Topology - HW

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Exercise 1. *By definition a subset $U \subset \mathbb{R}$ is open if it is a union of open intervals. Now suppose $f : \mathbb{R} \to \mathbb{R}$ is any function. Show that $f^{-1}(a,b)$ is open whenever (a,b) is an open interval if and only if $f^{-1}(U)$ is open whenever U is open.*

Solution 1.

$\implies$: Suppose f is a real function, and $f^{-1}(a,b)$ is open for every open interval (a,b). Now if U is open in $\mathbb{R}$, then $U = \bigcup(a_i, b_i)$. Now we WTS $f^{-1}(\bigcup(a_i,b_i)) = \bigcup f^{-1}(a_i,b_i)$:

$$f^{-1}(\bigcup(a_i,b_i)) = \{x \in \mathbb{R} | f(x) \in \bigcup(a_i,b_i)\} =$$
$$\{x \in \mathbb{R} | \exists i s.t. f(x) \in (a_i,b_i)\} = \bigcup\{x \in \mathbb{R} | f(x) \in (a_i,b_i)\} = \bigcup f^{-1}(a_i,b_i)$$

So $f^{-1}(U)$ is a union of open intervals, thus open.

$\impliedby$: By an entirely similar argument we can show that the converse holds.

Exercise 2. *Let X be a space and each of $B_1, B_2, \ldots, B_n$ is closed. Show that $B_1 \cup B_2$ is closed, show that $\emptyset$ is closed and X is closed. Show that $\bigcap_{n=1}^{\infty} B_n$ is closed.*

Solution 2.

(1): B_i is closed, then B_i^c is open, and as union of open sets needs to be open, $B_1^c \cup B_2^c$ is open, hence it's complement is closed, so $B_1 \cup B_2$ is closed.

(2): X is closed because $X^c = \emptyset$ is open by definition. $\emptyset$ is closed because $\emptyset^c = X$ and X is open in X by definition.

(3): $\bigcap_{n=1}^{\infty} B_n$ is complement of $\bigcup_{n=1}^{\infty} B_n^c$ which is an open set, so our set is closed.

Exercise 3. *Is $\bar{B}$ closed? Why or why not?*

Solution 3.

$\bar{B}$ is cearly closed as it is intersection of closed sets, and is by definition the smallest closed set containing B.

Exercise 4. *Suppose X is a space and $B \subset X$. Show $l(B)$ is closed if $\{x\}$ is closed for every $x \in X$. Show $B \cup l(B)$ is closed. Show that $\bar{B} = B \cup l(B)$. Show that B is closed if and only if $l(B) \subset B$*

Solution 4.

First let's show that closed set contains all of it's limit points. Let X be a space and $S \subset X$ be closed. Let $x \in l(S)$, therefore $\forall U \in T_x$ s.t. $x \in U$ has the following property: $U \cap (S \setminus \{x\}) \neq \emptyset$. Now suppose $x \notin S$, then $U := X \setminus S$ is open and $x \in U$, but $U \cap (S \setminus \{x\}) = \emptyset$, giving us a contradiction.

Now we can show that $l(B)$ is closed if $\{x\} is closed \forall x \in X$. First we take $S := l(l(B))$ and show $S \subset l(B)$. Take $x \in S$, by definition $\forall open U$ contains $y \in l(B)$ s.t. $y \neq x$. Now $U \setminus \{x\}$ is an open neighborhood of y, so by applying definition of $l(B)$, we get $z \in ((U \setminus \{x\}) \cap B)$ s.t. $z \neq y$. As this holds for all open neighborhoods of x, we get that $x \in l(B)$, hence $S = l(l(B)) \subset l(B) \implies l(B)$ is closed.

Now let's show that (1): $B \cup l(B) \subset \bar{B}$ and (2): $\bar{B} \subset B \cup l(B)$:
(1): Take $x \in B \cup l(B)$, if $x \in B$ then, $x \in \bar{B}$ by definition so let $x \in l(B)$, and suppose $x \notin \bar{B}$ for contradiction. Now $xin\bar{B}^c$ which is an open set, hence $\exists$ open U s.t. $x \in UU \cap \bar{B} = \emptyset$, even further

$U \cap B = \emptyset$, giving us a contradiction.

(2): Take $x \in \bar{B}$, even further assume $x \in (\bar{B} \setminus B)$. Now let $x \notin l(B)$, then $\exists U$ s.t. $U \cap B = \{x\}$ so $x \in B$.

Exercise 5. *Suppose (X, d) is a metric space and τ_d is a topology generated by the open metric balls. Show that $U \in \tau_d$ if and only if U is a union of open metric balls.*

Solution 5.

Let every set $U \subset X$ have property O if it is a union of open balls. Every set with property O is in τ_d and thus it suffices to show that O sets compose a topology. Firstly $\emptyset$ is a union of no open balls and X is the union of all open balls. Now union of O sets is just a bigger O set by definition. Now to prove that intersection of two O sets U and V is an O set, take $x \in U \cap V$, then x is in the intersection of two open balls $B_1 = B_{\epsilon_1}(x_1)$ and $B_2 = B_{\epsilon_2}(x_2)$, so now define $a := min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2))$. Now, if $y \in B_a(x)$ then $d(y, x_1) \leq d(y, x) + d(x, x_1) \leq a$ so $y \in B_1$ and by similar argument $y \in B_2$. So $y \in B_1 \cap B_2$ and $B_a(x) \subset B_1 \cap B_2$. Now as this is true for every x we get that the intersection is a union of open balls.

Exercise 6. *Let $X = \mathbb{R}$ and $d_1(x, y) = |x - y|$ and $d_2(x, y) = 2|x - y|$ be metrics. Show that $\tau_{d1} = \tau_{d2}$.*

Solution 6.

Let's show that open sets in d_1 and d_2 are the same. Take U open in d_1, now $\forall x \in U \exists \varepsilon_1$ s.t. $B_{\varepsilon 1}(x) \subset U$. Now to prove that U is open in d_2 just take $\varepsilon_2 = \varepsilon_1/2$ and your "epsilon ball" is in U.

