Spatial Statistics Exercises Lecture 1

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Exercise 2

Consider a binomial process on a Borel set $S \subseteq \mathbb{R}^d$ with n points, where each point follows a pdf f.
a Show that the intensity measure is given by

$$\mu(B) = n \int_B f(x)\mathrm{d}x$$

(a) Intensity Measure $\mu(B)$

The intensity measure $\mu(B)$ of a point process represents the expected number of points in a Borel set $B \subseteq S$:

$$\mu(B) = \mathbb{E}[N(B)],$$

where $N(B)$ is the number of points in B.

Step 1: Probability of a Single Point Being in B Each point in the binomial process is independently drawn from S with pdf $f(x)$. The probability that a single point X_i falls inside B is:

$$P(X_i \in B) = \int_B f(x)dx.$$

Step 2: Expected Number of Points in B Since the binomial process consists of n independent points, the number of points in B follows a binomial distribution:

$$N(B) \sim \text{Bin}(n, p), \quad \text{where } p = \int_B f(x)dx.$$

The expectation of a binomial random variable is:

$$\mathbb{E}[N(B)] = np = n \int_B f(x)dx.$$

Thus, the intensity measure is:

$$\mu(B) = n \int_B f(x)dx.$$

b Show that the void probability is given by

$$v(B) = \left(1 - \int_B f(x)\mathrm{d}x\right)^n$$

for any bounded Borel set $S \subseteq \mathbb{R}^d$.

(b) Void Probability $v(B)$

The void probability $v(B)$ is the probability that no points fall inside the Borel set B:

$$v(B) = P(N(B) = 0).$$

Step 1: Using the Binomial Distribution Since $N(B) \sim \mathrm{Bin}(n, p)$, the probability of observing zero points in B is given by the binomial probability mass function:

$$P(N(B) = 0) = \binom{n}{0} p^0 (1-p)^n = (1-p)^n.$$

Substituting $p = \int_B f(x)dx$, we obtain:

$$v(B) = \left(1 - \int_B f(x)dx\right)^n.$$

Exercise 3

Why is it impossible for the binomial process to be stationary *hint: think about if the distribution of a point in the process then would be welldefined?*

A process is stationary if its statistical properties remain the same when shifted spatially. For example, if you move the entire point pattern a certain distance in any direction, the distribution of points should look the same.

However, in a binomial process, the points are defined over a fixed and finite region. Thus, if we shift the region, points may move outside the original boundary, altering the statistical properties of the process. Which is the reason why the binomial process is not stationary.

4. Which of the following examples lead to the binomial process being isotropic:

(a) $f(x) = \frac{1}{|B(0,1)|}$, where $S = B(0,1)$ is a unit ball in $\mathbb{R}^d$ with the center at the origin and $|B(0,1)| = \int_{B(0,1)} du$.

(b) $f(x) = a^{-d}$ and S is a d-dimensional box centered at the origin with side lengths $a > 0$.

(c) $f(x)$ is the density of a d-dimensional normal distribution with mean vector 0 and covariance matrix $\sigma^2 I$, where I is the identity matrix and $\sigma > 0$, and where $S = \mathbb{R}^d$.

5. A small exercise in measures:

(a) Show that the intensity measure given by

$$\mu(B) = \int_B \rho(x)\, dx$$

for a non-negative function ρ and $B \in \mathcal{B}$ is indeed a measure (more precisely, ρ needs to be a so-called Borel function, which means that $\rho^{-1}(I)$ is a Borel set for any bounded interval I, but we don't care about such measure theoretical details in this course).

(b) On the other hand, argue why the void probability $v(B)$ for $B \in \mathcal{B}_0$ is not a measure on S. For instance, give a counter-example (hint: consider a binomial process).

6. A creative exercise:

(a) Come up with your own example of a data type which is a point pattern.

(b) Would you expect it to be clustered, regular, neither, or maybe some combination of the two?

(c) Would you expect it to be homogeneous or inhomogeneous?

(d) Can you come up with some marks that could be associated with the point pattern?

(e) Are there any covariates which might influence the point pattern?