UM6P University- Academic Year 2025-2026
Bachelor Mathematics - Numerical Analysis Lab

Lab 1: Approximation of Function Zeros

Introduction

In this lab, we present methods for approximating zeros of functions. The goal is to implement these methods and compare them when applied to simple examples.

1 Numerical Methods

1.1 Bisection Method

Let f be a continuous function on an interval $[a, b]$ such that $f(a)f(b) \leq 0$. By the Intermediate Value Theorem, there exists a real number $c \in [a, b]$ such that $f(c) = 0$. To approximate a root c, we construct two sequences $(a_n)_n$ and $(b_n)_n$ by the following recurrence:

- Set $a_0 = a$ and $b_0 = b$

- Compute $c_n = \dfrac{a_n + b_n}{2}$

- If $f(a_n)f(c_n) \leq 0$, then define:

$$a_{n+1} = a_n,$$
$$b_{n+1} = c_n$$

  Otherwise:

$$a_{n+1} = c_n,$$
$$b_{n+1} = b_n$$

1.2 Fixed Point Method

Let g be a continuous function on an interval $[a, b]$. We aim to approximate a fixed point of g on $[a, b]$, assuming conditions ensuring its existence are satisfied. To approximate a fixed point, we construct a sequence $(x_n)_{n \geq 0}$ by the following recurrence:

- Choose $x_0 \in [a, b]$

- For $n \geq 0$, set:
$$x_{n+1} = g(x_n)$$

To solve an equation of type $f(x) = 0$, we can use the fixed point method on the function g_α defined below, with a non-zero coefficient α to be chosen:

$$g_\alpha(x) = x - \alpha f(x)$$

1.3 Newton's Method

Let f be a differentiable function on $\mathbb{R}$. We aim to approximate a root of f. To approximate a root, we construct a sequence $(x_n)_{n \geq 0}$ by the following recurrence:

- Choose x_0 such that $f'(x_0) \neq 0$

- For $n \geq 0$, set:
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method is efficient when x_0 is chosen sufficiently close to a root of f, and when the derivative of f is not close to 0 at x_0. It is also well-defined when the derivative of f never vanishes at the points x_n.

1.4 Secant Method

The secant method is the analog of Newton's method when the derivative of the function f is not computable. To approximate a root, we construct a sequence $(x_n)_{n \geq 0}$ by the following recurrence:

- Choose x_0 and x_1 such that $f(x_0) \neq f(x_1)$

- For $n \geq 1$, set:
$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

This method is only defined if we always have $f(x_n) \neq f(x_{n-1})$ for all $n \geq 1$.

2 Applications

Exercise 1

1. Implement each of these methods as Python functions. Each function should take as arguments: the function whose root or fixed point is sought, the first terms of the approximation sequences, the maximum allowed error, and the maximum number of allowed iterations. Each function should also return: the approximate value of the sought solution, a vector containing all terms of the constructed approximation sequences, and the number of iterations needed to reach a stopping criterion.

2. Test each of the functions on the following equation to solve:

$$x^2 - 2 = 0,$$

in order to obtain an approximation of $\sqrt{2}$ and compare the results.

3. Illustrate each method graphically.

Some indications:

- For the fixed point method, we will initially choose $\alpha = 0.5$, and then we can change the value of this variable to test the convergence of the algorithm.

- Ensure to choose similar initial data for comparing the results.