Topology - HW

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Exercise 1. *By definition a subset $U \subset \mathbb{R}$ is open if it is a union of open intervals. Now suppose $f : \mathbb{R} \to \mathbb{R}$ is any function. Show that $f^{-1}(a, b)$ is open whenever (a, b) is an open interval if and only if $f^{-1}(U)$ is open whenever U is open.*

Solution 1.

$\implies$: Suppose f is a real function, and $f^{-1}(a, b)$ is open for every open interval (a, b). Now if U is open in $\mathbb{R}$, then $U = \bigcup(a_i, b_i)$. Now we WTS $f^{-1}(\bigcup(a_i, b_i)) = \bigcup f^{-1}(a_i, b_i)$:

$$f^{-1}(\bigcup(a_i, b_i)) = \{x \in \mathbb{R} | f(x) \in \bigcup(a_i, b_i)\} =$$
$$\{x \in \mathbb{R} | \exists i \text{s.t.} f(x) \in (a_i, b_i)\} = \bigcup\{x \in \mathbb{R} | f(x) \in (a_i, b_i)\} = \bigcup f^{-1}(a_i, b_i)$$

So $f^{-1}(U)$ is a union of open intervals, thus open.

$\impliedby$: By an entirely similar argument we can show that the converse holds.

Exercise 2. *Let X be a space and each of $B_1, B_2, \ldots, B_n$ is closed. Show that $B_1 \cup B_2$ is closed, show that $\emptyset$ is closed and X is closed. Show that $\bigcap_{n=1}^{\infty} B_n$ is closed.*

Solution 2.

(1): B_i is closed, then B_i^c is open, and as union of open sets needs to be open, $B_1^c \cup B_2^c$ is open, hence it's complement is closed, so $B_1 \cup B_2$ is closed.

(2): X is closed because $X^c = \emptyset$ is open by definition. $\emptyset$ is closed because $\emptyset^c = X$ and X is open in X by definition.

(3): $\bigcap_{n=1}^{\infty} B_n$ is complement of $\bigcup_{n=1}^{\infty} B_n^c$ which is an open set, so our set is closed.

Exercise 3. *Suppose X is a space and $B \subset X$. Show $l(B)$ is closed if $\{x\}$ is closed for every $x \in X$. Show $B \cup l(B)$ is closed. Show that $\bar{B} = B \cup l(B)$. Show that B is closed if and only if $l(B) \subset B$*

Solution 3.

First let's show that closed set contains all of it's limit points. Let X be a space and $S \subset X$ be closed. Let $x \in l(S)$, therefore $\forall U \in T_x$ s.t. $x \in U$ has the following property: $U \cap (S \setminus \{x\}) \neq \emptyset$. Now suppose $x \notin S$, then $U := X \setminus S$ is open and $x \in U$, but $U \cap (S \setminus \{x\}) = \emptyset$, giving us a contradiction.

Now we can show that $l(B)$ is closed if $\{x\} is closed \forall x \in X$. First we take $S := l(l(B))$ and show $S \subset l(B)$. Take $x \in S$, by definition $\forall \text{open} U$ contains $y \in l(B)$ s.t. $y \neq x$. Now $U \setminus \{x\}$ is an open neighborhood of y, so by applying definition of $l(B)$, we get $z \in ((U \setminus \{x\}) \cap B)$ s.t. $z \neq y$. As this holds for all open neighborhoods of x, we get that $x \in l(B)$, hence $S = l(l(B)) \subset l(B) \implies l(B)$ is closed.

Now let's show that (1): $B \cup l(B) \subset \bar{B}$ and (2): $\bar{B} \subset B \cup l(B)$:
(1): Take $x \in B \cup l(B)$, if $x \in B$ then, $x \in \bar{B}$ by definition so let $x \in l(B)$, and suppose $x \notin \bar{B}$ for contradiction. Now $xin\bar{B}^c$ which is an open set, hence $\exists$ open U s.t. $x \in UU \cap \bar{B} = \emptyset$, even further $U \cap B = \emptyset$, giving us a contradiction.

(2): Take $x \in \bar{B}$, even further assume $x \in (\bar{B} \setminus B)$. Now let $x \notin l(B)$, then $\exists U$ s.t. $U \cap B = \{x\}$ so $x \in B$.

