Nonlinear Equations

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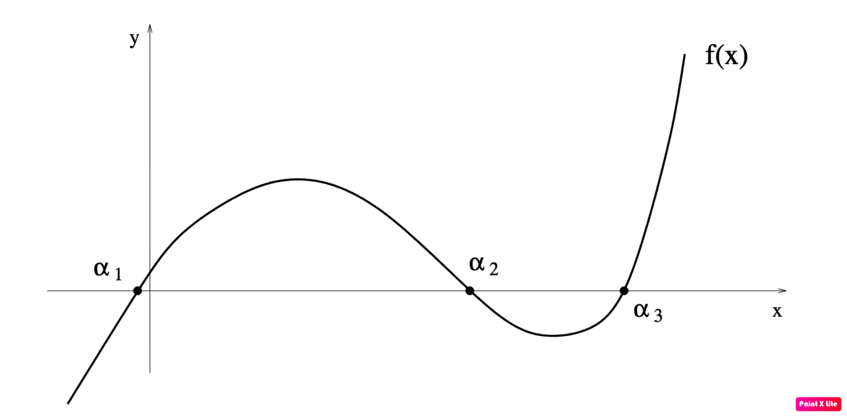
Plan

Introduction

Goal: Finding the zeros of nonlinear functions (or systems), i.e., the values $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$.



Problem Statement

Goal: Calculate the average annuity rate I of an investment fund over several years.

Given Data

- Annual investment: $v = 1000$ CHF
- Period: $n = 5$ years
- Final amount: $M = 6000$ CHF

Governing Formula

The relationship between the final amount M, the annual investment v, the rate I, and the number of years n is:

$$M = v \sum_{k=1}^{n} (1 + I)^k$$

Mathematical Formulation

The geometric series can be simplified to a closed-form expression:

$$M = v \sum_{k=1}^{n} (1 + I)^k = v \, \frac{1 + I}{I} \left[(1 + I)^n - 1 \right]$$

Core Problem:

Root-Finding Equation

Find the rate I such that the function $f(I)$ equals zero:

$$f(I) = M - v \, \frac{1 + I}{I} \left[(1 + I)^n - 1 \right] = 0 \quad (1)$$

Analysis and Conclusion

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- There is **no explicit algebraic formula** to solve for I directly.
- An **exact analytical solution** is not attainable by standard methods.
- This is a classic candidate for a **numerical solution method**.

Takeaway

To find the annuity rate I satisfying $f(I) = 0$, we must employ a numerical algorithm,

Introduction: Scalar Case

Problem Statement

Given a function $f : \mathbb{R}^p \to \mathbb{R}^p$, find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}) = \mathbf{0}. \quad (1)$$

Outline of This Chapter

- We will **first treat the scalar case** $(p = 1)$.
- The **end of the chapter will be devoted to the general vector case**.

Bisection method

Numerical Resolution of Equations

The Numerical Problem

Solving the equation $f(x) = 0$ numerically means finding x^* such that

$$|f(x^*)| \leq \varepsilon, \quad \text{with } \varepsilon \text{ very small.}$$

Fundamental Assumption: Separable Root

We will assume in the following that the root x^* is **separable**. That is, there exists an interval $[a, b]$ such that x^* is the *only* root within that interval.

Intermediate Value Theorem

Theorem

Let f be a continuous function on $[a, b]$, and let y be a number between $f(a)$ and $f(b)$ (i.e., $y \in [\min(f(a), f(b)), \max(f(a), f(b))]$). Then, there exists a point $x \in [a, b]$ such that $f(x) = y$.

Example

Crucial Consequence: Suppose there exists an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$. Since the function changes sign, by the Intermediate Value Theorem, there exists at least one point $\bar{x}$ where $f(\bar{x}) = 0$.

The Bisection Method (Dichotomy)

Core Assumptions

- The root x^* is **separable** (isolated) within the initial interval $[a, b]$.
- The function f is **continuous** on $[a, b]$.
- The initial interval satisfies the **sign change condition**: $f(a) \cdot f(b) < 0$.

