

PGSS Discrete Math
Solutions to Problem Set #1

Note: \square signifies the end of a problem, and \blacksquare signifies the end of a proof.

1. How many odd numbers with distinct digits and no leading zeroes can be constructed from the digits 0, 1, 2, 3, 4, 5, 6?

Solution: Since the digits have to be distinct, we see that the numbers can have anywhere between 1 and 7 digits. The digit in the ones place must be odd, so there are 3 choices for it. The leading digit can't be 0, so (for numbers with at least 2 digits) there are 5 choices for it (because it can't be the ones digit, and the ones digit can't be 0). Then we pick the remaining places in order from the remaining available digits. This leads us to:

$$\begin{aligned} 3 &+ (3)(5) \\ &+ (3)(5)(5) \\ &+ (3)(5)(5)(4) \\ &+ (3)(5)(5)(4)(3) \\ &+ (3)(5)(5)(4)(3)(2) \\ &+ (3)(5)(5)(4)(3)(2)(1) \\ &= 4893. \square \end{aligned}$$

2. How many five-digit numbers without repeated digits and no leading zeroes are there that include the pair of adjacent numbers 12 in that order?

Solution: The pair 12 can occupy any of 4 positions in the number. When they occur first, the 1 is the leading digit, and there is no concern about choosing a zero among the remaining three digits, so the remaining digits can be chosen in $(8)(7)(6)$ ways. In the other three cases the leading digit can be chosen in 7 ways, and the other two numbers can be chosen in $(7)(6)$ ways. So the answer is:

$$(8)(7)(6) + 3(7)(7)(6) = 1218. \square$$

3. Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

using induction.

Solution: Let $P(n)$ denote the formula above. Our base case is $n = 1$: since $1^2 = 1 = \frac{1(2)(3)}{6} = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$, $P(1)$ is true.

For the inductive hypothesis, suppose that $P(1), \dots, P(m)$ are all true. Then

$$\begin{aligned}
 1^2 + 2^2 + \dots + m^2 + (m+1)^2 &= (1^2 + 2^2 + \dots + m^2) + (m+1)^2 \\
 &= \left(\frac{m(m+1)(2m+1)}{6} \right) + (m+1)^2 \\
 &= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6} \\
 &= \frac{(m+1)(m(2m+1) + 6(m+1))}{6} \\
 &= \frac{(m+1)(2m^2 + 7m + 6)}{6} \\
 &= \frac{(m+1)(m+2)(2m+3)}{6},
 \end{aligned}$$

exactly what we get for the RHS of $P(m+1)$, so $P(m+1)$ is true, and by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. ■

4. Guess a formula for

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \dots + \frac{1}{n(n+1)}$$

by considering the first few cases. Then prove that your formula is correct.

Solution: For $n = 1, 2, 3$, we get $1/2$, $2/3$, and $3/4$, which suggests that the sum equals $n/(n+1)$, so we'll use induction to prove it: when $n = 1$, we get $1/(1 \cdot 2) = 1/2$ for the sum and $1/(1+1) = 1/2$, so the base case is true.

Suppose now that the formula holds for $n = 1, 2, \dots, m$. Then

$$\begin{aligned}
 \sum_{i=1}^{m+1} \frac{1}{i(i+1)} &= \left(\sum_{i=1}^m \frac{1}{i(i+1)} \right) + \left(\frac{1}{(m+1)(m+2)} \right) \\
 &= \left(\frac{m}{m+1} \right) + \left(\frac{1}{(m+1)(m+2)} \right) \\
 &= \left(1 - \frac{1}{m+1} \right) + \left(\frac{1}{m+1} - \frac{1}{m+2} \right) \\
 &= 1 - \frac{1}{m+2} \\
 &= \frac{m+1}{m+2},
 \end{aligned}$$

so the formula holds when $n = m+1$, proving the inductive hypothesis, and therefore $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. ■

5*. Using the fact that $(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$, find a formula for

$$1^3 + 2^3 + \cdots + n^3,$$

and prove that it is correct.

Solution: Let $S(n) = 1^3 + 2^3 + \cdots + n^3$. The trick is to use the formula given above to create an expression that includes $S(n)$; we do this by summing:

$$\sum_{i=1}^n ((i+1)^4 - i^4) = \sum_{i=1}^n (4i^3 + 6i^2 + 4i + 1)$$

Since addition is commutative and associative, we can change the order of the terms in the sums and how we group them, like so:

$$\sum_{i=1}^n (i+1)^4 - \sum_{i=1}^n i^4 = \sum_{i=1}^n 4i^3 + \sum_{i=1}^n 6i^2 + \sum_{i=1}^n 4i + \sum_{i=1}^n 1.$$

Since

$$\sum_{i=1}^n (i+1)^4 = 2^4 + 3^4 + \cdots + (n+1)^4 = \sum_{i=2}^{n+1} i^4,$$

the LHS becomes

$$\sum_{i=2}^{n+1} i^4 - \sum_{i=1}^n i^4,$$

and the terms corresponding to $i = 2, 3, \dots, n$ all cancel, leaving only $(n+1)^4 - 1^4$.

Looking at the RHS, we see that

$$\sum_{i=1}^n 4i^3 = 4 \cdot 1^3 + 4 \cdot 2^3 + \cdots + 4 \cdot n^3 = 4(1^3 + 2^3 + \cdots + n^3) = 4S(n),$$

and using this idea and what we already know from class and Problem 3, we have

$$\sum_{i=1}^n 6i^2 = n(n+1)(2n+1), \quad \sum_{i=1}^n 4i = 2(n)(n+1), \quad \text{and} \quad \sum_{i=1}^n 1 = n.$$

So

$$(n+1)^4 - 1 = 4S(n) + n(n+1)(2n+1) + 2(n)(n+1) + n,$$

and solving for $S(n)$ gives us

$$S(n) = \frac{n^4 + 2n^3 + n^2}{4} = \left(\frac{n(n+1)}{2} \right)^2. \quad \blacksquare$$