ENGR20005
Numerical Methods in Engineering

Workshop 9

Part A: Pre-Lecture Problems

Please try to attempt 9.1 before Lecture 17 and 9.2 before Lecture 18.
Don't worry if you cannot finish them. I plan to go through them in the "live" lectures

9.1 Question 8.2 in Workshop 8, went through steps you needed to take to construct derivative matrices to calculate derivatives in the x direction when data from the two-dimensional function $u(x, y)$ is arranged in the form of a column vector. For example, for the case where $N_x = 4$ and $N_y = 3$, the matrix operation took on the following form

$$
\begin{Bmatrix}
u_x(x_0,y_0)\\
u_x(x_1,y_0)\\
u_x(x_2,y_0)\\
u_x(x_3,y_0)\\
u_x(x_0,y_1)\\
u_x(x_1,y_1)\\
u_x(x_2,y_1)\\
u_x(x_3,y_1)\\
u_x(x_0,y_2)\\
u_x(x_1,y_3)\\
u_x(x_2,y_4)\\
u_x(x_3,y_5)
\end{Bmatrix}
=\frac{1}{2\Delta_x}
\begin{bmatrix}
\mathbf{-2} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\mathbf{-1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mathbf{-1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mathbf{-2} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathbf{-2} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathbf{-1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \mathbf{-1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & \mathbf{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & \mathbf{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & 0 & \mathbf{1} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & 0 & \mathbf{1}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & \mathbf{2}
\end{bmatrix}
\begin{Bmatrix}
u(x_0,y_0)\\
u(x_1,y_0)\\
u(x_2,y_0)\\
u(x_3,y_0)\\
u(x_0,y_1)\\
u(x_1,y_1)\\
u(x_2,y_1)\\
u(x_3,y_1)\\
u(x_0,y_2)\\
u(x_1,y_2)\\
u(x_2,y_2)\\
u(x_3,y_2)
\end{Bmatrix}
$$

$$\{u_x\} = [D_x]\{u\}.$$

$\{u_x\}$ is the column vector of the partial derivative of $\{u\}$ in the x direction. In this question, you will learn how to construct derivative matrices to calculate partial derivatives in the y direction

(a) Assuming an equally spaced grid, Δ_y, and using 2nd order CDA for the interior nodes and first order FDA and BDA for the end nodes, show that the derivative matrix in the y direction can be written as

$$\{u_y\} = \begin{Bmatrix} u_y(x, y_0) \\ u_y(x, y_1) \end{Bmatrix} = \frac{1}{\Delta_y} \begin{bmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix} \begin{Bmatrix} u(x, y_0) \\ u(x, y_1) \end{Bmatrix} \tag{9.1}$$

for $N_y = 2$ and

$$\{u_y\} = \begin{Bmatrix} u_y(x, y_0) \\ u_y(x, y_1) \\ u_y(x, y_2) \end{Bmatrix} = \frac{1}{2\Delta_y} \begin{bmatrix} \mathbf{-2} & \mathbf{2} & 0 \\ \mathbf{-1} & 0 & \mathbf{1} \\ 0 & \mathbf{-2} & \mathbf{2} \end{bmatrix} \begin{Bmatrix} u(x, y_0) \\ u(x, y_1) \\ u(x, y_2) \end{Bmatrix} \tag{9.2}$$

for $N_y = 3$.

