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Renormalization and symmetry: a review for non-specialists
(1971)

1 Introduction

I suppose that as good a way as any of explaining the contents of this lecture is to explain the title. By 'renormalization' I mean the removal of infinities for Feynman amplitudes, in perturbation theory, for Lagrangian field theories with polynomial interactions. In particular non-perturbative renormalization (the work of Jaffe, Glimm, etc.) is outside the scope of this lecture, as are the properties of non-polynomial interactions (the work of Efimov, Salam, Lehmann, etc.). By 'renormalization and symmetry' I mean that we will be concerned not only with the renormalization of scattering amplitudes, but also with the renormalization of the matrix elements of conserved and partially conserved currents. In particular, we will discuss some fairly recent results of Symanzik, Benjamin Lee, Preparata, Weisberger, and others. By 'a review for non-specialists' I mean that I hope that this talk will be intelligible to people who can do nothing more complicated than remove the divergences from the self-energy of the electron.

Since renormalization theory has a well-deserved reputation for complexity, it is obvious that I will be able to do all this in a single lecture only by cheating. To be precise, I will explain a very powerful theorem due to Klaus Hepp, but not prove it (this is the cheat); then I will show how a wide variety of results can be obtained from this master theorem by elementary methods.[1]

2 Bogoliubov's method and Hepp's theorem

For simplicity, we will restrict ourselves to field theories involving spin-zero and spin-one-half fields only, which we will call Bose and Fermi fields, respectively. We will write the Lagrangian for such a theory in the form

$$\mathscr{L} = \mathscr{L}_0 + \sum_i \mathscr{L}_i, \tag{1}$$

where $\mathscr{L}_0$ is a sum of free Lagrangians of standard form, one for each field, and each $\mathscr{L}_i$ is a monomial in the fields and their derivatives. For future use, it will be convenient to establish some notation, and denote by f_i the number of Fermi fields in $\mathscr{L}_i$, by b_i the number of Bose fields, and by d_i the number of derivatives. Thus, for example, the ps-ps meson–nucleon interaction

$$g\bar{\psi}\gamma_5\psi\phi,$$

has $f=2$, $b=1$, and $d=0$, while the ps-pv interaction

$$f\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi,$$

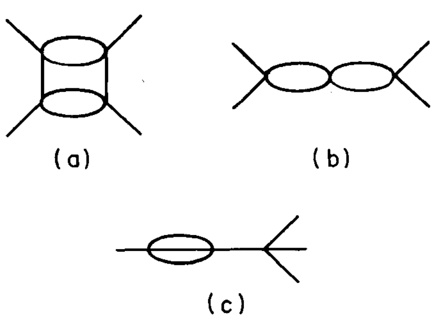
has $f=2$, $b=1$, and $d=1$.

If we attempt to calculate scattering amplitudes with such a Lagrangian, following the conventional Feynman rules, we soon encounter divergent diagrams, that is to say, infinite Feynman integrals. I will assume that we have cut off the theory in some way (say, by modifying the propagators) so that instead of divergent amplitudes we have cutoff-dependent ones. The renormalization procedure of Bogoliubov[2] consists of adding to the Lagrangian extra terms, the so-called renormalization counter-terms, whose function is to cancel the cutoff-dependence of the amplitude. First I will explain how these extra terms are constructed; later I will explain their physical meaning.

To explain the construction, three definitions are needed:

(1) One-particle-irreducible diagrams. A Feynman diagram is said to be one-particle-irreducible (abbreviated IPI) if it is connected and cannot be disconnected by cutting any one internal line. Fig. 1 shows three Feyn-

Fig. 1



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man diagrams in ϕ^4 theory. The first two are IPI; the third is not. (If the horizontal line is cut, the diagram falls into two pieces.)