Exercise 7. *Suppose $f : X \to Y$ is a continuous and* $\lim_{n\to\infty} x_n = x$. *Show* $\lim_{n\to\infty} f(x_n) = f(x)$.

Solution 7.

Suppose $U \in T_x$ is an open set containing x, by definition of convergence $\exists N$ s.t. $[n > N] \implies x_n \in U$. Now, as f is continuous, we get $f(U) \in T_y$ and $f(x) \in U$, now we want to find N_2 s.t. $[n > N] \implies f(x_n) \in f(U)$. So take $N_2 = N$ and plug it in. Now $[n > N_2] \implies x_n \in U \implies f(x_n) \in f(U)$, as we wanted.

Exercise 8. *Consider $\mathbb{R}$ with following weird topology. Declare $U \subset \mathbb{R}$ open if and only if $\mathbb{R} \setminus U$ is empty, finite or countably infinite(or if $U = \emptyset$). Show that this is a topology. Now let $Y = \{0, 1\}$ be a set with discrete topology. Define $f : \mathbb{R} \to Y$ so that $f(x) = 0$ iff $x \leq 0$. Is f continuous? Suppose $x_n \to x$. Must $\{x_n\}$ be eventually constant? Is it true that $x_n \to x \implies f(x_n) \to f(x)$?*

Solution 8.

First let's show that $T_{\mathbb{R}}$ is a topology. $\emptyset \in T_{\mathbb{R}}$ by definition and $\mathbb{R} \in T_mathbbR$ as $\mathbb{R} \setminus \mathbb{R} = \emptyset$, unions of open sets are open as if $\mathbb{R} \setminus U$ is finite and $\mathbb{R} \setminus V$ is finite, then the $\mathbb{R} \setminus V \cup U$ is either smaller or equal to those differences, hence open. Now for intersections: it is important to note that two open sets in this topology can't be disjoint, hence the complement of intersection will be less than the union of two countable sets, so it will be countable, hence this is a topology.
Now to prove f is continuous. Let $U = \{0\}$, U is closed in $\mathbb{R}$, but $U = f^{-1}(\{0\})$ which is open in Y, so f isn't continuous.
If we take a sequence $x_n \to x$ in $\mathbb{R}$ it does not have to be constant at any point as we always get arbitrarily closer to x as open sets in our topology are pretty big.

Exercise 9. *Suppose X is metrizable, Y is any space and $f : X \to Y$ is a function. Show f is continuous if and only if $x_n \to x \implies f(x_n) \to f(x)$.*

Solution 9.

$\implies$: already proven in exercise 7.
$\impliedby$: Suppose f preserves convergent sequences. We hope to show that f is continuous. Suppose $B \subset Y$ is closed, and to get a contradiction suppose $A = f^{-1}(B)$ is not closed. Get a convergent sequence $a_n \to x$ in A and $x \notin A$. We know $f(a_n) \to f(x)$. Since each $f(a_n) \in B$ and since $f(a_n) \to f(x)$ we have $f(x) \in B$. Thus $x \in A$, giving us the contradiction.

Exercise 10. *Show that convergent sequences have unique limits in T_2 spaces.*

Solution 10.

Let x_n be a convergent sequence and suppose $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$. Now take $U, V \in \tau$ such that $x \in U$ and $y \in V$ but $U \cap V = \emptyset$. Now $x_n \in U$ infinitely often and $x_n \in V$ infinitely often, thus $x_n \in (U \cap V)$ infinitely often, contradicting properties of T_2 space.

Exercise 11. *Show that space X is T_1 iff $\{x\}$ is closed $\forall x \in X$.*

Solution 11.

$\implies$: Suppose X is T_1. Then $\forall y \in \{x\}^c\ \exists U_y \in \tau$ s.t. $y \in U_y, x \notin U_y$. Now $\bigcup U_y$ is open and is exactly the complement of $\{x\}$, so $\{x\}$ is closed.
$\impliedby$: Suppose $\{x\}$ is closed $\forall x \in X$, then $\{x\}^c \in \tau$ and $y \in \{x\}^c, \forall y \neq x$ and $x \notin \{x\}^c$ so space is T_1.

Exercise 12. *Let $X = \{a, b\}$ with the following open sets:$\emptyset, \{a\}, \{a, b\}$. Is X T_0? Is X T_1?*

Solution 12.

X is T_0 as we can find $U \in \tau$, such that $a \in U$, but $b \notin U$, just take $U = \{a\}$. On the other hand X isn't T_1 as we can't find a set in τ containing b that doesn't contain a.

Exercise 13. *Find a space that is not T_0.*

Solution 13.

Just take any two point space with indiscrete topology. It is not T_0 as we can't get $U \in \tau$ containing just one point but not the other.

Exercise 14. *(Check that $h : \mathbb{R} \to \mathbb{R}$ defined so that $h(x) = 3x$ is a homeomorphism.*

Solution 14.

First let's check if h is continuous. As we're working in $\mathbb{R}$, all open sets are unions of open intervals, now if we take $U \in \tau$ in the codomain, and assume U is of the form $\bigcup_{i \in I}(a_i, b_i)$, then $f^{-1}(U) = \bigcup_{i \in I}(\frac{a_i}{3}, \frac{b_i}{3})$, which is obviously open. By similar argument the invers is continuous and hence h is homeomorphism.

Exercise 15. *Show that (A, τ_A) is a space, assuming $A \subset X$ and τ_A is a subspace topology.*

Solution 15.

To show that $(A.\tau_A)$ is a space we need to show $A \in \tau_A$ and $\emptyset \in \tau_A$. To show this we have to find $U \in \tau_X$ such that $U \cap A = A$, and $V \in \tau_X$ such that $V \cap A = \emptyset$. Let's take $U = X$, and $V = \emptyset$. Now to show that unions and intersections are in τ_A. Take U and V to be any open sets in X, then we need to show that $(V \cap A) \cap (U \cap A)$ is in τ_A. To do this we just need to show that this is the same as $A \cap (V \cap U)$ as this is for sure open in subspace topology. Start by taking $x \in LHS$, thus the x is in both U and V and it is also in A so it is in RHS. Now when we do the converse we get the closedness under intersections. Now to prove that infinite unions are in the topology we just do the same set theory manipulation.

