

# Topology - HW

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**Exercise 1.** By definition a subset  $U \subset \mathbb{R}$  is open if it is a union of open intervals. Now suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is any function. Show that  $f^{-1}(a, b)$  is open whenever  $(a, b)$  is an open interval if and only if  $f^{-1}(U)$  is open whenever  $U$  is open.

**Solution 1.**

$\implies$  : Suppose  $f$  is a real function, and  $f^{-1}(a, b)$  is open for every open interval  $(a, b)$ . Now if  $U$  is open in  $\mathbb{R}$ , then  $U = \bigcup (a_i, b_i)$ . Now we WTS  $f^{-1}(\bigcup (a_i, b_i)) = \bigcup f^{-1}(a_i, b_i)$ :

$$\begin{aligned} f^{-1}\left(\bigcup (a_i, b_i)\right) &= \{x \in \mathbb{R} \mid f(x) \in \bigcup (a_i, b_i)\} = \\ &= \{x \in \mathbb{R} \mid \exists \text{ is.t. } f(x) \in (a_i, b_i)\} = \bigcup \{x \in \mathbb{R} \mid f(x) \in (a_i, b_i)\} = \bigcup f^{-1}(a_i, b_i) \end{aligned}$$

So  $f^{-1}(U)$  is a union of open intervals, thus open.

$\impliedby$  : By an entirely similar argument we can show that the converse holds.

**Exercise 2.** Let  $X$  be a space and each of  $B_1, B_2, \dots, B_n$  is closed. Show that  $B_1 \cup B_2$  is closed, show that  $\emptyset$  is closed and  $X$  is closed. Show that  $\bigcap_{n=1}^{\infty} B_n$  is closed.

**Solution 2.**

(1):  $B_i$  is closed, then  $B_i^c$  is open, and as union of open sets needs to be open,  $B_1^c \cup B_2^c$  is open, hence its complement is closed, so  $B_1 \cup B_2$  is closed.

(2):  $X$  is closed because  $X^c = \emptyset$  is open by definition.  $\emptyset$  is closed because  $\emptyset^c = X$  and  $X$  is open in  $X$  by definition.

(3):  $\bigcap_{n=1}^{\infty} B_n$  is complement of  $\bigcup_{n=1}^{\infty} B_n^c$  which is an open set, so our set is closed.

**Exercise 3.** Is  $\bar{B}$  closed? Why or why not?

**Solution 3.**

$\bar{B}$  is clearly closed as it is intersection of closed sets, and is by definition the smallest closed set containing  $B$ .

**Exercise 4.** Suppose  $X$  is a space and  $B \subset X$ . Show  $l(B)$  is closed if  $\{x\}$  is closed for every  $x \in X$ . Show  $B \cup l(B)$  is closed. Show that  $\bar{B} = B \cup l(B)$ . Show that  $B$  is closed if and only if  $l(B) \subset B$ .

**Solution 4.**

First let's show that closed set contains all of its limit points. Let  $X$  be a space and  $S \subset X$  be closed. Let  $x \in l(S)$ , therefore  $\forall U \in T_x$  s.t.  $x \in U$  has the following property:  $U \cap (S \setminus \{x\}) \neq \emptyset$ . Now suppose  $x \notin S$ , then  $U := X \setminus S$  is open and  $x \in U$ , but  $U \cap (S \setminus \{x\}) = \emptyset$ , giving us a contradiction.

Now we can show that  $l(B)$  is closed if  $\{x\}$  is closed  $\forall x \in X$ . First we take  $S := l(l(B))$  and show  $S \subset l(B)$ . Take  $x \in S$ , by definition  $\forall \text{ open } U$  contains  $y \in l(B)$  s.t.  $y \neq x$ . Now  $U \setminus \{x\}$  is an open neighborhood of  $y$ , so by applying definition of  $l(B)$ , we get  $z \in ((U \setminus \{x\}) \cap B)$  s.t.  $z \neq y$ . As this holds for all open neighborhoods of  $x$ , we get that  $x \in l(B)$ , hence  $S = l(l(B)) \subset l(B) \implies l(B)$  is closed.

Now let's show that (1):  $B \cup l(B) \subset \bar{B}$  and (2):  $\bar{B} \subset B \cup l(B)$ :

(1): Take  $x \in B \cup l(B)$ , if  $x \in B$  then,  $x \in \bar{B}$  by definition so let  $x \in l(B)$ , and suppose  $x \notin \bar{B}$  for contradiction. Now  $x \in \bar{B}^c$  which is an open set, hence  $\exists$  open  $U$  s.t.  $x \in U \cap \bar{B} = \emptyset$ , even further

$U \cap B = \emptyset$ , giving us a contradiction.

(2): Take  $x \in \bar{B}$ , even further assume  $x \in (\bar{B} \setminus B)$ . Now let  $x \notin l(B)$ , then  $\exists U$  s.t.  $U \cap B = \{x\}$  so  $x \in B$ .

**Exercise 5.** Suppose  $(X, d)$  is a metric space and  $\tau_d$  is a topology generated by the open metric balls. Show that  $U \in \tau_d$  if and only if  $U$  is a union of open metric balls.

**Solution 5.**

Let every set  $U \subset X$  have property  $O$  if it is a union of open balls. Every set with property  $O$  is in  $\tau_d$  and thus it suffices to show that  $O$  sets compose a topology. Firstly  $\emptyset$  is a union of no open balls and  $X$  is the union of all open balls. Now union of  $O$  sets is just a bigger  $O$  set by definition. Now to prove that intersection of two  $O$  sets  $U$  and  $V$  is an  $O$  set, take  $x \in U \cap V$ , then  $x$  is in the intersection of two open balls  $B_1 = B_{\epsilon_1}(x_1)$  and  $B_2 = B_{\epsilon_2}(x_2)$ , so now define  $a := \min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2))$ . Now, if  $y \in B_a(x)$  then  $d(y, x_1) \leq d(y, x) + d(x, x_1) \leq a$  so  $y \in B_1$  and by similar argument  $y \in B_2$ . So  $y \in B_1 \cap B_2$  and  $B_a(x) \subset B_1 \cap B_2$ . Now as this is true for every  $x$  we get that the intersection is a union of open balls.

**Exercise 6.** Let  $X = \mathbb{R}$  and  $d_1(x, y) = |x - y|$  and  $d_2(x, y) = 2|x - y|$  be metrics. Show that  $\tau_{d_1} = \tau_{d_2}$ .

**Solution 6.**

Let's show that open sets in  $d_1$  and  $d_2$  are the same. Take  $U$  open in  $d_1$ , now  $\forall x \in U \exists \epsilon_1$  s.t.  $B_{\epsilon_1}(x) \subset U$ . Now to prove that  $U$  is open in  $d_2$  just take  $\epsilon_2 = \epsilon_1/2$  and your "epsilon ball" is in  $U$ .

**Exercise 7.** Suppose  $f : X \rightarrow Y$  is a continuous and  $\lim_{n \rightarrow \infty} x_n = x$ . Show  $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ .

**Solution 7.**

Suppose  $U \in T_x$  is an open set containing  $x$ , by definition of convergence  $\exists N$  s.t.  $[n > N] \implies x_n \in U$ . Now, as  $f$  is continuous, we get  $f(U) \in T_y$  and  $f(x) \in U$ , now we want to find  $N_2$  s.t.  $[n > N] \implies f(x_n) \in f(U)$ . So take  $N_2 = N$  and plug it in. Now  $[n > N_2] \implies x_n \in U \implies f(x_n) \in f(U)$ , as we wanted.

**Exercise 8.** Consider  $\mathbb{R}$  with following weird topology. Declare  $U \subset \mathbb{R}$  open if and only if  $\mathbb{R} \setminus U$  is empty, finite or countably infinite (or if  $U = \emptyset$ ). Show that this is a topology. Now let  $Y = \{0, 1\}$  be a set with discrete topology. Define  $f : \mathbb{R} \rightarrow Y$  so that  $f(x) = 0$  iff  $x \leq 0$ . Is  $f$  continuous? Suppose  $x_n \rightarrow x$ . Must  $\{x_n\}$  be eventually constant? Is it true that  $x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$ ?

