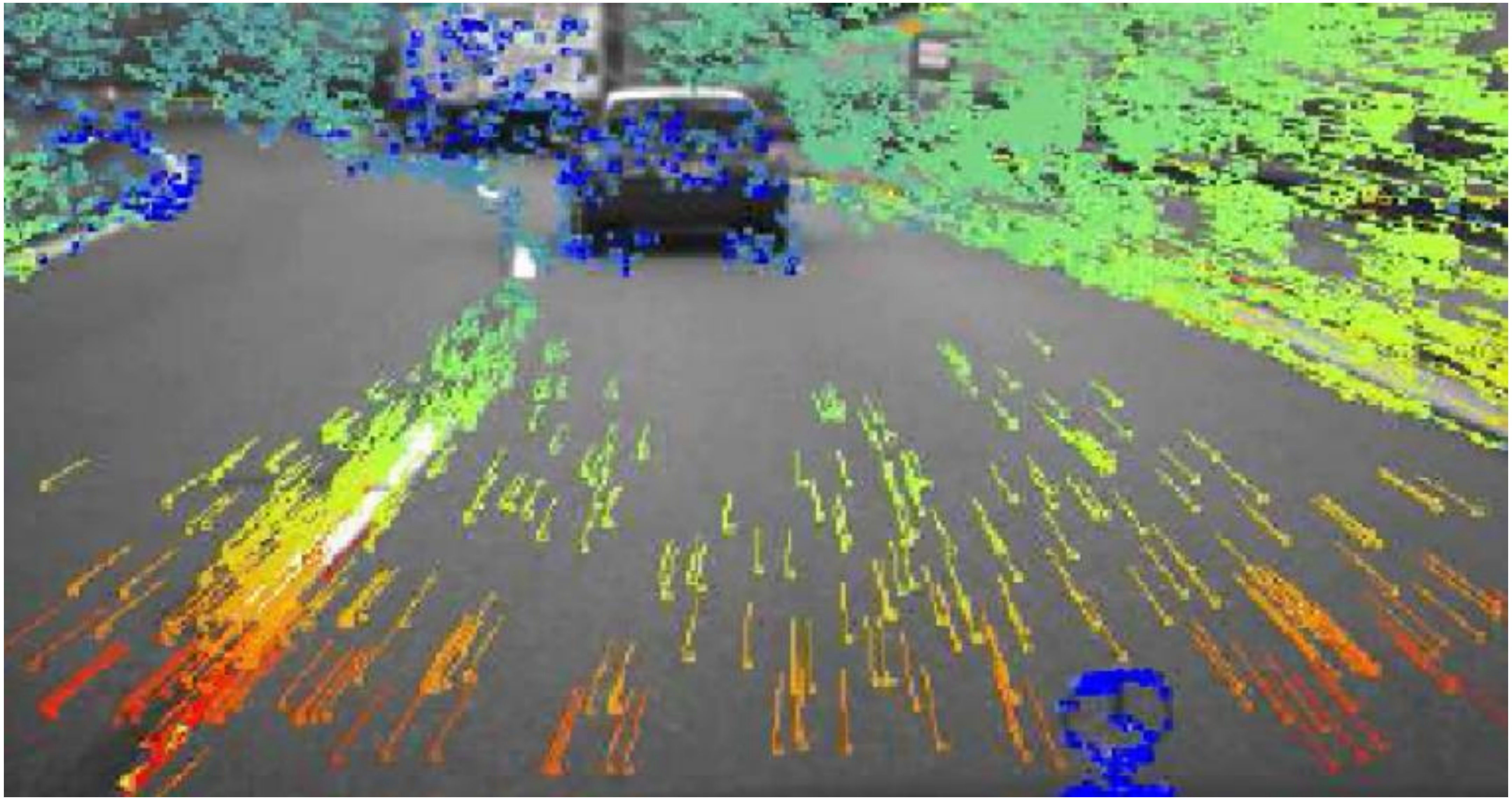
Optical flow



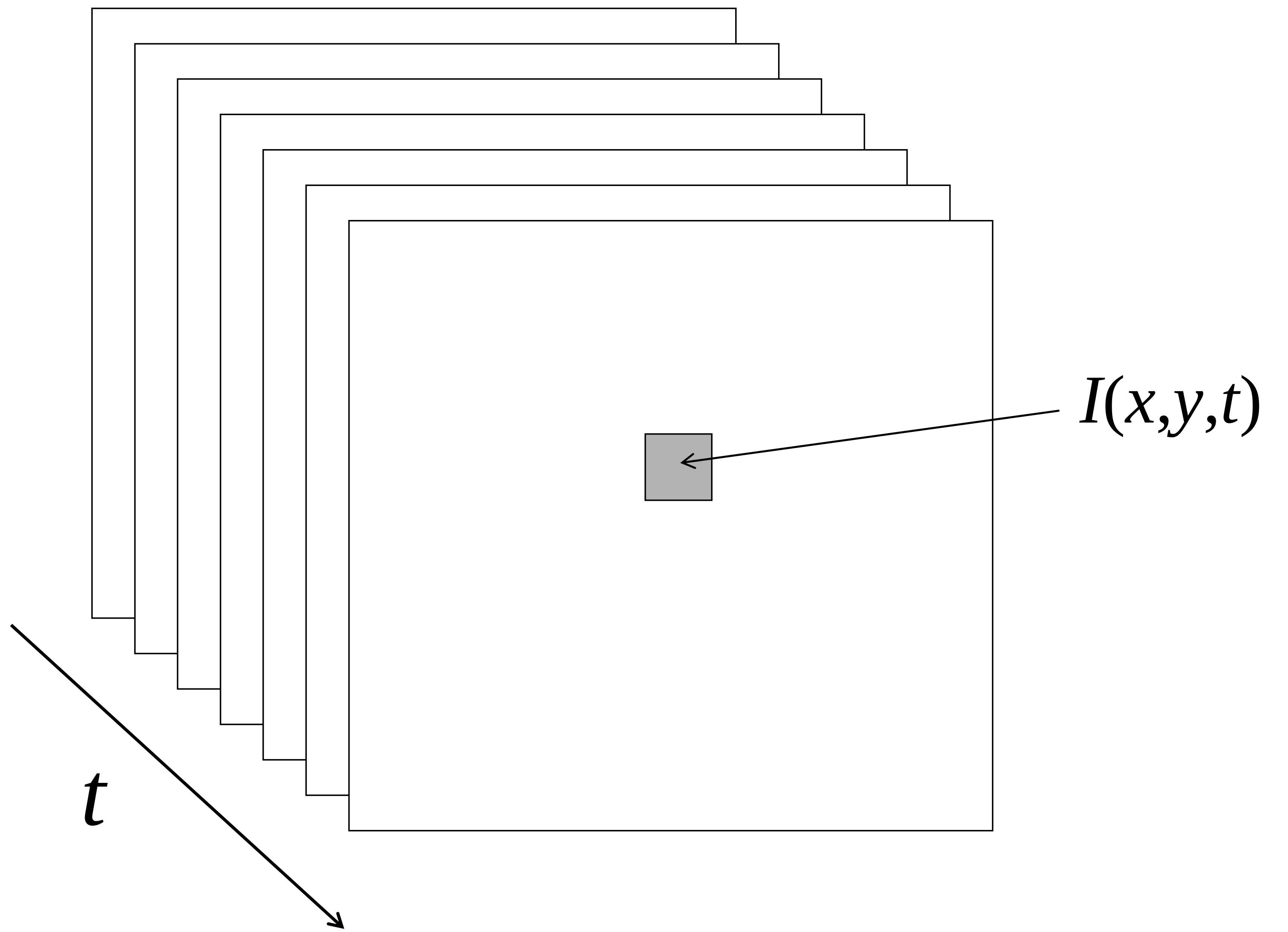
http://www.cs.cmu.edu/~16385/

16-385 Computer Vision
Spring 2019, Lecture 21

Overview of today's lecture

- Quick intro to vision for video.
- Optical flow.
- Constant flow.
- Horn-Schunck flow.

