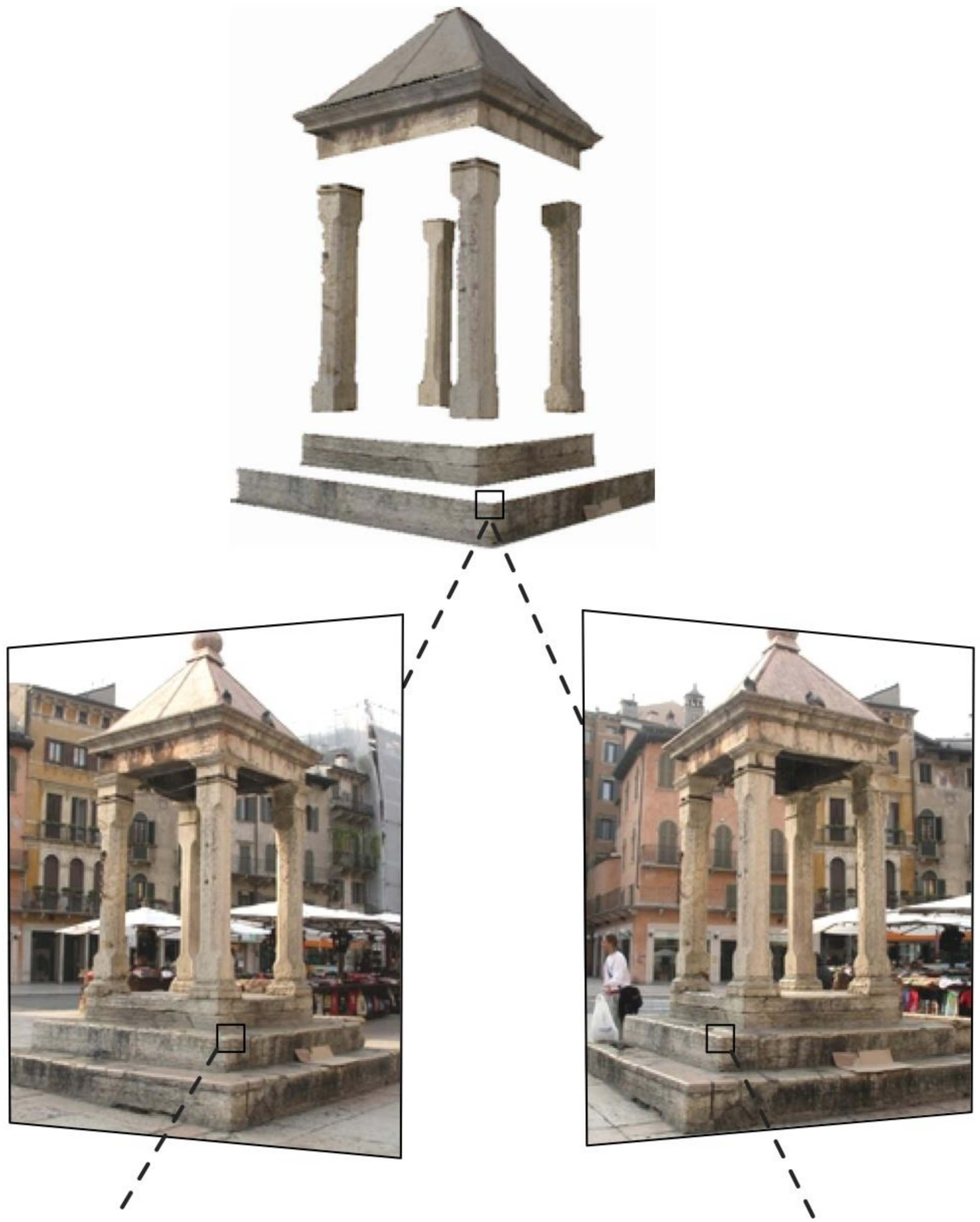
# Two-view geometry



16-385 Computer Vision
Fall 2019, Lecture 10

Slide Credits: Kris Kitani, Ioannis Gkioulekas

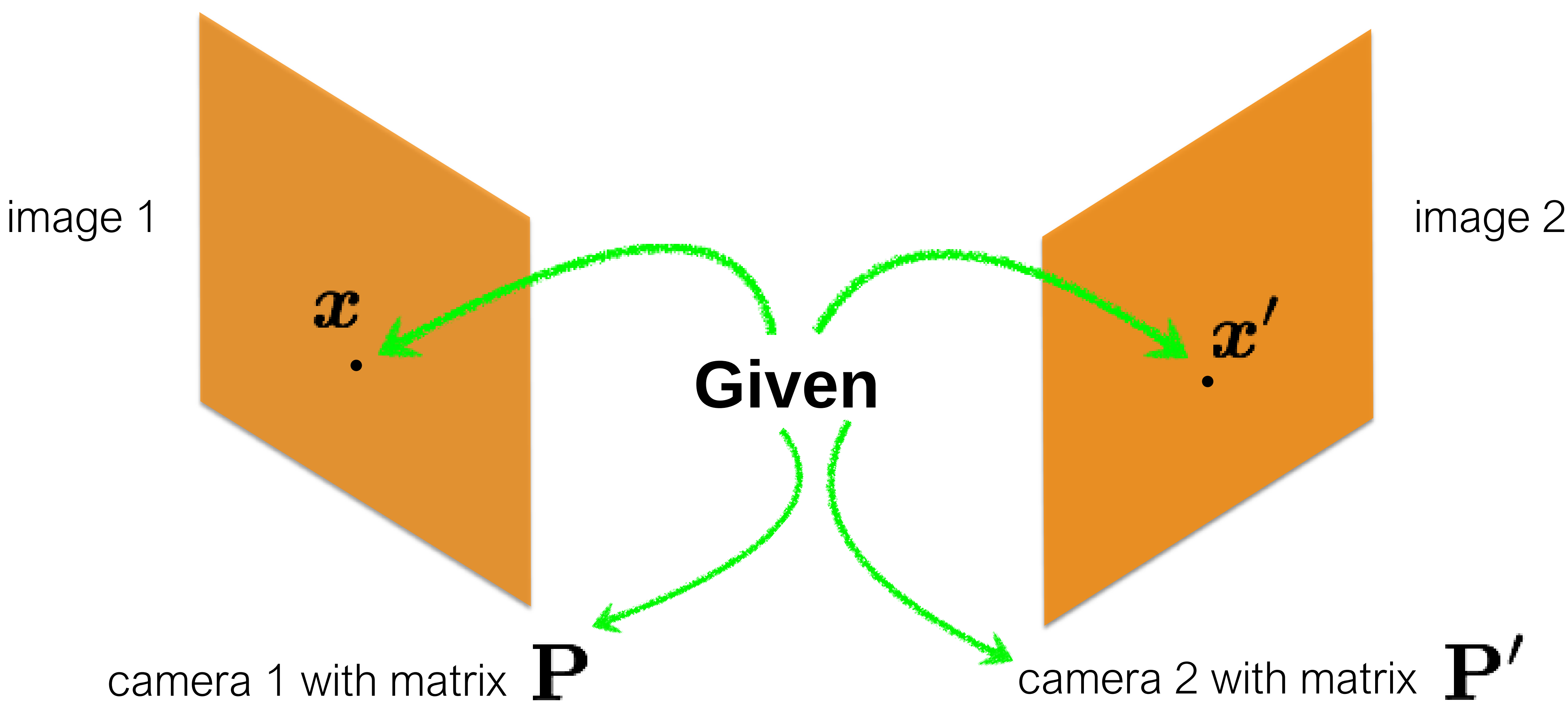
# Overview of today’s lecture

- Leftover from previous lecture: Other types of cameras, calibration.
- Triangulation.
- Epipolar geometry.
- Essential matrix.
- Fundamental matrix.
- 8-point algorithm.

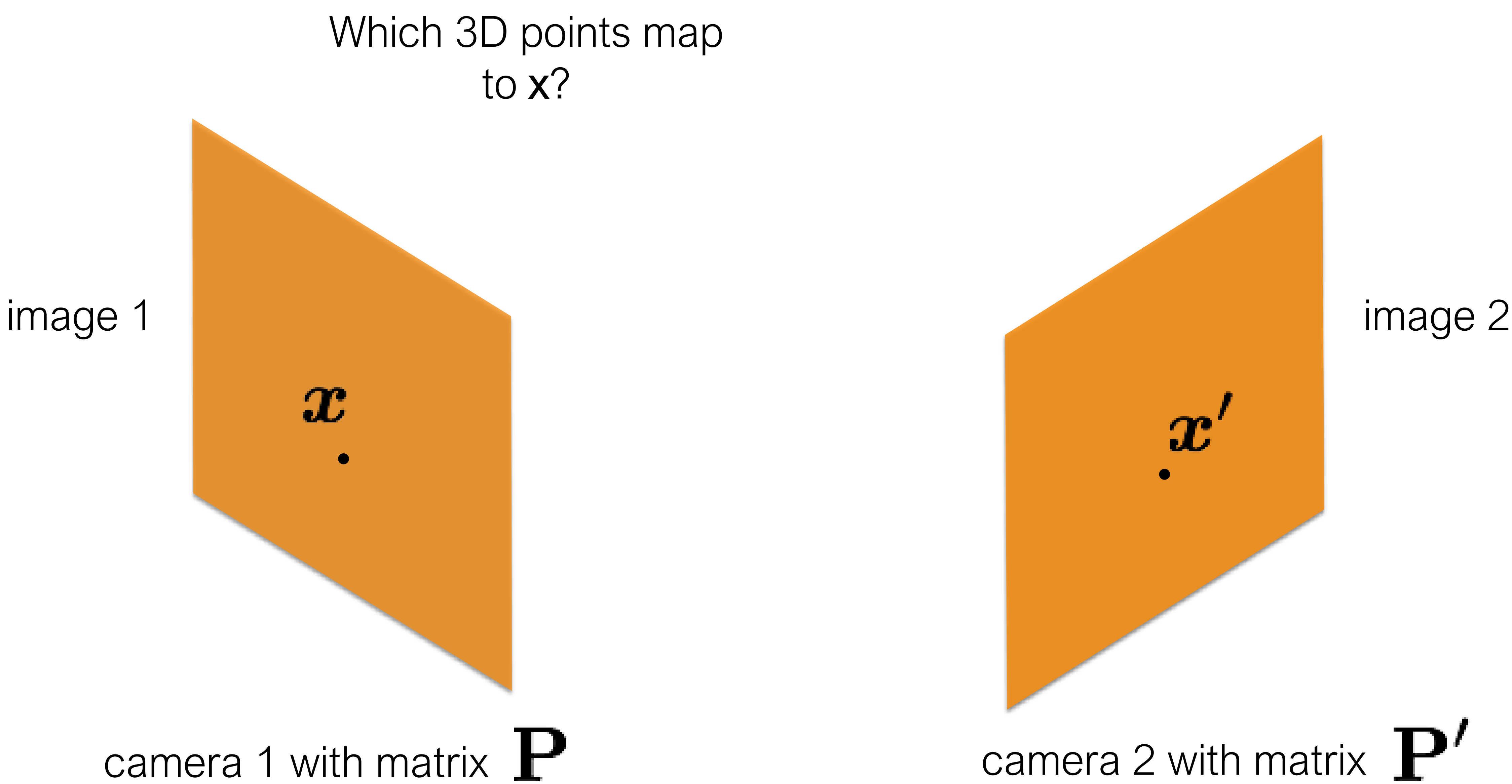
# Triangulation

| | Structure (scene geometry) | Motion (camera geometry) | Measurements |
|---|---|---|---|
| Pose Estimation | known | estimate | 3D to 2D correspondences |
| Triangulation | estimate | known | 2D to 2D coorespondences |
| Reconstruction | estimate | estimate | 2D to 2D coorespondences |

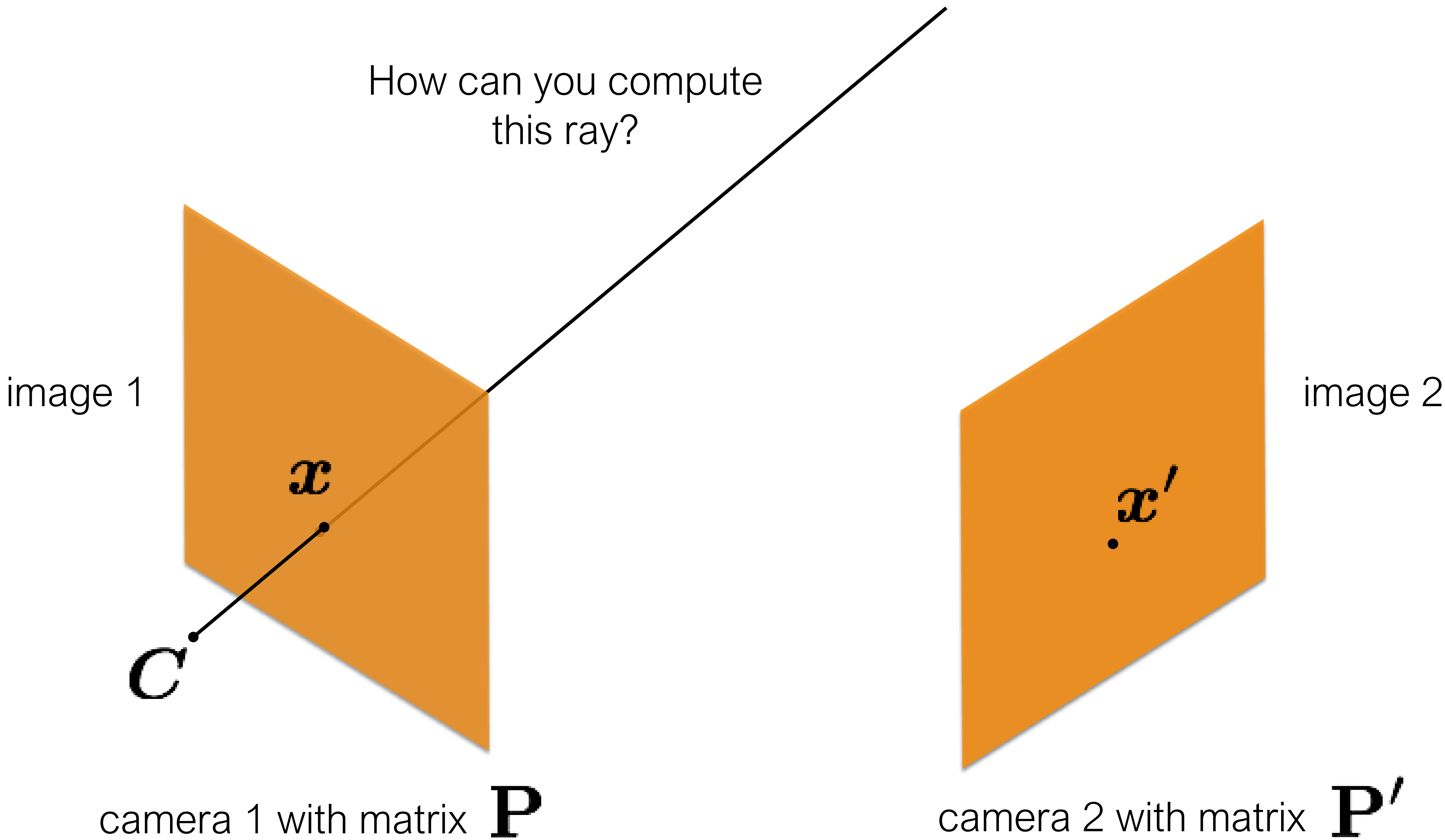
# Triangulation



# Triangulation

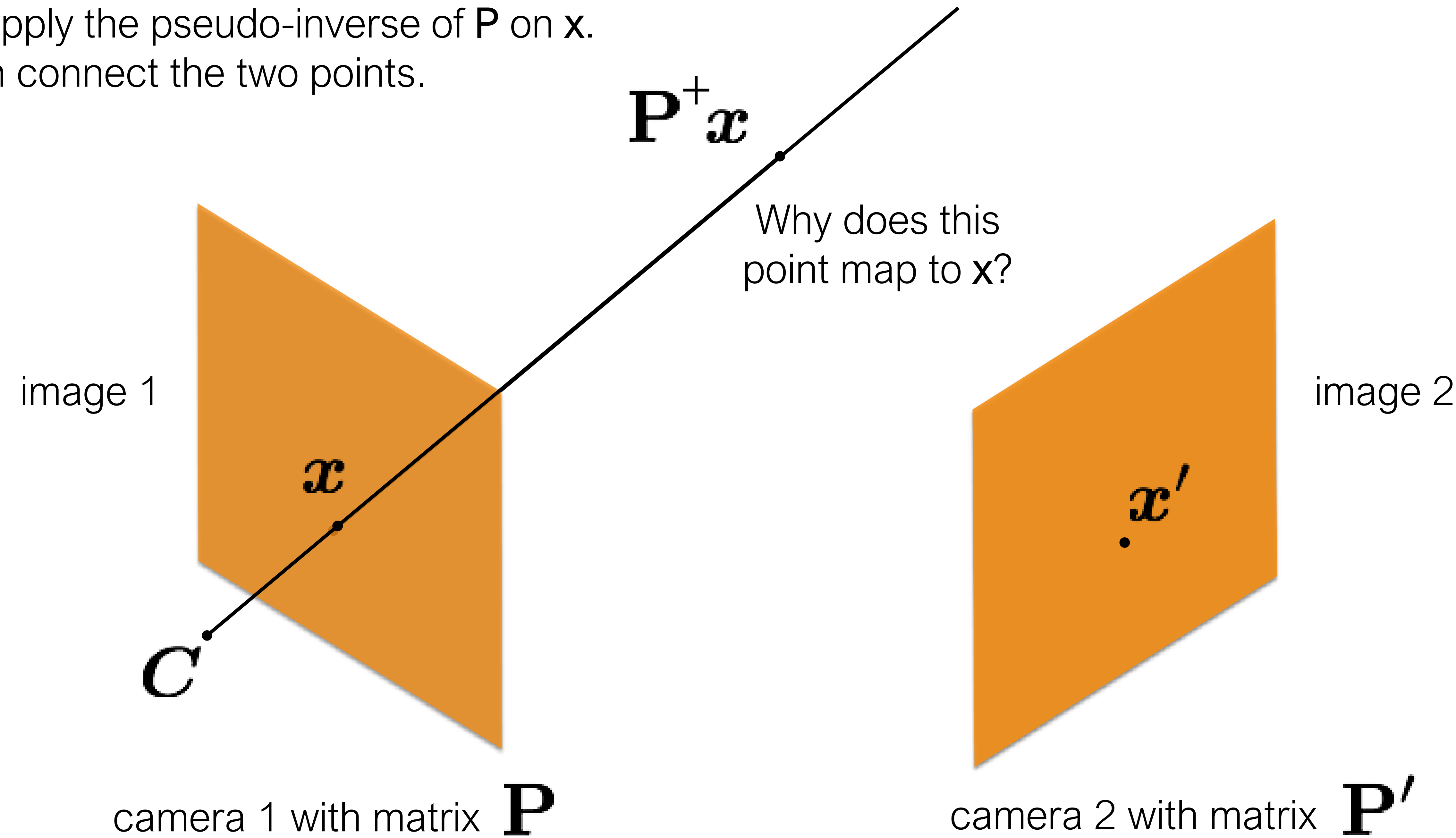


# Triangulation

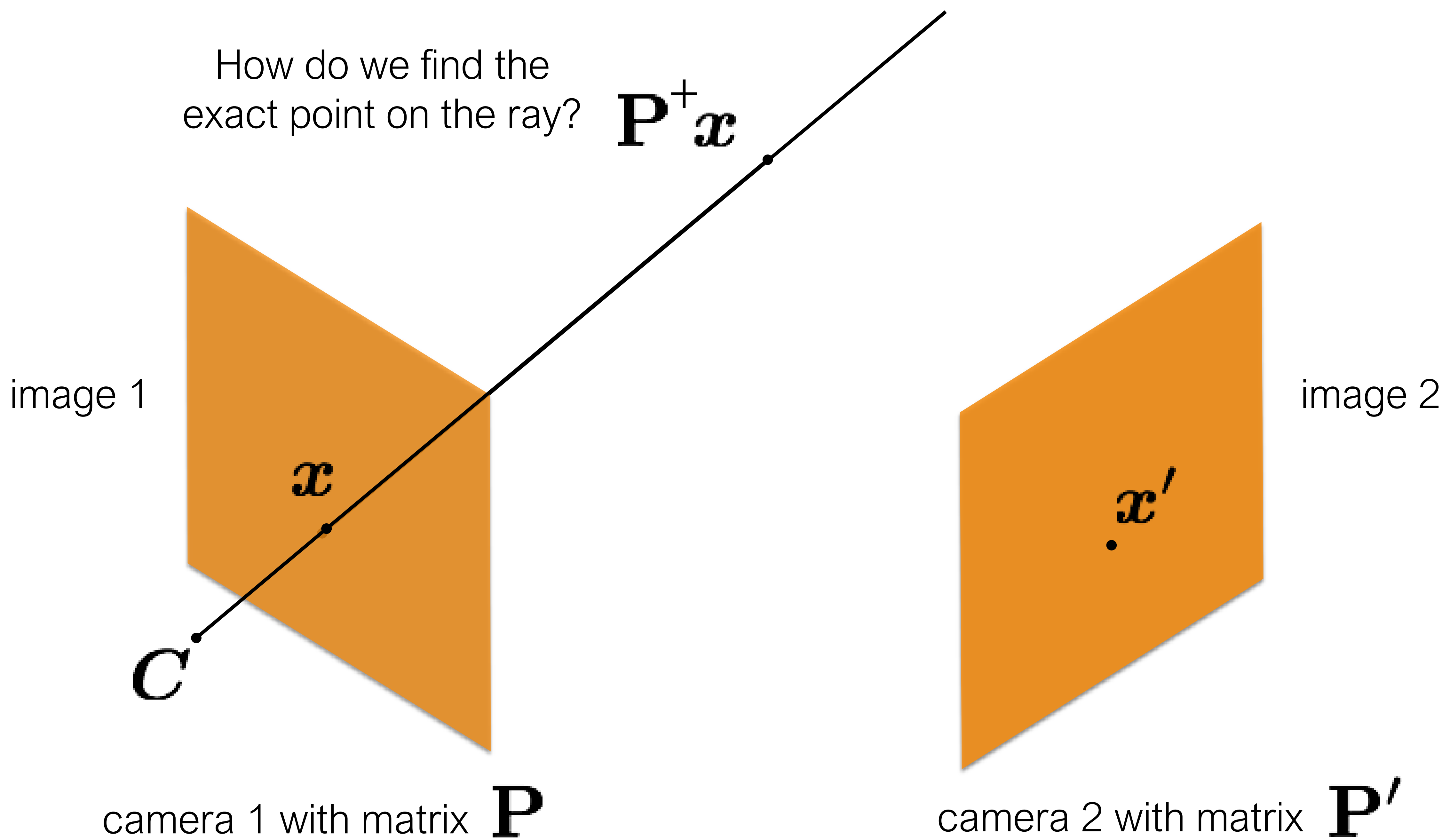


# Triangulation

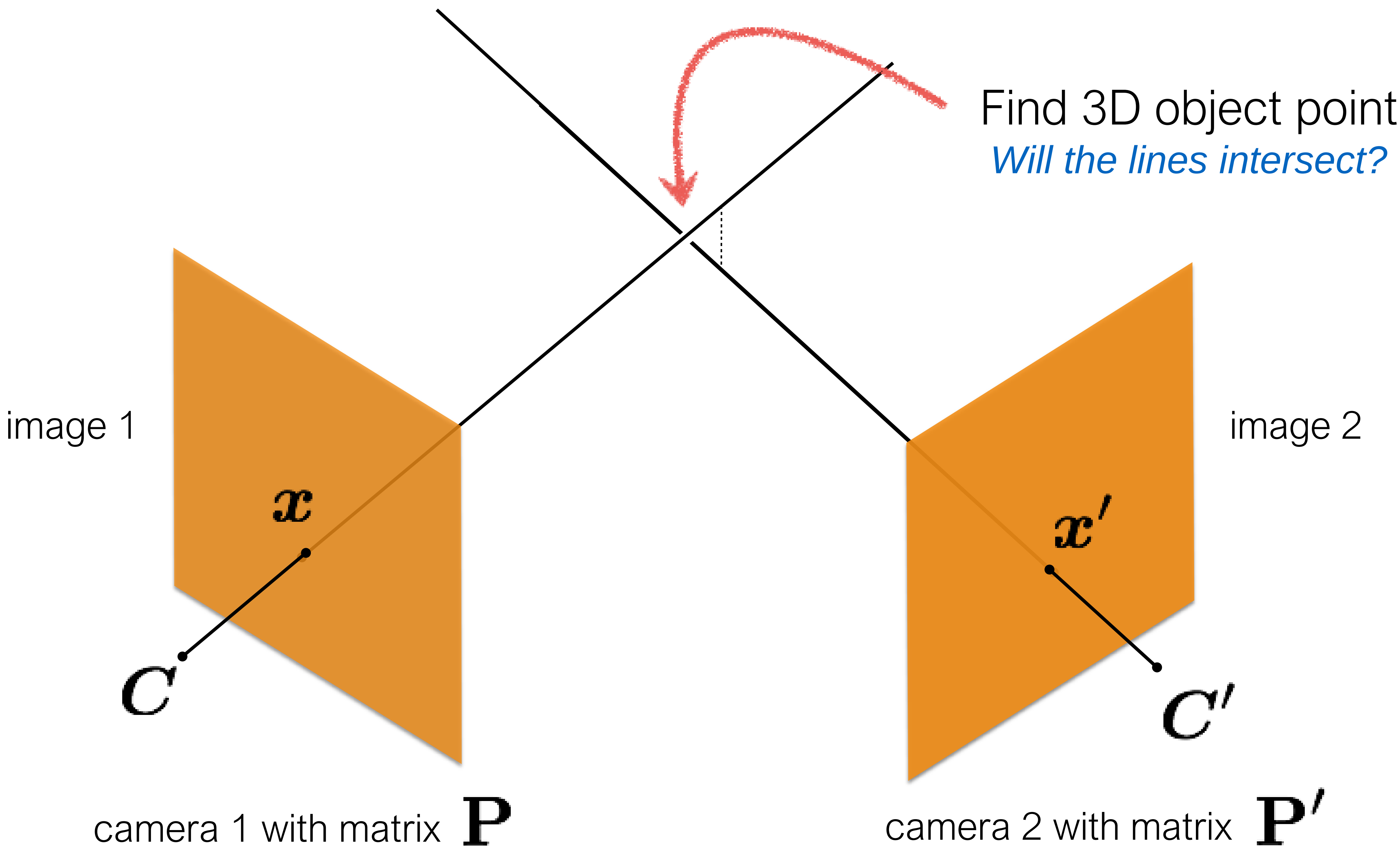
Create two points on the ray:
1) find the camera center; and
2) apply the pseudo-inverse of P on x.
Then connect the two points.



