Higher-degree First-order Equations

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Course: ENPM 667 (Control of Robotic Systems)

Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Introduction

- Higher-degree First-order equations do not occur often in physical systems. They do however appear in connection with geometrical problems.
- Higher-degree First-order equations can be written as

$$F(x, y, dy/dx) = 0$$

- The most general standard form is

$$p^n + a_{n-1}(x, y)p^{n-1} + \cdots + a_1(x, y)p + a_0(x, y) = 0$$

where for ease of notation we write $p = \dfrac{dy}{dx}$

- If the equation can be solved for one of x, y or p then either an explicit or a parametric solution can sometimes be obtained.
- We present the main types of such equations in this lecture.

Equations soluble for p

- Sometimes the LHS of the equation

$$p^n + a_{n-1}(x,y)p^{n-1} + \cdots + a_1(x,y)p + a_0(x,y) = 0$$

can be factorised into the form

$$(p - F_1)(p - F_2)\cdots(p - F_n) = 0$$

where $F_i = F_i(x,y)$

- We are then left with solving the n first-degree equations $p = F_i(x,y)$

- Writing the solutions to these first-degree equations as $G_i(x,y) = 0$, the general solution is given by:

$$G_1(x,y)G_2(x,y)\cdots G_n(x,y) = 0$$

Equations soluble for x

- Equations that can be solved for x such that they can be written in the form

$$x = F(y, p)$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to y so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy}$$

- This results in an equation of the form $G(y, p) = 0$, which can be used together with $x = F(y, p)$ to eliminate p and obtain the general solution.

- Note that often a singular solution to the equation will be found at the same time.

Equations soluble for y (1)

- Equations that can be solved for y are such that they maybe written in the form

$$y = F(x, p)$$

and can be reduced to first-degree first-order equations in p by differentiating both sides with respect to x so that

$$\frac{dy}{dx} = p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{dp}{dx}$$

- This results in an equation of the form $G(x, p) = 0$ which can be used with the equation $y = F(x, p)$ to eliminate p and give the general solution.

- Note that often a singular solution to the equation will be found at the same time as well.

Equations soluble for y (2)

- **Example:** Solve the equation

$$xp^2 + 2xp - y = 0$$

We can write this equation as $y = xp^2 + 2xp$. Differentiating both sides of the equation with respect to x we find that

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p$$

we can factorise this equation to get

$$(p+1)\left(p + 2x\frac{dp}{dx}\right) = 0$$

To obtain the general solution of the equation we consider the factor containing dp/dx. This first-degree first-order equation in p has the solution $xp^2 = c$. We plug this value into the original equation to eliminate p.

Equations soluble for y (3)

Rewriting the original equation as:

$$xp^2 - y = -2xp$$

Squaring on both sides and using $xp^2 = c$, we find that the general solution to the equation is given by:

$$(y - c)^2 = 4cx$$

If we instead use the factor $(p + 1) = 0$ we get the solution $p = -1$. Substituting this into the equation we get:

$$x + y = 0$$

which is a singular solution to the equation.

Clairaut's Equation (1)

- Clairaut's equation is wrriten in the following form

$$y = px + F(p)$$

  which is a special case of equations soluble for y.
- Clairaut's equation can be solved by a similar method to that used for solving equations soluble in y. However, the form of the general solution to Clairaut's equation is really simple.
- Differentiating with respect to x we find that

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF}{dp}\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}\left(\frac{dF}{dx} + x\right) = 0$$

Clairaut's Equation (2)

- Consider the first factor containing dp/dx to get

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad y = c_1x + c_2$$

- Since $p = \dfrac{dy}{dx} = c_1$, if we substitute into the equation we find that $c_1x + c_2 = c_1x + F(c_1)$.
- Therefore, the constant c_2 is given by $F(c_1)$ and the general solution to Clairaut's equation is given by:

$$y = c_1x + F(c_1)$$

- It should be noted that the general solution to Clairaut's equation can be obtained by replacing p in the ODE by the arbitrary constant c_1. Now consider the second factor we get

$$\frac{dF}{dp} + x = 0$$

Clairaut's Equation (3)

The second factor gives us the form

$$G(x, p) = 0$$

This relation can be used to eliminate p from the ODE to get a singular solution.