(2) ***Taylor expansions about the point zero.*** A Feynman amplitude with n external lines is a function of $n-1$ independent four-momenta. Furthermore, if there are no massless particles in the theory (as we shall assume from now on) it is an analytic function of these momenta in some neighbourhood of the point zero, the point where *all* external momenta vanish. Thus, it may be expanded in a Taylor series in these variables. For example, the third-order vertex diagram of ps-ps meson–nucleon theory, shown in Fig. 2, has an expansion of the form

$$\begin{aligned} &a\gamma_5 \\ &+b\gamma_5\gamma_\mu p^\mu + c\gamma_5\gamma_\mu p'^\mu \\ &+d\gamma_5 p^2 + e\gamma_5 p'^2 + f\gamma_5 p\cdot p' \\ &+\cdots, \end{aligned}$$

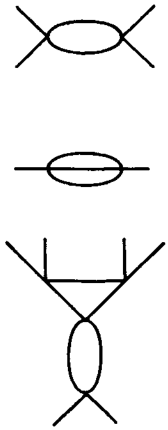
where a, b, c, etc. are constants. The term on the first line is called a term of zeroth order, those on the second line terms of first order, those on the third line terms of second order, etc.

Fig. 2

(3) ***Superficial degree of divergence.*** A Feynman amplitude is, in general, a multiple integral. The superficial degree of divergence of such an integral is the difference between the number of momenta in the numerator of the integral (arising from loop integration variables and from explicit momenta at vertices due to derivative interactions) and the number of momenta in the denominator (arising from propagators). Fig. 3 shows three Feynman diagrams from ϕ^4 theory, with their superficial degrees of divergences (denoted by D). The contribution from numerator and denominator are separately displayed. If $D=0$, we say the diagram is superficially logarithmically divergent, if $D=1$, that it is superficially linearly divergent, etc. If D is less than zero, we say it is superficially convergent.

Fig. 3(c) demonstrates the reason for the pejorative adjective 'superficial'. Although the diagram is superficially convergent, it is in fact divergent; the integration along the lower loop is logarithmically divergent no matter what happens in the rest of the diagram.

Fig. 3



(a) D = 4 − 4 = 0

(b) D = 8 − 6 = 2

(c) D = 8 − 10 = −2

It will be convenient later to have a general expression for the superficial degree of divergence of a connected Feynman diagram. For such a diagram let

B be the number of external boson lines,
IB be the number of internal boson lines,
F be the number of external fermion lines,
IF be the number of internal fermion lines, and
n_i be the number of vertices of the ith type, i.e. those that come from the ith term in the Lagrangian (1).

There is an elementary relation between these numbers. Since a vertex of the ith type has b_i boson line ends attached to it, and since every internal boson line has two ends attached to vertices and every external boson line has one, we can readily deduce that

$$B+2(IB)=\sum n_i b_i,$$

'the law of conservation of boson ends'. By the same reasoning, we can deduce 'the law of conservation of fermion ends',

$$F+2(IF)=\sum n_i f_i.$$

It is also elementary to compute the superficial degree of divergence:

$$D=\sum n_i d_i+2(IB)+3(IF)-4\sum n_i+4.$$

The five terms in this formula have the following origins. (1) Every derivative in an interaction puts a momentum in the numerator of the Feynman integral. (2) Every internal boson line puts four integration momenta in the numerator and two propagator momenta in the denominator. (3) Every internal fermion line puts four integration momenta in the numerator and one in the denominator. (4) Every vertex has a four-dimensional delta-function attached to it, which, upon integration, cancels four integration momenta, (5) except for one delta-function that is left over to give overall four-momentum conservation.

Putting all of this together, we find that

$$D = -B - \tfrac{3}{2}F + 4 + \sum n_i \delta_i, \tag{2}$$

where δ_i, 'the index of divergence of $\mathscr{L}_i$', is given by

$$\delta_i = b_i + \tfrac{3}{2} f_i + d_i - 4. \tag{3}$$

It is worth remarking that, for the cases we are considering,

$$\delta_i = \dim \mathscr{L}_i - 4, \tag{4}$$

where $\dim \mathscr{L}_i$ is the dimension of $\mathscr{L}_i$, in the usual sense of dimensional analysis, in units of mass. (This is, however, special to the theories we are considering; eq. (4) is not true, for example, for the interactions of a vector meson coupled to a non-conserved current.)

