Simultaneous Linear Equations

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Basics (1)

- Consider the situation where we have M equations in N unknowns, $x_1, ..., x_N$, of the form:

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \ldots + A_{1N}x_N &= b_1, \\
A_{21}x_1 + A_{22}x_2 + \ldots + A_{2N}x_N &= b_2, \\
&\vdots \\
A_{M1}x_1 + A_{M2}x_2 + \ldots + A_{MN}x_N &= b_M,
\end{aligned}
$$

  where the A_{ij} and b_i have known values.
- If all the b_i are zero then the system of equations is called homogeneous, otherwise it is called inhomogeneous.
- Depending on the given values, this set of the equations for the N unknowns may have a unique solution, no solution, or infinitely many solutions.
- The set of equations can be expressed as a single matrix equation $Ax = b$

Basics (2)

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix}$$

- We can interpret the equation $A\mathbf{x} = \mathbf{b}$ as representing in some basis the linear transformation, $\mathcal{A}\mathbf{x} = \mathbf{b}$, of a vector $\mathbf{x}$ in an N-dimensional vector space V into a vector $\mathbf{b}$ in some other M-dimensional vector space W.
- The operator $\mathcal{A}$ will map any vector in V into a particular subspace of W. This subspace is called the **Range space** of $\mathcal{A}$ (or A). Its dimension is equal to the rank of A.
- If A is singular then there exists some subspace of V that is mapped onto the zero vector in W. So any vector $\mathbf{y}$ in this subspace satisfies $A\mathbf{y} = \mathbf{0}$.This subspace is called the **Null space** of A.

- The dimensions of the range space and the null space of A are related by the following relationship:
$$rank(A) + dim(Null(A)) = N$$
where N is the dimension of the vector space V

Case 1: No Solution

It should be noted that the system of equations possesses no solution unless $\mathbf{b}$ lies in the range space of A. If we write the columns of A as vectors $\mathbf{v_1}, ..., \mathbf{v_N}$ then this requires that $\mathbf{b}, \mathbf{v_1}, ..., \mathbf{v_N}$ to have the same span as $\mathbf{v_1}, ..., \mathbf{v_N}$. This requirement is equivalent to the condition that the matrices A and M:

$$M = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} & b_1 \\ A_{21} & A_{22} & \dots & A_{2N} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{M1} & A_{M2} & \dots & A_{MN} & b_M \end{pmatrix}$$

have the same rank.

Basics (4)

If they do not have same rank then there is no solution to the equation.

Case 2: Unique Solution
If $\mathbf{b}$ lies in the range space of A and if $Rank(A) = r = N$ then the vectors $\mathbf{v_1}, ..., \mathbf{v_N}$ are linearly independent and the equation has a unique solution.

Case 3: Infinitely Many Solutions
If $\mathbf{b}$ lies in the range of A and if the $Rank(A) = r < N$ then only r of the vectors $\mathbf{v_1}, ..., \mathbf{v_N}$ are linearly independent. We can therefore choose the coefficients for the $N - r$ vectors in an arbitrary way and still satisfy the equation for some $x_1, ..., x_N$. Therefore, there are infinitely many solutions which span an $N - r$ dimensional vector space.

N Simultaneous Equations in N unknowns

- A special situation occurs when $M = N$. In this case the matrix A is square. The equations will have a solution if $\mathbf{b}$ lies in the range space of A. In fact, we have a unique solution if A in a non-singular matrix and infinitely many solutions if A is a singular matrix.
- One method of solving these equaitons is Gaussian Elimination.
- We will cover the methods using LU factorization, Cramer's rule, and Singular Value Decomposition.
- If A is invertible then clearly a solution to the equations can be obtained by the direct inversion method: $\mathbf{x} = A^{-1}\mathbf{b}$

LU Decomposition (1)

