ENPM 667 Project 2

Thomas Birney

1 Nonlinear State Derivations

1.1 Deriving Euler Lagrange Equation

K_c and P_c are Kinetic and Potential Energy of the Cart

K_1 and P_1 are Kinetic and Potential Energy of mass 1

K_2 and P_2 are Kinetic and Potential Energy of mass 2

$$\begin{aligned} K_c &= \frac{1}{2}m_c\dot{x}^2 \\ P_c &= 0 \end{aligned} \tag{1.1}$$

$$\begin{aligned} K_1 &= \frac{1}{2}m_1(\dot{x}^2 - l_1\dot{\theta}_1\cos(\theta_1))^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1\sin(\theta_1))^2 \\ K_1 &= \frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1\cos(\theta_1) + l_1^2\dot{\theta}_1^{\,2}\cos(\theta_1)^2 + (l_1\dot{\theta}_1\sin(\theta_1))^2) \\ P_1 &= m_1 g(l_1 - l_1\cos(\theta_1)) \\ P_1 &= m_1 g l_1(1 - \cos(\theta_1)) \end{aligned} \tag{1.2}$$

$$\begin{aligned} K_2 &= \frac{1}{2}m_2(\dot{x}^2 - l_2\dot{\theta}_2\cos(\theta_2))^2 + \frac{1}{2}m_2(l_2\dot{\theta}_2\sin(\theta_2))^2 \\ K_2 &= \frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot{\theta}_2 l_2\cos(\theta_2) + l_2^2\dot{\theta}_2^{\,2}\cos(\theta_2)^2 + (l_2\dot{\theta}_2\sin(\theta_2))^2) \\ P_2 &= m_2 g(l_2 - l_2\cos(\theta_2)) \\ P_2 &= m_2 g l_2(1 - \cos(\theta_2)) \end{aligned} \tag{1.3}$$

$$\begin{aligned} K &= K_c + K_1 + K_2 \\ P &= P_c + P_1 + P_2 \\ L &= K - P \\ &= \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1\cos(\theta_1) + l_1^2\dot{\theta}_1^{\,2}\cos(\theta_1)^2 + (l_1\dot{\theta}_1\sin(\theta_1))^2) \\ &+ \frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot{\theta}_2 l_2\cos(\theta_2) + l_2^2\dot{\theta}_2^{\,2}\cos(\theta_2)^2 + (l_2\dot{\theta}_2\sin(\theta_2))^2) \\ &- m_1 g l_1(1 - \cos(\theta_1)) - m_2 g l_2(1 - \cos(\theta_2)) \end{aligned} \tag{1.4}$$

1.2 Finding Initial Equation to $\ddot{X}$

$$F = \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \frac{\partial L}{\partial X} \tag{1.5}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{X}} &= \frac{\partial}{\partial \dot{X}}\frac{1}{2}m_c\dot{x}^2 \\
&+ \frac{\partial}{\partial \dot{X}}\frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1\cos(\theta_1) + l_1^2\dot{\theta_1}^2\cos(\theta_1)^2 + (l_1\dot{\theta}_1\sin(\theta_1))^2) \\
&+ \frac{\partial}{\partial \dot{X}}\frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot{\theta}_2 l_2\cos(\theta_2) + l_2^2\dot{\theta_2}^2\cos(\theta_2)^2 + (l_2\dot{\theta}_2\sin(\theta_2))^2) \\
&- \frac{\partial}{\partial \dot{X}}m_1 g l_1(1-\cos(\theta_1)) - \frac{\partial}{\partial \dot{X}}m_2 g l_2(1-\cos(\theta_2)) \\
\frac{\partial L}{\partial \dot{X}} &= m_c\dot{x} + m_1(\dot{x} - l_1\dot{\theta}_1\cos(\theta_1)) + m_2(\dot{x} - l_2\dot{\theta}_2\cos(\theta_2))
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} &= \frac{d}{dt}m_c\dot{x} + m_1(\dot{x} - l_1\dot{\theta}_1\cos(\theta_1)) + m_2(\dot{x} - l_2\dot{\theta}_2\cos(\theta_2)) \\
&= m_c\ddot{x} + m_1(\ddot{x} - l_1(-\dot{\theta}_1\sin(\theta_1)\dot{\theta}_1 + \cos(\theta_1)\ddot{\theta}_1)) + m_2(\ddot{x} - l_2(-\dot{\theta}_2\sin(\theta_2)\dot{\theta}_2 + \cos(\theta_2)\ddot{\theta}_2)) \\
&= m_c\ddot{x} + m_1(\ddot{x} + l_1\sin(\theta_1)\dot{\theta_1}^2 - l_1\cos(\theta_1)\ddot{\theta}_1) + m_2(\ddot{x} + l_2\sin(\theta_2)\dot{\theta_2}^2 - l_2\cos(\theta_2)\ddot{\theta}_2) \\
&= \ddot{x}(m_c + m_1 + m_2) + m_1 l_1\sin(\theta_1)\dot{\theta_1}^2 - m_1 l_1\cos(\theta_1)\ddot{\theta}_1 + m_2 l_2\sin(\theta_2)\dot{\theta_2}^2 - m_2 l_2\cos(\theta_2)\ddot{\theta}_2
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
\frac{\partial L}{\partial X} &= \frac{\partial}{\partial X}\frac{1}{2}m_c\dot{x}^2 \\
&+ \frac{\partial}{\partial X}\frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1\cos(\theta_1) + l_1^2\dot{\theta_1}^2\cos(\theta_1)^2 + (l_1\dot{\theta}_1\sin(\theta_1))^2) \\
&+ \frac{\partial}{\partial X}\frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot{\theta}_2 l_2\cos(\theta_2) + l_2^2\dot{\theta_2}^2\cos(\theta_2)^2 + (l_2\dot{\theta}_2\sin(\theta_2))^2) \\
&- \frac{\partial}{\partial X}m_1 g l_1(1-\cos(\theta_1)) - \frac{\partial}{\partial X}m_2 g l_2(1-\cos(\theta_2)) \\
&= 0
\end{aligned} \tag{1.8}$$

$$\begin{aligned}
F &= \ddot{x}(m_c + m_1 + m_2) + m_1 l_1\sin(\theta_1)\dot{\theta_1}^2 - m_1 l_1\cos(\theta_1)\ddot{\theta}_1 + m_2 l_2\sin(\theta_2)\dot{\theta_2}^2 - m_2 l_2\cos(\theta_2)\ddot{\theta}_2 \\
\ddot{x} &= \frac{1}{m_c + m_1 + m_2}(F - m_1 l_1\sin(\theta_1)\dot{\theta_1}^2 + m_1 l_1\cos(\theta_1)\ddot{\theta}_1 - m_2 l_2\sin(\theta_2)\dot{\theta_2}^2 + m_2 l_2\cos(\theta_2)\ddot{\theta}_2)
\end{aligned} \tag{1.9}$$

1.3 Deriving Initial Equation to $\ddot{\theta}_1$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \tag{1.10}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot\theta_1} &= \frac{\partial}{\partial \dot\theta_1}\frac{1}{2}m_c\dot{x}^2 \\
&+ \frac{\partial}{\partial \dot\theta_1}\frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot\theta_1 l_1\cos(\theta_1) + l_1^2\dot\theta_1^{\,2}\cos(\theta_1)^2 + (l_1\dot\theta_1\sin(\theta_1))^2) \\
&+ \frac{\partial}{\partial \dot\theta_1}\frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot\theta_2 l_2\cos(\theta_2) + l_2^2\dot\theta_2^{\,2}\cos(\theta_2)^2 + (l_2\dot\theta_2\sin(\theta_2))^2) \\
&- \frac{\partial}{\partial \dot\theta_1}m_1 g l_1(1-\cos(\theta_1)) \\
&- \frac{\partial}{\partial \dot\theta_1}m_2 g l_2(1-\cos(\theta_2)) \\
&= \frac{1}{2}m_1(-2\dot{x}l_1\cos(\theta_1) + 2l_1^2\dot\theta_1\cos(\theta_1)^2 + 2l_1^2\dot\theta_1\sin(\theta_1)^2) \\
&= \frac{1}{2}m_1(-2\dot{x}l_1\cos(\theta_1) + 2l_1^2\dot\theta_1) \\
\frac{\partial L}{\partial \dot\theta_1} &= -m_1 l_1\dot{x}\cos(\theta_1) + m_1 l_1^2\dot\theta_1
\end{aligned}
\tag{1.11}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} &= \frac{d}{dt} - m_1 l_1\dot{x}\cos(\theta_1) + m_1 l_1^2\dot\theta_1 \\
&= -m_1 l_1(-\dot{x}sin(\theta_1)\dot\theta_1 + \ddot{x}cos(\theta_1)) + m_1 l_1^2\ddot\theta_1 \\
&= m_1 l_1\dot{x}sin(\theta_1)\dot\theta_1 - m_1 l_1\ddot{x}cos(\theta_1) + m_1 l_1^2\ddot\theta_1
\end{aligned}
\tag{1.12}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= \frac{\partial}{\partial \theta_1}\frac{1}{2}m_c\dot{x}^2 \\
&+ \frac{\partial}{\partial \theta_1}\frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot\theta_1 l_1\cos(\theta_1) + l_1^2\dot\theta_1^{\,2}\cos(\theta_1)^2 + (l_1\dot\theta_1\sin(\theta_1))^2) \\
&+ \frac{\partial}{\partial \theta_1}\frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot\theta_2 l_2\cos(\theta_2) + l_2^2\dot\theta_2^{\,2}\cos(\theta_2)^2 + (l_2\dot\theta_2\sin(\theta_2))^2) \\
&- \frac{\partial}{\partial \theta_1}m_1 g l_1(1-\cos(\theta_1)) - \frac{\partial}{\partial \theta_1}m_2 g l_2(1-\cos(\theta_2)) \\
&= \frac{1}{2}m_1(2\dot{x}l_1\dot\theta_1 sin(\theta_1) - 2l_1^2\dot\theta_1^{\,2}cos(\theta_1)\sin(\theta_1) + 2l_1^2\dot\theta_1^{\,2}\sin(\theta_1)\cos(\theta_1)) - m_1 l_1 g\sin(\theta_1) \\
\frac{\partial L}{\partial \theta_1} &= m_1 l_1\dot\theta_1\dot{x}sin(\theta_1) - m_1 l_1 g\sin(\theta_1)
\end{aligned}
\tag{1.13}
$$

