Diagonalisation of Matrices

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Diagonalisation of Matrices (1)

- Suppose that a linear operator $\mathcal{A}$ is represented in some basis $\mathbf{e}_i, i = 1, \ldots, N$ by the matrix A. Consider a new basis $\mathbf{x}^j$ which is given as follows:

$$\mathbf{x}^j = \sum_{i=1}^{N} S_{ij}\mathbf{e}_i$$

  where we select the $\mathbf{x}^j$ to be the eigen vectors of the linear operators $\mathcal{A}$ i-e

$$\mathcal{A}\mathbf{x}^j = \lambda_j \mathbf{x}^j$$

- In the new basis, $\mathcal{A}$ is represented by the matrix $A' = S^{-1}AS$, where the columns of S are the eigen vectors of the matrix A. In other words

$$S_{ij} = (x^j)_i$$

- Therefore, we can write the matrix A' as follows:

$$(S^{-1}AS)_{ij} = \sum_k \sum_l (S^{-1})_{ik} A_{kl} S_{lj}$$

Diagonalisation of Matrices (2)

- This implies that

$$\Rightarrow \; (S^{-1}AS)_{ij} = \sum_k \sum_l (S^{-1})_{ik} A_{kl} S_{lj} = \sum_k \sum_l (S^{-1})_{ik} A_{kl} (x^j)_l$$

$$= \sum_k (S^{-1})_{ik} \lambda_j (x^j)_k = \sum_k \lambda_j (S^{-1})_{ik} S_{kj} = \lambda_j \delta_{ij}$$

- Therefore, the matrix A' is diagonal with the eigen values of $\mathcal{A}$ as the diagonal elements.

$$A' = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix}$$

- Note that given a matrix A, if S is constructed such that the eigen vectors of A forms its columns then $A' = S^{-1}AS$ is diagonal and has eigen values of A on its diagonal.

Diagonalisation of Matrices (3)

- Since S is non-singular so the N eigen vectors of A must be linearly independent and form a basis for the N-dimensional vector space.
- Note that any matrix with distinct eigen values can be diagonalised by this procedure.
- In general if a square matrix has degenerate eigen values then it may or may not have N independent eigen vectors. If its does not have N independent eigen vectors then it cannot be diagonalized.
- As discussed previously for normal matrices the N eigen vectors are linear independent. Also, when normalised these eigen vectors form an orthonormal set. Hence the matrix S with these eigen vectors is unitary i-e $S^{-1} = S^{\dagger}$. For this case A can be diagonalised by

$$A' = S^{-1}AS = S^{\dagger}AS$$

- So any normal matrix can be diagonalised by a similarity transformation using a unitary transformation matrix S.

Diagonalisation of Matrices (4)

- **Example:** Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

- **Solution:** Since the matrix is symmetric and hence normal so we can diagonalise it by a transformation of the form $A' = S^\dagger AS$. We already know the eigen vectors of the matrix A from the lecture on eigen values and eigen vectors. Therefore, we get:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Note that the three eigen vectors are linearly independent.

$$S^\dagger AS = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

- **Solution:**
$$S^{\dagger}AS = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
which is a diagonal matrix which has the eigen values of A on its diagonal.
- If a matrix A is diagonalised by a similarity transformation $A' = S^{-1}AS$ such that $A' = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ then we can write:
$$\text{Tr}(A') = \text{Tr}(A) = \sum_{i=1}^{N} \lambda_i$$
$$\det(A') = \det(A) = \prod_{i=1}^{N} \lambda_i$$
- These results can be used to prove the trace formula:
$$\det\big(\exp(A)\big) = \exp\big(\text{Tr}(A)\big)$$