- Finding the inverse of a matrix can be a very computationally demanding problem. Instead of performing the full inversion of A, to solve $A\mathbf{x} = \mathbf{b}$, we can perform a decomposition.
- In a LU decomposition the matrix A is decomposed into a product of a square lower triangular matrix L and a square upper triangular matrix U such that
$$A = LU$$
- For convenience of notation, we present the LU decomposition of a 3×3 matrix as follow:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{12} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

LU Decomposition (2)

$$\Rightarrow \quad A = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{pmatrix}$$

- The elements of the matrix A can be equated with elements of the LU product to solve for the L and U matrices.

$$U_{11} = A_{11},\ U_{12} = A_{12},\ U_{13} = A_{13},\ L_{21}U_{11} = A_{21},\ L_{21}U_{12}+U_{22} = A_{22}$$

$$L_{21}U_{13} + U_{23} = A_{23},\ L_{31}U_{11} = A_{31},\ L_{31}U_{12} + L_{32}U_{22} = A_{32}$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = A_{33}$$

- Once the matrices L and U have been determined they can be used to solve the matrix equation $A\mathbf{x} = \mathbf{b}$.

LU Decomposition (3)

- We have $LU\mathbf{x} = \mathbf{b}$ and this can be written as:

$$L\mathbf{y} = \mathbf{b} \quad , \quad U\mathbf{x} = \mathbf{y}$$

  where $\mathbf{y}$ is a vector that can easily be determined by solving the first triangular set of equations. This is plugged into the second set of equations to solve for $\mathbf{x}$.

- We illustrate the LU decomposition with a simple example. Let A be given by:

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

  By equating components we get:

$$U_{11} = 2, U_{12} = 4, U_{13} = 3, L_{21}U_{11} = 1, L_{31}U_{11} = -3,$$

$$L_{31}U_{12} + L_{32}U_{22} = 3, L_{31}U_{13} + L_{32}U_{23} + U_{33} = 2$$

LU Decomposition (4)

Solving we get:

$$L_{21} = \frac{1}{2}, L_{31} = -\frac{3}{2}, U_{22} = -4, U_{23} = -\frac{7}{2}, L_{32} = -\frac{9}{4}, U_{33} = -\frac{11}{8}$$

Now we can write the matrix A as:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & -\frac{9}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 \\ 0 & -4 & -\frac{7}{2} \\ 0 & 0 & -\frac{11}{8} \end{pmatrix}$$

The set of triangular equations $L\mathbf{y} = \mathbf{b}$ are given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & -\frac{9}{4} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -7 \end{pmatrix}$$

From this set of equations we get:

$$y_1 = 4, y_2 = -2, y_3 = -7 - (-\frac{3}{2})(4) - (-\frac{9}{4})(-2) = -\frac{11}{2}$$

LU Decomposition (5)

Substituting $y_1 = 4, y_2 = -2, y_3 = -\dfrac{11}{2}$, into the equation $U\mathbf{x} = \mathbf{y}$ we get:

$$\begin{pmatrix} 2 & 4 & 3 \\ 0 & -4 & -\frac{7}{2} \\ 0 & 0 & -\frac{11}{8} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -\frac{11}{2} \end{pmatrix}$$

Solving this triangular set of equations we get:

$$x_1 = 2,\ x_2 = -3,\ x_3 = 4$$

- The LU decomposition can also be used to find the inverse of A. For this the system of equations $A\mathbf{x} = \mathbf{b}$ is solved repeatedly with $\mathbf{b} = \mathbf{e_i}, \mathbf{i} = \mathbf{1}, ..., \mathbf{N}$, where $\mathbf{e_i}$ is the vector with its ith component equal to unity and all other components equal to zero. The solution x in each case gives the corresponding column of A^{-1}.

LU Decomposition (6)

- The determinant of A can also be easily determined from the LU decomposition. We have

$$|A| = |L||U|$$

Since L and U are triangular matrices there determinants are equal to the product of their diagonal elements. $|L| = 1$, so we have have that:

$$|A| = \prod_{i=1}^{N} U_{ii}$$

- It should be noted that the LU factorization of A does not necessarily exist even if A is non-singular.