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} = 0 \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1} \\
&m_1 l_1\dot{x}sin(\theta_1)\dot\theta_1 - m_1 l_1\ddot{x}cos(\theta_1) + m_1 l_1^2\ddot\theta_1 = m_1 l_1\dot\theta_1\dot{x}sin(\theta_1) - m_1 l_1 g\sin(\theta_1) \\
&- m_1 l_1\ddot{x}cos(\theta_1) + m_1 l_1^2\ddot\theta_1 = -m_1 l_1 g\sin(\theta_1) \\
&m_1 l_1^2\ddot\theta_1 = m_1 l_1\ddot{x}cos(\theta_1) - m_1 l_1 g\sin(\theta_1) \\
&l_1\ddot\theta_1 = \ddot{x}cos(\theta_1) - g\sin(\theta_1)
\end{aligned}
\tag{1.14}
$$

1.4 Deriving Initial Equation to $\ddot{\theta}_2$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \tag{1.15}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= \frac{\partial}{\partial \dot{\theta}_2}\frac{1}{2}m_c\dot{x}^2 \\
&+ \frac{\partial}{\partial \dot{\theta}_2}\frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1\cos(\theta_1) + l_1^2\dot{\theta}_1^{\,2}\cos(\theta_1)^2 + (l_1\dot{\theta}_1\sin(\theta_1))^2) \\
&+ \frac{\partial}{\partial \dot{\theta}_2}\frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot{\theta}_2 l_2\cos(\theta_2) + l_2^2\dot{\theta}_2^{\,2}\cos(\theta_2)^2 + (l_2\dot{\theta}_2\sin(\theta_2))^2) \\
&- \frac{\partial}{\partial \dot{\theta}_2}m_1 g l_1(1-\cos(\theta_1)) \\
&- \frac{\partial}{\partial \dot{\theta}_2}m_2 g l_2(1-\cos(\theta_2)) \\
&= \frac{1}{2}m_2(-2\dot{x}l_2\cos(\theta_2) + 2l_2^2\dot{\theta}_2\cos(\theta_2)^2 + 2l_2^2\dot{\theta}_2\sin(\theta_2)^2) \\
&= \frac{1}{2}m_2(-2\dot{x}l_2\cos(\theta_2) + 2l_2^2\dot{\theta}_2) \\
\frac{\partial L}{\partial \dot{\theta}_2} &= -m_2 l_2\dot{x}\cos(\theta_2) + m_2 l_2^2\dot{\theta}_2
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= \frac{d}{dt} - m_2 l_2\dot{x}\cos(\theta_2) + m_2 l_2^2\dot{\theta}_2 \\
&= -m_2 l_2(-\dot{x}sin(\theta_2)\dot{\theta}_2 + \ddot{x}cos(\theta_2)) + m_2 l_2^2\ddot{\theta}_2 \\
&= m_2 l_2\dot{x}sin(\theta_2)\dot{\theta}_2 - m_2 l_2\ddot{x}cos(\theta_2) + m_2 l_2^2\ddot{\theta}_2
\end{aligned} \tag{1.17}$$

$$\begin{aligned}
\frac{\partial L}{\partial \theta_2} &= \frac{\partial}{\partial \theta_2}\frac{1}{2}m_c\dot{x}^2 \\
&+ \frac{\partial}{\partial \theta_2}\frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1\cos(\theta_1) + l_1^2\dot{\theta}_1^{\,2}\cos(\theta_1)^2 + (l_1\dot{\theta}_1\sin(\theta_1))^2) \\
&+ \frac{\partial}{\partial \theta_2}\frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}\dot{\theta}_2 l_2\cos(\theta_2) + l_2^2\dot{\theta}_2^{\,2}\cos(\theta_2)^2 + (l_2\dot{\theta}_2\sin(\theta_2))^2) \\
&- \frac{\partial}{\partial \theta_2}m_1 g l_1(1-\cos(\theta_1)) - \frac{\partial}{\partial \theta_2}m_2 g l_2(1-\cos(\theta_2)) \\
&= \frac{1}{2}m_2(2\dot{x}l_2\dot{\theta}_2 sin(\theta_2) - 2l_2^2\dot{\theta}_2^{\,2}cos(\theta_2)\sin(\theta_2) + 2l_2^2\dot{\theta}_2^{\,2}\sin(\theta_2)\cos(\theta_2)) - m_2 l_2 g\sin(\theta_2) \\
\frac{\partial L}{\partial \theta_2} &= m_2 l_2\dot{\theta}_2\dot{x}sin(\theta_2) - m_2 l_2 g\sin(\theta_2)
\end{aligned} \tag{1.18}$$

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} - \frac{\partial L}{\partial \theta_2} = 0 \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} = \frac{\partial L}{\partial \theta_2} \\
&m_2 l_2 \dot{x} sin(\theta_2)\dot{\theta_2} - m_2 l_2 \ddot{x} cos(\theta_2) + m_2 l_2^2 \ddot{\theta_2} = m_2 l_2 \dot{\theta_2} \dot{x} sin(\theta_2) - m_2 l_2 g \sin(\theta_2) \\
&- m_2 l_2 \ddot{x} cos(\theta_2) + m_2 l_2^2 \ddot{\theta_2} = -m_2 l_2 g \sin(\theta_2) \\
&m_2 l_2^2 \ddot{\theta_2} = m_2 l_2 \ddot{x} cos(\theta_2) - m_2 l_2 g \sin(\theta_2) \\
&l_2 \ddot{\theta_2} = \ddot{x} cos(\theta_2) - g \sin(\theta_2)
\end{aligned} \tag{1.19}
$$

1.5 Plugging Values Back Into $\ddot{X}$

$$
\begin{aligned}
&F = \ddot{x}(m_c + m_1 + m_2) + m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 l_1 \cos(\theta_1)\ddot{\theta_1} + m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 l_2 \cos(\theta_2)\ddot{\theta_2} \\
&F = \ddot{x}(m_c + m_1 + m_2) + m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 \cos(\theta_1)(\ddot{x} cos(\theta_1) - g \sin(\theta_1)) \\
&\quad + m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 \cos(\theta_2)(\ddot{x} cos(\theta_2) - g \sin(\theta_2)) \\
&F = \ddot{x}(m_c + m_1 + m_2) + m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 \cos(\theta_1)^2 \ddot{x} + m_1 \cos(\theta_1) g \sin(\theta_1) \\
&\quad + m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 \cos(\theta_2)^2 \ddot{x}) + m_2 \cos(\theta_2) g \sin(\theta_2) \\
&F = \ddot{x}(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2)) + m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 + m_1 \cos(\theta_1) g \sin(\theta_1) \\
&\quad + m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 + m_2 \cos(\theta_2) g \sin(\theta_2) \\
&\quad \ddot{x} = \frac{1}{m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2)}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 \\
&\quad - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))
\end{aligned} \tag{1.20}
$$

