From percolation on triangular lattice
to SLE

Ref Conformally invariant scaling limits: an overview and a collection of problem

by Schramm (ICM 2006).

The setting

$\tilde{G}$: triangular lattice on $\mathbb{Z}^2$.

As $(\tilde{G})^* =$ honeycomb lattice,

the site perc. on $\tilde{G}$ is equivalent to

"face" perc. on $G := (\tilde{G})^*$.

Consider Bernoulli perc. on G, from now on.

Thm (Kesten) $\quad p_c(G) = \frac{1}{2}$.

From now on, consider $p = p_c$.

Smirnov's result

For a region D,

fix points $a, b \in D$

denote by ∂_+ : arc $\widehat{ab}$ (counter-clockwise oriented)

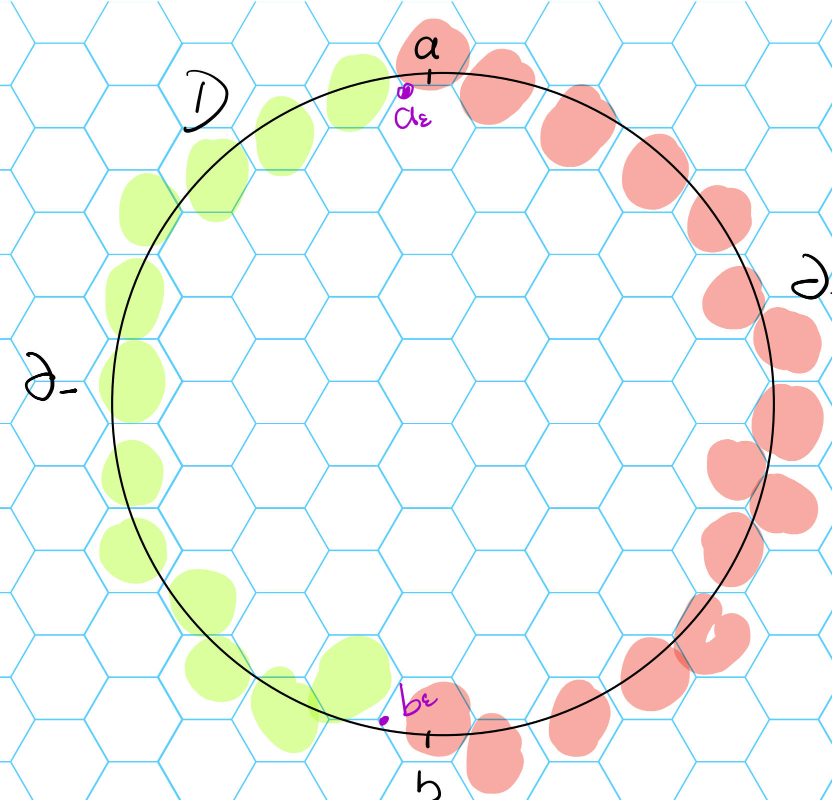
∂_- : arc $\widehat{ba}$

consider $\varepsilon G \subset D$,

denote by a_ε and b_ε

the nearest point in $\varepsilon G \cap D$

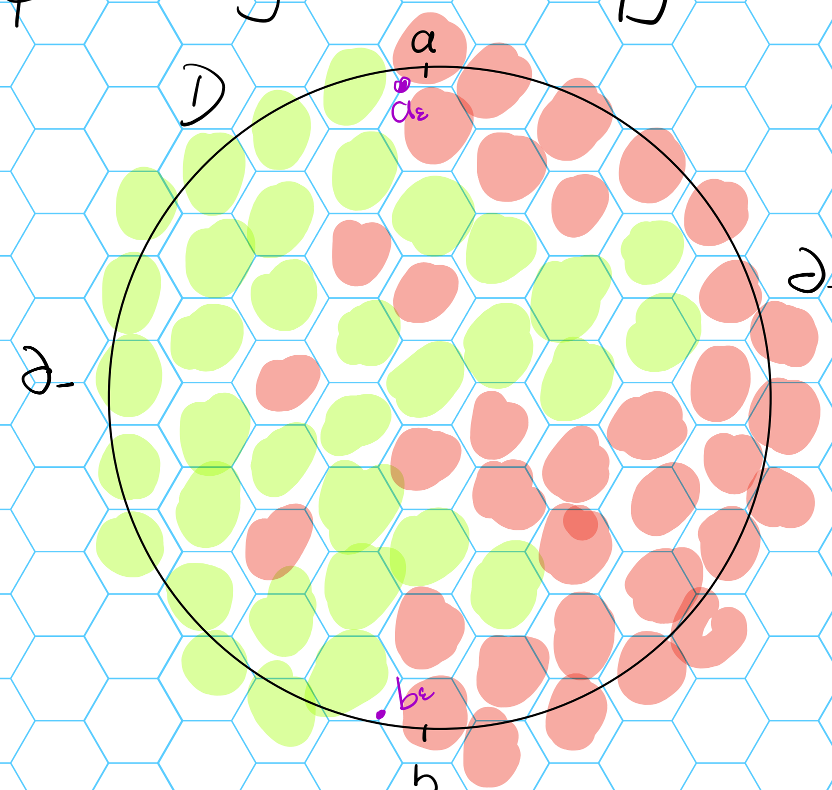
to a and b, resp.



Step 1 (Boundary condition)

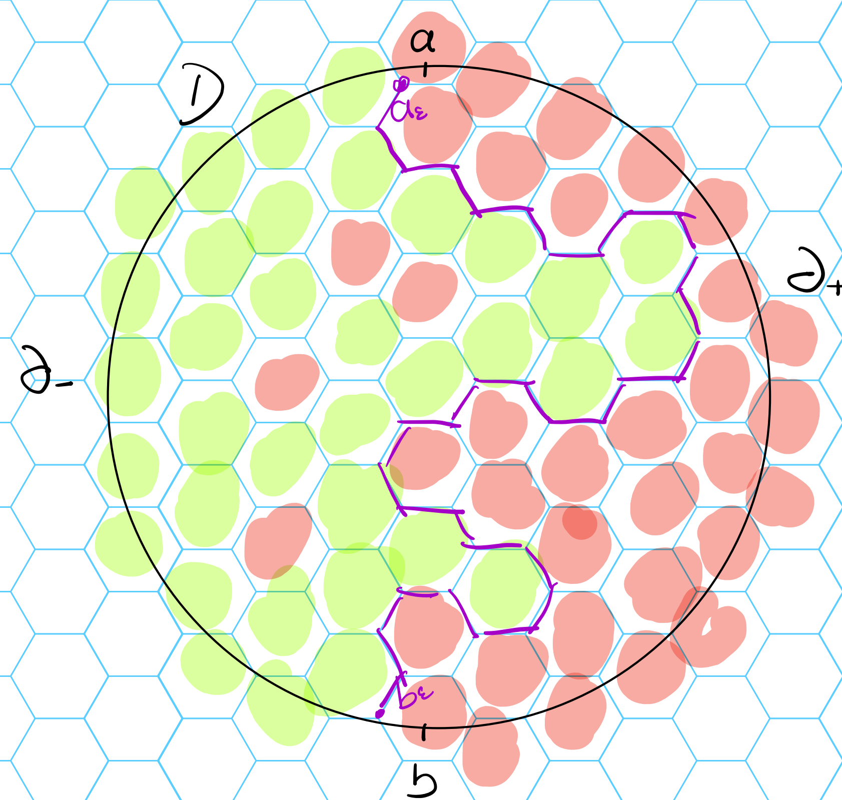
Mark all grids intersecting ∂D in εG according to if it's in ∂_+ or ∂_-.

Step 2 Then generate a config inside $\varepsilon G \cap D$.



Step 3 The interface is then defined to be

$$\{\ \} = \{\ \} \cap \{\ \}$$



Suppose this specific interface is a curve γ_ε.