The Bisection Algorithm

Initialization: Set $a_0 = a, \; b_0 = b$ where $f(a)f(b) < 0$
For $k = 0, 1, 2, \ldots$ until convergence **do:**

1. Compute the midpoint: $x_k = \dfrac{a_k + b_k}{2}$
2. **Stopping Test Check:** Evaluate $|f(a_k)| \leq \epsilon$
3. **Interval Update:**

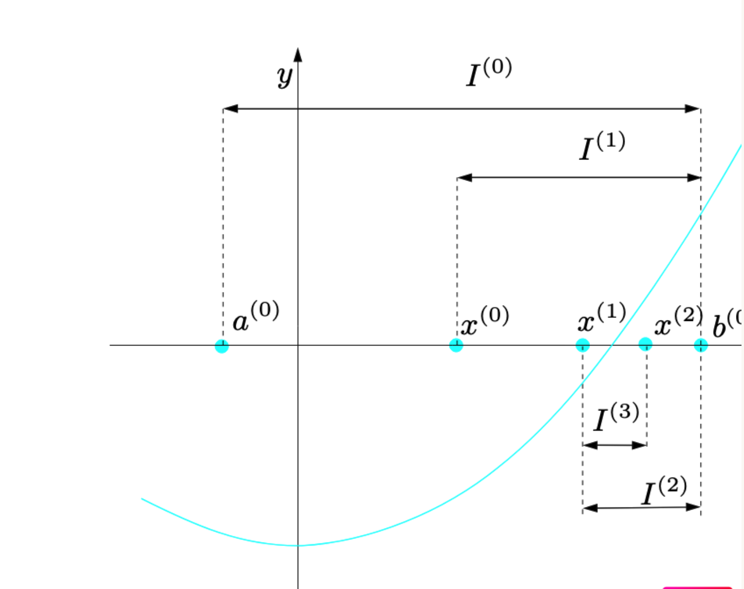
$$\begin{cases} \text{If } f(a_k) \cdot f(x_k) < 0: & b_{k+1} = x_k; \;\; a_{k+1} = a_k \\ \text{Else:} & a_{k+1} = x_k; \;\; b_{k+1} = b_k \end{cases}$$

End For. The approximate root is the final x_k.

Geometric Interpretation

At each iteration, the algorithm **halves the search interval** $[a_k, b_k]$ by evaluating the midpoint. It retains the subinterval where the sign change (and hence the root) is guaranteed by the Intermediate Value Theorem.

Bisection Method: Example Visualization: Function $f(x) = x^2 - 1$ with bisection intervals



For example, in the case shown in above Figure , which corresponds to

$$f(x) = x^2 - 1,$$

taking $a^{(0)} = -0.25$ and $b^{(0)} = 1.25$, we obtain

$$I^{(0)} = (-0.25,\ 1.25), \quad x^{(0)} = 0.5,$$

$$I^{(1)} = (0.5,\ 1.25), \quad x^{(1)} = 0.875,$$
$$I^{(2)} = (0.875,\ 1.25), \quad x^{(2)} = 1.0625,$$
$$I^{(3)} = (0.875,\ 1.0625), \quad x^{(3)} = 0.96875.$$

Note that each subinterval $I^{(k)}$ contains the root α. Moreover, the sequence $\{x^{(k)}\}$ necessarily converges to α since at each step the length

$$|I^{(k)}| = b^{(k)} - a^{(k)}$$

of the interval $I^{(k)}$ is divided by two. Since

$$|I^{(k)}| = \left(\frac{1}{2}\right)^k |I^{(0)}|,$$

the error at iteration k satisfies

$$|e^{(k)}| = |x^{(k)} - \alpha| < \frac{1}{2}|I^{(k)}| = \left(\frac{1}{2}\right)^{k+1}(b-a).$$

To guarantee that $|e^{(k)}| < \varepsilon$ for a given tolerance ε, it is sufficient to perform $k_{\min}$ iterations, where $k_{\min}$ is the smallest integer such that

$$k_{\min} > \log_2\left(\frac{b-a}{\varepsilon}\right) - 1. \tag{2.6}$$

Note that this inequality is general: it does not depend on the choice of the function f.

Bisection Method: Step-by-Step Example

Solving $f(x) = x^2 - 1 = 0$ on $I^{(0)} =]-0.25, 1.25[$

1. Problem Definition

Find the positive root of the equation:

$$f(x) = x^2 - 1 = 0$$

- Known exact roots: $x^* = 1$ and $x^* = -1$.
- Initial interval: $I^{(0)} =]a^{(0)}, b^{(0)}[=]-0.25, 1.25[$
- Verification: $f(-0.25) < 0$, $f(1.25) > 0 \Rightarrow$ Sign change confirmed.

2. Bisection Iterations (First 4 Steps)

k	Interval $I^{(k)}$	Midpoint $x^{(k)}$	$f(x^{(k)})$	Update Rule
0	$]-0.25,\ 1.25[$	0.5000	-0.7500	$f(a^{(0)})f(x^{(0)}) > 0 \Rightarrow a^{(1)} = x^{(0)}$
1	$]0.5000,\ 1.25[$	0.8750	-0.2344	$f(a^{(1)})f(x^{(1)}) > 0 \Rightarrow a^{(2)} = x^{(1)}$
2	$]0.8750,\ 1.25[$	1.0625	$+0.1289$	$f(a^{(2)})f(x^{(2)}) < 0 \Rightarrow b^{(3)} = x^{(2)}$
3	$]0.8750,\ 1.0625[$	0.96875	-0.0610	$f(a^{(3)})f(x^{(3)}) > 0 \Rightarrow a^{(4)} = x^{(3)}$

- **Logic:** Interval updates based on sign of $f(a^{(k)}) \cdot f(x^{(k)})$.
- **Progress:** Interval width halves each step: $1.5 \to 0.75 \to 0.375 \to 0.1875$.