**Solution 8.**

First let's show that  $T_{\mathbb{R}}$  is a topology.  $\emptyset \in T_{\mathbb{R}}$  by definition and  $\mathbb{R} \in T_{\mathbb{R}}$  as  $\mathbb{R} \setminus \mathbb{R} = \emptyset$ , unions of open sets are open as if  $\mathbb{R} \setminus U$  is finite and  $\mathbb{R} \setminus V$  is finite, then the  $\mathbb{R} \setminus (U \cup V)$  is either smaller or equal to those differences, hence open. Now for intersections: it is important to note that two open sets in this topology can't be disjoint, hence the complement of intersection will be less than the union of two countable sets, so it will be countable, hence this is a topology.

Now to prove  $f$  is continuous. Let  $U = \{0\}$ ,  $U$  is closed in  $\mathbb{R}$ , but  $U = f^{-1}(\{0\})$  which is open in  $Y$ , so  $f$  isn't continuous.

If we take a sequence  $x_n \rightarrow x$  in  $\mathbb{R}$  it does not have to be constant at any point as we always get arbitrarily closer to  $x$  as open sets in our topology are pretty big.

**Exercise 9.** Suppose  $X$  is metrizable,  $Y$  is any space and  $f : X \rightarrow Y$  is a function. Show  $f$  is continuous if and only if  $x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$ .

**Solution 9.**

$\implies$  : already proven in exercise 7.  
 $\impliedby$  : Suppose  $f$  preserves convergent sequences. We hope to show that  $f$  is continuous. Suppose  $B \subset Y$  is closed, and to get a contradiction suppose  $A = f^{-1}(B)$  is not closed. Get a convergent sequence  $a_n \rightarrow x$  in  $A$  and  $x \notin A$ . We know  $f(a_n) \rightarrow f(x)$ . Since each  $f(a_n) \in B$  and since  $f(a_n) \rightarrow f(x)$  we have  $f(x) \in B$ . Thus  $x \in A$ , giving us the contradiction.

**Exercise 10.** Show that convergent sequences have unique limits in  $T_2$  spaces.

**Solution 10.**

Let  $x_n$  be a convergent sequence and suppose  $x_n \rightarrow x$  and  $x_n \rightarrow y$  s.t.  $x \neq y$ . Now take  $U, V \in \tau$  such that  $x \in U$  and  $y \in V$  but  $U \cap V = \emptyset$ . Now  $x_n \in U$  infinitely often and  $x_n \in V$  infinitely often, thus  $x_n \in (U \cap V)$  infinitely often, contradicting properties of  $T_2$  space.

**Exercise 11.** Show that space  $X$  is  $T_1$  iff  $\{x\}$  is closed  $\forall x \in X$ .

**Solution 11.**

$\implies$  : Suppose  $X$  is  $T_1$ . Then  $\forall y \in \{x\}^c \exists U_y \in \tau$  s.t.  $y \in U_y, x \notin U_y$ . Now  $\bigcup U_y$  is open and is exactly the complement of  $\{x\}$ , so  $\{x\}$  is closed.

$\impliedby$  : Suppose  $\{x\}$  is closed  $\forall x \in X$ , then  $\{x\}^c \in \tau$  and  $y \in \{x\}^c, \forall y \neq x$  and  $x \notin \{x\}^c$  so space is  $T_1$ .

**Exercise 12.** Let  $X = \{a, b\}$  with the following open sets:  $\emptyset, \{a\}, \{a, b\}$ . Is  $X$   $T_0$ ? Is  $X$   $T_1$ ?

**Solution 12.**

$X$  is  $T_0$  as we can find  $U \in \tau$ , such that  $a \in U$ , but  $b \notin U$ , just take  $U = \{a\}$ . On the other hand  $X$  isn't  $T_1$  as we can't find a set in  $\tau$  containing  $b$  that doesn't contain  $a$ .

**Exercise 13.** Find a space that is not  $T_0$ .

**Solution 13.**

Just take any two point space with indiscrete topology. It is not  $T_0$  as we can't get  $U \in \tau$  containing just one point but not the other.

**Exercise 14.** ( Check that  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined so that  $h(x) = 3x$  is a homeomorphism.

**Solution 14.**

First let's check if  $h$  is continuous. As we're working in  $\mathbb{R}$ , all open sets are unions of open intervals, now if we take  $U \in \tau$  in the codomain, and assume  $U$  is of the form  $\bigcup_{i \in I} (a_i, b_i)$ , then  $f^{-1}(U) = \bigcup_{i \in I} (\frac{a_i}{3}, \frac{b_i}{3})$ , which is obviously open. By similar argument the inverse is continuous and hence  $h$  is homeomorphism.

**Exercise 15.** Show that  $(A, \tau_A)$  is a space, assuming  $A \subset X$  and  $\tau_A$  is a subspace topology.

**Solution 15.**

To show that  $(A, \tau_A)$  is a space we need to show  $A \in \tau_A$  and  $\emptyset \in \tau_A$ . To show this we have to find  $U \in \tau_X$  such that  $U \cap A = A$ , and  $V \in \tau_X$  such that  $V \cap A = \emptyset$ . Let's take  $U = X$ , and  $V = \emptyset$ . Now to show that unions and intersections are in  $\tau_A$ . Take  $U$  and  $V$  to be any open sets in  $X$ , then we need to show that  $(V \cap A) \cap (U \cap A)$  is in  $\tau_A$ . To do this we just need to show that this is the same as  $A \cap (V \cap U)$  as this is for sure open in subspace topology. Start by taking  $x \in LHS$ , thus the  $x$  is in both  $U$  and  $V$  and it is also in  $A$  so it is in  $RHS$ . Now when we do the converse we get the closedness under intersections. Now to prove that infinite unions are in the topology we just do the same set theory manipulation.

**Exercise 16.** Suppose  $(X, \tau_X)$  and  $(Y, \tau_Y)$  are spaces,  $f : X \rightarrow Y$  is a continuous and  $A \subset X$ . Let  $(A, \tau_A)$  be the space with subspace topology. Show  $f|_A$  is continuous.

**Solution 16.**

Let  $U \in \tau_Y$  and let  $V \in \tau_X$  be  $f^{-1}(U)$ . Now by definition  $(f|_A)^{-1} = A \cap V$  and hence open in the subspace topology, so  $f|_A$  is continuous.

**Exercise 17.** Is discrete topology in fact a topology? Is coarse topology in fact a topology?

**Solution 17.**

(1): Discrete topology is a topology because  $X$  and  $\emptyset$  are for sure in there, and also if we take unions or intersection over subsets of  $X$  we will still get a subset of  $X$ .

(2): Coarse topology is a topology because by definition it only contains empty set and the whole set, so if we take a union we get the whole set back (which is in the topology) and if we take an intersection we get back the empty set.

**Exercise 18.** Show that the space  $(X, \tau_X)$  has the discrete topology if and only if  $\{x\}$  is open in  $X \forall x \in X$  if and only if  $\forall (Y, \tau_Y)$ , and all functions  $f : X \rightarrow Y$  are continuous.

**Solution 18.**

$\implies$  : Let  $X$  be a space with discrete topology, thus every  $U \subset X$  is open, and  $\{x\} \subset X$ , hence  $\{x\}$  is open. Now take  $v \in \tau_Y$  and any  $f : X \rightarrow Y$ , now  $f^{-1}(V) \subset X \implies f^{-1}(V) \in \tau_X$ , so  $f$  is continuous.

$\impliedby$  : Suppose any function  $f : X \rightarrow Y$  is continuous. That means that if  $\forall y \in Y, \{y\} \in \tau_Y$ ,  $f^{-1}(y) = \{x\}$  is open, so  $\{x\} \in \tau_X$ . But now, any  $\bigcup_{i \in I} \{x_i\} \in \tau_X$ , so any subset of  $X$  is open, so  $X$  has discrete topology.

