}}$, in terms of Feynman graphs or in terms of functional integrals. The description in terms of graphs is shown in Fig. 4. The first graph gives the term linear in $\sigma$; the second graph gives a term quadratic in $\sigma$, which must be added to the third graph, the quadratic term already present in Eq. (2.5); the fourth graph gives the cubic term; etc. In terms of functional integrals, the quantum theory is defined by integrating the exponential of i$S$, the classical action, over all configurations of all fields in the theory. The effective action is obtained by integrating over the $\phi$s only:

$$e^{iS_{\text{eff}}(\sigma)} = \int \prod_a [d\phi^a] e^{iS(\phi^a,\sigma)}. \tag{2.6}$$

**Fig. 3**



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$S$ is quadratic in the $\phi$s, so the integral is a Gaussian one, and can be done in closed form. Of course, 'closed form' is a hoax; the answer is a functional determinant that, for general $\sigma$, can be evaluated only by doing the Feynman graphs of Fig. 4.

Whichever way we describe $S_{\text{eff}}$, one thing about it is obvious; every term in it is proportional to $N$:

$$S_{\text{eff}}(\sigma, N) = N S_{\text{eff}}(\sigma, 1). \tag{2.7}$$

This makes counting powers of $N$ very easy. Consider a graph in our effective field theory with $E$ external lines, $I$ internal lines, $V$ vertices, and $L$ independent loop integrations. These quantities are not independent. For a connected graph,

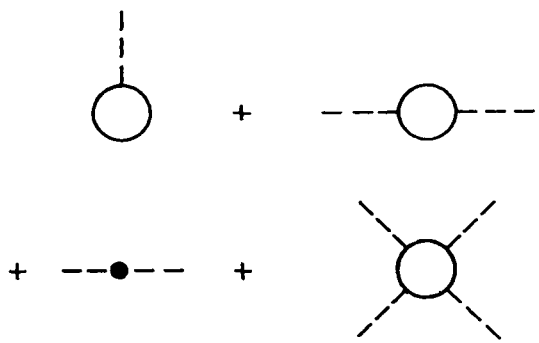
$$L = I - V + 1. \tag{2.8}$$

That is to say, we have one integration momentum for each internal line, but we also have one delta-function for each vertex; each delta-function cancels one momentum, except for one delta-function that is left over for overall momentum conservation. The power of $N$ associated with a graph can be expressed in terms of these quantities. The propagator is obtained by inverting the quadratic part of the Lagrangian; thus each external and internal line carries a factor of $1/N$. (We put propagators on the external $\sigma$ lines because we want eventually to attach them to external $\phi$ lines.) On the other hand, each vertex carries a factor of $N$. Thus the net power of $N$ associated with a given graph is

$$N^{V-I-E} = N^{-E-L+1} \tag{2.9}$$

by Eq. (2.8). Thus, the smallest power of $1/N$ is obtained from graphs with no loops (tree graphs) and with the minimum number of external lines required to connect the external $\phi$ lines. In the case we began by studying,

**Fig. 4**



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meson–meson scattering, two external $\sigma$ lines are required, and thus the leading power is $1/N$.

This is no great surprise, of course; we hardly needed all this formalism to get this piddling result. However, we are almost in a position to compute meson–meson scattering to $O(1/N)$ in closed form. I say 'almost' because $S_{\text{eff}}$ has the awkward feature of containing a term linear in $\phi$. In the presence of such a term, there are an infinite number of tree graphs with two external lines; all one has to do is build a tree graph of arbitrary complexity, and then terminate all but two of its external lines on linear vertices. The cure for this problem is well known. We define a new, shifted field,

$$\sigma' \equiv \sigma - \sigma_0, \tag{2.10}$$

where $\sigma_0$ is a constant chosen such that $\sigma = \sigma_0$ is a stationary point of $S_{\text{eff}}$,

$$\left.\frac{\delta S_{\text{eff}}}{\delta \sigma'}\right|_{\sigma'=0} = 0. \tag{2.11}$$

(I will shortly show that $\sigma_0$ exists.) In terms of $\sigma'$, there are no linear vertices.

The easiest way to construct $S_{\text{eff}}(\sigma')$ is to express $\mathscr{L}$ in terms of $\sigma'$. From Eq. (2.5),

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\mu_1^2 \phi^a \phi^a + \frac{1}{2}\frac{N}{g_0}\sigma'^2 - \frac{1}{2}\sigma' \phi^a \phi^a + \frac{N}{g_0}\sigma_0 \sigma' \tag{2.12}$$

plus an irrelevant constant, where

$$\mu_1^2 \equiv \mu_0^2 + \sigma_0. \tag{2.13}$$

$\mu_1$ is the $\phi$ mass, to leading (zeroth) order in $1/N$; it will be convenient to use it as an independent parameter of the theory instead of $\mu_0$. The graphical construction of $S_{\text{eff}}(\sigma')$ is now the same as that shown in Fig. 4, with two exceptions. (1) The internal $\phi$ lines now carry a mass $\mu_1$ rather than $\mu_0$. (2) There is an additional linear vertex, coming from the last term in Eq. (2.12). We use this to cancel the linear vertex from the first graph in Fig. 4, thus at one stroke fixing $\sigma_0$ and eliminating it from all future computations.

We are now in a position to compute whatever we want. For example, let me sketch out the computation of $\phi$–$\phi$ scattering, to leading order. There are only three graphs that can contribute, shown in Fig. 5. I have put a shaded blob on the $\sigma'$ propagator to remind you that it is not just $ig_0/N$, but the full propagator obtained from inverting the quadratic

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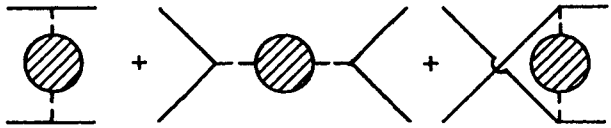
**Fig. 5**



term in $S_{\text{eff}}$, the sum of the second and third graphs in Fig. 4. In momentum space,

$$D^{-1}(p) = -N\left[ig_0^{-1} + \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-\mu_1^2+i\varepsilon)} \frac{1}{([p+k]^2-\mu_1^2+i\varepsilon)}\right]. \tag{2.14}$$

where $d$ is the number of space-time dimensions. Note that for $d=4$, the integral is logarithmically divergent, but that its divergence can be absorbed in the bare coupling constant, $g_0$.

Of course, this is just the beginning. We could evaluate this integral, study the properties of the scattering amplitude it defines, investigate the interesting case in which we choose $\mu_1^2$ to be negative (spontaneous symmetry breakdown?), worry about higher-order corrections, etc. However, as I warned you at the beginning, I am going to stop the discussion of $\phi^4$ theory half-way, and go on to investigate other models. If you want to find out more about this theory, you will have to go to the literature[1] (or work it out yourself – it's not that hard).

## 2.2 *The Gross–Neveu model*[2]

The Gross–Neveu model is a renormalizable field theory that admits a $1/N$ expansion and displays asymptotic freedom, dynamical symmetry breakdown, and dimensional transmutation. This is the good news; the bad news is that it is a field theory in two space-time dimensions.

The fundamental dynamical variables of the theory are a set of Dirac fields, $\psi^a$, $a=1 \ldots N$. In two dimensions, Dirac fields have only two components and the Dirac matrices are $2\times 2$ matrices. In standard representation,

$$\gamma^0=\sigma_z,\ \gamma^1=i\sigma_y,\ \gamma_5=\gamma^0\gamma^1=\sigma_x, \tag{2.15}$$

where the $\sigma$s are the Pauli spin matrices. In all other ways, conventions are the same as in four dimensions.

The model is defined by

$$\mathcal{L}=\bar{\psi}^a i\partial_\mu\gamma^\mu\psi^a+\frac{g_0}{N}(\bar{\psi}^a\psi^a)^2. \tag{2.16}$$

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We see from the first term that $\psi^a$ has dimensions of (length)$^{-1/2}$, so $g_0$ is dimensionless and the interaction should be renormalizable. A mass term is excluded by the discrete chiral symmetry,

$$\psi^a \to \gamma_5 \psi^a, \ \bar{\psi}^a \to -\bar{\psi}^a \gamma_5. \tag{2.17}$$

As we shall see, it is this symmetry (not the continuous U($N$) symmetry) which suffers spontaneous breakdown.

The construction of the $1/N$ approximation is a duplicate of that for $\phi^4$ theory. First we add a term to $\mathscr{L}$ that has no effect on the physics, involving an auxiliary field, $\sigma$,

$$\begin{aligned} \mathscr{L} &\to \mathscr{L} - \frac{N}{2g_0}\left(\sigma - \frac{g_0}{N}\bar{\psi}^a\psi^a\right)^2 \\ &= \bar{\psi}^a i\partial_\mu \gamma^\mu \psi^a - \frac{N}{2g_0}\sigma^2 + \sigma\bar{\psi}^a\psi^a. \end{aligned} \tag{2.18}$$

Next, we integrate over $\psi$ loops to obtain $S_{\text{eff}}(\sigma)$. This is shown diagrammatically in Fig. 6. We only have even powers of $\sigma$ because the trace of an odd number of Dirac matrices vanishes (alternatively, because $\sigma$ changes sign under the discrete symmetry (2.17)).

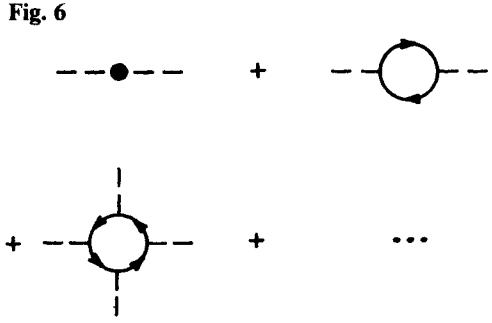
At first glance, it would seem that there is no need for the third step in the analysis, shifting the $\sigma$ field; $S_{\text{eff}}$ is even in $\sigma$, and thus $\sigma = 0$ is automatically a stationary point. But this is begging the question; if we are interested in spontaneous breakdown of the discrete symmetry, the issue is precisely whether there are stationary points other than $\sigma = 0$.

Fortunately, to settle this issue we do not need to compute $S_{\text{eff}}$ for general $\sigma$, merely for constant $\sigma$. In this case, we can put the universe in a box of spatial extent $L$ and temporal extent $T$, and define

$$-V(\sigma) = \lim_{L,T\to\infty} S_{\text{eff}}(\sigma)/LT. \tag{2.19}$$

Each stationary point of $V$ is a possible starting point for a $1/N$ expansion, defines a possible vacuum state of the theory to leading order in $1/N$.

**Fig. 6**



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The energy densities of these vacua are easily computed. If we denote a vacuum energy density by $\mathscr{E}$, and if we denoted by $\Sigma$ the sum of all connected vacuum-to-vacuum Feynman graphs, then a general formula of time-dependent perturbation theory states that

$$-i\mathscr{E} = \lim_{L,T\to\infty} \Sigma/LT. \tag{2.20}$$

To lowest (minus first) order in $1/N$, $\Sigma$ is given by the sum of all connected tree graphs with no external lines. This set consists of precisely one graph, with only one vertex, the term in $S_{\text{eff}}$ that contains no powers of the shifted field. That is to say, $V$ at the stationary point *is* the vacuum energy density (to leading order); if there are several stationary points, only those of minimum $V$ are true vacua.[3]

Diagrammatically, $V$ is given by the sum of Feynman diagrams in Fig. 6, with all external lines carrying zero two-momentum, and with the momentum-conserving delta-functions left off. (These give the factor of $LT$.) The summation of these graphs has been done countless times in the literature. I will bore you by doing it once more:

$$-iV = -i\frac{N}{2g_0}\sigma^2 - \sum_{n=1}^{\infty}\frac{N}{2n}\operatorname{Tr}\int\frac{d^2p}{(2\pi)^2}\left(\frac{-\not{p}\sigma}{p^2+i\varepsilon}\right)^{2n}. \tag{2.21}$$

The terms in this series have the following origins. (a) From Eq. (2.18), every vertex carries a factor of i and every propagator a factor of $i\not{p}/(p^2+i\varepsilon)$. (b) There is an $N$ for the $N$ Fermi fields and a $(-1)$ for the Fermi loop. (c) Cyclic permutation of the external lines recreates the same graph; thus the $1/2n!$ in Dyson's formula is incompletely cancelled and we have a left-over factor of $1/2n$. It is trivial to do the trace, sum the series, and rotate the integration to Euclidean two-momentum, $p_{\text{E}}$. We thus obtain

$$V = N\left[\frac{\sigma^2}{2g_0} - \int\frac{d^2p_{\text{E}}}{(2\pi)^2}\ln\left(1+\frac{\sigma^2}{p_{\text{E}}^2}\right)\right]. \tag{2.22}$$

The momentum integral is ultraviolet divergent; we cut it off by restricting the integral to $p_{\text{E}}^2 \leqslant \Lambda^2$, with $\Lambda$ some large number. We find

$$V = N\left[\frac{\sigma^2}{2g_0} + \frac{1}{4\pi}\sigma^2\left(\ln\frac{\sigma^2}{\Lambda^2} - 1\right)\right] \tag{2.23}$$

We wish to rewrite this in terms of a (conveniently defined) renormalized coupling constant, $g$. I will pick an arbitrary renormalization mass, $M$, and define $g$ by

$$\frac{1}{g} \equiv N^{-1}\left.\frac{d^2V}{d\sigma^2}\right|_M = \frac{1}{g_0} + \frac{1}{2\pi}\ln\frac{M^2}{\Lambda^2} + \frac{1}{\pi}. \tag{2.24}$$

Note that $g$ is $g_0 + O(g_0^2)$, as a good renormalized coupling constant should be. If you do not like my choice of $M$ and want to use another, $M'$, you are

free to do so. Your coupling constant is connected to mine by

$$\frac{1}{g'}=\frac{1}{g}+\frac{1}{2\pi}\ln\frac{M'^2}{M^2}. \tag{2.25}$$

$V$ is now given by

$$V=N\left[\frac{\sigma^2}{2g}+\frac{1}{4\pi}\sigma^2\left(\ln\frac{\sigma^2}{M^2}-3\right)\right]. \tag{2.26}$$

Two of the announced properties of the model are now manifest. Firstly, it is renormalizable, at least in the order to which we are working; Eq. (2.26) is totally free of cutoff-dependence. Secondly, the theory is asymptotically free. This can be seen in the usual two equivalent ways. (1) In Eq. (2.24), if we hold $g$ and $M$ fixed, and let $\Lambda$ go to infinity, $g_0$ goes to zero. In a fixed theory, the bare coupling constant vanishes for infinite cutoff. (2) In Eq. (2.25), if we hold $g$ and $M$ fixed, and let $M'$ go to infinity, $g'$ goes to zero. In a fixed theory, the renormalized coupling constant vanishes for infinite renormalization mass.

