Spatial Statistics Exercises Lecture 2

Tanja Bugajski

April 3, 2025

Exercise 1

Consider a Poisson process X, on the unit square, with intensity function $\rho(x, y) = 400y$. Calculate the mean number of points.
Which of the following point patterns could reasonably be a simulation of X. Which Poisson processes could the others be coming from?

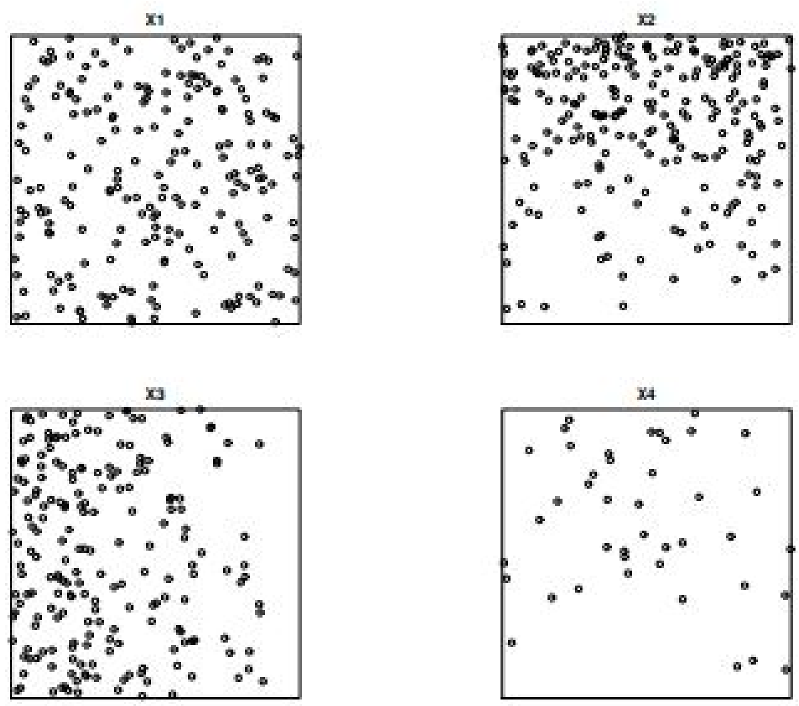

Figure 1: Examples for Exercises.

(1)

The mean number of points, is found by integrating the intensity function over the region $[0,1] \times [0,1]$:

$$\mu = \int_0^1 \int_0^1 \rho(x,y) \mathrm{d}x \mathrm{d}y,$$

Insert:

$$\mu = \int_0^1 \int_0^1 400y \mathrm{d}x \mathrm{d}y =$$

Integrate:

$$\mu = \int_0^1 \int_0^1 400y \mathrm{d}x \mathrm{d}y = \int_0^1 400y \mathrm{d}y = 400 \int_0^1 y \mathrm{d}y = 400 \left[1/2y^2\right]_0^1 = 200.$$

Thus, the mean number of points is 200.

A reasonable simulation would have a higher concentration of points near the top of the unit square and fewer points near the bottom, reflecting the increasing intensity function $\rho(x,y) = 400y$. Thus, the answer is $x2$.

For the other plots:

- $x1$ is a uniform Poission process, meaning that the intensity function is constant everywhere i.e.,

$$\rho(x,y) = \lambda \quad \text{for } \lambda > 0.$$

- $x3$ is concentrated to the left, thus $\rho(x,y) = 400(1-x)$.
- $x4$ is concentrated around the top, but with fewer points, thus $\rho(x,y) = 1/4 \cdot 400y$.

Exercise 2

An insurance agent is trying to estimate the number of car accidents paid by the insurance company during a given year. Let $t \in [0,12)$ denote time during the year and assume an inhomogeneous Poisson process of car accidents on this interval with intensity function

$$\rho(t) = \begin{cases} \alpha, & \text{if } t < 3 \text{ or } t \geq 10, \\ \beta, & \text{if } 3 \leq t < 10, \end{cases}$$

where $\alpha > \beta$ due to the possibility of slippery roads.

(a) Using this model, find the mean number of car accidents in the Spring ($t \in [2,5)$), if $\alpha = 20$ and $\beta = 10$.

(b) What is the probability that no accidents occur in December ($t \in [11, 12)$)?

(c) Make an algorithm for simulating this Poisson process (if you like to, you can implement this in R or some other software, but you can also just specify the algorithm on paper).

(d) Do you find the model realistic, or do you have suggestions for improving the model?

(a)

We find the mean number by

$$\mathbb{E}[N(B)] = \int_B \rho(t)\mathrm{d}t$$

Thus,

$$\mathbb{E}[N([2,5)])] = \int_2^3 \alpha \mathrm{d}t + \int_4^5 \beta \mathrm{d}t = 20\int_2^3 \mathrm{d}t + 10\int_4^5 \mathrm{d}t = 20(3-2) + 10(5-3) = 40$$

(b)

For a Poisson process, the probability of zero events in an interval B is given by the void probability formula:

$$P(N(B = 0)) = \exp(-\mu(B)) = \exp(-\int_B \rho(t)\mathrm{d}t).$$

Fro Dec. we have $t \in [11, 12)$ and $\rho(t) = \alpha = 20$:

$$\mu([11, 12)]) = -\int_{11}^{12} 20\mathrm{d}t = 20$$

Thus;

$$\exp(-20) \approx 2.06 \times 10^{-9},$$

close to zero.

(C)

Algorithm for simulating this Poisson process: We simulate an inhomogeneous Poisson process on the interval $[0, 12)$ with intensity function

$$\rho(t) = \begin{cases} \alpha, & t < 3 \text{ or } t \geq 10, \\ \beta, & 3 \leq t < 10. \end{cases}$$

where $\alpha > \beta$ due to the possibility of slippery roads.

Steps:

1. Set $t = 0$ and initialise an empty list for event times.
2. Define $\rho_{\max} = \alpha$ as the upper bound for the thinning algorithm.
3. **While** $t < 12$:
   (a) Generate an interarrival time $U \sim \text{Exp}(\rho_{\max})$ and set $t \leftarrow t + U$.
   (b) If $t \geq 12$, stop.
   (c) Generate a uniform random variable $V \sim U(0, 1)$.
   (d) Compute the acceptance probability:
   $$p(t) = \frac{\rho(t)}{\rho_{\max}}.$$
   (e) If $V \leq p(t)$, accept t as an event time.
4. Return the list of accepted event times.

(d)

Pros:

- The model captures seasonality, with higher accident rates during the winter months.
- It is simple and computationally efficient.

Cons and Possible Improvements:

- **No daily variations:** Accidents may follow daily traffic patterns, such as peak hours. A time-dependent function $\rho(t)$ capturing rush hours could improve realism.
- **No spatial effects:** Accidents may be clustered in specific areas, such as highways or intersections. A spatial Poisson process could improve accuracy.
- **No weather dependence:** If real data were available, $\rho(t)$ could be modified based on actual weather conditions, such as snowstorms.
- **Possible model refinement:** A more sophisticated model might define $\rho(t)$ as a smooth function, for example, using a spline or sinusoidal variation, instead of piecewise constants.