(b) Use the matrices specified by Eqs. (9.1) and (9.2) to calculate $\partial u/\partial y$ respectively. Compare with the analytical solution

$$\frac{\partial u}{\partial y} = 3y^2 e^{-x^2}. \tag{9.3}$$

For this part of the question, assume that the information about the function $u(x, y)$ is organised as a matrix with the data for increasing y organised in rows and increasing x in columns

$$[u] = \begin{bmatrix} u(x_0, y_0) & u(x_1, y_0) & .... & u(x_{Nx-1}, y_0) \\ u(x_0, y_1) & u(x_1, y_1) & .... & u(x_{Nx-1}, y_1) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ u(x_0, y_{Ny-1}) & u(x_1, y_{Ny-1}) & .... & u(x_{Nx-1}, y_{Ny-1}) \end{bmatrix}.$$

For $N_x = 4$ and $N_y = 3$, the matrix is

$$[u] = \begin{bmatrix} u(x_0, y_0) & u(x_1, y_0) & u(x_2, y_0) & u(x_3, y_0) \\ u(x_0, y_1) & u(x_1, y_1) & u(x_2, y_1) & u(x_3, y_1) \\ u(x_0, y_2) & u(x_1, y_2) & u(x_2, y_2) & u(x_3, y_2) \end{bmatrix}. \tag{9.4}$$

Compare with the analytical solution shown in Eq. (9.3).

(c) Now we would like to put the matrix $[u]$ in vector form, similar to what was done in question 8.2(d). Assuming $N_x = 4$ and $N_y = 3$, convince yourself that the structure of the derivative matrix to calculate u_y if $u(x, y)$ is organised as a vector is

$$
\begin{Bmatrix} u_y(x_0,y_0) \\ u_y(x_1,y_0) \\ u_y(x_2,y_0) \\ u_y(x_3,y_0) \\ u_y(x_0,y_1) \\ u_y(x_1,y_1) \\ u_y(x_2,y_1) \\ u_y(x_3,y_1) \\ u_y(x_0,y_2) \\ u_y(x_1,y_2) \\ u_y(x_2,y_2) \\ u_y(x_3,y_2) \end{Bmatrix} = \frac{1}{2\Delta_y}
\begin{bmatrix}
\mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 & 0 \\
\mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\
0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & \mathbf{2}
\end{bmatrix}
\begin{Bmatrix} u(x_0,y_0) \\ u(x_1,y_0) \\ u(x_2,y_0) \\ u(x_3,y_0) \\ u(x_0,y_1) \\ u(x_1,y_1) \\ u(x_2,y_1) \\ u(x_3,y_1) \\ u(x_0,y_2) \\ u(x_1,y_2) \\ u(x_2,y_2) \\ u(x_3,y_2) \end{Bmatrix}
$$

(d) The result above can be generalized to calculate the partial derivatives on a $N_x \times N_y$ mesh

$$
\begin{aligned}
u_y(x_j, y_i) &= [D] \otimes [I]\{u\} \\
\{u_y\} &= [D] \otimes [I]\{u\}
\end{aligned}
$$

where $[D_y] = [I] \otimes [D]$. $\otimes$ is some operation on the one-dimensional derivative matrix $[D]$ and the identity matrix $[I]$ that produces the correct two-dimensional derivative matrix. The operation $\otimes$ on two matrices is called the *Kronecker product* which is implemented in MATLAB using the `kron()` function. For an $m \times n$ matrix $[A]$ and a $p \times q$ matrix $[B]$ is defined as

$$
[A] \otimes [B] = \begin{pmatrix} a_{11}[B] & \cdots & a_{1n}[B] \\ \vdots & \ddots & \vdots \\ a_{n1}[B] & \cdots & a_{nn}[B]. \end{pmatrix} \tag{9.5}
$$

Write a computer program that calculates the u_y if the data for $u(x, y)$ is arranged in column vector form. Prove that this program works by computing

$$
\{u_y\} = [D_y]\{u\}
$$

and compare with Eq. (9.3).