We can now search for spontaneous symmetry breakdown.

$$\frac{dV}{d\sigma}=N\left[\frac{\sigma}{g}+\frac{\sigma}{2\pi}\left(\ln\frac{\sigma^2}{M^2}-2\right)\right]. \tag{2.27}$$

This vanishes at

$$\sigma^2=\sigma_0^2\equiv M^2\exp\left(2-\frac{2\pi}{g}\right). \tag{2.28}$$

At this point

$$V=-N\sigma_0^2/4\pi. \tag{2.29}$$

This is negative, that is to say, less than $V(0)$. The third announced property of the model is now manifest. The discrete chiral symmetry suffers spontaneous breakdown, and the massless fermions acquire a mass. To leading order, this mass is just $\sigma_0$.

I stated earlier that one could choose the renormalization mass, $M$, arbitrarily; a change in $M$ could always be compensated for by an appropriate change in $g$. I will now use this freedom to choose $M$ to be $\sigma_0$. By Eq. (2.28) this fixes $g$:

$$g=\pi. \tag{2.30}$$

This is the fourth announced property, dimensional transmutation. We began with a theory that apparently depended on only one continuous parameter, $g_0$. We have arrived at a theory that depends on only one continuous parameter, $\sigma_0$. The surprise is that we began with a dimensionless parameter, on which we would expect observable quantities to

depend in a complicated way, while we arrived at a dimensionful parameter, on which observable quantities must depend in a trivial way, given by dimensional analysis.

Of course, dimensional transmutation is an inevitable feature of any renormalizable field theory depending only on a single dimensionless coupling constant. Renormalization trades the single bare coupling constant, $g_0$, for the pair $(g, M)$, the renormalized coupling constant and the renormalization point. But this is a redundant pair; we still have only a one-parameter theory; the $(g, M)$ plane is the union of curves such that any two points on the same curve define the same theory. One way of parametrizing these curves is by that value of $M$ at which they pass through some fixed value of $g$, like $1/2$ or $\pi$. Our one-parameter family of theories are now labeled by a single parameter, and it is a mass. Nevertheless, even though we expect dimensional transmutation to occur in very general circumstances, it is still pleasant to have a model in which we can explicitly see it happening.

## 2.3 *The $\mathbb{C}P^{N-1}$ model*[4]

Like the Gross–Neveu model, the $\mathbb{C}P^{N-1}$ model is a two-dimensional renormalizable field theory which displays dimensional transmutation. Also like the Gross–Neveu model, the theory contains a set of particles that are massless in perturbation theory but which acquire a mass in the leading $1/N$ approximation. However, this is not due to spontaneous symmetry breakdown, but to its reverse. The particles are massless because they are the Goldstone bosons of a spontaneously broken symmetry, and they acquire a mass because the symmetry is dynamically restored. This should be no surprise. In two dimensions there can be no spontaneous breakdown of a symmetry associated with a local conserved current; if the $1/N$ approximation had not predicted symmetry restoration, we would have known it was a lie.

More interestingly, the massive particles are confined; there is a linear potential between particle and antiparticle which prevents the components of a pair from being separated indefinitely. Of course, a linear potential is not as difficult to achieve in two space-time dimensions as in four; on a line, the classical electric force between oppositely charged particles is independent of distance. However, in this case the linear potential arises in a theory without any fundamental gauge fields; this is astonishing.

The $\mathbb{C}P^{N-1}$ model is a generalization of the nonlinear sigma model. I will first remind you of this latter theory and then go on to describe the generalization.

The linear sigma model is a theory of $N$ scalar fields, assembled into an

$N$-vector, $\phi$, with dynamics defined by

$$\mathscr{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} - \frac{\lambda}{8}(\boldsymbol{\phi}\cdot\boldsymbol{\phi} - a^2)^2, \tag{2.31}$$

where $\lambda$ and $a$ are positive numbers. This theory is SO($N$)-invariant, but, at least in perturbation theory, the symmetry spontaneously breaks down to SO($N-1$); the ground states of the theory are constant fields lying on the $(N-1)$-dimensional sphere,

$$\boldsymbol{\phi}\cdot\boldsymbol{\phi} = a^2. \tag{2.32}$$

The nonlinear sigma model is the formal limit of this theory as $\lambda$ goes to infinity. The fields in general, not just in their ground state, are restricted to obey Eq. (2.32). The Lagrange density then simplifies to

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi}\partial^\mu\boldsymbol{\phi}. \tag{2.33}$$

Of course, this simple form does not mean that the dynamics is simple. The $N$ components of $\phi$ are not independent, and Eq. (2.33) in fact describes a highly complicated nonlinear theory, as would be manifest were we to write $\phi$ as a function of $N-1$ independent variables (say, angles on the sphere). The nonlinear model can be thought of as a stripped-down version of the linear model, with only the Goldstone bosons retained.

The role of the parameter $a$ can be clarified by rescaling the fields,

$$\boldsymbol{\phi} \to a\boldsymbol{\phi}. \tag{2.34}$$

Under this transformation, the constraint becomes

$$\boldsymbol{\phi}\cdot\boldsymbol{\phi} = 1, \tag{2.35}$$

while

$$\mathscr{L} = \frac{a^2}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}. \tag{2.36}$$

From this we see that $1/a$ is a coupling constant. (See the discussion of powers of $N$ following Eq. (2.7).) This makes sense; if $a$ were infinite, we would not be able to tell the difference between the sphere defined by Eq. (2.32) and ordinary flat space, for which Eq. (2.33) would define a free field theory.

I emphasize that the passage to the nonlinear model is purely a formal limit. For example, the linear model is renormalizable in four dimensions or less, while the nonlinear model is renormalizable only in two dimensions (where scalar fields are dimensionless) or less. We have thrown away some important physics (at least at short distances) by throwing away the non-Goldstone modes. Nevertheless, once we have the nonlinear model, we can certainly study it as a theory in its own right. Indeed, I could have constructed the model directly, as a field theory where the field variables

lie in a nonlinear space; I chose to build it from the linear model only for reasons of pedagogy.

The $\mathbb{C}P^{N-1}$ model can likewise be constructed directly as a field theory in a nonlinear space. However, again for reasons of pedagogy, I will obtain it as the formal limit of a linear theory.

The linear theory is a theory of $N^2-1$ scalar fields, assembled into an $N \times N$ traceless Hermitian matrix, $\phi$, with dynamics defined by

$$\mathscr{L} = \tfrac{1}{2}\mathrm{Tr}\, \partial_\mu \phi\, \partial^\mu \phi - \lambda\, \mathrm{Tr}\, P(\phi), \tag{2.37}$$

where $\lambda$ is a positive number and $P$ is some polynomial in $\phi$. This theory is invariant under SU($N$):

$$\phi \to U\phi U^\dagger,\ U \in \mathrm{SU}(N). \tag{2.38}$$

It is possible to choose $P$ such that the minima of Tr $P$ are matrices with $N-1$ equal eigenvalues and one unequal eigenvalue; SU($N$) then breaks down spontaneously to SU$(N-1)\otimes$U(1). In equations, the ground states of the theory are constant fields of the form

$$\phi = g_0^{-1}[N^{\frac{1}{2}} z z^\dagger - N^{-\frac{1}{2}} I], \tag{2.39}$$

where $z$ is an $N$-dimensional column vector of unit length,

$$z^\dagger z = 1; \tag{2.40}$$

and $g_0$ is some parameter derived from $P(\phi)$. I will assume that $P$ has $N$ dependence such that $g_0$ remains fixed as $N$ goes to infinity. As we shall see shortly, this is necessary to get a $1/N$ expansion.

The $\mathbb{C}P^{N-1}$ model is the formal limit of this theory as $\lambda$ goes to infinity. The fields in general, not just in their ground state, are restricted to obey Eq. (2.39). The Lagrange density then simplifies to

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{2}\mathrm{Tr}\, \partial_\mu \phi\, \partial^\mu \phi \\ &= (N/g_0^2)(\partial_\mu z^\dagger\, \partial^\mu z - j_\mu j^\mu),\end{aligned} \tag{2.41}$$

where

$$j_\mu \equiv (2\mathrm{i})^{-1}[z^\dagger \partial_\mu z - (\partial_\mu z^\dagger) z]. \tag{2.42}$$

It is convenient to rescale $z$,

$$z \to g_0 N^{-\frac{1}{2}} z. \tag{2.43}$$

The Lagrange density then becomes

$$\mathscr{L} = \partial_\mu z^\dagger\, \partial^\mu z - g_0^2 N^{-1} j_\mu j^\mu, \tag{2.44}$$

while the constraint equation is

$$z^\dagger z = N/g_0^2. \tag{2.45}$$

We see that we have (in perturbation theory) a theory of massless particles with short-range interactions between them. The theory is slightly more

complex than the nonlinear sigma model; there are not only interactions induced by the constraint but also explicit interactions in the Lagrange density.

It is interesting to count the number of particles in the theory. At first glance, it looks like we have $N$ complex fields with one real constraint, yielding $2N-1$ real fields. On the other hand, if we count Goldstone bosons, we would expect the number of real fields to be

$$\dim \mathrm{SU}(N) - \dim \mathrm{U}(N-1) = N^2 - 1 - (N-1)^2 = 2N-2. \tag{2.46}$$

The second count is the correct one. The first count ignored the fact that the transformation

$$z \to e^{i\theta} z, \tag{2.47}$$

does nothing to $\phi$; the overall phase of $z$ is not a dynamical variable and should not have been counted. The manifold in which our fields lie is the set of complex $N$-vectors with fixed length, *and* with $N$-vectors differing only by a multiplicative phase factor identified. This is complex projective $N-1$ space, $\mathbb{C}P^{N-1}$.

I will now solve the model to leading order in $1/N$, in two space-time dimensions, where it is renormalizable. The first step is standard, eliminating the quartic interaction by introducing an auxiliary field. Since the quartic interaction is of the form vector times vector, the auxiliary field must be a vector field. Thus, we change $\mathscr{L}$ by

$$\begin{aligned} \mathscr{L} &\to \mathscr{L} + g_0^2 N^{-1} (j_\mu + g_0^{-2} N A_\mu)^2 \\ &= \partial_\mu z^\dagger \partial^\mu z + 2 j^\mu A_\mu + g_0^{-2} N A_\mu A^\mu. \end{aligned} \tag{2.48}$$

Using the constraint, Eq. (2.45), this can be rewritten in the amusing form,

$$\mathscr{L} = (\partial_\mu - iA_\mu) z^\dagger (\partial_\mu + iA_\mu) z. \tag{2.49}$$

This looks like a piece of a gauge field theory, with the gauge transformation of the fields, Eq. (2.47), canceled by the gauge transformation of the vector potential,

$$A_\mu \to A_\mu - \partial_\mu \theta. \tag{2.50}$$

Of course, the 'gauge invariance' is a hoax, just a reflection of the fact that we are describing the theory in terms of highly redundant variables.[5]

The next step is to get the constraint into the Lagrange density. We do this with another auxiliary field, $\sigma$,

$$\mathscr{L} = (\partial_\mu - iA_\mu) z^\dagger (\partial_\mu + iA_\mu) z - \sigma [z^\dagger z - g_0^{-2} N]. \tag{2.51}$$

The new field is a Lagrange multiplier; its Euler–Lagrange equation is the constraint. Equivalently, performing the functional integral over $\sigma$ yields a delta-function at each point of space-time which enforces the constraint.

We now have a Lagrange density that is quadratic in $z$, so we can proceed

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as before to integrate out the internal $z$ loops and obtain an effective action, a functional of $\sigma$ and $A_\mu$. The first few terms in the graphical expansion of $S_{\text{eff}}$ are shown in Fig. 7; the directed lines are $z$s, the dashed lines $\sigma$s, and the wiggly lines $A$s. We see that we have pure $\sigma$ terms (the first two lines), pure $A$ terms (the third line) and mixed terms (the fourth line); however, all the terms are proportional to $N$, just as before.