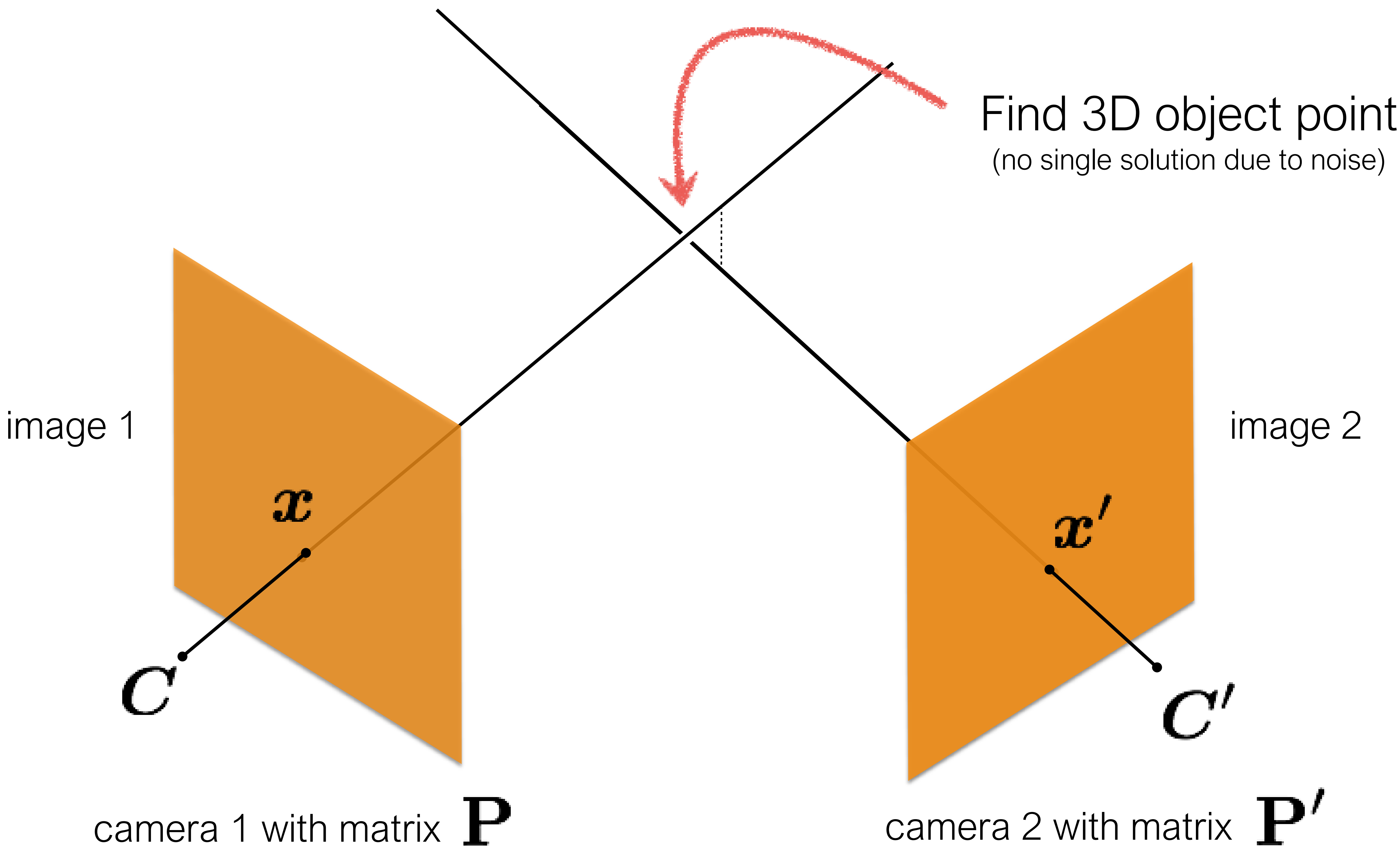
# Triangulation



# Triangulation



# Triangulation



# Triangulation

Given a set of (noisy) matched points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

and camera matrices

$$\mathbf{P}, \mathbf{P}'$$

Estimate the 3D point

$$\mathbf{X}$$

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(heterogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

*How do we solve for unknowns in a similarity relation?*

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

*How do we solve for unknowns in a similarity relation?*

Remove scale factor, convert to linear system and solve with

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

*How do we solve for unknowns in a similarity relation?*

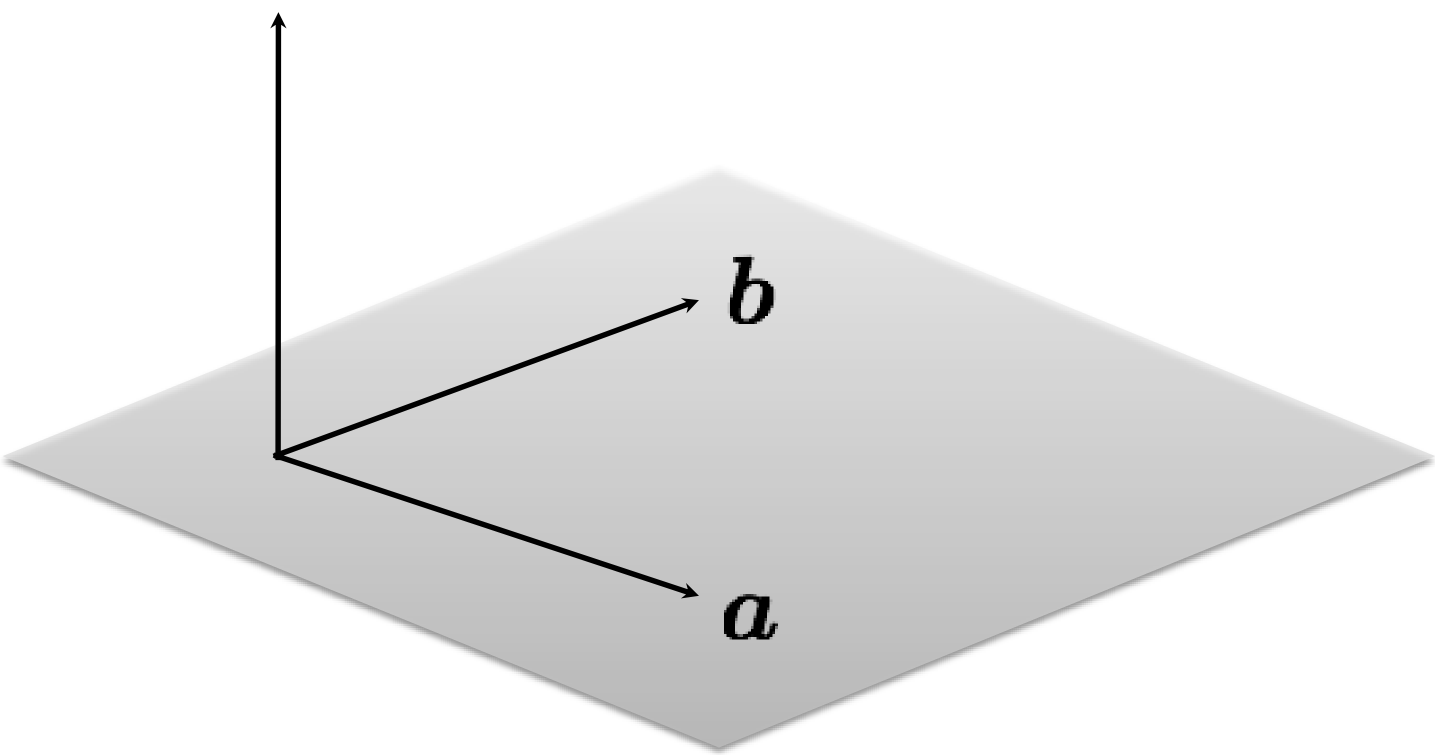
Remove scale factor, convert to linear system and solve with SVD!

# Recall: Cross Product

**Vector (cross) product**

takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$




$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

cross product of two vectors in the same direction is zero

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

remember this!!!

$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

$$\mathbf{x} = \alpha \mathbf{P} \boldsymbol{X}$$

Same direction but differs by a scale factor

$$\mathbf{x} \times \mathbf{P} \boldsymbol{X} = \mathbf{0}$$

Cross product of two vectors of same direction is zero
(this equality removes the scale factor)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{——} & \boldsymbol{p}_1^\top & \text{——} \\ \text{——} & \boldsymbol{p}_2^\top & \text{——} \\ \text{——} & \boldsymbol{p}_3^\top & \text{——} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{——} & \boldsymbol{p}_1^\top & \text{——} \\ \text{——} & \boldsymbol{p}_2^\top & \text{——} \\ \text{——} & \boldsymbol{p}_3^\top & \text{——} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you [redacted] equations

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_i \boldsymbol{X} = \mathbf{0}$$

Now we can make a system of linear equations
(two lines for each 2D point correspondence)

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

*sanity check! dimensions?*

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

*How do we solve homogeneous linear system?*

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

*How do we solve homogeneous linear system?*

S V D !

# Recall: Total least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{TLS}} = \sum_i (\boldsymbol{a}_i \boldsymbol{x})^2$$

$$= \|\mathbf{A}\boldsymbol{x}\|^2$$ (matrix form)

$$\|\boldsymbol{x}\|^2 = 1$$ constraint

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

→ minimize $\dfrac{\|\mathbf{A}\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2}$

(Rayleigh quotient)

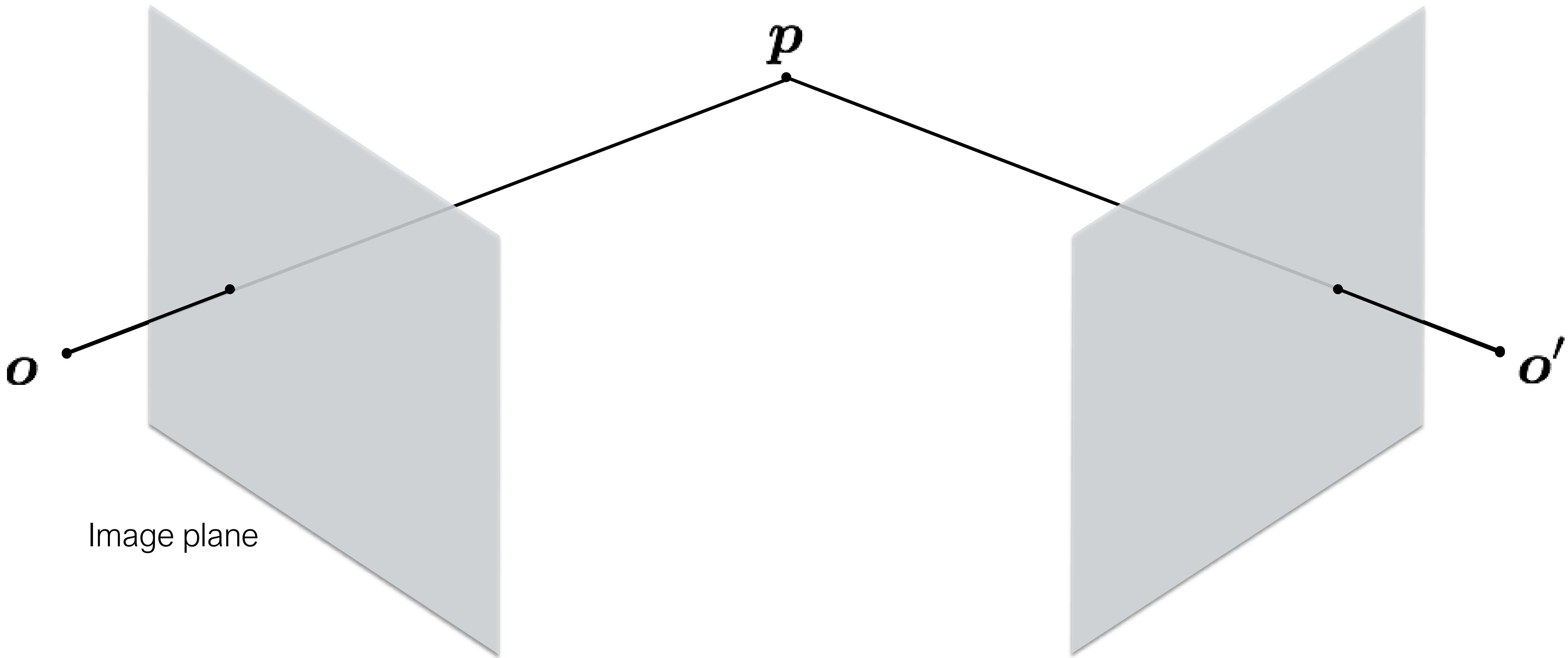
Solution is the eigenvector corresponding to smallest eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

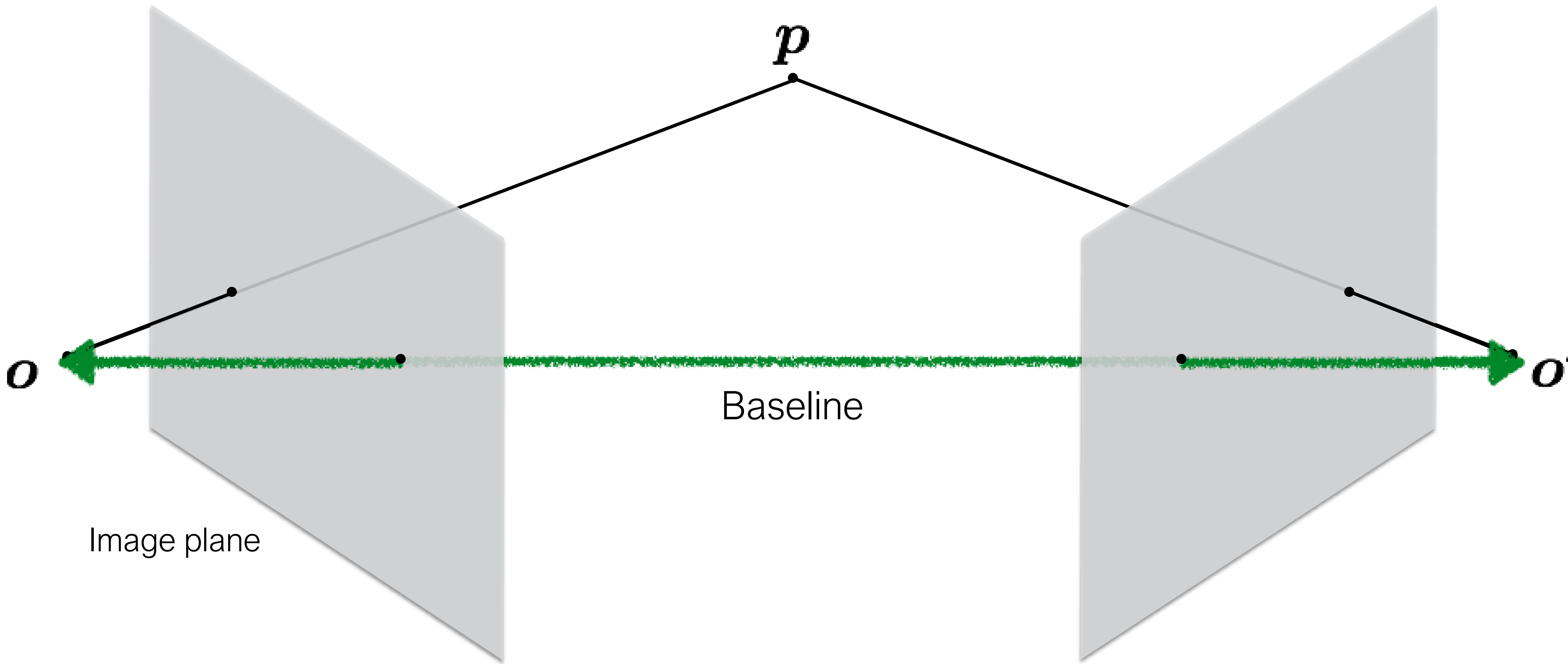
| | Structure (scene geometry) | Motion (camera geometry) | Measurements |
|---|---|---|---|
| Pose Estimation | known | estimate | 3D to 2D correspondences |
| Triangulation | estimate | known | 2D to 2D coorespondences |
| Reconstruction | estimate | estimate | 2D to 2D coorespondences |