3. Practical Example

Let $a = 1$, $b = 2$, and $\varepsilon = 10^{-4}$. The required iterations are:

$$k \geq \log_2\left(\frac{2-1}{10^{-4}}\right) - 1 = \log_2\left(10^4\right) - 1 \approx \log_2(10000) - 1 \approx 12.29.$$

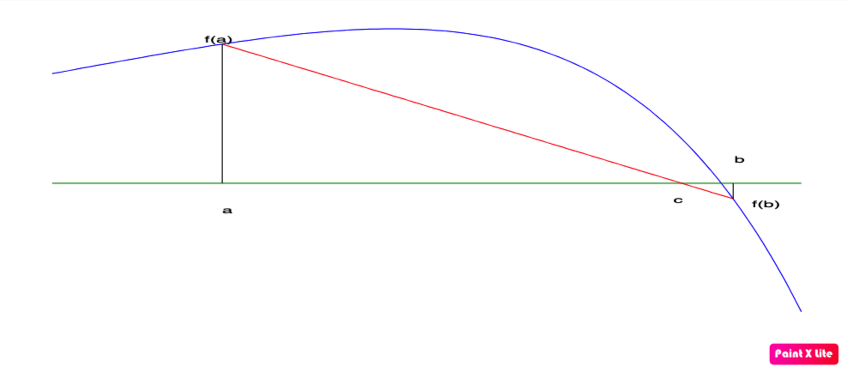
$$\boxed{k \geq 13}$$

- This shows that **13 iterations are sufficient** to guarantee an error less than 10^{-4}, independent of the specific function (as long as it's continuous and has a sign change).

Bisection Method: First Variant (Regula-Falsi Method)

Bisection Method: First Variant

Instead of taking x_k as the midpoint of $[a_k, b_k]$, we will first draw the line passing through the two points $(a_n, f(a_k))$ and $(b_k, f(b_k))$. The new point x_k will therefore be the intersection of this line with the Ox axis.


f(a)
b
c
f(b)
a
Paint X lite

Bisection Method: First Variant

The line passing through the points $(a_k, f(a_k))$ and $(b_k, f(b_k))$ has the equation

$$y = f(a_k) + (x - a_k)\frac{f(b_k) - f(a_k)}{b_k - a_k}.$$

We therefore deduce that x_k is given by

$$x_k = \frac{a_k f(b_k) - b_k f(a_k)}{f(b_k) - f(a_k)}.$$

By modifying the previous algorithm we obtain

Regula Falsi (False Position) Method

Hypothesis: x^* is separable (isolated) in $[a, b]$. Furthermore, suppose that $f(a)f(b) < 0$.

We define the following algorithm:

False Position Algorithm (Regula-Falsi)

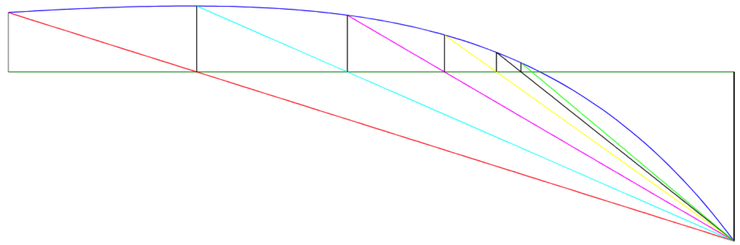
Initialization: $a_0 = a$, $b_0 = b$ with $f(a)f(b) < 0$

Iterations: For $k = 0, 1, 2, \ldots$

1. Compute: $x_k = \dfrac{a_k f(b_k) - b_k f(a_k)}{f(b_k) - f(a_k)}$
2. Condition Check: Evaluate $f(a_k) \cdot f(x_k)$
3. Update interval:

$$\begin{cases} \text{If } f(a_k) \cdot f(x_k) < 0: & a_{k+1} = a_k,\ b_{k+1} = x_k \\ \text{Otherwise:} & a_{k+1} = x_k,\ b_{k+1} = b_k \end{cases}$$

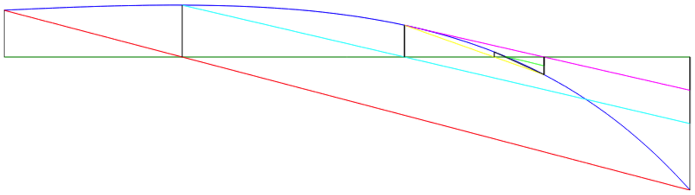
End For.