**Exercise 19.** Show that the space  $(X, \tau_X)$  has course topology if and only if for all spaces  $(Y, \tau_Y)$  and all functions  $f : Y \rightarrow X$ ,  $f$  is continuous.

**Solution 19.**

$\implies$  : Assume  $(X, \tau_X)$  has course topology, and let  $(Y, \tau_Y)$  be any space and  $f : Y \rightarrow X$  be any function. Now to show that  $f$  is continuous we need to check that  $X$  and  $\emptyset$  have open preimages, however this is easy as  $f^{-1}(X) = Y$  and  $f^{-1}(\emptyset) = \emptyset$  which are open by definition of a topology.

$\impliedby$  : Suppose  $(Y, \tau_Y), (X, \tau_X)$  are spaces and any  $f : Y \rightarrow X$  is continuous. Suppose  $X$  has a not course topology, then we can always construct an  $f_i$  that sends a closed set in  $Y$  to an open set in  $X$ , except for  $X$  and  $\emptyset$  as  $f^{-1}(X) = Y$  by definition and  $f^{-1}(\emptyset) = \emptyset$  which are open in  $Y$ , so  $X$  has course topology.

**Exercise 20.** If  $(X, \tau_a)$  is finer than  $(X, \tau_b)$ , which of the following functions is guaranteed to be continuous?  $idx : (X, \tau_a) \rightarrow (X, \tau_b)$  or  $jdx : (X, \tau_b) \rightarrow (X, \tau_a)$ ? Is either guaranteed to be homeomorphism?

**Solution 20.**

Assume,  $\tau_b \subset \tau_a$ , now this guarantees that  $idx$  is continuous, but  $jdx$  isn't continuous, and thus neither is a homeomorphism.

**Exercise 21.** If  $(X, \tau_a)$  is finer than  $(X, \tau_b)$ , which space is likely to have more convergent sequences?

**Solution 21.**

Coarser topology is likely to have more convergent sequences as all convergent sequences in  $\tau_a$  converge in  $\tau_b$ , but the converse doesn't need to hold.

**Exercise 22.** Suppose  $(X, \tau_X)$  is a space,  $A \subset X$ , and  $(A, \tau_A)$  has the subspace topology. Show that if  $F \subset A$ , then  $F$  is closed in  $(A, \tau_A)$  if and only if  $F = A \cap C$ , and  $C$  is closed in  $X$ .

**Solution 22.**

$\implies$  : Let  $F$  be closed in  $A$ , then  $F^c \in \tau_A$  and  $\exists U \in \tau_X$  such that  $F^c = A \cap U$ , so now  $F = A \cap U^c$  and  $U^c$  is closed in  $X$  exactly like we wanted.

$\impliedby$  : Take  $F = U \cap A$  for some closed  $U \in X$ , then  $F^c = U^c \cap A$ , and since  $U^c \in \tau_X$  we know  $F^c \in \tau_A$ , and hence  $F$  is closed in  $A$ .

**Exercise 23.**  $\mathbb{R}$  with usual topology is connected.

**Solution 23.**

Let  $A \subset \mathbb{R}$  be open, nonempty and  $A \neq \mathbb{R}$ , now we need to show  $A$  is not closed. Let  $a \in A$  and take  $c \notin A, a < c$ . Now define a set  $Z = \{x | x \in \mathbb{R}, [a, x] \subset A\}$ , and let  $b = \sup(Z)$ . By definition of  $Z, b \notin Z$ , but  $b \in \bar{Z}$ , so  $b \in l(Z)$ . Now,  $b \notin Z \implies b \notin A$  and  $b \in \bar{Z} \implies b \in \bar{A}$ , so  $l(A) \not\subset A$ , hence  $A$  is not closed.

**Exercise 24.** Suppose  $X$  is a space and  $A \subset X$  is connected. Show that  $\bar{A}$  is connected.

**Solution 24.**

Take  $Y \subset X = \bar{A}$ , now  $A$  is dense in  $Y$ . Take nonempty  $U \subset Y$  and assume  $U$  is clopen in  $Y$ , now we want to show  $U = Y$ . By definition  $U$  is clopen in  $A$ , further, either  $U \cap A = A$  or  $U \cap A = \emptyset$ . To show that intersection is nonempty take  $x \in U$ , now if  $x \notin A$ , then  $x \in \bar{A} \implies x \in l(A)$ . Now by definition of a limit point  $\forall S \in \tau$  s.t.  $x \in S, S \cap A \neq \emptyset$ , and as  $U \in \tau$ , we can't have  $U \cap A = \emptyset$ , so  $U \cap A = A$ . Now:

$$A \subset U \implies \bar{A} \subset \bar{U} \implies Y \subset \bar{U} \implies Y = U$$

As we wanted to show.

**Exercise 25.** Suppose  $X$  is a space and let  $a \in X$ . Now assume that  $A_i$  is connected and  $a \in A_i \forall i \in I$ . Show that if  $\bigcup_{i \in I} A_i = X$ ,  $X$  is connected.

**Solution 25.**

Suppose  $U \subset X$  is clopen, and  $U \neq \emptyset$ , now by definition,  $U^c$  is clopen. Assume  $a \in U$ , now we want to prove  $X \subset U$ . To do this, let's get  $x \in A_i$ , note that  $A_i \cap U$  is clopen and  $a \in A_i \cap U$ , hence  $A_i \cap U = A_i$ , so  $x \in U$ . Hence,  $X = U$ ,  $\forall$  clopen and nonempty  $U \in X$ , so  $X$  is connected.

**Exercise 26.** *If  $X$  is a space and  $a \in X$ , then there is a unique component  $A \subset X$ , such that  $a \in A$ .*

**Solution 26.**

Take  $a \in X$  and let  $A_i$  be the union of all connected subsets  $B$  such that  $a \in B$ . Now  $A_i$  is connected. Furthermore  $A_i$  is maximal by its definition. Now if  $A$  and  $B$  are components and  $A \cap B \neq \emptyset$ , then  $A \cup B$  is connected, hence  $A = B$ . Thus the component containing any  $x \in X$  is unique.

**Exercise 27.** *Suppose  $(X, \tau_X)$  is connected and  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is a continuous surjection. Show that  $(Y, \tau_Y)$  is connected.*

**Solution 27.**

Take a clopen, nonempty  $U \subset Y$  and show that  $U = Y$ . Assume opposite for contradiction, now  $f^{-1}(U)$  and  $(f^{-1}(U))^c$  are both open, but the only nonempty set satisfying that is  $X$ , so now  $f^{-1}(U) = X$ , and as  $f$  is surjective that means that  $U = Y$ . So  $Y$  is connected.

**Exercise 28.** *Suppose  $X$  is a space and  $A \subset X$  is a component of  $X$ . Why is  $A$  closed?*

**Solution 28.**

By definition  $A$  is a maximal connected set and is therefore clopen, and any clopen set is closed.

**Exercise 29.** *Suppose  $X$  is a space and  $A \subset X$  is closed,  $B \subset X$  is closed and  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  are continuous such that  $f|_{A \cap B} = g|_{A \cap B}$ . Then  $h = f \cup g : A \cup B \rightarrow Y$  is continuous.*

**Solution 29.**

Firstly, let  $K \subset Y$  be closed, then  $h^{-1}(K) = (h^{-1}(K) \cap A) \cup (h^{-1}(K) \cap B) = h^{-1}(K) \cap (A \cup B)$ . So now  $h^{-1}(K)$  is the union of two closed subsets of  $X$ . Thus by subspace topology,  $h^{-1}$  is closed in  $A \cup B$ .