Cramer's Rule (1)

- An alternative scheme is to use Cramer's rule which also gives some insight into the nature of the solutions. Using this rule we can either obtain the solution to the equation $A\mathbf{x} = \mathbf{b}$ or determine that there are no solutions.
- For convenience of notation we consider the case when we have three equations given as follows:

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1$$
$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2$$
$$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3$$

  Since the determinant of a matrix is invariant to the addition of a constant multiple of a column to another column we have that $|A|$ remains unchanged by the following addition to the first column $\frac{x_2}{x_1} \times$ (second column of A) $+ \frac{x_3}{x_1} \times$ (third column of A)

Cramer's Rule (2)

So we can write:

$$|A| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{11} + \frac{x_2}{x_1}A_{12} + \frac{x_3}{x_1}A_{13} & A_{12} & A_{13} \\ A_{21} + \frac{x_2}{x_1}A_{22} + \frac{x_3}{x_1}A_{23} & A_{22} & A_{23} \\ A_{31} + \frac{x_2}{x_1}A_{32} + \frac{x_3}{x_1}A_{33} & A_{32} & A_{33} \end{vmatrix}$$

we can subsittitute $\frac{b_i}{x_i}$ for the ith entry in the first column to obtain

$$|A| = \begin{vmatrix} \frac{b_1}{x_1} & A_{12} & A_{13} \\ \frac{b_2}{x_1} & A_{22} & A_{23} \\ \frac{b_3}{x_1} & A_{32} & A_{33} \end{vmatrix} = \frac{1}{x_1} \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix} = \frac{1}{x_1}\Delta_1$$

The determinant Δ_1 is known as the Cramer determinant. Similar manipulations of the second and third column of the matrix yield x_2 and x_3:

$$x_1 = \frac{\Delta_1}{|A|}, \quad x_2 = \frac{\Delta_2}{|A|}, \quad x_3 = \frac{\Delta_3}{|A|}$$

Cramer's Rule (3)

The Cramer determinants are given by:

$$\Delta_1 = \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix}, \Delta_2 = \begin{vmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{vmatrix}, \Delta_3 = \begin{vmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ A_{31} & A_{32} & b_3 \end{vmatrix}$$

It should be noted that if $|A| \neq 0$ then we get a unique solution for the equation by Cramer's rule. We further illustrate Cramer's rule by an example. Consider the following system of equations:

$$\begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -7 \end{pmatrix}$$

The determinant of the matrix A is given by $|A| = 11$.

The Cramer determinants are given by:

$$\Delta_1 = \begin{vmatrix} 4 & 4 & 3 \\ 0 & -2 & -2 \\ -7 & 3 & 2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 0 & -2 \\ -3 & -7 & 2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} 2 & 4 & 4 \\ 1 & -2 & 0 \\ -3 & 3 & -7 \end{vmatrix}$$

Solving the determinants we get:

$$\Delta_1 = 22, \Delta_2 = -33, \Delta_3 = 44$$

The solution to the equation is given by:

$$x_1 = \frac{\Delta_1}{|A|} = \frac{22}{11} = 2, \quad x_2 = \frac{-33}{11} = -3, \quad x_3 = \frac{44}{11} = 4$$

Cramer's Rule (5)

Using Cramer's rule we can write the following properties for the solutions of the system of equations $A\mathbf{x} = \mathbf{b}$:

(i) $|A| \neq 0, \mathbf{b} \neq \mathbf{0}$: Then there is only one unique solution and can be computed by using the Cramer's rule technique.

(ii) $|A| \neq 0, \mathbf{b} = \mathbf{0}$: There is only one trivial solution, $\mathbf{x} = \mathbf{0}$

(iii) $|A| = 0, \mathbf{b} \neq \mathbf{0}$, Cramer Determinants all zero: There is an infinite number of solutions

(iv) $|A| = 0, \mathbf{b} \neq \mathbf{0}$, Cramer Determinants not all zero: There are no solutions to the system of equations.

(v) $|A| = 0, \mathbf{b} = \mathbf{0}$: There is an infinite number of solutions.