This completes our three definitions (plus one long digression). We are now in a position to state the renormalization prescription of Bogoliubov.[2] As advertised, this is an iterative procedure; as we calculate in perturbation theory, to each order we change the Lagrangian, adding to it extra terms. The procedure is as follows:

(1) Calculate in perturbation theory until you encounter an IPI diagram whose superficial degree of divergence, D, is greater than or equal to zero.

(2) Add to the Lagrangian extra terms (the counterterms) chosen to precisely cancel, to this order, all terms in the Taylor expansion of this diagram of order D or less.*

As an example of this procedure, let us consider $\lambda\phi^4$ theory, for which the Lagrangian (1) is

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\mu^2 \phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5}$$

In order λ^2, we encounter the divergent diagrams 3(a) and 3(b). For the first of these, $D=0$, for the second $D=2$. Thus we change the Lagrangian

* Please note that it follows from this and Eqs. (2) and (3) that the counterterms induced have index of divergence, δ, less than or equal to the sum of the indices of divergence of the interactions occurring in the diagram. This observation has been stuck in a footnote because it is not important now, but it will be useful later.

(5) by adding to it extra terms

$$\mathscr{L} \to \mathscr{L} - \frac{A_2}{4!}\phi^4 + \frac{1}{2}B_2\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}C_2\phi^2. \tag{6}$$

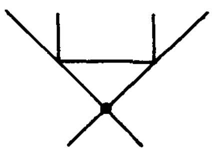
(The subscript 2 is to remind you that these terms are of second order in λ; they are also cutoff-dependent, but that is not important at the moment). The A_2 term is chosen to cancel the zeroth-order term in the Taylor expansion of 3(a); the B_2 and C_2 terms to cancel the zeroth and second order terms in the Taylor expansion of 3(b). (There is no need for a first-order counterterm because Lorentz-invariance forbids a first-order term in the Taylor expansion.)

(3) Continue computing, now using the corrected Lagrangian.

Theorem (Hepp).[3] This procedure eliminates all divergences. That is to say, the resultant perturbation expansion is independent of the cutoff in the limit of infinite cutoff.[4]

For the moment, I would like you to think of this purely as a mathematical theorem about Feynman expansions; we will try to understand its physical significance shortly. However, there is one point I would like to make now – we can already begin to see why it is the superficial degree of divergence, rather than the true degree of divergence, that is important. Remember the order λ^4 diagram 3(c), which has $D = -2$. By our prescription, even though this diagram is in fact divergent, it does not induce a counterterm. We can now see the reason for this: there is another diagram of order λ^4, shown in Fig. 4, where the heavy dot is the A_2 term in eq. (6), the counterterm that was added to the Lagrangian in order λ^2. This diagram *automatically* cancels the divergence of Fig. 3(c). Speaking very roughly, we only need new counterterms at a given order of perturbation theory to take care of new divergences; old divergences, divergences caused by lower-order diagrams hiding inside higher-order ones, as 3(a) is hiding inside 3(c), are taken care of by old counterterms.

Fig. 4



3 Renormalizable and non-renormalizable interactions

Let us look a little more closely at the theory defined by eq. (5). We have already classified the counterterms that arise in order λ^2; what