1.6 Plugging Back Into $\ddot{\theta_1}$ and $\ddot{\theta_2}$

$$
\begin{aligned}
&l_1 \ddot{\theta_1} = \ddot{x} cos(\theta_1) - g \sin(\theta_1) \\
&\quad \ddot{\theta_1} = \frac{F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2)}{l_1(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2))} \cos(\theta_1) \\
&\quad - \frac{g \sin(\theta_1)}{l_1}
\end{aligned} \tag{1.21}
$$

$$
\begin{aligned}
&l_2 \ddot{\theta_2} = \ddot{x} cos(\theta_2) - g \sin(\theta_2) \\
&\quad \ddot{\theta_2} = \frac{F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2)}{l_2(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2))} \cos(\theta_2) \\
&\quad - \frac{g \sin(\theta_2)}{l_2}
\end{aligned} \tag{1.22}
$$

1.7 Defining the state

We define Our Nonlinear State using

$$X = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}, \text{and } U = F \tag{1.23}$$

And using our definitions for the state variables we can say

$$\dot{X} = f(X, U)$$

where f is a collection of nonlinear functions

2 Linearizing the System

We linearize the system around the equilibrium point

$$x = 0, \dot{x} = 0, \theta_1 = 0, \dot{\theta}_1 = 0, \theta_2 = 0, \dot{\theta}_2 = 0, F = 0$$

By using the Jacobian of the nonlinear function, we linearize the state space by saying

$$\dot{X} = \nabla_x f(x, u)|_{x=0,u=0} X(t) + \nabla_U f(x, u)|_{x=0,u=0} U(t)$$

Since

$$f = \begin{bmatrix} f_{\dot{x}} \\ f_{\ddot{x}} \\ f_{\dot{\theta}_1} \\ f_{\ddot{\theta}_1} \\ f_{\dot{\theta}_2} \\ f_{\ddot{\theta}_2} \end{bmatrix} \tag{2.1}$$

$$
\nabla_X = \begin{bmatrix}
\frac{\partial f_{\dot{x}}}{\partial x} & \frac{\partial f_{\dot{x}}}{\partial \dot{x}} & \frac{\partial f_{\dot{x}}}{\partial \theta_1} & \frac{\partial f_{\dot{x}}}{\partial \dot{\theta}_1} & \frac{\partial f_{\dot{x}}}{\partial \theta_2} & \frac{\partial f_{\dot{x}}}{\partial \dot{\theta}_2} \\
\frac{\partial f_{\ddot{x}}}{\partial x} & \frac{\partial f_{\ddot{x}}}{\partial \dot{x}} & \frac{\partial f_{\ddot{x}}}{\partial \theta_1} & \frac{\partial f_{\ddot{x}}}{\partial \dot{\theta}_1} & \frac{\partial f_{\ddot{x}}}{\partial \theta_2} & \frac{\partial f_{\ddot{x}}}{\partial \dot{\theta}_2} \\
\frac{\partial f_{\dot{\theta}_1}}{\partial x} & \frac{\partial f_{\dot{\theta}_1}}{\partial \dot{x}} & \frac{\partial f_{\dot{\theta}_1}}{\partial \theta_1} & \frac{\partial f_{\dot{\theta}_1}}{\partial \dot{\theta}_1} & \frac{\partial f_{\dot{\theta}_1}}{\partial \theta_2} & \frac{\partial f_{\dot{\theta}_1}}{\partial \dot{\theta}_2} \\
\frac{\partial f_{\ddot{\theta}_1}}{\partial x} & \frac{\partial f_{\ddot{\theta}_1}}{\partial \dot{x}} & \frac{\partial f_{\ddot{\theta}_1}}{\partial \theta_1} & \frac{\partial f_{\ddot{\theta}_1}}{\partial \dot{\theta}_1} & \frac{\partial f_{\ddot{\theta}_1}}{\partial \theta_2} & \frac{\partial f_{\ddot{\theta}_1}}{\partial \dot{\theta}_2} \\
\frac{\partial f_{\dot{\theta}_2}}{\partial x} & \frac{\partial f_{\dot{\theta}_2}}{\partial \dot{x}} & \frac{\partial f_{\dot{\theta}_2}}{\partial \theta_1} & \frac{\partial f_{\dot{\theta}_2}}{\partial \dot{\theta}_1} & \frac{\partial f_{\dot{\theta}_2}}{\partial \theta_2} & \frac{\partial f_{\dot{\theta}_2}}{\partial \dot{\theta}_2} \\
\frac{\partial f_{\ddot{\theta}_2}}{\partial x} & \frac{\partial f_{\ddot{\theta}_2}}{\partial \dot{x}} & \frac{\partial f_{\ddot{\theta}_2}}{\partial \theta_1} & \frac{\partial f_{\ddot{\theta}_2}}{\partial \dot{\theta}_1} & \frac{\partial f_{\ddot{\theta}_2}}{\partial \theta_2} & \frac{\partial f_{\ddot{\theta}_2}}{\partial \dot{\theta}_2}
\end{bmatrix} \tag{2.2}
$$

$$
\nabla_U = \begin{bmatrix}
\frac{\partial f_{\dot{x}}}{\partial F} \\
\frac{\partial f_{\ddot{x}}}{\partial F} \\
\frac{\partial f_{\dot{\theta}_1}}{\partial F} \\
\frac{\partial f_{\ddot{\theta}_1}}{\partial F} \\
\frac{\partial f_{\dot{\theta}_2}}{\partial F} \\
\frac{\partial f_{\ddot{\theta}_2}}{\partial F}
\end{bmatrix} \tag{2.3}
$$

2.1 Finding Partial Derivatives

I will be linearizing with respect to the equilibrium point as we go

2.1.1 For $\ddot{x}$

$$\begin{aligned}\ddot{x} = & \frac{1}{m_c + m_1 sin(\theta_1) + m_2\sin(\theta_2)}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 \\ & - m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g\cos(\theta_2)\sin(\theta_2))\end{aligned} \tag{2.4}$$

$\frac{\partial \ddot{x}}{\partial x}$ and $\frac{\partial \ddot{x}}{\partial \dot{x}}$ are zero because these variables are not present in the equation

Once we have the partial derivative formula we can substitute in the equilibrium values into the non differentiated components

$$\begin{aligned}\frac{\partial \ddot{x}}{\partial \theta_1} = & \\ & (m_c)^{-1}\frac{\partial}{\partial \theta_1}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g\cos(\theta_2)\sin(\theta_2)) \\ & + F\frac{\partial}{\partial \theta_1}(m_c + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2))^{-1} \\ & = (m_c)^{-1}(-m_1 l_1\cos(\theta_1)\dot{\theta_1}^2 - m_1 g(\cos(\theta_1) - \sin^2(\theta_1))) \\ & - F(m_c + m_1\sin^2(\theta_2) + m_2\sin^2(\theta_2))^{-2} 2m_1\sin(\theta_1)\cos(\theta_1) \\ \frac{\partial \ddot{x}}{\partial \theta_1} = & \frac{-m_1 g}{m_c}\end{aligned} \tag{2.5}$$

$$\begin{aligned}\frac{\partial \ddot{x}}{\partial \dot{\theta}_1} = & (m_c + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2))^{-1} \\ & \frac{\partial}{\partial \dot{\theta}_1}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g\cos(\theta_2)\sin(\theta_2)) \\ & + (F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g\cos(\theta_2)\sin(\theta_2)) \\ & \frac{\partial}{\partial \dot{\theta}_1}(m_c + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2))^{-1}\end{aligned} \tag{2.6}$$

$$\begin{aligned}\frac{\partial \ddot{x}}{\partial \dot{\theta}_1} = & \\ & (m_c)^{-1}\frac{\partial}{\partial \dot{\theta}_1}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g\cos(\theta_2)\sin(\theta_2)) \\ & + F\frac{\partial}{\partial \dot{\theta}_1}(m_c + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2))^{-1} \\ & = (m_c)^{-1}(-2m_1 l_1\sin(\theta_1)\dot{\theta_1}^2) + F(0) = 0\end{aligned} \tag{2.7}$$

$$\frac{\partial \ddot{x}}{\partial \theta_2} = (m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-1}$$
$$\frac{\partial}{\partial \theta_2}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))$$
$$+ (F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))$$
$$\frac{\partial}{\partial \theta_2}(m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-1} \tag{2.8}$$

Once we have the partial derivative formula we can substitute in the equilibrium values into the non differentiated components

$$\frac{\partial \ddot{x}}{\partial \theta_2} =$$
$$(m_c)^{-1}\frac{\partial}{\partial \theta_2}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))$$
$$+ F\frac{\partial}{\partial \theta_2}(m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-1}$$
$$= (m_c)^{-1}(-m_2 l_2 \cos(\theta_2)\dot{\theta_2}^2 - m_2 g(\cos(\theta_2) - \sin^2(\theta_2)))$$
$$- F(m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-2} 2 m_2 \sin(\theta_2)\cos(\theta_2)$$
$$\frac{\partial \ddot{x}}{\partial \theta_2} = \frac{-m_2 g}{m_c} \tag{2.9}$$

$$\frac{\partial \ddot{x}}{\partial \dot{\theta}_2} = (m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-1}$$
$$\frac{\partial}{\partial \dot{\theta}_2}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))$$
$$+ (F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))$$
$$\frac{\partial}{\partial \dot{\theta}_2}(m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-1} \tag{2.10}$$

$$\frac{\partial \ddot{x}}{\partial \dot{\theta}_2} =$$
$$(m_c)^{-1}\frac{\partial}{\partial \dot{\theta}_2}(F - m_1 l_1 \sin(\theta_1)\dot{\theta_1}^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2 - m_2 g \cos(\theta_2)\sin(\theta_2))$$
$$+ F\frac{\partial}{\partial \dot{\theta}_2}(m_c + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2))^{-1}$$
$$= (m_c)^{-1}(-2 m_2 l_2 \sin(\theta_2)\dot{\theta_2}^2) + F(0) = 0 \tag{2.11}$$