Also just as before, to eliminate the linear terms, we must shift to a stationary point of $V(\sigma)$. The computation of $V$ is essentially a rerun of that for the Gross–Neveu model. The relevant graphs are those in the first two lines of Fig. 7; the only difference in the computation is that the Fermi minus sign is missing and that we now have $\sigma$ where before we had $\sigma^2$. Thus,

$$V = -N\left[\frac{\sigma}{g_0^2} + \frac{\sigma}{4\pi}\left(\ln\frac{\sigma}{\Lambda^2} - 1\right)\right], \tag{2.52}$$

where $\Lambda$ is the cutoff. To renormalize this, I pick an arbitrary renormalization mass, $M$, and define the renormalized coupling constant, $g$, by

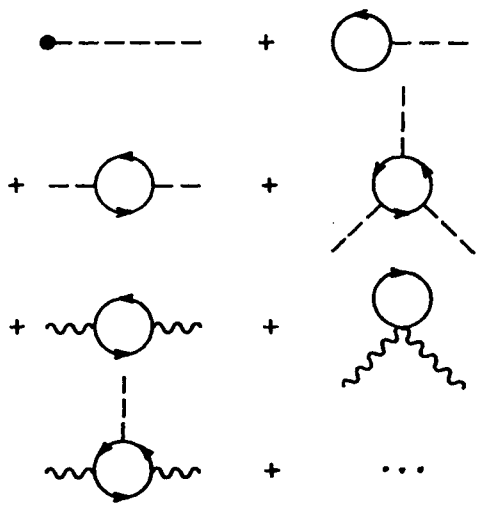
$$\frac{1}{g^2} \equiv -N^{-1}\left.\frac{dV}{d\sigma}\right|_{M^2} = \frac{1}{g_0^2} + \frac{1}{4\pi}\ln\frac{M^2}{\Lambda^2}. \tag{2.53}$$

Note that $g$ is $g_0 + O(g_0^2)$, as it should be. We can now rewrite $V$ as

$$V = -N\left[\frac{\sigma}{g^2} + \frac{\sigma}{4\pi}\left(\ln\frac{\sigma}{M^2} - 1\right)\right]. \tag{2.54}$$

As before, the theory is renormalizable; all reference to the cutoff has

**Fig. 7**



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disappeared. As before, it is asymptotically free; for fixed $g$ and $M$, $g_0$ vanishes as $\Lambda$ goes to infinity.

We can now search for stationary points.

$$\frac{dV}{d\sigma} = -N\left[\frac{1}{g^2} + \frac{1}{4\pi}\left(\ln\frac{\sigma}{M^2}\right)\right]. \tag{2.55}$$

This has a unique zero, at

$$\sigma = \sigma_0 \equiv M^2 \exp(-4\pi/g^2). \tag{2.56}$$

In terms of $\sigma_0$,

$$V = -\frac{N\sigma}{4\pi}\left(\ln\frac{\sigma}{\sigma_0} - 1\right). \tag{2.57}$$

Once again, dimensional transmutation has occurred; the dimensionless coupling constant, $g$, has disappeared from the theory, to be replaced by the dimensionful parameter $\sigma_0$.

From Eq. (2.51), $\sigma_0$ is the squared mass of the $z$-particles, to lowest (zeroth) order in $1/N$. Something remarkable has happened. We started out, in our linear model, with a set of fields transforming according to the adjoint representation of SU($N$). By sending $\lambda$ to infinity, we replaced these by a set of Goldstone bosons which transformed nonlinearly under the action of the group. These have now turned into an ordinary set of massive mesons, transforming linearly, according to the fundamental representation of the group. We are used to making mesons out of quarks; here we have made (bosonic) quarks out of mesons, fundamental representations out of adjoint ones.

Something even more remarkable happens when we study the long-range force between a pair of $z$s. By the same arguments as were given in $\phi^4$ theory, to leading order the force is given by graphs like those shown in Fig. 5. To compute these graphs, we need the $\sigma$–$\sigma$ propagator, the $\sigma$–$A_\mu$ propagator, and the $A_\mu$–$A_\nu$ propagator. By current conservation and Lorentz invariance, the $\sigma$–$A_\mu$ propagator vanishes. At zero momentum transfer, the $\sigma$–$\sigma$ propagator is given by

$$D_{\sigma\sigma}(0) = -i\left[\frac{d^2V}{d\sigma^2}\bigg|_{\sigma_0}\right]^{-1} = \frac{4\pi i}{N}. \tag{2.58}$$

This is not infinite; thus $\sigma$ exchange gives no long-range force.

The $A_\mu$–$A_\nu$ propagator is a different story. The term in $S_{\text{eff}}$ quadratic in $A$ is obtained from the graphs on the third line of Fig. 7. These are just the standard second-order photon self-energy graphs; to compute their sum is a trivial exercise in Feynman-graph technology. The answer, in

momentum space, is

$$-\frac{iN}{4\pi}[g_{\mu\nu}p^2-p^\mu p^\nu]\int_0^1 \frac{dx(1-2x)^2}{\sigma_0^2-p^2x(1-x)-i\varepsilon}. \tag{2.59}$$

If we are only interested in long-range forces, that is to say, in small momenta, we may neglect the $p^2$ in the integrand. We thus obtain

$$-\frac{iN}{12\pi\sigma_0^2}[g_{\mu\nu}p^2-p_\mu p_\nu]. \tag{2.60}$$

This corresponds to a term in the effective action of the form

$$S_{\text{eff}}=-\frac{N}{48\pi\sigma_0^2}\int d^2x(\partial_\mu A_\nu-\partial_\nu A_\mu)^2. \tag{2.61}$$

Aside from a trivial normalization, this is the action for the free electromagnetic field. This is the most astonishing feature of the model; a genuine gauge field has been dynamically generated, produced as a result of radiative corrections in a theory that perturbatively has only short-range interactions. The gauge field now produces a linear potential that confines the $z$s. Not only does the theory have (bosonic) quarks, it has confined quarks.

There is much more that can be said about the $\mathbb{C}P^{N-1}$ model. For example, the classical theory admits instantons (just as in chromodynamics), and there is no infrared cutoff on instanton sizes (again as in chromodynamics). Thus the model can be used as a laboratory for instanton physics (=arena for bloody controversies). Unfortunately, I do not have the time to go into any of this here, and must once again refer you to the literature.[6]

## 3 Adjoint representations, or, chromodynamics

### 3.1 *The double-line representation and the dominance of planar graphs*[7]

The $1/N$ expansion is vastly more difficult for SU($N$) gauge theories than for any of the theories of Section 2. The source of the difficulty has nothing to do with the traditional problems of chromodynamics, the intricacies of gauge invariance or the uncontrollable infrared divergences. It is just that we have to deal with fields that transform according to the adjoint representation rather than the vector representation, objects that carry two group indices rather than one. $\phi^4$ theory is just as difficult if $\phi$ is in the adjoint representation.

The dynamical variables of the theory are a set of Dirac fields, $\psi^a$, and a set of gauge fields, $A^a_{\mu b}$, where $a$ and $b$ run from 1 to $N$. The Dirac fields can be thought of as elements of an SU($N$) column vector, the gauge

fields as those of a traceless Hermitian matrix,

$$A^a_{\mu b} = A^{b\dagger}_{\mu a}, \qquad A^a_{\mu a} = 0. \tag{3.1}$$

From the gauge fields we define

$$F^a_{\mu\nu b} = \partial_\mu A^a_{\nu b} + \mathrm{i} A^a_{\mu c} A^c_{\nu b} - (\mu \leftrightarrow \nu). \tag{3.2}$$

The dynamics of the theory is defined by

$$\mathscr{L} = \frac{N}{g^2}\left[-\tfrac{1}{4} F^a_{\mu\nu b} F^{\mu\nu b}_a + \bar{\psi}_a(\mathrm{i}\partial_\mu + A^a_{\mu b})\gamma^\mu \psi^b - m\bar{\psi}_a \psi^a\right], \tag{3.3}$$

where $g$ and $m$ are real numbers. For $N = 3$, this is the Lagrange density of quantum chromodynamics; the Dirac fields are quarks and the gauge fields gluons. I will retain this nomenclature for general $N$.

***Remarks.*** (1) I have put the coupling constant in front of the total Lagrangian. Of course, by rescaling the fields, as we did in Section 2.3, we can remove it from the quadratic terms in Eq. (3.3) and put it in its conventional position, a factor of $g/N^{\frac{1}{2}}$ multiplying the cubic terms and one of $g^2/N$ multiplying the quartic terms. (2) Equation (3.3) is incomplete; I have left out gauge-fixing terms, ghost couplings, renormalization counterterms, and the possibility of more than one flavor. I have done this to keep my equations as simple as possible. Practically all of my analysis will be purely combinatoric, hardly dependent at all on the detailed form of the interactions; thus, the extension to include all these neglected effects will be trivial. (3) I have taken advantage of my knowledge of how things are going to turn out to put the factor of $N$ in the right place from the very beginning. As we shall see shortly, it is the theory defined by Eq. (3.3) that admits a non-trivial $1/N$ expansion, and not, for example, the one with an $N^2$ in place of the $N$.

To take proper account of factors of $1/N$, we must keep proper track of the indices within a Feynman graph. Let us begin our analysis with the propagators. The quark propagator is

$$\overbrace{\psi^a(x)\bar{\psi}_b(y)} = \delta^a_b S(x-y), \tag{3.4}$$

where $S$ is the propagator for a single Dirac field. Thus there is no trouble following indices along a quark line; the index at the beginning is the same as the index at the end. The gluon propagator is

$$\overbrace{A^a_{\mu b}(x) A^c_{\nu d}(y)} = \left(\delta^a_d \delta^c_b - \frac{1}{N}\delta^a_b \delta^c_d\right) D_{\mu\nu}(x-y), \tag{3.5}$$

where $D_{\mu\nu}$ is the propagator for a single gauge field. The term proportional to $1/N$ is there because the gluon field is traceless; it would not be present if our gauge group were U($N$) rather than SU($N$). However, precisely because this term is proportional to $1/N$, we can drop it, even for SU($N$),

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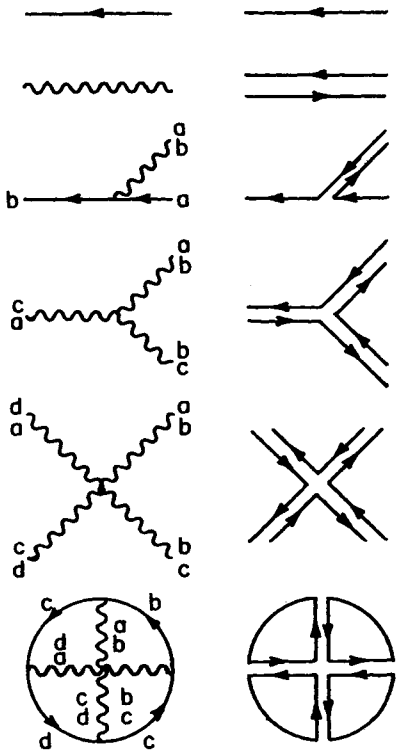
if we are only interested in the leading order in $1/N$. (Of course, we must remember we have dropped it if we want to compute subleading orders. See Appendix 3.) There is now no problem following indices along a gluon line; the index pair at the beginning is the same as the index pair at the end. As far as the index structure goes, a gluon propagates like a quark–anti-quark pair.

This observation is at the root of the ingenious double-line representation of 't Hooft. This is an alternative way of drawing Feynman graphs in which we draw one line for each index rather than one line for each virtual particle. Thus a quark propagator is represented by a single index line, because a quark carries only one index, but a gluon propagator is represented by two index lines, a double line, because a gluon carries two indices. Fig. 8 is a translation dictionary from the old single-line representation on the left to the new double-line representation on the right. The figure shows the translations of propagators, vertices, and a typical vacuum-to-vacuum graph. The great advantage of the double-line representation is

**Fig. 8**



immediately obvious: we do not have to clutter our graphs with little letters to show where the indices go; to follow the indices all we have to do is follow the arrows. Phrased in another way, to each double-line graph there corresponds a single-line graph with indices assigned to the lines. Thus, if there is more than one way of assigning indices to the lines in a given single-line graph, there will be more than one double-line graph associated with it.

I will now show that the power of $1/N$ carried by a double-line graph is determined by certain topological properties of the graph. For simplicity, I will begin by restricting myself to vacuum-to-vacuum graphs, graphs with no external lines. I will later extend the analysis as needed.

Because the graph has no external lines, every index line must close to make an index loop. Let us imagine each index loop to be the perimeter of a polygon. The double-line graph can then be read as a prescription for fitting together these polygons. To be more precise, we identify one edge of one polygon with one edge of another if they both lie on the same double line (gluon propagator). In this way, we construct a two-dimensional surface.

We can give an orientation to each polygon by the direction of the arrows around its perimeter and the right-hand rule. Because the two halves of a double line are always oppositely directed, this orientation is consistent as we travel about the surface; we have constructed an oriented surface. Thus we can get spheres or toruses, but not Klein bottles. (If we were doing the parallel analysis for SO($N$) rather than SU($N$), quark and antiquark would transform equivalently, our lines would not carry arrows, and we could get Klein bottles.)

It is easy to count the power of $N$ associated with this surface. Let the surface have $V$ vertices, $E$ edges, and $F$ faces. Every vertex is an interaction vertex of a Feynman graph, and carries a factor of $N$, by Eq. (3.3). Every edge is a propagator, either quark or gluon, and carries a factor of $1/N$, again by Eq. (3.3). Every face is an index loop, and thus yields a factor of $N$ when we sum over all possible values of the index. Thus the graph is proportional to

$$N^{F-E+V} \equiv N^{\chi}. \tag{3.6}$$

$\chi$ is the Euler characteristic. It is a famous topological invariant, and can be computed in quite another way. Every two-dimensional oriented surface is topologically equivalent to a sphere with some number of holes cut out of it and some number of handles stuck on to it. For example, a torus is a sphere with one handle; a disc is a sphere with one hole; a cylinder (without end caps) is a sphere with two holes; a loving cup is a

sphere with one hole and two handles; etc. Let $H$ be the number of handles and $B$ (for boundary) be the number of holes. Then the Euler characteristic is given by

$$\chi = 2 - 2H - B. \tag{3.7}$$

(If you are not familiar with the Euler characteristic, a quick and dirty proof of this formula is given in Appendix 1.)

Thus the leading connected vacuum-to-vacuum graphs are proportional to $N^2$, and the associated surface has the topology of a sphere. What does this mean in terms of our original single-line graphs? The boundary of a hole is a loop of unpaired index lines, that is to say, a quark loop; thus, the leading graphs involve only gluons. (This is not a deep result. It takes just as many powers of the coupling constant to make a quark pair as to make a gluon pair, but there are $N$ times more gluons than quarks to sum over.) Let us remove one randomly selected face from our spherical surface and project the remainder of the surface onto a plane. We thus obtain a planar graph. If we collapse the double lines to single lines, the graph remains planar. Conversely, given a planar graph made up only of gluon lines, we can always associate a double-line graph of the desired type with it. A planar graph divides the portion of the plane it occupies into regions. All we need do is draw a clockwise index line just inside the boundary of each of these regions and a counterclockwise index line just outside the boundary of the whole graph.