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happens in an arbitrary order of perturbation theory? Equations (2) and (3) give us the answer; for the special case of a ϕ^4 interaction, eq. (3) becomes

$$\delta=4+0+0-4=0,$$

and eq. (4) becomes

$$D=4-B.$$

Thus the only superficially divergent diagrams are those with B equal to two or four (diagrams with odd numbers of external lines vanish because of the symmetry of the Lagrangian under $\phi\to-\phi$), and their superficial degrees of divergence are the same in a general order as in second order. The only effect of renormalization, to any order, is to change (5) into

$$\mathscr{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}\mu^2\phi^2-\frac{\lambda}{4!}\phi^4-\frac{A}{4!}\phi^4+\frac{B}{2}(\partial_\mu\phi)^2-\frac{1}{2}C\phi^2, \tag{7}$$

where the constants A, B, and C are power series in λ, with coefficients that are, in general, cutoff-dependent. We can now see the physical meaning of the renormalization procedure; for if we define

$$\phi_u=(1+B)^{\frac{1}{2}}\phi, \tag{8a}$$

$$\mu_0^2=(\mu^2+C)(1+B)^{-1}, \tag{8b}$$

$$\lambda_0=(\lambda+A)(1+B)^{-2}, \tag{8c}$$

then we may rewrite the Lagrangian (7) as

$$\mathscr{L}=\frac{1}{2}(\partial_\mu\phi_u)(\partial^\mu\phi_u)-\frac{1}{2}\mu_0^2\phi_u^2-\frac{\lambda_0}{4!}\phi_u^4.$$

This is of the same form as our starting Lagrangian (5), except that the coefficients have been changed. The field ϕ_u is called the unrenormalized field; it obeys canonical commutation relations, but has cutoff-dependent matrix elements (because of the cutoff-dependent quantity B in eq. (8a)). The quantities μ_0 and λ_0 are called the bare mass and bare coupling constants. Thus, for this theory the content of Hepp's theorem is that if we choose the bare mass and coupling constants in an appropriate cutoff-dependent fashion, and rescale the fields in an appropriate cutoff-dependent way, all the divergences disappear, order by order, in perturbation theory. A Lagrangian that has this property is said to be renormalizable.

The field ϕ and the quantities μ and λ are not the renormalized field, mass, and coupling constants as usually defined; this is because they are defined in terms of Green's functions at the point zero, rather than at some astutely chosen mass-shell point. However, they are cutoff-independent parameters that characterize the theory; the usual parameters can be computed in terms of them to any order of perturbation theory, and, if one wishes, these expressions can be inverted in the standard way to obtain a perturbation theory in terms of the usual parameters. For our purposes they are more convenient than the usual parameters because it is easier to do power series expansions about the point zero than about mass-shell points, where one has to worry about possible singularities of Feynman integrals. To distinguish them from the usual parameters we will refer to μ and λ as the intermediate mass and coupling constants; likewise, we will refer to the Green's functions associated with the field ϕ as intermediate Green's functions.

Not all Lagrangians are renormalizable. For example, if we had added a ϕ^5 interaction to our starting Lagrangian, this would have δ equal to one, and the renormalization procedure would inexorably add to the Lagrangian, as we computed higher and higher orders of perturbation theory, higher and higher order monomials in the field and its derivatives. Once such expressions appear in the counterterms, there is no physical reason to exclude them from the starting Lagrangian. (After all, it was only mathematical convenience that made us choose the point zero for our renormalization prescription; if we had chosen a different point (or even a separate point for every Green's function, or different points in different orders of perturbation theory) we would have obtained different values for the coefficients of the counterterms.) Thus we would be led to a theory with an infinite number of parameters. Such theories are called non-renormalizable.

4 Symmetry and symmetry-breaking: Symanzik's rule.

We are now ready to begin pulling interesting results out of Hepp's theorem. I would like to begin by taking the observation made in a footnote to Sect. 2 and raising it to the dignified status of a

Lemma. The counterterms induced by a given Feynman diagram have index of divergence, δ, less than or equal to the sum of the indices of divergence of all the interactions in the diagram.

I would also like to adopt a somewhat more stringent definition of renormalizability than usual: I will call a Lagrangian renormalizable only if all the counterterms induced by the renormalization procedure can be absorbed into a redefinition of the parameters in the Lagrangian. Thus, by this strict test, the theory of a single nucleon field interacting with a single pseudoscalar meson field through Yukawa coupling, $\bar{\psi}\gamma_5\psi\phi$, is *not* renormalizable, because renormalization induces a ϕ^4 counterterm, not present in the original Lagrangian. However, the same theory, *with* a ϕ^4 interaction in the original Lagrangian, *is* renormalizable, because now all the counterterms are of the same form as terms originally present.