**Exercise 30.** *Suppose  $(Y, \tau_Y)$  is a space and  $X \subset Y$ . Show that the following are equivalent. The subspace  $(X, \tau_X)$  is compact. If  $\{V_i\}$  is a collection of open sets in  $\tau_Y$  covering  $X$ , there exists finitely many sets  $\{V_1, V_2, \dots, V_n\} \subset \{V_i\}$  such that  $V \subset \bigcup_{j=1}^n V_j$ .*

**Solution 30.**

$\implies$  : If  $X$  is compact then every open cover yields a finite subcover. Now take  $\{V_i\}$  to be the open cover and by definition it will yield a finite subcover, i.e  $\{V_1, V_2, \dots, V_n\}$ . Converse is entirely similar.

**Exercise 31.**  $[0, 1]$  is compact using the subspace topology of  $\mathbb{R}$ .

**Solution 31.**

Since every open  $U$  in  $\mathbb{R}$  is a union of open intervals, it suffices to prove the special case when  $[0, 1]$  is covered by open intervals. Suppose  $[0, 1] \subset \bigcup_{i \in I} (a_i, b_i)$ .

Now let  $K = \{x \in [0, 1] \mid [0, x] \text{ can be covered by finitely many } (a_i, b_i)\}$ .  $0 \in K \implies K \neq \emptyset$ . If  $x \in K$  and  $0 < y < x$ , then  $y \in K$ , and since  $K \subset [0, 1]$ , we can express  $K$  as either  $[0, b]$  or  $[0, b)$ . Now let  $b$  be  $\sup(K)$ , so  $b \in K$ , now if we take  $b < 1$ , we can always find another  $a \in K$  such that  $a > b$ , contradicting maximality of  $b$ , hence  $b = 1$ ,  $K = [0, 1]$ ,  $[0, 1]$  is compact.

**Exercise 32.** *Suppose  $(X, \tau_X)$  is a compact space, and  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is a continuous surjection. Show that  $(Y, \tau_Y)$  is a compact space.*

**Solution 32.**

Take a family of open covers  $O \in \tau_Y$  covering  $Y$  and prove that it yields a finite subcover. Firstly note that  $f^{-1}(O) \in \tau_X$  and thus yields a finite subcover, thus the image of that finite subcover is open in  $Y$  and covers  $Y$ . so  $Y$  is compact.

**Exercise 33.** *Suppose  $(X, \tau_X)$  is a compact space and  $A \subset X$  is closed. Prove that  $(A, \tau_A)$  is compact.*

**Solution 33.**

Cover  $A$  by open sets from  $X$  and throw in the open set  $V = X \setminus A$ , that gives us an open cover of  $X$ , now take the finite subcover guaranteed by compactness of  $X$ , and throw out  $V$ , notice we have covered  $A$ .

**Exercise 34.** Let  $X = \{1, 2, 3, \dots\}$  with a funny topology. Let's call it the closed finite topology, a topology where every finite set is closed. Which of the  $T_1, T_2$  axioms are true? Do convergent sequences have unique limits? Is  $X$  compact? Are compact subsets closed?

**Solution 34.**

Let's check for  $T_1$  first, as only finite sets are closed, this is pretty easy to confirm as for two distinct points  $x, y \in X$ , we can take  $U_x = X \setminus \{y\}$  and  $U_y = X \setminus \{x\}$ , both will be open as they're infinite and they satisfy the  $T_1$  axiom.

Now for  $T_2$ , we firstly take sets  $E, O \in \tau_X$  such that  $E$  contains all even numbers and  $O$  contains all the even ones, it is obvious that  $E \cap O = \emptyset$ , now for any two distinct points  $x, y$ , take  $U_x = E \setminus \{y\} \cup \{x\}$  if  $y \in E$  or  $U_x = E \cup \{x\}$  otherwise, now for  $U_y$  take  $U_y = O \setminus \{x\} \cup \{y\}$  if  $x \in O$  or  $U_y = O \cup \{y\}$  otherwise. Obviously  $U_x$  and  $U_y$  are open and disjoint, hence satisfying the properties of  $T_2$  space. Now by definition convergent sequences have unique limits.

Now to check if  $X$  is compact. Let  $K = \{x \in X \mid \{1, \dots, x\} \text{ can be covered by a finite amount of open sets}\}$ , now show that  $K = X$ . As long as  $\{1, 2, \dots, x\}$  is finite, it is compact, so now  $K$  is unbounded, so  $K = X$ .

**Exercise 35.** Show that  $W \subset X \times Y$  is open iff  $W$  is a union of sets of the form  $U_i \times V_i$  with  $U_i$  open in  $X$  and  $V_i$  open in  $Y$ .

**Solution 35.**

$\Rightarrow$  : Let  $W$  be open in  $X \times Y$ , now by definition of topology  $W$  is either a cartesian product  $U \times V$  where  $U \in X, V \in Y$  or the union of such sets.

**Exercise 36.** Given functions  $f : W \rightarrow X$  and  $g : W \rightarrow Y$  define  $(f, g) : W \rightarrow X \times Y$  as  $(f, g)(w) = (f(w), g(w))$ . Show that  $(f, g)$  is continuous if and only if  $f$  and  $g$  are continuous.

**Solution 36.**

$\Rightarrow$  : Assume  $(f, g)$  is continuous, that means that if  $U$  is open in  $X \times Y$ , then  $f(U)$  is open in  $X$  and  $g(U)$  is open in  $Y$  as by the definition of product topology, so in order for  $(f, g)^{-1}(U)$  to be open in  $W$ , then  $f^{-1}(U)$  and  $g^{-1}(U)$  have to be open in  $W$ , thus  $f$  and  $g$  are continuous.

$\Leftarrow$  : Assume  $f$  and  $g$  are continuous and let  $U, V$  be open in  $X, Y$  respectively. Now  $Z = U \times V$  is open in  $X \times Y$ , and  $f^{-1}(U), g^{-1}(V)$  are open in  $W$ , and by definition of  $(f, g)$ ,  $(f, g)^{-1}(Z)$  is open in  $Z$  so  $(f, g)$  is continuous.

**Exercise 37.** Note that sets of the form  $V_{i1} \cap V_{i2} \cap \dots \cap V_{in}$  must be open. Show that any open set  $V \subset \prod_{i \in I} X_i$  is a union of sets of the previous format.

**Solution 37.**

Not important for this exercise but what does  $V_{i1} \cap V_{i2} \dots \cap V_{in}$  look like? Ignoring the order that we write down the indices, it should be of the form  $U = U_{i_1} \times U_{i_2} \dots \times U_{i_n} \times \prod_{i \neq i_k} X_i$ .

Formally  $U = \prod_{i \in I} U_i$  so that  $U_i \subset X_i$  open in  $X_i$ , and with finitely many exceptions  $U_i = X_i$ . Such a  $U$  is a **basic open set** (every open set in  $\prod X_i$  is a union of sets of the format  $U = \prod_{i \in I} U_i$ ).

By the proof that follows every open set in  $\prod X_i$  is a union of sets of the format  $U_{i_1} \times U_{i_2} \dots \times U_{i_n} \times \prod_{i \neq i_k} X_i$ .

(From subbasis to basis to topology.)

Notice a much more general claim. Suppose  $X$  is any set and  $S \subset 2^X$  is a collection of sets so that  $\emptyset \in S$  and  $X \in S$  and so that if  $\{V_1, V_2, \dots, V_n\} \subset S$  then  $\bigcap_{i=1}^n V_i \in S$ . What topology does  $S$  generate? Let  $\tau_S$  be the collection of all sets  $U \subset X$  so that  $U$  is a union of sets in  $S$ . Keep in my we are trying to understand the smallest (coursest) topology  $\tau_X$  so that  $S \subset \tau_X$ . Notice  $\tau_S \subset \tau_X$ .

$\emptyset \in \tau_S$

$X \in \tau_S$

$\tau_S$  is closed under unions (if for each  $i \in I$  the set  $U_i$  is a union of sets in  $S$ . Then  $\bigcup_{i \in I} U_i$  is a union of sets in  $S$  why? if  $x \in \bigcup_{i \in I} U_i$  then  $x \in U_i$  for some  $i$  and hence  $x \in A \subset U_i$  for some  $A \in S$ ).