Exercise 4. *Suppose* (X, d) *is a metric space and* τ_d *is a topology generated by the open metric balls. Show that* $U \in \tau_d$ *if and only if* U *is a union of open metric balls.*

Solution 4.

$\implies$: Suppose $U \in \tau_d$ is an open set, and assume it's nonempty. Now if $U = X$, U is a union of all open balls in τ_d as $X \in \tau_d$ because X is an open ball in d for any metric d. Now if $U \neq X$, but $U in \tau_d$ that means that U is an open ball or a union of open balls, as union of sets in τ_d is still in τ_d.

$\impliedby$: If U is a union of open metric balls, then each component of U is in τ_d, hence $U \in \tau_d$.

Exercise 5. *s Let* $X = \mathbb{R}$ *and* $d_1(x, y) = |x - y|$ *and* $d_2(x, y) = 2|x - y|$ *be metrics. Show that* $\tau_{d1} = \tau_{d2}$.

Solution 5.

Let's show that open sets in d_1 and d_2 are the same. Take U open in d_1, now $\forall x \in U \exists \varepsilon_1$ s.t. $B_{\varepsilon 1}(x) \subset U$. Now to prove that U is open in d_2 just take $\varepsilon_2 = \varepsilon_1/2$ and your "epsilon ball" is in U.

Exercise 6. *Suppose* $f : X \to Y$ *is a continuous and* $\lim_{n\to\infty} x_n = x$. *Show* $\lim_{n\to\infty} f(x_n) = f(x)$.

Solution 6.

Suppose $U \in T_x$ is an open set containing x, by definition of convergence $\exists N$ s.t. $[n > N] \implies x_n \in U$. Now, as f is continuous, we get $f(U) \in T_y$ and $f(x) \in U$, now we want to find N_2 s.t. $[n > N] \implies f(x_n) \in f(U)$. So take $N_2 = N$ and plug it in. Now $[n > N_2] \implies x_n \in U \implies f(x_n) \in f(U)$, as we wanted.

Exercise 7. *Consider* $\mathbb{R}$ *with following weird topology. Declare* $U \subset \mathbb{R}$ *open if and only if* $\mathbb{R} \setminus U$ *is empty, finite or countably infinite(or if* $U = \emptyset$*). Show that this is a topology. Now let* $Y = \{0, 1\}$ *be a set with discrete topology. Define* $f : \mathbb{R} \to Y$ *so that* $f(x) = 0$ *iff* $x \leq 0$*. Is* f *continuous? Suppose* $x_n \to x$*. Must* $\{x_n\}$ *be eventually constant? Is it true that* $x_n \to x \implies f(x_n) \to f(x)$*?*

Solution 7.

First let's show that $T_{\mathbb{R}}$ is a topology. $\emptyset \in T_{\mathbb{R}}$ by definition and $\mathbb{R} \in T_mathbbR$ as $\mathbb{R} \setminus \mathbb{R} = \emptyset$, so yes it is a topology.
Now to prove f is continuous. Let $U = \{0\}$, U is closed in $\mathbb{R}$, but $U = f^{-1}(\{0\})$ which is open in Y, so f isn't continuous.

Exercise 8. *Suppose* X *is metrizable,* Y *is any space and* $f : X \to Y$ *is a function. Show* f *is continuous if and only if* $x_n \to x \implies f(x_n) \to f(x)$.

Solution 8.

$\implies$:

Exercise 9. *Show that convergent sequences have unique limits in* T_2 *spaces.*

Solution 9.

Let x_n be a convergent sequence and suppose $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$. Now take $U, V \in \tau$ such that $x \in U$ and $y \in V$ but $U \cap V = \emptyset$. Now $x_n \in U$ infinitely often and $x_n \in V$ infinitely often, thus $x_n \in (U \cap V)$ infinitely often, contradicting properties of T_2 space.

Exercise 10. *Show that space* X *is* T_1 *iff* $\{x\}$ *is closed* $\forall x \in X$.

Solution 10.

