Linear Equations with Constant Coefficients

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Course: ENPM 667 (Control of Robotic Systems)

Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Higher-order Ordinary Differential Equations (1)

- It is an empirical fact that when put into mathematical form many natural processes appear as higher-order linear ODEs especially second-order.
- It should be noted that a linear ODE of general order n has the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

- If $f(x) = 0$ then the equation is called homogeneous otherwise it is called inhomogeneous.
- The general solution to this equation will contain n arbitrary constants which maybe determined if n boundary conditions are also provided.
- In order to solve any equation of the form given above we must first find the solution of the equation obtained by setting $f(x) = 0$. This is called the complementary equation and is given by:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Higher-order Ordinary Differential Equations (2)

- To determine the general solution of the complementary equation we must find n linearly independent functions that satisfy it.
- The general solution of the complementary equation is given by a linear superposition of these n functions. If the n solutions are $y_1(x), y_2(x), \cdots, y_n(x)$ then the general solution is given by the linear superposition

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

  where $c_1, c_2, \cdots, c_n$ are arbitrary constants that may be determined if n boundary conditions are provided.
- The linear combination $y_c(x)$ is called the complementary function of the linear ODE of general order n.
- If the original equation has $f(x) = 0$ then the complementary function $y_c(x)$ is already the general solution.

Higher-order Ordinary Differential Equations (3)

- If the equation has $f(x) \neq 0$ then $y_c(x)$ is only part of the solution. The general solution is then given by:

$$y(x) = y_c(x) + y_p(x)$$

  where $y_p(x)$ is the particular integral which is any function, linearly independent of $y_c(x)$, that satisfies the ODE of order n directly.

- It should be noted that any such function $y_p(x)$ is equally valid in forming the general solution to the ODE.

- It is important to note that this method for finding the general solution to an ODE by superposing particular solutions assumes that the ODE is linear. For non-linear equations this method cannot be used.

Linear Equations with Constant Coefficients (1)

- If the $a_1, \cdots, a_n$ in the linear ODE of general order n are constants rather than functions of x then we have

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

- Equations like this find wide application in practical problems. The method of their solution falls into two parts i-e (i) finding the complementary function $y_c(x)$, (ii) and finding $y_p(x)$.
- If $f(x) = 0$ then we do not have to find $y_p(x)$ and the complementary function itself is the general solution.
- We know that $y_c(x)$ is given by:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

where the individual solutions are linearly independent. But how do we determine linear independence.

Linear Equations with Constant Coefficients (2)

- For n functions to be linearly indepedent over an interval, there must not exist any set of constants $c_1, c_2, \cdots, c_n$ such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

over that interval except the trivial solution $c_i = 0, i = 1, ..., n$

- An alternative way to determine linear independence is given by the Wronskian. The n functions $y_1(x), y_2(x), \cdots, y_n(x)$ are linearly independent over an interval if

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & \ldots & \ldots & y_n^{n-1} \end{vmatrix} \neq 0$$

over that interval. $W(y_1, \ldots, y_n)$ is called the Wronskian for the set of functions.

Linear Equations with Constant Coefficients (3)

- The standard method to finding $y_c(x)$ is to try a solution of the form $y = Ae^{\lambda x}$. Substituting this into the equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

After dividing the resulting equation by $Ae^{\lambda x}$ we are left with a polynomial equation in λ of order n.

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

which is called the auxiliary equation.

- The auxiliary equation has n roots $\lambda_1, \cdots, \lambda_n$. In some of these cases the roots maybe repeated or complex. There are three main cases.

(i) *All roots real and distinct:* In this case the n solutions are $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$. By using the Wronskian it can be shown that if the roots are distinct then these solutions are linearly independent.

Linear Equations with Constant Coefficients (4)

(i) *All roots real and distinct:* We can therefore linearly superpose the solutions $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ to form the complementary function

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$$

(ii) *Some roots complex:* For the special case that all the coefficients $a_1, \ldots, a_n$ are real if one of the roots of the auxiliary equation is complex say $\alpha + i\beta$ then its complex conjugate $\alpha - i\beta$ is also a root. In this case:

$$c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\big(d_1 \cos(\beta x) + d_2 \sin(\beta x)\big)$$

$$= Ae^{\alpha x} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\beta x + \phi)$$

where A and ϕ are arbitrary constants.