From the perc. setting, the event

$$\{ \text{interface } a_\varepsilon \to b_\varepsilon \text{ is } \gamma_\varepsilon \}$$

has a prob which is determined.

Put $\Omega_{D,\partial_+} = \{ \text{close subsets of } \overline{D} \}$

and give it a Hausdorff metric.

Formally, we have a prob measure

on Ω_{D,∂_+}:

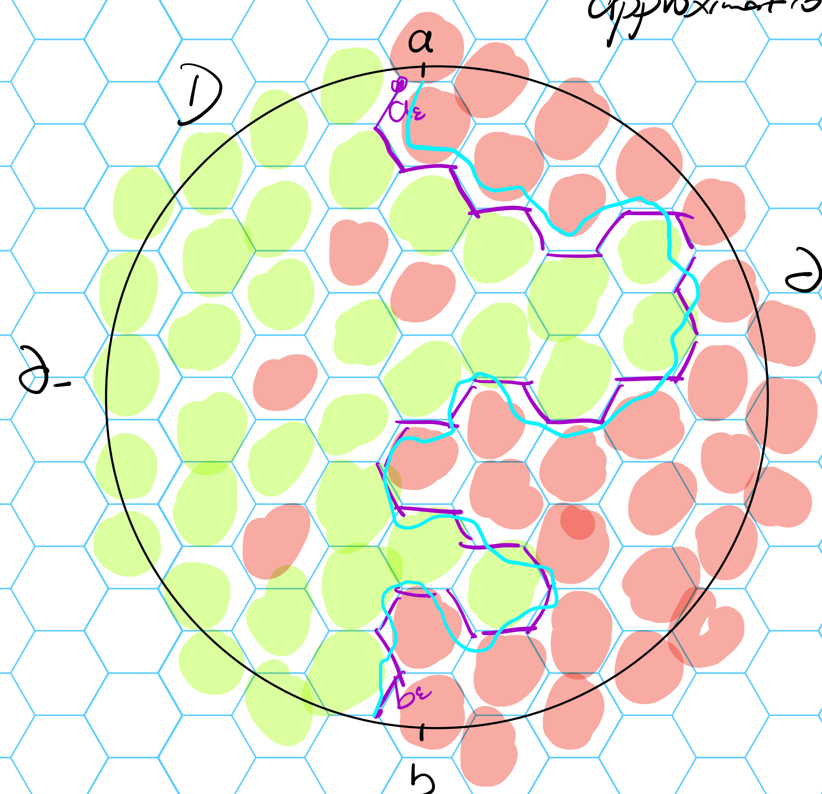
$$\mu_{D,\partial_+,\varepsilon} : \Omega_{D,\partial_+} \longrightarrow [0,1],$$

where

$$\mu_{D,\partial_+,\varepsilon}(S) =$$

$$P_{p_c,\varepsilon G}\{\omega : \text{the interface determined by } \omega \text{ is the best approximation in Hausdorff metric sense for some "curve" (as point) } s \in S\}$$

ex. here I give a curve to which the purple interface is the best approximation.



Question How $\mu_{D,\partial+,\varepsilon}$ will be when $\varepsilon \downarrow 0$?

Can it "represent" in an approximate sense, the "actual" continuous curves?

Thm (Smirnov).

$\exists$ measure $\mu_{D,\partial+}$ on Ω_D,

such that

$\mu_{D,\partial+,\varepsilon}$ weakly converges to $\mu_{D,\partial+}$, (μ ‖)

and μ is conformally invariant, in following sense:

Consider D' another region in $\mathbb{R}^2$ and $f: \overline{D} \to \overline{D'}$ a conformal mapping,

and consider the following ε-approximation over D', then the diagram commutes:

$$\begin{tikzcd} \Omega_{D,\partial_+} \arrow[r, "f"] \arrow[dr, "\mu_{D,\partial_+}"'] & \Omega_{D',f(\partial_+)} \arrow[d, "\mu_{D',\partial_+}"] \\ & {[0,1]} \end{tikzcd}$$

From Smirnov to SLE

This time, take $D = \mathbb{H}$, $\partial_+ = \mathbb{R}_+$.