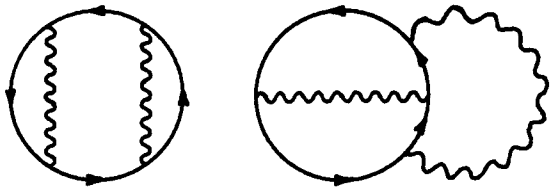
This analysis will shortly become important to us, so I summarize it in the following: *First Result – The leading connected vacuum-to-vacuum graphs are of order $N^2$. They are planar graphs made up only of gluons.*

We might be interested in the leading vacuum-to-vacuum graphs that have a nontrivial dependence on quark parameters, like quark masses or the number of flavors. These graphs must involve at least one quark loop; we see from Eq. (3.7) that the leading graphs have only one quark loop (= one hole) and no handles. Thus one of these graphs defines a spherical surface with one face removed. We can project this onto a plane, just as we did before. The only difference from the preceding case is that the outer boundary of the resultant planar graph is the perimeter of the hole, the quark loop.

Thus we obtain: *Second Result – The leading connected vacuum-to-vacuum graphs with quark lines are of order $N$. They are planar graphs with only one quark loop; the loop forms the boundary of the graph.* Thus the first graph in Fig. 9 is leading, but the second is not, even though it is planar.

As we shall see immediately, a very large amount of meson phenomenology is implicit in these two results.

**Fig. 9**



3.2 *Topology and phenomenology*

The naive quark model deals with mesons (particles made of a quark and an antiquark) and baryons (particles made of three quarks). Somewhat more sophisticated models worry about glueballs (particles made of gluons) and exotics, in particular exotic mesons (particles made of more than one quark–antiquark pair). Because it takes $N$ quarks to make a color singlet, baryons are a special problem for the $1/N$ expansion, and I will postpone their study to Section 3.4. However, we do have all the machinery needed to study mesons, glueballs, and exotic mesons.

Our method will be to study the states made by applying certain gauge-invariant local operators to the vacuum. We will restrict ourselves to monomials in $\psi$, $\bar{\psi}$, $F_{\mu\nu}$, and their covariant derivatives, and, further, to monomials that cannot be written as the product of two gauge-invariant monomials of lower degree. (Practically everything I say will be valid even for non-local non-polynomial gauge-invariant operators, like $\bar{\psi}_a(x)U^a_b\psi^b(y)$, where $U$ is the ordered exponential integral of the gauge fields over the line from $x$ to $y$; as far as counting powers of $N$ goes, the key point is that the operator cannot be decomposed into a product of gauge-invariant operators.)

To begin with, let us study quark bilinears, operators involving one $\psi$ and one $\bar{\psi}$. Let $B_1 \ldots B_n$ be a string of such bilinears, each at some point, and let $\langle B_1 \ldots B_n\rangle_C$ be the connected Green's function for this string. If we modify the action of our theory by an additional term,

$$S \to S + N \sum_i b_i B_i, \tag{3.8}$$

where the $b$s are numbers, and if $W$ is the sum of connected vacuum-to-vacuum graphs, then

$$\langle B_1 \ldots B_n\rangle_C = (iN)^{-n} \frac{\partial^n W}{\partial b_1 \ldots \partial b_n}\bigg|_{b_i=0}. \tag{3.9}$$

The reason for doing things this way is that all the analysis of Section 3.1 applies to the action (3.8) without a word of alteration; every interaction

vertex carries a factor of $N$, and, in the double-line representation, every interaction vertex becomes a vertex of a polyhedron. (Note that this last point would not be true if one of the $B$s was a product of two gauge-invariant monomials of lower degree. In that case we would get two polyhedral vertices from a single interaction vertex.)

Thus, from Result 2 of Section 3.1, we immediately know that the leading graphs are planar graphs with one quark loop, the boundary of the graph. Of course, all the bilinears must appear as insertions on the quark loop. Fig. 10 shows a leading graph for a three-bilinear Green's function; the bilinears are indicated by crosses. (We see that $B_3$ is linear in gluon fields as well as bilinear in quark fields.) We also know from Result 2 that the contribution of these graphs to $W$ is proportional to $N$; thus, to leading order,

$$\langle B_1 \dots B_n \rangle_C \propto N^{(1-n)}. \tag{3.10}$$

We will also have use for gauge-invariant local operators made up exclusively of gauge fields. I will denote such operators by $G_i$. The Green's function for a mixed string of $B$s and $G$s can be studied by the same method as before. Once again the leading graphs are planar graphs with only one quark loop, the boundary of the graph. Of course, the $G$s can appear as insertions anywhere in the graph. Again, the contribution to $W$ is proportional to $N$; thus, to leading order

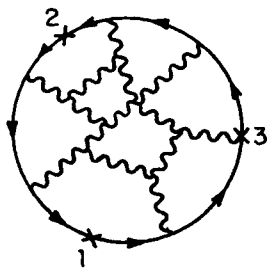
$$\langle B_1 \dots B_n G_1 \dots G_m \rangle_C \propto N^{(1-n-m)}. \tag{3.11}$$

The situation is slightly different for a string made up of $G$s alone. Here Result 1 of Section 3.1 applies; the leading graphs are planar graphs with no quark loops, and their contribution to $W$ is proportional to $N^2$. Thus, to leading order,

$$\langle G_1 \dots G_m \rangle_C \propto N^{(2-m)}. \tag{3.12}$$

We can derive much interesting physics from these equations if we make one assumption, that chromodynamics confines for arbitrarily

**Fig. 10**



large $N$, that all states made by applying strings of gauge-invariant operators to the vacuum are states composed of SU($N$)-singlet particles. I stress this is pure assumption. Our main reason for believing in confinement is that we can see quarks within hadrons but we cannot liberate them; this is an experimental reason, not a theoretical one, and experiment exists only for $N=3$. Nevertheless, the assumption is not unreasonable. To the small extent to which we do have a theoretical understanding of confinement (for example, from strong-coupling lattice gauge theories), there does not seem to be anything special about small $N$. But the best reason for assuming large-$N$ chromodynamics confines is that, if it does not confine, it bears no resemblance to reality, and the $1/N$ expansion is hopeless. Thus, we might as well make the assumption and see where it leads us.

I will define a meson to be a one-particle state made by applying a quark bilinear to the vacuum. From Eq. (3.10), our bilinears are not properly normalized to create mesons with $N$-independent amplitudes. Therefore, we renormalize them, and define

$$B'_i = N^{\frac{1}{2}} B_i. \tag{3.13}$$

For these,

$$\langle B'_1 \ldots B'_n \rangle \propto N^{(2-n)/2}. \tag{3.14}$$

Meson scattering amplitudes are obtained from these Green's functions by the reduction formula; thus a scattering amplitude with $n$ legs is proportional to $N^{(2-n)/2}$. For large $N$, mesons interact weakly; $N^{-\frac{1}{2}}$ sets the scale of meson interactions just as $e$ sets the scale of the interactions of electrons and photons.

Indeed, the parallel is exact. In quantum electrodynamics, in lowest non-vanishing order in perturbation theory, the tree approximation, an $n$-field Green's function is proportional to $e^{(n-2)}$. As far as dependence on $N$ is concerned, it is as if our bilinears were linear functions of fundamental fields in some field theory with coupling constant proportional to $N^{-\frac{1}{2}}$, as if the leading order in $1/N$ were the tree approximation in this theory.

Of course, the tree approximation is characterized by more than just its dependence on the coupling constant. Green's functions in the tree approximation have very simple analytic structures in momentum space; their only singularities are poles. I will now argue that Eq. (3.14) implies that the same holds for our Green's functions.

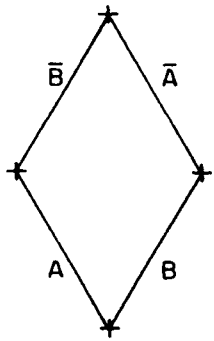
I will begin with the two-point function. To show that the only singularities are poles is to show that a bilinear applied to the vacuum produces only single-meson states (to leading order). The proof is by contradiction. For example, let us assume a renormalized bilinear produces a pair of

singlet particles ($A$, $B$), with an amplitude of order unity. We can then construct the sequence of events shown in Fig. 11. This is not a Feynman diagram, but a drawing of events in space-time (time runs upward). Initially a bilinear produces a pair; the components separate until they are each reflected by a bilinear; when they come back to the same point, they are absorbed by yet another bilinear. Every vertex in the figure is obtained from the assumed initial vertex by crossing; thus all the vertices are of order unity and the process produces a physical-region singularity of order unity in the four-bilinear connected Green's function. But this is impossible; Eq. (3.14) tells us that this Green's function is proportional to $1/N$. This argument generalizes instantly from a pair to a multi-particle state; all we need to do is declare that each line in the figure represents a cluster of particles.

The argument also generalizes to higher Green's functions. For example, let us consider a four-point function, and let us assume it has a two-particle cut, in a two-bilinear subenergy, in leading (first) order in $1/N$. Then there must be a connected amplitude for two bilinears to produce a pair, proportional to $N^{-\frac{1}{2}}$. If we reproduce the reasoning of the preceding paragraph, with the cross in Fig. 11 now denoting the double bilinear, we deduce that there is a singularity in an eight-bilinear Green's function proportional to $1/N^2$. But this is impossible; this Green's function is proportional to $1/N^3$. And so on.

There is one way in which the large-$N$ theory does not resemble a conventional weakly coupled field theory; there is an infinite number of mesons. We know this is so because asymptotic freedom is not spoiled in the large-$N$ limit; thus two-point functions must behave logarithmically for large spacelike momenta. This cannot be achieved if the only singularities of these functions are poles and if all poles lie within some bounded

**Fig. 11**



region. Thus there must be an infinite number of mesons of ever-increasing mass. We can sharpen this argument a bit: because we can build bilinears of any spin, there must be an infinite number of mesons of each spin. Such an infinite tower of stable mesons makes sense only if all mesonic $S$-matrix elements vanish in the large-$N$ limit; otherwise, a heavy meson could decay into light ones. As we have seen, they do indeed vanish.

All of our mesonic analysis can be extended trivially to glueballs, particles made by applying gluonic operators to the vacuum. By Eq. (3.12), the $G$s are already properly normalized and need no renormalization. From the same equation, a glueball scattering amplitude with $n$ legs is proportional to $N^{(2-n)}$. Thus glueballs interact even more weakly than mesons; $N^{-\frac{1}{2}}$ in meson dynamics is replaced by $N^{-1}$ in glueball dynamics.

If glueballs interact more weakly than mesons, they are not mesons. This assertion can be checked by an independent line of argument. From Eq. (3.11),

$$\langle B'_1 \ldots B'_n G_1 \ldots G_m \rangle_C \propto N^{(1-m-\frac{1}{2}n)}. \tag{3.15}$$

Thus, glueball–meson mixing vanishes like $N^{-\frac{1}{2}}$ as $N$ becomes infinite. Also, in meson–meson scattering, glueball production is suppressed; to replace a final-state meson by a glueball costs a factor of $N^{-\frac{1}{2}}$ in amplitude.

Up to now, flavor has been irrelevant to our discussion. This will not be the case for our next (and last) two topics, the validity of Zweig's rule and the existence of exotic mesons. With only one flavor of quark, we would be hard pressed to distinguish Zweig-allowed from Zweig-forbidden processes, or to tell exotic mesons from ordinary ones.

The usual statement of Zweig's rule is that for any mesonic scattering graph, it is impossible to divide the meson legs into two sets unconnected by quark lines. We already have this, in the form of the statement that in leading graphs all bilinears must appear as insertions on a single quark loop. Graphs for Zweig-forbidden processes must involve at least two quark loops, and thus are down in amplitude by at least one factor of $1/N$. Thus we have all the usual consequences of the rule: in the limit of strict SU(3), mesons must fall into nonets, an $s\bar{s}$ meson cannot decay into a final state free of strange quarks, etc.

To show the nonexistence of exotics requires a little more work. Exotic mesons, if they existed, would be states created from the vacuum by the application of local gauge-invariant quark quadrilinears. Every such object is the sum of products of local gauge-invariant bilinears. (That is to say, the only way to make an SU($N$) scalar is to take the inner product of each quark column vector with an antiquark row vector, perhaps with an

intervening SU($N$) matrix made of $F_{\mu\nu}$s and covariant derivatives.) With no loss of generality we can study the states made by a single product,

$$Q(x) = B'_1(x)B'_2(x), \tag{3.16}$$

where, for purposes of this argument, I have restored explicit space-time dependence. For simplicity, I will assume we have chosen the flavors of these operators so that $B_1$, $B_2$, and $Q$ all have vanishing vacuum expectation values. Then,

$$\begin{aligned}\langle Q^\dagger(x)Q(y)\rangle = &\langle B'^\dagger_1(x)B'_1(y)\rangle\langle B'^\dagger_2(x)B'_2(y)\rangle \\ &+ \langle B'^\dagger_1(x)B'_2(y)\rangle\langle B'^\dagger_2(x)B'_1(y)\rangle \\ &+ \langle B'^\dagger_2(x)B'^\dagger_2(x)B'_1(y)B'_2(y)\rangle_{\mathrm{C}}.\end{aligned} \tag{3.17}$$

The first two terms on the right are of order one, while the third is of order $1/N$, and thus should be dropped in the large-$N$ limit. But the first two terms simply describe the independent propagation of two mesons from $x$ to $y$. In the large-$N$ limit, quadrilinears make meson pairs and nothing else.

### 3.3 *The 't Hooft model*[8]

The arguments of the preceding section have been powerful but abstract; it would be nice to have a concrete example in which we could see them at work. Such an example is provided by the 't Hooft model, large-$N$ chromodynamics in two space-time dimensions. The model is almost exactly soluble; the simplest Green's functions can be found in closed form (in an appropriate gauge), and, although the computation of the particle spectrum requires numerical analysis, is is of a sort that can be carried out on a pocket calculator. The model also serves to eliminate a worrisome possibility, that the arguments of Section 3.2 are internally inconsistent, that confinement cannot exist for arbitrarily large $N$.