Computer vision for video

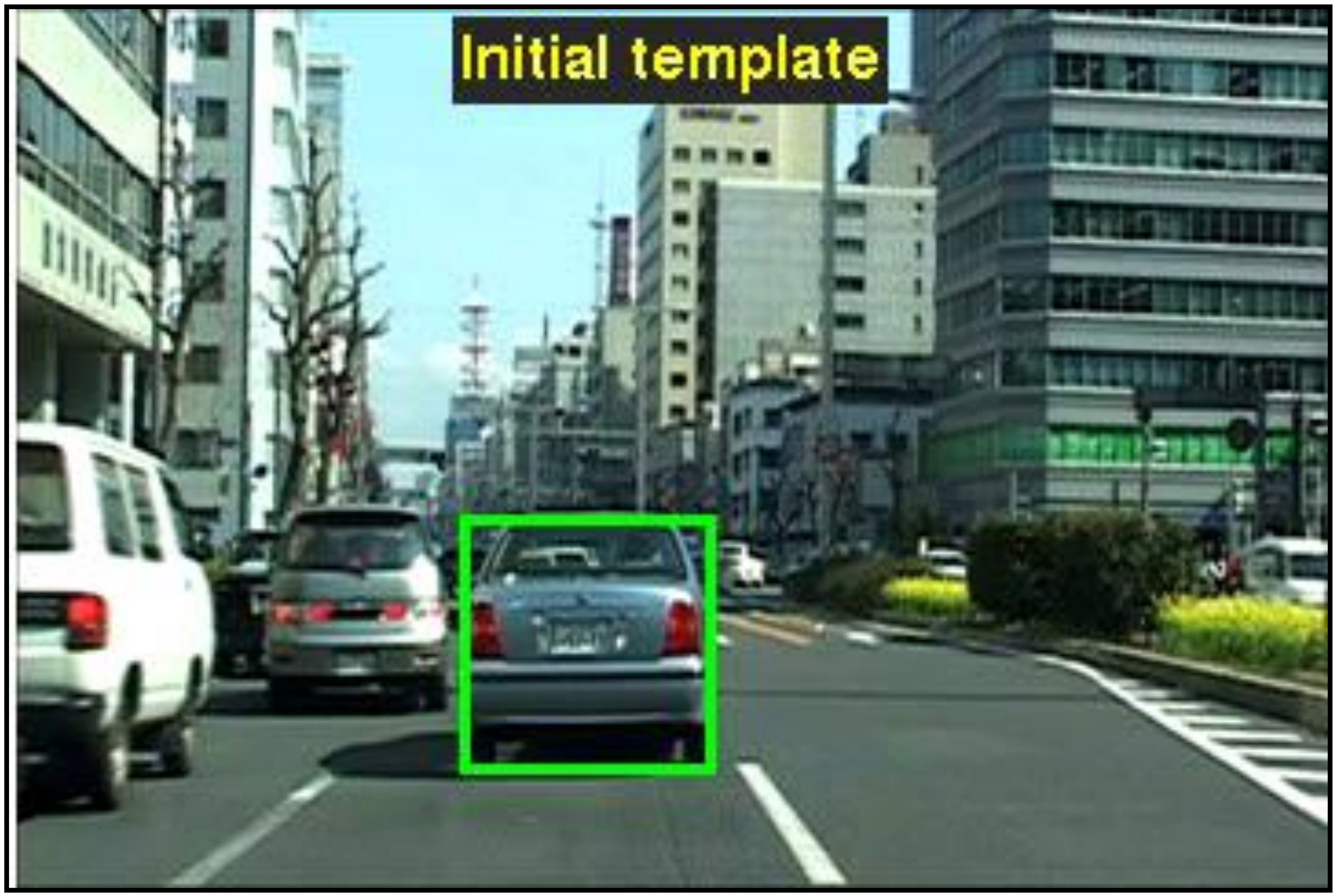


Optical Flow and Motion

- We are interested in finding the movement of scene objects from time-varying images (videos).