Bisection Method: Second Variant (Modified Regula-Falsi Method)

Interpretation of the Modified Regula-Falsi Method

We will remedy this small problem by changing the ordinate (we will divide it by two) of the point causing the stagnation. If we refer back to the previous figure, we now obtain:



Algorithm: Modified Regula-Falsi (Illinois Algorithm)

Hypothesis: x^* is isolated in $[a, b]$ with $f(a)f(b) < 0$.
Input: Function f, interval $[a, b]$, tolerances ε_x, ε_f, maximum iterations $N_{\max}$.
Output: Approximate root w, error code *ier*, iterations N.

1. Initialize: $a_0 \leftarrow a,\ b_0 \leftarrow b,\ f_a \leftarrow f(a),\ f_b \leftarrow f(b),\ w \leftarrow a,\ f_w \leftarrow f_a$.
2. For $N = 1$ to $N_{\max}$:
   - **2.1** If $|b - a| < \varepsilon_x$: set $ier = 0$; **return** (converged by interval).
   - **2.2** If $|f_w| \leq \varepsilon_f$: set $ier = 1$; **return** (converged by residual).
   - **2.3** Compute new point: $w \leftarrow \dfrac{a \cdot f_b - b \cdot f_a}{f_b - f_a}$.
   - **2.4** Save previous value: $f_{wp} \leftarrow f_w$; evaluate $f_w \leftarrow f(w)$.
   - **2.5** **Update interval:**
     - If $f_a \cdot f_w > 0$:
       $$a \leftarrow w,\ f_a \leftarrow f_w; \quad \text{and if } (f_w \cdot f_{wp} > 0) \text{ then } f_b \leftarrow f_b/2.$$
     - Else:
       $$b \leftarrow w,\ f_b \leftarrow f_w; \quad \text{and if } (f_w \cdot f_{wp} > 0) \text{ then } f_a \leftarrow f_a/2.$$
3. $ier = 2$ (maximum iterations reached); **return**.

Newton Method

Newton Method

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $x^{(0)}$ be a given starting point.

- Consider the line $y(x)$ passing through the point $(x^{(k)}, f(x^{(k)}))$ with slope $f'(x^{(k)})$:

$$y(x) = f'(x^{(k)})(x - x^{(k)}) + f(x^{(k)}).$$

- Define $x^{(k+1)}$ as the point where this line intersects the x-axis, i.e., $y(x^{(k+1)}) = 0$.

Solving for $x^{(k+1)}$ yields the Newton-Raphson iteration formula:

$$\boxed{x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}}, \quad k = 0, 1, 2, \ldots$$

Newton Algorithm

Input Parameters:

- x_0: initial approximation
- ε: desired tolerance
- `itemax`: maximum number of iterations

Algorithm:

Given: An initial point x_0 and a tolerance ε. Set $k = 0$.

While: $|f(x_k)| > \varepsilon$ **and** $k \leq \mathtt{itemax}$, **do:**

Step: Compute the Newton update:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Increment: $k \leftarrow k + 1$.

nd While.

Newton's Method: First Iterations for a Specific Function

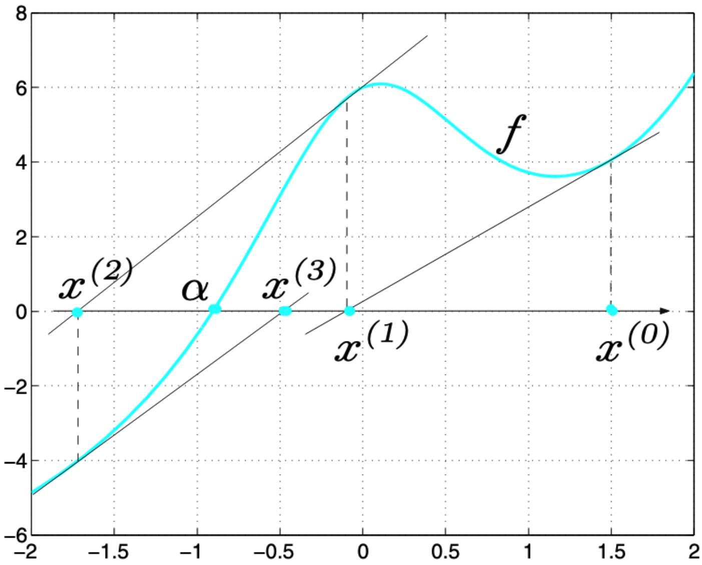
Consider the function:

$$f(x) = x + e^x + \frac{10}{1 + x^2} - 5.$$

The first iterations obtained with Newton's method, starting from an initial guess $x^{(0)}$, follow the iteration formula:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}.$$

A complete iteration table would require computing $f'(x)$ and evaluating at successive $x^{(k)}$.