With this definition, we can now state our:

First result. Given a set of spin-zero and spin-one-half fields, the most general Lagrangian constructed from this set containing all terms with δ less than or equal to zero (equivalently, with dimension less than or equal to four) is renormalizable.

This is a trivial consequence of the Lemma.

Second result. If we restrict the Lagrangians defined in the first result to only contain parity-conserving terms, they are still renormalizable. Likewise, if we restrict them to preserve some internal symmetry, such as isotopic spin, they are still renormalizable.

This is also trivial. Unless we have been so stupid as to introduce parity violation into our cutoff procedure, Feynman diagrams computed from a parity-conserving Lagrangian will be parity-conserving. Thus, they will have no parity-violating terms in their Taylor expansions about the point zero, and hence no parity-violating counterterms will be induced by the renormalization procedure. Ditto for internal symmetries. (In fact, ditto for chiral symmetries, such as those of the σ-model, although here one must be more clever than usual to construct a cutoff procedure that does not break the symmetry.)[5]

Third result. (Symanzik's rule for symmetry-breaking):[6] If we generalize the preceding set of Lagrangians to include symmetry-breaking terms, but only with dimensions less than or equal to n, where n is either 3, 2, or 1, they are still renormalizable.

Although this is our first 'new' (1970) result, it is also trivial.[7] The symmetric terms in the Lagrangian have $\delta \leqslant 0$; the symmetry-breaking terms have $\delta \leqslant n-4 < 0$. A symmetry-breaking counterterm can arise only from a diagram that involves at least one symmetry-breaking interaction. By the Lemma, this must also have $\delta \leqslant n-4$.

Thus, for example, if, in the standard isospin-symmetric theory of pions and nucleons, we choose to break isospin only by giving the charged and neutral pions different masses, then renormalization will not force us to change our intention and also introduce symmetry-breaking Yukawa couplings. Remember, though, that we are speaking here of the intermediate coupling constants. The physical renormalized coupling constants *do* display the effects of symmetry-breaking; the new terms we have added to the Lagrangian *do* affect the three-particle Green's functions. It is just that these effects are not divergent, and hence do not require counterterms. If we look at the equations that define the bare masses and coupling constants, discussed in the preceding section, we see that another way of stating this result is to say that the constraint that the internal symmetry be broken only by the bare masses, while the bare coupling constants remain symmetric, does not introduce any divergences. (Unfortunately it is the opposite case – equal bare masses but asymmetric coupling (to electromagnetism) – that is of greatest physical interest, for this is the problem of the electromagnetic mass differences within isotopic multiplets. Alas, we have to go, one way or another, beyond conventional renormalized field theory, to solve this problem.)

The most important special case of Symanzik's rule is the renormalization of the outstanding example of a Lagrangian field theory obeying PCAC, the σ model. This can be characterized as the theory of the interactions of pions, sigmas (scalar isoscalar mesons) and nucleons, such that the chiral symmetry group $SU_2 \times SU_2$ is broken only by terms of dimension one (i.e., linear in the σ field). Symanzik's rule then immediately says that this model is renormalizable.[5]

5 Symmetry and symmetry-breaking: currents

Field theories with internal symmetries have the famous feature of possessing conserved currents, and frequently the matrix elements of these currents are objects of great physical interest (e.g. electromagnetic form factors). These currents are typically bilinear forms in *unrenormalized* fields and their derivatives. Thus, one would naively expect them to be doubly divergent – divergent because the unrenormalized fields are themselves divergent, and divergent also because we are bringing two fields together at the same space-time point. Thus the following result is as surprising as it is beautiful:

Fourth result. In a renormalizable field theory with internal symmetry, the matrix elements of the conserved currents associated with the symmetry are cutoff-independent in every order of perturbation theory.[8]