Singular Value Decomposition (1)

- **Unitary Matrix:** A matrix A is called unitary if it satisfies the following property:

$$A^{\dagger} = A^{-1}$$

- Singular Value Decomposition (SVD) is a very powerful technique which can be applied whether or not the number of simultaneous equations M is equal to the number of unknowns N.
- Let A be an $M \times N$ complex matrix and suppose that we can write A as follows:

$$A = USV^{\dagger}$$

U, S, and V have the following properties:

(i) The square matrix U has dimensions $M \times M$ and is unitary.

(ii) The square matrix V has dimensions $N \times N$ and is unitary.

Singular Value Decomposition (2)

(iii) The matrix S has dimensions $M \times N$ and is diagonal in the sense that $S_{ij} = 0, i \neq j$. The diagonal elements will be denoted by $s_i, i = 1, .., p$, where $p = \min(M, N)$. It should be noted that the $s_1, ..., s_p$ are called the **singular values** of A.

The matrices U, S, V can be constructed from the matrix A as follows:

- From the matrix A, construct two Hermitian square matrices $A^\dagger A$ with dimensions $N \times N$ and $AA^\dagger$ with dimensions $M \times M$ respectively.

$$A^\dagger A = VS^\dagger U^\dagger USV^\dagger = VS^\dagger SV^\dagger$$

$$AA^\dagger = USV^\dagger VS^\dagger U^\dagger = USS^\dagger U^\dagger$$

here $S^\dagger S$ and $SS^\dagger$ are diagonal matrices with dimensions $N \times N$ and $M \times M$ respectively. The first p elements of each of these diagonal matrices are $s_i^2, i = 1, ..., p$, and the rest are zero. $p = \min(N, M)$

Singular Value Decomposition (3)

- The columns of V are given by the normalized eigen vectors $v^i, i = 1, ..., N$ of the matrix $A^\dagger A$.
- Similarly the columns of U are given by the normalized eigen vectors $u^j, j = 1, ..., M$ of the matrix $AA^\dagger$.
- The singular values s_i must satisfy $s_i^2 = \lambda_i$, where the λ_i are the eigen values of the smaller of the two matrices $A^\dagger A$ or $AA^\dagger$. Since these matrices are Hermitian so the eigen values are real and hence the singular values are real and non-negative.
- In order to make this decomposition unique we write the singular values in decreasing order $s_1 \geq s_2 \geq ... \geq s_p$.

Singular Value Decomposition (4)

- Now we can write the matrix A in terms of the normalized eigen vectors u^i and v^i as follows:

$$A = \sum_{i=1}^{p} s_i u^i (v^i)^\dagger$$

- It should be noted that some of the singular values are zero as a result of the degeneracies in the equation $Ax = b$. Suppose that there are r non-zero singular values. As a result of our convention the singular values $s_1, ..., s_r$ are non-zero and the singular values $s_{r+1}, ..., s_p$ are zero. We can write A as:

$$A = \sum_{i=1}^{r} s_i u^i (v^i)^\dagger$$

- Now we can write: $Ax = \sum_{i=1}^{r} s_i u^i (v^i)^\dagger x$. Since $(v^i)^\dagger x$ is just a number so the vectors u^i must span the range space of A.

- The vectors u^i actually form an orthonormal basis for the range space of A. Since this subspace is r-dimensional so $Rank(A) = r$.
- Now it can be easily proved that $Av^i = 0$ for $i = r+1, ..., N$. Thus the $N - r$ vectors $v^i, i = r+1, ..., N$, form an orthonormal basis for the Null space of A.

We illustrate singular value decomposition by an example. Let A be given as follows:

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{pmatrix}$$

The matrix has dimensions 3×4, $M = 3$ and $N = 4$. We can form the 3×3 matrix $AA^\dagger$ and the 4×4 matrix $A^\dagger A$. We compute the eigen values of the smaller matrix $AA^\dagger$.

