Vector Spaces

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Vector Spaces (1)

A set of vectors **a,b,c,...** is said to form a (linear) vector space if it satisfies the following properties:

- The set is closed under commutative and associative addition:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$
$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

- The set is closed under mutiplication by a scalar, the operation being both distributive and associative:

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$$
$$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$$
$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$$
$$\lambda \text{ and } \mu \text{ are arbitrary scalars}$$

- There exists a null vector $\mathbf{0}$ such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$, for all $\mathbf{a}$
- Multiplication by unity leaves any vector unchanged: $\mathbf{a} \times \mathbf{1} = \mathbf{a}$
- Every vector $\mathbf{a}$ has a corresponding additive inverse $-\mathbf{a}$ such that: $\mathbf{a} + (-\mathbf{a}) = 0$

Vector Spaces (2)

- It should be noted that if the scalars are restricted to be real then we obtain a *real vector space* otherwise a *complex vector space* is obtained.
- **Span of a set of Vectors**: The span of a set of vectors, $v_1, ..., v_n$, is defined as the set of all vectors that can be written as a linear combination of the original set:

$$\alpha_1 v_1 + ... + \alpha_n v_n$$

  where $\alpha_1, ..., \alpha_n$ are arbitrary scalars.
- **Linearly Indepedent:** A set of vectors, $v_1, ..., v_n$ is said to be lineary independent if the equation

$$\alpha_1 v_1 + ... + \alpha_n v_n = 0$$

  has only the trivial solution $\alpha_i = 0, i = 1, ..., n$

Vector Spaces (3)

- A set of vectors that is not linear idependent is called linearly dependent and atleast one vector in the set can be expressed as linear combination of the others.
- **Dimensionality of a Vector Space:** If in a given vector space there exist N linearly independent vectors but no set of $N+1$ linearly independent vectors then the vector space is said to be N-dimensional.
- **Basis of a Vector Space:** If V is an N-dimensional vector space then any set of N linearly independent vectors $e_1, ..., e_n$ in V forms a basis for V. If x is an arbitrary vector lying in V then it can be written as a linear combination of basis vectors:

$$x = \sum_{i=1}^{N} x_i e_i$$

It should be noted x_i are the components of x with respect to the e_i basis vectors. These components are unique with respect to these basis vectors.

Inner Product (1)

- The inner product of two vectors **a** and **b**, denoted by $\langle \mathbf{a}|\mathbf{b}\rangle$, is a scalar function of **a** and **b**. It generalizes the the notion of a dot product to more abstract spaces.
- The inner product satisfies the following properties:

$$\langle \mathbf{a}|\mathbf{b}\rangle = \langle \mathbf{b}|\mathbf{a}\rangle^*$$
$$\langle \mathbf{a}|\lambda\mathbf{b} + \mu\mathbf{c}\rangle = \lambda\langle \mathbf{a}|\mathbf{b}\rangle + \mu\langle \mathbf{a}|\mathbf{c}\rangle$$

  where λ and μ are scalars. For a complex vector space these two properties imply that

$$\langle \lambda\mathbf{a} + \mu\mathbf{b}|\mathbf{c}\rangle = \lambda^*\langle \mathbf{a}|\mathbf{c}\rangle + \mu^*\langle \mathbf{b}|\mathbf{c}\rangle$$
$$\langle \lambda\mathbf{a}|\mu\mathbf{b}\rangle = \lambda^*\mu\langle \mathbf{a}|\mathbf{b}\rangle$$

- Two vectors in a general vector space are defined to be orthogonal if $\langle \mathbf{a}|\mathbf{b}\rangle = \mathbf{0}$
- The norm of a vector **a** is given by $\|\mathbf{a}\| = \langle \mathbf{a}|\mathbf{a}\rangle^{\frac{1}{2}}$

Inner Product (2)

- A basis, $\hat{\mathbf{e}}_{\mathbf{1}}, ...\hat{\mathbf{e}}_{\mathbf{n}}$, of an N-dimensional vector space is said to be orthonormal if:

$$\langle \hat{\mathbf{e}}_{\mathbf{i}}, \hat{\mathbf{e}}_{\mathbf{j}} \rangle = \delta_{ij}, \quad \text{where } \delta_{ij} = \begin{Bmatrix} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{Bmatrix}$$

  In other words the basis vectors are mutually orthogonal and have a unit norm.
- Consider two vectors **a** and **b** which can be written using the orthonormal basis as:

$$\mathbf{a} = \sum_{i=1}^{N} a_i \hat{\mathbf{e}}_{\mathbf{i}}, \quad \mathbf{b} = \sum_{i=1}^{N} b_i \hat{\mathbf{e}}_{\mathbf{i}}$$

  In such an othonormal basis for any vector **a**:

$$\langle \hat{\mathbf{e}}_{\mathbf{j}} | \mathbf{a} \rangle = \sum_{i=1}^{N} \langle \hat{\mathbf{e}}_{\mathbf{j}} | a_i \hat{\mathbf{e}}_{\mathbf{i}} \rangle = \sum_{i=1}^{N} a_i \langle \hat{\mathbf{e}}_{\mathbf{j}} | \hat{\mathbf{e}}_{\mathbf{i}} \rangle = a_j$$

Inner Product (3)

- The inner product of two vectors **a** and **b** in terms of their components in an orthonormal basis can be written as:

$$\langle \mathbf{a} | \mathbf{b} \rangle = \sum_{i=1}^{N} a_i^* b_i$$

- The aforementioned result can be generalized to the case when do not have an orthonormal basis. Let $\mathbf{e_1}, ..., \mathbf{e_n}$ be a basis for an N-dimensional vector space. The vectors **a** and **b** can be written as:

$$\mathbf{a} = \sum_{i=1}^{N} a_i \mathbf{e}_i, \quad \mathbf{b} = \sum_{i=1}^{N} b_i \mathbf{e_i}$$

Then the inner product of these vectors is given by:

$$\langle \mathbf{a} | \mathbf{b} \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* b_j \langle \mathbf{e_i} | \mathbf{e_j} \rangle$$

Some Useful Inequalities

- **Cauchy Schwarz's Inequality:** $|\langle \mathbf{a}|\mathbf{b}\rangle| \leq \|\mathbf{a}\|\|\mathbf{b}\|$
  where the equality holds when **a** is a scalar multiple of **b**, $\mathbf{a} = \lambda\mathbf{b}$
- **Triangle Inequality:** The triangle inequality states that
$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$
- **Parallelogram equality:** This result follows directly from the properties of an inner product and states that
$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)$$
- **Bessel's Inequality:** Consider an orthonormal basis, $\hat{\mathbf{e}}_1, ..., \hat{\mathbf{e}}_n$, in an N-dimensional vector space then for any vector **a**
$$\|\mathbf{a}\|^2 \geq \sum_{\mathbf{i}} |\langle \hat{\mathbf{e}}_\mathbf{i}|\mathbf{a}\rangle|^2$$
  where the equality holds if the sum is over all the N basis vectors. This inequality can also be written as: $\langle \mathbf{a}|\mathbf{a}\rangle \geq \sum_i |a_i|^2$