$\implies$: Suppose X is T_1. Then $\forall y \in \{x\}^c\ \exists U_y \in \tau$ s.t. $y \in U_y, x \notin U_y$. Now $\bigcup U_y$ is open and is exactly the complement of $\{x\}$, so $\{x\}$ is closed.
$\impliedby$: Suppose $\{x\}$ is closed $\forall x \in X$, then $\{x\}^c \in \tau$ and $y \in \{x\}^c, \forall y \neq x$ and $x \notin \{x\}^c$ so space is T_1.

Exercise 11. *Let* $X = \{a, b\}$ *with the following open sets:*$\emptyset, \{a\}, \{a, b\}$*. Is* X T_0*? Is* X T_1*?*

Solution 11.

X is T_0 as we can find $U \in \tau$, such that $a \in U$, but $b \notin U$, just take $U = \{a\}$. On the other hand X isn't T_1 as we can't find a set in τ containing b that doesn't contain a.

Exercise 12. *Find a space that is not T_0.*

Solution 12.

Just take any two point space with indiscrete topology. It is not T_0 as we can't get $U \in \tau$ containing just one point but not the other.

Exercise 13. *(Check that $h : \mathbb{R} \to \mathbb{R}$ defined so that $h(x) = 3x$ is a homeomorphism.*

Solution 13.

First let's check if h is continuous. As we're working in $\mathbb{R}$, all open sets are unions of open intervals, now if we take $U \in \tau$ in the codomain, and assume U is of the form $\bigcup_{i \in I}(a_i, b_i)$, then $f^{-1}(U) = \bigcup_{i \in I}(\frac{a_i}{3}, \frac{b_i}{3})$, which is obviously open. By similar argument the invers is continuous and hence h is homeomorphism.

Exercise 14. *Show that (A, τ_A) is a space, assuming $A \subset X$ and τ_A is a subspace topology.*

Solution 14.

To show that $(A.\tau_A)$ is a space we need to show $A \in \tau_A$ and $\emptyset \in \tau_A$. To show this we have to find $U \in \tau_X$ such that $U \cap A = A$, and $V \in \tau_X$ such that $V \cap A = \emptyset$. Let's take $U = X$, and $V = \emptyset$, by definition U, V satisfy our needs, so (A, τ_A) is a space with a subspace topology.

Exercise 15. *Suppose (X, τ_X) and (Y, τ_Y) are spaces, $f : X \to Y$ is a continuous and $A \subset X$. Let (A, τ_A) be the space with subspace topology. Show $f|A$ is continuous.*

Solution 15.

Let $U \in \tau_Y$ and let $V \in \tau_X$ be $f^{-1}(U)$. Now by definition $(f|A)^{-1} = A \cap V$ and hence open in the subspace topology, so $f|A$ is continuous.

Exercise 16. *Is discrete topology in fact a topology? Is course topology in fact a topology?*

Solution 16.

(1): Discrete topology is a topology because $X \in 2^X$ and $\emptyset \in 2^X$ and $\tau_X = 2^X$.
(2): Course topology is a topology because by definition it only contains emppty set and the whole set.

Exercise 17. *Show that the space (X, τ_X) has the discrete topology if and only if $\{x\}$ is open in $X \forall x \in X$ if and only if $\forall (Y, \tau_Y)$, and all functions $f : X \to Y$ are continuous.*

Solution 17.

$\implies$: Let X be a space with discrete topology, thus every $U \subset X$ is open, and $\{x\} \subset X$, hence $\{x\}$ is open. Now take $v \in \tau_Y$ and any $f : X \to Y$, now $f^{-1}(V) \subset X \implies f^{-1}(V) \in \tau_X$, so f is continuous.

$\impliedby$: Suppose any function $f : X \to Y$ is continuous. That means that if $\forall y \in Y$, $\{y\} \in \tau_Y$, $f^{-1}(y) = \{x\}$ is open, so $\{x\} \in \tau_X$. But now, any $\bigcup_{i \in I}\{x_i\} \in \tau_X$, so any subset of X is open, so X has discrete topology.

Exercise 18. *Show that the space (X, τ_X) has course topology if and only if for all spaces (Y, τ_Y) and all functions $f : Y \to X$, f is continuous.*

Solution 18.