Is  $\tau_S$  closed under finite intersections

It is adequate to show that if  $U \in \tau_S$  and  $V \in \tau_S$  then  $U \cap V \in \tau_S$ .

$U = (\bigcup_{i \in I} A_i)$  with  $A_i \in S$ . And  $V = (\bigcup_{j \in J} B_j)$  with  $B_j \in S$ .

$U \cap V = (\bigcup_{i \in I} A_i) \cap (\bigcup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} (A_i \cap B_j)$ . The last set is  $\tau_S$  since each  $A_i \cap B_j \in S$ . (This proof is from the notes obviously i just wanted to have it handy as i liked it)

**Exercise 38.** Take a function  $f : W \rightarrow \prod_{i \in I} X_i$  and show it is continuous iff each function  $\pi_i : W \rightarrow X_i$  is continuous.

**Solution 38.**

$\implies$  : Suppose  $f$  is continuous and construct  $id_j : \prod_{i \in I} X_i \rightarrow X_j$ , by definition each  $id_j$  is continuous. Now note that  $f \circ id_j = \pi_j$  for each  $j$ , and as composition of two continuous functions is continuous, each  $\pi_j$  is continuous.

$\impliedby$  : Suppose each  $\pi_i$  is continuous. Let  $S = \prod_{i \in I} U_i$  be open in  $\prod_{i \in I} X_i$ , then  $f^{-1}(S) = \bigcap_{i \in I} \pi_i^{-1}(U_i)$ , a countable intersection of open sets, hence countable.

**Exercise 39.** Show what are convergent sequences in the countable product  $X_1 \times X_2 \times \dots$ .

**Solution 39.**

My best guess is that convergent sequences in countable product  $\prod_{i \in I} X_i$  are sequences  $x_n$  such that  $\pi_i(x_n)$  converges in  $X_i$ .

$\implies$  :

**Exercise 40.** If  $X_1$  is not compact, prove that  $X_1 \times X_2 \dots$  is not compact.

**Solution 40.**

Suppose that the product is compact for contradiction, then for every open cover  $O$  in the product, we can find a finite subcover  $O_\lambda$ , as  $O_\lambda$  is open, it is a union of open sets, and thus its components are open in each  $X_i$ , so  $O_1$  but can't be finite in  $X_1$ , thus  $O$  doesn't yield an open cover, so the product is not compact.

**Exercise 41.** Show that if each  $X_n$  is sequentially compact, then the countable product  $\prod_{i=1}^n X_i$  is sequentially compact.

**Solution 41.**

We consider a sequence of sequences, where each is a subsequence of the last. Define  $x_0 = a_1, a_2, \dots$  to be our original sequence, and construct  $x_1 = a_{11}, a_{12}$  to be a sequence where our first component converges (guaranteed by sequential compactness).

Now similarly build  $x_n$  to be a subsequence of  $x_{n-1}$  such that the  $n$ -th component converges. Now consider:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

More specifically consider the sequence on the diagonal of the "matrix", each  $a_{mm}$  converges in the  $X_m$  and thus the whole sequence converges in the product and it is the subsequence of  $x_0$ , hence the product is sequentially compact.

**Exercise 42.** if  $X \times Y$  are spaces using the product topology, define  $f : X \times Y \rightarrow X$  as  $f(x, y) = x$ . Which among the following are guaranteed? The function  $f$  is continuous? The function  $f$  is open and closed?

**Solution 42.**

None of these is guaranteed as open set  $U \subset X$  can have multiple elements in its preimage, some of which will be neither open nor closed if  $V \subset Y$  is closed.

**Exercise 43.** If  $X \times Y$  is compact, is  $X$  compact?

**Solution 43.**

By previous exercise, we know that if one component is not compact, the whole product is not compact. So in order for the product to be compact we need  $X$  to be compact.

**Exercise 44.** Prove that every metric space is Hausdorff ( $T_2$ ).

**Solution 44.**

Let  $X$  be a metric space with metric  $d$ . Get two distinct points  $x, y \in X$ , To find  $U$  containing  $x$  but not  $y$  just let  $U$  be the open ball with radius  $\frac{d(x,y)}{2}$  centered at  $x$ , and do the same for  $y$ .

**Exercise 45.** If  $X$  is  $T_2$  and  $Y$  is  $T_2$ , prove that  $X \times Y$  is  $T_2$ .

**Solution 45.**

Take points  $(x, y)$  and  $(a, b)$  in  $X \times Y$ , now if we take a set  $U \subset X$  such that  $x \in U$  but  $a \notin U$ , existence of  $U$  is given to us by  $X$  being  $T_2$  and we find a similar  $V \subset Y$  for  $y$  and  $b$ ,  $U \times V$  will be open in product space and  $(x, y) \in U \times V$  but  $(a, b) \notin U \times V$ . now we can find the same set containing  $(a, b)$ , thus  $X \times Y$  is  $T_2$ . Is  $X$  compact, connected, locally connected.

**Exercise 46.** Since  $[0, 1]$  is compact and metrizable, and since the finite product  $[0, 1] \times \cdots \times [0, 1]$  is compact and  $T_2$ , what can you say about the relationship between the closed and compact subsets of  $[0, 1] \times \cdots \times [0, 1]$ ?

**Solution 46.**

**Claim:** Every closed subset of  $[0, 1] \times \cdots \times [0, 1]$  is compact. To prove this just note that if we take a closed subset  $V$ , we get a free open set  $V^c$  now take any open cover of  $V$  and add  $V^c$  in order to cover the whole product, now by compactness you get a finite subcover of the product, and you have covered  $V$ .

**Claim:** Every compact subset of  $[0, 1] \times \cdots \times [0, 1]$  is closed. To prove this we will assume that a compact subset  $U$  is open for contradiction. Then the set is of the form  $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ , but by previous exercise, if any of the multiples is not compact, the product is not compact, so we just need to prove that  $(a, b) \subset [0, 1]$  is not compact. To do that take the open cover made by open intervals  $(0, \frac{1}{n})$ ,  $\frac{1}{n} \leq b$ , this doesn't yield a finite subcover Thus, every compact subset is closed.

**Exercise 47.** According to a legend, the compact subsets of the Euclidean metric space  $\mathbb{R}^n$  are precisely the closed and bounded subsets of  $\mathbb{R}^n$ . Prove this.

**Solution 47.**

Let  $X$  be closed and bounded in  $\mathbb{R}^n$ , then  $X = \prod_{i=1}^n [a_i, b_i]$ , now every  $[a_i, b_i]$  is compact and we know that product of two compact sets is compact, so by simple induction the whole countable product is compact so closed bounded sets are compact in  $\mathbb{R}^n$ .

For converse assume  $X \subset \mathbb{R}$  is compact but for contradiction suppose it is open, then it is of the form  $(a_1, b_1) \times (a_2, b_2) \times \cdots$ , and as we previously established, we need only one component of the product to fail in order for product to fail to be compact, and open intervals aren't compact in  $\mathbb{R}$  which finishes the proof.

**Exercise 48.** Suppose  $X$  is compact and  $f : X \rightarrow Y$  is continuous and  $Y$  is  $T_2$ . Prove  $f$  is a closed map.

**Solution 48.**

Take a closed  $C \subset X$ , we want to prove  $f(C)$  is closed. First note that  $C$  is compact as it is a subset of a compact space. Hence  $f(C)$  is a compact subset of a  $T_2$  space. Now to prove that  $K = f(C)$  is closed take  $y \in Y \setminus K$ , since  $Y$  is  $T_2$  there are disjoint open sets  $U_y$  and  $V_z$  such that  $y \in U_y$  and  $z \in V_z$  for every  $z \in K$ . Now  $\bigcup_{z \in K} V_z$  is an open cover of  $K$  and yields a finite subcover  $\{V_z | z \in L\}$  where  $L$  is finite. Now  $\bigcap_{z \in L} U_y$  is an open neighborhood of  $y$  disjoint from  $K$ . Since  $y$  is arbitrary  $K$  is closed.

