First-degree First-Order Equations

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Reference:
K. Riley, M. Hobson, S. Bence "Mathematical Methods for Physics and Engineering", 2006

First-degree First-Order Equations

- First-degree first-order equations contain only dy/dx equated to some function of x and y and can be written in either of two equivalent forms:
$$\frac{dy}{dx} = F(x, y), \qquad A(x, y)dx + B(x, y)dy = 0$$
where $F(x, y) = -A(x, y)/B(x, y)$ and $F(x, y), A(x, y)$ and $B(x, y)$ are in general functions of both x and y.

- Which of the two forms is the more useful for finding a solution depends on the type of the equation being considered.

- There are several different types of First-degree first-order equations that are of interest in the physical sciences. We will discuss some of these in this lecture.

Separable Variable Equations (1)

- A separable-variable equation is one which maybe written in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

  where $f(x)$ and $g(y)$ are functions of x and y respectively. This also includes cases in which $f(x)$ or $g(y)$ is simply a constant.
- Rearranging this equation so that the terms depending on x and y appear on opposite sides and integrating we obtain:

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

- Finding the solution $y(x)$ that satisfies the equation depends only on the ease with which the integrals can be evaluated.
- It should also be noted that ODE's that at first sight do not appear to be in the conventional form can be made separable by an appropriate factorisation.

- **Example:** Solve the equation

$$\frac{dy}{dx} = x + xy$$

  The RHS of this equation can be factorised to give $x(1+y)$, the equation becomes separable and we obtain $\int \frac{dy}{1+y} = \int x dx$. Integrating both sides separately we obtain

$$\ln(1+y) = \frac{x^2}{2} + c \;\Rightarrow\; 1+y = \exp\left(\frac{x^2}{2}+c\right) = A\exp\left(\frac{x^2}{2}\right)$$

  where c and hence A is an arbitrary constant.

- **Solution Method:** *Factorise the equation so that it becomes separable. Rearrange it so that the terms depending on x and those depending on y appear on opposite sides and then integrate directly.*

Exact Equations (1)

- An exact first-degree frist-order ODE is of the form

$$A(x,y)dx + B(x,y)dy = 0 \quad \text{and for which} \quad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

- In this case $A(x,y)dx + B(x,y)dy$ is an exact differential say $dU(x,y)$. In other words

$$Adx + Bdy = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

which gives us that

$$A(x,y) = \frac{\partial U}{\partial x}, \quad B(x,y) = \frac{\partial U}{\partial y}$$

Since $\dfrac{\partial^2 U}{\partial x \partial y} = \dfrac{\partial^2 U}{\partial y \partial x}$ we therefore require

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

Exact Equations (2)

- If $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$ then the ODE can be written as $dU(x,y) = 0$ which has the solution $U(x,y) = c$ where c is a constant and by using $\dfrac{\partial U}{\partial x} = A(x,y)$, we get

$$U(x,y) = \int A(x,y)dx + F(y)$$

- The function $F(y)$ can be found from equation $B(x,y) = \dfrac{\partial U}{\partial y}$ by differentiating the expression for $U(x,y)$ with respect to y and equating it to $B(x,y)$.
- **Example:** Solve

$$x\frac{dy}{dx} + 3x + y = 0$$

Rearranging the equation we get:

$$(3x + y)dx + xdy = 0$$

Exact Equations (3)

It should be noted that $A(x, y) = 3x + y$ and $B(x, y) = x$. Since

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} = 1$$

Therefore, the equation is exact. So the solution is given by:

$$U(x, y) = \int (3x + y)dx + F(y) = c_1 \quad \Rightarrow \quad \frac{3x^2}{2} + yx + F(y) = c_1$$

Differentiating $U(x, y)$ with respect to y and equating it to $B(x, y) = x$ we obtain $dF/dy = 0$ which integrates to give $F(y) = c_2$. Letting $c = c_1 - c_2$, the solution is given by:

$$\frac{3x^2}{2} + xy = c$$

Inexact Equations: Method of Integrating Factors (1)

- Equations that maybe written in the form

$$A(x,y)dx + B(x,y)dy = 0 \quad \text{but for which} \quad \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

  are known as inexact equations.
- However, the differential can always be made exact by multiplying by an integrating factor $\mu(x,y)$ which obeys

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}$$

- For an integrating factor that is a function of both x and y, $\mu = \mu(x,y)$ there exists no general method for finding it. In such cases it might be found by inspection.
- If an integrating factor exists that is a function of either x or y alone then the equation above can be used to solve it.

