ENGR20005
Numerical Methods in Engineering

Workshop 7

Part A: Pre-Lecture Problems

Please try to attempt 7.1 before Lecture 13 and 7.2 before Lecture 14.

7.1 Consider the function

$$f(x) = e^{-(x-10)^2}\cos(6(x-10)) \tag{7.1}$$

(a) Find the analytical derivative of this function.

(b) Use Lecture13M03.m and use CDA, FDA or BDA to calculate the derivative of this function using equally and unequally spaced grid in the domain $x \in [0, 10]$. Note that all the 'action' of this function is confined to closer to $x = 10$ so you should construct an appropriate mesh when you use an unequally spaced grid.

7.2 Write a MATLAB program to compute the derivative matrix on an equally spaced grid. Use CDA for the interior points and FDA and BDA at the end points. Use this matrix to compute the derivative of Eq. (7.1)

Part B: MATLAB Livescripts

7.3 The livescript *ENGR20005_Workshop7p3.mlx* runs through the use of MATLAB functions to integrate functions.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to compute the integral

$$\int_0^{10} f(x)\, dx \tag{7.2}$$

where $f(x)$ is given by

i. $f(x) = x^3 + 6x + 9$

ii. $f(x) = \sin^2(x)\tan(x^2)$

7.4 The livescript *ENGR20005_Workshop7p4.mlx* runs through numerical integration.

(a) Read through the livescript and make sure you understand what each line of code does.

(b) Modify the livescript to evaluate the integrals in Question 7.3.

Part C: Problems

7.5 Consider the function

$$f(x) = 6x^3 + x^2 - 11x + 4 \tag{7.3}$$

What is the error in evaluating the integral

$$\int_0^1 f(x)\, dx$$

using both the Trapezoidal and Simpson's rule with a single interval.

7.6 The temperature distribution on a rectangular plate can be approximated by

$$T(x, y) = x^4 + 2y^2x^2. \tag{7.4}$$

The average temperature on this 2m wide and 6m tall plate can be written as

$$T_{ave} = \frac{1}{6 \times 2} \int_0^6 \int_0^2 T(x, y) dx dy. \tag{7.5}$$

Write a MATLAB program that uses the Simpson's rule to compute the double integral and find T_{ave} by following the steps below. You might like to start with the live scripts given in lectures as answers to Workshop 6.1 and 6.2 (*ENGR20005Workshop06p01.mlx* and *ENGR20005Workshop06p02.mlx*)

(a) Calculate and plot the temperature averaged in the x direction

$$g(y) = \frac{1}{2} \int_0^2 T(x, y) dx \tag{7.6}$$

and compare with the analytical answer $g(y) = 32/10 + 8y^2/3$.

(b) Extend your code in 7.6a and apply the Simpson's rule again (but now in y direction) to calculate

$$T_{ave} = \frac{1}{6} \int_0^6 g(y) dy. \tag{7.7}$$

What is the average temperature, T_{ave}, on this plate?

7.7 * In applied mathematics, the *error function*

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} \exp\left(-\tau^2\right) d\tau \tag{7.8}$$

often appears in statistics as well as phenomena involving diffusion.

(a) Write a MATLAB function (`.m` file) that computes Eq. (7.8), using Gaussian–Legendre quadrature with k points. Your function should take x and k as inputs, and output the value of $\mathrm{erf}(x)$.

To obtain the quadrature points and weights, the function `gauss_legendre_points.m` has been provided.

Compare your function with MATLAB's inbuilt `erf()` function.

(b) The evolution of a flow created by a infinitely long plate accelerating instantly from rest to a velocity of U is given by

$$u = U\left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right] \tag{7.9}$$

Use your answer to (a) to plot u within the domain $(y, t) \in [0, 2] \times [0, 5]$, assuming a viscosity of $\nu = 1$ m^2/s and a plate velocity of $U = 1$ m/s.

7.8 While Gauss–Legendre quadrature is useful most of the time, sometimes it's necessary to include the end nodes at $x = -1, 1$.

Assume that the quadrature formula is given by

$$I = \int_{-1}^{1} f(x)\, dx = w_0 f(-1) + w_1 f(x_1) + \cdots + w_{n-1} f(x_{n-1}) + w_n f(1) \tag{7.10}$$

(a) Determine the quadrature weights and points for the two, three, four, and five-point quadrature formulas.

(b) What is the highest order of polynomial that Eq. (7.10) will integrate exactly.

(c) i. Show that the Legendre differential equation may be written as

$$\frac{d}{dt}\left[(1 - t^2)\frac{dP_k}{dt}(t)\right] = -k(k+1)P_k(t) \tag{7.11}$$

ii. Integrate Eq. (7.11) once and show that

$$\hat{Lo}_{k+1}(t) = (1 - t^2)\frac{dP_k}{dt} = -k(k+1)\int_{-1}^{t} P_k(\tau)\, d\tau$$

which are known as the *Completed Lobatto polynomials*.

iii. Determine the Complete Lobatto polynomials when $k = 1, 2, 3, 4$.

iv. Determine the roots of the polynomials you found in part iii.

(d) Use what you found in part (c) to determine higher order Gauss–Lobatto quadrature rules.