EigenValues and EigenVectors

Waseem A. Malik

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Basics (1)

- Suppose that a linear operator $\mathcal{A}$ transforms a vector $\mathbf{x}$ in a vector space to another vector $\mathcal{A}\mathbf{x}$ in the same vector space. Consider the special case when the linear operator tranforms the vector to a multiple of itself:

$$\mathcal{A}\mathbf{x} = \lambda\mathbf{x}, \qquad \lambda \text{ is a scalar}$$

  Any non-zero vector $\mathbf{x}$ that satisfies this equation is an eigenvector of the linear operator and λ is the corresponding eigenvalue.
- The linear operator has N independent eigen vectors $\mathbf{x^i}$ and corresponding eigen values λ_i. The λ_i need not be distinct.
- If we fix the basis then we can write the aforementioned equation as a matrix equation as follows:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

  where $\mathbf{A}$ is an $N \times N$ matrix.

Basics (2)

- It should be noted that if $\mathbf{x}$ is an eigenvector of A then any scalar multiple of $\mathbf{x}$ will also be an eigenvector. We therefore use normalized eigenvectors satisfying: $\mathbf{x}^{\dagger}\mathbf{x} = \mathbf{I}$
- Eigenvalues and eigenvectors play an important role in characterizing a matrix
- The inverse of the matrix, A^{-1} has the same eigen vectors $\mathbf{x^i}$ as A but the eigen values have the form $\frac{1}{\lambda^i}$. This can be shown as follows:

$$\text{We know that } A\mathbf{x^i} = \lambda_i \mathbf{x^i}$$

$$\Rightarrow A^{-1}A\mathbf{x^i} = \lambda \mathbf{x^i}$$

$$\text{Rearranging we get: } A^{-1}\mathbf{x^i} = \frac{1}{\lambda^i}\mathbf{x^i}$$

- **Normal Matrices:** A matrix that commutes with its Hermitian conjugate is called a normal matrix:

$$A^\dagger A = AA^\dagger$$

- For a normal matrix A, the eigenvalues of $A^\dagger$ are the complex conjugates of the eigenvalues of A.
  If $\mathbf{x}$ is an eigenvector of a normal matrix A then: $A\mathbf{x} = \lambda\mathbf{x}$.
  $$\Rightarrow (A - \lambda I)\mathbf{x} = 0$$
  Let $B = A - \lambda I$. Then we have $B\mathbf{x} = 0$. Taking the Hermitian conjugate we get:

  $$(B\mathbf{x})^\dagger = \mathbf{x}^\dagger B^\dagger = 0 \quad \Rightarrow \quad \mathbf{x}^\dagger B^\dagger B\mathbf{x} = 0$$

  Since A is normal so the product $B^\dagger B$ is given by:

  $$B^\dagger B = (A - \lambda I)^\dagger (A - \lambda I) = AA^\dagger - \lambda^* A - \lambda A^\dagger + \lambda\lambda^\dagger = BB^\dagger$$

  Hence B is also normal.

Basics (4)

Now using the equations above we get:

$$\mathbf{x}^\dagger B^\dagger B\mathbf{x} = \mathbf{x}^\dagger BB^\dagger \mathbf{x} = (B^\dagger \mathbf{x})^\dagger B^\dagger \mathbf{x} = 0$$

Therefore, we can conclude that:

$$B^\dagger \mathbf{x} = (A^\dagger - \lambda^* I)\mathbf{x} = 0 \quad \Rightarrow \quad A^\dagger x = \lambda^* \mathbf{x}$$

This completes the proof.

- If all N eigen values of a normal matrix A are distinct then all N eigenvectors of A are mutually orthogonal. Even if some eigenvalues are degenerate a set of mutually orthogonal eigenvectors can be constructed.
- A normal matrix A can be written in terms of its eigenvalues λ^i and othonormal eigenvectors $\mathbf{x^i}$ as follows:

$$A = \sum_{i=1}^{N} \lambda^i \mathbf{x^i} (\mathbf{x^i})^\dagger$$

Determination of EigenValues and EigenVectors (1)

Consider a $N \times N$ matrix A with corresponding eigenvalue λ and eigenvector $\mathbf{x}$. Then we can re-write the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$ as follows:

$$A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = 0$$

From basic linear algebra, we know that the equation $(A - \lambda I)\mathbf{x} = 0$ has a nontrivial solution if and only if the determinant of $A - \lambda I$ is zero.

$$|A - \lambda I| = 0$$

This is known as the characteristic equation of A. This equation is a ploynomial of degree N in λ. The N roots of this equation give the eigenvalues $\lambda_i, i = 1, ..., N$, of A. Corresponding to each eigenvalue λ_i there will be a column vector $\mathbf{x^i}$ which will be the eigenvector of A. This can be determined from the equation: $A\mathbf{x^i} = \lambda_\mathbf{i}\mathbf{x^i}$.

Determination of EigenValues and EigenVectors (2)

Example: Determine the eigen values and normalized eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}$$

From the characteristic equation we get:

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 1-\lambda & -3 \\ 3 & -3 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(1-\lambda)(-3-\lambda)-9\} - 1\{-3-\lambda+9\} + 3\{-3-3(1-\lambda)\} = 0$$

Simplifying we get:

$$\Rightarrow \lambda^3 + \lambda^2 - 24\lambda + 36 = \lambda^3 + 3\lambda^2 - 2\lambda^2 - 6\lambda - 18\lambda + 36 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda + 6) = 0$$

Determination of EigenValues and EigenVectors (2)

So $\lambda_1 = 2, \lambda_2 = 3$, and $\lambda_3 = -6$ are the eigen values of A. Now for eigenvalue $\lambda_1 = 2$ we determine the eigen vector $\mathbf{x^1}$ with elements x_1, x_2, x_3. Using the equation $A\mathbf{x^1} = \mathbf{2x^1}$ we get:

$$
\begin{aligned}
x_1 + x_2 + 3x_3 &= 2x_1 \\
x_1 + x_2 - 3x_3 &= 2x_2 \\
3x_1 - 3x_2 - 3x_3 &= 2x_3
\end{aligned}
$$

Solving we get $x_1 = x_2 = k$, where k in any non-zero number, and $x_3 = 0$. A suitable eigenvector is $x^1 = (k \;\; k \;\; 0)^T$. For normalization we require: $k^2 + k^2 + 0 = 1 \;\Rightarrow\; k = \frac{1}{\sqrt{2}}$. So the normalized eigenvector is given by:

$$\mathbf{x^1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$$

Similarly we can determine:

$$\mathbf{x^2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T, \; \mathbf{x^3} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -1 & -2 \end{pmatrix}^T$$