Now we temporarily forget the percolation setting, instead, we only consider a priori a curve $\beta: [0,+\infty) \to \mathbb{H}$ with $\beta(0) = 0$, without thinking of it as a percolation interface.

For any $t \in [0,+\infty)$, Comparing $\overline{\mathbb{H}} \setminus \beta([0,t])$ and $\overline{\mathbb{H}}$ which are both open, we have

Thus (Riemann mapping theorem) $\exists$ conformal mapping

$$g_t : \mathbb{H} \setminus \beta([0,t]) \to \mathbb{H},$$

and under some additional conditions, g_t is unique.

Remark To make it unique we put

1. $g_t(\infty) = \infty$;

and as g_t is analytic at ∞,

consider its power series at $1/z$:

(for $|z|$ very large)

$$g_t(z) = \underbrace{a_1 z}_{\text{to make sure } g_t(\infty)=\infty} + a_0 + \underbrace{a_{-1} z^{-1} + a_{-2} z^{-2} + \cdots}_{\text{converges}},$$

As g_t maps boundary to boundary,

$g_t(z) \in \mathbb{R}$ for z very large.

$\Rightarrow$ as $a_k = \dfrac{g_t^{(k)}(z)}{k!}$,

every a_k is real.

Then because $g_t(z) \in \mathbb{H}$

($\operatorname{Im} g_t(z) > 0$),

$a_1 > 0$.

② We can then ask

$$a_1 = 1, \quad a_0 = 0$$

Now consider each a_j as func of t.

Prop. $a_{-1}(t)$ is continuous and $\nearrow$.

Proof: By intuition. $\square$

Now reparametrize β so that

$$a_{-1}(t) = 2t,$$

which is called half-plane capacity parametrization of β.

Thm (Loewner's equation)

for z such that $g_t(z)$ is defined

we have

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - W(t)},$$

where $W(t) := g_t(\beta(t)) = \lim_{z \to \beta(t)} g_t(z)$

is called the Loewner driving term.

Rmk ① $W(t)$ is defined by Carathéodory's thm.

② Meaning:

transforms β: a path in $\mathbb{H}$

into

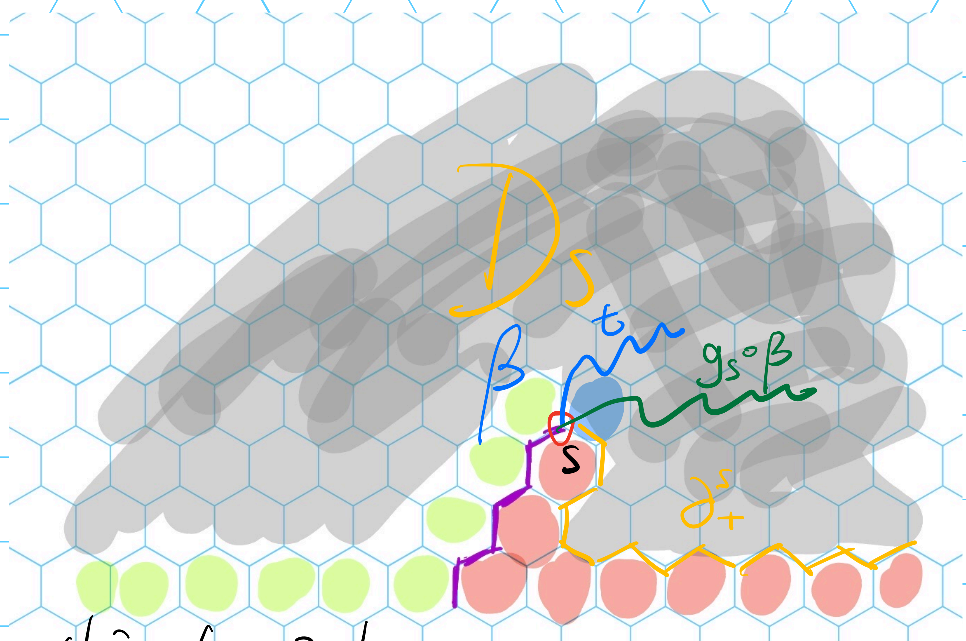
$W(t)$: a path in $\partial\mathbb{H} = \mathbb{R}$.

