Topology - HW

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Exercise 1. *By definition a subset $U \subset \mathbb{R}$ is open if it is a union of open intervals. Now suppose $f : \mathbb{R} \to \mathbb{R}$ is any function. Show that $f^{-1}(a,b)$ is open whenever (a,b) is an open interval if and only if $f^{-1}(U)$ is open whenever U is open.*

Solution 1.

$\Longrightarrow$: Suppose f is a real function, and $f^{-1}(a,b)$ is open for every open interval (a,b). Now if U is open in $\mathbb{R}$, then $U = \bigcup(a_i,b_i)$. Now we WTS $f^{-1}(\bigcup(a_i,b_i)) = \bigcup f^{-1}(a_i,b_i)$:

$$f^{-1}(\bigcup(a_i,b_i)) = \{x \in \mathbb{R} | f(x) \in \bigcup(a_i,b_i)\} =$$
$$\{x \in \mathbb{R} | \exists i \text{s.t.} f(x) \in (a_i,b_i)\} = \bigcup\{x \in \mathbb{R} | f(x) \in (a_i,b_i)\} = \bigcup f^{-1}(a_i,b_i)$$

So $f^{-1}(U)$ is a union of open intervals, thus open.

$\Longleftarrow$: By an entirely similar argument we can show that the converse holds.

Exercise 2. *Let X be a space and each of $B_1, B_2, \dots, B_n$ is closed. Show that $B_1 \cup B_2$ is closed, show that $\emptyset$ is closed and X is closed. Show that $\bigcap_{n=1}^{\infty} B_n$ is closed.*

Solution 2.

(1): B_i is closed, then B_i^c is open, and as union of open sets needs to be open, $B_1^c \cup B_2^c$ is open, hence it's complement is closed, so $B_1 \cup B_2$ is closed.

(2): X is closed because $X^c = \emptyset$ is open by definition. $\emptyset$ is closed because $\emptyset^c = X$ and X is open in X by definition.

(3): $\bigcap_{n=1}^{\infty} B_n$ is complement of $\bigcup_{n=1}^{\infty} B_n^c$ which is an open set, so our set is closed.

Exercise 3. *Suppose X is a space and $B \subset X$. Show $l(B)$ is closed if $\{x\}$ is closed for every $x \in X$. Show $B \cup l(B)$ is closed. Show that $\bar{B} = B \cup l(B)$. Show that B is closed if and only if $l(B) \subset B$*

Solution 3.

First let's show that closed set contains all of it's limit points. Let X be a space and $S \subset X$ be closed. Let $x \in l(S)$, therefore $\forall U \in T_x$ s.t. $x \in U$ has the following property: $U \cap (S \setminus \{x\}) \neq \emptyset$. Now suppose $x \notin S$, then $U := X \setminus S$ is open and $x \in U$, but $U \cap (S \setminus \{x\}) = \emptyset$, giving us a contradiction.

Now we can show that $l(B)$ is closed if $\{x\} is closed \forall x \in X$. First we take $S := l(l(B))$ and show $S \subset l(B)$. Take $x \in S$, by definition $\forall \text{open} U$ contains $y \in l(B)$ s.t. $y \neq x$. Now $U \setminus \{x\}$ is an open neighborhood of y, so by applying definition of $l(B)$, we get $z \in ((U \setminus \{x\}) \cap B)$ s.t. $z \neq y$. As this holds for all open neighborhoods of x, we get that $x \in l(B)$, hence $S = l(l(B)) \subset l(B) \implies l(B)$ is closed.

Now let's show that (1): $B \cup l(B) \subset \bar{B}$ and (2): $\bar{B} \subset B \cup l(B)$:
(1): Take $x \in B \cup l(B)$, if $x \in B$ then, $x \in \bar{B}$ by definition so let $x \in l(B)$, and suppose $x \notin \bar{B}$ for contradiction. Now $x in \bar{B}^c$ which is an open set, hence $\exists$ open U s.t. $x \in U U \cap \bar{B} = \emptyset$, even further $U \cap B = \emptyset$, giving us a contradiction.

(2): Take $x \in \bar{B}$, even further assume $x \in (\bar{B} \setminus B)$. Now let $x \notin l(B)$, then $\exists U$ s.t. $U \cap B = \{x\}$ so $x \in B$.

Exercise 4. *Suppose (X, d) is a metric space and τ_d is a topology generated by the open metric balls. Show that $U \in \tau_d$ if and only if U is a union of open metric balls.*

Solution 4.

$\implies$: Suppose $U \in \tau_d$ is an open set, and assume it's nonempty. Now if $U = X$, U is a union of all open balls in τ_d as $X \in \tau_d$ because X is an open ball in d for any metric d. Now if $U \neq X$, but $U in \tau_d$ that means that U is an open ball or a union of open balls, as union of sets in τ_d is still in τ_d.