Of course, it is no surprise to find confinement in a two-dimensional gauge theory. As a warm-up for the 't Hooft model, let me remind you how confinement occurs in an even simpler theory, two-dimensional quantum electrodynamics. This theory is defined by

$$\mathscr{L} = \tfrac{1}{2}(F_{01})^2 + \bar{\psi}(i\partial_\mu\gamma^\mu - eA_\mu\gamma^\mu - m)\psi, \tag{3.18}$$

where

$$F_{01} = \partial_0 A_1 - \partial_1 A_0. \tag{3.19}$$

As always in the analysis of a gauge theory, the first step is to pick a gauge. I will choose axial gauge,

$$A_1 = 0. \tag{3.20}$$

In this gauge,

$$\mathscr{L} = \tfrac{1}{2}(\partial_1 A_0)^2 + \bar{\psi}(\mathrm{i}\partial_\mu \gamma^\mu - eA_0\gamma^0 - m)\psi. \tag{3.21}$$

No time derivatives of $A$ appear in this equation; $A$ is not a dynamical variable at all, but a constrained variable, one that must be eliminated from the theory before we can write it in canonical form.

The equation that determines $A$ is

$$\partial_1^2 A_0 = -e\psi^\dagger\psi \equiv -ej^0. \tag{3.22}$$

The general solution of this is

$$A_0(x^0, x^1) = -\frac{e}{2}\int \mathrm{d}y^1 |x^1 - y^1| j^0(x^0, y^1) + Bx^1 + C, \tag{3.23}$$

where $B$ and $C$ are constants. $C$ is irrelevant; it can always be eliminated by a gauge transformation; for simplicity, I will set it to zero. $B$ is relevant; non-zero $B$ corresponds to the existence of a constant background electric field, such as would be caused by classical charges at spatial infinity. In the Abelian case that occupies us at the moment such a background field has a real effect on the physics, and an interesting one. In the non-Abelian case we are heading for, it turns out that the corresponding object has no physical effect whatsoever. This is a fascinating byway, but it is a byway, and I do not want to spend time on it here. Thus, I will assume for this investigation that there is no background field, and set $B$ to zero.

Thus, we can eliminate $A_0$ and write the Lagrangian as

$$L = L_{0\mathrm{f}} + \frac{e^2}{4}\int \mathrm{d}x^1\, \mathrm{d}y^1\, j_0(x^0, x^1)|x^1 - y^1| j_0(x^0, y^1), \tag{3.24}$$

where $L_{0\mathrm{f}}$ is the free fermion Lagrangian. A linear potential has appeared between charges; confinement is manifest, at least for small coupling, where we can trust perturbation theory.

It will be convenient for our later work to express the current–current interaction in Eq. (3.24) as the effect of exchange of a photon propagator,

$$\begin{aligned} \mathrm{D}_{\mu\nu}(k) &= -\frac{\mathrm{i}}{2}\delta_{\mu 0}\delta_{\nu 0}\int \mathrm{d}^2x\, \mathrm{e}^{\mathrm{i}k\cdot x}|x^1|\delta(x^0) \\ &= \mathrm{i}\delta_{\mu 0}\delta_{\nu 0}\frac{\mathrm{P}}{(k_1)^2}, \end{aligned} \tag{3.25}$$

where P is the principal-value symbol,

$$\mathrm{P}\frac{1}{z^2} = \frac{1}{2}\left[\frac{1}{(z+\mathrm{i}\varepsilon)^2} + \frac{1}{(z-\mathrm{i}\varepsilon)^2}\right]. \tag{3.26}$$

Of course, we could have obtained the momentum-space propagator

directly from the Lagrange density, Eq. (3.21), by standard methods; I detoured through position space to justify the somewhat unusual principal-value prescription at the pole.

It requires no work to generalize this chromodynamics. The nonlinear terms in $F_{01}$ are proportional to the product of $A_0$ and $A_1$. Thus they vanish in axial gauge, and with them vanishes one of the characteristic complications of chromodynamics, the self-coupling of the gauge field. Thus the only difference between chromodynamics and electrodynamics is a sprinkling of indices here and there; following the derivation that led to Eq. (3.24), we find

$$L = L_{0f} + \frac{g^2}{N} \int dx'\, dy'\, j^b_{0a}(x^0, x^1)|x^1 - y^1| j^a_{0b}(x^0, y^1), \tag{3.27}$$

where

$$j^b_{0a} = \psi^\dagger_a \psi^b - \frac{\delta^b_a}{N} \psi^\dagger_c \psi^c, \tag{3.28}$$

and I have rescaled the fields to put the coupling constant in its conventional location.

The elimination of gluon self-coupling drastically diminishes the number of graphs that contribute in the large-$N$ limit. Fig. 12 shows the set of graphs that contribute to a Green's function for two quark bilinears; the shaded blobs represent quark propagators. The simple structure of the graphs arises because a gluon line that connects the upper quark line to the lower quark line forms an impassable barrier. No gluon line can cross it without interaction, because this would violate planarity; no gluon line can cross it with interaction, because there are no interactions. The same simplicity of structure appears in Fig. 13, the equations for the

**Fig. 12**

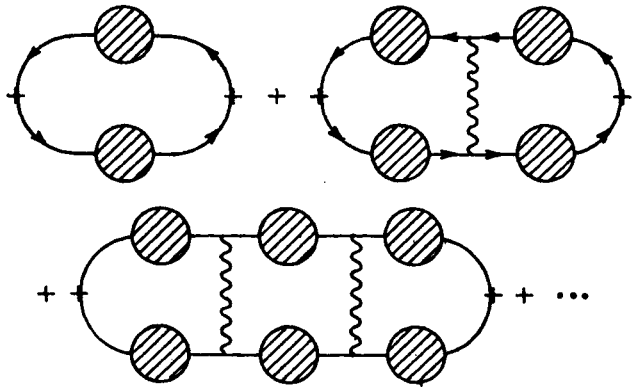


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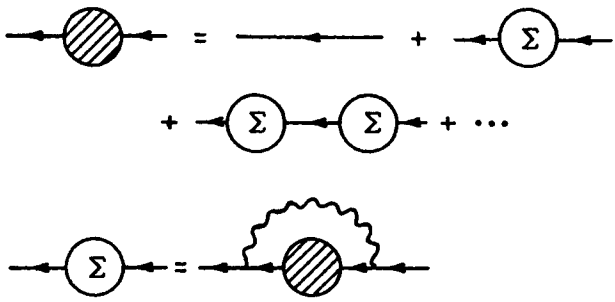
**Fig. 13**




quark self-energy, Σ. The first gluon line to leave the quark must be the last to return, for it forms an impassable barrier.

Things are still not as simple as they could be. For one thing, we still have to keep track of the two components of the quark field; Fig. 13 defines a matrix equation. For another, we are working in a non-covariant gauge, so we do not have the advantages of manifest Lorentz invariance. Both these problems can be eliminated if we switch from axial gauge to light-cone gauge.

Light-cone coordinates are defined by

$$x^{\pm} = (x^0 \pm x^1)/\sqrt{2}. \tag{3.29}$$

In these coordinates,

$$g^{+-} = g^{-+} = g_{+-} = g_{-+} = 1. \tag{3.30}$$

All other components of the metric tensor vanish. Note that this implies a peculiar Dirac algebra,

$$(\gamma^+)^2 = (\gamma^-)^2 = 0, \{\gamma^+, \gamma^-\} = 2. \tag{3.31}$$

Light-cone gauge is defined by

$$A_- = A^+ = 0. \tag{3.32}$$

This condition is Lorentz-invariant (but not parity-invariant). A rerun of the derivation of Eq. (3.25) leads to the photon propagator,

$$D_{\mu\nu}(k) = i\delta_{\mu+}\delta_{\nu+}\frac{P}{(k_-)^2}. \tag{3.33}$$

At every interaction vertex, only the matrix $\gamma^+$ appears. Thus, if we consider a quark line joining two interaction vertices, most of its matrix structure is annihilated; from Eq. (3.31),

$$\gamma^+ \begin{Bmatrix} 1 \\ \gamma^+ \\ \gamma^- \end{Bmatrix} \gamma^+ = 2\gamma^+ \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{3.34}$$

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Hence, all the Lorentz-index structure of our graphs is trivial. The photon propagator has only one non-zero component, and the interaction vertex is always proportional to a single Dirac matrix, as is the only surviving part of the quark propagator. There is no point in keeping track of these unvarying structures; we might as well drop them and make the substitutions

$$D_{\mu\nu} \to i \frac{P}{(k_-)^2}, \tag{3.35a}$$

$$-\frac{ig\gamma^+}{N^{\frac{1}{2}}} \to -\frac{2ig}{N^{\frac{1}{2}}}, \tag{3.35b}$$

and

$$i\frac{p_+\gamma^+ + p_-\gamma^- + m}{2p_+p_- - m^2 + i\varepsilon} \to \frac{ip_-}{2p_+p_- - m^2 + i\varepsilon}. \tag{3.35c}$$

The internal-index structure of our graphs is also trivial, by our earlier analysis; for each graph there is a unique way of distributing the internal indices, and the net effect of this distribution is to cancel the factor of $1/N^{\frac{1}{2}}$ in Eq. (3.35b).

Thus the first part of Fig. 13, the equation for the quark propagator, $S(p)$, becomes

$$S(p) = \frac{ip_-}{2p_+p_- - m^2 - p_-\Sigma(p) + i\varepsilon}. \tag{3.36}$$

while the second part of Fig. 13, the equation for the quark self-energy, $\Sigma$, becomes

$$-i\Sigma = -i4g^2 \int \frac{dk_+\, dk_-}{(2\pi)^2} S(p-k) \frac{P}{(k_-)^2}. \tag{3.37}$$

If, in this equation, we make the shift of integration variables,

$$p_+ - k_+ \to -k_+, \tag{3.38}$$

we eliminate all reference to $p_+$. $\Sigma$ is a function of $p_-$ only; by Lorentz invariance, it must be a constant multiple of $1/p_-$. Thus,

$$m^2 - p_-\Sigma \equiv M^2, \tag{3.39}$$

is a constant. To leading order in $1/N$, the sole effect of the interaction is to replace the bare quark mass, $m$, by the 'renormalized quark mass', $M$. (The quotation marks are to remind you that we are working with gauge-dependent entities, entities which do not necessarily have any physical meaning. I will return to this point shortly.) It is now straightforward to evaluate $M^2$. The details of the computation are in Appendix 2; the answer

is

$$M^2 = m^2 - (g^2/\pi). \tag{3.40}$$

Now that we have the quark propagator, we can evaluate the sum of ladder graphs in Fig. 12, and discover the spectrum of meson states. I will only state the answer here (again, the details are in Appendix 2). There is a bound state of mass $\mu$ for every eigenvalue of the integral equation

$$\mu^2\phi(x) = \left(\frac{M^2}{x} + \frac{M^2}{1-x}\right)\phi(x) - \frac{g^2}{\pi}\int_0^1 \mathrm{d}y\, \frac{\mathrm{P}}{(x-y)^2}\,\phi(y), \tag{3.41}$$

where $\phi$ is a function defined on the interval [0, 1] and vanishing at the end points of the interval.

Although it has been derived from field theory, Eq. (3.41) may be read as an equation in particle mechanics; to be more precise, it is a two-particle time-independent light-cone Schrödinger equation.

In the normal Schrödinger formalism, the state of the system is given at fixed $x^0$, and dynamics is defined by the operator that generates $x^0$-translations, $P_0$. This commutes with the generator of $x^1$-translations, $P_1$, and thus we can simplify dynamical problems by going to an eigenspace of $P_1$. In the light-cone Schrödinger formalism, the state of the system is given at fixed $x^+$, and dynamics is defined by the operator that generates $x^+$-translations, $P_+$. This commutes with the generator of $x^-$-translations, $P_-$, and thus we can simplify dynamical problems by going to an eigenspace of $P_-$. For example, for a single free particle of mass $M$,

$$2P_+ = M^2/P_-, \tag{3.42}$$

and dynamics is totally diagonal in a $P_-$ basis, just as it is totally diagonal in the normal formalism in a $P_1$ basis. An important difference between the two formalisms is that while the spectrum of $P_1$ is the entire real line, that of $P_-$ is the positive half-line only.

For a two-particle system, it is convenient to work in an eigenspace of total $P_-$ with eigenvalue one. Thus, if we denote the $P_-$ operator for one of the particles by $x$, that of the other is $(1-x)$. Since each $P_-$ must be positive, $x$ must lie between 0 and 1. For two non-interacting particles of equal mass,

$$2P_+ = \frac{M^2}{x} + \frac{M^2}{1-x}. \tag{3.43}$$

Because $P_-$ is one, the eigenvalues of this operator are the squared masses of the two-particle system. In terms of a momentum-space

Schrödinger wave function, the eigenvalue equation is

$$\mu^2\phi(x)=\left(\frac{M^2}{x}+\frac{M^2}{1-x}\right)\phi(x). \tag{3.44}$$

This is almost Eq. (3.41); all that is missing is the last term. But such a convolution integral in a momentum-space Schrödinger equation is a familiar object; it corresponds to an ordinary potential back in position space, in the case at hand, to a linear potential. Once we strip away the heavy disguise of the light-cone momentum-space formalism, Eq. (3.41) is revealed to be the simplest meson model of all, two quarks interacting through a linear potential.