- Applications
  - Track object behavior
  - Correct for camera jitter (stabilization)
  - Align images (mosaics)
  - 3D shape reconstruction
  - Special effects

Tracking – Rigid Objects



(Simon Baker, CMU)

Tracking – Non-rigid Objects

(Comaniciu et al, Siemens)

Face Tracking

(Simon Baker et al, CMU)

Applications of Face Tracking

- User Interfaces:
  - Mouse Replacement: Head Pose and Gaze Estimation
  - Automotive: Windshield Displays, Smart Airbags, Driver Monitoring
- Face Recognition:
  - Pose Normalization
  - Model-Based Face Recognition
- Lipreading/Audio-Visual Speech Recognition
- Expression Recognition and Deception Detection
- Rendering and Animation:
  - Expression Animations and Transfer
  - Low-Bandwidth Video Conferencing
  - Audio-Visual Speech Synthesis

(Simon Baker, CMU)

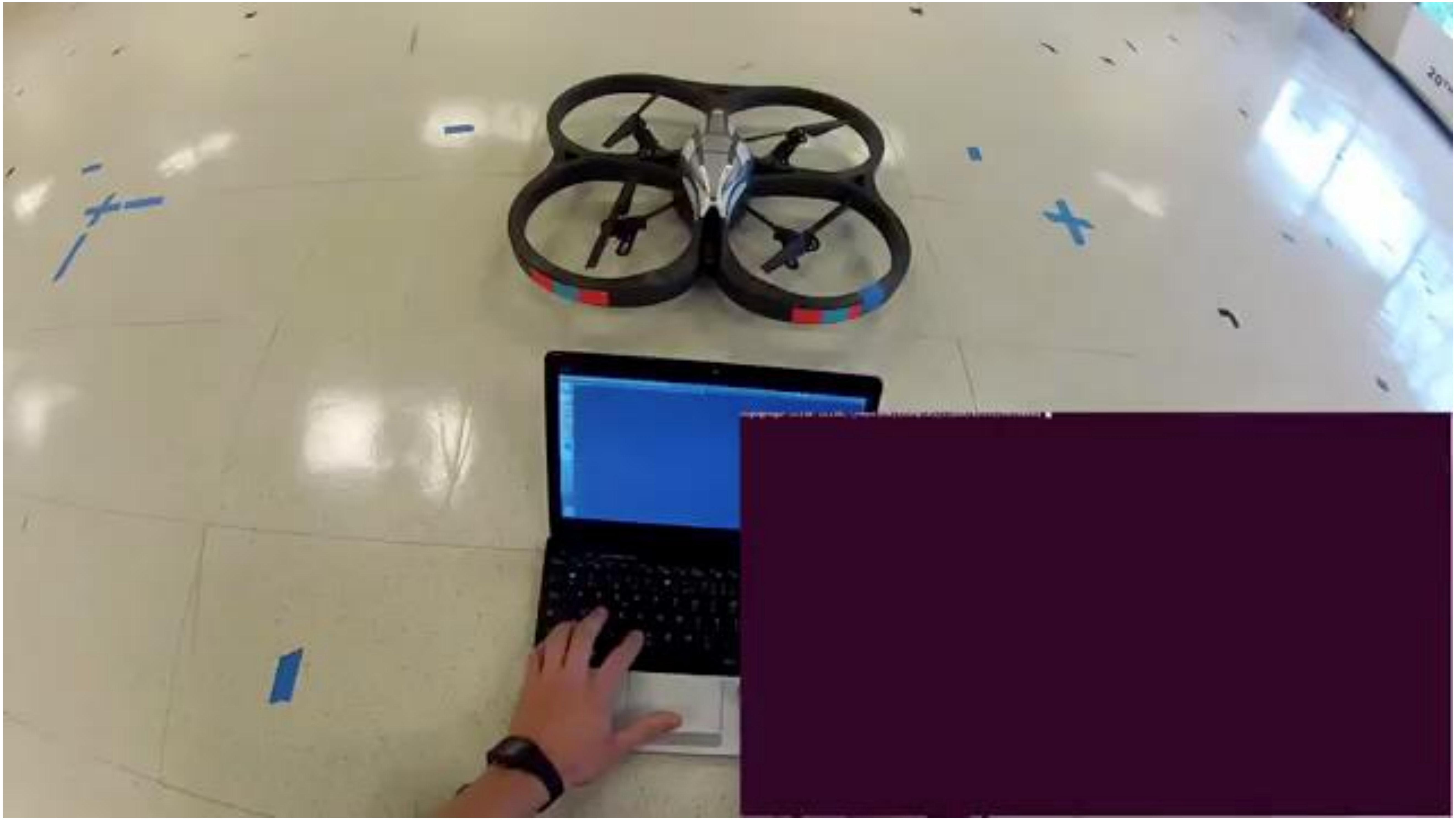
Behavior Analysis

Query

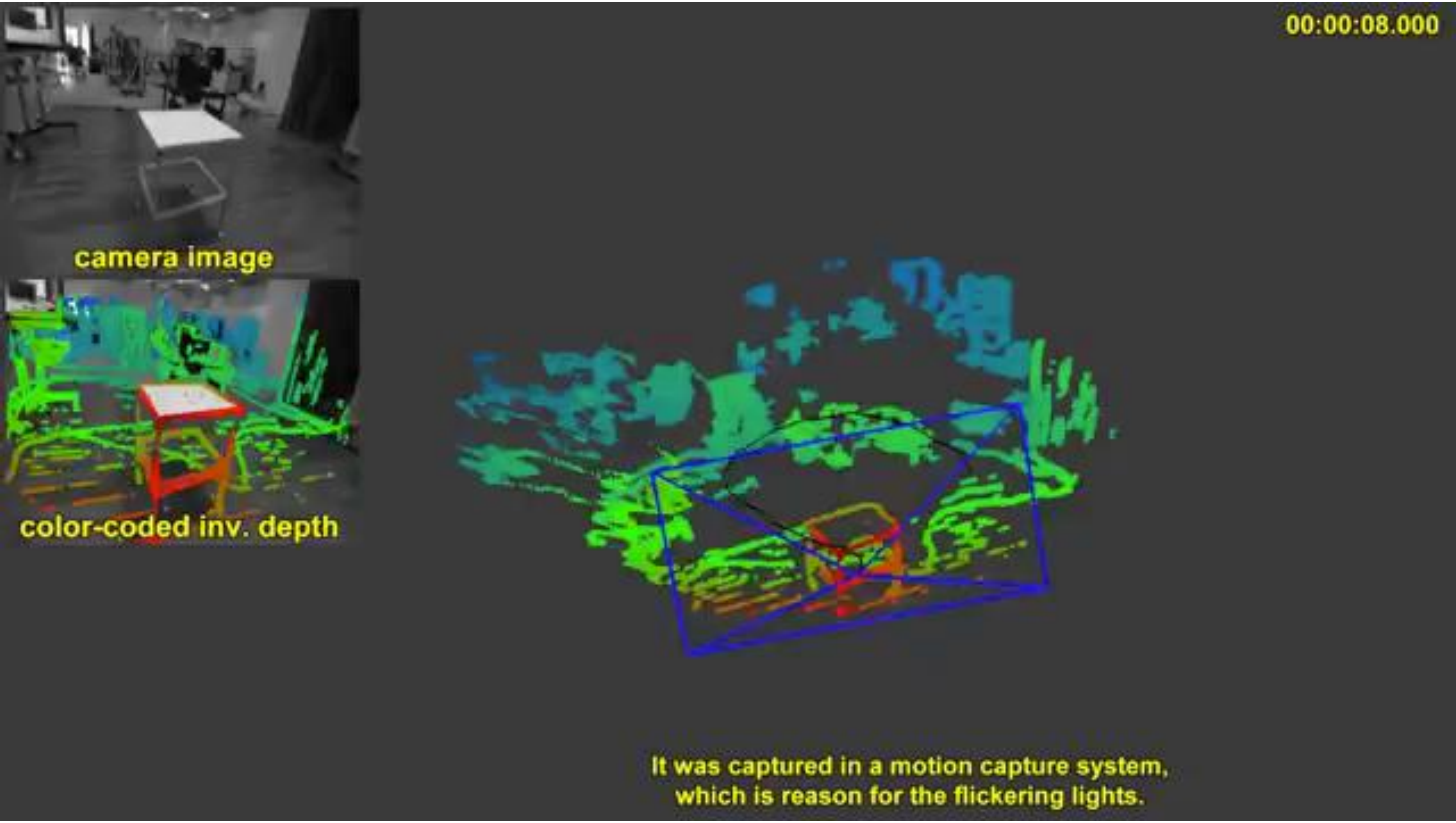
Result

(Michal Irani, Weizmann)