Exercise 16. *Suppose (X, τ_X) and (Y, τ_Y) are spaces, $f : X \to Y$ is a continuous and $A \subset X$. Let (A, τ_A) be the space with subspace topology. Show $f|A$ is continuous.*

Solution 16.

Let $U \in \tau_Y$ and let $V \in \tau_X$ be $f^{-1}(U)$. Now by definition $(f|A)^{-1} = A \cap V$ and hence open in the subspace topology, so $f|A$ is continuous.

Exercise 17. *Is discrete topology in fact a topology? Is course topology in fact a topology?*

Solution 17.

(1): Discrete topology is a topology because X and $\emptyset$ are for sure in there, and also if we take unions or intersection over subsets of X we will still get a subset of X.
(2): Course topology is a topology because by definition it only contains emppty set and the whole set, so if we take a union we get the whole set back (which is in the topology) and if we take an intersection we get back the empty set.

Exercise 18. *Show that the space (X, τ_X) has the discrete topology if and only if $\{x\}$ is open in $X \forall x \in X$ if and only if $\forall (Y, \tau_Y)$, and all functions $f : X \to Y$ are continuous.*

Solution 18.

$\Longrightarrow$: Let X be a space with discrete topology, thus every $U \subset X$ is open, and $\{x\} \subset X$, hence $\{x\}$ is open. Now take $v \in \tau_Y$ and any $f : X \to Y$, now $f^{-1}(V) \subset X \implies f^{-1}(V) \in \tau_X$, so f is continuous.

$\Longleftarrow$: Suppose any function $f : X \to Y$ is continuous. That means that if $\forall y \in Y$, $\{y\} \in \tau_Y$, $f^{-1}(y) = \{x\}$ is open, so $\{x\} \in \tau_X$. But now, any $\bigcup_{i \in I}\{x_i\} \in \tau_X$, so any subset of X is open, so X has discrete topology.

Exercise 19. *Show that the space (X, τ_X) has course topology if and only if for all spaces (Y, τ_Y) and all functions $f : Y \to X$, f is continuous.*

Solution 19.

$\Longrightarrow$: Assume (X, τ_X) has course topology, and let (Y, τ_Y) be any space and $fY \to X$ be any function. Now to show that f is continuous we need to check that X and $\emptyset$ have open preimages, however this is easy as $f^{-1}(X) = Y$ and $f^{-1}(\emptyset) = \emptyset$ which are open by definition of a topology.
$\Longleftarrow$: Suppose $(Y, \tau_Y), (X, \tau_X)$ are spaces and any $f : Y \to X$ is continuous. Suppose X has a not course topology, then we can always construct an f_i that sends a closed set in Y to an open set in X, except for X and $\emptyset$ as $f^{-1}(X) = Y$ by definition and $f^{-1}(\emptyset) = \emptyset$ which are open in Y, so X has course topology.

Exercise 20. *If (X, τ_a) is finer than (X, τ_b), which of the following functions is guaranteed to be continuous? $idx : (X, \tau_a) \to (X, \tau_b)$ or $jdx : (X, \tau_b) \to (X, \tau_a)$? Is either guaranteed to be homeomorphism?*

Solution 20.

Assume, $\tau_b \subset \tau_a$, now this guarantees that idx is continuous, but jdx isn't continuous, and thus neither is a homeomorphism.

Exercise 21. *If (X, τ_a) is finer than (X, τ_b), which space is likely to have more convergent sequences?*

Solution 21.

Courser topology is likely to have more convergent sequences as all convergent sequences in τ_a converge in τ_b, but the converse doesn't need to hold.

Exercise 22. *Suppose (X, τ_X) is a space, $A \subset X$, and (A, τ_A) has the subspace topology. Show that if $F \subset A$, then F is closed in (A, τ_A) if and only if $F = A \cap C$, and C is closed in X.*

Solution 22.

$\Longrightarrow$: Let F be closed in A, then $F^c \in \tau_A$ and $\exists U \in \tau_X$ such that $F^c = A \cap U$, so now $F = A \cap U^c$ and U^c is closed in X exactly like we wanted.
$\Longleftarrow$: Take $F = U \cap A$ for some closed $U \in X$, then $F^c = U^c \cap A$, and since $U^c \in \tau_X$ we know $F^c \in \tau_A$, and hence F is closed in A.

Exercise 23. $\mathbb{R}$ *with usual topology is connected.*

Solution 23.

Let $A \subset \mathbb{R}$ be open, nonempty and $A \neq \mathbb{R}$, now we need to show A is not closed. Let $a \in A$ and take $c \notin A$, $a < c$. Now define a set $Z = \{x | x \in \mathbb{R}, [a, x] \subset A\}$, and let $b = sup(Z)$. By definition of Z, $b \notin Z$, but $b \in \bar{Z}$, so $b \in l(Z)$. Now, $b \notin Z \implies b \notin A$ and $b \in \bar{Z} \implies b \in \bar{A}$, so $l(A) \not\subset A$, hence A is closed.

Exercise 24. *Suppose X is a space and $A \subset X$ is connected. Show that $\bar{A}$ is connected.*

Solution 24.

Take $Y \subset X = \bar{A}$, now A is dense in Y. Take nonempty $U \subset Y$ and assume U is clopen in Y, now we want to show $U = Y$. By definition U is clopen in A, further, either $U \cap A = A$ or $U \cap A = \emptyset$. To show that intersection is nonempty take $x \in U$, now if $x \notin A$, then $x \in \bar{A} \implies x \in l(A)$. Now by definition of a limit point $\forall S \in \tau$ s.t. $x \in S$, $S \cap A \neq \emptyset$, and as $U \in \tau$, we can't have $U \cap A = \emptyset$, so $U \cap A = A$. Now:

$$A \subset U \implies \bar{A} \subset \bar{U} \implies Y \subset \bar{U} \implies Y = U$$

As we wanted to show.