**Exercise 49.** If  $X$  is metrizable and  $A$  is not closed. Then if  $x \in \bar{A} \setminus A$  there exists a sequence  $a_n$  converging to  $x$  and for all  $n$  we have  $a_n \in A$

**Solution 49.**

Get a metric  $d$  such that every open set in  $X$  is a union of open metric balls. Get the wanted  $x \in \bar{A} \setminus A$ .  $x$  is a limit point of  $A$ . For each  $n \in \mathbb{N}$  obtain  $a_n \in A$  such that  $a_n \in B(x, \frac{1}{2^n})$ . Now  $x$  is in said open ball because  $d$  is a metric and  $x$  is in every open ball around  $x$ . Note that  $x$  is a limit point of  $A$  but  $x \notin A$  and the open ball is open and contains  $x$  therefore the mentioned  $a_n$  exists. Now to show that  $a_n$  converges to  $x$ . Suppose  $U$  is an open set containing  $x$ . We must find  $N$  such that if  $n \geq N$  then  $a_n \in U$ . Since  $x \in U$  and  $U$  is open,  $U$  is a union of open balls of  $d$ . Thus, there exists  $y \in X$  and  $\epsilon \geq 0$  such that  $x$  in open  $\epsilon$ -ball around  $y$ , call it  $B_y$ , note that  $B_y \subset U$ . Thus  $d(x, y) < \epsilon$ . Now take  $\delta = \epsilon - d(x, y)$ . Get  $N$  such that if  $n > N$  then  $d(x, a_n) < \delta$ . Now when  $n \geq N$  we get  $d(a_n, y) \leq d(a_n, x) + d(x, y) < \delta + (\epsilon - \delta) = \epsilon$ . Thus  $a_n$  converges to  $x$ .

**Exercise 50.** Suppose  $X$  is any space and  $B \subset X$  is closed. Suppose  $b_n \in B$  for each  $n$  and suppose the sequence  $b_n \rightarrow x$ . Then  $x \in B$ .

**Solution 50.**



Suppose  $x \notin B$ . Then  $x$  is not a limit point of  $B$ . Let  $U$  be open and contain  $x$ . Since  $b_n \rightarrow x$ , get  $N$  such that  $n > N \implies b_n \in U$ . Since  $b_n \in B$  and  $x \notin B$ , this means that  $x$  is a limit point of  $B$  yielding us a contradiction.

**Exercise 51.** *If  $X$  is path connected then  $X$  is connected*

**Solution 51.**

Fix  $a \in X$ , now get a path  $\alpha_b$  for each  $b \in X$ . Now notice that image of every  $\alpha_b$  is connected and thus the union of all images is connected, that union is the space  $X$  and thus  $X$  is connected.

**Exercise 52.** *If  $X$  is nonempty, connected and locally path connected then  $X$  is path connected.*

**Solution 52.**

Fix  $a \in X$ . Define  $U_a = \{x \in X \mid \exists \text{ a path } \alpha : [0; 1] \rightarrow X \text{ so that } \alpha(0) = a \text{ and } \alpha(1) = x\}$ . Hope to show:  $U_a = X$ . Obviously  $U_a$  is nonempty as constant path  $\alpha_a$  exists so  $a \in U_a$ . Now to show that  $U_a$  is open: fix  $x \in U_a$  and choose a subset  $U$  containing  $x$ . Now all  $u \in U$  are path connected to  $x$  and thus path connected to  $a$  so  $U \subset U_a$  and  $U_a$  is open. To show that  $U_a$  is closed look at the closure of  $U_a$ . Look at  $y \in \bar{U}_a$ , and choose an open path connected set  $V$  containing  $y$ . Now as  $V \cap U_a \neq \emptyset$  hence take  $z$  in the intersection which is path connected to  $y$  and thus  $y$  is path connected to  $a$ , so now  $U_a = \bar{U}_a$ , thus  $\bar{U}_a$  is closed. Now as  $U_a$  is clopen and nonempty so it is actually our space  $X$  and thus our space is path connected.

**Exercise 53.** *Consider the following subset  $X \subset \mathbb{R}^2$ . Let  $X = [(0, 0), (0, 1)] \cup [(0, 0), (1, 0)] \cup [\bigcup_{n=2}^{\infty} (\frac{1}{n}, 0), (\frac{1}{n}, 1)]$ . Is  $X$  compact? Connected? Locally connected?*

**Solution 53.**

Can't think about this right now it seems hard, will do it later

**Exercise 54.** *Prove that  $T_4 \implies T_3 \implies T_2$ .*

**Solution 54.**

Suppose  $X$  is  $T_4$ , then for each disjoint pair of closed sets  $A$  and  $B$  we can find disjoint pair of open sets  $A \subset U$  and  $B \subset V$ , so now in order for our set to be  $T_3$  we will take  $a \in A$  and keep the set  $U$  and now  $a$  and  $B$  suffice the definition of  $T_3$ . To further this into a  $T_2$  argument just take  $b \in B$  while keeping the open set  $V$  and we're done.

**Exercise 55.** *Suppose  $X$  is compact and  $T_2$ . Show that  $X$  is  $T_3$ .*

**Solution 55.**

Let  $X$  be a compact  $T_2$  space, it is obviously compact so in order to show it is  $T_3$  we just need to show that it is regular. Take  $x \in X$  and let  $B$  be a closed set in  $X$  not containing  $x$ ,  $B$  is compact. Since  $X$  is  $T_2$  we can find  $U_x$  and  $U_b$  for every  $b \in B$  such that  $U_x \cap U_b = \emptyset$ . Now consider the cover of  $B$ :  $\mathcal{B} = \{U_b \mid b \in B\}$ , by compactness of  $B$  we can reduce this to a finite subcover  $\mathcal{B}' = \{U_{b_1}, U_{b_2}, \dots, U_{b_n}\}$ . Now let  $V = \bigcup_{i=1}^n U_{b_i}$ , it is clearly open and contains  $b$  but not  $x$ . Now let  $U = \bigcap_{i=1}^n U_{x_i}$ ,  $U$  is open and contains  $x$  but not  $b$  and thus  $U \cap V = \emptyset$ . So now we have satisfied regularity and  $X$  is  $T_3$ .

**Exercise 56.** *Suppose  $X$  is compact and  $T_2$ . Show that  $X$  is  $T_3$ .*

**Solution 56.**

Let  $X$  be a compact  $T_2$  space, then for  $T_4$  we just need to show normality of  $X$  as it is already  $T_1$ . Take disjoint closed sets  $A$  and  $B$  in  $X$ . We have proven that  $X$  is  $T_3$  so, for every  $a \in A$  exists  $U_a$  and  $V_a$  satisfying the properties of  $T_3$ . Now consider the following open cover of  $A$ :  $\mathcal{A} = \{U_a \mid a \in A\}$ , clearly this open cover yields a finite subcover  $\mathcal{A}' = \{U_{a_1}, \dots, U_{a_n}\}$ . Now let  $U$  be the union of sets in  $\mathcal{A}'$ ,  $A \subset U$ . Now let  $V$  be the finite intersection of corresponding  $V_{x_i}$ .  $V$  is open in  $X$  and  $B \subset V$ , moreover by the construction  $U \cap V = \emptyset$  satisfying the properties of  $T_4$ .

**Exercise 57.** *Suppose  $X$  is metrizable, show that  $X$  is  $T_3$*

**Solution 57.**

Suppose  $a \in X$  and take closed  $B$  not containing  $a$ . Get a  $d$  metric on  $X$ . Now there exists  $\epsilon > 0$  such that  $B_\epsilon(a) \cap B = \emptyset$ , then let  $U = B_{\frac{\epsilon}{2}}(a)$  and let  $V = \bigcup_{b \in B} B_{\frac{\epsilon}{2}}(b)$ . Clearly  $U \cap V = \emptyset$  sufficing conditions of  $T_3$ .