2.1.2 For $\dot{\theta}_1$

$$\begin{aligned}
\frac{\partial \dot{\theta}_1}{\partial x} &= 0 \\
\frac{\partial \dot{\theta}_1}{\partial \dot{x}} &= 0 \\
\frac{\partial \dot{\theta}_1}{\partial \theta_1} &= 1 \\
\frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} &= 0 \\
\frac{\partial \dot{\theta}_1}{\partial \theta_2} &= 0 \\
\frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2} &= 0
\end{aligned} \tag{2.12}$$

2.2 For $\ddot{\theta}_1$

$$\begin{aligned}
\ddot{\theta}_1 = &\frac{F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^{\,2} - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g \cos(\theta_2)\sin(\theta_2)}{l_1(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2)} \cos(\theta_1) \\
&- \frac{g \sin(\theta_1)}{l_1}
\end{aligned} \tag{2.13}$$

$$\frac{\partial \ddot{\theta}_1}{\partial \ddot{X}}, \frac{\partial \ddot{\theta}_1}{\partial \dot{X}} = 0$$

because they do not appear in the function

$$
\begin{aligned}
&\frac{\partial \ddot{\theta}_1}{\partial \theta_1} = (l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&\frac{\partial}{\partial \theta_1}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1))\frac{\partial \ddot{\theta}_1}{\partial \theta_1}(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&\frac{\partial \ddot{\theta}_1}{\partial \theta_1} - \frac{g\sin(\theta_1)}{l_1} \\
&= (l_1 m_c)^{-1} \\
&\frac{\partial}{\partial \theta_1}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1))\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&+ (F)\frac{\partial \ddot{\theta}_1}{\partial \theta_1}(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&- \frac{g\cos(\theta_1)}{l_1} \\
&= (l_1 m_c)^{-1} \\
&(-F\sin(\theta_1) - m_1 l_1(-\sin^2(\theta_1) + \cos^2(\theta_1))\dot{\theta}_1{}^2 - m_1 g(\cos^3(\theta_1) - 2\sin^2(\theta_1)\cos(\theta_1)) \\
&+ m_2 l_2 \sin(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 + m_2 g \sin(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&- (F)(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-2}(2\sin(\theta_1)\cos(\theta_1)) \\
&- \frac{g\cos(\theta_1)}{l_1} \\
&= \frac{m_1 g}{l_1 m_c} - \frac{g}{l_1}
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&\frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} = (l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&\frac{\partial}{\partial \dot{\theta}_1}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1))\frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1}(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&- \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1}\frac{g\sin(\theta_1)}{l_1} \\
&= (l_1 m_c)^{-1} \\
&\frac{\partial}{\partial \dot{\theta}_1}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1))\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&= (l_1 m_c)^{-1}(-m_1 l_1 \sin(\theta_1)\cos(\theta_1))\dot{\theta}_1) \\
&= 0
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&\frac{\partial \ddot{\theta}_1}{\partial \theta_2} = (l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&\frac{\partial}{\partial \theta_2}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1))\frac{\partial \ddot{\theta}_1}{\partial \theta_2}(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&- \frac{\partial \ddot{\theta}_1}{\partial \theta_2}\frac{g\sin(\theta_1)}{l_1} \\
&= (l_1 m_c)^{-1} \\
&\frac{\partial}{\partial \theta_2}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1))\dot{\theta}_1{}^2 - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&+ (F)\frac{\partial \ddot{\theta}_1}{\partial \theta_2}(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&= (l_1 m_c)^{-1} \\
&(-m_2 l_2 \cos(\theta_1)\cos(\theta_2)\dot{\theta}_2{}^2 - m_2 g \cos(\theta_1)(\cos^2(\theta_2 - \sin^2(\theta_2))) \\
&- 2(F)(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-2} 2\sin(\theta_2)\cos(\theta_2) \\
&= \frac{-m_2 g}{l_1 m_c}
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
&\frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} = (l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&\frac{\partial}{\partial \dot{\theta}_2}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1^{\,2} - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1)\dot{\theta}_1^{\,2} - m_1 g \cos^2(\theta_1)\sin(\theta_1))\frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2}(l_1 m_c + l_1 m_1 \sin^2(\theta_1) + l_1 m_2 \sin^2(\theta_2))^{-1} \\
&- \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2}\frac{g\sin(\theta_1)}{l_1} \\
&= (l_1 m_c)^{-1} \\
&\frac{\partial}{\partial \dot{\theta}_2}(F\cos(\theta_1) - m_1 l_1 \sin(\theta_1)\cos(\theta_1))\dot{\theta}_1^{\,2} - m_1 g \cos^2(\theta_1)\sin(\theta_1) \\
&- m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g \cos(\theta_1)\cos(\theta_2)\sin(\theta_2)) \\
&= (l_1 m_c)^{-1} \\
&(-2 m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2) \\
&= 0
\end{aligned}
\tag{2.17}
$$

2.2.1 For $\dot{\theta}_2$

$$
\begin{aligned}
\frac{\partial \dot{\theta}_2}{\partial x} &= 0 \\
\frac{\partial \dot{\theta}_2}{\partial \dot{x}} &= 0 \\
\frac{\partial \dot{\theta}_2}{\partial \theta_1} &= 0 \\
\frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_1} &= 0 \\
\frac{\partial \dot{\theta}_2}{\partial \theta_2} &= 0 \\
\frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} &= 1
\end{aligned}
\tag{2.18}
$$

2.2.2 For $\ddot{\theta}_2$

$$
\begin{aligned}
\ddot{\theta}_2 = &\frac{F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^{\,2} - m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g \cos(\theta_2)\sin(\theta_2)}{l_2(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2))}\cos(\theta_2) \\
&- \frac{g\sin(\theta_2)}{l_2}
\end{aligned}
\tag{2.19}
$$

$$
\frac{\partial \ddot{\theta}_2}{x}, \frac{\partial \ddot{\theta}_2}{\dot{x}} = 0
$$

$$
\begin{aligned}
&\frac{\partial\ddot{\theta}_2}{\partial\theta_1}\\
&=(l_2m_c+m_1l_2sin(\theta_1)+m_2l_2\sin(\theta_2))^{-1}\\
&\frac{\partial\ddot{\theta}_2}{\partial\theta_1}(F\cos(\theta_2)-m_1l_1\sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\ 2}-m_1g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2)\\
&-m_2l_2\sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\ 2}-m_2g\cos^2(\theta_2)\sin(\theta_2))\\
&+(F\cos(\theta_2)-m_1l_1\sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\ 2}-m_1g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2)\\
&-m_2l_2\sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\ 2}-m_2g\cos^2(\theta_2)\sin(\theta_2))\\
&\frac{\partial\ddot{\theta}_2}{\partial\theta_1}(l_2m_c+m_1l_2sin(\theta_1)+m_2l_2\sin(\theta_2))^{-1}\\
&+\frac{\partial\ddot{\theta}_2}{\partial\theta_1}\frac{g\sin(\theta_2)}{l_2}\\
&=(l_2m_c)^{-1}\\
&(-F\sin(\theta_2)+m_1l_1\sin(\theta_1)\sin(\theta_2)\dot{\theta}_1^{\ 2}+m_1g\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)-\\
&m_2l_2(-\sin^2(\theta_2)+\cos^2(\theta_2))\dot{\theta}_2^{\ 2}-m_2g(-2\sin^2(\theta_2)\cos(\theta_2)+cos^2(\theta_2)))\\
&-2(F)(l_2m_c+m_1l_2sin(\theta_1)+m_2l_2\sin(\theta_2))^{-2}m_2l_2\sin(\theta_2)\cos(\theta_2)\\
&=\frac{-m_2g}{l_2m_c}
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} \\
&= (l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1}(F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\;2} - m_1 g \cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\;2} - m_2 g \cos^2(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\;2} - m_1 g \cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\;2} - m_2 g \cos^2(\theta_2)\sin(\theta_2)) \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1}(l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&- \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1}\frac{g\sin(\theta_2)}{l_2} \\
&= (l_2 m_c)^{-1} \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1}(F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\;2} - m_1 g \cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\;2} - m_2 g \cos^2(\theta_2)\sin(\theta_2)) \\
&+ (F) \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1}(l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&= (l_2 m_c)^{-1}(-2 m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\;2}) \\
&= 0
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
&\frac{\partial\ddot{\theta}_2}{\partial\theta_2} \\
&= (l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&\frac{\partial\ddot{\theta}_2}{\partial\theta_2}(F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\ 2} - m_1 g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\ 2} - m_2 g\cos^2(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\ 2} - m_1 g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\ 2} - m_2 g\cos^2(\theta_2)\sin(\theta_2)) \\
&\frac{\partial\ddot{\theta}_2}{\partial\theta_2}(l_2 m_c + m_1 l_2 sin(\theta_1) - m_2 l_2 \sin(\theta_2))^{-1} \\
&- \frac{\partial\ddot{\theta}_2}{\partial\theta_2}\frac{g\sin(\theta_2)}{l_2} \\
&= (l_2 m_c)^{-1} \\
&\frac{\partial\ddot{\theta}_2}{\partial\theta_2}(F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\ 2} - m_1 g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\ 2} - m_2 g\cos^2(\theta_2)\sin(\theta_2)) \\
&+ (F)\frac{\partial\ddot{\theta}_2}{\partial\theta_2}(l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&- \frac{\partial\ddot{\theta}_2}{\partial\theta_2}\frac{g\sin(\theta_2)}{l_2} \\
&= (l_2 m_c)^{-1} \\
&(-F\sin(\theta_2) + m_1 l_1 \sin(\theta_1)\sin(\theta_2)\dot{\theta}_1^{\ 2} + m_1 g\cos(\theta_1)\sin(\theta_1)\sin(\theta_2) \\
&- m_2 l_2(-\sin^2(\theta_2) + \cos(\theta_2))\dot{\theta}_2^{\ 2} - m_2 g(\cos^3(\theta_2) - \sin^2(\theta_2))) \\
&- 2(F)(l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-2}\sin(\theta_2)\cos(\theta_2) \\
&- \frac{g\cos(\theta_2)}{l_2} \\
&= \frac{-m_2 g}{l_2 m_c} - \frac{g}{l_2}
\end{aligned}
\tag{2.22}
$$

