Change of Basis and Similarity Transformations

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

Change of Basis (1)

- Consider a basis $\mathbf{e_1}, ..., \mathbf{e_N}$ of an N-dimensional vector space and let $\mathbf{x}$ belong to this vector space. We can then write:

$$\mathbf{x} = \sum_{i=1}^{N} x_i \mathbf{e_i}$$

- The $x_1, ..., x_n$ are the components of $\mathbf{x}$. In this section, we analyze how these components change as a result of changing the basis.
- Consider a new basis $\mathbf{e}'_\mathbf{1}, ..., \mathbf{e}'_\mathbf{N}$ which is related to the former basis by the equation:

$$\mathbf{e}'_\mathbf{j} = \sum_{i=1}^{N} S_{ij} \mathbf{e_i}$$

where the coefficent S_{ij} is the ith component of $\mathbf{e}'_\mathbf{j}$ with respect to the former basis. The vector $\mathbf{x}$ can now be written as:

$$\mathbf{x} = \sum_{i=1}^{N} x_i \mathbf{e_i} = \sum_{j=1}^{N} x'_j \mathbf{e}'_\mathbf{j} = \sum_{j=1}^{N} x'_j \sum_{i=1}^{N} S_{ij} \mathbf{e_i}$$

Change of Basis (2)

The relationship between the components of $\mathbf{x}$ in the two bases can be represented as:

$$x_i = \sum_{j=1}^{N} S_{ij} x'_j$$

which can be written in matrix form as follows:

$$x = Sx', \quad x = (x_1, \ldots, x_n)^T, \; x' = (x'_1, \ldots, x'_n)^T$$

here S is a transformation matrix associated with the change of basis. Since, the vectors $\mathbf{e_j}$ are linearly independent, therefore the matrix S is non-singular.

$$\Rightarrow \quad x' = S^{-1} x$$

Note that the components transform inversely when compared to how the basis vectors transform. This is required so that the vector, $\mathbf{x}$, remains unchanged.

Change of Basis (3)

- We can also find the transformation law for the components of a linear operator under the same change of basis. The operator equation, $\mathbf{y} = \mathcal{A}\mathbf{x}$, can be written as a matrix equation in each of the two bases:

  $$\mathbf{y} = A\mathbf{x}, \quad \mathbf{y}' = A^{'}\mathbf{x}'$$

  Using the transformation matrix equation, $x = Sx'$, we can write:

  $$y = Ax \;\Rightarrow\; Sy' = ASx' \;\Rightarrow\; y' = S^{-1}ASx'$$

  Now combining these two equations we get that the components of the linear operator transform as follows: $A^{'} = S^{-1}AS$
  This is an example of a similarity transformation.

Change of Basis (4)

Given the square matrix A, we can interpret it as representing the linear operator $\mathcal{A}$ in a certain basis $\mathbf{e_i}$. However, the matrix $A' = S^{-1}AS$ also represents the same linear operator but in a new basis $\mathbf{e'_j}$.
Therefore, any property of the matrix A that represents some (basis-independent) property of the linear operator will be shared by A'.

1. If $A = I$ then $A' = I$
2. The value of the determinant is unchanged:
   $|A'| = |S^{-1}AS| = |S^{-1}||A||S| = |A||S^{-1}S| = |A|$
3. The characteristic determinant and hence the eigen value of A and A' are the same: $|A' - \lambda I| = |S^{-1}AS - \lambda I| = |S^{-1}(A - \lambda I)S| = |A - \lambda I|$
4. The trace is unchanged; Tr(A) = Tr(A')