# Epipolar geometry

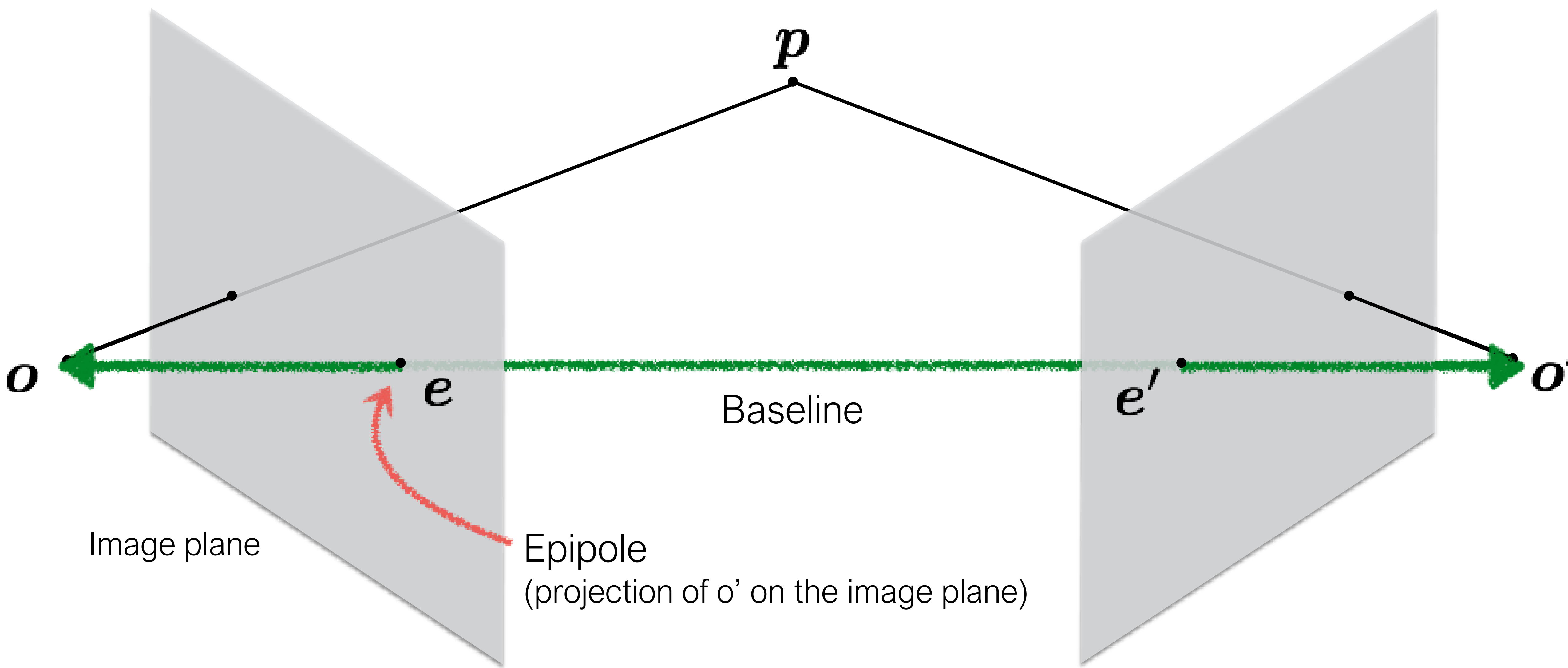
# Epipolar geometry



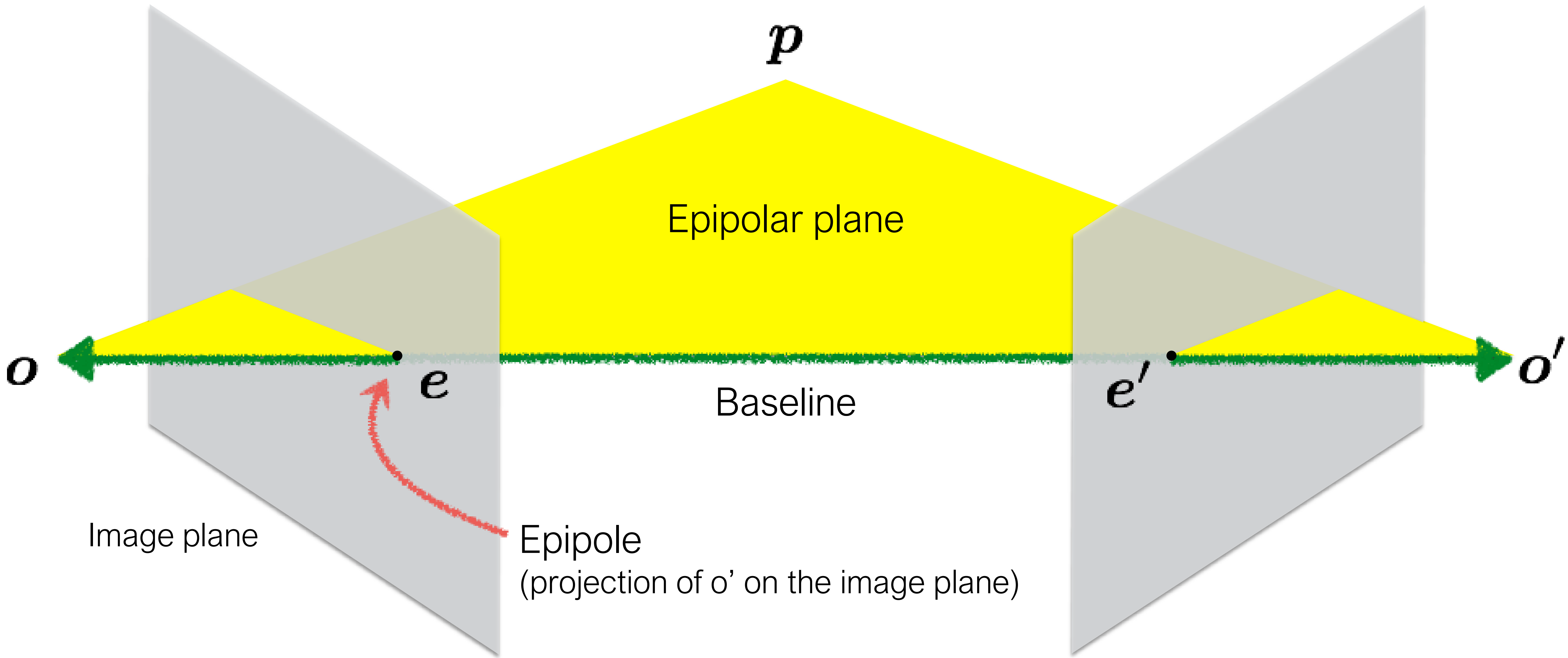
# Epipolar geometry



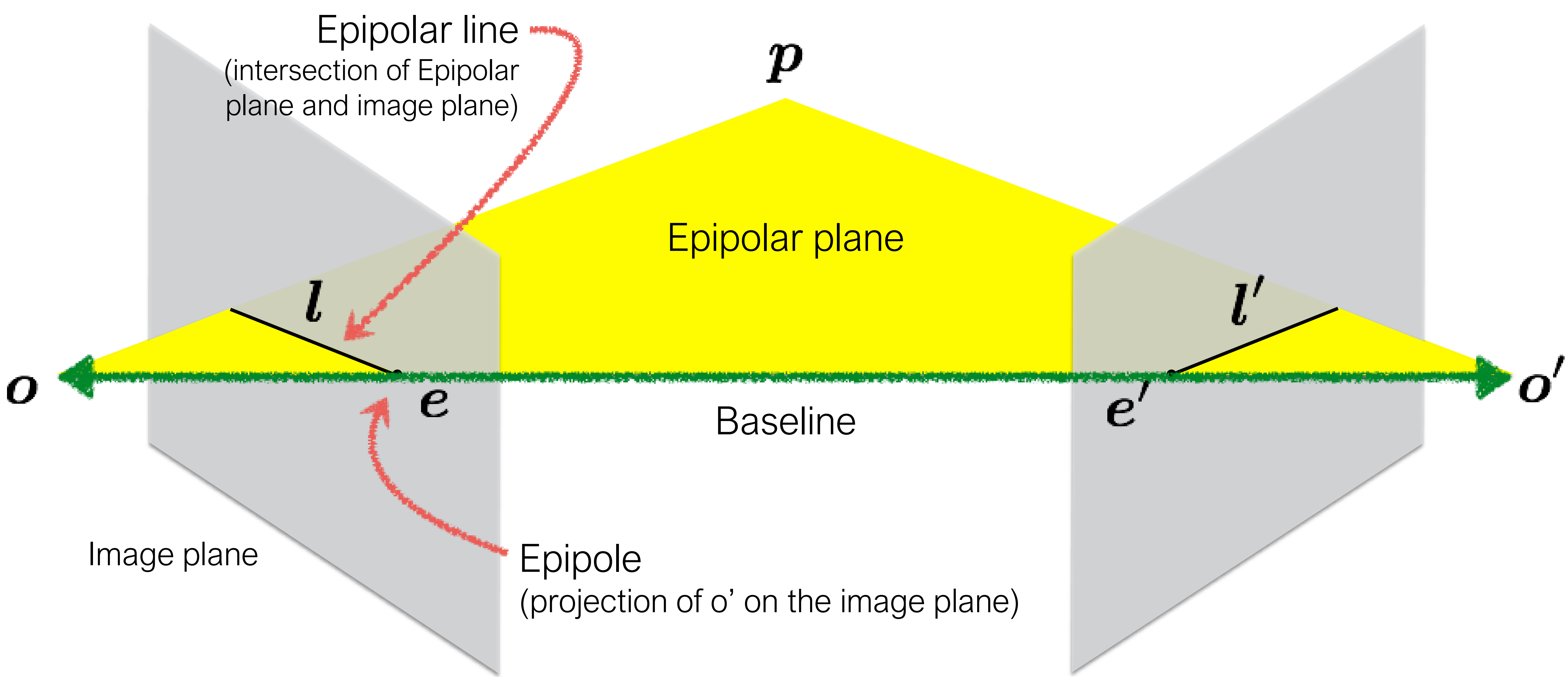
# Epipolar geometry



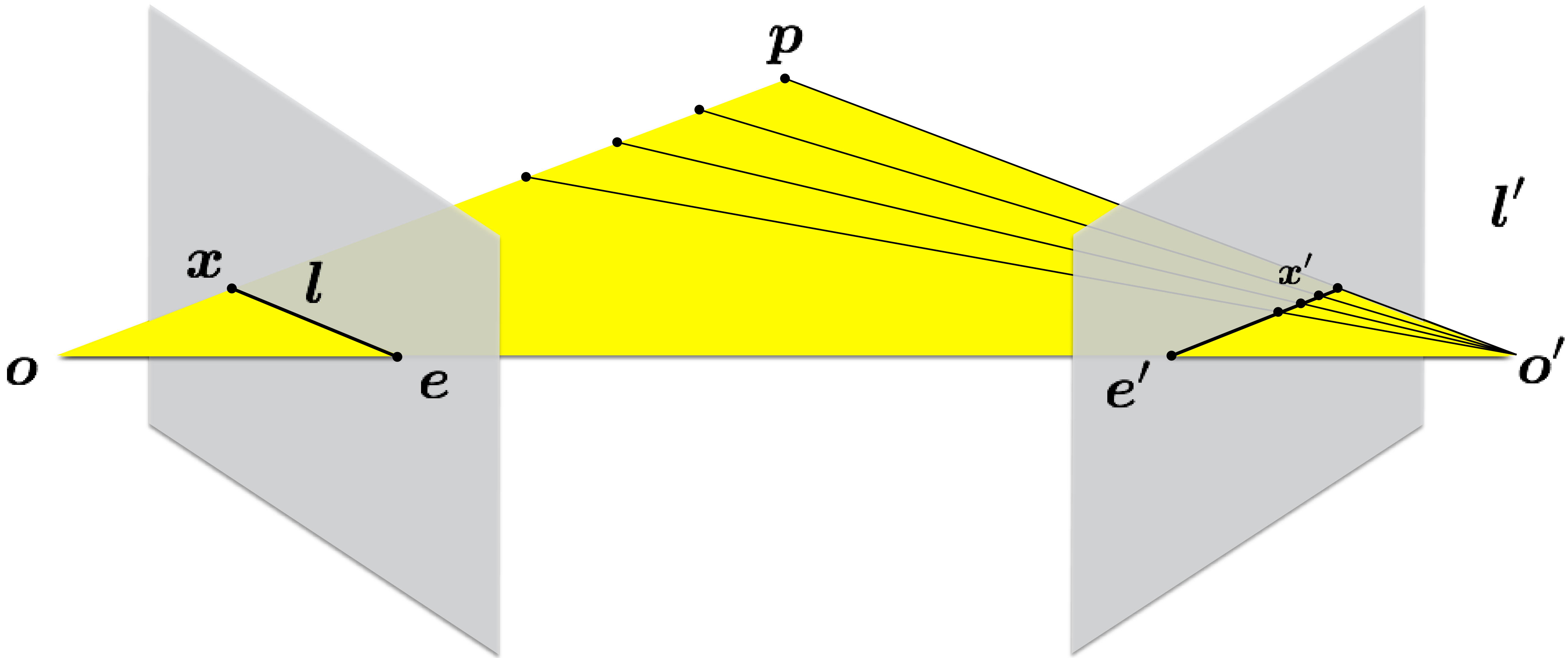
# Epipolar geometry



# Epipolar geometry

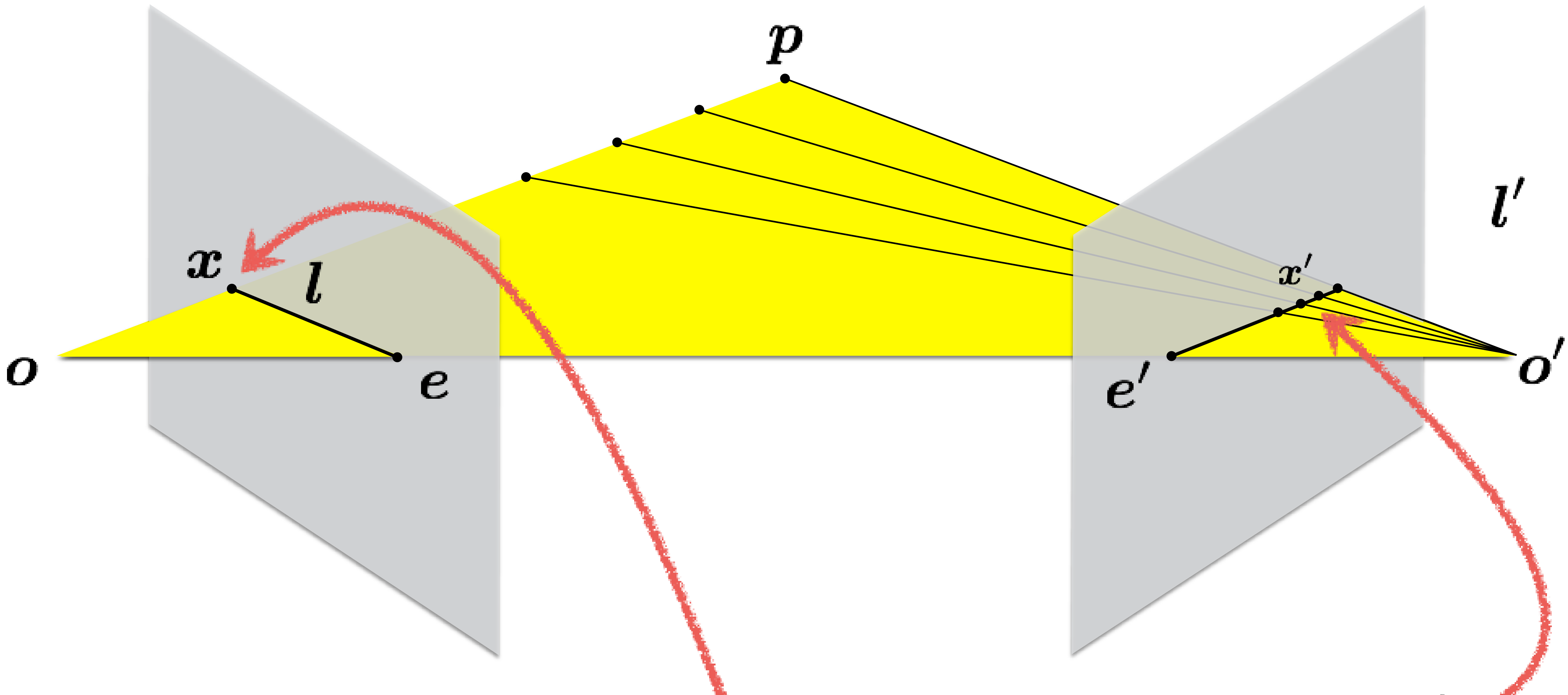


# Epipolar constraint



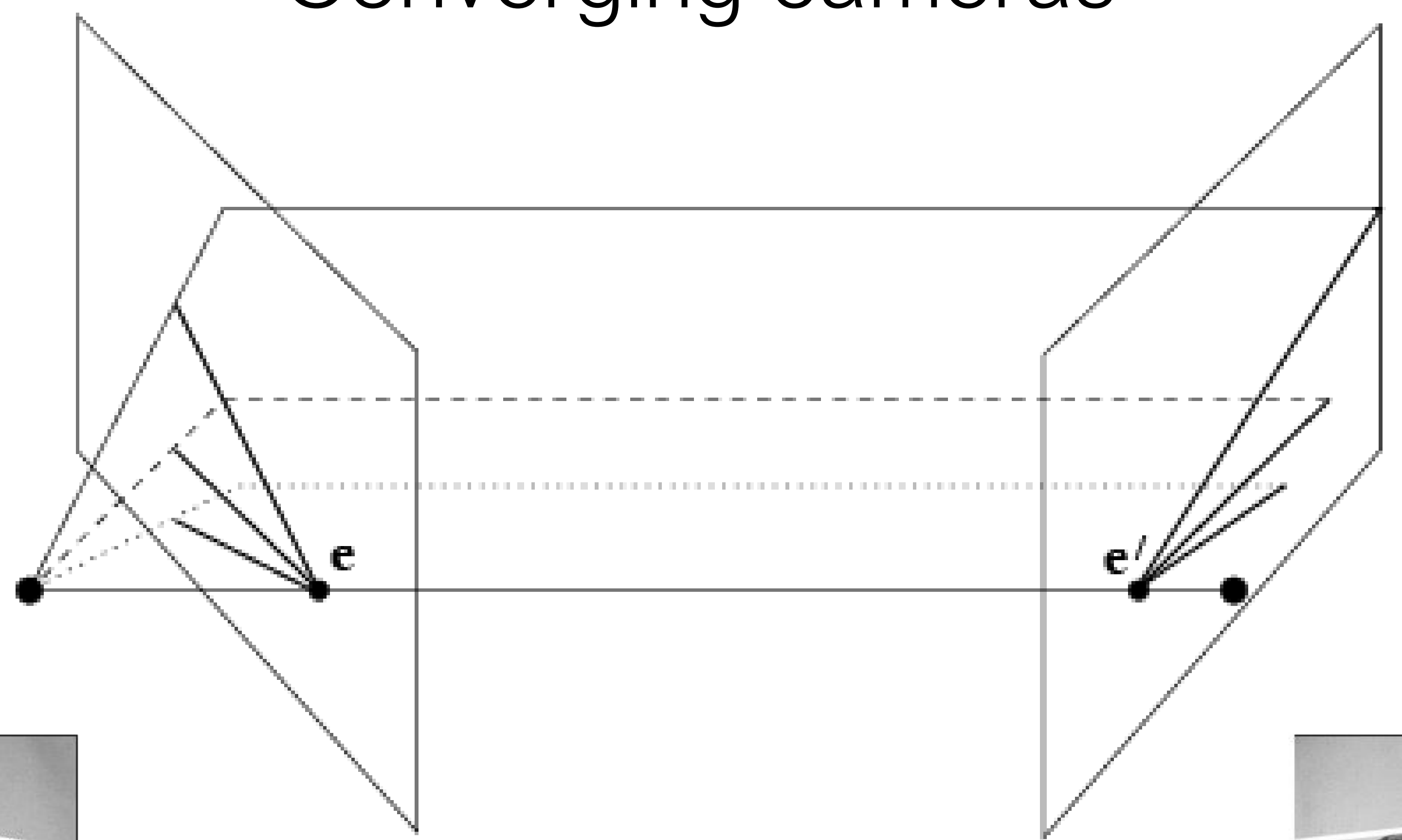

Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

# Epipolar constraint



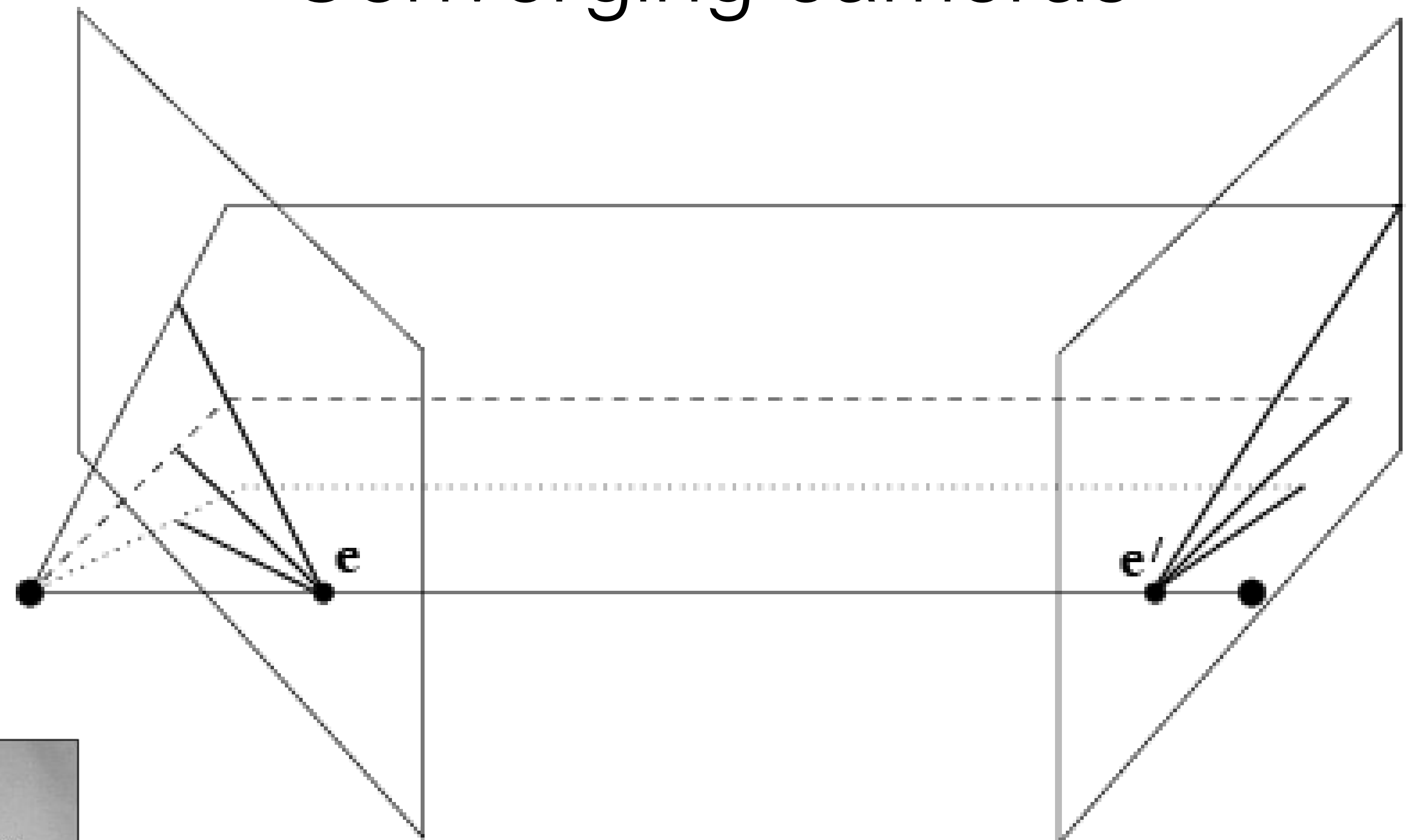

Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

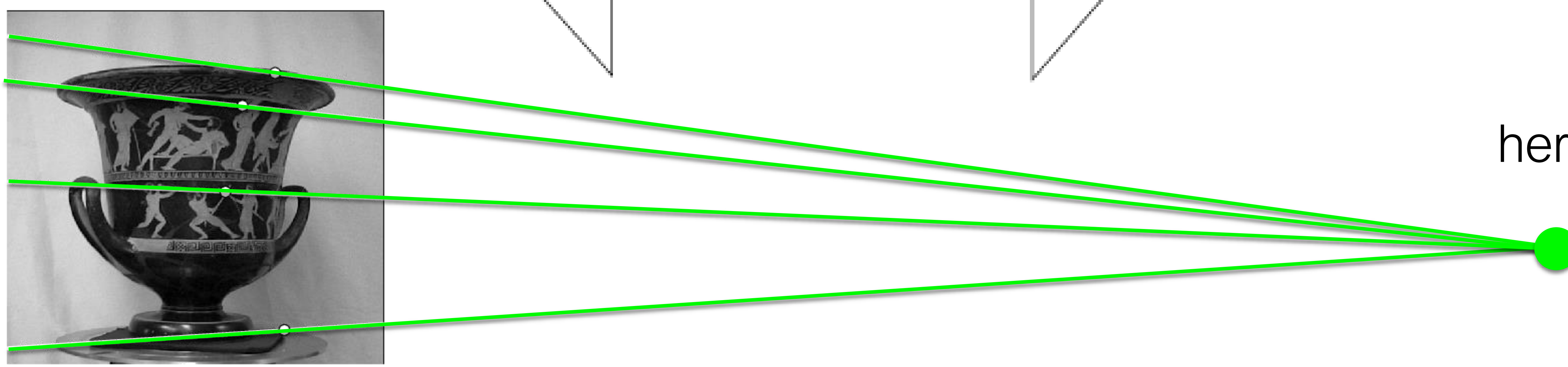
# Converging cameras




*Where is the epipole in this image?*

# Converging cameras



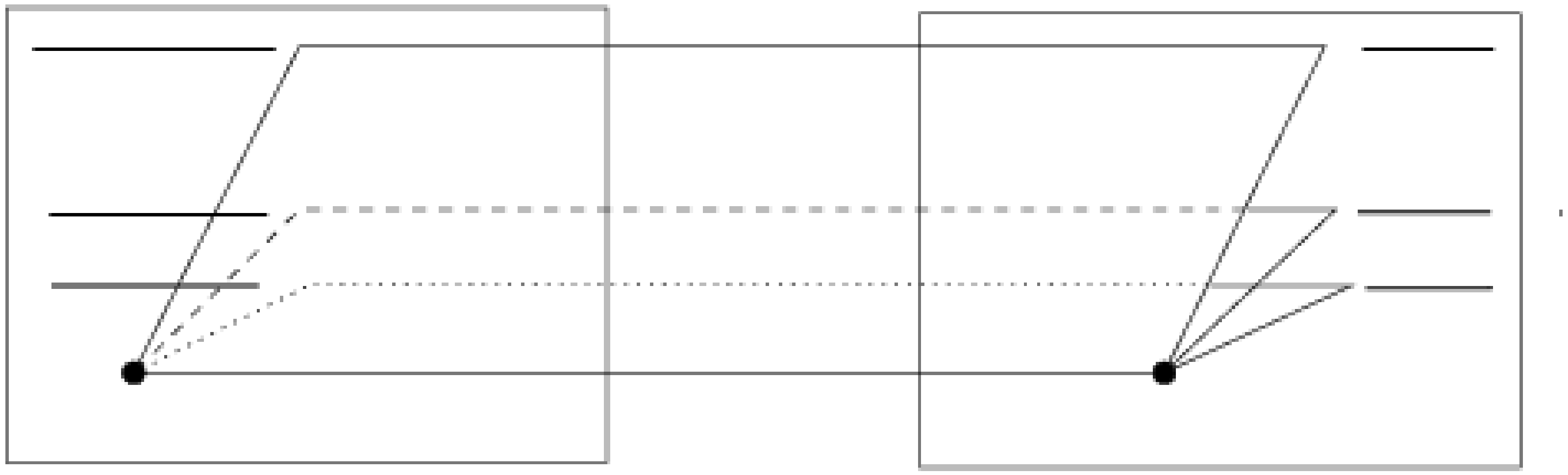



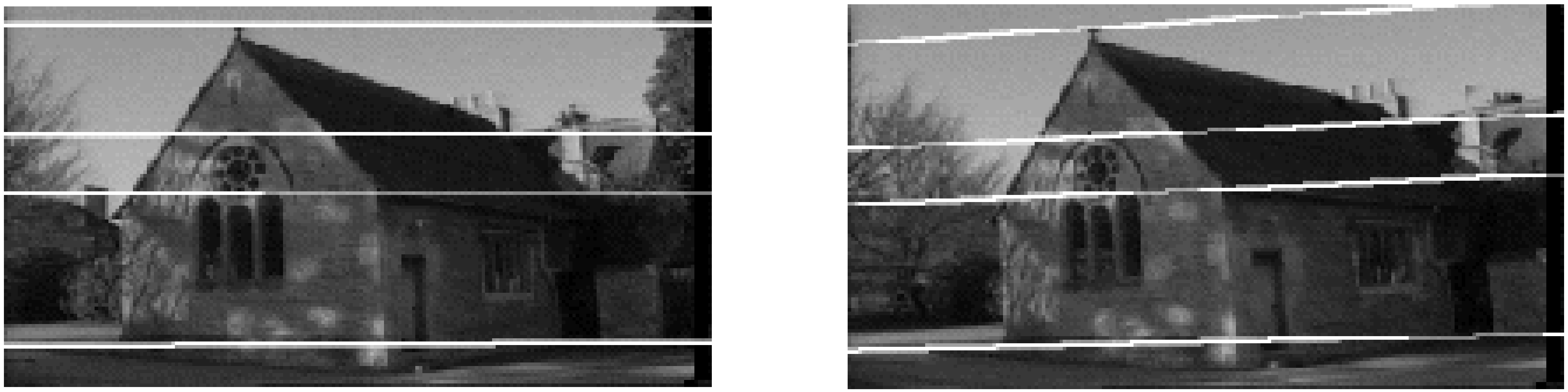
here!