Inexact Equations: Method of Integrating Factors (2)

- If we assume that the integrating factor is a function of x alone, $\mu = \mu(x)$ then

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x} \quad \Rightarrow \quad \mu\frac{\partial A}{\partial y} = \mu\frac{\partial B}{\partial x} + B\frac{d\mu}{dx}$$

  Rearranging we get:

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx = f(x)dx$$

  where we require $f(x)$ also to be a function of x only.

- This provides a general method of determining whether the integrating factor μ is a function of x alone. This integrating factor is then given by:

$$\mu(x) = \exp\left\{\int f(x)dx\right\} \text{ where } f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)$$

Inexact Equations: Method of Integrating Factors (3)

- Similarly if the integrating factor is a function of y alone $\mu = \mu(y)$
$$\mu(y) = \exp\left\{ \int g(y)dy \right\} \text{ where } g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$$

- **Example:** Solve $\dfrac{dy}{dx} = -\dfrac{2}{y} - \dfrac{3y}{2x}$
Rearranging we have
$$(4x + 3y^2)dx + 2xydy = 0$$
where $A = 4x + 3y^2$ and $B = 2xy$. Computing the partial derivatives we get:
$$\frac{\partial A}{\partial y} = 6y, \quad \frac{\partial B}{\partial x} = 2y$$
Clearly the ODE is not exact. However, we observe that
$$f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x}$$

Inexact Equations: Method of Integrating Factors (4)

- Therefore, an integration factor exists that is a function of x alone. Ignoring the arbitrary constant of integration, $\mu(x)$ is given by:

$$\mu(x) = \exp\left\{2\int \frac{dx}{x}\right\} = \exp(2\ln x) = x^2$$

Multiplying by the integrating factor we obtain the equation

$$(4x^3 + 3x^2y^2)dx + 2x^3ydy = 0$$

Using the methodology for solving exact equations we get:

$$U(x, y) = \int 4x^3 + 3x^2y^2dx + F(y) = x^4 + x^3y^2 + F(y) = c_1$$

Taking the derivative of $U(x, y)$ with respect to y and setting it equal to $B(x, y)$ we get $F(y) = c_2$. Letting $c = c_1 - c_2$ we get the solution

$$x^4 + x^3y^2 = c$$

Linear Equations (1)

- Linear first-order ODEs are a special case of inexact ODEs and can be written in the conventional form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- Such an equation can be made exact by multiplication by an appropriate integrating factor.
- In this case the integrating factor is a function of x alone and can be expressed in a particularly simple form. The integrating factor $\mu(x)$ must be such that:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

which can be integrated directly to give:

$$\mu(x)y = \int \mu(x)Q(x)dx$$

Linear Equations (2)

- We can also write

$$\frac{d}{dx}(\mu y) \;=\; \mu\frac{dy}{dx} + \frac{d\mu}{dx}y \;=\; \mu\frac{dy}{dx} + \mu P y$$

This gives the simple relation:

$$\frac{d\mu}{dx} = \mu(x)P(x) \quad \Rightarrow \quad \mu(x) = \exp\left\{\int P dx\right\}$$

- **Solution Method:** Rearrange the given equation into the conventional form. Multiply by the integrating factor given by $\mu(x) = \exp\left\{\int P dx\right\}$. Then the solution is given by the equation $\mu(x)y = \int \mu(x)Q(x)dx$

Homogeneous Equations (1)

- Homogeneous equations are ODEs that maybe written in the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

where $A(x,y)$ and $B(x,y)$ are homogeneous functions of the same degree.

- A function $f(x,y)$ is homogeneous of degree n if for any λ

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

for example if $A = x^2(y) - xy^2$ then we see that A is a homogeneous function of degree 3.

- The RHS of a homogeneous equation can be written as a function of y/x.

Homogeneous Equations (2)

- The equation can be solved by making the substitution $y = vx$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v)$$

- This is now a separable equation and can be integrated directly to give

$$\int \frac{1}{F(v) - v} dv = \int \frac{1}{x} dx$$

Bernoulli's Equation

- Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{where } n \neq 0, 1$$

- This equation is very similar in form to the linear equation but is in fact non-linear due to the extra y^n factor on the RHS.
- This equation can be made linear by substituting $v = y^{1-n}$ to get

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}$$

- Now subsituting this into the equation and dividing throughout by y^n we find

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

which is a linear equation and can be solved by using the integrating factor technique.