We would expect such a system to have a purely discrete spectrum. The easiest way to see that this is the case is to reinterpret Eq. (3.41), to think of $x$ as a position operator and the conjugate variable as a momentum operator, $p$. The operator version of Eq. (3.41) then becomes

$$2P_+=\frac{M^2}{x}+\frac{M^2}{1-x}+g^2|p|. \tag{3.45}$$

Aside from a trivial multiplicative constant, this is the ordinary Hamiltonian for a mass-zero particle moving in a potential, *and* restricted to the box [0, 1]. It is this last condition that guarantees that the spectrum is purely discrete, that in fact our space of states does not contain any particles that correspond to two free quarks.

We can also use our reinterpretation to get a quantitative idea of the meson spectrum. For a particle moving in a potential and restricted to a box, we would expect the potential to be irrelevant for sufficiently high excited states, and the eigenstates to be those of a free particle in a box,

$$\phi_n=\sin \pi n x,\ n=1, 2, \ldots, \tag{3.46}$$

with associated eigenvalues

$$\mu_n^2=g^2\pi n. \tag{3.47}$$

An easy perturbative calculation shows that this expectation is correct, at least for large $n$; the corrections to Eq. (3.47) are $O(\log n/n)$.

Of course, Eq. (3.47) is no good for the low-lying mesons. Indeed, since $M^2$ can be negative for sufficiently large $g/m$, one might fear that for large $g$ the low-lying spectrum might go crazy. This fear is groundless. From Eq. (3.41) and the identity

$$\int_0^1 \frac{dy\ P}{(x-y)^2}=-\left[\frac{1}{x}+\frac{1}{1-x}\right], \tag{3.48}$$

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it follows that

$$\mu^2 \int_0^1 |\phi|^2 \, dx = m^2 \int_0^1 |\phi|^2 \left(\frac{1}{x} + \frac{1}{1-x}\right) dx + \frac{g^2}{2\pi} \int_0^1 dx \int_0^1 dy \, \frac{|\phi(x) - \phi(y)|^2}{(x-y)^2}. \tag{3.49}$$

Thus if $m^2$ is positive, $\mu^2$ is positive, no matter how large $g^2$ is; tachyonic quarks do not make tachyonic mesons.

To go on requires numerical analysis, so I will stop our investigation here. However, before I leave the model altogether, I would like to make two points.

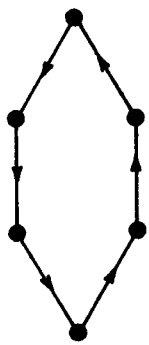
(1) Everything worked out as we expected it to. Confinement and the large-$N$ limit are not in contradiction.

(2) The structure of the quark propagator tells us nothing about confinement. Here we have a reasonable theory solved in a reasonable approximation; the quark propagator is that of a free particle (sometimes a tachyon); nevertheless, the theory contains no free quarks and no tachyons.

This point needs expansion. Why should the quark propagator be irrelevant? The first answer is that it is a gauge-variant object, and thus not an observable, but this is insufficient; the same might be said about the electron propagator in electrodynamics, and we know the location of the pole here tells us the electron mass, very much an observable quantity. However, it is observable only because it governs the singularity structure of gauge-invariant Green's functions.

For example, Fig. 14 shows a Feynman graph that contributes to a six-current Green's function in electrodynamics; the dots denote the

**Fig. 14**



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currents. This graph has a singularity that corresponds to a reading of it as a process going on in space-time (time runs upward). The initial current creates an electron–positron pair, which are then bounced about by widely separated external fields until they recombine. The location of the corresponding singularity is obviously governed by the electron mass. (This is an idealization of an actual measurement of the mass of a charged particle; the external fields are idealized bending magnets or counters.)

Of course, Fig. 14 is just a lowest-order graph, and we must be sure that higher-order corrections cannot destroy the singularity. We need not worry about propagator corrections, where a photon returns to the same electron line from which it emerged; these merely renormalize the electron mass. Nor need we worry about vertex corrections, where the photon goes from one side of a dot to another; these merely renormalize the strength of the external field. Nor need we worry about corrections where the photon connects electron lines separated by two or more dots; because all the external currents are widely separated, the photon ends are necessarily widely separated, and the electrodynamic interaction is negligible at large distances.

Oh.

We have reached the crux of the matter. The argument that the location of the pole in the quark propagator is an observable quantity rests upon the assumption that there is no confinement. Those who look for confinement in the singularities of the quark propagator are like the man who settled in Casablanca for the waters. They have been misinformed.

### 3.4 *Witten's theory of baryons*[9]

Baryons present a special problem for the $1/N$ expansion. The method we used in mesodynamics, the study of fixed Feynman graphs in the large-$N$ limit, cannot be applied here. It takes $N$ quarks (in a totally antisymmetric color state) to make a baryon, and thus we must study graphs with more and more quark lines as $N$ grows larger and larger. This does not mean that there is no hope, that baryons do not obey simple scaling laws. As Witten discovered, the proper procedure is to break the problem into two parts, to first use graphical methods to study $n$-quark forces in the large-$N$ limit, and then to use other methods to study the effects of these forces on an $N$-body state.

Defining an $n$-body force is a delicate matter in a quantum field theory. For a two-body force, we must use the Bethe–Salpeter equation; for higher values of $n$, Faddeev equations. Fortunately, for our purposes, we need worry about none of these niceties. All we want to do is count

powers of $N$, and for this all we need to know is that the $n$-body interaction kernel is obtained by summing up some family of graphs with $n$ quark lines entering and $n$ quark lines leaving. We can always imagine obtaining such graphs by breaking open $n$ internal quark lines in a vacuum-to-vacuum graph. By result 2 of Section 3.1, the leading vacuum-to-vacuum graphs with internal quark lines are the planar graphs bounded by a single quark loop; these are proportional to $N$. We want an interaction that will be effective in a totally antisymmetric color state; thus each quark line should carry a different color index, and breaking the quark lines costs us a factor of $N^{-n}$ from lost index sums. Hence, the $n$-quark interaction is proportional to $N^{1-n}$.

This completes the first part of the analysis. We must now study the effects of these interactions. From this point on I will assume that the states we are studying are non-relativistic, so we can use ordinary particle mechanics and treat the interactions as ordinary $n$-body potentials. I am embarrassed by the necessity of this assumption. Detailed dynamical assumptions should not be needed just to count powers of $1/N$; it should be possible to do the whole analysis in an elegant relativistic formalism. Unfortunately, I have not been able to find such a formalism, so we will just have to plug along with particle mechanics.

We wish to study a bound state made up of a very large number of particles interacting very weakly. This is the traditional domain of the Hartree approximation: each particle is treated as moving independently of the others in a common potential, which in turn is determined self-consistently from the motions of all the particles. Let me remind you of the justification for this approximation by estimating the sources of error. Firstly, the approximation neglects the fact that as each particle moves, it changes the state of the other particles, and thus the potential which it feels. This effect is proportional to the square of the interaction strength, and is thus a negligible correction to the Hartree potential (directly proportional to the interaction strength) if the interaction is weak. Secondly, the approximation neglects the fact that each particle feels not the potential caused by all the particles, but the potential caused by all the particles but itself. The correction this makes to the Hartree potential is inversely proportional to the number of particles and is negligible if this number is large.

The Hartree approximation leads to many-particle energy eigenfunctions that are products of single-particle wave functions. In our case, the many-particle wave function is antisymmetric in color, and thus symmetric in the remaining quark variables: space, spin, and flavor. Thus, if

we factor out the color part of the wave function, the quarks act like identical bosons; in the baryon ground state, all the quarks will be in the same state, the ground state of the Hartree potential.

Let me make this more quantitative. For notational simplicity, I will ignore flavor and spin. The baryon Hamiltonian is then

$$H=\frac{1}{2m}\sum_a |\mathbf{p}_a^2| + \frac{1}{2N}\sum_{a\neq b} V^{(2)}(\mathbf{r}_a, \mathbf{r}_b) + \frac{1}{6N^2}\sum_{a\neq b\neq c} V^{(3)}(\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c) + \cdots, \tag{3.50}$$

where the $V$s are functions independent of $N$. The approximate ground state wave function is of the form

$$\psi(\mathbf{r}_1 \cdots \mathbf{r}_N) = \prod_a \phi(\mathbf{r}_a) \tag{3.51}$$

We find the best choice of $\phi$ by the variational method. We compute

$$\langle\psi|H|\psi\rangle = N\left[\frac{1}{2m}\int d^3\mathbf{r}|\nabla\phi|^2 + \frac{1}{2}\int d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, V^{(2)}(\mathbf{r}_1, \mathbf{r}_2)|\phi(r_1)\phi(r_2)|^2 + \frac{1}{6}\int d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, d^3\mathbf{r}_3\, V^{(3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)|\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3)|^2 + \cdots\right], \tag{3.52}$$

plus terms of $O(1)$, which we neglect. We now minimize this as a function of $\phi$, subject to the constraint

$$\int d^3\mathbf{r}\,|\phi|^2 = 1. \tag{3.53}$$

We find

$$\left[-\frac{\nabla^2}{2m} + V(\mathbf{r})\right]\phi = \varepsilon\phi, \tag{3.54}$$

where $\varepsilon$ is the Lagrange multiplier associated with the constraint, and $V$ is the Hartree potential,

$$V = \int d^3\mathbf{r}_1\, V^{(2)}(\mathbf{r}, \mathbf{r}_1)|\phi(\mathbf{r}_1)|^2 + \frac{1}{2}\int d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, V^{(3)}(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2)|\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)|^2 + \cdots. \tag{3.55}$$

Equations (3.54) and (3.55) define the Hartree approximation; note that they are independent of $N$. This is a consequence of the fact that every term in the expression for the total energy, Eq. (3.52), is proportional to $N$; the $n$-quark interaction is $O(N^{1-n})$, but there are $O(N^n)$ distinct $n$-quark clusters. As we shall see, this lucky cancellation of powers of $N$ leads to simple scaling laws for baryon physics.

Let me begin with the static properties of baryons. $\phi$ is independent of $N$; thus so is the shape of the ground-state baryon, as measured, for example, by its charge or mass distribution. This is just like the situation for a meson. In contrast to a meson, though, the energy of the baryon grows with $N$. This is essential to the resolution of what would otherwise be a problem for the approximation, that the ground state is not a momentum eigenstate. Precisely because the Hamiltonian is translationally invariant, any spatial translation of a Hartree ground state is also a Hartree ground state. Thus we can form linear combinations of these states that are momentum eigenstates; more carefully phrased, the problem is not that the energy eigenstates are not momentum eigenstates but that the momentum eigenstates are degenerate in energy. But this is as it should be, if the mass of the states is $O(N)$; in the non-relativistic approximation, for example, the momentum dependence of the energy is proportional to (momentum)$^2$/mass; this is $O(1/N)$, and should not be seen in leading order.

Low-lying excited baryons are obtained by placing a few quarks in excited energy eigenstates of Eq. (3.54). Note that, to leading order, there is no need to change $V$; the change in the Hartree potential caused by exciting only a few quarks is an effect of $O(1/N)$. Thus, for example, if $\phi_1$ is the first excited state of Eq. (3.54) and $\varepsilon_1$ is the associated eigenvalue, the first excited baryon is given by

$$\psi_1 = \frac{1}{N^{\frac{1}{2}}} \sum_a \phi_1(\mathbf{r}_a) \prod_{b \neq a} \phi(\mathbf{r}_b). \tag{3.56}$$

Its energy exceeds that of the ground-state baryon by $\varepsilon_1 - \varepsilon$. Thus, although the baryon spectrum begins high, the spacing between successive baryons is $O(1)$, just like the spacing of mesons.

So much for the static properties of baryons. Now let me turn to their interactions with mesons. As explained in Section 3.2, mesons are created and annihilated by quark bilinears, like

$$B = N^{-\frac{1}{2}} \bar{\psi}^a \psi^a. \tag{3.57}$$

Here the $\psi$s are quark fields, normalized such that they obey canonical commutation relations, that is to say, such that in the non-relativistic

approximation they create and annihilate quarks with amplitudes of order unity. Let us consider the matrix element of such a bilinear between two ground-state baryons. Each of the $N$ terms in Eq. (3.57) can annihilate and recreate a quark; thus the baryon–meson–baryon vertex is $O(N/N^{\frac{1}{2}}) = O(N^{\frac{1}{2}})$. Of course, because the baryon energy is not changed, the meson must carry energy zero, and cannot be real. However, it can be a virtual meson, for example, one exchanged between baryon and meson in a meson–baryon scattering graph. Because the trilinear meson vertex is $O(1/N^{\frac{1}{2}})$, this contribution to meson–baryon scattering is $O(1)$.

Indeed, in general the meson–baryon scattering amplitude is $O(1)$. To prove this, let us consider the one-baryon expectation value of the time-ordered product of two quark bilinears. There are two classes of terms that contribute to this expression. (1) One of the bilinears can annihilate and recreate a quark; the other can do the same. In this case, the two quarks can be of different colors; the sum over colors gives a factor of $N^2$, and the total contribution is proportional to $N$. However, since the intermediate state is the ground-state baryon, the bilinears must each carry energy zero, and this contribution vanishes on the meson mass shell. (2) One of the bilinears can annihilate a ground-state quark and replace it with an excited quark; the other can then reverse the process. In this case, all the quarks must be of the same color, so the sum over colors only gives a factor of $N$, and the total contribution is $O(1)$. However, this contribution does not vanish on the meson mass shell.

We can also study the matrix element of a bilinear between a ground-state baryon and an excited baryon, like the one given by Eq. (3.56). Here each term in Eq. (3.57) must match up with the corresponding term in Eq. (3.56); thus, as before, we only get a factor of $N$ from the color sum, and the amplitude for mesonic decay of an excited baryon is $O(1)$. By similar reasoning, the amplitude for the process meson + baryon → meson + excited baryon is $O(N^{-\frac{1}{2}})$.

Because the meson–baryon scattering amplitude is $O(1)$, the contribution of continuum states to the absorptive part of the amplitude is of the same order as the contribution of excited baryons. This is in striking contrast to the situation for meson–meson scattering. If, in the real world, meson–baryon scattering is well approximated by a sum over narrow resonances, the explanation of this phenomenon does not lie in the $1/N$ expansion.