*Where is the epipole in this image?*

It's not always in the image

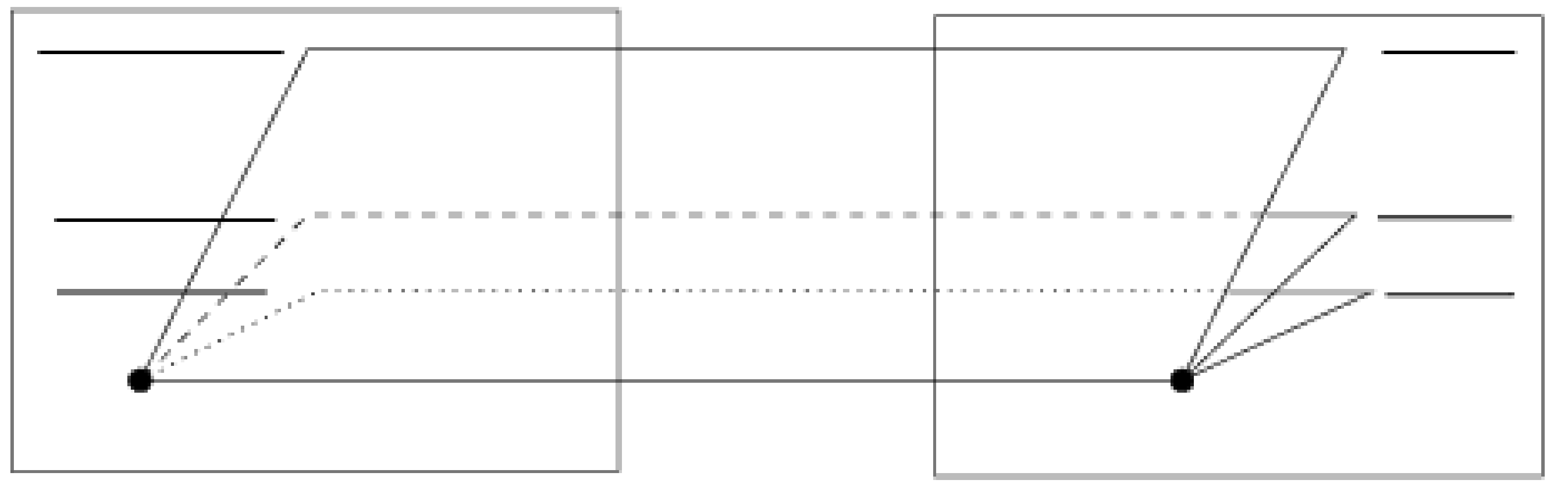
# Parallel cameras





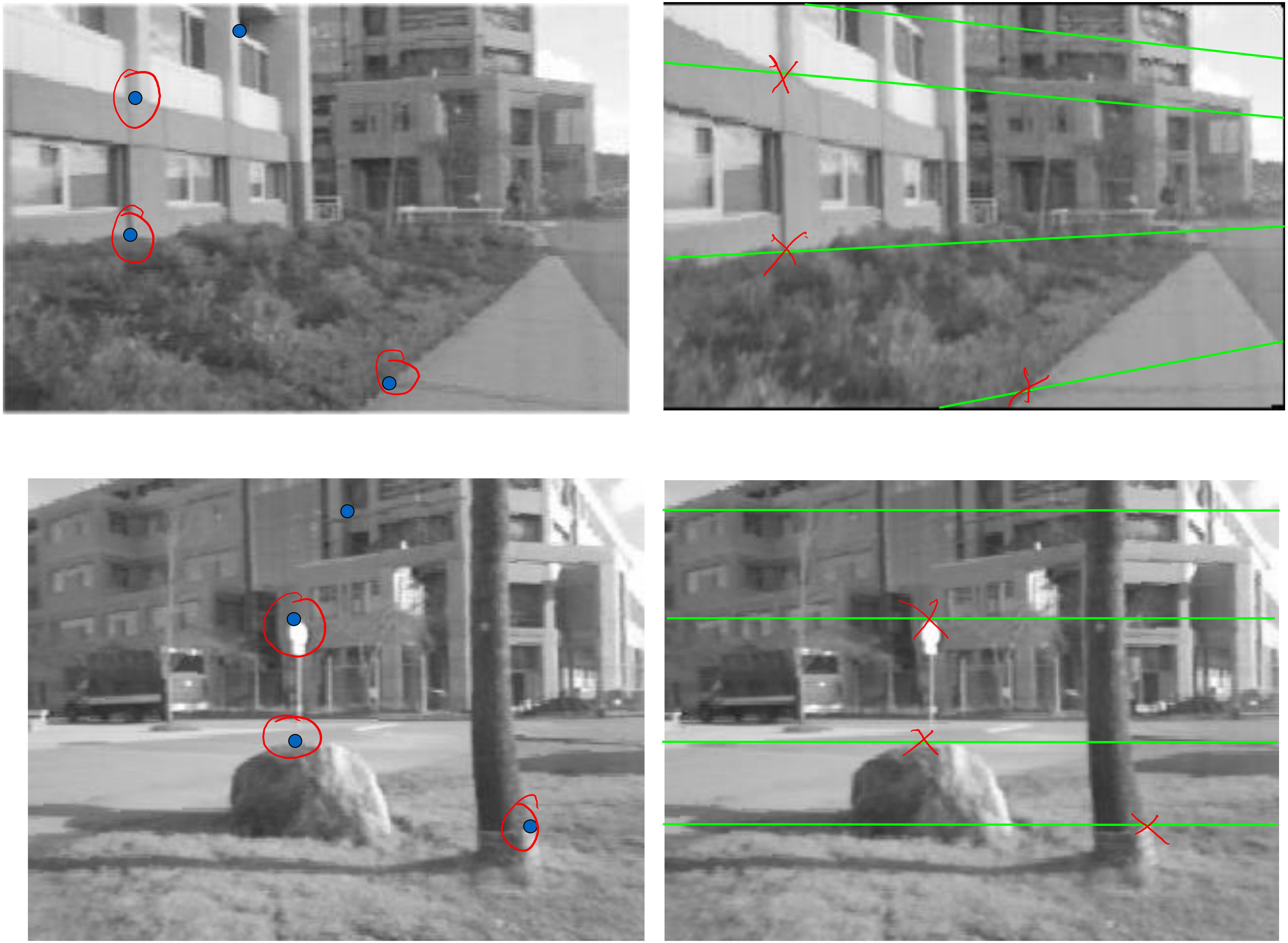
*Where is the epipole?*

# Parallel cameras

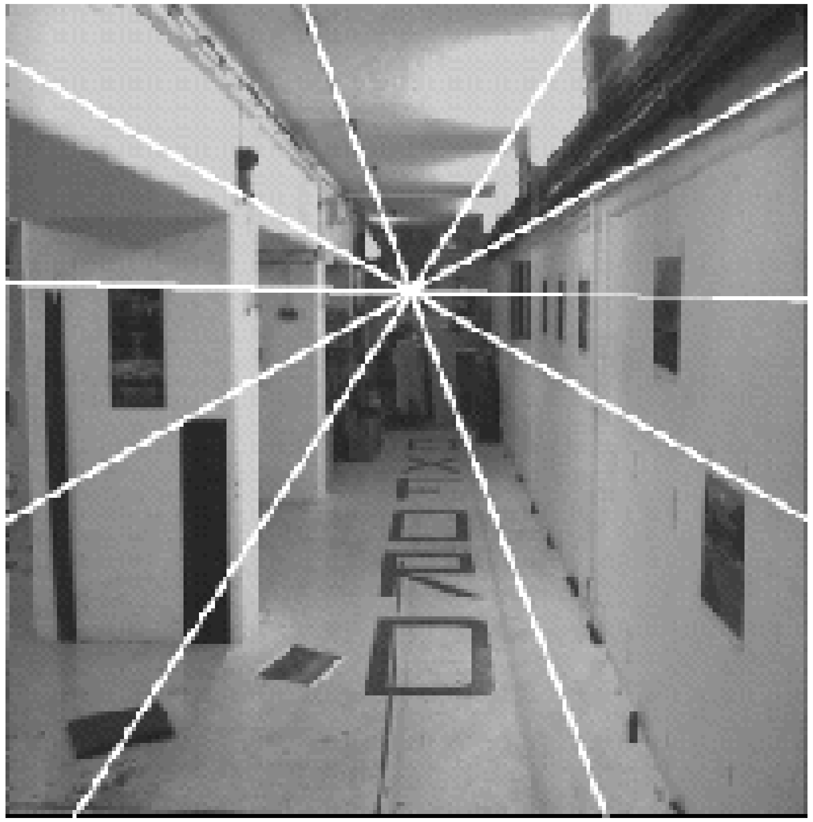


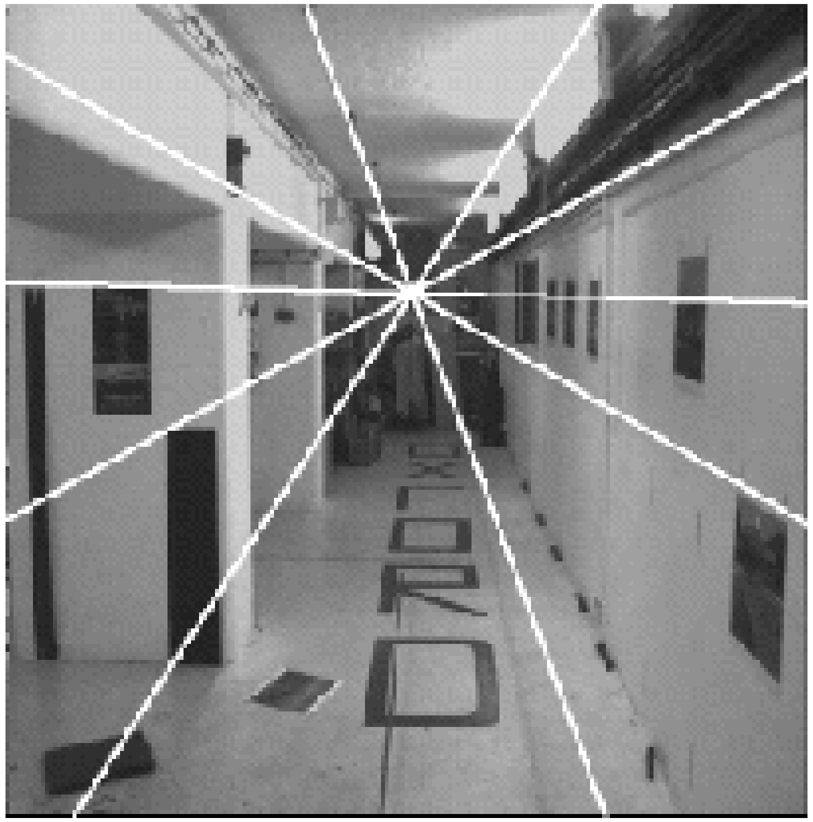
epipole at infinity

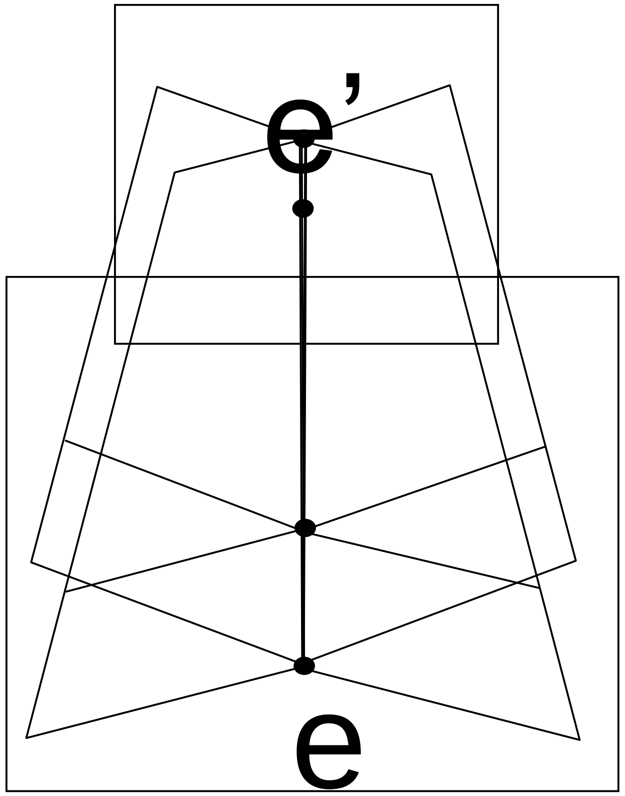
# Epipolar Geometry Example



# Forward Camera Motion

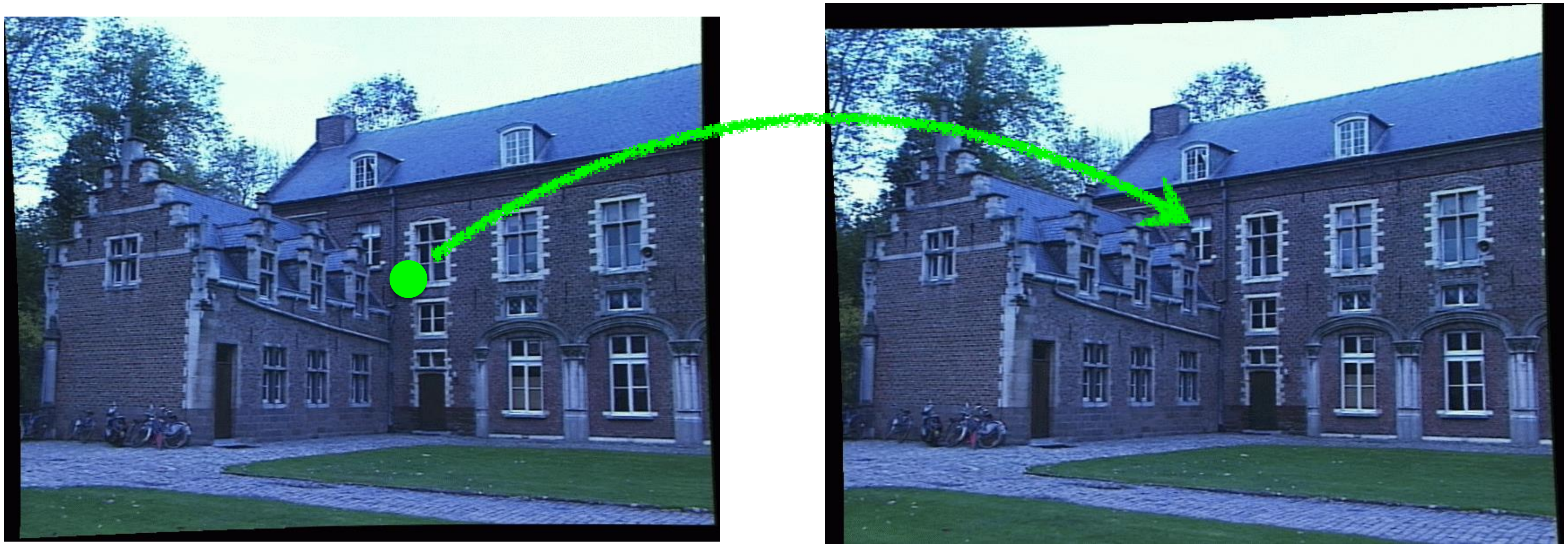






The epipolar constraint is an important concept for stereo vision

**Task:** Match point in left image to point in right image

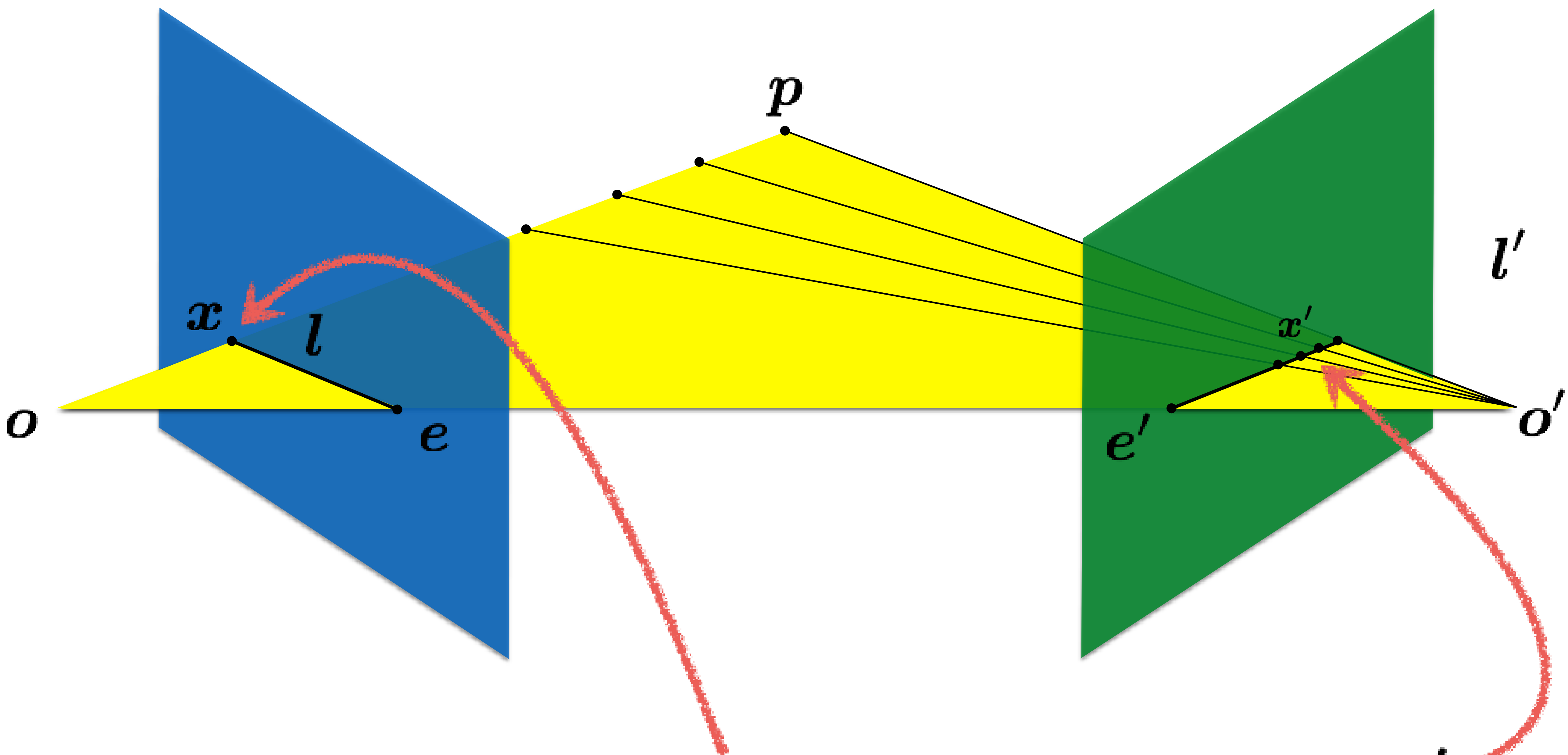


Left image

Right image

*How would you do it?*

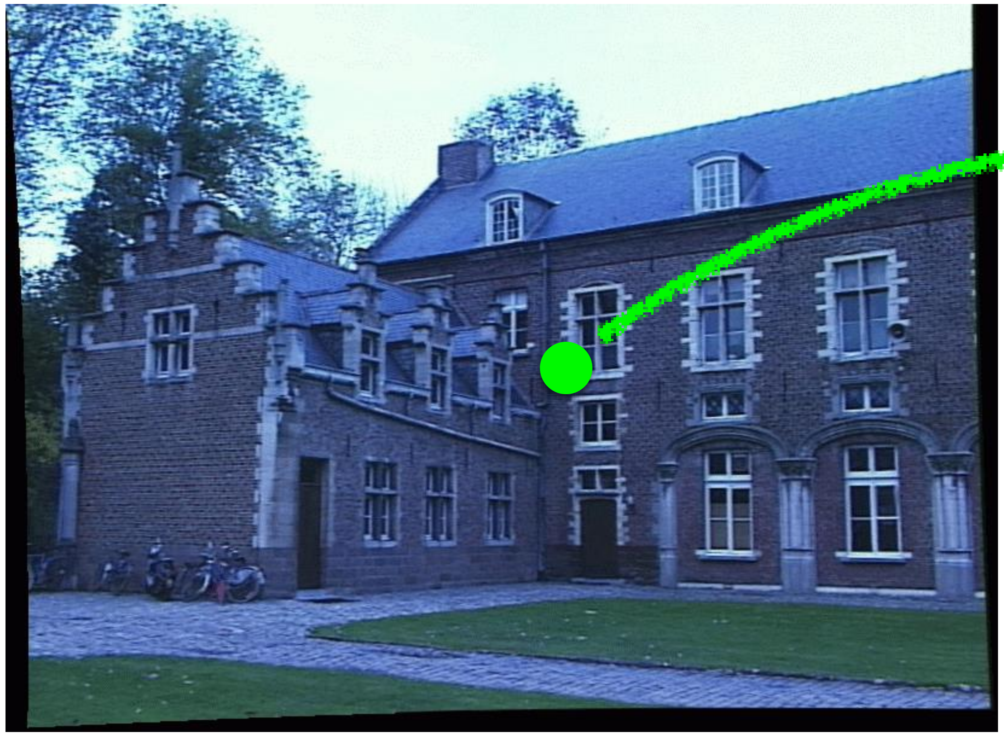
# Recall:Epipolar constraint



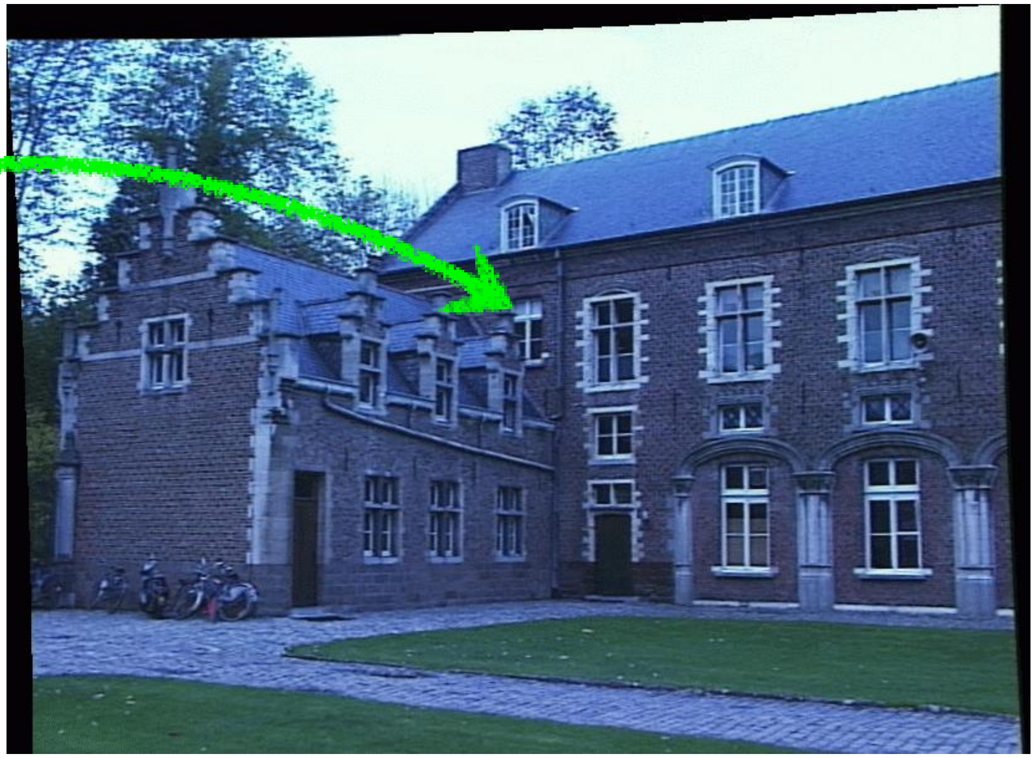

Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

The epipolar constraint is an important concept for stereo vision

**Task:** Match point in left image to point in right image



Left image



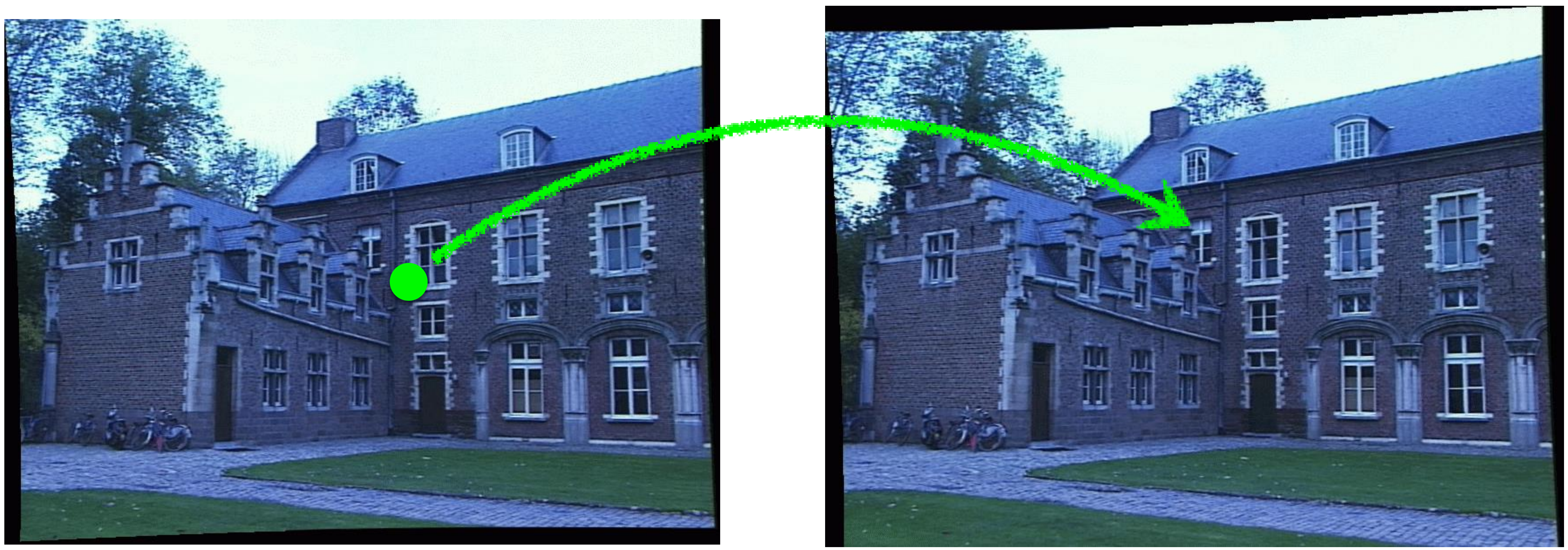
Right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line