Exercise 25. *Suppose X is a space and let $a \in X$. Now assume that A_i is connected and $a \in A_i$ $\forall i \in I$. Show that if $\bigcup_{i \in I} A_i = X$, X is connected.*

Solution 25.

Suppose $U \subset X$ is clopen, and $U \neq \emptyset$, now by definition, U^c is clopen. Assume $a \in U$, now we want to prove $X \subset U$. To do this, let's get $x \in A_i$, note that $A_i \cap U$ is clopen and $a \in A_i \cap U$, hence $A_i \cap U = A_i$, so $x \in U$. Hence, $X = U$, $\forall$ clopen and nonempty $U \in X$, so X is connected.

Exercise 26. *If X is a space and $a \in X$, then there is a unique component $A \subset X$, such that $a \in A$.*

Solution 26.

Take $a \in X$ and let A_i be the union of all connected subsets B such that $a \in B$. Now A_i is connected. Furthermore A_i is maximal by it's definition. Now if A and B are components and $A \cap B \neq \emptyset$, then $A \cup B$ is connected, hence $A = B$. Thus the component containing any $x \in X$ is unique.

Exercise 27. *Suppose (X, τ_X) is connected and $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous surjection. Show that (Y, τ_Y) is connected.*

Solution 27.

Take a clopen, nonempty $U \subset Y$ and show that $U = Y$. Assume opposite for contradiction, now $f^{-1}(U)$ and $(f^{-1}(U))^c$ are both open, but the only nonempty set satisfying that is X, so now $f^{-1}(U) = X$, and as f is surjective that means that $U = Y$. So Y is connected.

Exercise 28. *Suppose X is a space and $A \subset X$ is a component of X. Why is A closed?*

Solution 28.

By definition A is a maximal connected set and is therefore clopen, and any clopen set is closed.

Exercise 29. *Suppose X is a space and $A \subset X$ is closed, $B \subset X$ is closed and $f : A \to Y$ and $g : B \to Y$ are continuous such that $f|A \cap B = g|A \cap B$. Then $h = f \cup g : A \cup B \to Y$ is continuous.*

Solution 29.

Firstly, let $K \subset Y$ be closed, then $h^{-1}(K) = (h^{-1}(K) \cap A) \cup (h^{-1}(K) \cap B) = h^{-1}(K) \cap (A \cup B)$. So now $h^{-1}(K)$ is the union of two closed subsets of X. Thus by subspace topology, h^{-1} is closed in $A \cup B$.

Exercise 30. *Suppose (Y, τ_Y) is a space and $X \subset Y$. Show that the following are equivalent. The subspace (X, τ_X) is compact. If $\{V_i\}$ is a collection of open sets in τ_Y covering X, there exists finitely many sets $\{V_1, V_2, \ldots, V_n\} \subset \{V_i\}$ such that $V \subset \bigcup_{j=1}^{n} V_j$.*

Solution 30.

$\implies$: If X is compact then every open cover yields a finite subcover. Now take $\{V_i\}$ to be the open cover and by definition it will yield a finite subcover, i.e $\{V_1, V_2, \ldots, V_n\}$. Converse is entirely similar.

Exercise 31. $[0, 1]$ *is compact using the subspace topology of* $\mathbb{R}$.

Solution 31.

Since every open U in $\mathbb{R}$ is a union of open intervals, it suffices to prove the special case when $[0, 1]$ is covered by open intervals. Suppose $[0, 1] \subset \bigcup_{i \in I}(a_i, b_i)$.
Now let $K = \{x \in [0, 1] | [0, x]$ can be covered by finitely many $(a_i, b_i)\}$. $0 \in K \implies K \neq \emptyset$. If $x \in K$ and $0 < y < x$, then $y \in K$, and since $K \subset [0, 1]$, we can express K as either $[0, b]$ or $[0, b)$. Now let b be $sup(K)$, so $b \in K$, now if we take $b < 1$, we can always find another $a \in K$ such that $a > k$, contradicting maximality of b, hence $b = 1$, $K = [0, 1]$, $[0, 1]$ is compact.

Exercise 32. *Suppose (X, τ_X) is a compact space, and $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous surjection. Show that (Y, τ_Y) is a compact space.*

Solution 32.

Take a family of open covers $O \in \tau_Y$ covering Y and prove that it yields a finite subcover. Firstly note that $f^{-1}(O) \in \tau_X$ and thus yields a finite subcover, thus the image of that finite subcover is open in Y and covers Y. so Y is compact.

Exercise 33. *Supoose (X, τ_X) is a compact space and $A \subset X$ is closed. Prove that A, τ_A is compact.*

Solution 33.

Cover A by open sets from X and throw in the open set $V = X \setminus A$, that gives us an open cover of X, now take the finite subcover guaranteed by compactness of X, and throw out V, notice we have covered A.

Exercise 34. *Let $X = \{1, 2, 3, \dots\}$ with a funny topology. Let's call it the closed finite topology, a topology where every finite set is closed. Which of the T_1, T_2 axioms are true? Do convergent sequences have unique limits? Is X compact? Are compact subsets closed?*

Solution 34.

Let's check for T_1 first, as only finite sets are closed, this is pretty easy to confirm as for two distinct points $x, y \in X$, we can take $U_x = X \setminus \{y\}$ and $U_y = X \setminus \{x\}$, both will be open as they're infinite and they satisfy the T_1 axiom.
Now for T_2, we firstly take sets $E, O \in \tau_X$ such that E contains all even numbers and O contains all the even ones, it is obvious that $E \cap O = \emptyset$, now for any two distinct points x, y, take $U_x = E \setminus \{y\} \cup \{x\}$ if $y \in E$ or $U_x = E \cup \{x\}$ otherwise, now for U_y take $U_y = O \setminus \{x\} \cup \{y\}$ if $x \in O$ or $U_y = O \cup \{y\}$ otherwise. Obviously U_x and U_y are open and disjoint, hence satisfying the properties of T_2 space. Now by definition convergent sequences have unique limits.
Now to check if X is compact. Let $K = \{x \in X | \{1, \dots, x\}$ can be covered by a finite amount of open sets$\}$, now show that $K = X$. As long as $\{1, 2, \dots, x\}$ is finite, it is compact, so now K is unbounded, so $K = X$.