Baryon–baryon scattering has a special feature because the baryon mass increases with $N$; if one studies scattering at fixed center-of-mass energy and momentum transfer, one soon finds oneself below threshold.

The solution is to study scattering at fixed center-of-mass velocity and scattering angle. This implies that momentum transfer grows linearly with $N$, so it is not profitable to study the scattering process in terms of one-meson exchange, or, indeed, in terms of exchange of any finite number of mesons. The proper strategy is to directly compute the baryon–baryon interaction. When we first worked through the Hartree approximation, we found the interaction of one quark in a baryon with all the rest of the baryon was $O(1)$. For a two-baryon system, by the same reasoning, the interaction of a quark in one baryon with the entirety of the other baryon is $O(1)$. Thus the baryon–baryon interaction is proportional to $N$, just like the baryon energies, and $N$ factors neatly out of the baryon–baryon scattering equation. Baryon–baryon scattering, at fixed velocity and scattering angle is $O(1)$.

Let me summarize our results. The meson spectrum is independent of $N$, and meson–meson scattering is $O(1/N)$. The baryon spectrum begins at an energy proportional to $N$, but, after it begins, the spacing of baryons is independent of $N$. The sizes and shapes of the baryons are also independent of $N$, as are the amplitudes for meson–baryon scattering and baryon–baryon scattering (at fixed velocities and angles).

As Witten has pointed out, we have heard this tune before. Certain classical field theories admit finite-energy time-independent solutions of the field equations, like the soliton in the sine–Gordon equation and the monopoles that arise in many unified electroweak and grand unified theories.[10] These lumps of energy become particles in the quantum versions of these theories, and, for small coupling, it is possible to study the properties of these particles. They are, word for word, the properties of the baryons enunciated in the preceding paragraph, with $1/N$ replaced by the small coupling constant ($e^2$ in the electroweak theories). This is a tantalizing parallelism; it strongly suggests that there should be some way of formulating the $1/N$ expansion such that the baryons appear directly as lumps. Unfortunately, at this moment, I know of no such formulation.

### 3.5 *The master field*

It would be good to know the leading term in the $1/N$ expansion of chromodynamics. We have been able to go far without this knowledge, but with it we could go much farther, and much faster. A direct approach, an attempt to compute and sum all planar graphs, is hopeless; some indirect method is needed. In this section, I will explain an indirect method recently proposed by Witten.[11] Witten's program has not yet been brought to completion. Nevertheless, I think it is worth talking about even in its

incomplete state; it may well succeed, and, even if it fails, it involves such novel insights that it may inspire you to discover some other, better method.

I will first describe the method for pure gauge theory, and then go on to explain how to assimilate quarks. I think the clearest way to explain things is by drawing a parallel with the classical limit. Feynman's path integral formula tells us that Green's functions for a quantum theory are obtained by integrating over all possible classical motions. However, as $\hbar$ goes to zero, the measure in function space becomes more and more sharply concentrated about the solution to the classical equations of motion; in the limit of vanishing $\hbar$, all quantities are given by their values at the classical solution. A very similar statement applies to the large-$N$ limit. There is a classical gauge-field configuration, which I will call the master field, such that the large-$N$ limits of all gauge-invariant Green's functions are given by their values at the master field. I emphasize that once things are evaluated at the master field, there are no further steps; in particular, no integrations, functional or other, need to be done.

I will first comment on the method and then give the proof of the existence of the wonder-working master field.

***Comments.*** (1) The master field is a field for the large-$N$ limit of chromodynamics, that is to say, it is a gauge field for gauge group $U(\infty)$. (2) The master field is not unique; because we are interested only in gauge-invariant quantities, any gauge transform of a master field is also a master field. However, this is the end of the non-uniqueness; two gauge-inequivalent fields assign different values to some gauge-invariant quantity, and thus cannot both be master fields. A purist would thus speak not of 'the master field' but of 'the master orbit of the gauge group'. (3) For the classical limit, we not only know that everything is dominated by a single field configuration, we have an algorithm for finding it, solution of the classical equations of motion. This is the missing element in Witten's program. We know that the master field exists but we have no algorithm for finding it. (4) Nevertheless, we can say some things about the master field. We expect large-$N$ Green's functions to be translationally invariant; thus the master field should be translationally invariant also. That is to say, we expect that, in an appropriate gauge, $A_\mu$ should be independent of space-time. (Note that this does not mean that the master field is trivial, that $F_{\mu\nu}$ vanishes; the components of $A_\mu$ need not commute.) Thus, to find the master field we need only find four matrices. True, these are infinity-by-infinity matrices, so this is not necessarily an easy task. Nevertheless,

this is a remarkable reformulation of the problem of summing all planar graphs.

***Proof.*** We wish to show that for large $N$ the measure in function space becomes concentrated on a single orbit of the gauge group. This is equivalent to showing that the probability of finding any gauge-invariant quantity away from its expectation value goes to zero as $N$ goes to infinity. Because all gauge-invariant quantities are sums of products of the $G$s of Section 3.2, it suffices to prove the proposition for an arbitrary $G$. Of course, before we begin, we must normalize $G$ such that its expectation value has a large-$N$ limit, that is to say, is of order unity. Thus we define

$$G' = G/N. \tag{3.58}$$

Now let us estimate the probability of $G'$ departing from $\langle G' \rangle$ by computing the variance:

$$\begin{aligned} \langle (G' - \langle G' \rangle)^2 \rangle &= \langle G'G' \rangle - \langle G' \rangle \langle G' \rangle \\ &= \langle G'G' \rangle_C \\ &= O(1/N^2), \end{aligned} \tag{3.59}$$

by Eq. (3.12). Q.E.D.

All of this has been for pure gauge field theory, but the method can readily be extended to the computation of Green's functions involving quark bilinears. For simplicity, let me assume that we are only interested in the bilinear $\bar{\psi}\psi$. We can compute Green's functions for strings of these operators by giving the quarks a space-time dependent mass, $m(x)$, computing the vacuum-to-vacuum amplitude, and then functionally differentiating this with respect to $m$. We know that in the large-$N$ limit, the dominant vacuum-to-vacuum graphs are those with one quark loop. In a given external gauge field, the sum of all these graphs is given by a famous expression,

$$\text{Tr ln } [i\partial\!\!\!/ - A\!\!\!/ - m]. \tag{3.60}$$

But this is a gauge-invariant function of gauge fields only; thus, in the large-$N$ limit, when we integrate over gauge fields, it is given by its value with $A_\mu$ replaced by the master field.

You may find these arguments a bit too slick and abstract, and yearn for a concrete example in which one can explicitly find the master field. Fortunately, such an example exists. As I have said, the master field has not been found for four-dimensional chromodynamics. It has not even been found for the vastly simpler 't Hooft model, two-dimensional chromodynamics. However, it has been found for zero-dimensional

chromodynamics. This is quite a come-down from field theory; instead of functional integrals over matrix-valued fields we have ordinary integrals over ordinary matrices, and the master field is just a master matrix. Nevertheless, even though the dynamics has been trivialized, the combinatorics retains much of its four-dimensional horror, and the integrals evaluated easily with the master matrix would be nightmares if attempted by the summation of planar graphs. These matrix integrals were first evaluated by this method in a brilliant paper by the Saclay group.[12] I will follow their analysis closely in what follows.

We wish to evaluate integrals of functions of an $N \times N$ Hermitian matrix, $H$. To begin, we must define integration over $H$:

$$dH = \prod_{a,b} dH_{ab},\ a, b = 1 \ldots N, \tag{3.61}$$

where integration over complex variables is defined in the usual way,

$$dH_{ab} dH_{ba} = d(\mathrm{Re}\ H_{ab}) d(\mathrm{Im}\ H_{ab}). \tag{3.62}$$

This measure is invariant under zero-dimensional gauge transformations,

$$H \to U^\dagger H U, \tag{3.63}$$

where $U$ is a unitary matrix. We wish to study the zero-dimensional version of the chromodynamic formula for the expectation value of a gauge-invariant operator,

$$\langle \mathrm{Tr}\ g \rangle \equiv \frac{\int dH\, e^{-S(H)}\, \mathrm{Tr}\ g(H)}{\int dH\, e^{-S(H)}}, \tag{3.64}$$

where $g$ is some function, and

$$S(H) = N\ \mathrm{Tr}\ f(H), \tag{3.65}$$

for some function $f$. (Note that the positioning of the factor of $N$, and thus the combinatoric analysis, is the same as in four-dimensional chromodynamics.)

We want to evaluate (3.64) for large $N$. We begin by writing $H$ in canonical form,

$$H = U^\dagger D U, \tag{3.66}$$

where $D$ is a diagonal matrix, with eigenvalues $\lambda_1 \ldots \lambda_N$. We can rewrite the integration measure in terms of the $\lambda$s and $U$. By gauge invariance, it must be of the form

$$dH = \left( \prod_a d\lambda_a \right) h(\lambda_1 \ldots \lambda_N)\, dU \tag{3.67}$$

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where $dU$ is the invariant measure on U($N$) and $h$ is a function we shall find immediately. To determine $h$, we compare the two sides of Eq. (3.67) in the neighborhood of the identity in U($N$). Here,

$$U = 1 + i\varepsilon, \tag{3.68}$$

where $\varepsilon$ is a Hermitian matrix. At this point, Eq. (3.66) becomes

$$H = D - i[\varepsilon, D], \tag{3.69}$$

or, written out in components,

$$H_{ab} = \lambda_a \delta_{ab} + i\varepsilon_{ab}(\lambda_a - \lambda_b), \tag{3.70}$$

where there is no sum on the repeated indices. Hence,

$$dH = \left(\prod_a d\lambda_a\right) \prod_{a \neq b} i(\lambda_a - \lambda_b) d\varepsilon_{ab}. \tag{3.71}$$

Aside from a possible (but irrelevant) multiplicative constant, the term involving $\varepsilon$ is $dU$. Thus,

$$h = \prod_{a \neq b} (\lambda_a - \lambda_b). \tag{3.72}$$

The integration over U($N$) factors out of Eq. (3.64). Thus we are left with

$$\langle \mathrm{Tr}\, g \rangle = \frac{\int \prod_a d\lambda_a \sum_b g(\lambda_b) e^{-S_{\mathrm{eff}}}}{\int \prod_a d\lambda_a\, e^{-S_{\mathrm{eff}}}}, \tag{3.73}$$

where

$$S_{\mathrm{eff}} = N \sum_a f(\lambda_a) - \sum_{a \neq b} \ln|\lambda_a - \lambda_b|. \tag{3.74}$$

If we count both explicit factors of $N$ and factors of $N$ arising from the number of terms in a sum, we see that both terms in $S_{\mathrm{eff}}$ are $O(N^2)$. This can be made more apparent by introducing

$$\rho(\lambda) \equiv \frac{1}{N} \sum_a \delta(\lambda - \lambda_a). \tag{3.75}$$

The factor of $N$ has been introduced so $\rho$ obeys an $N$-independent normalization condition,

$$\int d\lambda\, \rho(\lambda) = 1. \tag{3.76}$$

The integral of $\rho$ over any interval gives the fraction of the total number of eigenvalues that lie on that interval; $\rho$ is the fractional density of eigenvalues. For any finite $N$, $\rho$ is a spiky sum of delta-functions; however, as

we shall see shortly, it has a continuous limit as $N$ goes to infinity. In terms of $\rho$,

$$S_{\text{eff}} = N^2 \left[ \int d\lambda\, \rho(\lambda) f(\lambda) - \int d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda') \ln |\lambda - \lambda'| \right]. \tag{3.77}$$

All factors of $N$ have now been made explicit, and the character of the large-$N$ limit is now clear. The integral is dominated by the $\rho$ which minimizes $S_{\text{eff}}$; to leading order,

$$\langle \text{Tr}\, g \rangle = N \int d\lambda\, \rho(\lambda) g(\lambda). \tag{3.78}$$

This can be thought of as Tr $g(H)$, where $H$ is a master matrix, a matrix whose density of eigenvalues is given by the minimizing $\rho$. We can find the minimizing $\rho$ by searching for the stationary points of Eq. (3.77),

$$f(\lambda) - 2 \int d\lambda'\, \rho(\lambda') \ln |\lambda - \lambda'| = \text{constant}, \tag{3.79}$$

where the constant is the Lagrange multiplier associated with the constraint equation, (3.76). It is convenient to eliminate the constant by differentiating with respect to $\lambda$. We find

$$f'(\lambda) - 2 \int d\lambda'\, \rho(\lambda') \frac{\text{P}}{\lambda - \lambda'} = 0. \tag{3.80}$$

This is the equation that must be solved to find the master matrix. There is an important technical point: because $\rho$ is restricted to be positive, Eq. (3.80) holds only within the support of $\rho$, the region where $\rho$ is non-zero. The easiest way to see this is to enforce positivity by writing $\rho$ as $\sigma^2$; $\delta\rho$ is then $2\sigma\delta\sigma$, and deriving Eq. (3.80) outside the support of $\rho$ involves an illegitimate division by zero.

I could go on to solve Eq. (3.80) for special choices of $f$, but I would prefer to stop here; if you want more, you can find it in the literature.[12] The point has been made. There is nothing wrong with the general arguments; for zero-dimensional chromodynamics, the master field exists, and we have found an algorithm for constructing it, Eq. (3.80). The unsolved problem is to find the appropriate generalization of Eq. (3.80) to four dimensions.

### 3.6 *Retrospect and prospect*

Where are we?

For mesons, things are wonderful. Only an enthusiast who has spent too much time studying dual resonance models and too little time studying reality would claim that the properties we found in Section 3.2 form an accurate portrait of the mesons. They form a caricature. But it is a recognizable caricature; we look upon it and cry, 'These are the mesons!'

I know of no method other than the $1/N$ expansion in which the lineaments of the mesons emerge so clearly and unambiguously from chromodynamics.