**Exercise 58.** *Suppose  $X$  is metrizable, show that  $X$  is  $T_4$*

**Solution 58.** Let  $A$  and  $C$  be disjoint closed subsets in  $X$ . Now for each  $x \in A$  find  $\epsilon_x$  such that open metric ball  $B(x, \epsilon_x)$  is disjoint with  $C$ , similarly find  $\epsilon_y$  for every  $y \in C$ . Now let  $U = \bigcup_{i \in I} B(y_i, \epsilon_{y_i}/3)$  and  $V = \bigcup_{i \in I} B(x_i, \epsilon_{x_i}/3)$ . Clearly,  $V$  and  $U$  are open in  $X$ , and  $A \subset V$ ,  $C \subset U$ .

Now suppose  $z \in U \cap V$ , then  $d(x, z) < \frac{\epsilon_x}{3}$  and  $d(y, z) < \frac{\epsilon_y}{3}$ . That would imply that  $d(x, y) \leq \frac{\epsilon_y}{3} + \frac{\epsilon_x}{3} < \epsilon_x$ , assuming  $\epsilon_x > \epsilon_y$ . But then  $y \in B(x, \epsilon_x)$ , yielding a contradiction.

**Exercise 59.** Suppose  $X$  is compact, show that  $X$  is locally compact.

**Solution 59.**

$X$  is open and is an open neighborhood of every point in the space, and as  $\bar{X} = X$  it's closure is compact, so  $X$  is locally compact.

**Exercise 60.** Suppose each of  $X$  and  $Y$  are locally compact. Prove that  $X \times Y$  is locally compact.

**Solution 60.**

Let  $(x, y)$  be a point in  $X \times Y$ , now as  $X$  is locally compact, there exists an open  $U$  containing  $x$  such that  $\bar{U}$  is compact, similarly we find an open  $V$  with compact closure in  $Y$ . Now  $U \times V$  is open in the product topology and  $\bar{U} \times \bar{V}$  is compact in the product space as it is a product of compact sets, therefore we found our open set with compact closure and  $X \times Y$  is locally compact.

**Exercise 61.** Show that countable product  $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$  is not locally compact.

**Solution 61.**

Consider a point  $x^n = (x_1, x_2, \dots, x_n)$  in the countable product. Every open set  $U$  containing  $x^n$  is of the form  $(a_1, b_1) \times (a_2, b_2) \times \cdots \times \mathbb{R} \times \mathbb{R}$  so its closure won't be compact as  $\mathbb{R}$  is not compact, thus sufficing to show the countable product is not compact.

**Exercise 62.** Suppose  $X$  is linearly ordered space with at least two elements  $a < b$ . Suppose  $m$  is minimal in  $X$  and  $m < x \in X$ . Must the half open interval  $[m, x)$  be open? Suppose  $M$  is maximal in  $X$ . If  $x < M$  must the half open interval  $(x, M]$  be open in  $X$ ? Suppose  $X$  has only one point. What are the open rays? What are the open sets in  $X$ ?

**Solution 62.**

Half open interval  $[m, x)$  is actually just left open ray  $L_x$  and is therefore open in  $X$ . Similarly  $(x, M]$  is  $R_x$ . If  $X$  has only one point  $x$ , then  $L_x = R_x = x$  and only open sets are  $X$  and  $\emptyset$ .

**Exercise 63.** Does every linearly ordered space  $(X, <)$  satisfy the  $T_2$  axiom?

**Solution 63.**

Suppose  $a, b \in X$  and  $a \neq b$ , then  $\{a, b\} \subset X$ . Assume  $a < b$ , then we have two cases:

- 1) If there exists  $c$  such that  $a < c < b$ , then we can just take  $L_c$  and  $R_c$  and thus satisfy the  $T_2$  axiom
- 2) If there is no point between  $a$  and  $b$ , then just take  $R_a$  and  $L_b$  and satisfy the  $T_2$  axiom.

**Exercise 64.** Suppose  $(X, <)$  is a well ordered space with  $x_1 \geq x_2 \geq x_3 \dots$ . Must  $\lim_{n \rightarrow \infty}$  exist? Is it possible that all terms of  $x_n$  are distinct?

**Solution 64.**

The limit must exist as we have a minimal element and our sequence has to become constant at some point, thus not all entries can be distinct.

**Exercise 65.** Suppose  $(X, <)$  is an infinite well ordered space with maximal element  $M$  and suppose  $x_n$  is a sequence in  $X$ . Show that  $x_n$  has a constant subsequence  $x_{n_1} \leq x_{n_2} \leq \dots$ . Must  $x_{m_j}$  converge?

**Solution 65.** If there exists a constant subsequence it immediately solves both of our problems. So let's assume such a subsequence doesn't exist. For starters take only distinct terms of  $x_n$ . Then let  $x_{n_1} = m_1 = \min\{x_1, x_2, \dots\}$ . Let  $m_2 = x_{n_2}$  be the minimum of the terms with index bigger than  $n_1$  and so on. This creates a subsequence  $m_1 < m_2 < \dots$ .

Now we have  $m_n < M$  for all  $n$ . Let  $B = \{x \in X \mid m_n < x \forall n\}$ . Let  $b = \min(B)$ . Now clearly  $m_n < b$ , so let  $U$  be an open set containing  $b$ . If  $b \in L_x$  for some  $x$  then we have  $m_n \in L_x$ . If  $b \in R_x$  then  $x \notin B$  and hence there exists some  $n$  for which  $x < m_n$ . Thus  $x < m_{n+k}$ , this shows that  $m_n \rightarrow b$ .

**Exercise 66.** Suppose  $X$  is a space and  $Y$  is a set and  $f : X \rightarrow Y$  is a surjection. Declare  $U \subset Y$  open iff  $f^{-1}(U)$  is open. Show this is really a topology on  $Y$ .

**Solution 66.**

First to show that the set  $Y$  is open in  $Y$ , it is obvious it is as  $f^{-1}(Y) = X$ . Similarly  $\emptyset$  is open in  $Y$  as it is its own preimage. So now we have to show that if  $U_1, U_2, \dots$  are open in  $Y$  so is their union. For this just take in consideration that  $f^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} f^{-1}(U_i)$  and as every individual preimage is open so will be the union of preimages and thus our union is open. Now we want to show that  $U \cap V$  is open assuming both  $U$  and  $V$  are open. To prove this just consider that  $f^{-1}(V \cap U) = f^{-1}(U) \cap f^{-1}(V)$  and as both preimages are open so is their intersection and thus  $U \cap V$  is open finishing the proof.

**Exercise 67.** Every closed map is a quotient map

**Solution 67.**

Take spaces  $X$  and  $Y$  and a closed map  $f : X \rightarrow Y$ . Now if we take a closed set  $K$  in  $Y$ , we know that  $f^{-1}(K)$  is closed and thus  $(f^{-1}(K))^c$  is open and mapped to  $K^c$  and thus open sets are mapped to open sets and our map is a quotient map.

**Exercise 68.** Every open map is a quotient map

**Solution 68.**

Take spaces  $X$  and  $Y$  and open map  $f$ , then any open set  $U$  in  $Y$  has open preimage  $V$  in  $X$ , similarly any open set  $V$  is mapped to an open  $U$  finishing the proof.

**Exercise 69.** Every homeomorphism is a quotient map.

**Solution 69.**

Take to spaces  $X$  and  $Y$  and homeomorphism  $h$ . Now  $h$  being continuous guarantees that preimage of every open  $U$  in  $Y$  is open in  $X$ . Now as  $h$  is homeomorphic that gives us continuous  $h^{-1}$  which guarantees us that each open  $V$  in  $X$  is mapped to an open set in  $Y$ , hence open sets in  $Y$  are exactly the ones whose preimage is open.