$$\begin{aligned}
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2} \\
&= (l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}(F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\,2} - m_1 g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\,2} - m_2 g\cos^2(\theta_2)\sin(\theta_2)) \\
&+ (F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\,2} - m_1 g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\,2} - m_2 g\cos^2(\theta_2)\sin(\theta_2)) \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}(l_2 m_c + m_1 l_2 sin(\theta_1) - m_2 l_2 \sin(\theta_2))^{-1} \\
&- \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}\frac{g\sin(\theta_2)}{l_2} \\
&= (l_2 m_c)^{-1} \\
&\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}(F\cos(\theta_2) - m_1 l_1 \sin(\theta_1)\cos(\theta_2)\dot{\theta}_1^{\,2} - m_1 g\cos(\theta_1)\sin(\theta_1)\cos(\theta_2) \\
&- m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\,2} - m_2 g\cos^2(\theta_2)\sin(\theta_2)) \\
&+ (F)\frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}(l_2 m_c + m_1 l_2 sin(\theta_1) + m_2 l_2 \sin(\theta_2))^{-1} \\
&- \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}\frac{g\sin(\theta_2)}{l_2} \\
&= (l_2 m_c)^{-1}(-2 m_2 l_2 \sin(\theta_2)\cos(\theta_2)\dot{\theta}_2^{\,2}) \\
&= 0
\end{aligned} \tag{2.23}$$

Which leaves the linearized A Matrix As

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-g m_1}{m_c} & 0 & \frac{-g m_1}{m_c} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{-g m_1}{l_1 m_c} - \frac{g}{l_1} & 0 & \frac{-g m_1}{l_1 m_c} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{-g m_1}{l_2 m_c} & 0 & \frac{-g m_1}{l_1 m_c} - \frac{g}{l_2} & 0
\end{bmatrix} \tag{2.24}$$

2.3 Linearizing by Input

$$\begin{aligned}
\ddot{x} &= \frac{1}{m_c + m_1 sin(\theta_1) + m_2\sin(\theta_2)}(F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^{\,2} \\
&- m_1 g\cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g\cos(\theta_2)\sin(\theta_2)) \\
\frac{\partial \ddot{x}}{\partial F} &= \frac{1}{m_c}
\end{aligned} \tag{2.25}$$

$$\ddot{\theta}_1 = \frac{F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^{\,2} - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g \cos(\theta_2)\sin(\theta_2)}{l_1(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2)}\cos(\theta_1)$$
$$- \frac{g\sin(\theta_1)}{l_1}$$
$$\frac{\partial\ddot{\theta}_1}{\partial F} = \frac{\cos(\theta_1)}{l_1 m_c} = \frac{1}{l_1 m_c} \tag{2.26}$$

$$\ddot{\theta}_2 = \frac{F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^{\,2} - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^{\,2} - m_2 g \cos(\theta_2)\sin(\theta_2)}{l_2(m_c + m_1 sin(\theta_1) + m_2 \sin(\theta_2)}\cos(\theta_2)$$
$$- \frac{g\sin(\theta_2)}{l_2}$$
$$\frac{\partial\ddot{\theta}_2}{\partial F} = \frac{\cos(\theta_2)}{l_2 m_c} = \frac{1}{l_2 m_c} \tag{2.27}$$

Which Leaves the linearized B matrix as

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_c} \\ 0 \\ \frac{1}{l_1 m_c} \\ 0 \\ \frac{1}{l_2 m_c} \end{bmatrix} \tag{2.28}$$

2.3.1 State Space Representation

Therefore we have the linearized state space representation as

$$\dot{(X(t))} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{m_c} & 0 & \frac{-gm_1}{m_c} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{l_1 m_c} - \frac{g}{l_1} & 0 & \frac{-gm_1}{l_1 m_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{l_2 m_c} & 0 & \frac{-gm_1}{l_1 m_c} - \frac{g}{l_2} & 0 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ \frac{1}{m_c} \\ 0 \\ \frac{1}{l_1 m_c} \\ 0 \\ \frac{1}{l_2 m_c} \end{bmatrix} U(t) \tag{2.29}$$

This is controllable when $l_1 \neq l_2$

2.4 LQR

Setting the appropriate values for A and B give us

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -11/20 & 0 & -1/20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1/10 & 0 & -11/10 & 0 \end{bmatrix} \tag{2.30}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20000} \\ 0 \\ \frac{1}{10000} \end{bmatrix} \tag{2.31}$$

Controllability is determined though the rank of the controllability matrix

$$\begin{bmatrix} B^k & AB^K & A^2B^K & A^3B^K & A^4B^K & A^5B^K \end{bmatrix}$$

Which when evaluated gives us

$$\begin{bmatrix} 0 & \frac{1}{1000} & 0 & -\frac{3}{20000} & 0 & \frac{59}{400000} \\ \frac{1}{1000} & 0 & -\frac{3}{20000} & 0 & \frac{59}{400000} & 0 \\ 0 & \frac{1}{10000} & 0 & -\frac{23}{200000} & 0 & \frac{519}{4000000} \\ \frac{1}{10000} & 0 & -\frac{23}{200000} & 0 & \frac{519}{4000000} & 0 \\ 0 & \frac{1}{20000} & 0 & -\frac{13}{400000} & 0 & \frac{189}{8000000} \\ \frac{1}{20000} & 0 & -\frac{13}{400000} & 0 & \frac{189}{8000000} & 0 \end{bmatrix} \tag{2.32}$$