Singular Value Decomposition (6)

$AA^\dagger$ is given by:

$$AA^\dagger = \begin{pmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{15} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{pmatrix}$$

The characteristic equation to determine the eigen values of this matrix gives:

$$\begin{vmatrix} 16-\lambda & 0 & 0 \\ 0 & \frac{29}{15}-\lambda & \frac{12}{5} \\ 0 & \frac{12}{5}-\lambda & \frac{36}{5} \end{vmatrix} = (16-\lambda)(\lambda^2 - 13\lambda + 36) = 0$$

Thus the eigen values are given by: $\lambda_1 = 16, \lambda_2 = 9$, and $\lambda_3 = 4$. The singular values are given by $s_i = \sqrt{\lambda_i}$.
$s_1 = 4, s_2 = 3, s_3 = 2$ are the singular values, respectively.

Singular Value Decomposition (7)

The matrix S is given by:

$$S = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

The matrix U is given by the normalised eigenvectors of $AA^\dagger$. These eigenvectors are given as follows:

$$\lambda_1 = 16 \quad \Rightarrow \quad u^1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

$$\lambda_2 = 9 \quad \Rightarrow \quad u^2 = \begin{bmatrix} 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}^T$$

$$\lambda_3 = 4 \quad \Rightarrow \quad u^3 = \begin{bmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}^T$$

$$\Rightarrow \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Singular Value Decomposition (8)

The 4×4 matrix $A^\dagger A$ is given by:

$$A^\dagger A = \frac{1}{4}\begin{pmatrix} 29 & 21 & 3 & 11 \\ 21 & 29 & 11 & 3 \\ 3 & 11 & 29 & 21 \\ 11 & 3 & 21 & 29 \end{pmatrix}$$

The nonzero eigenvalues of this matrix are the same as $AA^\dagger$. The corresponding normalised eigenvectors are given as follows:

$$\lambda_1 = 16 \quad \Rightarrow \quad v^1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$$

$$\lambda_2 = 9 \quad \Rightarrow \quad v^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}^T$$

$$\lambda_3 = 4 \quad \Rightarrow v^3 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & -1 \end{pmatrix}^T$$

$$\lambda_4 = 0 \quad \Rightarrow v^4 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$$

Singular Value Decomposition (9)

The matrix V can be written as:

$$V = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

So the singular value decomposition of A is given by:

$$A = USV^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Singular Value Decomposition (10)

- Consider the use of SVD to solve a set of M simultaneous linear equations in N unknowns, $Ax = b$.
- $b = 0$: If A is square and non-singular then we clearly have the unique trivial solution $x = 0$. Otherwise, any of the vectors $v^i, i = r + 1, ..., N$ or any linear combination of them will be a solution.
- For the case that $b \neq 0$, the set of equations possess a solution if b lies in the range space of A.

Consider the $N \times M$ matrix $\bar{S}$ which is constructed by taking the transpose of S and by replacing each non-zero singular value s_i by $\dfrac{1}{s_i}$. So $S\bar{S}$ is a $M \times M$ diagonal matrix whose diagonal entries are either zero or 1.

- The solution to the equation $Ax = b$ is given by:

$$x^* = V\bar{S}U^\dagger b$$

- This solution is unique if the rank r equals N, so that the matrix A does not posses a Null space. This only happens if A is square and non-singular.
- We know that the $N - r$ vectors $v^i, i = r + 1, ..., N$, form an orthonormal basis for the Null space of A. So we can add any linear combination of these vector to the solution x^* and get another solution to the equation $Ax = b$. In this case there exists an infinity of solutions.

We can illustrate the application of this method with an example. Let $b = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ and let A be the matrix whose SVD was determined in the previous slides.

Singular Value Decomposition (12)

- The solution to the equation is given by: $x^* = V\bar{S}U^\dagger b$
- In this case the matrix $\bar{S}$ can be shown to be:

$$\bar{S} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

- Computing the values in the aforementioned equation we get:

$$x^* = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$$

It should be noted that this solution is not unique as the matrix A posseses a Null space. So there is an infinite number of solutions.