8
6
4
2
0
−2
−4
−6
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
f
$x^{(2)}$
α
$x^{(3)}$
$x^{(1)}$
$x^{(0)}$

Newton's Method: Important Remarks

- The function f must be **differentiable**.
- x_{k+1} may not be computable if:
  - $f'(x_k) = 0$ (division by zero).
  - x_k is not in the domain of definition of f.
- Each iteration requires:
  - One evaluation of f.
  - One evaluation of f'.
- This method is also often called the **Newton-Raphson method**.
- Newton's method is a **fixed-point method**, since x_{k+1} can be written in the form $x_{k+1} = g(x_k)$ with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Newton's Method: Example 1 (Computing a^{-1})

Let a be a positive number. To compute a^{-1}, we apply Newton's method to the function

$$f(x) = \frac{1}{x} - a.$$

The Newton iteration for this function takes the following form:

$$\boxed{x_{k+1} = 2x_k - ax_k^2}.$$

Taking an initial guess $x_0 \in]0, \frac{1}{a}[$, we have the following properties:

Newton's Method (Example 1): Properties

The sequence $\{x_k\}$ is well-defined since $x_k \in]0, \frac{1}{a}[$.
$\{x_k\}$ is an increasing sequence bounded above by $\frac{1}{a}$; it therefore converges to $\frac{1}{a}$.
$\{x_k\}$ is a fixed-point sequence with

$$x_{k+1} = 2x_k - ax_k^2 = g(x_k).$$

Let e_k be the method's error, i.e., $e_k = x_k - \frac{1}{a}$. It is easy to see that:

Quadratic Convergence

$$\frac{e_{k+1}}{e_k^2}$$

Newton's Method: Example 2 (Computing $\sqrt{A}$)

Let A be a positive number. To compute $\sqrt{A}$, we apply Newton's method to the function

$$f(x) = x^2 - A.$$

The Newton iteration for this function takes the following form:

$$\boxed{x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right)}.$$

Taking an initial guess $x_0 \in]\sqrt{A}, \infty[$, we have the following properties:

Newton's Method (Example 2): Properties

The sequence $\{x_k\}$ is well-defined since $x_k \in]\sqrt{A}, \infty[$.
$\{x_k\}$ is a decreasing sequence bounded below by $\sqrt{A}$; it is therefore convergent.
$\{x_k\}$ is a fixed-point sequence with $x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right)$, which converges to $\sqrt{A}$.
Let e_k be the method's error, i.e., $e_k = x_k - \sqrt{A}$. It is easy to see that:

Quadratic Convergence

$$\frac{e_{k+1}}{e_k^2} = \frac{1}{2x_k} \quad \text{and} \quad \lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{2\sqrt{A}}.$$

Secant Method: Introduction

Although Newton's method is widely used in practice, its main drawback is the need to use the derivative at each iteration. When the function f is not defined explicitly, we do not always have access to its derivative.

This is why we will now look at the **secant method**, which does not use the derivative of f.

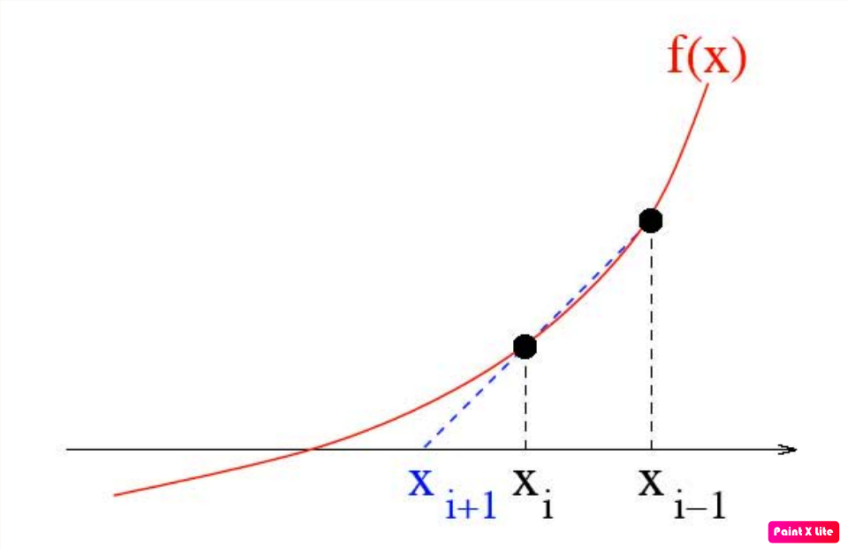
The idea of this method is to approximate the derivative $f'(x_k)$ by a divided difference:

$$f'(x_k) \approx [x_{k-1}, x_k] = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

Secant Method

The iteration of the secant method is therefore written as:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}}.$$


f(x)
X i+1
X i
X i−1
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Secant Method: Example 1 (Computing a^{-1})

Let $f(x) = \frac{1}{x} - a$.
The secant iteration for this function takes the following form:

$$\boxed{x_{k+1} = x_k + x_{k-1} - a x_k x_{k-1}}.$$

Taking x_0 and x_1 in $]0, \frac{1}{a}[$, we have the following properties:

- The sequence $\{x_k\}$ is well-defined since $x_k \in]0, \frac{1}{a}[$.