Which we can see has rank 6, so it is controllable

In order to find gains for the LQR controller I solve for P using

$$A^T P + PA - PB_K R - 1B_K^T P = -Q$$

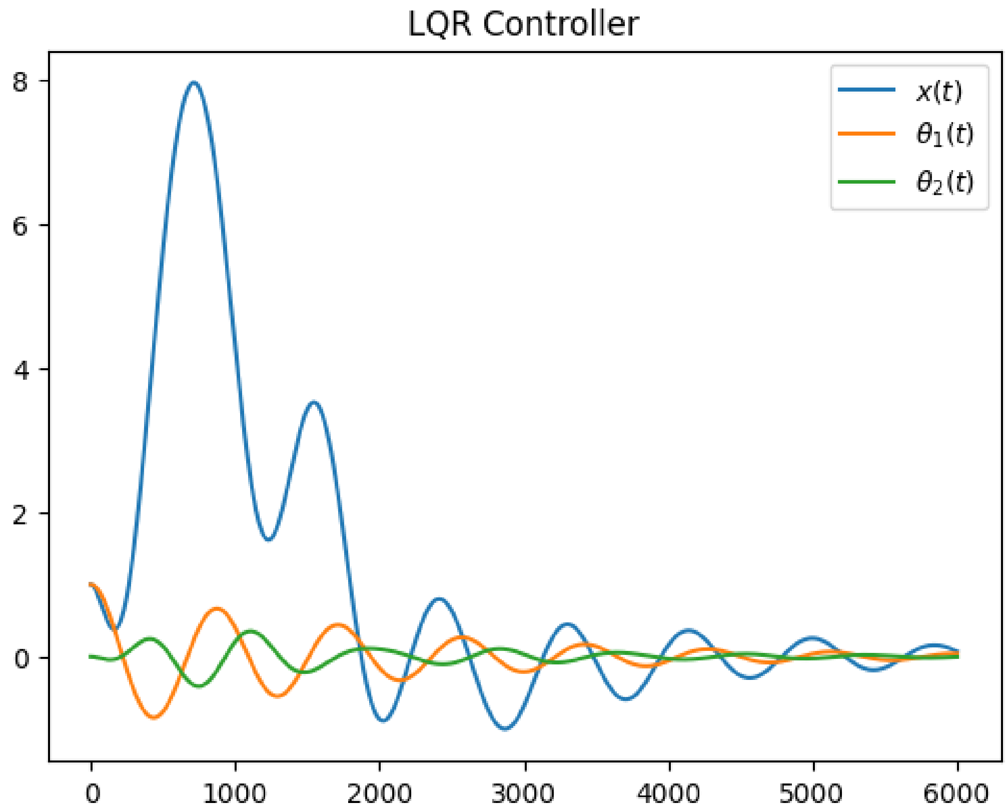
and

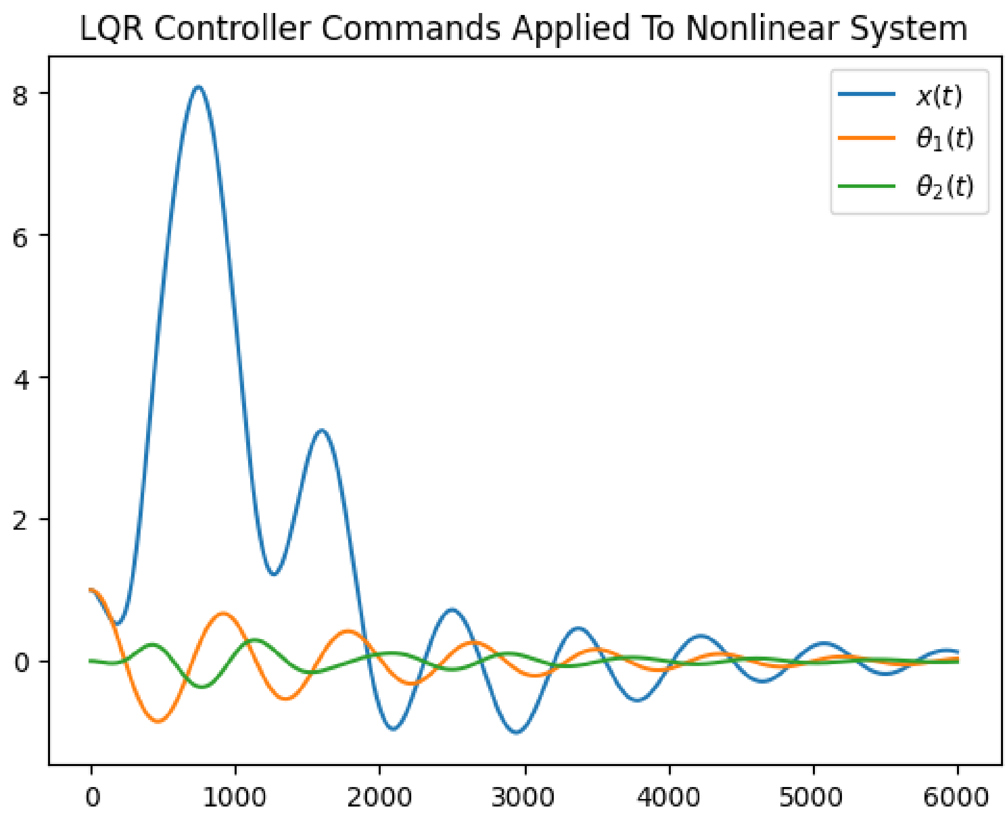
$$K = -R^{-1} B_k^T P$$

My Q and R matrices are set as

$$Q = \begin{bmatrix} 50 & 0 & 0 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix}, R = 0.02 \tag{2.33}$$

This does provide a solution which linearizes the system to the equilibrium point. The Following are graphs for the states over time





2.5 Luenberger Observer

A set (A, C) is controllable if $\begin{bmatrix} C^T, A^T C^T & (A^T)^2 C^T & (A^T)^3 C^T & (A^T)^4 C^T & (A^T)^5 C^T \end{bmatrix}$

Has full rank

For our system, if we only want to observe x(t), then

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$obsv = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0.65 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0.65 \\ 0 & 0 & -1 & 0 & 1.15 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1.15 \end{bmatrix} \tag{2.34}$$

Which is full rank, so observable

If we only want to observe $\theta_1(t), \theta_2(2)$, then

$$c = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$obsv = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -0.65 & 0 & 0.4725 & 0 \\ 0 & 1 & 0 & -0.65 & 0 & 0.4725 \\ 1 & 0 & -1.15 & 0 & 1.2975 & 0 \\ 0 & 1 & 0 & -1.15 & 0 & 1.2975 \end{bmatrix} \tag{2.35}$$

Which is not full rank, so not observable

If we only want to observe $x(t), \theta_2(t)$, then

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$obsv = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.1 & 0 & 0.815 & 0 \\ 0 & 0 & 0 & -1.1 & 0 & 0.815 \\ 1 & 0 & -2.1 & 0 & 2.365 & 0 \\ 0 & 1 & 0 & -2.1 & 0 & 2.365 \end{bmatrix} \tag{2.36}$$

Which is full rank, so observable

If we only want to observe $x(t), \theta_1(t), \theta_2(t)$, then

$$c = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$obsv = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.1 & 0 & 0.815 & 0 \\ 0 & 0 & 0 & -1.1 & 0 & 0.815 \\ 1 & 0 & -2.1 & 0 & 2.365 & 0 \\ 0 & 1 & 0 & -2.1 & 0 & 2.365 \end{bmatrix} \tag{2.37}$$

