Assignment 1 part 2

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For $A, B \in \mathbb{C}^{d\times d}$, define

$$\langle A|B\rangle_{\mathrm{Tr}} = \mathrm{Tr}(A^\dagger B)\,.$$

Prove that the above map defines an inner product on the vector space $\mathbb{C}^{d\times d}$. (In the literature, this inner product is called the trace inner product or Hilbert-Schmidt inner product.)

We will use proof by definition. Assuming the above map to be true, it must have the following attributes:

1. It must have linearity in the first argument
2. Conjugate-commutativity
3. Non-negativity

We can see that $\langle ., .\rangle$ is linear in the first argument since for every $a, b \in \mathbb{C}$ and $A, B, C \in \mathbb{C}^{d\times d}$

$$\begin{aligned}\langle aA + bB, C\rangle &= \mathrm{Tr}((aA + bB)C^*)\\ &= \mathrm{Tr}(aAC^* + bBC^*)\\ &= a\mathrm{Tr}(AC^*) + b\mathrm{Tr}(BC^*)\\ &= a\langle A, C\rangle + b\langle B, C\rangle\end{aligned} \tag{1}$$

We can see $\langle ., .\rangle$ has conjugate-commutativity by

$$\langle A, B\rangle = \mathrm{Tr}(AB^*) = \mathrm{Tr}((BA^*)^*) = \overline{\mathrm{Tr}(AB^*)} = \overline{\langle B, A\rangle} \tag{2}$$

Lastly, we can prove non-negativity by using the definition of matrix mulitplication:

$$(A^TA)_{ij} = \sum_k (A^T)_{ik}A_{kj} = \sum_k A_{ki}A_{kj} \tag{3}$$

And:

$$\mathrm{Tr}(A^TA) = \sum_k (A^TA)_{ii} = \sum\sum_k (A_{ki})^2 \tag{4}$$