Let e_k be the method's error, i.e., $e_k = x_k - \frac{1}{a}$. It is easy to see that:

Super-linear Convergence

$$\frac{e_{k+1}}{e_k e_{k-1}} = -a.$$

Secant Method: Example 2 (Computing $\sqrt{A}$)

Let $f(x) = x^2 - A$.

The secant iteration for this function takes the following form:

$$\boxed{x_{k+1} = \frac{x_k x_{k-1} + A}{x_k + x_{k-1}}}.$$

Let e_k be the method's error, i.e., $e_k = x_k - \sqrt{A}$. It is easy to see that:

Super-linear Convergence

$$\frac{e_{k+1}}{e_k e_{k-1}} = \frac{1}{2x_k} \quad \text{and} \quad \lim_{k\to\infty} \frac{e_{k+1}}{e_k e_{k-1}} = \frac{1}{2\sqrt{A}}.$$

Definition: Rates of Convergence

Let $x^* \in \mathbb{R}$, $x_k \in \mathbb{R}$, $k = 1, 2, \ldots$. Define the error as $e_k = x_k - x^*$.

- **Linear Convergence:** There exists a constant $c \in [0, 1)$ and an integer $k_0 \geq 0$ such that for all $k \geq k_0$,
$$|e_{k+1}| \leq c|e_k|.$$
- **Super-linear Convergence:** There exists a sequence $\{c_k\}$ that tends to 0, such that for all k,
$$|e_{k+1}| \leq c_k|e_k|.$$
- **Convergence of order $p > 1$:** There exist constants $p > 1$, $c \geq 0$, and $k_0 \geq 0$ such that for all $k \geq k_0$,
$$|e_{k+1}| \leq c|e_k|^p.$$

  If $p = 2$ or $p = 3$, the convergence is called **quadratic** or **cubic**, respectively.

Fixed-Point Method

A general procedure for finding the roots of a nonlinear equation $f(x) = 0$ consists of transforming it into an equivalent problem $x - \varphi(x) = 0$, where the auxiliary function $\varphi : [a, b] \to \mathbb{R}$ must have the following property:

$$\varphi(\alpha) = \alpha \quad \text{if and only if} \quad f(\alpha) = 0.$$

The point α is then called a **fixed point** of the function φ. Approximating the zeros of f is thus reduced to the problem of determining the fixed points of φ.

Idea: We will construct sequences satisfying $x^{(k+1)} = \varphi(x^{(k)}),\ k \geq 0$. Indeed, if $x^{(k)} \to \alpha$ and φ is continuous on $[a, b]$, then the limit α satisfies

$$\varphi(\alpha) = \alpha.$$

Fixed-Point Method: Application Example (Population Equilibrium)

Modeling Population Dynamics as a Fixed-Point Problem

1. General Population Model

When studying populations (e.g., bacteria), we want to link the number of individuals in one generation x to the number in the next generation x_+:

$$x_+ = \varphi(x) = xR(x), \qquad (4)$$

where $R(x)$ represents the growth (or decay) rate of the population.

2. Common Growth Models for $R(x)$

Several mathematical models exist for $R(x)$:

1. Malthusian Model

(Thomas Malthus, 1766–1834)

$$R_1(x) = r, \quad r > 0$$

$$\varphi_1(x) = rx$$

Constant growth rate, **unlimited** resources.

2. Logistic Growth

(Pierre François Verhulst, 1804–1849)

$$R_2(x) = \frac{r}{1 + x/K}, \quad r, K > 0$$

$$\varphi_2(x) = \frac{rx}{1 + x/K}$$

Limited resources, carrying capacity K.