For the baryons, things are not so good. Witten's theory is an analytical triumph but a phenomenological disaster. It is true that in significant ways baryon phenomenology is qualitatively different from meson phenomenology. To take one famous example, duality plus no exotics works wonderfully for mesons, but leads to contradictions for baryons. Baryons are different from mesons, but not as different as they are in the $1/N$ expansion. Baryons are not much heavier than mesons, and baryon resonances are not much broader than meson ones. If our picture of the mesons is a good caricature, our picture of the baryons is a bad one.

There are two possibilities. One is that the $1/N$ expansion is a better approximation for mesons than for baryons. This statement is not as silly as it seems. A reasonable person might agree that a quark and an antiquark is a quark and an antiquark, pretty much the same no matter how many colors there are, while a three-quark baryon is very different from a hundred-quark baryon, no matter how we adjust our coupling constants. The second possibility is that the $1/N$ expansion is terrible for both mesons and baryons. After all, most of our meson phenomenology was derived only from the dominance of graphs with a single quark loop. Although this is certainly a consequence of the $1/N$ expansion, it is not inconceivable that it might also be a consequence of some other principle altogether, and thus be valid even though the $1/N$ expansion is not.

Of course, we would know which of these possibilities is correct, and know much more, if we had an explicit expression for the leading approximation. As Witten has stressed, to seek such an expression is not ridiculously ambitious. One is not searching for a portrait of nature in all its fine shadings, not all phase shifts at all energies, but just for a recognizable caricature of mesodynamics, a table of resonance masses and couplings.

In these lectures I have discussed one attempt to find such an explicit expression. This particular attempt may succeed or fail, but, in any event, I feel future progress in this field rests upon constructing the leading approximation. It is amazing how far we have gone while avoiding this problem, but I do not think we can go much farther without solving it.

## Appendix 1. The Euler characteristic

Given a surface composed of polygons, with $F$ faces, $E$ edges, and $V$ vertices, the Euler characteristic is defined by

$$\chi = F - E + V. \tag{A.1.1}$$

In this appendix I give (very sloppy) proofs of two propositions. (1) The

Euler characteristic is a topological invariant. (2) If our polygonal surface is topologically equivalent to a sphere with $B$ holes cut out of it and $H$ handles stuck on to it, then

$$\chi = 2 - 2H - B. \tag{A.1.2}$$

***Proof of* (1).** One can convince oneself that there are only three fundamental ways one can change a polygonal surface continuously. (i) One can distort the surface without changing either $F$, $E$, or $V$. Of course, this does not change $\chi$. (ii) One can shrink an edge to a point. This eliminates one edge, merges two vertices, causing a net loss of one vertex, and does not change the number of faces. Thus $\chi$ does not change. The same argument applies to the reverse process. (iii) One can shrink a face to a point. If the face is a polygon with $n$ sides, this procedure eliminates one face, $n$ edges, and $n-1$ vertices. Again $\chi$ does not change. The same argument applies to the reverse process. Processes like conversion of a face to an edge can be obtained as (iii) followed by the reverse of (ii) and do not require independent analysis.

***Proof of* (2).** The argument goes in stages. First we prove the equation for a sphere, then for a sphere with holes, and finally for a sphere with holes and handles. (i) A sphere can be constructed by taking two $n$-sided polygons and identifying their perimeters. We thus obtain a surface with $n$ edges, $n$ vertices, and two faces, for which

$$\chi = 2. \tag{A.1.3}$$

(ii) If we have a polygonal surface that is topologically equivalent to a sphere, we cut holes out of it by removing faces. Thus, each hole reduces $\chi$ by one, and

$$\chi = 2 - B. \tag{A.1.4}$$

(iii) To make a handle, we cut two holes that are both $n$-sided polygons (reducing $\chi$ by two) and then identify the perimeters of the two polygons (reducing both $E$ and $V$ by $n$ and not changing $\chi$ at all). Thus,

$$\chi = 2 - 2H - B. \tag{A.1.5}$$

## Appendix 2. The 't Hooft equations

This appendix gives the computations promised in Section 3.3. I will begin with the quark self-energy. We need a preliminary identity:

$$\int \frac{dx}{x \pm i\varepsilon} = \int dx \left[ \frac{P}{x} \mp i\pi\delta(x) \right] = \mp i\pi. \tag{A.2.1}$$

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Hence,

$$\int \frac{dp_+p_-}{2p_+p_- - a + i\varepsilon} = -i\frac{\pi}{2}\,\mathrm{sgn}\,p_-, \tag{A.2.2}$$

and Eq. (3.37) becomes

$$\Sigma = \frac{g^2}{2\pi}\int dk_-\ \mathrm{sgn}(p_- - k_-)\frac{P}{(k_-)^2}$$
$$= -\frac{g^2}{\pi p_-}. \tag{A.2.3}$$

I now turn to the eigenvalue equation for meson masses. If the Green's function of Fig. 12 has a meson pole, then standard arguments lead to the Bethe–Salpeter equation shown in Fig. 15. Here all momenta are oriented to the right, and the shaded blob on the right is the matrix element of the time-ordered product of two quark fields between the vacuum and the meson state. If we denote the Fourier transform of this matrix element by $\psi$, then the Bethe–Salpeter equation is

$$\psi(p, q) = -4g^2 iS(p-q)S(-q)\int \frac{d^2k}{(2\pi)^2}\frac{P}{(k_- - q_-)^2}\psi(p, k). \tag{A.2.4}$$

If we define

$$\phi(p, q_-) = \int dq_+\,\psi(p, q), \tag{A.2.5}$$

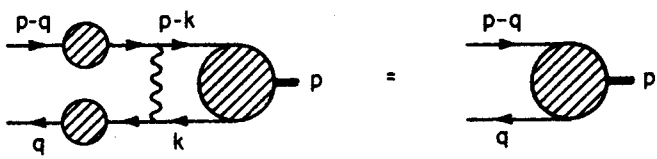
then

$$\phi(p, q_-) = -\frac{ig^2}{\pi^2}\int dq_+\,S(p-q)S(-q)\int dk_-\frac{P}{(k_- - q_-)^2}\phi(p, k_-). \tag{A.2.6}$$

The $q_+$ integral is an integral over known rational functions and can be done explicitly:

$$\int dq_+\,S(p-q)S(-q) \equiv I(p, q_-)$$
$$= \int dq_+\,\frac{1}{2(p+q)_+ - [(M^2 - i\varepsilon)/(p-q)_-]}\,\frac{1}{2q_+ - [(M^2 - i\varepsilon)/q_-]}. \tag{A.2.7}$$

**Fig. 15**



If $q_-$ is outside the interval $[0, p_-]$, the two poles of the integrand are on the same side of the real axis, and the integral vanishes. This implies that $\phi$ vanishes outside this interval. For $q_-$ within the interval, the integral may be done trivially by closing the contour; the result is

$$I = -\pi i/[2p_+ - M^2/q_- - M^2/(p_- - q_-)]. \tag{A.2.8}$$

Equation (A.2.6) thus becomes

$$[2p_+ - M^2/q_- - M^2/(p_- - q_-)]\phi(p, q_-) = -\frac{g^2}{\pi}\int_0^{p_-} dk_- \frac{\mathrm{P}}{(k_- - q_-)^2}\phi(p, k_-). \tag{A.2.9}$$

If we make the substitutions,

$$2p_+ = \mu^2/p_-,\ q_- = xp_-,\ k_- = yp_-, \tag{A.2.10}$$

this becomes Eq. (3.41).

## Appendix 3. U($N$) as an approximation to SU($N$)

In Section 3.1, I dropped the second term in the gluon propagator, Eq. (3.5). To drop this term is to approximate SU($N$) gauge theory by U($N$) gauge theory; to restore this term is to correct the approximation by introducing a correction gluon, a negative-norm color-singlet gauge meson that cancels the positive-norm color-singlet gauge meson of the U($N$) theory. In this appendix I compute the powers of $1/N$ associated with graphs containing internal correction gluons.

As in Section 3, I will restrict myself to vacuum-to-vacuum graphs; from these, all Green's functions can be obtained by functional differentiation. If all we had in our theory were gauge fields, we would have made no error and we would need no corrections; in a U($N$) gauge theory, the U(1) gauge meson is completely decoupled from the SU($N$) gauge mesons. Thus we need only consider graphs where all correction gluons terminate on quarks. Because the correction gluon carries no color indices, we can always imagine constructing such a graph by adding correction gluons to graphs without them. How many powers of $1/N$ do we introduce by this process? For each quark line, we introduce one extra propagator and one extra vertex; the $1/N$ for the propagator cancels the $N$ from the vertex. However, the correction gluon itself carries $1/N$ because it is a propagator and an additional $1/N$ from the explicit factor of $1/N$ in Eq. (3.5). Thus the net effect of adding a correction gluon is $O(1/N^2)$. This is all we need to know, because we already know how to compute powers of $1/N$ for graphs without correction gluons.

As an example, let me compute the power of $1/N$ associated with the leading connected vacuum-to-vacuum graphs containing a correction gluon. We must add the correction gluon to graphs that contain quark loops. The leading graphs are those in which the correction gluon connects two disconnected graphs each of which contains a quark loop; these are $O(N \times N \times 1/N^2) = O(1)$. However, these graphs are trivial; they serve only to exactly cancel the corresponding gluon-exchange graphs in the uncorrected theory. The leading non-trivial graphs are those in which a correction gluon is added to a connected graph containing a quark loop; these are $O(N \times 1/N^2) = O(1/N)$.

## Notes and references

1. To my knowledge, the first to observe that $\phi^4$ theory became simple for large $N$ was K. Wilson, *Phys. Rev.* **D7**, 2911 (1973). Wilson drew on ideas developed in statistical mechanics, especially Stanley's work on the spherical model (H. E. Stanley, *Phys. Rev.* **176**, 718 (1968)). The auxiliary-field method used here was also developed in statistical mechanics (but not in the context of large-$N$ expansions), by R. L. Statonovich, *Doklady Akad. Nauk. S.S.S.R.* **115**, 1097 (1957). The treatment of $\phi^4$ theory given here follows that of S. Coleman, R. Jackiw, and H. D. Politzer, *Phys. Rev.* **D10**, 2491 (1974). Some points left confused in this paper are clarified in L. Abbott, J. Kang, and H. Schnitzer, *Phys. Rev.* **D13**, 2212 (1976).
2. D. Gross and A. Neveu, *Phys. Rev.* **D10**, 3235 (1974).
3. This note is for the cognoscenti only; it is written in shorthand. You may know that you can define an effective potential, much like $V$, in theories with fundamental scalar fields, and that, in such theories, $V$ can be interpreted as an energy density for general arguments, not just stationary points. This is important; for example, it implies that if $V$ is unbounded below the theory is sick, no matter how nicely behaved $V$ is at its stationary points. *There is no such interpretation of the effective potential for composite fields.* For example, in the case at hand, if we add a source term to the Lagrange density,

   $$\mathscr{L} \to \mathscr{L} + J(x)\sigma,$$

   this corresponds to adding a term to the Hamiltonian density,

   $$\mathscr{H} \to \mathscr{H} - J\phi^a\phi^a + \frac{g_0}{2N} J^2.$$

   The $J^2$ term has no analog for a fundamental scalar field, and destroys the standard energy arguments (except at stationary points of $V$, where it vanishes).
4. The model was devised by H. Eichenherr, *Nucl. Phys.* **B146**, 215 (1978) and V. Golo and A. Perelomov, *Phys. Lett.* **79B**, 112 (1978). The $1/N$ expansion is worked out in A. D'Adda, M. Lüscher, and P. Di Vecchia, *Nucl. Phys.* **B146**, 63 (1978), and **B152**, 125 (1979).
5. This is a bit too slick. After all, in electrodynamics, gauge invariance could also be thought of as a mere reflection of the presence of a redundant variable, the longitudinal part of the vector potential. The real difference is slightly more subtle. In electrodynamics, it is not possible to eliminate the redundant variables and still have a local theory, as is shown by the Bohm–Aharonov effect; in the case at hand, it is possible to do just this, as is shown by our

derivation of the model. As we shall see, this distinction will disappear when we sum up the radiative corrections.

6. E. Witten, *Nucl. Phys.* **B149**, 285 (1979); A. Jevicki, *Phys. Rev.* **D20**, 3331 (1979); I. Affleck, *Phys. Lett.* **92B**, 149 (1980); *Nucl. Phys.* **B162**, 461 (1980); **B171**, 420 (1980).
7. (This note covers both this subsection and the next.) The large-$N$ expansion for chromodynamics was invented by G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974). There are numerous parallels and connections with the topological expansions of $S$-matrix theory; see G. Veneziano, *Nucl. Phys.* **B117**, 519 (1976) and G. Chew and C. Rosenzweig, *Phys. Rep.* **41C**, 263 (1978).
8. G. 't Hooft, *Nucl. Phys.* **B75**, 461 (1974). This paper has spawned a large literature, with two branches. (1) Papers which investigate the model in more detail (for example, by computing form factors) and/or use the model to gain insight into four-dimensional chromodynamics. Some examples: C. G. Callan, N. Coote, and D. J. Gross, *Phys. Rev.* **D13**, 1649 (1976); M. B. Einhorn, *Phys. Rev.* **D14**, 3451 (1976); R. Brower, J. Ellis, M. Schmidt, and J. Weis, *Nucl. Phys.* **B128**, 131, 175 (1977). (2) Papers which attempt to clean up 't Hooft's original derivations and/or to find inconsistencies in the model. To my mind, the paper of this kind that does things best and settles all the problems is I. Bars and M. B. Green, *Phys. Rev.* **D17**, 537 (1978).
9. E. Witten, *Nucl. Phys.* **B160**, 57 (1979).
10. For a review of lumps, see Chapter 6 in this volume.
11. The proposal was made in a lecture given at Harvard in the Spring of 1979 E. Witten in *Recent Developments in Gauge Theories*, 1979 Cargese Lectures, edited by G. 't Hooft *et al.* Plenum (1980).
12. E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, *Comm. Math. Phys.* **59**, 35 (1978).