Exercise 35. *Show that $W \subset X \times Y$ is open iff W is a union of sets of the form $U_i \times V_i$ with U_i open in X and V_i open in Y.*

Solution 35.

$\implies$: Let W be open in $X \times Y$, now by definition of topology W is either a cartesian product $U \times V$ where $U \in X$, $V \in Y$ or the union of such sets.

Exercise 36. *Given functions $f : W \to X$ and $g : W \to Y$ define $(f, g) : W \to X \times Y$ as $(f, g)(w) = (f(w), g(w))$. Show that (f, g) is continuous if and only if f and g are continuous.*

Solution 36.

$\implies$: Assume (f, g) is continuous, that means that if U is open in $X \times Y$, then $f(U)$ is open in X and $g(U)$ is open in Y as by the definition of product topology, so in order for $(f, g)^{-1}(U)$ to be open in W, then $f^{-1}(U)$ and $g^{-1}(U)$ have to be open in W, thus f and g are continuous.
$\impliedby$: Assume f and g are continuous and let U, V be open in X, Y respectively. Now $Z = U \times V$ is open in $X \times Y$, and $f^{-1}(U), g^{-1}(V)$ are open in W, and by definition of (f, g), $(f, g)^{-1}(Z)$ is open in Z so (f, g) is continuous.

Exercise 37. *Note that sets of the form $V_{i1} \cap V_{i2} \cap \cdots \cap V_{in}$ must be open. Show that any open set $V \subset \prod_{i \in I} X_i$ is a union of sets of the previous format.*

Solution 37.

Not important for this exercise but what does $V_{i_1} \cap V_{i_2} ... \cap V_{i_n}$ look like? Ignoring the order that we write down the indices, it should be of the form $U = U_{i_1} \times U_{i_2} .... \times U_{i_n} \times \Pi_{i \neq i_k} X_i$.

Formally $U = \Pi_{i \in I} U_i$ so that $U_i \subset X_i$ open in X_i, and with finitely many exceptions $U_i = X_i$. Such a U is a **basic open set** (every open set in ΠX_i is a union of sets of the format $U = \Pi_{i \in I} U_i$).

By the proof that follows every open set in ΠX_i is a union of sets of the format $U_{i_1} \times U_{i_2} .... \times U_{i_n} \times \Pi_{i \neq i_k} X_i$.

(From subbasis to basis to topology.)

Notice a much more general claim. Suppose X is any set and $S \subset 2^X$ is a collection of sets so that $\emptyset \in S$ and $X \in S$ and so that if $\{V_1, V_2, .., V_n\} \subset S$ then $\cap_{i=1}^n V_i \in S$. What topology does S generate? Let τ_S be the collection of all sets $U \subset X$ so that U is a union of sets in S. Keep in my we are trying to understand the smallest (coursest) topology τ_X so that $S \subset \tau_X$. Notice $\tau_S \subset \tau_X$.

$\emptyset \in \tau_S$

$X \in \tau_S$

τ_S is closed under unions (if for each $i \in I$ the set U_i is a union of sets in S. Then $\cup_{i \in I} U_i$ is a union of sets in S why? if $x \in \cup_{i \in I} U_i$ then $x \in U_i$ for some i and hence $x \in A \subset U_i$ for some $A \in S$).

Is τ_S closed under finite intersections

It is adequate to show that if $U \in \tau_S$ and $V \in \tau_S$ then $U \cap V \in \tau_S$.

$U = (\cup_{i \in I} A_i)$ with $A_i \in S$. And $V = (\cup_{j \in J} B_j)$ with $B_j \in S$.

$U \cap V = (\cup_{i \in I} A_i) \cap (\cup_{j \in J} B_j) = \cup_{I \times J} (A_i \cap B_j)$. The last set is τ_S since each $A_i \cap B_j \in S$. (This proof is from the notes obviously i just wanted to have it handy as i liked it)

Exercise 38. *Take a function $f : W \to \prod_{i\in I} X_i$ and show it is continuous iff each function $\pi_i : W \to X_i$ is continuous.*

Solution 38.

$\implies$: Suppose f is continuous and construct $id_j : \prod_{i\in I} X_i \to X_j$, by definition each id_j is continuous. Now note that $f \circ id_j = \pi_j$ for each j, and as composition of two continuous functions is continuous, each π_j is continuous.
$\impliedby$: Suppose each pi_i is continuous. Let $S = \prod_{i\in I} U_i$ be open in $\prod_{i\in I} X_i$, then $f^{-1}(S) = \bigcap_{i\in I} \pi_i^{-1}(U_i)$, a countable intersection of open sets, hence countable.

Exercise 39. *Show what are convergent sequences in the countable product $X_1 \times X_2 \times \dots$.*

Solution 39.

My best guess is that convergent sequences in countable product $\prod_{i\in I} X_i$ are sequences x_n such that $\pi_i(x_n)$ converges in X_i.
$\implies$:

Exercise 40. *If X_1 is not compact, prove that $X_1 \times X_2 \dots$ is not compact.*

Solution 40.

Suppose that the product is compact for contradiction, then for every open cover O in the product, we can find a finite subcover O_λ, as O_λ is open, it is a union of open sets, and thus its components are open in each X_i, so O_1 but can't be finite in X_1, thus O doesn't yield an open cover, so the product is not compact.

Exercise 41. *Show that if each X_n is sequentially compact, then the countable product $\prod_{i=1}^{n} X_i$ is sequentially compact.*

Solution 41.

We consider a sequence of sequences, where each is a subsequence of the last. Define $x_0 = a_1, a_2, \dots$ to be our original sequence, and construct $x_1 = a_1 1, a_1 2$ to be a sequence where our first component converges(guaranteed by sequential compactness).
Now similarly build x_n to be a subsequence of x_{n-1} such that the n-th component converges. Now consider:

More specifically consider the sequence on the diagonal of the "matrix", each $a_m m$ converges in the X_m and thus the whole sequence converges in the product and it is the subsequence of x_0, hence the product is sequentially compact.

