Matrices: Determinant, Rank, Inverse

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Linear Operators (1)

- A linear operator $\mathcal{A}$ associates with every vector $\mathbf{x}$ another vector $\mathbf{y}$
$$\mathbf{y} = \mathcal{A}\mathbf{x}$$
in such a way that for two vectors $\mathbf{a}$ and $\mathbf{b}$
$$\mathcal{A}(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda\mathcal{A}\mathbf{a} + \mu\mathcal{A}\mathbf{b}$$
- Note that the action of a linear operator is indepedent of any basis and can be thought of as transforming one geometrical entity to another.
- If a basis is introduced then the action of the linear operator on each basis vector is to produce a linear combination as follows:
$$\mathcal{A}\mathbf{e_j} = \sum_{i=1}^{N} A_{ij}\mathbf{e_i}$$
It should be noted that A_{ij} is the ith component of the vector $\mathcal{A}\mathbf{e_j}$ in this basis. Collectively A_{ij} are called the components of the linear operator in the $\mathbf{e_1}, ..., \mathbf{e_n}$ basis.

Linear Operators (2)

- In this basis we can express the relation $\mathbf{y} = \mathcal{A}\mathbf{x}$ in component form as follows:
$$\mathbf{y} = \sum_{i=1}^{N} y_i \mathbf{e_i} = \mathcal{A} \sum_{j=1}^{N} x_j \mathbf{e_j} = \sum_{j=1}^{N} x_j \sum_{i=1}^{N} A_{ij} \mathbf{e_i}$$
Therefore, we can write the component y_i as follows
$$y_i = \sum_{j=1}^{N} A_{ij} x_j$$
- It should be noted that the components of both the vectors and the linear operators are with respect to a certain basis of the vector space.

Matrices (1)

- Consider the case where the linear operator maps the vector **x** to a vector **y** that belongs to a different M-dimensional vector space. Let $\mathbf{f_1}, ..., \mathbf{f_M}$ be a basis for this vector space. Then we can write

$$\mathcal{A}\mathbf{e_j} = \sum_{i=1}^{N} A_{ij}\mathbf{f_i}$$

  In this case the components, $\mathcal{A}_{ij}$, of the linear operator relate to both of the bases $\mathbf{e_j}$ and $\mathbf{f_i}$
- These components can be displayed as an array of numbers called a matrix. So if the linear operator transforms vector in an N-dimensional vector space to vectors in an M-dimensional vector space then we can respresent the operator $\mathcal{A}$ by the matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{bmatrix}$$

- **Complex Conjugate** The complex conjugate of a matrix A, denoted by A^*, is obtained by taking the complex conjugate of each of the elements of A

$$(A^*)_{ij} = (A_{ij})^*$$

Example: The complex conjugate of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3i \\ 1+i & 1 & 0 \end{bmatrix}$$

is given by:

$$A^* = \begin{bmatrix} 1 & 2 & -3i \\ 1-i & 1 & 0 \end{bmatrix}$$

- **Hermitian Conjugate:** The Hermitian conjugate or adjoint of a matrix is the transpose of its complex conjugate or equivalently the complex conjugate of its transpose.

$$A^\dagger = (A^*)^T = (A^T)^* \tag{1}$$

- **Trace of a Matrix:** The trace of a square $N \times N$ matrix, denoted $Tr(A)$, is defined as the sum of the diagonal elements of the matrix

$$Tr(A) = \sum_{i=1}^{N} A_{ii}$$

- The trace of a matrix is a linear operation, for example

$$Tr(A \pm B) = Tr(A) \pm Tr(B)$$

- The trace of the product of matrices is independent of the order of their multiplication, $Tr(AB) = Tr(BA)$.
- $Tr(A^T) = Tr(A)$ and $Tr(A^\dagger) = Tr(A^*)$

Determinant (1)

- Like the trace, the determinant of a matrix is only defined for square matrices.
- **Minor:** The minor M_{ij} of the element A_{ij} of an $N \times N$ matrix A is the determinant of the $(N-1) \times (N-1)$ matrix obtained by removing all the elements of the ith row and jth column of A.
- **Cofactor:** The cofactor C_{ij} is found by mutliplying the minor M_{ij} by $(-1)^{i+j}$, $C_{ij} = (-1)^{i+j} M_{ij}$.
- The determinant is defined as the sum of the products of the elements of any row or column and their corresponding cofactors. This sum is called a Laplace expansion.

