ENGR20005
Numerical Methods in Engineering

Workshop 11

Part A: Pre-Lecture Problems

Please try to attempt 11.1 before Lecture 21 and 11.2 before Lecture 22.

11.1 Consider the system of equation shown below

$$\frac{d}{dt}\begin{Bmatrix} x_0 \\ x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -3 & 6 & 9 \\ 1 & -9 & 2 \\ 2 & -15 & -9 \end{bmatrix}\begin{Bmatrix} x_0 \\ x_1 \\ x_2 \end{Bmatrix} \tag{11.1}$$

(a) Perform stability analysis and explore the values of Δt that can be used in order for your solution to be stable if you use Explicit Euler or RK-2 to obtain the numerical solution for Eq. (11.1). Can you expect to get a solution for $\Delta t = 0.02$, 0.1 and 0.2? What is the approximate maximum value of Δt that must be used in the computations for both the Explicit Euler or RK-2 methods.

(b) Solve Eq. (11.1) using Explicit Euler and RK-2 for $t \in [0, 2]$. You are given that $x_0(t = 0) = 0$ and $x_1(t = 0) = x_2(t = 0) = 1$. Plot $x_0(t)$ and comment on the solution obtained using $\Delta t = 0.02$, 0.1 and 0.2 with respect to your analysis in (a).

11.2 Write a MATLAB program that uses the Implicit Euler method to solve

$$\frac{d^2\phi}{dt^2} + \phi + \phi^3 = 0 \tag{11.2}$$

in the domain $t \in [0, 50]$ given $\phi(0) = 1$, $\phi'(0) = 0$. You might like to start with *Lecture22M04.m* given in Lecture 22.

Part B: MATLAB Livescripts

11.3 The livescript *ENGR20005_Workshop11p3.mlx* runs through the solution of initial value problems using implicit methods.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve the the initial value problem

$$\frac{dx}{dt} = -\exp x \tag{11.3}$$

with the initial conditions $x(0) = 1$.

11.4 The livescript *ENGR20005_Workshop11p4.mlx* runs through the solution of initial value problems using Runge–Kutta methods.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to solve Eq. (11.3)

Part C: Applications

11.5 In Workshop 10, we looked at the *logistic equation* for modelling population growth in a finite community

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{11.4}$$

(a) Using the code snippets in *ENGR20005_Workshop11p4.mlx*, apply the RK2 method to solve Eq. (11.4). Assume that the population is initially $N(0) = 10$, the growth rate is $r = 2$, and the carrying capacity is $K = 1000$.

(b) i. To apply the Crank–Nicolson method to Eq. (11.4), show that at time step t_n we need to solve the nonlinear equation

$$N_{n+1} = N_n + \frac{\Delta t}{2}\left[rN_n\left(1 - \frac{N_n}{K}\right) + rN_{n+1}\left(1 - \frac{N_{n+1}}{K}\right)\right] \tag{11.5}$$

ii. Apply the Newton–Raphson method to Eq. (11.5) and show that at each step $N_{n+1} = p$ is given by the iterative formula

$$p^{(l+1)} = p^{(l)} - \frac{p^{(l)} - \frac{\Delta t}{2}rp^{(l)}\left(1 - \frac{p^{(l)}}{K}\right) - N_n - \frac{\Delta t}{2}rN_n\left(1 - \frac{N_n}{K}\right)}{1 - \frac{\Delta t}{2}\left(r - \frac{2rp^{(l)}}{K}\right)} \tag{11.6}$$

iii. Modify you script from part (a) to apply Eq. (11.6).

11.6 * Consider the initial value problem

$$\frac{dx}{dt} = x^{1/5} \tag{11.7}$$

with the initial condition $x(0) = 0$.

(a) Show that the analytical solution is given by

$$x(t) = \left(\frac{4t}{5}\right)^{5/4}$$

(b) Use the explicit Euler method with a time step of $\Delta t = 0.01$ to solve Eq. (11.7) and show that it fails to find the correct solution. Why is this the case?

(c) i. Apply the implicit Euler method to Eq. (11.7) and show that at each step, you have the root finding problem

$$h(p) = p - \Delta t p^{1/5} - x_n = 0 \tag{11.8}$$

where $p = x_{n+1}$.

ii. Apply the Newton–Raphson method to Eq. (11.8). Hence argue that if we choose the initial guess $p^{(0)} = x_n$, the method will fail to converge to the correct root.

iii. Instead of using Newton–Raphson to find the root of Eq. (11.8), we'll now consider using fixed point iteration.

A. Show that possible fixed point formulae for Eq. (11.8) are

I. $g(p) = x_n + \Delta t p^{1/5}$

II. $g(p) = \frac{x_n}{1-\Delta t p^{-4/5}}$

B. To see which fixed point formula we should use, consider the initial time step $n = 0$ so that $x_0 = 0$. We'll also consider the same step size as in part (b) $\Delta t = 0.01$. Apply your knowledge from the livescript *ENGR20005_Workshop2p4.mlx* to show that only

$$g(p) = x_n + \Delta t p^{1/5}$$

will converge to the correct value.

Hint: The solutions to $h(p) = 0$ in this case are $p = 0$ and $p = 1/(100\sqrt{10})$.

iv. Modify the code snippets on *ENGR20005_Workshop11p3.mlx* to solve Eq. (11.7).

(d) Using your answer to part (c), how can you modify your script in part (b) to provide an accurate solution to Eq. (11.7).

11.7 The Explicit Euler, Implicit Euler, and Crank–Nicolson methods are special cases of the very creatively named *theta method*

$$x_{n+1} = x_n + \Delta t[\theta f(x_n, t_n) + (1-\theta) f(x_{n+1}, t_{n+1})] \tag{11.9}$$

where $0 \leq \theta \leq 1$.

Note that we recover the explicit Euler method for $\theta = 1$, the Crank–Nicolson method for $\theta = 1/2$, and the implicit Euler method for $\theta = 0$.

(a) Determine the truncation error of the *theta method* and use this result to determine the order of the method for $0 \leq \theta \leq 1$.

(b) Find an expression for the region of absolute stability for arbitrary θ.

11.8 One of the main difficulties with the Crank–Nicolson method

$$x_{n+1} = x_n + \frac{\Delta t}{2}[f(x_n, t_n) + f(x_{n+1}, t_{n+1})] \tag{11.10}$$

is that it is an implicit method, meaning that for nonlinear ODEs, we have to solve a set of nonlinear equations.

One possible way of making Eq. (11.10) explicit is to use the explicit Euler method to guess the value of f at t_{n+1}, giving *Heun's method*

$$\begin{aligned} \tilde{x}_{n+1} &= x_n + \Delta t f(t_n, x_n) \\ x_{n+1} &= x_n + \frac{\Delta t}{2}[f(t_n, x_n) + f(t_{n+1}, \tilde{x}_{n+1})] \end{aligned} \tag{11.11}$$

(a) Determine the order of accuracy of Eq. (11.11).

(b) Determine the region of absolute stability of Eq. (11.11).

(c) Compare Eq. (11.11) with the general RK2 method and determine the constants a_1, a_2, p_1, and q_{11}.