Exercise 42. *if $X \times Y$ are spaces using the product topology, define $f : X \times Y \to X$ as $f(x, y) = x$. Which among the following are guaranteed? The function f is continuous? The function f is open and closed?*

Solution 42.

None of these is guaranteed as open set $U \subset X$ can have multiple elements in it's preimage, some of which will be neither open nor closed if $V \subset Y$ is closed.

Exercise 43. *If $X \times Y$ is compact, is X compact?*

Solution 43.

By previous exercise, we know that if one component is not compact, the whole product is not compact. So in order for the product to be compact we need X to be compact.

Exercise 44. *Prove that every metric space is Hausdorff(T_2).*

Solution 44.

Let X be a metric space with metric d. Get two distinct points $x, y \in X$, To find U containing x but not y just let U be the open ball with radius $\frac{d(x,y)}{2}$ centered at x, and do the same for y.

Exercise 45. *If X is T_2 and Y is T_2, prove that $X \times Y$ is T_2.*

Solution 45.

Take points (x, y) and (a, b) in $X \times Y$, now if we take a set $U \subset X$ such that $x \in U$ but $a \notin U$, existance of U is given to us by X being T_2 and we find a similar $V \subset Y$ for y and b, $U \times V$ will be open in product space and $(x, y) \in U \times V$ but $(a, b) \notin U \times V$. now we can find the same set containing (a, b), thus $X \times Y$ is T_2. Is X compact, connected, locally connected.

Exercise 46. *Since* $[0, 1]$ *is compact and metrizable, and since the finite product* $[0, 1] \times \cdots \times [0, 1]$ *is compact and* T_2*, what can you say about the relationship between the closed and compact subsets of* $[0, 1] \times \cdots \times [0, 1]$*?*

Solution 46.

Claim: Every closed subset of $[0, 1] \times \cdots \times [0, 1]$ is compact. To prove this just note that if we take a closed subset V, we get a free open set V^c now take any open cover of V and add V^c in order to cover the whole product, now by compactness you get a free finite subcover of the product, and you have covered V.
Claim: Every compact subset of $[0, 1] \times \cdots \times [0, 1]$ is closed. To prove this we will assume that a compact subset U is open for contradiction. Then the set is of the form $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$, but by previous exercise, if any of the multiples is not compact, the product is not compact, so we just need to prove that $(a, b) \subset [0, 1]$ is not compact. To do that take the open cover made by open intervals $(0, \frac{1}{n})$, $\frac{1}{n} \leq b$, this doesn't yield a finite subcover Thus, every compact subset is closed.

Exercise 47. *According to a legend, the compact subsets of the Eucledian metric space* $\mathbb{R}^n$ *are preciesly the closed and bounded subsets of* $\mathbb{R}^n$*. Prove this.*

Solution 47.

Let X be closed and bounded in $\mathbb{R}^n$, then $X = \prod_{i=1}^{n} [a_i, b_i]$, now every $[a_i, b_i]$ is compact and we know that product of two compact sets is compact, so by simple induction the whole countable product is compact so closed bounded sets are compact in $\mathbb{R}^n$.
For converse assume $X \subset \mathbb{R}$ is compact but for contradiction suppose it is open, then it is of the form $(a_1, b_1) \times (a_2, b_2) \times \cdots$, and as we previously established, we need only one component of the product to fail in order for product to fail to be compact, and open intervals aren't compact in $\mathbb{R}$ which finishes the proof.

Exercise 48. *Suppose* X *is compact and* $f : X \to Y$ *is continuous and* Y *is* T_2*. Prove* f *is a closed map.*

Solution 48.

Take a closed $C \subset X$, we want to prove $f(C)$ is closed. First note that C is compact as it is a subset of a compact space. Hence $f(C)$ is a compact subset of a T_2 space. Now to prove that $K = f(C)$ is closed take $y \in Y \setminus K$, since Y is T_2 there are disjoint open sets U_y and V_z such that $y \in U_y$ and $z \in V_z$ for every $z \in Y$. Now $\bigcup_{i \in I} V_{zi}$ is an open cover of K and yields a finite subcover $\{V_z | z \in L\}$ where L is finite. Now $\bigcap_{z \in F} U_y$ is an open neighborhood of y disjoint from K. Since y is arbitrary K is closed.

Exercise 49. *If* X *is metrizable and* A *is not closed. Then if* $x \in \bar{A} \setminus A$ *there exists a sequence* a_n *converging to* x *and for all* n *we have* $a_n \in A$

Solution 49.

Get a metric d such that every open set in X is a union of open metric balls. Get the wanted $x \in \bar{A} \setminus A$. x is a limit point of A. For each $n \in \mathbb{N}$ obtain $a_n \in A$ such that $a_n \in B(x, \frac{1}{2^n})$. Now x is in said open ball because d is a metric and x is in every open ball around x. Note that x is a limit point of A but $x \notin A$ and the open ball is open and contains x therefore the mentioned a_n exists. Now to show that a_n converges to x. Suppose U is and open set containing x. We must find N such that if $n \geq N$ then $a_n \in U$. Since $x \in U$ and U is open, U is a union of open balls of d. Thus, there exists $y \in X$ and $\epsilon \geq 0$ such that x in open ϵ-ball around around y, call it B_y, note that $B_y \subset U$. Thus $d(x, y) < \epsilon$. Now take $\delta = \epsilon - d(x, y)$. Get N such that if $n > N$ then $d(x, a_n) < \delta$. Now when $n \geq N$ we get $d(a_n, y) \leq d(a_n, x) + d(x, y) < \delta + (\epsilon - \delta) = \epsilon$. Thus a_n converges to x.

Exercise 50. *Suppose* X *is any space and* $B \subset X$ *is closed. Suppose* $b_n \in B$ *for each* n *and suppose the sequence* $b_n \to x$*. Then* $x \in B$.

Solution 50.

Suppose $x \not{i}nB$. Then x is not a limit point of B. Let U be open and contain x. Since $b_n \to x$, get N such that $n > N \implies b_n \in U$. Since $b_n \in B$ and $x \notin B$, this means that x is a limit point of B yielding us a contradiction.