(e) Now use the derivative matrices to calculate

$$
\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} &= \\
\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)u &\approx \\
\left([D_x]+[D_x]\right)\{u\} &
\end{aligned}
$$

Compare with the analytical expression

$$
\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=y^2e^{-x^2}\left(3-2xy\right). \tag{9.6}
$$

(f) Now see if you can get back $u(x,y)$ by solving

$$
\left([D_x]+[D_x]\right)\{u\}=\{p\}
$$

where $\{p\}=\{u_x\}$ is the vector values of the right hand side of Eq. (9.6). For this question you will need to implement the boundary condition

$$
\begin{aligned}
u(x=0,y) &= y^3 \\
u(x,y=0) &= 0.
\end{aligned}
$$

9.2 Compute the approximate solutions to

$$
\frac{dx}{dt}=-x^2
$$

using the Euler's method and Taylors method order 2 in the domain $t\in[0,10]$. Use the intial condition $x(t=0)=1$. Compare with the analytical solution

$$
x=\frac{1}{(t+1)}.
$$

Part B: MATLAB Livescripts

9.3 The livescript *ENGR20005_Workshop9p3.mlx* runs through the solution of boundary value problems in MATLAB.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve the the boundary value problem

$$\frac{d^2y}{dx^2} = 4(y - x) \tag{9.7}$$

with the boundary conditions $y(0) = 0$ and $y(1) = 2$.

Compare your solution with the analytical solution

$$y(x) = e^2(e^4 - 1)^{-1}(e^{2x} - e^{-2x}) + x$$

9.4 The livescript *ENGR20005_Workshop9p4.mlx* runs through the solution of boundary value problems using finite difference methods.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve Eq. (9.7).

9.5 The livescript *ENGR20005_Workshop9p5.mlx* runs through the different types of boundary conditions commonly used in the solution of boundary value problems. Read through the livescript and make sure you understand how to implement each type.

9.6 The livescript *ENGR20005_Workshop9p6.mlx* runs through the solution of boundary value problems using spectral collocation methods.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve Eq. (9.7). Compare your answer with the finite difference method in Question 9.4.

Part C: Problems

9.7 Consider the following differential equation

$$\frac{d^2y}{dx^2} + y = 0 \tag{9.8}$$

(a) By pairing Eq. (9.8) with the following boundary conditions

i. $y(0) = 1$ and $y(\pi/2) = 1$.

ii. $y(0) = 1$ and $y(\pi) = 1$.

iii. $y(0) = 1$ and $y(2\pi) = 1$.

solve the boundary value problem using the finite difference method. Use as many points as you think is necessary.

(b) Repeat part (a) using the spectral collocation method.

(c) Solve Eq. (9.8) analytically with the boundary conditions listed in part (a). Compare the analytical solution with your answers in part (a).

9.8 The flow of fluid between two plates, where one of them is stationary and the other is moving with velocity U, is given by the boundary value problem

$$\frac{d^2u}{dy^2} + \alpha = 0 \tag{9.9}$$

with the boundary conditions $u(0) = 0$ and $u(1) = 1$.

NOTE: The dimensionless parameter $\alpha = \frac{Gd^2}{\mu U}$ measures the relative strength between pressure and viscous forces.

(a) Solve Eq. (9.9) analytically.

(b) Assuming that $\alpha = 1$, discretise Eq. (9.9) using the central difference scheme with 5 evenly spaced nodes.

(c) Solve the set equations you obtained in part (b). Compare your solution with your answer in part (a).

(d) Repeat the problem with the spectral collocation method with 5 Gauss–Lobatto nodes. Compare your answer with the finite difference method.

(e) Repeat the problem with $\alpha = 0.01$ and 100.

9.9 * In this question, you will learn the steps to be taken to calculate partial double derivatives of a two-dimensional function. This question combines everything that was taught in Workshop questions 8.2 and 9.1. You will learn how you can extend the derivative matrices idea from one-dimension to calculate partial derivatives of two-dimensional functions. We will first construct the matrix for one-dimensional function and then extend the analysis to two-dimensions. In this example, we will calculate the double partial derivatives of a two-dimensional function (in workshop questions 8.2 and 9.1 we calculated the first derivatives, you will find the the steps are very similar).