$\implies$: Assume (X, τ_X) has course topology, and let (Y, τ_Y) be any space and $fY \to X$ be any function. Now to show that f is continuous we need to check that X and $\emptyset$ have open preimages, however this is easy as $f^{-1}(X) = Y$ and $f^{-1}(\emptyset) = \emptyset$ which are open by definition of a topology.
$\impliedby$: Suppose $(Y, \tau_Y), (X, \tau_X)$ are spaces and any $f : Y \to X$ is continuous. Suppose X has a not course topology, then we can always construct an f_i that sends a closed set in Y to an open set in X, except for X and $\emptyset$ as $f^{-1}(X) = Y$ by definition and $f^{-1}(\emptyset) = \emptyset$ which are open in Y, so X has course topology.

Exercise 19. *If (X, τ_a) is finer than (X, τ_b), which of the following functions is guaranteed to be continuous? $idx : (X, \tau_a) \to (X, \tau_b)$ or $jdx : (X, \tau_b) \to (X, \tau_a)$? Is either guaranteed to be homeomorphism?*

Solution 19.

Assume, $\tau_b \subset \tau_a$, now this guarantees that idx is continuous, but jdx isn't continuous, and thus neither is a homeomorphism.

Exercise 20. *If (X, τ_a) is finer than (X, τ_b), which space is likely to have more convergent sequences?*

Solution 20.

Courser topology is likely to have more convergent sequences as all convergent sequences in τ_a converge in τ_b, but the converse doesn't need to hold.

Exercise 21. *Suppose (X, τ_X) is a space, $A \subset X$, and (A, τ_A) has the subspace topology. Show that if $F \subset A$, then F is closed in (A, τ_A) if and only if $F = A \cap C$, and C is closed in X.*

Solution 21.

$\implies$: Let F be closed in A, then $F^c \in \tau_A$ and $\exists U \in \tau_X$ such that $F^c = A \cap U$, so now $F = A \cap U^c$ and U^c is closed in X exactly like we wanted.
$\impliedby$: Take $F = U \cap A$ for some closed $U \in X$, then $F^c = U^c \cap A$, and since $U^c \in \tau_X$ we know $F^c \in \tau_A$, and hence F is closed in A.

Exercise 22. $\mathbb{R}$ *with usual topology is connected.*

Solution 22.

Let $A \subset \mathbb{R}$ be open, nonempty and $A \neq \mathbb{R}$, now we need to show A is not closed. Let $a \in A$ and take $c \notin A$, $a < c$. Now define a set $Z = \{x | x \in \mathbb{R}, [a, x] \subset A\}$, and let $b = sup(Z)$. By definition of Z, $b \notin Z$, but $b \in \bar{Z}$, so $b \in l(Z)$. Now, $b \notin Z \implies b \notin A$ and $b \in \bar{Z} \implies b \in \bar{A}$, so $l(A) \not\subset A$, hence A is closed.

Exercise 23. *Suppose X is a space and $A \subset X$ is connected. Show that $\bar{A}$ is connected.*

Solution 23.

Take $Y \subset X = \bar{A}$, now A is dense in Y. Take nonempty $U \subset Y$ and assume U is clopen in Y, now we want to show $U = Y$. By definition U is clopen in A, further, either $U \cap A = A$ or $U \cap A = \emptyset$. To show that intersection is nonempty take $x \in U$, now if $x \notin A$, then $x \in \bar{A} \implies x \in l(A)$. Now by definition of a limit point $\forall S \in \tau$ s.t. $x \in S$, $S \cap A \neq \emptyset$, and as $U \in \tau$, we can't have $U \cap A = \emptyset$, so $U \cap A = A$. Now:

$$A \subset U \implies \bar{A} \subset \bar{U} \implies Y \subset \bar{U} \implies Y = U$$

As we wanted to show.

Exercise 24. *Suppose X is a space and let $a \in X$. Now assume that A_i is connected and $a \in A_i$ $\forall i \in I$. Show that if $\bigcup_{i \in I} A_i = X$, X is connected.*

Solution 24.

Suppose $U \subset X$ is clopen, and $U \neq \emptyset$, now by definition, U^c is clopen. Assume $a \in U$, now we want to prove $X \subset U$. To do this, let's get $x \in A_i$, note that $A_i \cap U$ is clopen and $a \in A_i \cap U$, hence $A_i \cap U = A_i$, so $x \in U$. Hence, $X = U$, $\forall$ clopen and nonempty $U \in X$, so X is connected.