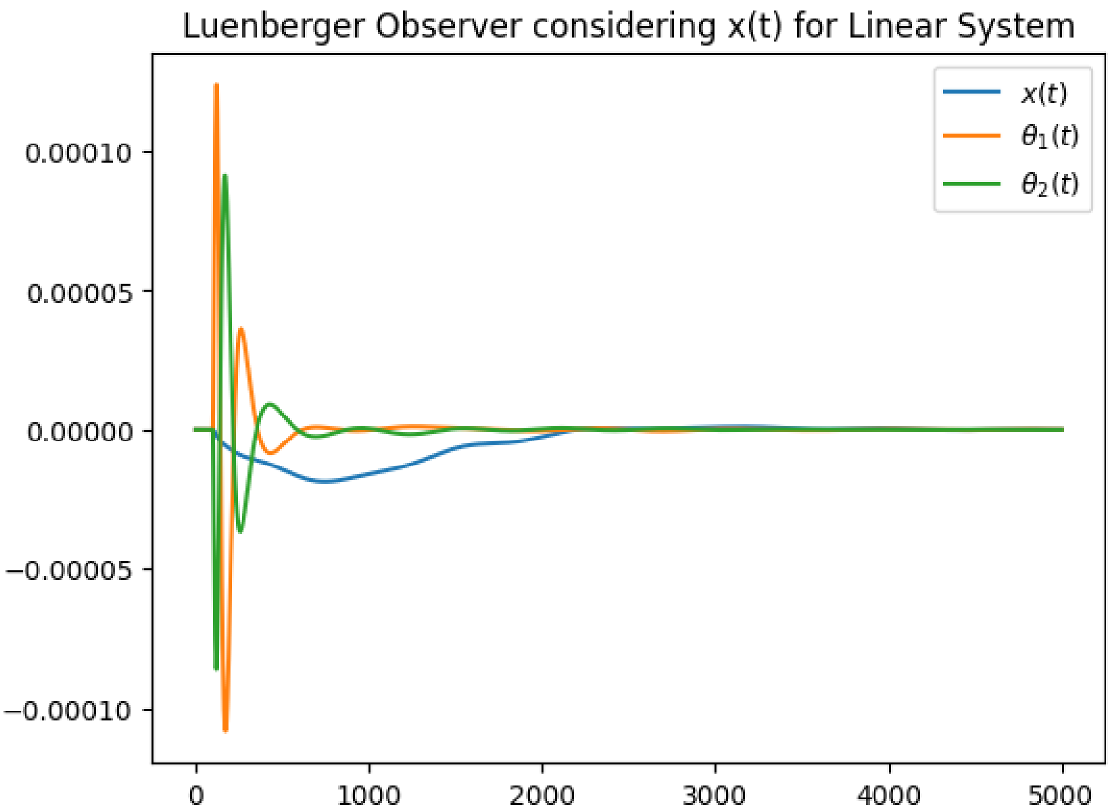
Which is full rank, so observable

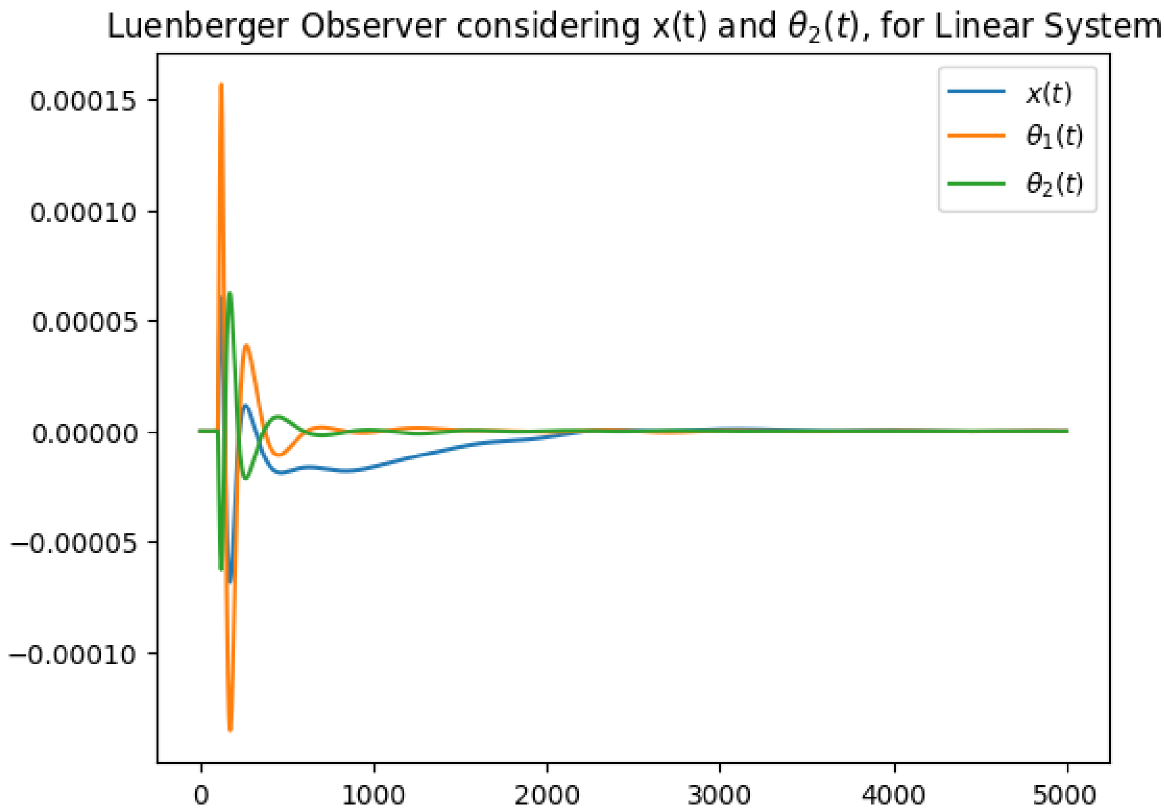
My "Best" Luenberger observer was just to have the poles of each (A-LC) be around integers from -1 to -6. I did this by treating L^T as the gain matrix for $A^T - C_T L^T$ since it has the same eigenvalues as $A - LC$.

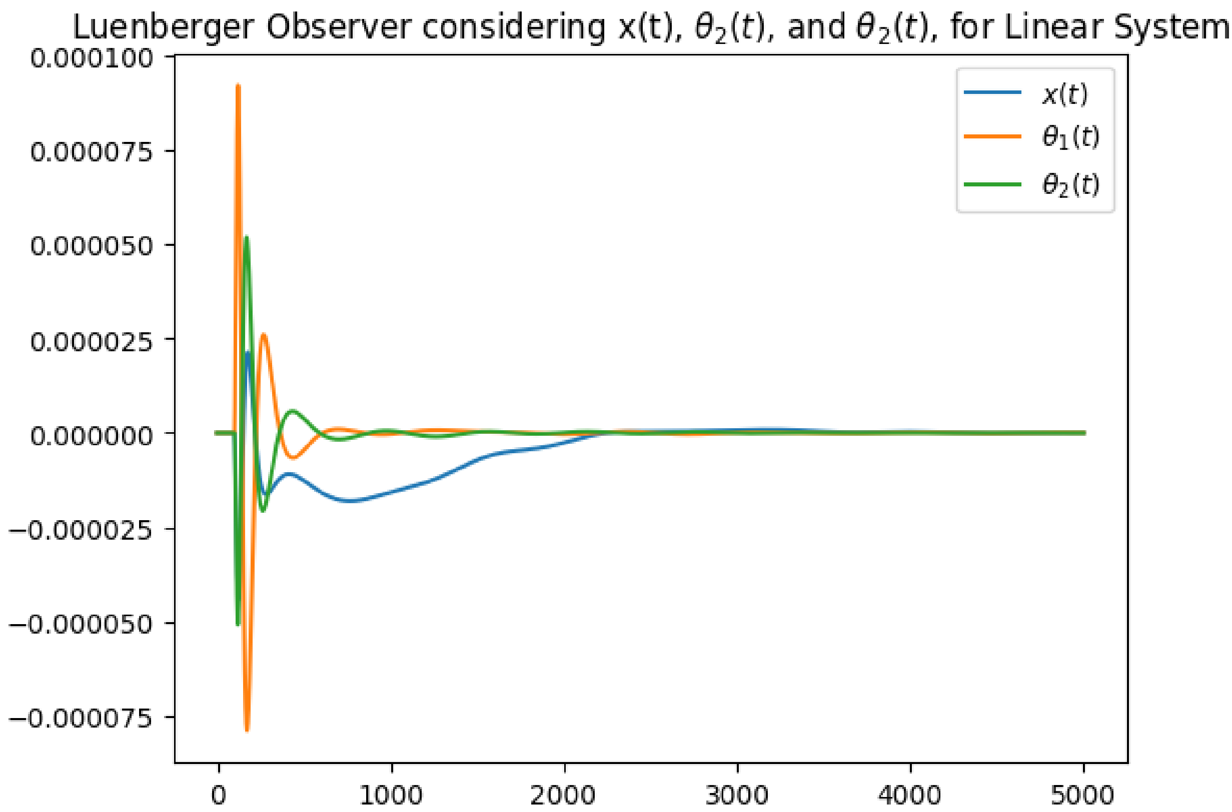
Using the unit step as a constant disturbance we use the standard form of

$$\begin{aligned}
\dot{x} &= (A + B_K K)x + B_D U \\
\dot{\hat{x}} &= (A + B_K K)\hat{x} + B_D U + LC(x - \hat{x}) \\
e &= x - \hat{x}
\end{aligned} \tag{2.38}$$

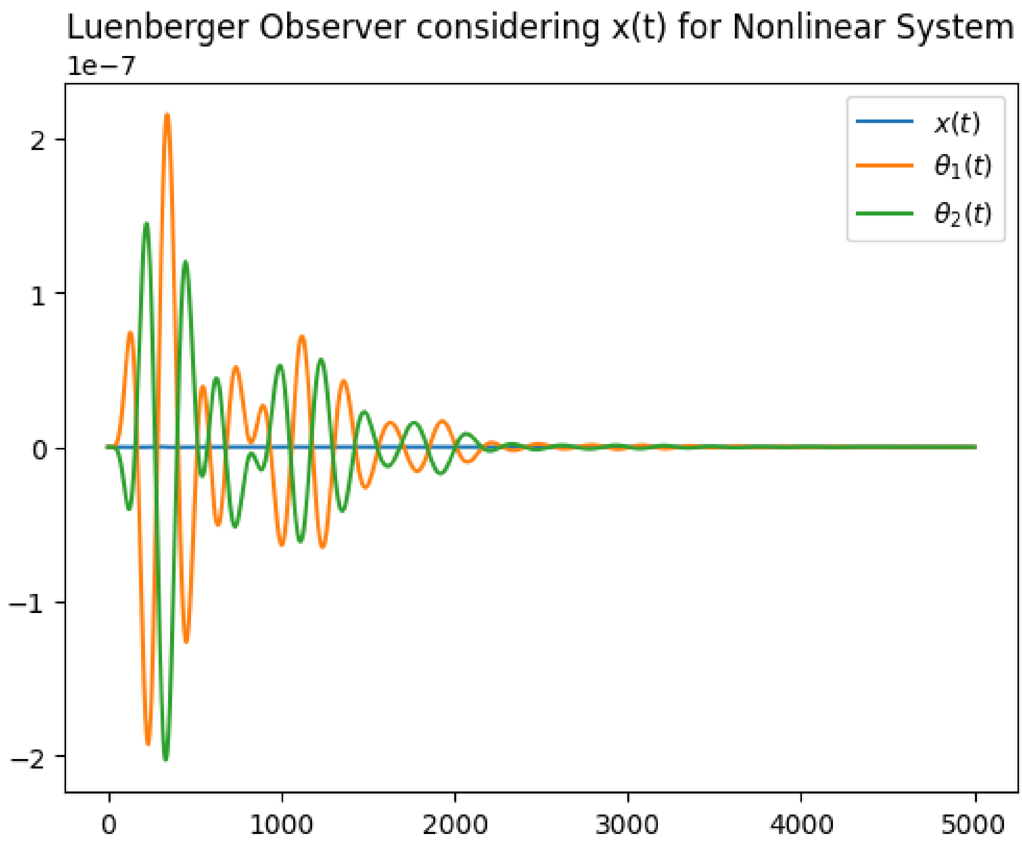
Below are the graphs for the plots of the linear Luenberger observers which track the error of the state

