Laplace Transform Method

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Laplace Transform Method (1)

- The method of Laplace tranform is very useful for solving linear ODEs with constant coefficients.
- Taking the Laplace transform of such an equation transforms it into a purely algebraic equation in terms of the Laplace transform of the required solution.
- Once the algebraic equation has been solved for this Laplace transform, the general solution to the original ODE can be obtained by performing an inverse Laplace transform.
- In order to apply this method we need only two results from the theory of Laplace transforms.
- First, the Laplace transform of a function $f(x)$ is defined by:

$$\bar{f}(s) = \int_0^\infty e^{-sx} f(x) dx$$

Laplace Transform Method (2)

- The second result concerns the Laplace tranform of the n^{th} derivative of $f(x)$ and is given by:

$$\overline{f^{(n)}}(s) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

- Using these relations along with the table of Laplace tranforms of standard functions we can solve linear ODEs with constant coefficients.

- **Example:** Solve the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{-x}$$

subject to the boundary conditions $y(0) = 2$ and $y'(0) = 1$

Laplace Transform Method (3)

- **Solution:** Taking the Laplace transform and using the table of standard results we obtain

$$s^2\bar{y}(s) - sy(0) - y'(0) - 3\big(s\bar{y}(s) - y(0)\big) + 2\bar{y}(s) = \frac{2}{s+1}$$

Simplifying we get:

$$(s^2 - 3s + 2)\bar{y}(s) - 2s + 5 = \frac{2}{s+1}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{2s^2 - 3s - 3}{(s+1)(s-1)(s-2)}$$

Using partial fractions we can write the Laplace transform as:

$$\bar{y}(s) = \frac{1}{3(s+1)} + \frac{2}{s-1} - \frac{1}{3(s-2)}$$

Laplace Transform Method (4)

Taking the inverse Laplace transform by using the table of tranform pairs we get the required solution

$$y(x) = \frac{1}{3}e^{-x} + 2e^{x} - \frac{1}{3}e^{2x}$$