The epipolar constraint is an important concept for stereo vision

**Task:** Match point in left image to point in right image



Left image

Right image

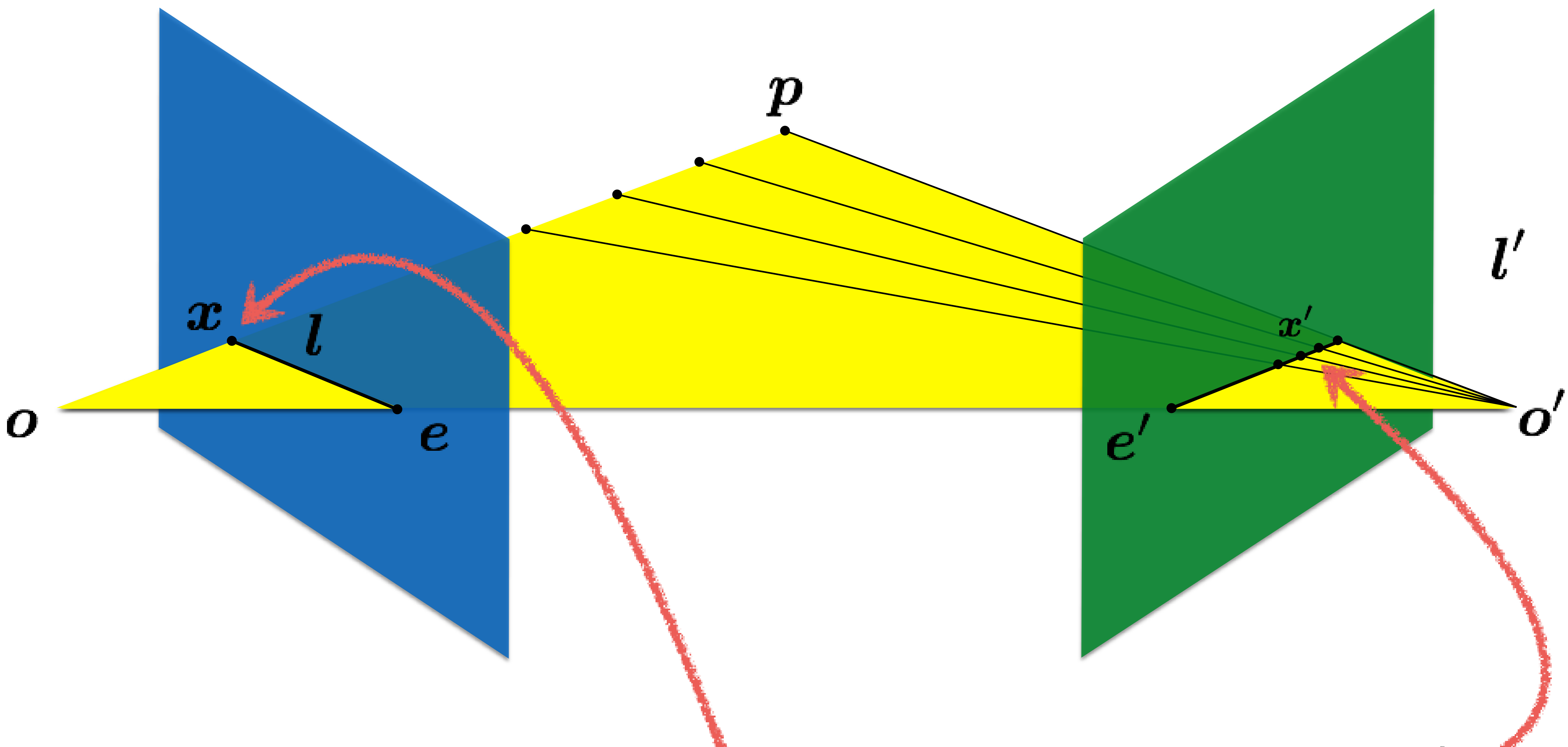
Want to avoid search over entire image

Epipolar constraint reduces search to a single line

*How do you compute the epipolar line?*

# The essential matrix

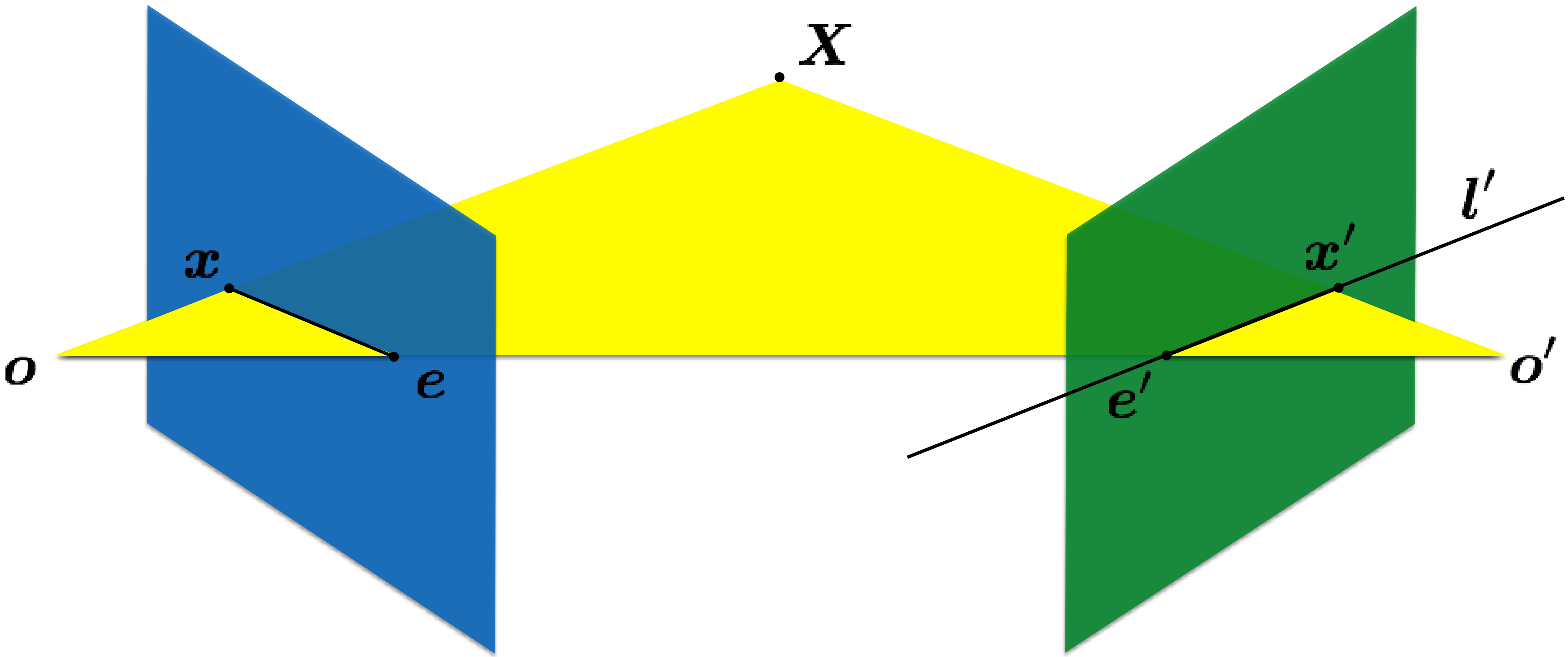
# Recall:Epipolar constraint




Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l'}$

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$

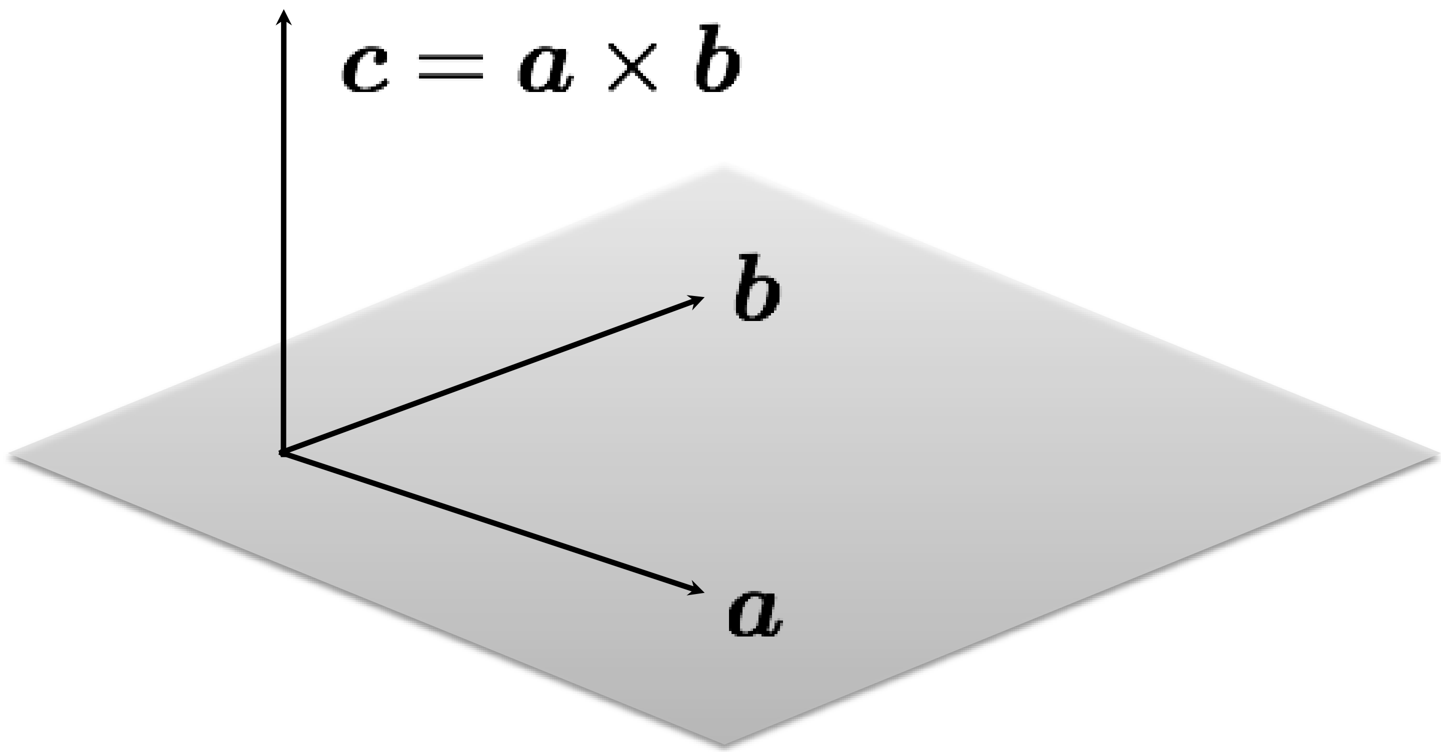


# Motivation

The Essential Matrix is a 3 x 3 matrix that encodes **epipolar geometry**

Given a point in one image,
multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

# Recall: Dot Product



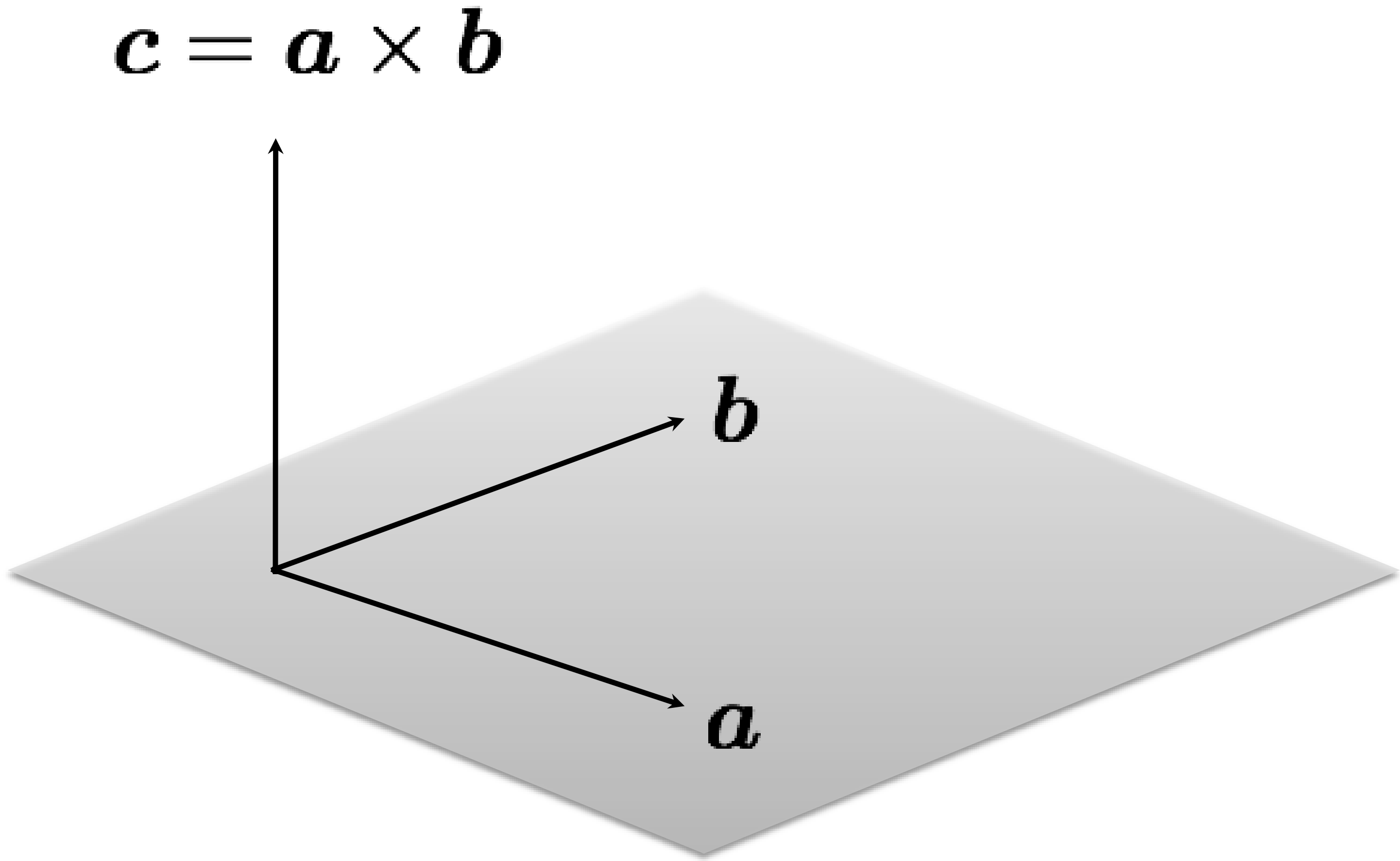

$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

**dot product of two orthogonal vectors is zero**

# Recall: Cross Product

**Vector (cross) product**

takes two vectors and returns a vector perpendicular to both




$\boldsymbol{c} \cdot \boldsymbol{a} = 0$ $\boldsymbol{c} \cdot \boldsymbol{b} = 0$

Cross product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Can also be written as a matrix multiplication

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

**Skew symmetric**

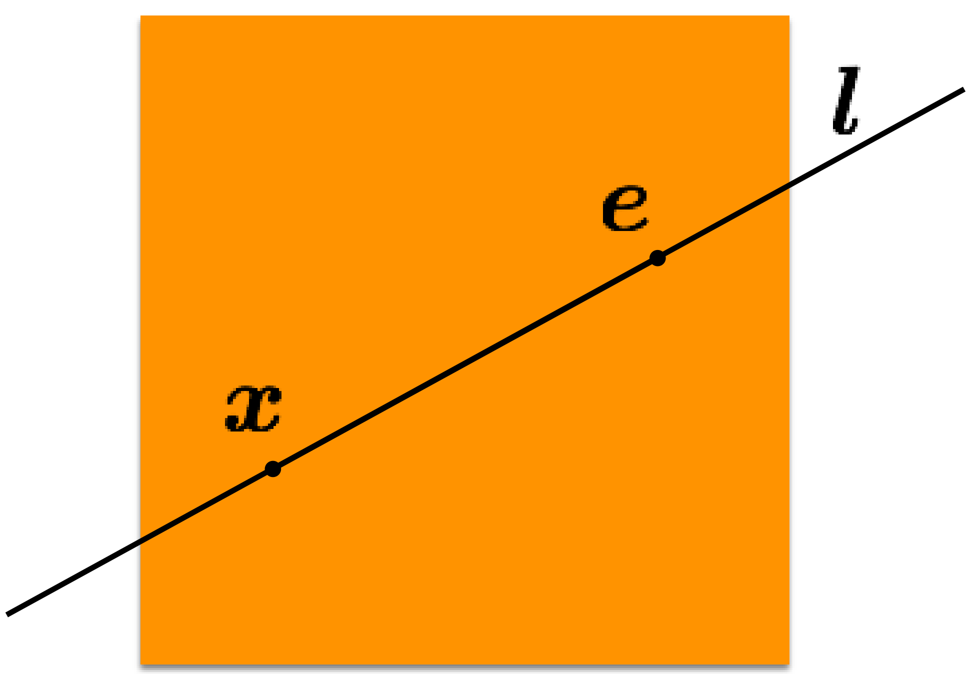
Representing the …

# Epipolar Line

$$ax + by + c = 0$$

in vector form

$$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

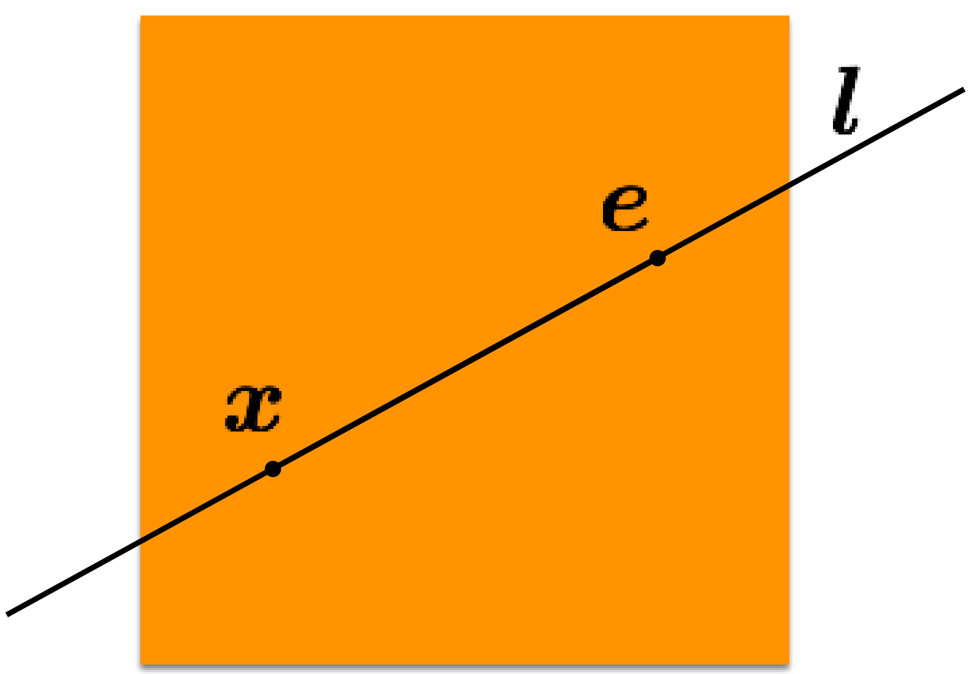



If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = \;?$$

# Epipolar Line

$$ax + by + c = 0$$ in vector form $$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

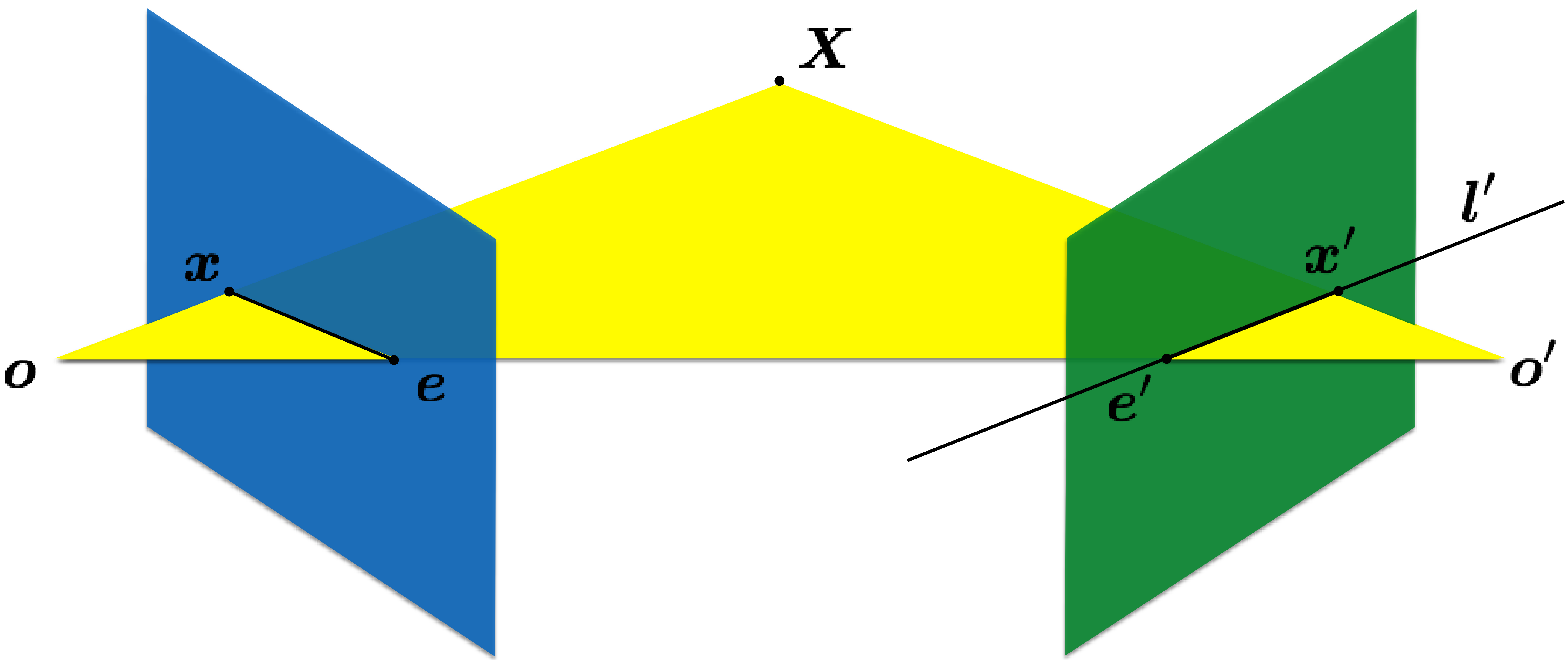



If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = 0$$

So if $\boldsymbol{x}^\top \boldsymbol{l} = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = \ ?$$



So if $\boldsymbol{x}^\top \boldsymbol{l} = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

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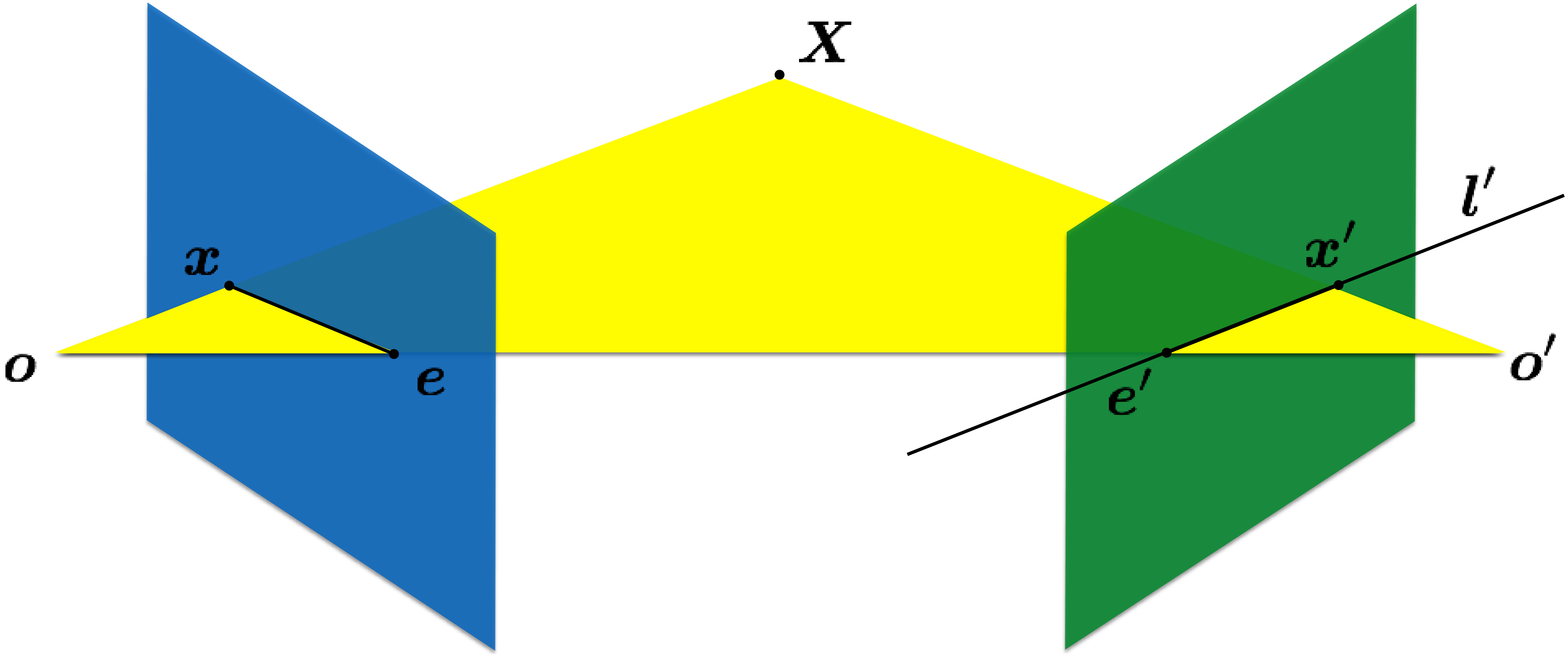


# Essential Matrix vs Homography

*What's the difference between the essential matrix and a homography?*

# Essential Matrix vs Homography

*What's the difference between the essential matrix and a homography?*

They are both 3 x 3 matrices but …

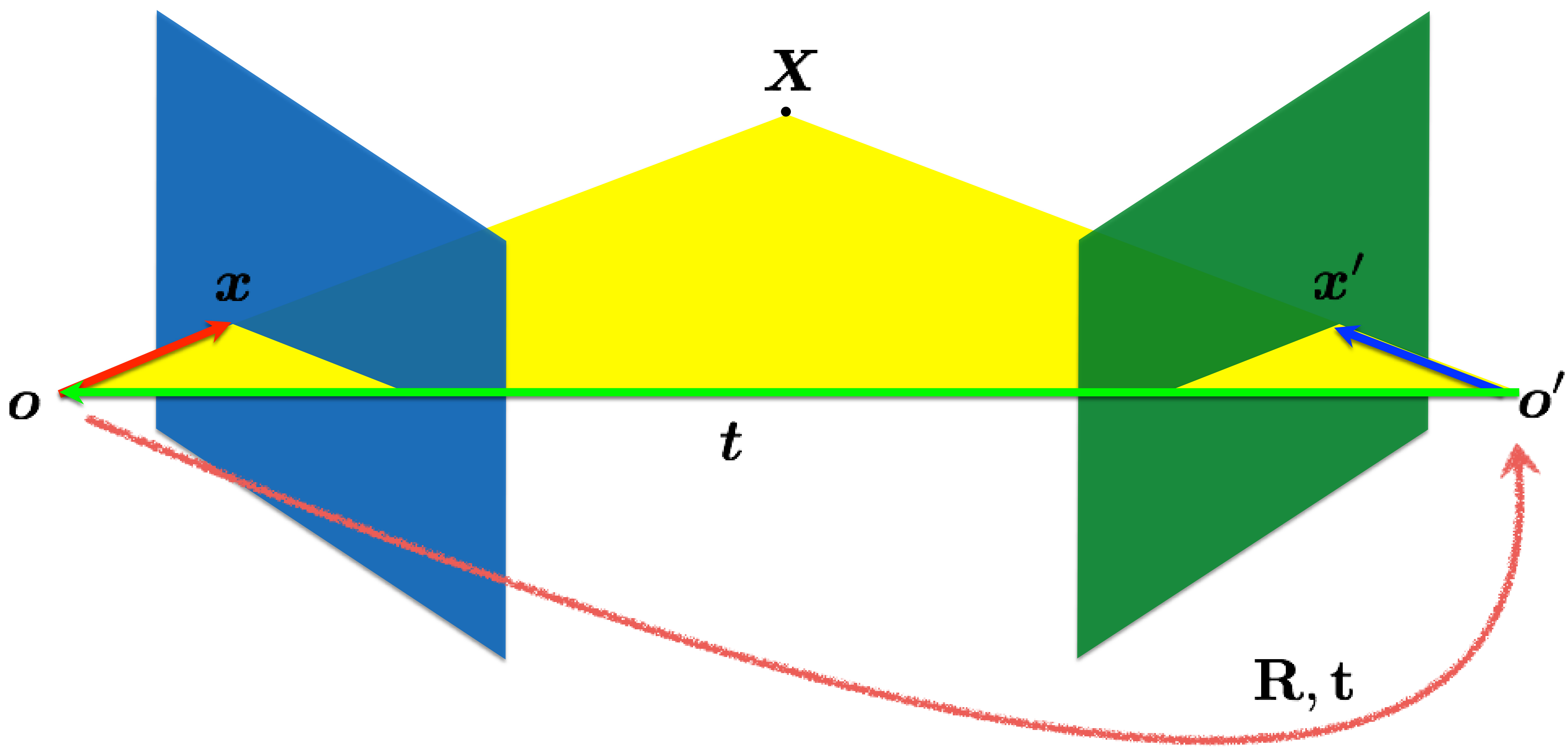
$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$$

Essential matrix maps a **point** to a **line**

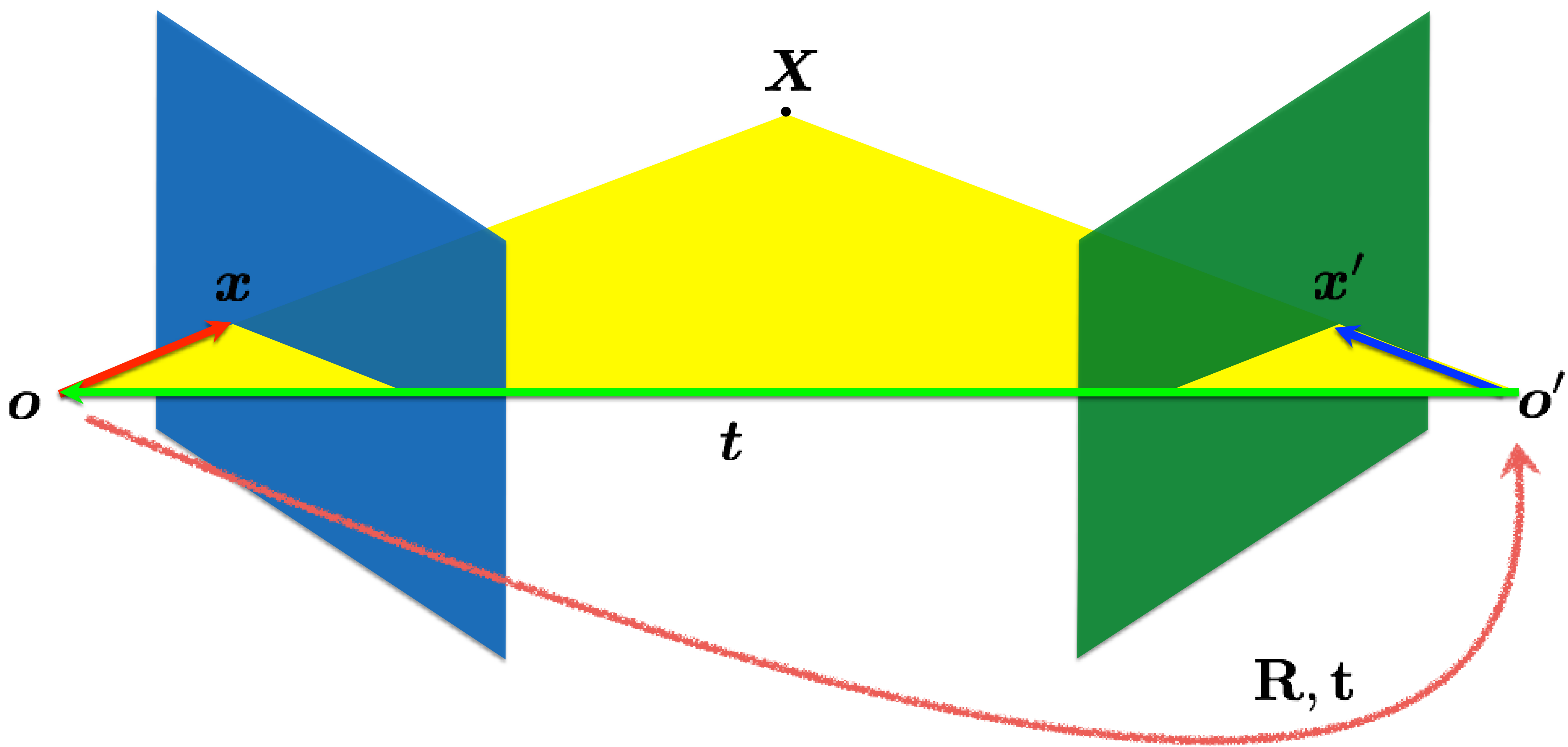
$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

Homography maps a **point** to a **point**

Where does the Essential matrix come from?

