

General Form of Solution

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Reference:

K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

General Form of Solution (1)

- Ordinary differential equations (ODE) maybe classified into different categories according to their general characteristics.
- The order of an ODE is simply the order of the highest derivative it contains. For example equations containing dy/dx are called first order whereas equations containing d^2y/dx^2 are called second order.
- The degree of an ODE is the power to which the highest-order derivative is raised after the equation has been rationalised to contain only integer powers of the derivatives. Therefore, the ODE:

$$\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2y = 0$$

is of third order and second degree, since after rationalisation it contains the term $(d^3y/dx^3)^2$

- The general solution to an ODE is the most general function that satisfies the equation. It contains constants of integration that must be determined by some application of suitable boundary conditions.

General Form of Solution (2)

- It is useful when evaluating the general form of the solution of an ODE to consider the inverse process i-e to determine an ODE from a given group of functions each of which is a solution of the ODE.
- Suppose the members of the group can be written as

$$y = f(x, a_1, a_2, \dots, a_n)$$

where each member is specified by a different set of values of the parameters a_i .

- Since an ODE is required for which any of the group is a solution, it clearly must not contain any of the a_i .
- Since there are n of the a_i in the expression so we must obtain $n + 1$ equations involving them so that by elimination we can obtain one final equation without them.
- If we differentiate the equation given above n times a total of $n + 1$ equations is obtained from which all of the a_i can be eliminated to give one ODE satisfied by all the group.

General Form of Solution (3)

- As a result of the n differentiations $d^n y/dx^n$ will be present in one of the $n + 1$ equations and hence in the final equation which will be of order n .
- For example consider the group of functions given by:

$$y = a_1 \sin(x) + a_2 \cos(x)$$

Taking derivatives we get the equations:

$$\frac{dy}{dx} = a_1 \cos(x) - a_2 \sin(x)$$

$$\frac{d^2 y}{dx^2} = -a_1 \sin(x) - a_2 \cos(x)$$

- Here the elimination of a_1 and a_2 is trivial resulting in a second order equation:

$$\frac{d^2 y}{dx^2} + y = 0$$

General Form of Solution (4)

- In summary, a group of functions with n parameters satisfies in general an n^{th} order ODE (although in some degenerate cases an ODE of less than n^{th} order is obtained).
- The converse of this is that the general solution of an n^{th} order ODE contains n arbitrary parameters. This will be assumed to be valid for our purposes.
- It should be noted that the boundary conditions on the solutions are the means of determining the parameters and hence specifying precisely which function is the required solution.
- In order to obtain a unique solution the number of boundary conditions must match the number of parameters and hence the order of the equation.
- Fewer independent boundary conditions will lead to a number of undetermined parameters in the solution whereas an excess will usually mean that no acceptable solution is possible.

General Form of Solution (5)

- For an n^{th} -order equation the required n boundary conditions can take many forms.
- For example the value of y at n different values of x or the value of any $n - 1$ of the n derivatives $dy/dx, d^2y/dx^2, \dots, d^ny/dx^n$ together with that of y all for the same value of x or for many intermediate combinations.