**Exercise 70.** Being homotopic is an equivalence relation on the set of maps  $C(X, Y)$ .

**Solution 70.**

- 1) Is  $f \sim f$ ? Define  $H : X \times [0, 1] \rightarrow Y$  as  $H(x, t) = f(x)$
- 2) Suppose  $f \sim g$ , is  $g \sim f$ ? Let  $G_t = H_{1-t}$  if  $H$  connects  $f$  to  $g$
- 3) Suppose  $f \sim g$  and  $g \sim h$ . Given the homotopies  $H_{fg}$  and  $H_{gh}$  and define  $H_{fh}(t)$  as  $H_{fg}(2t)$  for  $t \in (0, \frac{1}{2})$  and  $H_{gh}(2t - 1)$  for  $t \in (\frac{1}{2}, 1)$ .

**Exercise 71.** Homotopy equivalence is an equivalence relation.

**Solution 71.**

- 1) Is  $X \sim X$ ? Let  $f = g = id_X$ . Define  $H : X \times [0, 1] \rightarrow X$  as  $H(x, t) = x$
- 2) If  $X \sim Y$ , is  $Y \sim X$ ? True by definition of homotopy inverses
- 3) If  $X \sim Y$  and  $Y \sim Z$  is  $X \sim Z$ ? Start with  $f : X \rightarrow Y$ ,  $g : Y \rightarrow X$  and  $h : Y \rightarrow Z$ ,  $k : Z \rightarrow Y$ . Now we get  $hf : X \rightarrow Z$  and  $gk : Z \rightarrow X$

**Exercise 72.** Show that  $X$  is contractible if and only if  $X$  is homotopy equivalent to a one point space.

**Solution 72.**

$\Rightarrow$  : Assume  $X$  is contractible, that means that  $id_x$  is homotopic to a constant map to a one point subspace of  $X$ , will finish tomorrow I'm tired...

**Exercise 73.** Suppose  $(X, d)$  is a metric space and suppose the continuous surjection  $q : X \rightarrow Y$  is a quotient map. Show that  $Y$  is a sequential space.

**Solution 73.**

As we know that  $X$  is a metric space, we know that  $X$  is sequential, so take a nonclosed set  $A$  and a sequence  $a_1, a_2, \dots \rightarrow x$  where  $x \notin A$ . Now as we know  $A$  is nonclosed and  $q$  is a quotient map, we know that  $q(A)$  is nonclosed. Now as we know that  $q$  is continuous it will map converging sequences to converging sequences so  $q(a_1), q(a_2), \dots \rightarrow q(x)$  and  $q(x) \notin q(A)$  so  $Y$  is sequential.

**Exercise 74.** Suppose  $X$  is a sequential space and  $Z$  is any space and  $f : X \rightarrow Z$  is any map. Then  $f$  is continuous iff for each sequence  $\{x_1, x_2, \dots, x\}$  then  $f(x_n) \rightarrow f(x)$

**Solution 74.**

$\Leftarrow$  : Suppose that for an arbitrary sequence  $\{x_1, x_2, \dots, x\}$  in  $X$  we have  $f(x_n) \rightarrow f(x)$ . Now find an open  $K$  in  $Z$ ,

**Exercise 75.** Suppose for each  $i \in I$  that  $A_i$  is a closed compact subset of the compact space  $X$ . Then  $\emptyset \neq \bigcap_{i \in I} A_i$  if and only if  $A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k} \neq \emptyset$  for finite  $k$ .

**Solution 75.**

One direction is trivial. For the other direction let  $U_i = X \setminus A_i$  for each  $i$ . If  $\bigcap_{i \in I} A_i = \emptyset$  then  $\{U_i\}$  is an open cover admitting an open subcover of  $X$ , however, finite union of  $U_i$  would fail to cover a finite intersection of  $A_i$  giving us a contradiction.

**Exercise 76.** Show that if  $\dim(X) = 0$ , then  $\dim(X \times X \cdots) = 0$ .

**Solution 76.**

If  $\dim(X) = 0$  then the basis of  $X$  is composed of clopen sets, so now denote the basis as  $A_1, A_2, \dots$ , if we consider the product topology on the product, the base will be composed of the sets of the form  $\prod_{i=1}^n A_i \times X \times X \cdots$ , and thus be clopen as each component is clopen in  $X$ .

**Exercise 77.** Let  $C = \{0, 1\} \times \{0, 1\} \dots$ . Is  $C \times C$  homeomorphic to  $C$ ?

**Solution 77.**

Try  $h : C \rightarrow C \times C$  defined as  $h(s_1, s_2, \dots) = (s_1, s_3, s_5, \dots)(s_2, s_4, \dots)$ . To show that  $h$  is injective just take  $a$  and  $b$  in  $C$  such that  $h(a) = h(b)$ , that means that all their odd and even coordinates are the same, so  $a = b$ . To prove it's surjective just take any  $c \in C \times C$  and reconstruct the  $h^{-1}(c)$ . So now to finish we just have to prove  $h$  is continuous and we are done. To do this notice that open sets

**Exercise 78.** Must every homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $h(q) \in \mathbb{Q}$  for all  $q \in \mathbb{Q}$ ? Is it true if  $h$  isometry or linear.

**Solution 78.**

Consider  $h(x) = x + \sqrt{2}$  which is clearly injective, surjective and continuous but does not map  $\mathbb{Q}$  to  $\mathbb{Q}$ . this  $h$  is also an isometry so it works for that as well. To show that it doesn't work if  $h$  is linear just consider  $h(x) = \sqrt{2} \cdot x$ , it does everything we need.

**Exercise 79.** Suppose  $g : A \rightarrow B$  is a continuous bijection. Prove  $g$  is a homeomorphism iff  $g$  is a quotient map.

**Solution 79.**

One direction is obvious as  $g$  being a homeomorphism implies that  $g$  is a quotient map. For other direction assume  $g$  is a continuous, bijective quotient map, so now we have to prove that  $g^{-1}$  is continuous in order to get what we need. So let  $K$  be open in  $A$ , so now we want to know that  $(g^{-1})^{-1}(K)$  is open in  $B$ , but that is  $g(K)$  which we know is open as  $g$  is continuous.

**Exercise 80.** Suppose  $r : X \rightarrow A \subset X$  is a retraction. Then  $r(x) = x$  if and only if  $x \in A$ .

**Solution 80.**

If  $x \in A$  then we get what we want by definition of a retraction. If  $x \notin A$  then since  $\text{im}(r) \subset A$  and thus  $r(x) \in A$  so  $r(x) \neq x$ .

**Exercise 81.** Suppose  $X = [0, 1]$  and  $A = \{0, 1\}$ . Prove or disprove that  $A$  is a retract of  $X$ .

**Solution 81.**

Since  $X$  is connected and  $A$  isn't no function  $X \rightarrow A$  can be continuous.

**Exercise 82.** Suppose  $X$  is  $T_2$  and  $A \subset X$  and  $r : X \rightarrow A$  is a retraction. Prove that  $A$  is closed.

**Solution 82.**

Take  $x \notin A$ , so  $r(x) \in A$  and  $x \neq r(x)$ . So by  $X$  being  $T_2$  we can get disjoint open sets such that  $x \in U$  and  $r(x) \in V$ . Let  $W = U \cap r^{-1}(V)$ . We know  $x \in W$  so we WTS that  $x \notin A$ . If  $a \in U \cap A$ , then  $r(a) = a \notin V$ . Now as  $x \in r^{-1}(V)$  iff  $r(x) \in V$  so our  $a \notin r^{-1}(V)$ . This shows  $W \cap A = \emptyset$ .

**Exercise 83.**  $x \rightarrow x^2$  and  $x \rightarrow |x|$ . Is either a retraction?

**Solution 83.**

First function isn't a retraction as it has fixed point set  $\{0, 1\}$  which isn't connected so by previous exercise function can't be a retraction. Second function is a retraction with retract  $A = \mathbb{R}^{\geq 0}$ .