Linear Equations with Constant Coefficients (5)

(iii) *Some roots repeated:* If for example λ_1 occurs k times as a root of the auxiliary equation then we do not have n linearly independent solutions. We must find another $k-1$ solutions that are linearly independent of those already found and also of each other. By direct substitution into the complementary equation we find that

$$xe^{\lambda_1 x}, x^2 e^{\lambda_1 x}, \ldots, x^{k-1} e^{\lambda_1 x}$$

are also solutions and by using the Wronskian can be shown to satisfy our linear independence requirements. Therefore, we get:

$$y_c(x) = (c_1 + c_2 x + \ldots + c_k x^{k-1}) e^{\lambda_1 x} + c_{k+1} e^{\lambda_{k+1} x} + \\ + \; c_{k+2} e^{\lambda_{k+2} x} + \ldots + c_n e^{\lambda_n x}$$

If more than one root is repeated the above argument is easily extended. Suppose that λ_1 is a k times repeated root and that λ_2 is l times repeated.

(iii) *Some roots repeated:* The complementary function is then given by

$$y_c(x) = (c_1 + c_2 x + \ldots + c_k x^{k-1})e^{\lambda_1 x} + (c_{k+1} + c_{k+2} x + \ldots +$$

$$+ c_{k+l} x^{l-1})e^{\lambda_2 x} \; + \; c_{k+l+1} e^{\lambda_{k+l+1}} + \ldots + c_n e^{\lambda_n x}$$

- There is no generally applicable method for finding the particular integral $y_p(x)$ but for linear equations with constant coefficients $y_p(x)$ can often be found by inspection or by assuming a parameterised form similar to $f(x)$. The latter method is called the **Method of Undetermined Coefficients**

- If $f(x)$ contains only polynomial, exponential, sine or cosine terms then we consider a trial function for $y_p(x)$ of similar form which contains some undetermined parameters. This step is followed by substituting this trial function into the ODE to determine the parameters and hence deduce $y_p(x)$. We present some standard trial functions.

Linear Equations with Constant Coefficients (7)

(i) If $f(x) = ae^{rx}$ then try

$$y_p(x) = be^{rx}$$

(ii) If $f(x) = a_1 \sin(rx) + a_2 \cos(rx)$ (a_1 or a_2 may be zero) then try

$$y_p(x) = b_1 \sin(rx) + b_2 \cos(rx)$$

(iii) If $f(x) = a_0 + a_1 x + \cdots + a_N x^N$ (some a_m maybe zero) then try

$$y_p(x) = b_0 + b_1 x + \cdots + b_N x^N$$

(iv) If $f(x)$ is the sum or product of any of the above then try $y_p(x)$ as the sum or product of the corresponding individual trial functions.

Linear Equations with Constant Coefficients (8)

- It should be noted that this method fails if any term in the assumed trial function is also contained within the complimentary function $y_c(x)$. In such a case the trial function should be multiplied by the smallest integer power of x such that it will then contain no term that already appears in the complimentary function.
- Three further methods that are useful in finding the particular integral $y_p(x)$ are based on Green's functions, the variations of parameters, and a change in the dependent variable using knowledge of the complimentary function. These are discussed in the next Module.
- It should be noted that the general solution is contructed by adding the complementary function and any particular integral.

$$y(x) = y_c(x) + y_p(x)$$

- **Example:** Solve the equation

$$\frac{d^2y}{dx^2} + 4y = x^2 \sin(2x)$$

Now setting the RHS equal to zero we can express the following auxiliary equation:

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda^2 = \pm 2i$$

Therefore, the complementary function $y_c(x)$ is given by:

$$y_c(x) = d_1 \cos(2x) + d_2 \sin(2x)$$

We now compute the particular solution $y_p(x)$. Considering the list of trial functions discussed in previous slides we find that a first guess at a suitable trial function should be

$$(ax^2 + bx + c)\sin(2x) + (dx^2 + ex + f)\cos(2x)$$

Linear Equations with Constant Coefficients (10)

However, we observe that the trial function contains terms in $\sin(2x)$ and $\cos(2x)$ both of which already appear in the complementary function $y_c(x)$. We therefore multiply the trial function by the smallest integer power of x such that none of the resulting term appears in $y_c(x)$. Consider the trial function

$$(ax^3 + bx^2 + cx)\sin(2x) + (dx^3 + ex^2 + fx)\cos(2x)$$

Substituting this into the ODE we get:

$$\begin{aligned}\Big[-12dx^2 + (6a-8e)x + (2b-4f)\Big]\sin(2x) + \Big[12ax^2 + (6d+8b)x \\ + (2e+4c)\Big]\cos(2x) \quad &= \quad x^2\sin(2x)\end{aligned}$$

Solving we get:

$$y_p(x) = -\frac{x^3}{12}\cos(2x) \;+\; \frac{x^2}{16}\sin(2x) \;+\; \frac{x}{32}\cos(2x)$$

Linear Equations with Constant Coefficients (11)

- The general solution is given by:

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= d_1 \cos(2x) + d_2 \sin(2x) - \frac{x^3}{12}\cos(2x) + \frac{x^2}{16}\sin(2x) + \frac{x}{32}\cos(2x) \end{aligned}$$