As was shown in lectures, the double derivative of a function $u_{xx}(x)$ of a function $u(x)$ can be calculated using the second order central, forward and backward difference schemes

$$u_{xx}(x_i) = (u_{i-1} - 2u_i + u_{i+1})/\Delta_x^2$$
$$u_{xx}(x_0) = (u_0 - 2u_1 + u_2)/\Delta_x^2$$
$$u_{xx}(x_{N_x-1}) = (u_{N_x-1} - 2u_{N_x-2} + u_{N_x-3})/\Delta_x^2.$$

$u(x)$ is evaluated at equally spaced grid points $x = x_0, x_1, ..x_{Nx-1}$, i.e. $x_i - x_{i-1} = \Delta_x$. $u_i = u(x_i)$ and $i = 0..N_x - 1$ where N_x is the number of grid points in x. The above equations can be written more concisely in matrix form as $\{u_{xx}\} = [D_x^2]\{u\}$. The explicit structure for the matrix equations for $N_x = 5$ and $N_x = 9$ are shown below

$$\{u_{xx}\} = \begin{Bmatrix} u_{xx}(x_0) \\ u_{xx}(x_1) \\ u_{xx}(x_2) \\ u_{xx}(x_3) \\ u_{xx}(x_4) \end{Bmatrix} = \frac{1}{\Delta_x^2} \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{9.10}$$

$$\{u_{xx}\} = \begin{Bmatrix} u_{xx}(x_0) \\ u_{xx}(x_1) \\ u_{xx}(x_2) \\ u_{xx}(x_3) \\ u_{xx}(x_4) \\ u_{xx}(x_5) \\ u_{xx}(x_6) \\ u_{xx}(x_7) \\ u_{xx}(x_8) \end{Bmatrix} = \frac{1}{\Delta_x^2} \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{9.11}$$

(a) Write a MATLAB program to compute the $[D_x^2]$ matrix and use it to calculate and plot the double derivative of $u(x) = x^7$ by performing the matrix multiplication, $\{u_{xx}\} = [D_x^2] \times \{u\}$, in MATLAB. Use the domain $x \in [-1, 1]$. Check that your answer is correct by comparing with the exact solution $u_{xx}(x) = 42x^5$. How many grid points do you need to use to get a good approximation for $u_{xx}(x)$? Note that the one sided approximation of $u_{xx}(x)$ is not very good at the end points.

(b) To see how $[D_x^2]$ can be used to calculate the double derivative of a two-dimensional function $u_{ji} = u(x_i, y_j)$, we will write the two-dimensional matrix $u_{j,i}$ in vector form

$$\begin{aligned} \{u\} &= (u_0, u_1, \ldots, u_{N_x-1}, u_{N_x}, u_\ell, \ldots, u_{N_x\times N_y-1})^T \\ &= (u_{0,0}, u_{0,1}, \ldots, u_{0,N_x-1}, u_{1,0}, u_{i,j}, \ldots, u_{N_y-1,N_x-1})^T \end{aligned}$$

where $\ell = i + jN_x$, $i = 0..N_x - 1$ and $j = 0..N_y - 1$. Look at section L16.1 of the lecture slides if you are still unsure of how the data is organised.

(c) Assume that the mesh has 5×5 grid points. Show that the two-dimensional double partial derivative matrices in the x and y directions, $[D_x^2]$ and $\left[D_y^2\right]$ are given by Eqs. (9.21) and (9.22) (see the last two pages of this document) respectively.