On the other hand, given a $W(t)$ we get a curve in $\mathbb{H}$ by solving the equation!

Applying the percolation setting

Now consider a smaller $\varepsilon > 0$ for $D = \mathbb{H}$).

Assume we now have already found the first 6 steps of the interface curve:



Now for this terminal vertex s the algorithm to find the next step is:

① Determine all the 3 adjacent faces.

② Among the 3 faces, check one by one (if necessary, in ↺ orientation).

(At this example, only needs to

be checked).

③ in step 2, if find new segment of interface, stop and move to $(S+1)$-step.

the Probability argument

We start from $\beta(S)$ now, that is, we consider $D = D_S$ and ∂_S^+ as in the picture.

Then the Prob of $\beta(\tau_S, \infty)$

$$\sim \mu_{D_S, \partial_\tau^S, \varepsilon}.$$

By Smirnov's thm:

take $G: D_S \longrightarrow \mathbb{H}$ the conformal map,

then
$$\begin{tikzcd} \Omega_{D_S, \partial_T^S} \arrow[r, "G"] \arrow[dr, "\mu_{D_S, \partial_T^S}"'] & \Omega_{\mathbb{H}, G(\partial_T^S)} \arrow[d, "\mu_{\mathbb{H}, G(\partial_T^S)}"] \\ & {[0,1]} \end{tikzcd}$$

commutes and the ε-approximation says

$\mu_{D_S, \partial_T^S, \varepsilon}$ is close to

$\mu_{\mathbb{H}, G(\partial_T^S), \varepsilon}$.

Moreover G is close to g_S

i.e., $G(\partial_T^S)$ is close to

$[g_S(\beta(S)), \infty)$.

As at this time $\beta([0,S])$ is "known information"),

We get a "conditional probability".

that $$g_s \circ \beta([s,\infty)) \sim \mu_{\mathbb{H},\mathbb{R}_+}$$

Consider

$$t \longmapsto g_s \circ \beta(s+t).$$

$$\begin{tikzcd} \beta[s,s+t] & & g_s(\beta[s,s+t]) \arrow[d, phantom, "\cup"] \\ \mathbb{H}\setminus\beta([0,s]) \arrow[rr, "g_s"] \arrow[dd, "g_{s+t}"'] & & \mathbb{H} \arrow[ll, bend left, "g_s^{-1}"] \\ & & \mathbb{H}\setminus g_s(\beta[s,s+t]) \arrow[dll, "g_{s+t}\circ g_s^{-1}"] \\ \mathbb{H} & & \end{tikzcd}$$

$g_{s+t}\circ g_s^{-1}$ $\Vert$ gives the driving func.

$$\tilde{W}(t) = (g_{s+t} \circ g_s^{-1})(g_s \circ \beta[s, s+t])$$

$$= g_{s+t}(\beta[s, s+t])$$

$$= W(s+t)$$

We have the conclusion about $W(t)$:

Prop (Markov property) for $s > 0$,

Given $(W(t) : t \in [0, s])$,

the distribution of $W(s+t)$ $(t > 0)$

is the same as $W(t)$.

Thm $W(t) = B(kt)$ for some $k \geq 0$

and B the standard one-dim Brownian motion.

Proof. (Not easy, but you need to carefully check some probability properties of W). $\square$

This gives a motivation to define.

Def (chordal SLE).

For $\kappa \geq 0$, let g_t be the solution of Loewner's equation.

satisfying
$$\begin{cases} g_0 = Id \\ W(t) = B(\kappa t). \end{cases}$$

where B is standard one-dim Brownian motion.

Then g_t is called chordal SLE with parameter κ.