Exercise 51. *If X is path connected then X is connected*

Solution 51.

Fix $a \in X$, now get a path α_b for each $b \in X$. Now notice that image of every α_b is connected and thus the union of all images is connected, that union is the space X and thus X is connected.

Exercise 52. *If X is nonempty, connected and locally path connected then X is path connected.*

Solution 52.

Fix $a \in X$. Defne $U_a = \{x \in X | \exists a path \alpha : [0;1] \to X$ so that $\alpha(0) = a and \alpha(1) = x\}$. Hope to show: $U_a = X$. Obviouslt U_a is nonempty as constant path α_a exists so $a \in U_a$. Now to show that U_a is open: fix $x \in U_a$ and choose a subset U containing x. Now all $u \in U$ are path connected to x and thus path connected to a so $U \subset U_a$ and U_a is open. To show that U_a is closed look at the closure of U_a. Look at $y \in \bar{U_a}$, and choose an open path connected set V containing y. Now as $V \cap U_a \neq \emptyset$ hence take z in the intersection which is path connected to y and and thus y is path connected to a, so now $U_a = \bar{U_a}$, thus $\bar{U_a}$ is closed. Now as U_a is clopen and nonempty so it is actually our space X and thus our space is path connected.

Exercise 53. *Consider the following subset $X \subset \mathbb{R}^2$. Let $X = [(0,0),(0,1)] \cup [(0,0),(1,0)] \cup [\bigcup_{n=2}^{\infty}(\frac{1}{n},0),(\frac{1}{n},1)]$. Is X compact? Connected? Locally connected?*

Solution 53.

Can't think about this right now it seems hard, will do it later

Exercise 54. *Prove that $T_4 \implies T_3 \implies T_2$.*

Solution 54.

Suppose X is T_4, then for each disjoint pair of closed sets A and B we can find disjoint pair of open sets $A \subset U$ and $B \subset V$, so now in order for our set to be T_3 we will take $a \in A$ and keep the set U and now a and B suffice the definition of T_3. To further this into a T_2 argument just take $b \in B$ while keeping the open set V and we're done.

Exercise 55. *Suppose X is compact and T_2. Show that X is T_3.*

Solution 55.

Let X be a compact T_2 space, it is obviously compact so in order to show it is T_3 we just need to show that it is regular. Take $x \in X$ and let B be a closed set in X not containing x, B is compact. Since X is T_2 we can find U_x and U_b for every $b \in B$ such that $U_x \cap U_b = \emptyset$. Now consider the cover of B: $\mathcal{B} = \{U_b | b \in B\}$, by compactness of B we can reduce this to a finite subcover $\mathcal{B}' = \{U_{b1}, U_{b2}, \ldots, U_{bn}\}$. Now let $V = \bigcup_{i=1}^{n} U_{bi}$, it is clearly open and contains b but not x. Now let $U = \bigcap_{i=1}^{n} U_{xi}$, U is open and contains x but not b and thus $U \cap V = \emptyset$. So now we have satisfied regularity and X is T_3.

Exercise 56. *Suppose X is compact and T_2. Show that X is T_3.*

Solution 56.

Let X be a compact T_2 space, then for T_4 we just need to show normality of X as it is already T_1. Take disjoint closed sets A and B in X. We have proven that X is T_3 so, for every $a \in A$ exists U_a and V_a satisfying the properties of T_3. Now consider the following open cover of A: $\mathcal{A} = \{U_a | a \in A\}$, clearly this open cover yields a finite subcover $\mathcal{A}' = \{U_{a1}, \ldots, U_{an}\}$. Now let U be the union of sets in $\mathcal{A}'$, $A \subset U$. Now let V be the finite intersection of coresponding V_{xi}. V is open in X and $B \subset U$, moreover by the construction $U \cap V = \emptyset$ satisfying the properties of T_4.

Exercise 57. *Suppose X is metrizable, show that X is T_3*

Solution 57.

Suppose $a \in X$ and take closed B not containing a. Get a d metric on X. Now there exists $\epsilon > 0$ such that $B_\epsilon(a) \cap B = \emptyset$, then let $U = B_{\frac{\epsilon}{2}}(a)$ and let $V = \bigcup_{b \in B} B_{\frac{\epsilon}{2}}(b)$. Clearly $U \cap V = \emptyset$ sufficing conditions of T_3.

Exercise 58. *Suppose X is metrizable, show that X is T_4*

Solution 58. *Let A and C be disjoint closed subsets in X. Now for each $x \in A$ find ϵ_x such that open metric ball $B(x, \epsilon_x)$ is disjoint with C, similarly find ϵ_y for every $y \in C$. Now let $U = \bigcup_{i \in I} B(y_i, \epsilon_{yi}/3)$ and $V = \bigcup_{i \in I} B(x_i, \epsilon_{xi}/3)$. Clearly, V and U are open in X, and $A \subset V$, $C \subset U$.*
Now suppose $z \in U \cap V$, then $d(x,z) < \frac{\epsilon_x}{3}$ and $d(y,z) < \frac{\epsilon_y}{3}$. That would imply that $d(x,y) \leq \frac{\epsilon_y}{3} + \frac{\epsilon_x}{3} < \epsilon_x$, assuming $\epsilon_x > \epsilon_y$. But then $y \in B(x, \epsilon_x)$, yielding a contradiction.

Exercise 59. *Suppose X is compact, show that X is locally compact.*

Solution 59.

X is open and is an open neighborhood of every point in the space, and as $\bar{X} = X$ it's closure is compact, so X is locally compact.

Exercise 60. *Suppose each of X and Y are locally compact. Prove that $X \times Y$ is locally compact.*

Solution 60.

Let (x, y) be a point in $X \times Y$, now as X is locally compact, there exists an open U containing X such that $\bar{U}$ is compact, similaryly we find an open V with compact closure in Y. Now $U \times V$ is open in the product topology and$\bar{U} \times \bar{V}$ is compact in the product space as it is a product of compact sets, there fore we found our open set with compact closure and $X \times Y$ is locally compact.

Exercise 61. *Show that countable product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is not locally compact.*

Solution 61.