3. Predator–Prey with Saturation

$$R_3(x) = \frac{rx}{1 + (x/K)^2}$$

The dynamics of a population are defined by an iterative process, starting from a given initial state $\mathbf{x}^{(0)}$:

$$\mathbf{x}^{(k)} = \varphi(\mathbf{x}^{(k-1)}), \quad k > 0,$$

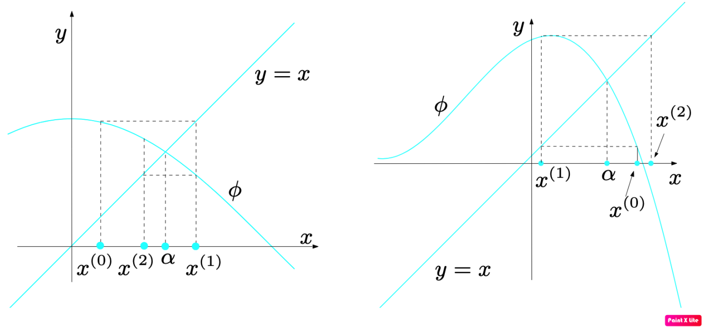
where $\mathbf{x}^{(k)}$ represents the number of individuals after k generations from the initial state. Furthermore, the stationary (equilibrium) states $\mathbf{x}^*$ of the considered population are identified by the following fixed-point problem:

$$\mathbf{x}^* = \varphi(\mathbf{x}^*), \tag{5}$$

or, equivalently,

$$\mathbf{x}^* = \mathbf{x}^* R(\mathbf{x}^*), \quad \text{i.e.,} \quad R(\mathbf{x}^*) = 1. \tag{6}$$

In both cases, the task is to solve a nonlinear problem. In particular, problem (5) is called a fixed-point problem.



y
$y = x$
ϕ
x
$x^{(0)}$
$x^{(2)}$
α
$x^{(1)}$
y
ϕ
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α
x
$x^{(0)}$
$y = x$
Paint X lite

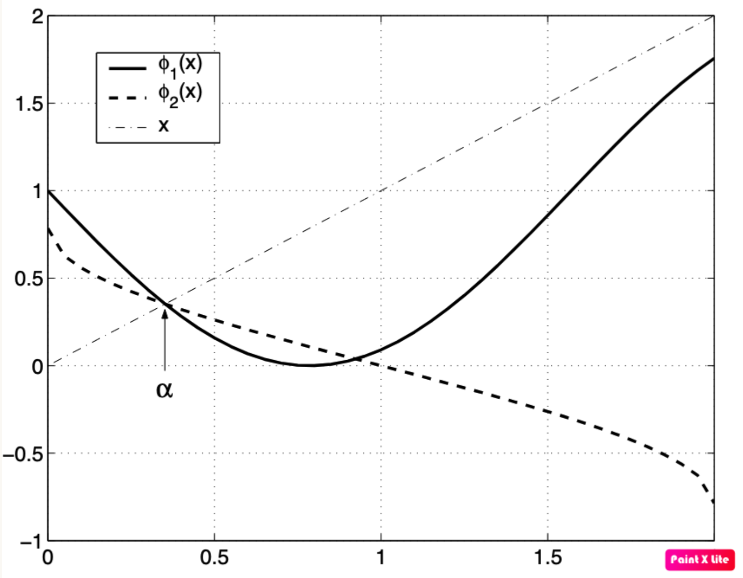
Example: We are still considering the equation

$$f(x) = \sin(2x) - 1 + x = 0.$$

We can rewrite it in two fixed-point forms:

$$x = \varphi_1(x) = 1 - \sin(2x),$$

$$x = \varphi_2(x) = \frac{1}{2}\arcsin(1 - x), \quad 0 \le x \le 1.$$



ϕ1(x)
ϕ2(x)
x
α
2
1.5
1
0.5
0
−0.5
−1
0
0.5
1
1.5
Paint X lite

Global Convergence Theorem for Fixed-Point Methods

Theorem (Global Convergence)

*Let $\varphi : [a, b] \to \mathbb{R}$ be a **continuous** and **differentiable** function on $[a, b]$ such that:*

$$\forall x \in [a, b], \quad \varphi(x) \in [a, b].$$

Then there exists at least one fixed point $\alpha \in [a, b]$ of φ.

Furthermore, suppose that:

$$\exists K < 1 \quad \text{such that} \quad |\varphi'(x)| \leq K, \quad \forall x \in [a, b].$$

Then:

1. *There exists a **unique** fixed point α of φ in $[a, b]$;*
2. *For any $x^{(0)} \in [a, b]$, the sequence $\{x^{(k)}\}$ defined by*

$$x^{(k+1)} = \varphi(x^{(k)}), \quad k \geq 0,$$

converges to α as $k \to \infty$; We have the **error estimate**:

$$|x^{(k+1)} - \alpha| \leq K\,|x^{(k)} - \alpha|, \quad \forall k \in \mathbb{N}.$$

Remark

For linear convergence, the constant C is called the *rate of convergence*. For higher orders ($p > 1$), C is the *asymptotic error constant*.