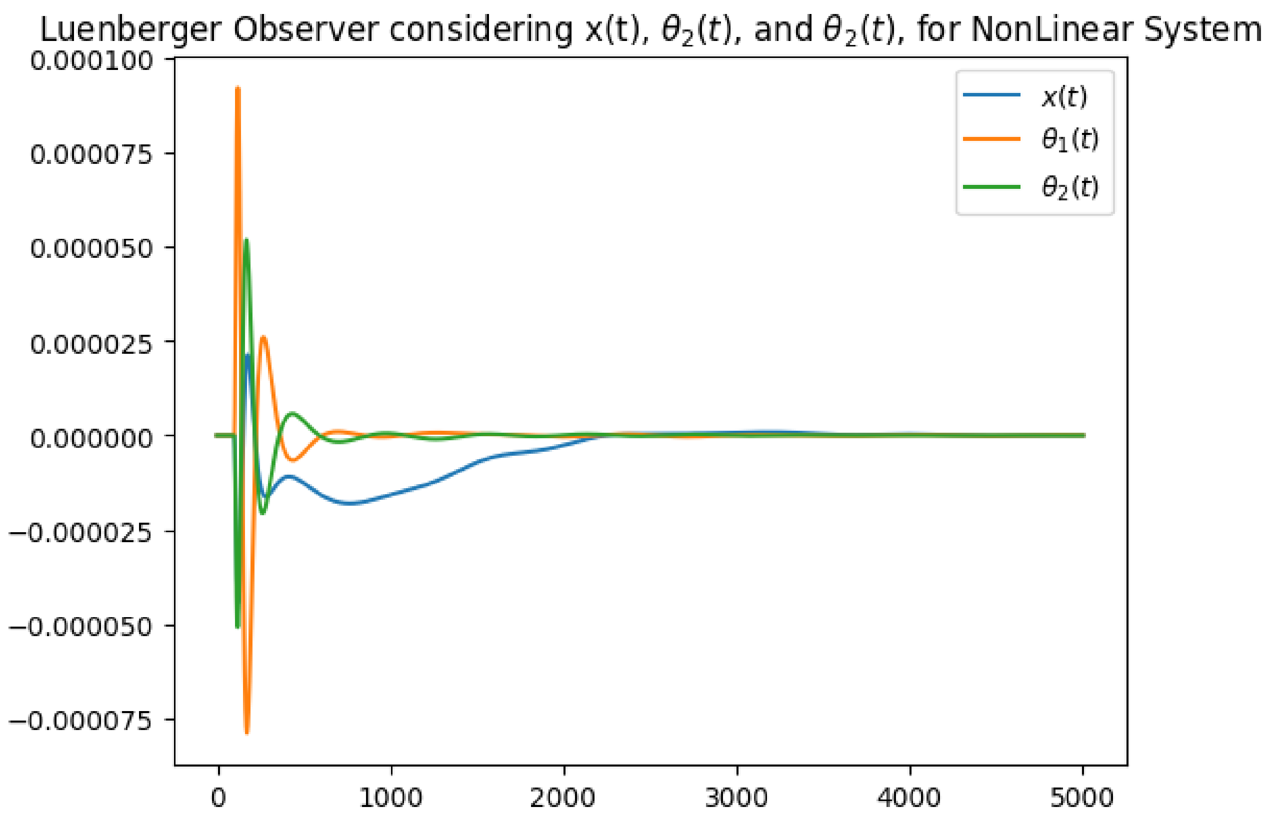






And here is the graph for the nonlinear Luenberger observer


Luenberger Observer considering x(t) for Nonlinear System
1e−7
2
1
0
−1
−2
0
1000
2000
3000
4000
5000
$x(t)$
$\theta_1(t)$
$\theta_2(t)$


Luenberger Observer considering x(t), $\theta_2(t)$, and $\theta_2(t)$, for NonLinear System
0.000100
0.000075
0.000050
0.000025
0.000000
−0.000025
−0.000050
−0.000075
0
1000
2000
3000
4000
5000
$x(t)$
$\theta_1(t)$
$\theta_2(t)$

2.6 LQG Controller

For the LQG Controller I used the same K matrix as in the LQR and Luenberger observer, bu the L was solved through

$$L = PC^T(Cov_d)^{-1} \tag{2.39}$$

Where P is the solution to

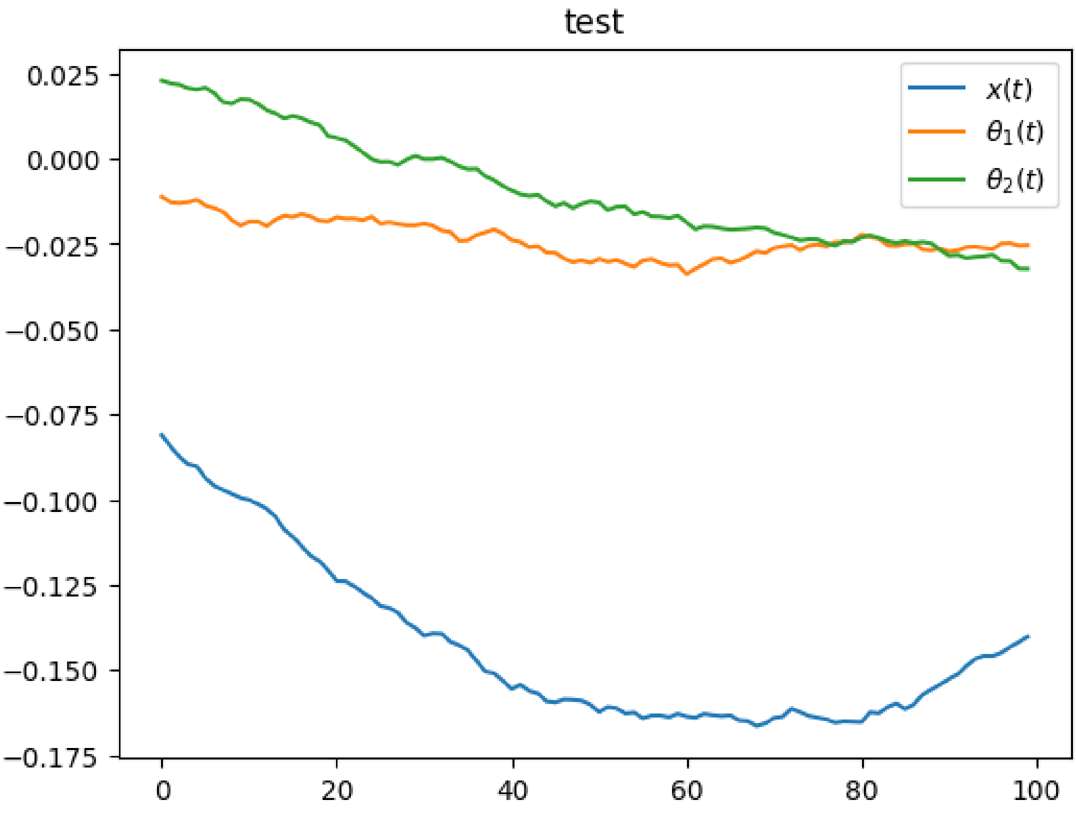
$$0 = A^T P + PA - PC^T(Cov_v)CP + Cov_d$$

Here $Cov_d = 0.01 \cdot I_6, Cov_v = 0.01$ are our process and measurement covariance matrices

If I were to implement a constant reference tracking to this then instead of having $(A + B_K K)X$ in our state space representation, I would use $(A + B_K K)(X - X_d)$ where d is out desired state. This wouldn't handle constant force disturbance unless you added a $B_d U_d$ term as well.

I was having issues with my solver and couldn't implement this in time.

Below shows the graph I was getting


test
0.025
0.000
−0.025
−0.050
−0.075
−0.100
−0.125
−0.150
−0.175
0
20
40
60
80
100
$x(t)$
$\theta_1(t)$
$\theta_2(t)$