ENGR20005
Numerical Methods in Engineering

Workshop 10

Part A: Pre-Lecture Problems

Please try to attempt 10.1 before Lecture 19 and 10.2 before Lecture 20.

10.1 Compute the approximate solutions to

$$\frac{dx}{dt} = -x^2$$

using the 2nd and 4th order Runge-Kutta method in the domain $t \in [0, 10]$. Use the intial condition $x(t = 0) = 1$. Compare with the analytical solution

$$x = \frac{1}{(t+1)}.$$

10.2 Repeat Question 10.1 but now compute the solution using the implicit Euler and the Crank-Nicolson methods.

Part B: MATLAB Livescripts

10.3 The livescript *ENGR20005_Workshop10p3.mlx* runs through the solution of initial value problems in MATLAB.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve the the initial value problem

$$\frac{dx}{dt} = \exp x \tag{10.1}$$

with the initial conditions $x(0) = 1$.

10.4 The livescript *ENGR20005_Workshop10p4.mlx* runs through the solution of initial value problems using explicit methods.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve Eq. (10.1).

10.5 The livescript *ENGR20005_Workshop10p5.mlx* runs through stability analysis. Read through the livescript and make sure you understand how to conduct this type of analysis.

Part C: Problems

10.6 A simple method of modelling population growth within a community with finite resources is to use the *logistic equation*

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{10.2}$$

Here, the term rN denotes the exponential growth that occurs in a community with infinite resources with a growth rate r, and the term $\left(1 - \frac{N}{K}\right)$ represents the decay once a specific carrying capacity K is exceeded.

(a) Discretise Eq. (10.2) using both the explicit Euler and Taylor's method of order 2.

(b) Using your answer to (a) write a function that determines the population over time.

(c) Assume that the population is initially $N(0) = 10$, the growth rate is $r = 2$, and the carrying capacity is $K = 1000$. Using your function from (b) determine when the population exceeds $N = 500$ and $N = 900$.

10.7 * A model for the growth of small perturbations in a fluid is given by the *Landau equation*

$$\tau\frac{dA}{dt} = \epsilon A - gA^3 \tag{10.3}$$

where A is the amplitude of the perturbation, τ is its typical time scale, ϵ is a dimensionless parameter, and $g < 0$ is a negative constant.

(a) Discretise Eq. (10.3) using both the explicit Euler and Taylor's method of order 2.

(b) Using your answer to (a), write a MATLAB function that computes the size of the perturbation as a function of time. Assume that $\tau = g = 1$ and the initial condition is $A(0) = 0.1$.

(c) Experimental results for Rayleigh–Bernard convection have shown that the steady-state amplitude A^* has a power law dependence on ϵ, $A^* \propto \epsilon^\beta$, where $\beta = 0.50 \pm 0.10$. Use your function from (b) to see how well the Landau equation predicts β.

10.8 Consider the test problem

$$\frac{dx}{dt} = -x \tag{10.4}$$

with the initial condition $x(0) = 1$.

(a) Solve Eq. (10.4) analytically and determine the value of x at $t = 1$.

(b) Use the explicit Euler method with a step size of $\Delta t = 1$ to determine the value of x at $t = 1$.

(c) Repeat (b) with step sizes of $\Delta t = 10^{-n}$, $n = 1, 2, 3, 4$.

(d) Plot the error $E = |x(1) - x_{\text{exact}}(1)|$ as a function of Δt on loglog scales and determine the order of convergence.

10.9 For the following differential equations, determine the maximum time-step that can be used for both the explicit Euler and second order Taylor's method.

(a) $\frac{dx}{dt} = -10x$

(b) $\frac{dx}{dt} = -4x + \sin(t)$

(c) $\frac{dx}{dt} = -x^2$

(d) $\frac{dx}{dt} = -|x^3|$

(e) $\frac{dx}{dt} = -1 - \exp(x)$

Verify your estimates by applying the explicit Euler and second order Taylor's methods with the initial condition $x(0) = 1$.