Local Convergence Theorem

Theorem (Local Convergence)

Let φ be differentiable on $[a, b]$ and α a fixed point of φ (i.e., $\varphi(\alpha) = \alpha$).
If

$$|\varphi'(\alpha)| < 1,$$

then there exists $\delta > 0$ such that for any $x^{(0)}$ satisfying $|x^{(0)} - \alpha| \leq \delta$, the sequence

$$\{x^{(k)}\} \quad \textit{defined by} \quad x^{(k+1)} = \varphi(x^{(k)}), \quad k \geq 0,$$

converges to α as $k \to \infty$.
Furthermore, we have the asymptotic rate:

$$\lim_{k\to\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \varphi'(\alpha).$$

Practical Interpretation

If $0 < |\varphi'(\alpha)| < 1$, then for any constant C satisfying

$$|\varphi'(\alpha)| < C < 1,$$

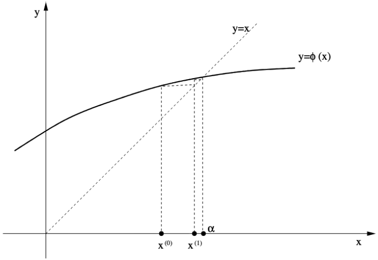
there exists k_0 such that for all $k \geq k_0$:

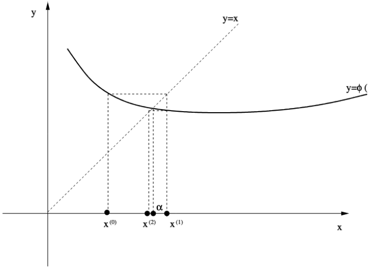
$$|x^{(k+1)} - \alpha| \leq C\,|x^{(k)} - \alpha|.$$

This means the convergence is **eventually linear** with rate approximately $|\varphi'(\alpha)|$.

Cas **convergent** :

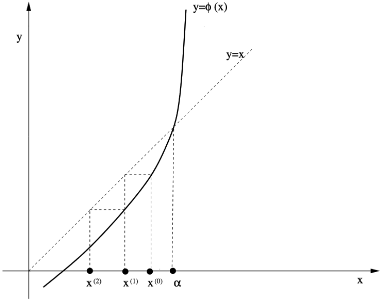
$$0 < \phi'(\alpha) < 1, \qquad -1 < \phi'(\alpha) < 0.$$

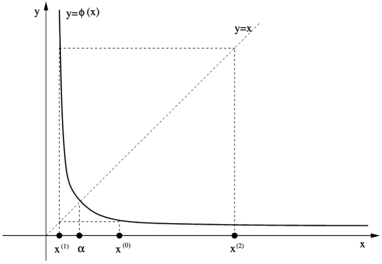




Cas **divergent** :

$$\phi'(\alpha) > 1, \qquad \phi'(\alpha) < -1.$$





Example: Fixed-Point Iteration for Population Models

Models and Iteration Functions

For a population x, the next generation is $x_+ = \varphi(x)$. Two classical models:

1. **Verhulst (Logistic Growth)**:

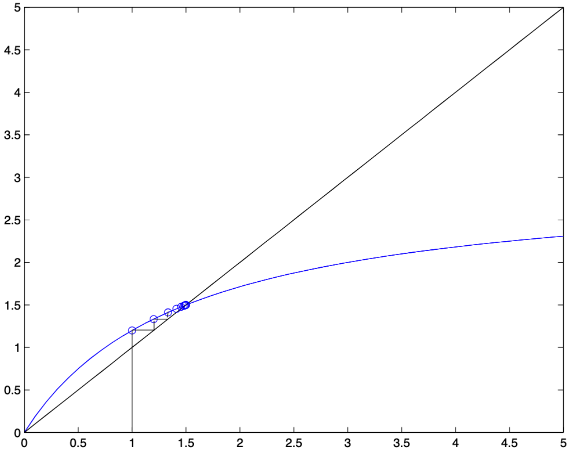
$$\varphi_2(x) = \frac{rx}{1 + x/K}, \quad r, K > 0$$

2. **Predator–Prey with Saturation**:

$$\varphi_3(x) = \frac{rx^2}{1 + (x/K)^2}, \quad r, K > 0$$

In this example: $r = 2, K = 1.5$, and starting point $x^{(0)} = 1.0$.

Fonction $\phi_2(x)$:



$$x^{(0)} = 1.0000, \qquad |x^{(0)} - \alpha_2| = 0.5000$$
$$x^{(1)} = 1.2000, \qquad |x^{(1)} - \alpha_2| = 0.3000$$
$$x^{(2)} = 1.3333, \qquad |x^{(2)} - \alpha_2| = 0.1667$$
$$x^{(3)} = 1.4118, \qquad |x^{(3)} - \alpha_2| = 0.0882$$