To prove this result we shall need two pieces of information. Firstly, we need to know how to compute the Green's functions for one current and a string of fields in a Lagrangian way, so we can apply Hepp's theorem, which is about Lagrangians. Fortunately, there is a standard trick for doing this: let $\mathbf{j}_\mu$ be the current, and let $\mathscr{I}_\mu(x)$ be an arbitrary c-number function of space and time. Change the Lagrangian of the theory by adding to it an extra term:

$$\mathscr{L} \to \mathscr{L} + \mathscr{I}_\mu(x)\mathbf{j}^\mu. \tag{9}$$

Compute all Green's functions to first order in the added term, and then functionally differentiate with respect to $\mathscr{I}_\mu$. The result is the Green's function with a current inserted.

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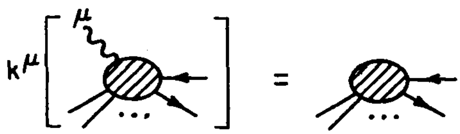
Secondly, we need the Ward identities for conserved currents. I have discussed these in some detail in my other lectures at this school. Here we need the Ward identities only in the somewhat sketchy form depicted in Fig. 5. The blob on the left is a Green's function for one current and a string of Bose and Fermi fields, represented by the solid lines, without and with arrows. The current is represented by a wiggly line; it carries momentum k, and vector index μ. The right-hand side of the Ward identity is some linear combination of Green's functions without a current, represented by the blob on the right.

Fig. 5



A crucial property of this equation is that it involves the same number of fields on the right as on the left, and is therefore true for any normalization of the fields that respects the internal symmetry. In particular, it is true for the intermediate fields we have been using, and therefore, for this choice of fields, the right-hand side of the Ward identity is independent of the cutoff in the limit of large cutoff. This is the essential fact we will use in the sequel.

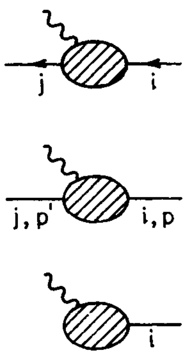
Now let us begin counting divergences. Let us, in the manner of eq. (9), add to the Lagrangian an extra term

$$\mathscr{L} \to \mathscr{L} + \mathscr{J}^{(x)}_\mu \Big(\sum_{i,j} \alpha_{ij} \bar{\psi}_i \gamma^\mu \psi_j + \sum_{i,j} \beta_{ij} \bar{\psi}_i \gamma^\mu \gamma^5 \psi_j + \sum_{i,j} \gamma_{ij} \phi_i \partial^\mu \phi_j + \sum \varepsilon_i \partial^\mu \phi_i \Big), \tag{10}$$

where the αs, βs, γs, and εs are numerical coefficients, and the sums run over all the Fermi (or Bose) fields in the theory.[9] The interactions that give us the Green's functions for the conserved currents are certainly of this form, with special choices for the numerical coefficients. However, for the moment, let us consider a general interaction of the form (10), without asking whether or not it is associated with a conserved current. Now let us follow the renormalization procedure for this new interaction (but only going to first order in $\mathscr{J}_\mu$). Since (10) is the most general Lorentz-covariant interaction linear in $\mathscr{J}_\mu$ and of dimension three or less, the counterterms induced will also be of the form (10). That is to say, starting with any interaction of the form (10), we can obtain a cutoff-independent interaction, (i.e. one that leads to cutoff-independent Green's functions; the actual numerical coefficients in the interaction will be, of course, cutoff-dependent) order by order in renormalized perturbation theory, by appropriately adjusting the numerical coefficients.

We can choose a certain subset of these interactions – say, those that are generated by starting with interactions (10) for which all but one of the numerical coefficients vanishes – as a linearly independent set. Then any interaction of the form (10) is a linear combination of these with some coefficients. The value of these coefficients is completely determined by certain terms in the Taylor expansions of certain Green's functions about the point zero. The relevant Green's functions, and their expansions, are shown in Fig. 6, where the latin letters label the fields. The one-to-one correspondence between these coefficients and the terms in (10) is evident.

