The Physical Model

We consider the 1D reaction–diffusion equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru(1-u)$$

Where:

- $u(x,t)$ = temperature (or concentration)
- D = diffusion coefficient
- $ru(1-u)$ = reaction (logistic growth)

Boundary condition (Neumann):

$$\frac{\partial u}{\partial x} = 0$$

Time Discretization

We discretize time:

$$\frac{u_i^{n+1} - u_i^n}{dt} = D\frac{\partial^2 u}{\partial x^2} + ru_i^n(1 - u_i^n)$$

We treat:

- Diffusion implicitly (stable)
- Reaction explicitly (simpler)

Space Discretization

Using finite differences:

$$\frac{u_i^{n+1} - u_i^n}{dt} = D\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{dx^2} + ru_i^n(1 - u_i^n)$$

Define:

$$\alpha = \frac{D\,dt}{dx^2}$$

Resulting Linear System

We obtain:

$$-\alpha u_{i-1}^{n+1} + (1 + 2\alpha)u_i^{n+1} - \alpha u_{i+1}^{n+1} = u_i^n + dt\, r u_i^n(1 - u_i^n)$$

For all grid points i.

This gives the matrix form:

$$Au^{n+1} = b$$

At each time step we must solve a linear system.

Structure of Matrix A

Matrix A is tridiagonal:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -\alpha & 1+2\alpha & -\alpha & \cdots \\ 0 & -\alpha & 1+2\alpha & -\alpha \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

Properties:

- Sparse
- Diagonally dominant
- Symmetric