Consider a 3×3 matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Determinant(2)

- The minor element M_{23} of this matrix is given by: $M_{23} = \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$
- The cofactor element C_{23} for this matrix is given by:

$$C_{23} = (-1)^{2+3} \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} = - \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

- The detrerminat of this matrix can be given by:
  $A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}$
  or by using some other row or column and their cofactor values.

Properties:

1. A matrix and its transpose have the same determinant: $|A| = |A^T|$
2. $|A^\dagger| = |(A^*)^T| = |A^*| = |A|^*$
3. If two rows (columns) of A are interchanged its determinant changes sign but is unaltered in magnitude.

Determinant(3)

4. If every element of a $N \times N$ matrix is multiplied by a constant λ then $|\lambda A| = \lambda^N |A|$
5. If A and B are square matrices of the same order then

$$|AB| = |A||B| = |BA|$$

   This result can be extended to the general case to show that the determinant is invariant under permutation of the matrices in a multiple product.

Matrix Inverse(1)

- A square matrix whose determinant is zero is called a singular matrix. Otherwise, it is called a non-singular matrix.
- Given that A is a non-singular matrix, we can define a matrix A^{-1} called the inverse of A which satisfies the property that:

$$AA^{-1} = A^{-1}A = I$$

  where I is the identity matrix.
- There are many ways to construct the inverse of a matrix. One way is to use the matrix C of cofactors of the elements of A. We have already seen that the cofactor C_{ij} of the element A_{ij} is given by: $C_{ij} = (-1)^{i+j}M_{ij}$. The elements of the inverse, A^{-1}, are given by:

$$(A^{-1})_{ik} = \frac{(C_{ik})^T}{|A|} = \frac{C_{ki}}{|A|}$$

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So A^{-1} is formed by taking the transpose of the cofactor matrix C and dividing it by the determinant of A. Consider the following matrix whose inverse we compute:

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{bmatrix}$$

Now using Laplace expansion, the determinant is given by:

$$|A| = 2\begin{vmatrix} -2 & -2 \\ 3 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} = 11$$

The cofactor matrix C is determined to be:

$$C = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{bmatrix}, \ C^T = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{bmatrix}$$

Then the inverse of A is given by:

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{bmatrix}$$

- For a 2×2 matrix the inverse has a simple form and is given by:

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

Matrix Inverse(4)

Properties:

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(A^\dagger)^{-1} = (A^{-1})^\dagger$
4. $(AB)^{-1} = B^{-1}A^{-1}$
5. $(AB...G)^{-1} = G^{-1}...B^{-1}A^{-1}$

Rank of a Matrix (1)

- Suppose that the columns of an $M \times N$ matrix can be interpreted as N vectors $v_1, ..., v_N$. The rank of A, denoted by R(A), is defined as the number of linearly independent vectors in the set $v_1, ..., v_N$ and equals the dimension of the vector space spanned by these column vectors.
- Alternatively the rank can also be shown to equal the number of linearly independent row vectors. Let $w_1, ..., w_M$ represent the row vectors of a matrix. Then the rank of A is equal to the number of linearly independent vectors in the set $w_1, ..., w_M$.
- The rank of a matrix can be determined by converting the matrix to its reduced echleon form. Another tecnhique is to use the concept of submatrices.
- A submatrix of any matrix is a matrix that is formed by ignoring a row or oclumn of the original matrix.
- For a general $M \times N$ matrix A, the rank is equal to the size of the largest square submatrix of A whose determinant is non-zero.

Rank of a Matrix (2)

- A $N \times N$ matrix A is non-singular if and only if $R(A) = N$.
- Elementary row and column operations are rank preserving.
- $R(A) = R(A^T)$

Consider the following 3×4 matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$$

Converting this matrix into its reduced echleon form we get:

$$ref(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there only two linearly independent columns (rows) so R(A) $= 2$.