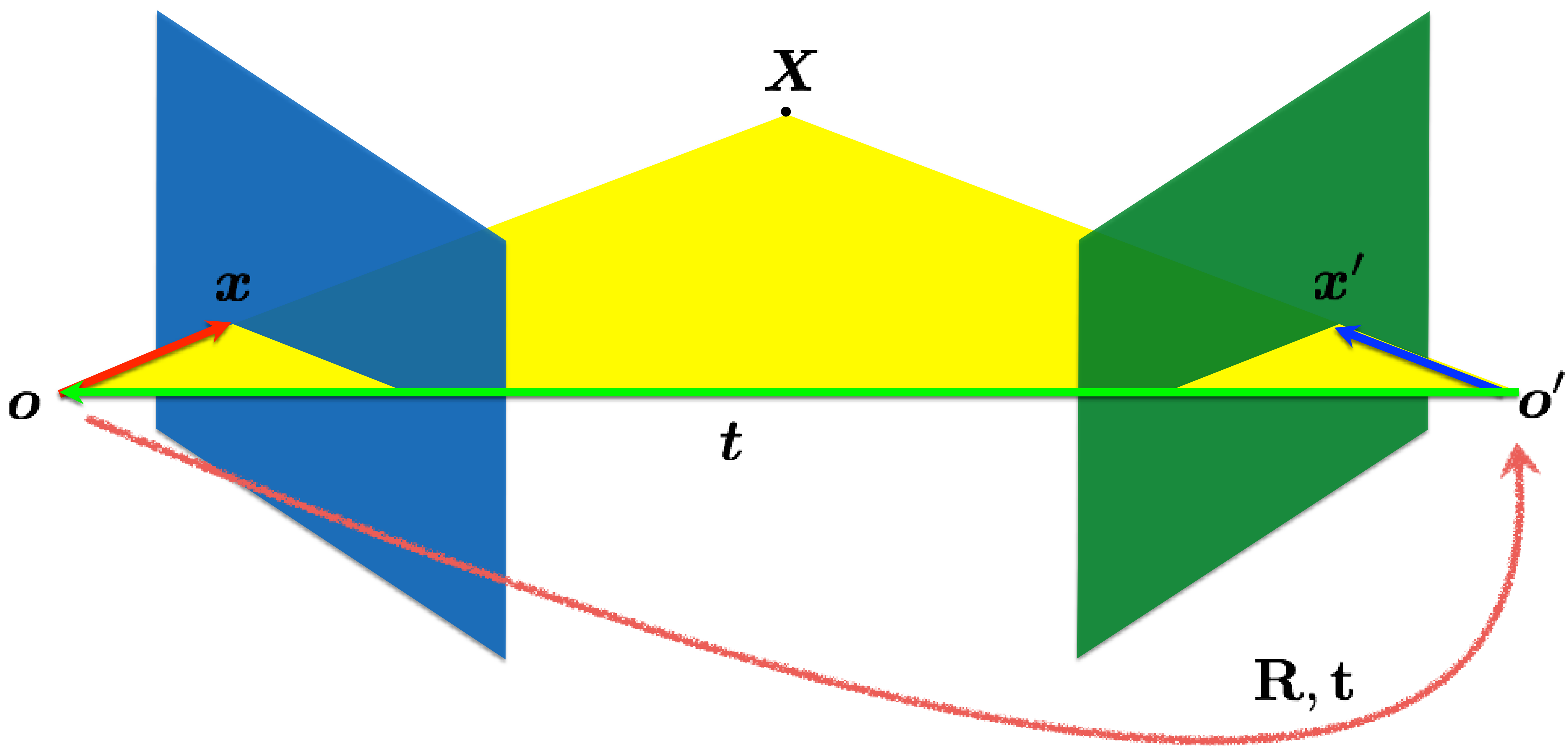



$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$



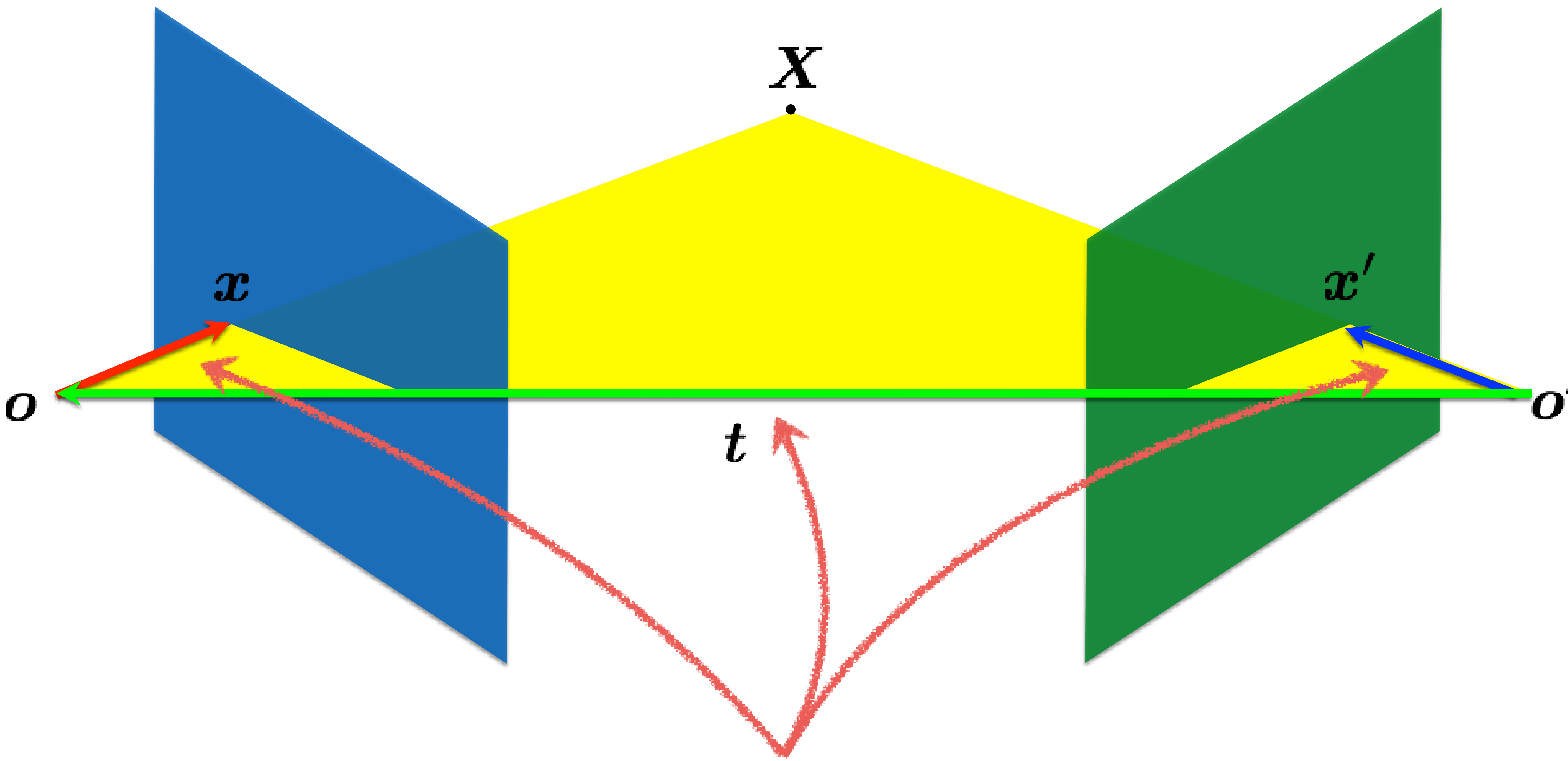

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

*Does this look familiar?*



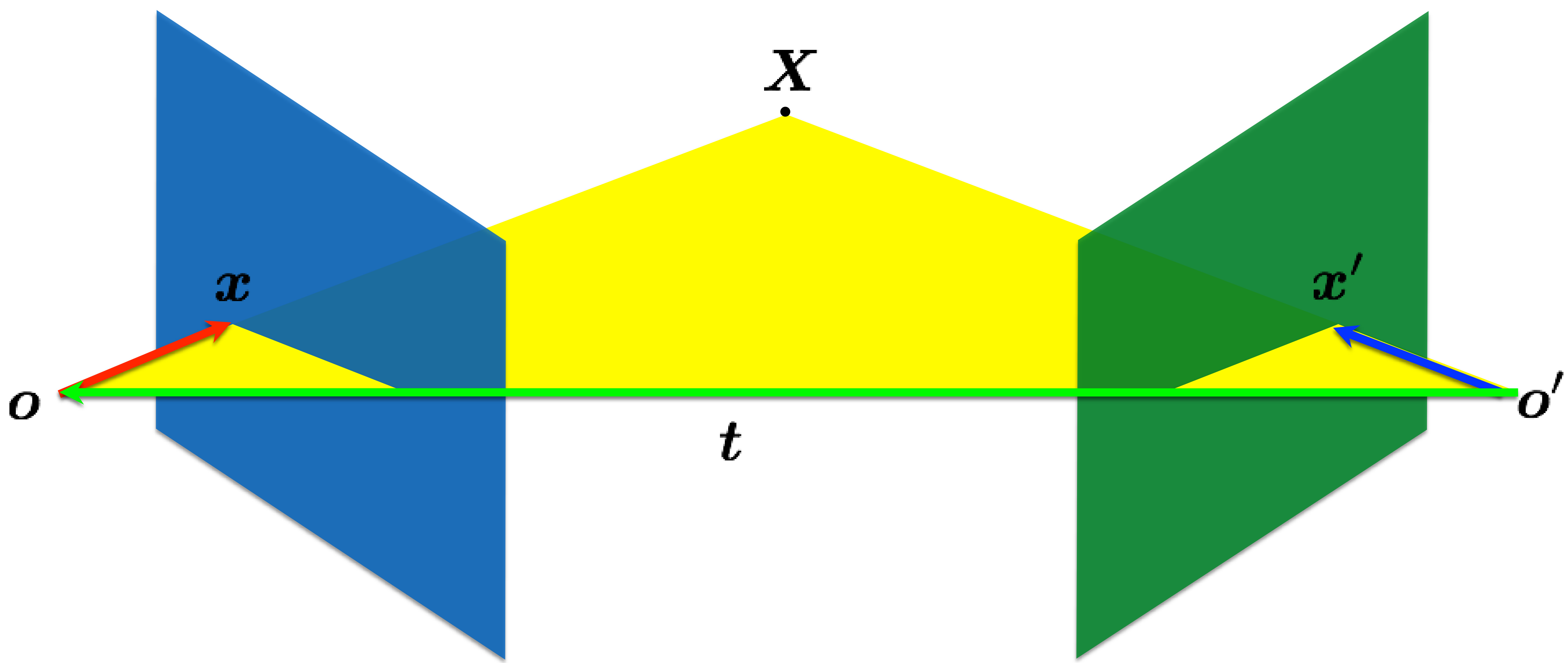

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

**Camera-camera** transform just like **world-camera** transform



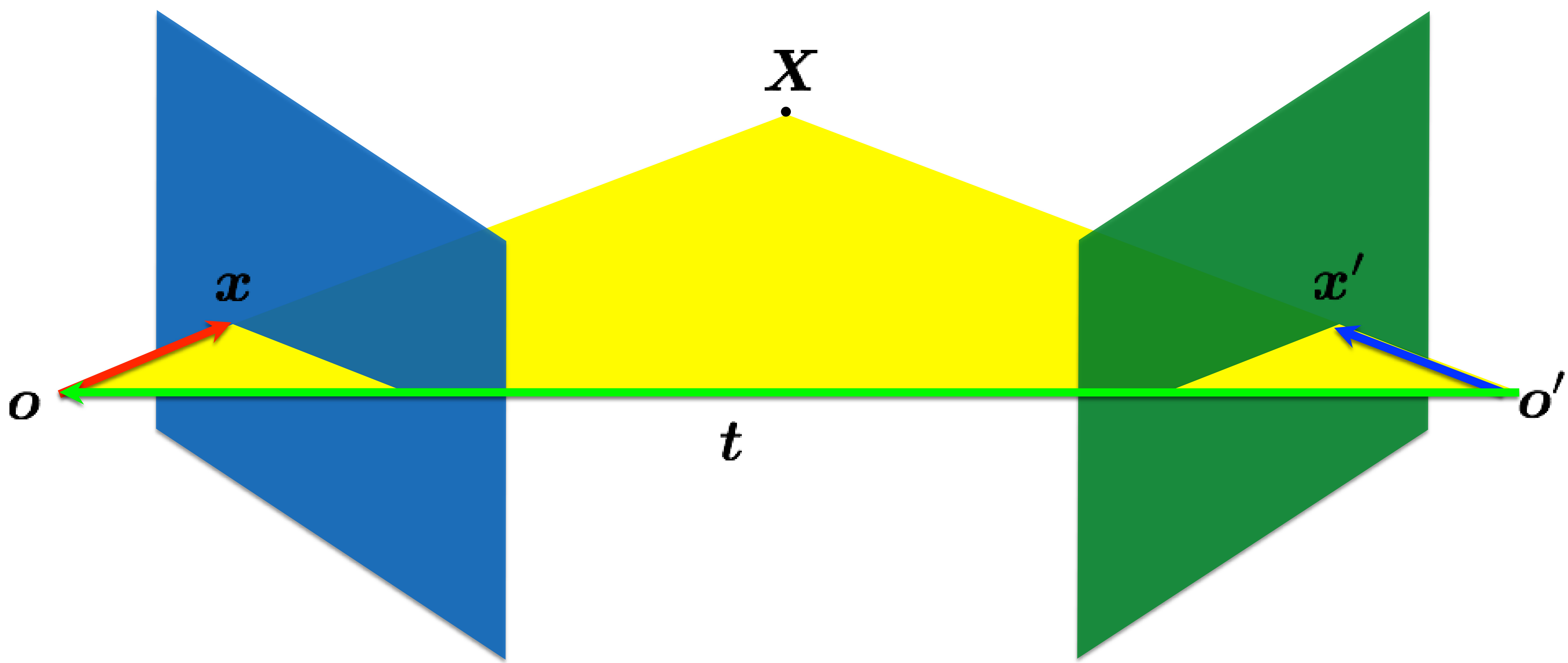

These three vectors are coplanar

$$\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$$




If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^\top (\boldsymbol{t} \times \boldsymbol{x}) = \ ?$$

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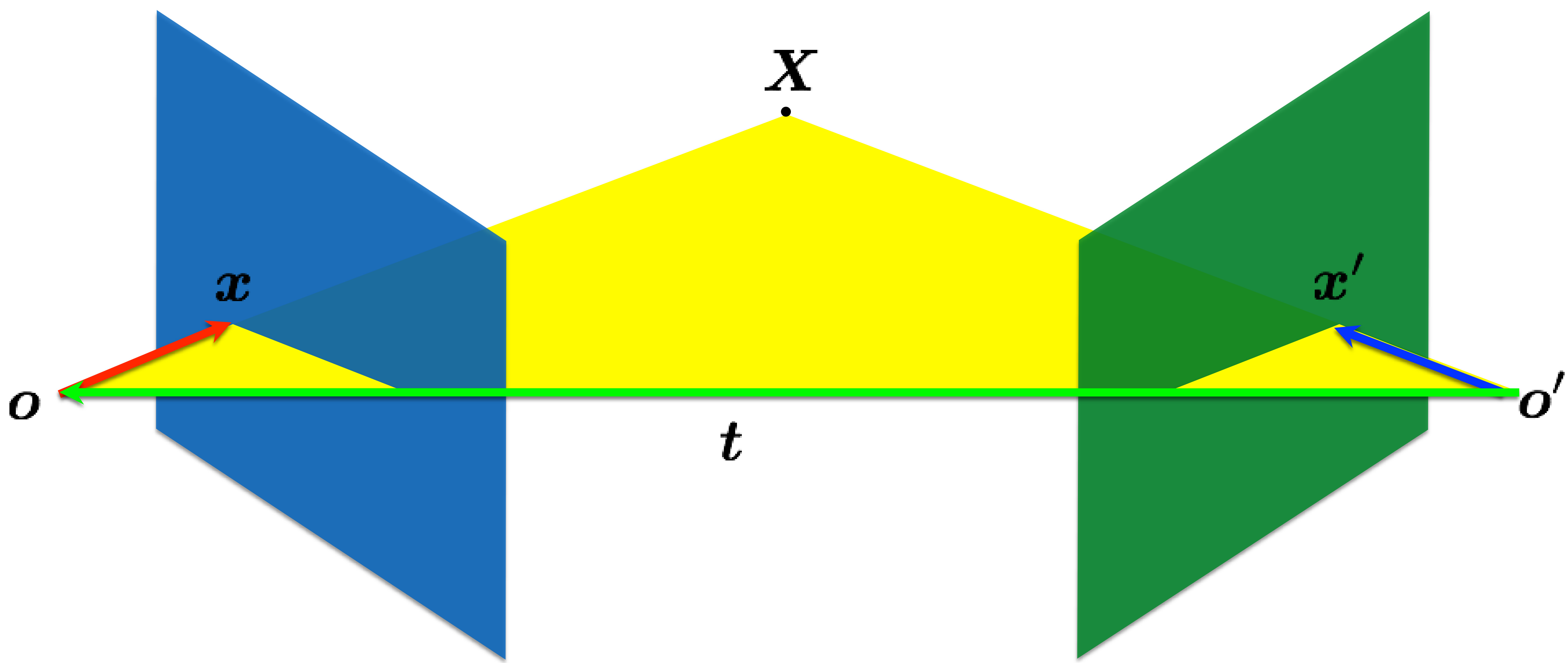
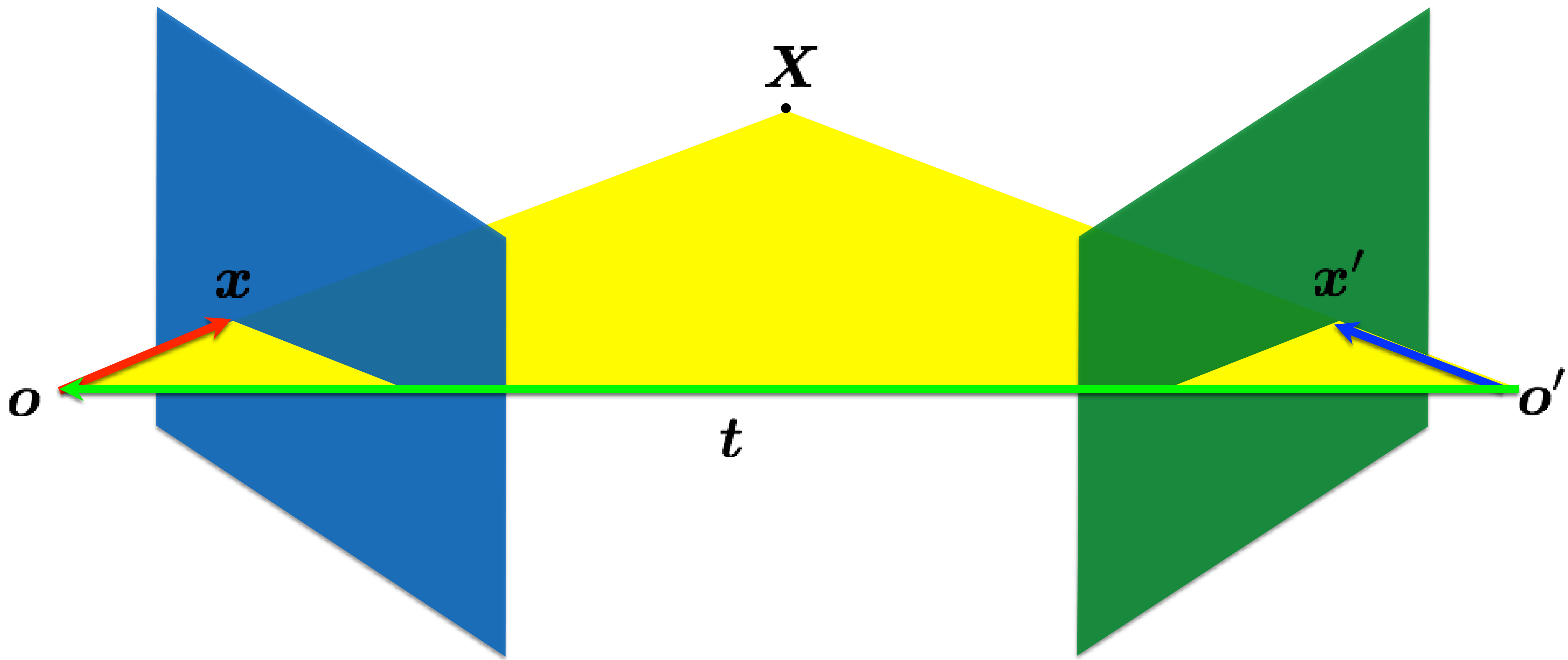



If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = \ ?$$




If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

# putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

# putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

# putting it together

rigid motion

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$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

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# putting it together

rigid motion

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$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boxed{\boldsymbol{x}'^\top \mathbf{E} \boldsymbol{x} = 0}$$

**Essential Matrix**
[Longuet-Higgins 1981]

# properties of the E matrix

Longuet-Higgins equation $\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$

(points in normalized coordinates)

# properties of the E matrix

Longuet-Higgins equation $\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$

Epipolar lines

$$\boldsymbol{x}^{\top}\boldsymbol{l} = 0 \qquad \boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$$

$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x} \qquad \boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$$

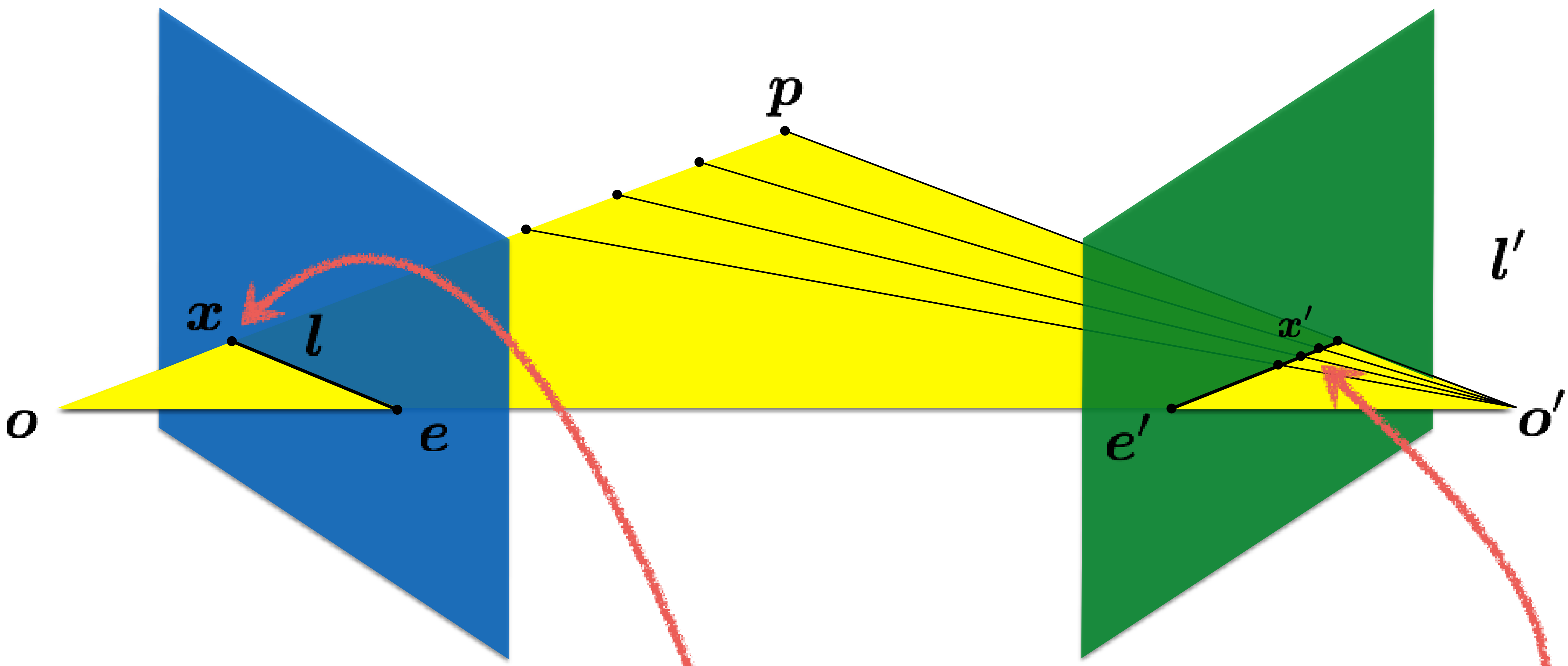
(points in normalized coordinates)

# properties of the E matrix

| | | |
|---|---|---|
| Longuet-Higgins equation | $\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$ | |
| Epipolar lines | $\boldsymbol{x}^{\top}\boldsymbol{l} = 0$ <br> $\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$ | $\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ <br> $\boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$ |
| Epipoles | $\boldsymbol{e}'^{\top}\mathbf{E} = \mathbf{0}$ | $\mathbf{E}\boldsymbol{e} = \mathbf{0}$ |

(points in normalized camera coordinates)

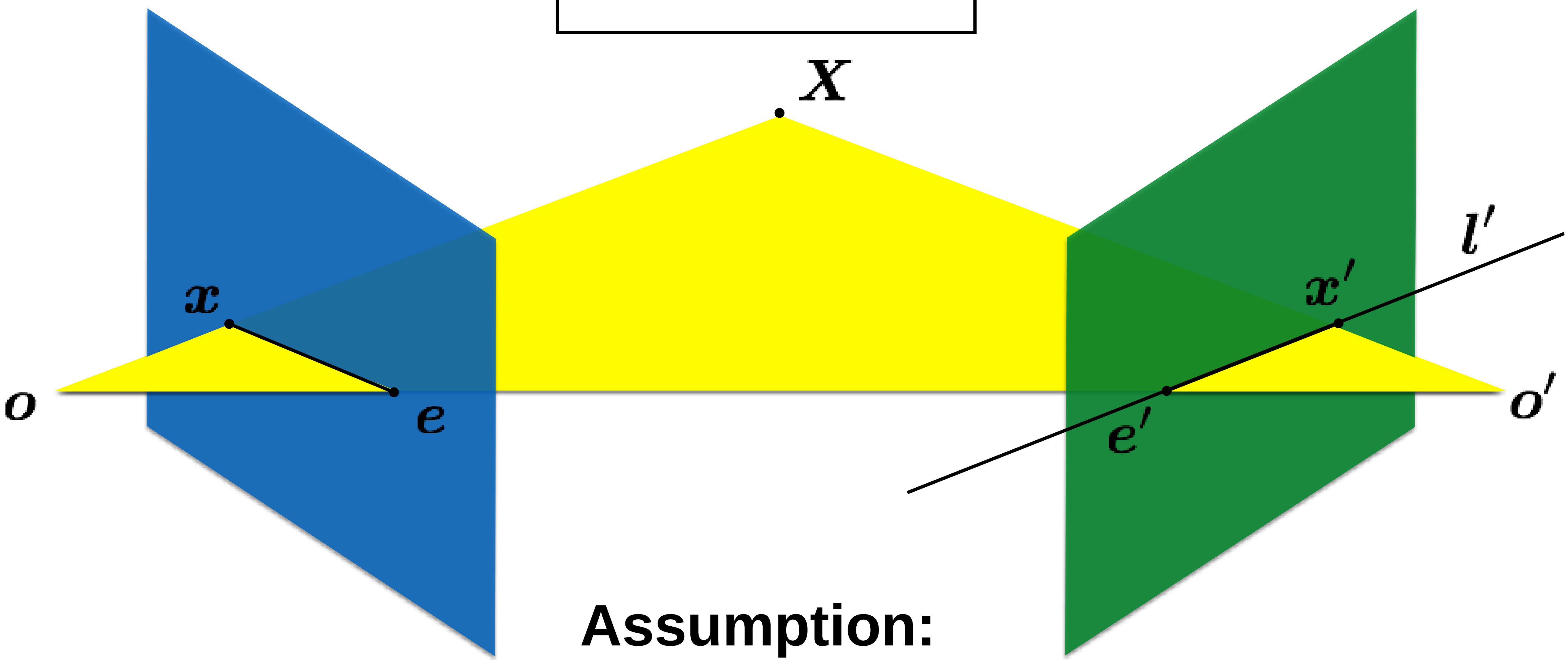
# Recall:Epipolar constraint




Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$




**Assumption:**
points aligned to camera coordinate axis (calibrated camera)

How do you generalize to uncalibrated cameras?