$\impliedby$: If U is a union of open metric balls, then each component of U is in τ_d, hence $U \in \tau_d$.

Exercise 5. *s Let $X = \mathbb{R}$ and $d_1(x, y) = |x - y|$ and $d_2(x, y) = 2|x - y|$ be metrics. Show that $\tau_{d1} = \tau_{d2}$.*

Solution 5.

Let's show that open sets in d_1 and d_2 are the same. Take U open in d_1, now $\forall x \in U \exists \varepsilon_1$ s.t.
$B_{\varepsilon 1}(x) \subset U$. Now to prove that U is open in d_2 just take $\varepsilon_2 = \varepsilon_1/2$ and your "epsilon ball" is in U.

Exercise 6. *Suppose $f : X \to Y$ is a continuous and* $\lim_{n\to\infty} x_n = x$. *Show* $\lim_{n\to\infty} f(x_n) = f(x)$.

Solution 6.

Suppose $U \in T_x$ is an open set containing x, by definition of convergence $\exists N$ s.t. $[n > N] \implies x_n \in U$. Now, as f is continuous, we get $f(U) \in T_y$ and $f(x) \in U$, now we want to find N_2 s.t. $[n > N] \implies f(x_n) \in f(U)$. So take $N_2 = N$ and plug it in. Now $[n > N_2] \implies x_n \in U \implies f(x_n) \in f(U)$, as we wanted.

Exercise 7. *Consider $\mathbb{R}$ with following weird topology. Declare $U \subset \mathbb{R}$ open if and only if $\mathbb{R} \setminus U$ is empty, finite or countably infinite(or if $U = \emptyset$). Show that this is a topology. Now let $Y = \{0, 1\}$ be a set with discrete topology. Define $f : \mathbb{R} \to Y$ so that $f(x) = 0$ iff $x \leq 0$. Is f continuous? Suppose $x_n \to x$. Must $\{x_n\}$ be eventually constant? Is it true that $x_n \to x \implies f(x_n) \to f(x)$?*

Solution 7.

First let's show that $T_{\mathbb{R}}$ is a topology. $\emptyset \in T_{\mathbb{R}}$ by definition and $\mathbb{R} \in T_m athbbR$ as $\mathbb{R} \setminus \mathbb{R} = \emptyset$, so yes it is a topology.
Now to prove f is continuous. Let $U = \{0\}$, U is closed in $\mathbb{R}$, but $U = f^{-1}(\{0\})$ which is open in Y, so f isn't continuous.

Exercise 8. *Suppose X is metrizable, Y is any space and $f : X \to Y$ is a function. Show f is continuous if and only if $x_n \to x \implies f(x_n) \to f(x)$.*

Solution 8.

$\implies$:

Exercise 9. *Show that convergent sequences have unique limits in T_2 spaces.*

Solution 9.

Let x_n be a convergent sequence and suppose $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$. Now take $U, V \in \tau$ such that $x \in U$ and $y \in V$ but $U \cap V = \emptyset$. Now $x_n \in U$ infinitely often and $x_n \in V$ infinitely often, thus $x_n \in (U \cap V)$ infinitely often, contradicting properties of T_2 space.

Exercise 10. *Show that space X is T_1 iff $\{x\}$ is closed $\forall x \in X$.*

Solution 10.

$\implies$: Suppose X is T_1. Then $\forall y \in \{x\}^c \; \exists U_y \in \tau$ s.t. $y \in U_y, x \notin U_y$. Now $\bigcup U_y$ is open and is exactly the complement of $\{x\}$, so $\{x\}$ is closed.
$\impliedby$: Suppose $\{x\}$ is closed $\forall x \in X$, then $\{x\}^c \in \tau$ and $y \in \{x\}^c, \forall y \neq x$ and $x \notin \{x\}^c$ so space is T_1.

Exercise 11. *Let $X = \{a, b\}$ with the following open sets:$\emptyset, \{a\}, \{a, b\}$. Is X T_0? Is X T_1?*

Solution 11.

X is T_0 as we can find $U \in \tau$, such that $a \in U$, but $b \notin U$, just take $U = \{a\}$. On the other hand X isn't T_1 as we can't find a set in τ containing b that doesn't contain a.

Exercise 12. *Find a space that is not T_0.*

Solution 12.

Just take any two point space with indiscrete topology. It is not T_0 as we can't get $U \in \tau$ containing just one point but not the other.

Exercise 13. *(Check that $h : \mathbb{R} \to \mathbb{R}$ defined so that $h(x) = 3x$ is a homeomorphism.*

Solution 13.

First let's check if h is continuous.