Consider a point $x^n = (x_1, x_2, \dots, x_n)$ in the countable product. Every open set U containing x^n is of the form $(a_1, b_1) \times (a_2, b_2) \times \cdots \times \mathbb{R} \times \mathbb{R}$ so it's closure won't be compact as $\mathbb{R}$ is not compact, thus sufficing to show the countable product is not compact.

Exercise 62. *Suppose X is linearly ordered space with at least two elements $a < b$. Suppose m is minimal in X and $m < x \in X$. Must the half open interval $[m, x)$ be open? Suppose M is maximal in X. If $x < M$ must the half open interval $(x, M]$ be open in X? Suppose X has only one point. What are the open rays? What are the open sets in X?*

Solution 62.

Half open interval $[m, x)$ is actually just left open ray L_x and is therefore open in X. Similarly $(x, M]$ is R_x. If X has only one point x, then $L_x = R_x = x$ and only open sets are X and $\emptyset$.

Exercise 63. *Does every linearly ordered space $(X, <)$ satisfy the T_2 axiom?*

Solution 63.

Suppose $a, b \in X$ and $a \neq b$, then $\{a, b\} \subset X$. Assume $a < b$, then we have two cases:
1) If there exists c such that $a < c < b$, then we can just take L_c and R_c and thus satisfy the T_2 axiom
2) If there is no point between a and b, then just take R_a and L_b and satisfy the T_2 axiom.

Exercise 64. *Suppose $(X, <)$ is a well ordered space with $x_1 \geq x_2 \geq x_3 \dots$. Must $\lim_{n \to \infty}$ exist? Is it possible that all terms of x_n are distinct?*

Solution 64.

The limit must exist as we have a minimal element and our sequence has to become constant at some point, thus not all entries can be distinct.

Exercise 65. *Suppose $(X, <)$ is an infinite well ordered space with maximal element M and suppose x_n is a sequence in X. Show that x_n has a constant subsequence $x_{n1} \leq x_{n2} \leq \cdots$. Must x_{mj} converge?*

Solution 65. *If there exists a constant subsequence it immediately solves both of our problems. So let's assume such a subsequence doesn't exist. For starters take only distinct terms of x_n. Then let $x_{n1} = m_1 = min\{x_1, x_2, \dots\}$. Let $m_2 = x_{n2}$ be the minimum of the terms with index bigger than n_1 and so on. This creates a subsequence $m_1 < m_2 < \dots$*
Now we have $m_n < M$ for all n. Let $B = \{x \in X | m_n < x \forall n\}$. Let $b = min(B)$. Now clearly $m_n < b$, so let U be an open set containing b. If $b \in L_x$ for some x then we have $m_n \in L_x$. If $b \in R_x$ then $x \notin B$ and hence there exists some n for which $x < m_n$. Thus $x < m_{n+k}$, this shows that $m_n \to b$.

Exercise 66. *Suppose X is a space and Y is a set and $f : X \to Y$ is a surjection. Declare $U \subset Y$ open iff $f^{-1}(U)$ is open. Show this is really a topology on Y.*

Solution 66.

First ti show that the set Y is open in Y, it is obvious it is as $f^{-1}(Y) = X$. Similarly $\emptyset$ is open in Y as it is its own preimage. So know we have to show that if $U_1, U_2, \ldots$ are open in Y so is their union. For this just take in consideration that $f^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} f^{-1}(U_i)$ and as every individual preimage is open so will be the union of preimages and thus our union is open. Now we want to show that $U \cap V$ is open assuming both U and V are open. To prove this just consider that $f^{-1}(V \cap U) = f^{-1}(U) \cap f^{-1}(V)$ and as both preimages are open so is their intersection and thus $U \cap V$ is open finishing the proof.

Exercise 67. *Every closed map is a quotient map*

Solution 67.

Take spaces X and Y and a closed map $f : X \to Y$. Now if we take a closed set K in Y, we know that $f^{-1}(K)$ is closed and thus $(f^{-1}(K))^c$ is open and mapped to K^c and thus open sets are mapped to open sets and our map is a quotient map.

Exercise 68. *Every open map is a quotient map*

Solution 68.

Take spaces X and Y and open map f, then any open set U in Y has open preimage V in X, similarly any open set V is mapped to an open U finishing the proof.

Exercise 69. *Every homeomorphism is a quotient map.*

Solution 69.

Take to spaces X and Y and homeomorphism h. Now h being continuous guarantees that preimage of every open U in Y is open in X. Now as h is homeomorphic that gives us continuous h^{-1} which guarantees us that each open V in X is mapped to an open set in Y, hence open sets in Y are exactly the ones whose preimage is open.

Exercise 70. *Being homotopic is an equivalence relation on the set of maps $C(X, Y)$.*

Solution 70.

1) Is f f? Define $H : X \times [0, 1] \to Y$ as $H(x, t) = f(x)$
2) Suppose $f \sim g$, is g f? Let $G_t = H_{1-t}$ if H connects f to g
3) Suppose $f \sim g$ and $g \sim h$. Given the homotopies H_{fg} and H_{gh} and define $H_{fh}(t)$ as $H_{fg}(2t)$ for $t \in (0, \frac{1}{2})$ and $H_{gh}(2t - 1)$ for $t \in (\frac{1}{2}, 1)$.

Exercise 71. *Homotopy equivalence is an equivalence relation.*

Solution 71.

1) Is $X \sim X$? Let $f = g = id_X$. Define $H : X \times [0, 1] \to X$ as $H(x, t) = x$
2) If $X \sim Y$, is $Y \sim X$? True by definition of homotopy inverses
3) If $X \sim Y$ and $Y \sim Z$ is $X \sim Z$? Start with $f : X \to Y$, $g : Y \to X$ and $h : Y \to Z$, $k : Z \to Y$. Now we get $hf : X \to Z$ and $gk : Z \to X$

Exercise 72. *Show that X is contractible if and only if X is homotopy equivalent to a one point space.*

Solution 72.

$\Longrightarrow$: Assume X is contractible, that means that id_x is homotopic to a constant map to a one point subspace of X, will finish tomorrow I'm tired...