Exercise 25. *If X is a space and $a \in X$, then there is a unique component $A \subset X$, such that $a \in A$.*

Solution 25.

Take $a \in X$ and let A_i be the union of all connected subsets B such that $a \in B$. Now A_i is connected. Furthermore A_i is maximal by it's definition. Now if A and B are components and $A \cap B \neq \emptyset$, then $A \cup B$ is connected, hence $A = B$. Thus the component containing any $x \in X$ is unique.

Exercise 26. *Suppose (X, τ_X) is connected and $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous surjection. Show that (Y, τ_Y) is connected.*

Solution 26.

Take a clopen, nonempty $U \subset Y$ and show that $U = Y$. Assume opposite for contradiction, now $f^{-1}(U)$ and $(f^{-1}(U))^c$ are both open, but the only nonempty set satisfying that is X, so now $f^{-1}(U) = X$, and as f is surjective that means that $U = Y$. So Y is connected.

Exercise 27. *Suppose X is a space and $A \subset X$ is a component of X. Why is A closed?*

Solution 27.

By definition A is a maximal connected set and is therefore clopen, and any clopen set is closed.

Exercise 28. *Suppose X is a space and $A \subset X$ is closed, $B \subset X$ is closed and $f : A \to Y$ and $g : B \to Y$ are continuous such that $f|A \cap B = g|A \cap B$. Then $h = f \cup g : A \cup B \to Y$ is continuous.*

Solution 28.

Firstly, let $K \subset Y$ be closed, then $h^{-1}(K) = (h^{-1}(K) \cap A) \cup (h^{-1}(K) \cap B) = h^{-1}(K) \cap (A \cup B)$. So now $h^{-1}(K)$ is the union of two closed subsets of X. Thus by subspace topology, h^{-1} is closed in $A \cup B$.

Exercise 29. *Suppose (Y, τ_Y) is a space and $X \subset Y$. Show that the following are equivalent. The subspace (X, τ_X) is compact. If $\{V_i\}$ is a collection of open sets in τ_Y covering X, there exists finitely many sets $\{V_1, V_2, \ldots, V_n\} \subset \{V_i\}$ such that $V \subset \bigcup_{j=1}^{n} V_j$.*

Solution 29.

$\implies$: If X is compact then every open cover yields a finite subcover. Now take $\{V_i\}$ to be the open cover and by definition it will yield a finite subcover, i.e $\{V_1, V_2, \ldots, V_n\}$. Converse is entirely similar.

Exercise 30. $[0,1]$ *is compact using the subspace topology of* $\mathbb{R}$.

Solution 30.

Since every open U in $\mathbb{R}$ is a union of open intervals, it suffices to prove the special case when $[0,1]$ is covered by open intervals. Suppose $[0,1] \subset \bigcup_{i \in I}(a_i, b_i)$. Now let $K = \{x \in [0,1] | [0,x]$ can be covered by finitely ma $0 \in K \implies K \neq \emptyset$. If $x \in K$ and $0 < y < x$, then $y \in K$, and since $K \subset [0,1]$, we can express K as either $[0,b]$ or $[0,b)$. Now let b be $sup(K)$, so $b \in K$, now if we take $b < 1$, we can always find another $a \in K$ such that $a > k$, contradicting maximality of b, hence $b = 1$, $K = [0,1]$, $[0,1]$ is compact.

Exercise 31. *Suppose (X, τ_X) is a compact space, and $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous surjection. Show that (Y, τ_Y) is a compact space.*

Solution 31.

Take a family of open covers $O \in \tau_Y$ covering Y and prove that it yields a finite subcover. Firstly note that $f^{-1}(O) \in \tau_X$ and thus yields a finite subcover, thus the image of that finite subcover is open in Y and covers Y. so Y is compact.

Exercise 32. *Supoose (X, τ_X) is a compact space and $A \subset X$ is closed. Prove that A, τ_A is compact.*

Solution 32.

Cover A by open sets from X and throw in the open set $V = X \setminus A$, that gives us an open cover of X, now take the finite subcover guaranteed by compactness of X, and throw out V, notice we have covered A.