Fig. 6



$$= a_{ij}\gamma_\mu + b_{ij}\gamma_\mu\gamma_5 + \cdots$$

$$= c_{ij}\,p_\mu - c_{ji}\,p'_\mu + \cdots$$
$$(k = p' - p)$$

$$= e_i\,k_\mu + \cdots$$

If, for the particular case of the Green's functions of a conserved current, we can show that these expansion coefficients are cutoff-independent, we will have shown that these Green's functions are linear combinations of cutoff-independent Green's functions with cutoff-independent coefficients, and we will have the desired result. But this is just where the Ward identities come in, for they tell us that k^μ dotted into the expansions shown in Fig. 6 must be cutoff-independent, and it is trivial to check that this is enough to tell us that the coefficients themselves are cutoff-independent.

(Please note that if we had had to go to higher orders in the Taylor expansion, the Ward identities would not have been sufficient. For example, they tell us nothing about the coefficient of the following term which can occur in the expansion of the second line of Fig. 6:

$$p_\mu k^2 - k_\mu p\cdot k,$$

because k^μ dotted into this expression vanishes. We need the divergence-

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counting of the renormalization procedure to tell us that all possible divergences are controlled if we can control only a few terms in the Taylor expansion. Only then can we use the Ward identities to control those terms.)

This completes the proof of our fourth result.

It is now fairly trivial to get a generalization.

Fifth result. The matrix elements of internal symmetry currents are cutoff-independent even if the symmetry is broken, provided: (1) it is broken in the manner described in the third result, that is to say, by terms of dimension three or less; and (2) the theory possesses no Bose fields with the same internal-symmetry transformation properties as the symmetry-breaking terms of dimension three. (This is a slight generalization of a result of Preparata and Weisberger.[10]

Here we proceed just as we did when establishing the third result. We treat the symmetry-breaking as a perturbation, and ask if it can introduce new divergences into current Green's functions – that is to say, whether it can induce new counterterms in the interaction (10). Since the symmetry-breaking has $\delta \leqslant -1$, and since (10) has dimension three or less, these new counterterms, if they exist, must be of dimension two or less. Thus, they must be proportional to the gradient of a Bose field. But such terms are excluded by hypothesis (2) above; they have the wrong internal-symmetry transformation properties.

Notes and references

1. Sometimes I will make further cheats. I will warn you about them in notes like this.
2. N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (Interscience, 1959), especially Chapter IV and references contained therein.
3. K. Hepp: *Comm. Math. Phys.* **1**, 95 (1965).
4. A cheat: we will treat this theorem as if it had been proved for general cutoff procedures; in fact it has been proved only for a restricted class of cutoffs.
5. Cheating again! Here I am blatantly ignoring the fact that the σ-model displays the Goldstone phenomenon, and that we are, therefore, not perturbing about the solution with manifest symmetry, but the one with a Goldstone boson. This cheat is not so bad, though. What we are really interested in is whether the counterterms spoil the Ward identities of chiral symmetry; these are independent of whether we are in the manifest-symmetry mode or in the Goldstone mode. See B. W. Lee, *Nucl. Phys.* **B9**, 649 (1969).
6. K. Symanzik in *Fundamental Interactions at High-Energies*, ed. by A. Perlmutter *et al.* (Gordon and Breach, 1970)
7. After it was done first by Symanzik.
8. Remember, we are discussing theories without vector mesons. The result is not true if the theory contains vector mesons with the same quantum numbers as the conserved currents, as does quantum electrodynamics.

9. As the γ_5 may indicate, these arguments work for chiral symmetries as well as for internal symmetries in the more usual sense. This may disturb those of you who know that the Ward identities for chiral theories sometimes contain anomalies, but don't worry – those with only one current have no anomalies, and those are the only ones we are using.

10. G. Preparata and W. Weisberger: *Phys. Rev.* **175**, 1973 (1968).