# The fundamental matrix

The

**Fundamental matrix**

is a

**generalization**

of the

**Essential matrix,**

where the assumption of

**calibrated cameras**

is removed

$$\hat{\boldsymbol{x}}'^{\top} \mathbf{E} \hat{\boldsymbol{x}} = 0$$

The Essential matrix operates on image points expressed in
**normalized coordinates**
(points have been aligned (normalized) to camera coordinates)

$$\hat{\boldsymbol{x}}' = \mathbf{K}^{-1} \boldsymbol{x}'$$

$$\underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1} \underbrace{\boldsymbol{x}}_{\text{image point}}$$

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

The Essential matrix operates on image points expressed in
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$$\underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

Writing out the epipolar constraint in terms of image coordinates

$$\boldsymbol{x}'^{\top}\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}(\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1})\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

Same equation works in image coordinates!

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

it maps pixels to epipolar lines

# properties of the ~~E~~ F matrix

| | | |
|---|---|---|
| Longuet-Higgins equation | $\boldsymbol{x}'^\top \mathbf{F} \boldsymbol{x} = 0$ | |
| Epipolar lines | $\boldsymbol{x}^\top \boldsymbol{l} = 0$ | $\boldsymbol{x}'^\top \boldsymbol{l}' = 0$ |
| | $\boldsymbol{l}' = \mathbf{F} \boldsymbol{x}$ | $\boldsymbol{l} = \mathbf{F}^T \boldsymbol{x}'$ |
| Epipoles | $\boldsymbol{e}'^\top \mathbf{F} = \mathbf{0}$ | $\mathbf{F} \boldsymbol{e} = \mathbf{0}$ |

(points in **image** coordinates)

Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

Depends on both intrinsic and extrinsic parameters

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$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_\times]\mathbf{R}\mathbf{K}^{-1}$$

Depends on both intrinsic and extrinsic parameters

*How would you solve for F?*

$$\boldsymbol{x}_m'^{\top}\mathbf{F}\boldsymbol{x}_m = 0$$

# The 8-point algorithm

Assume you have *M* matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

*How would you solve for the 3 x 3* ***F*** *matrix?*

Assume you have *M* matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

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S V D

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Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

*How would you solve for the 3 x 3 **F** matrix?*

Set up a homogeneous linear system with 9 unknowns

$$\boldsymbol{x}_m'^\top \mathbf{F} \boldsymbol{x}_m = 0$$

$$\begin{bmatrix} x_m' & y_m' & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

*How many equation do you get from one correspondence?*

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

ONE correspondence gives you ONE equation

$$\begin{aligned} x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 + \\ y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 + \\ x'_m f_7 + y'_m f_8 + f_9 = 0 \end{aligned}$$

$$
\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0
$$

Set up a homogeneous linear system with 9 unknowns

$$
\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_Mx'_M & x_My'_M & x_M & y_Mx'_M & y_My'_M & y_M & x'_M & y'_M & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix} = \mathbf{0}
$$

*How many equations do you need?*

Each point pair (according to epipolar constraint) contributes only one scalar equation

$$\boldsymbol{x}'^{\top}_{m} \mathbf{F} \boldsymbol{x}_{m} = 0$$

**Note:** This is different from the Homography estimation where each point pair contributes 2 equations.

We need at least 8 points

# Hence, the 8 point algorithm!

*How do you solve a homogeneous linear system?*

$$\mathbf{AX} = \mathbf{0}$$

*How do you solve a homogeneous linear system?*

$$\mathbf{A}\mathbf{X} = \mathbf{0}$$

**Total Least Squares**

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

*How do you solve a homogeneous linear system?*

$$\mathbf{AX} = \mathbf{0}$$

**Total Least Squares**

minimize $\quad \|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\quad \|\boldsymbol{x}\|^2 = 1$

SVD!

# Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

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How do we do this?

S V D !

# Enforcing rank constraints

**Problem:** Given a matrix F, find the matrix F' of rank k that is closest to F,

$$\min_{\substack{F' \\ \mathrm{rank}(F')=k}} \quad \|F - F'\|^2$$

**Solution:** Compute the singular value decomposition of F,

$$F = U\Sigma V^T$$

Form a matrix Σ' by replacing all but the k largest singular values in Σ with 0.

Then the problem solution is the matrix F' formed as,

$$F' = U\Sigma' V^T$$

# Eight-Point Algorithm

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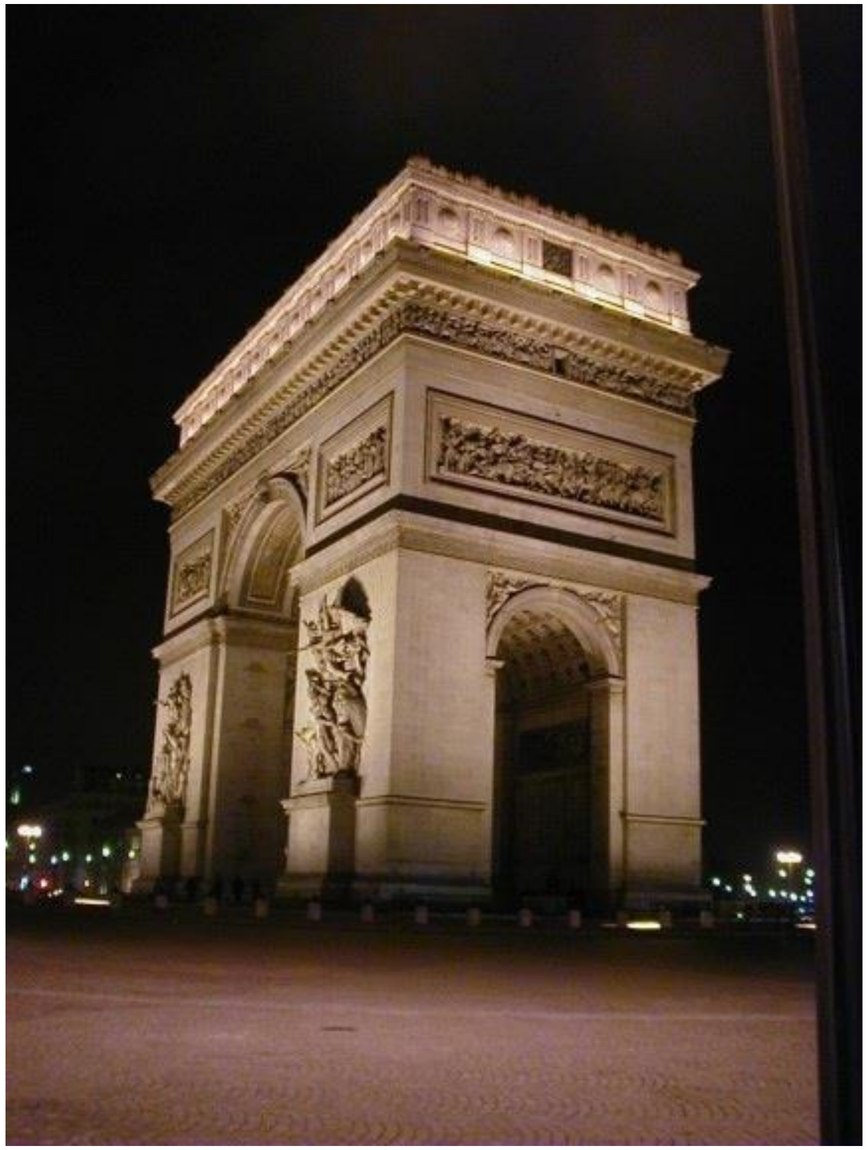
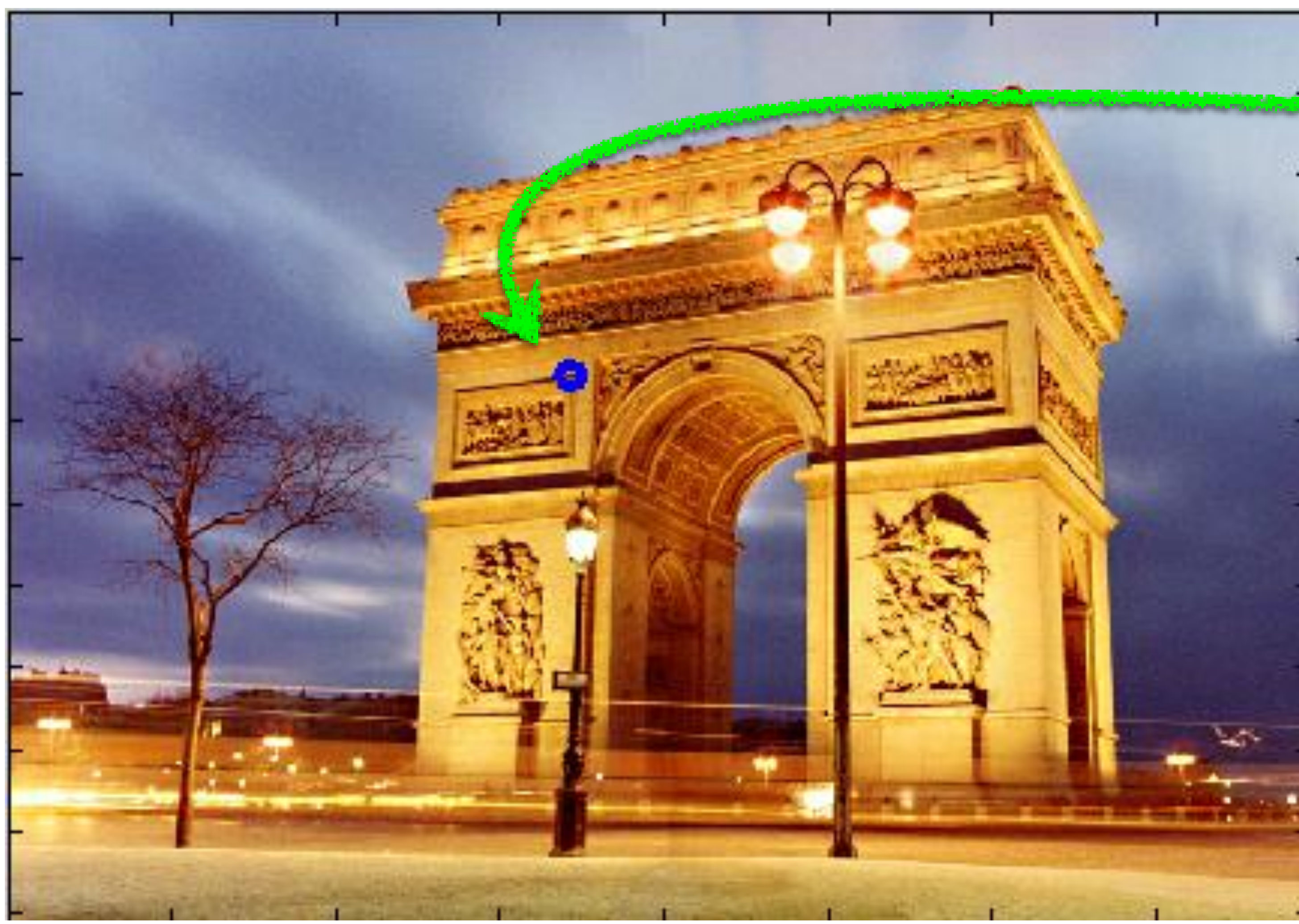
# Example



# epipolar lines


$$\mathbf{F} = \begin{bmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{bmatrix}$$

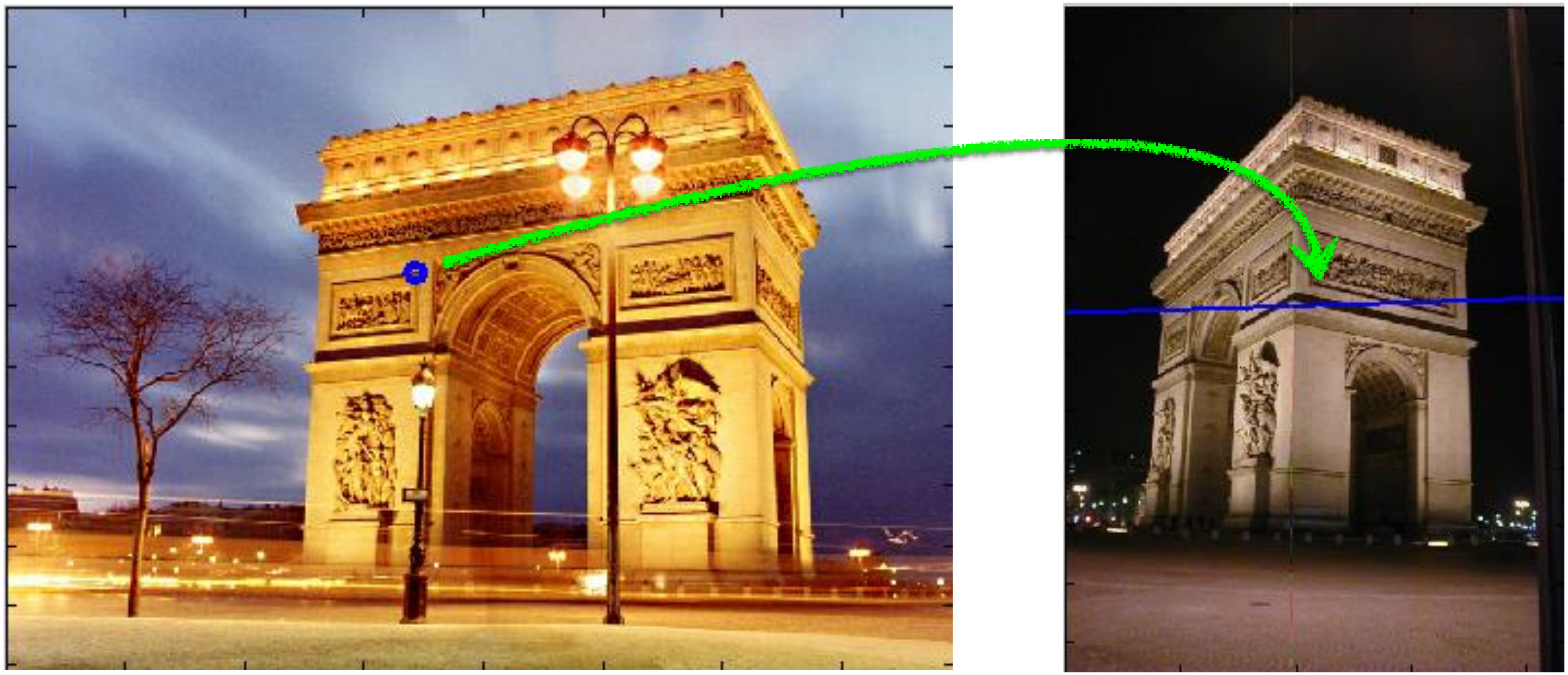


$$\boldsymbol{x} = \begin{bmatrix} 343.53 \\ 221.70 \\ 1.0 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$

$$
\begin{aligned}
\boldsymbol{l}' &= \mathbf{F}\boldsymbol{x} \\
&= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}
\end{aligned}
$$



# Where is the epipole?

*How would you compute it?*

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

*How would you solve for the epipole?*

(hint: this is a homogeneous linear system)

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The epipole is in the right null space of **F**

*How would you solve for the epipole?*

(hint: this is a homogeneous linear system)

S V D !

```
>> [u,d] = eigs(F' * F)
```

eigenvectors

```
u =
     -0.0013    0.2586   -0.9660
      0.0029   -0.9660   -0.2586
      1.0000    0.0032   -0.0005
```

eigenvalue

```
d = 1.0e8*
    -1.0000         0         0
          0   -0.0000         0
          0         0   -0.0000
```

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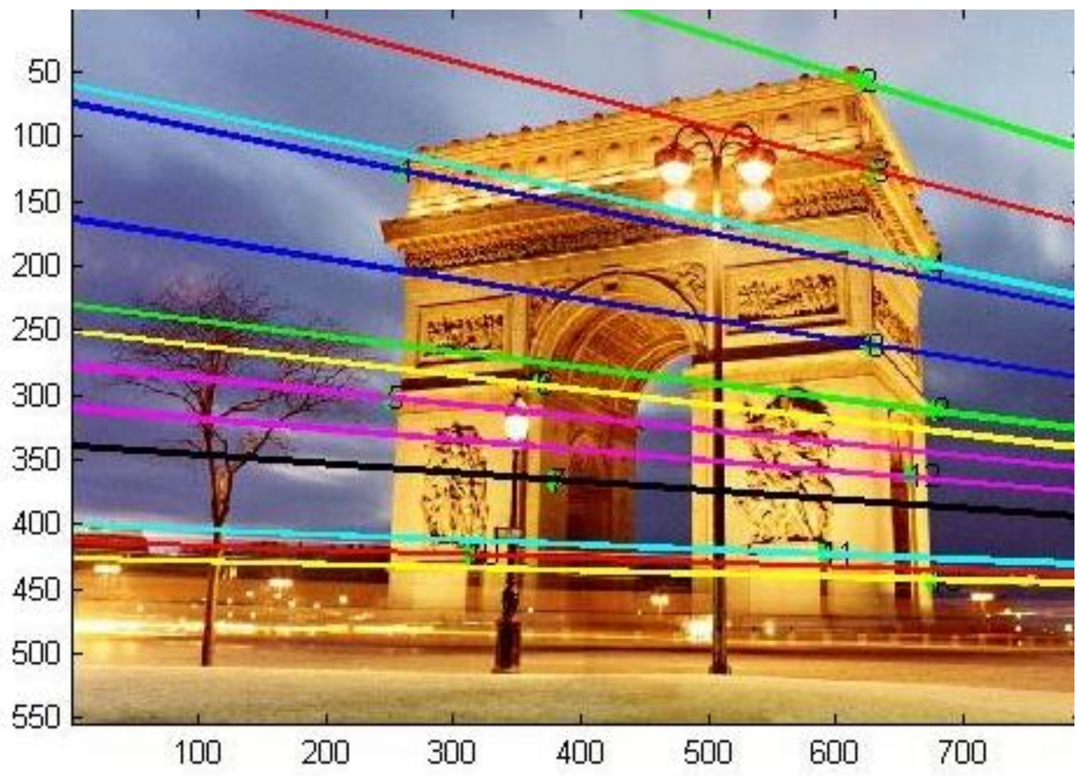
```

Eigenvector associated with smallest eigenvalue

```
>> uu = u(:,3)
(  -0.9660    -0.2586    -0.0005)
```

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>> uu = u(:,3)
(   -0.9660     -0.2586     -0.0005)
```
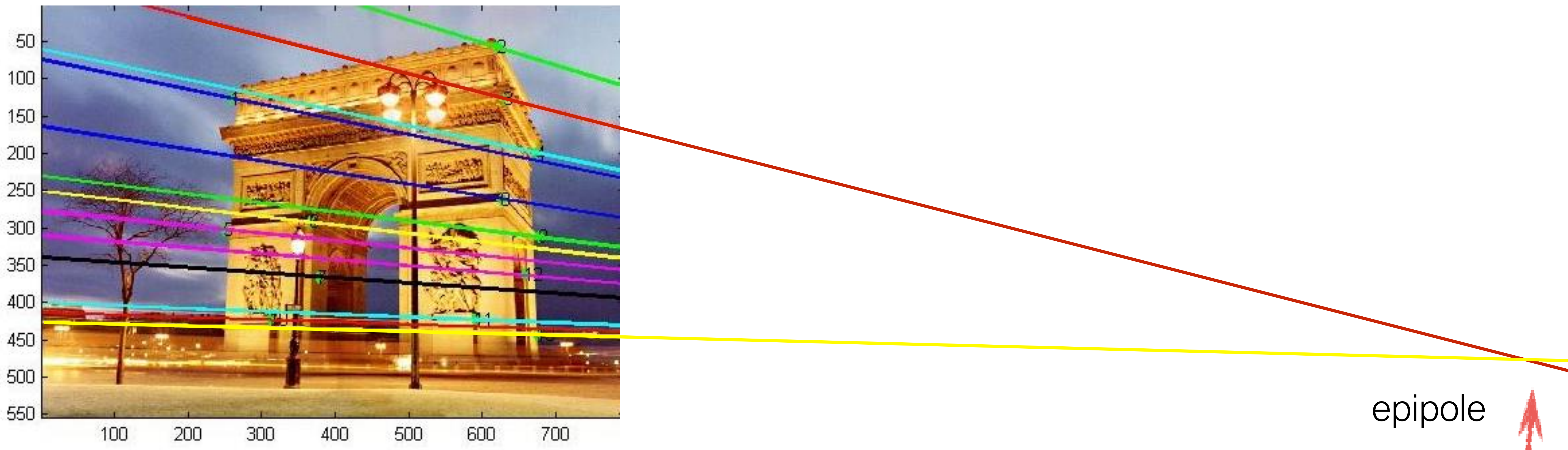

Epipole projected to image coordinates

```
>> uu / uu(3)
(1861.02      498.21     1.0)
```




Epipole projected to image coordinates

```
>> uu / uu(3)
(1861.02     498.21     1.0)
```

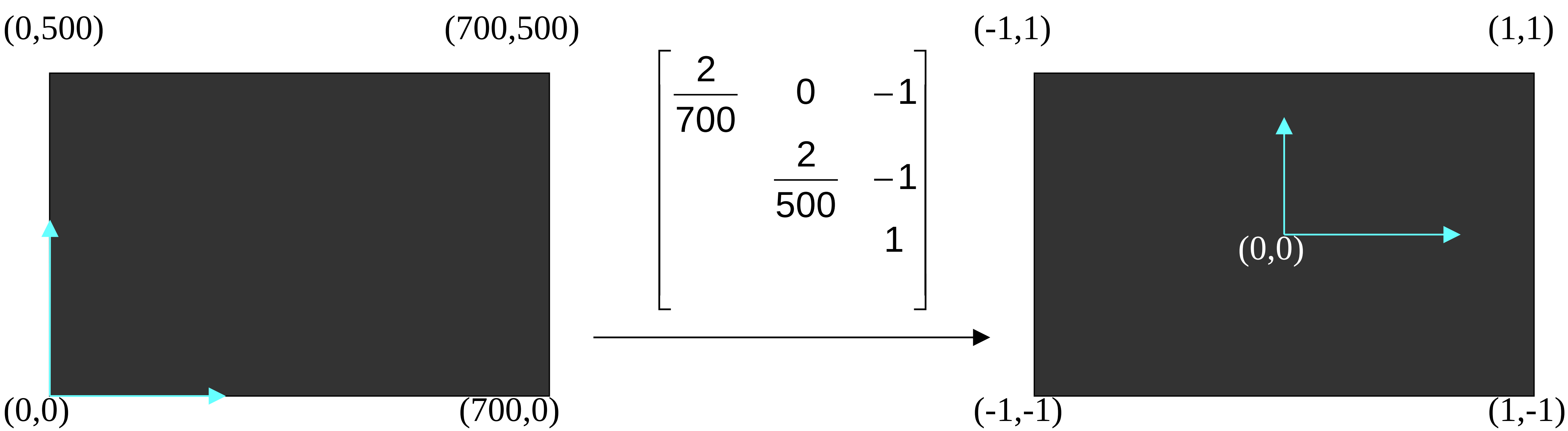
# The NOT normalized 8-point algorithm

$$
\begin{array}{c}
\begin{bmatrix}
x_1x_1' & y_1x_1' & x_1' & x_1y_1' & y_1y_1' & y_1' & x_1 & y_1 & 1 \\
x_2x_2' & y_2x_2' & x_2' & x_2y_2' & y_2y_2' & y_2' & x_2 & y_2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_nx_n' & y_nx_n' & x_n' & x_ny_n' & y_ny_n' & y_n' & x_n & y_n & 1
\end{bmatrix} \\
\begin{matrix} \sim 10000 & \sim 10000 & \sim 100 & \sim 10000 & \sim 10000 & \sim 100 & \sim 100 & \sim 100 & 1 \end{matrix}
\end{array}
\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0
$$