The result above can be extended to calculate the partial double derivatives on a $N_x \times N_y$ mesh

$$u_{xx}(x_i, y_i) = [I] \otimes [D^2]\{u\} \tag{9.12}$$

$$u_{yy}(x_i, y_i) = [D^2] \otimes [I]\{u\} \tag{9.13}$$

where $\otimes$ is some operation on the one-dimensional second derivative matrix $[D^2]$ and the identity matrix $[I]$ that produces the correct two-dimensional derivative matrix. The operation $\otimes$ on two matrices is called the *Kronecker product* which is implemented in MATLAB using the `kron()` function. For an $m \times n$ matrix $[A]$ and a $p \times q$ matrix $[B]$ is defined as

$$[A] \otimes [B] = \begin{pmatrix} a_{11}[B] & \cdots & a_{1n}[B] \\ \vdots & \ddots & \vdots \\ a_{n1}[B] & \cdots & a_{nn}[B]. \end{pmatrix} \tag{9.14}$$

(d) Use your matrices above to calculate

$$u_{xx}(x_j, y_i) \tag{9.15}$$

and

$$u_{yy}(x_j, y_i) \tag{9.16}$$

where $u(x,y) = \sin(y)x^5$ for $x \in [-1,1]$ and $y \in [0,2]$. Plot your computed u_{xx} and u_{yy} and compare with the exact solution

$$u_{xx} = 20\sin(y)x^3$$

and

$$u_{yy} = -\sin(y)x^5$$

(e) Combine the two operations above to calculate and plot

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{9.17}$$

where $u(x,y) = \sin(y)x^5$ for $x \in [-1,1]$ and $y \in [0,2]$.

9.10 An idealised model of the displacement of a vibrating drum is given by the *wave equation*

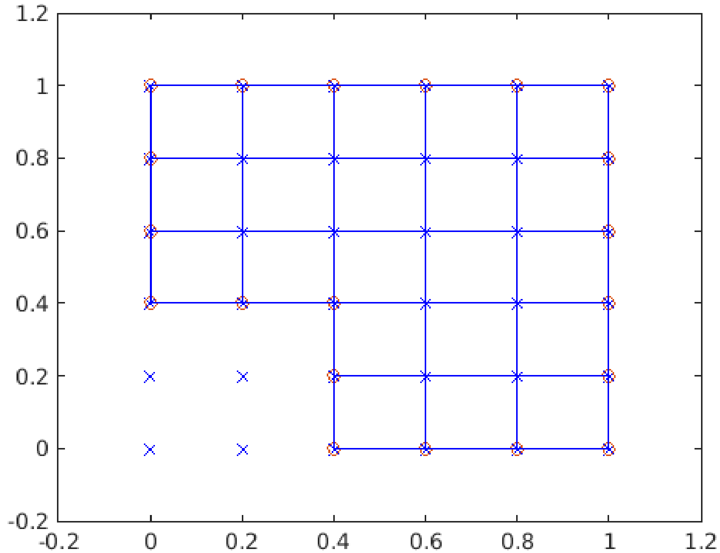
$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \tag{9.18}$$

Using some mathematical magic, part of the solution of Eq. (9.18) may be found by solving the *Helmholtz equation*

$$-\Big(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\Big) = k^2\phi \tag{9.19}$$

where k is a constant.

Consider an L-shaped domain with the following grid points



(a) Use the central difference method to discretise Eq. (9.19) on the L-shaped domain above. Assume that $\phi = 0$ along the boundary.

(b) Write the system of equations you found in part (a) in the form

$$-[L]\{\phi\} = k^2[I]\{\phi\} \tag{9.20}$$

Notice how this is an eigenvalue problem.

(c) Use `eig()` to determine the eigenvalues and eigenvectors of $[L]$.

These are the vibrational modes of an L-shaped drum, which are essentially the sounds that such a drum can make.

(d) Plot the first 3 eigenvectors of Eq. (9.20).

(e) It turns out the MATLAB logo is given by the eigenvector associated with the first eigenvalue. Repeat the problem with additional nodes and reproduce the MATLAB logo.

$$
[D^2]_x = \frac{1}{\Delta_x^2}
\left(\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
\mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1}
\end{array}\right)
\tag{9.21}
$$

$$
[D^2]_y = \frac{1}{\Delta_y^2}
\left(\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1}
\end{array}\right) \tag{9.22}
$$