! Orders of magnitude difference
Between column of data matrix
$\rightarrow$ least-squares yields poor results

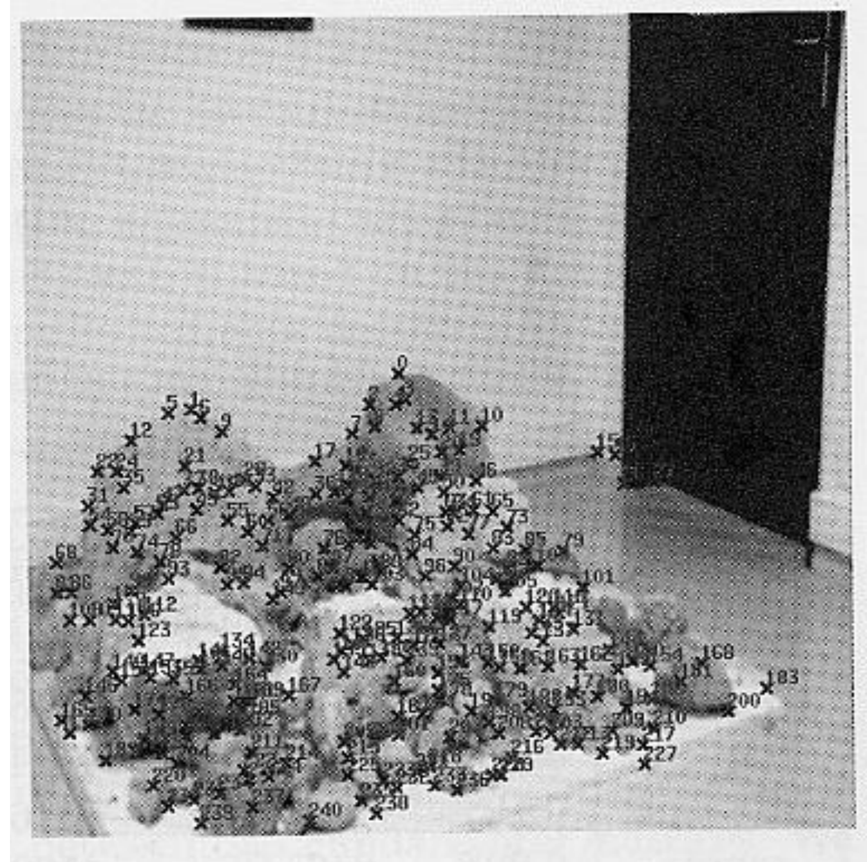
# The normalized 8-point algorithm

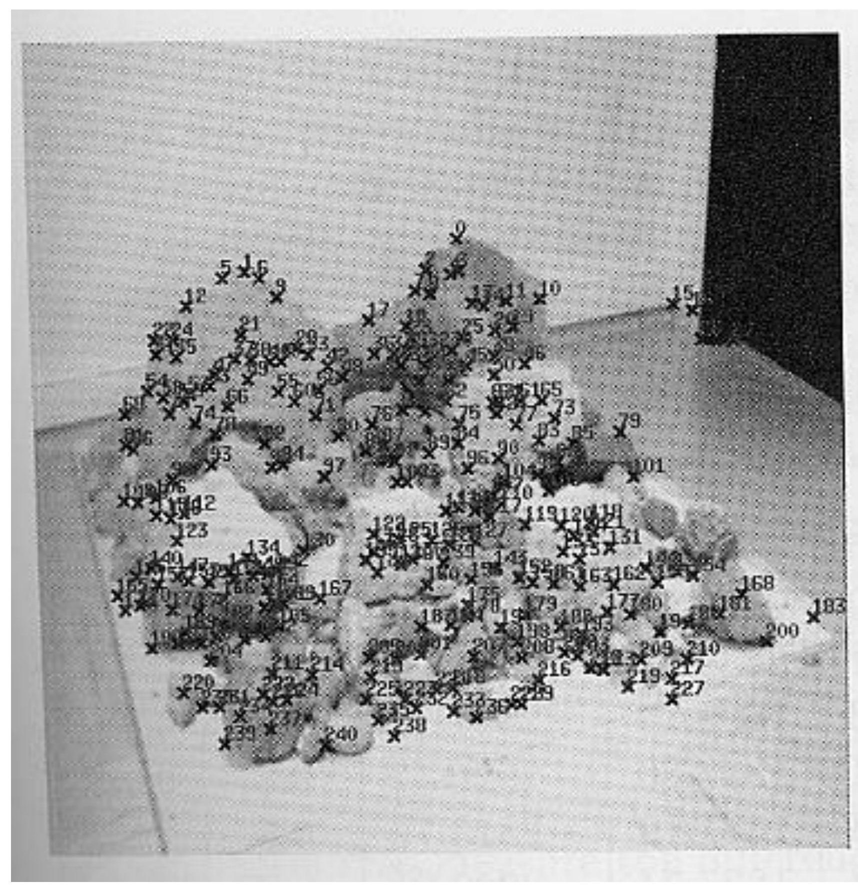
Transform image to ~[-1,1]x[-1,1]

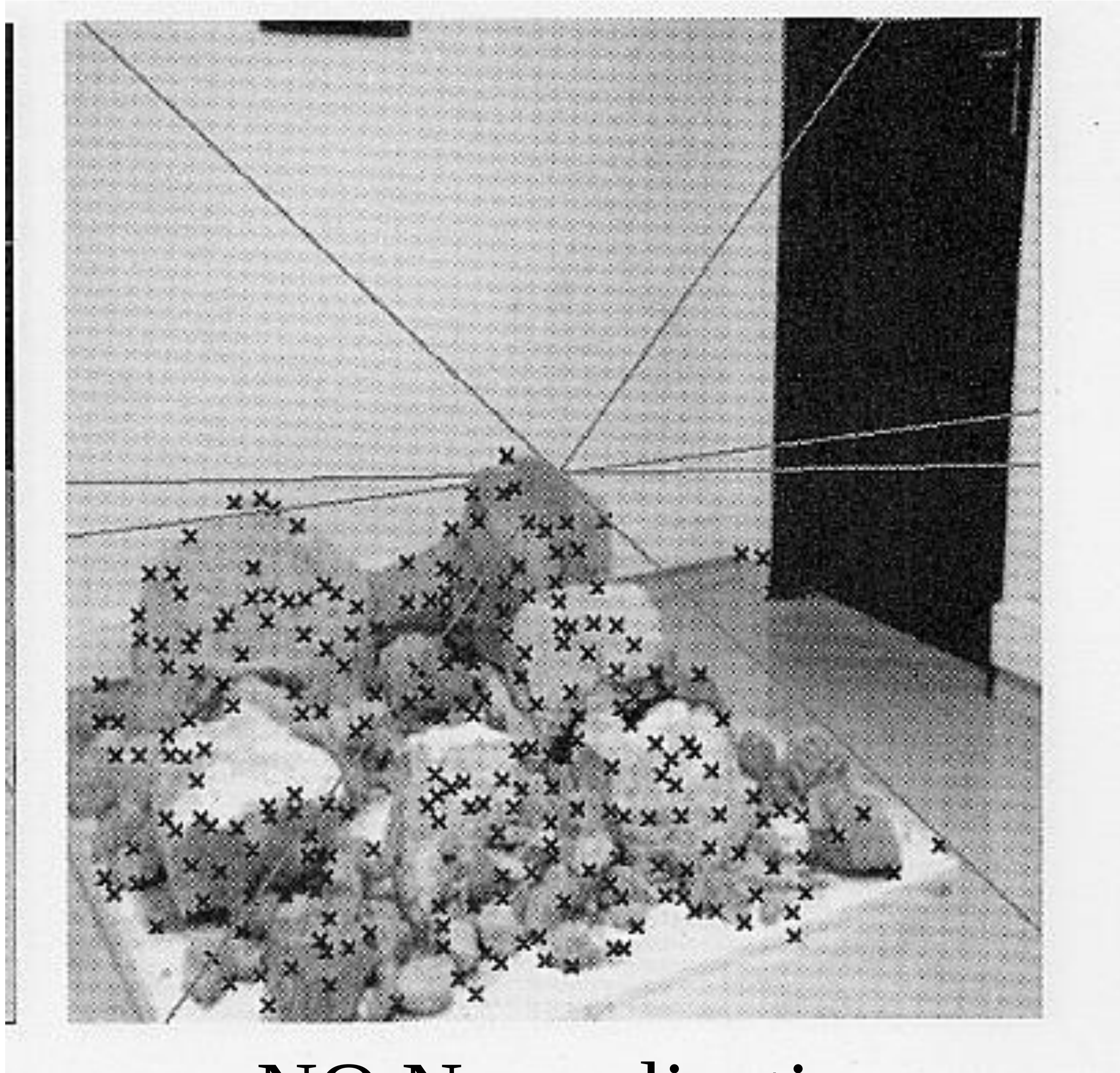



$$\begin{bmatrix} \frac{2}{700} & 0 & -1 \\ & \frac{2}{500} & -1 \\ & & 1 \end{bmatrix}$$

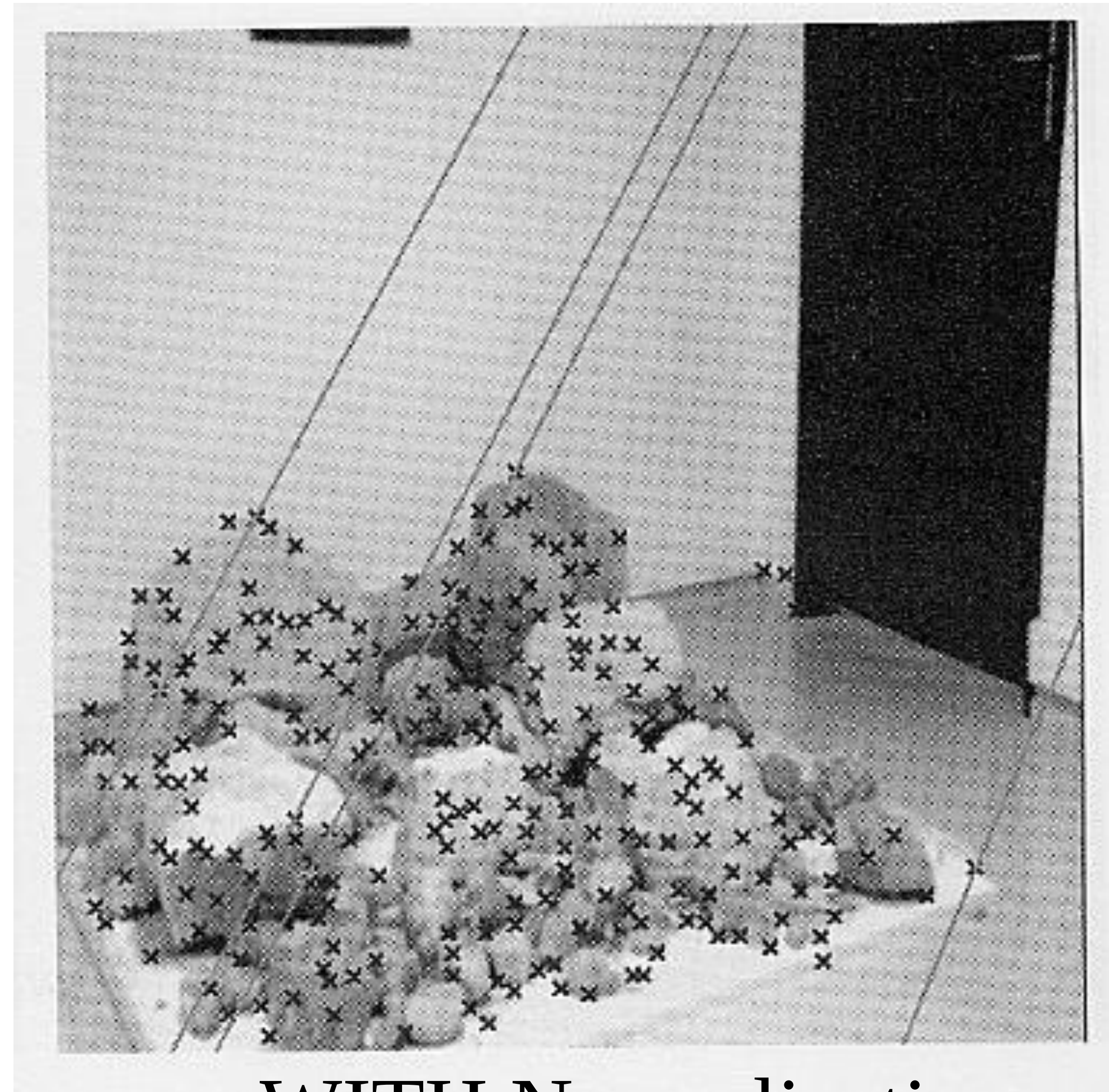
Least squares yields good results (Hartley, PAMI´97)







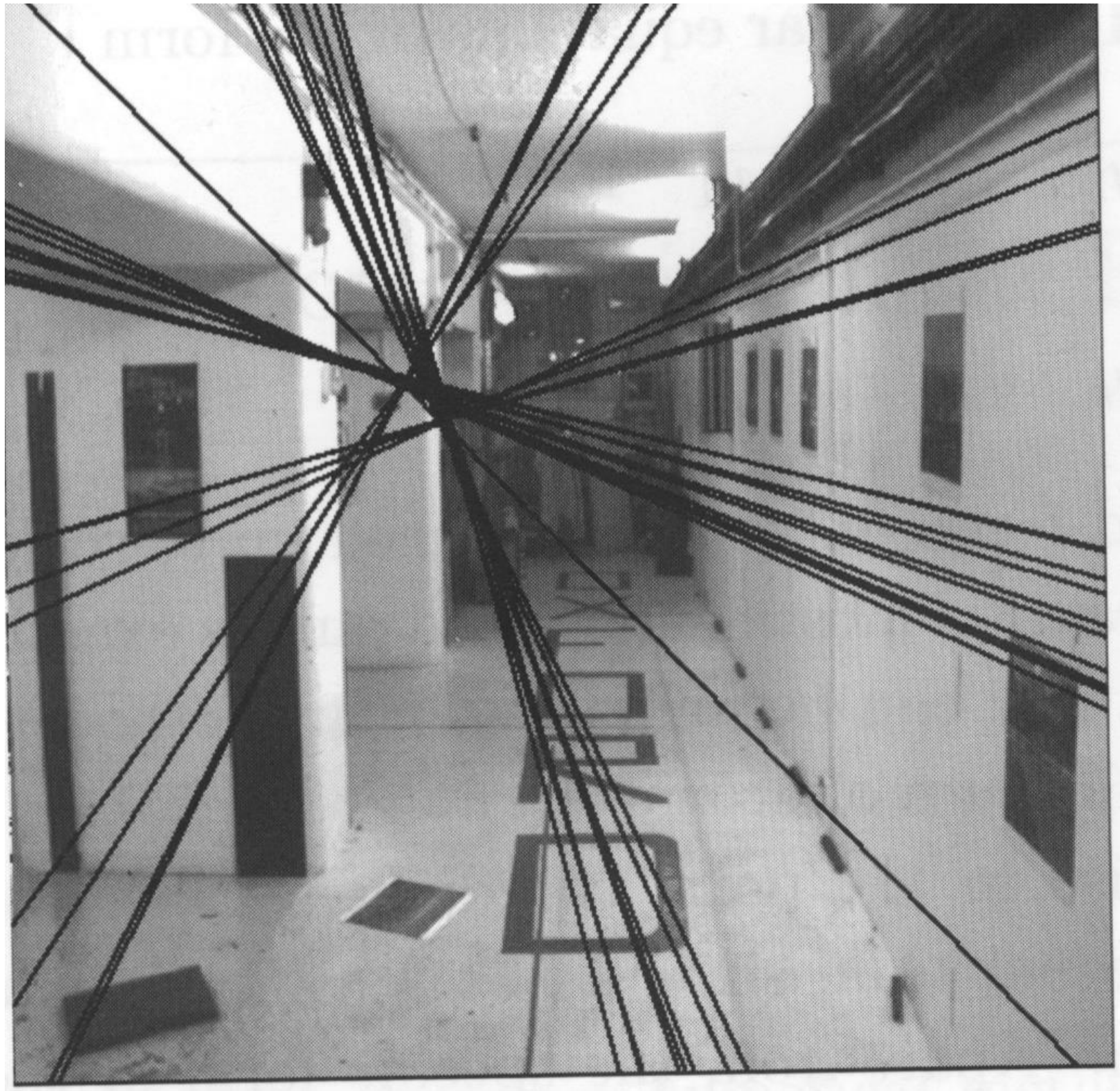
NO Normalization



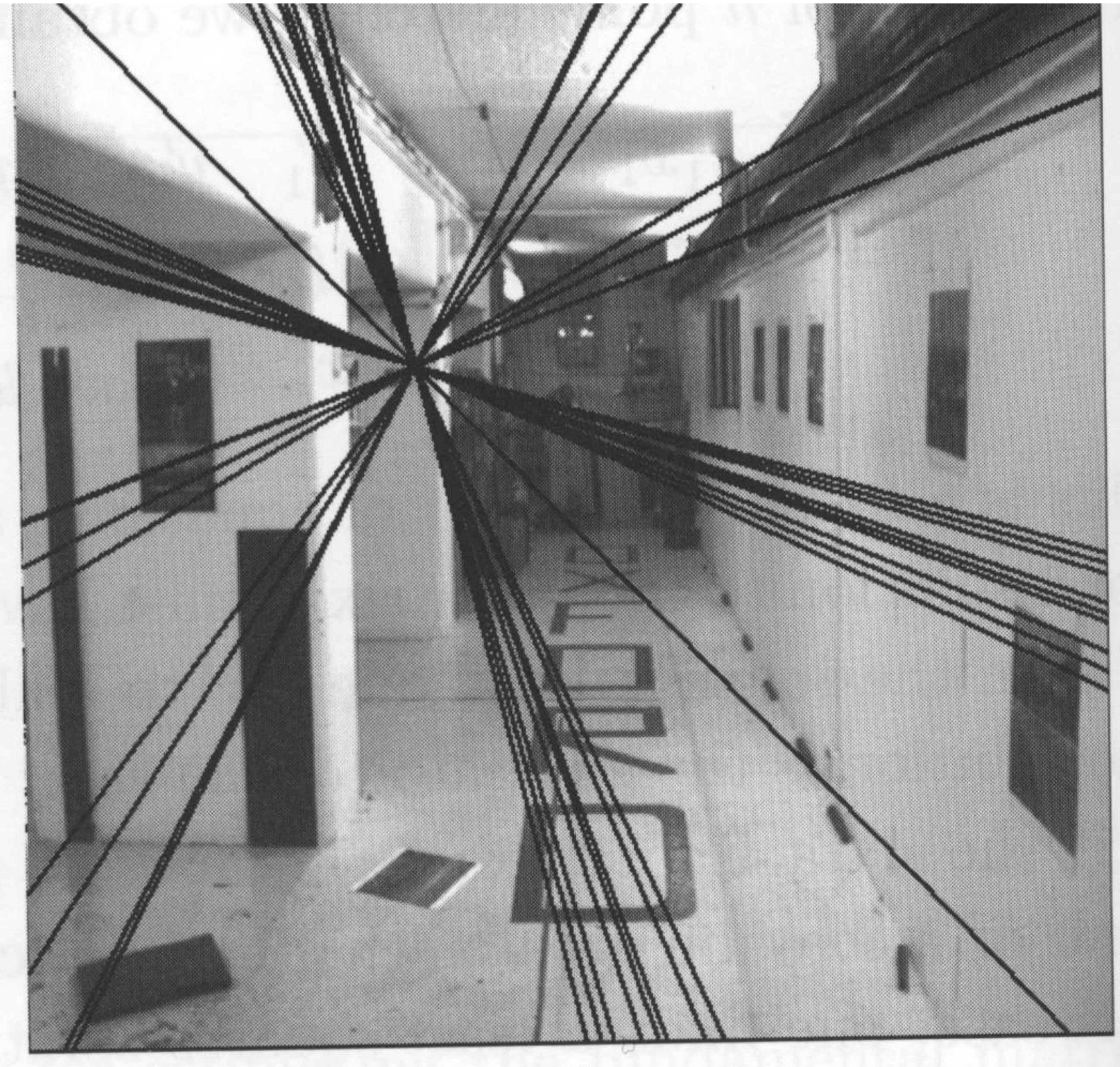
WITH Normalization

The minimization is numerically unstable → Normalize the coordinates to magnitude between 0 and 1

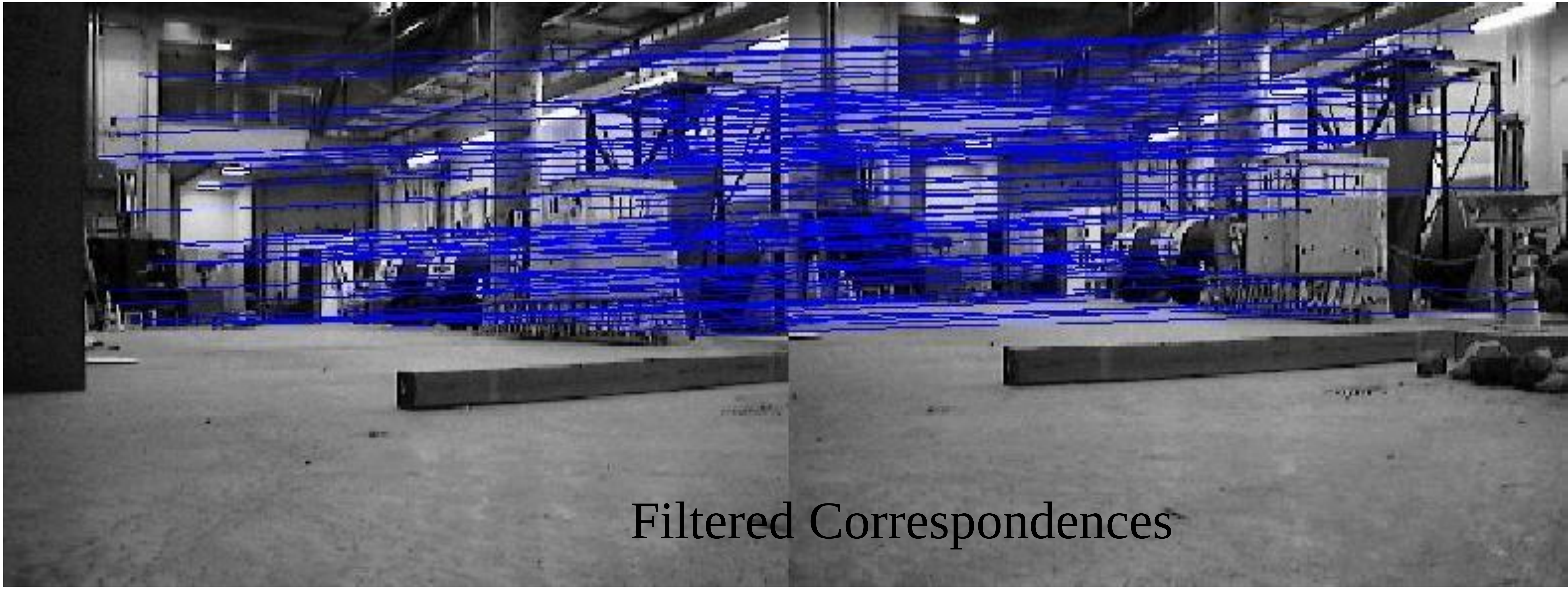
The importance of enforcing the constraint that **F** must be singular:



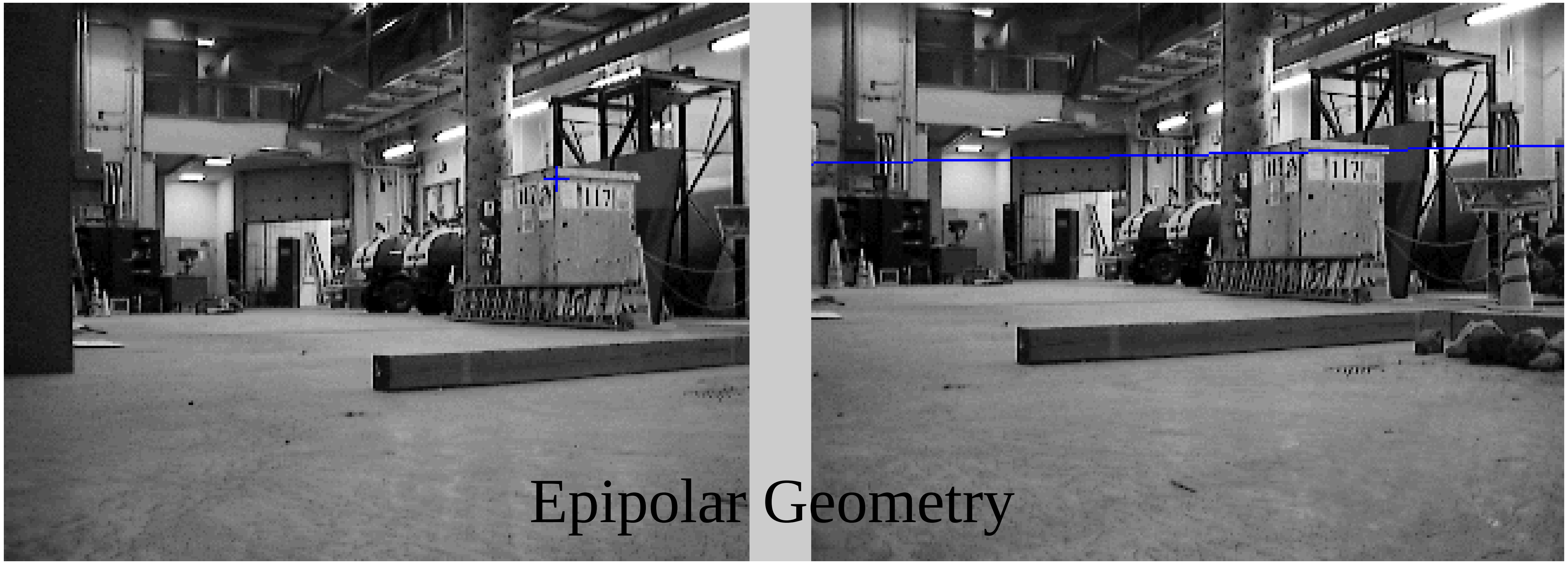
Incorrect **F** (non-singular matrix): The epipolar lines do not intersect



Correct **F**: All the epipolar lines Intersect at the epipole



Filtered Correspondences



Epipolar Geometry

# References

Basic reading:

- Szeliski textbook, Sections 7.1, 7.2, 11.1.
- Hartley and Zisserman, Chapters 9, 11, 12.