Optical flow used for feature tracking on a drone



optical flow used for motion estimation in visual odometry



Lucas-Kanade
(Forward additive)

Baker-Matthews
(Inverse Compositional)

Image Alignment

Optical flow

Optical Flow

Problem Definition

Given two consecutive image frames,
estimate the motion of each pixel

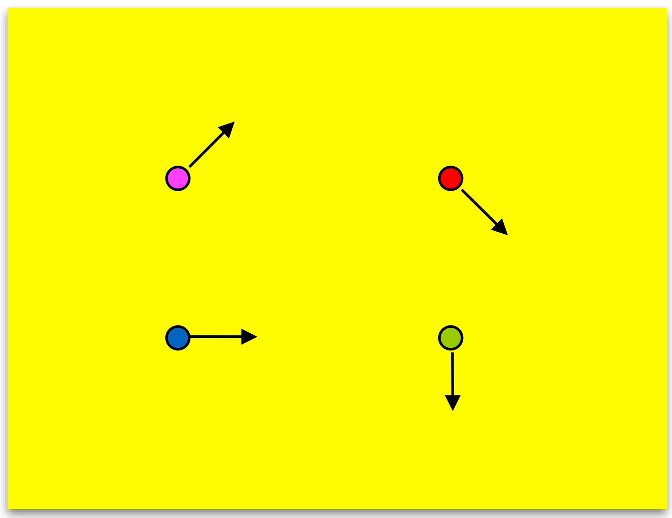
Assumptions

Brightness constancy

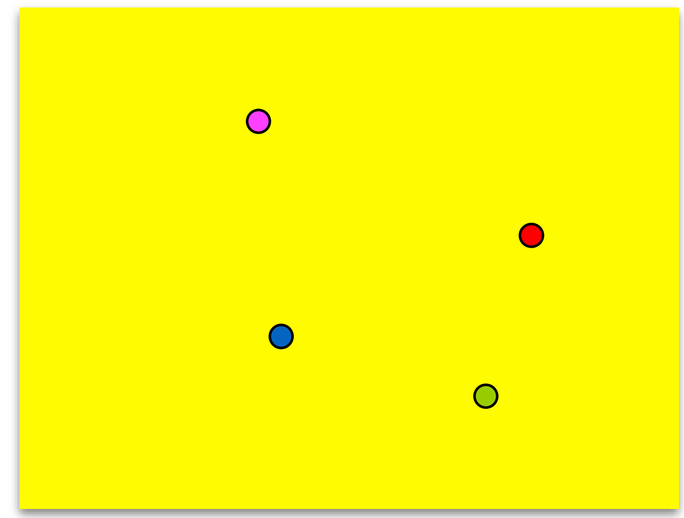
Small motion

Optical Flow

(Problem definition)



$I(x,y,t)$



$I(x,y,t')$

Estimate the motion (flow) between these two consecutive images

How is this different from estimating a 2D transform?

Key Assumptions

(unique to optical flow)

Color Constancy

(Brightness constancy for intensity images)

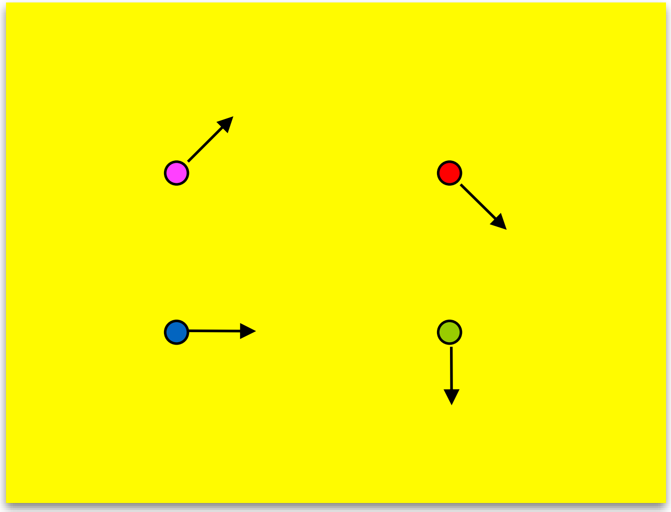
Implication: allows for pixel to pixel comparison
(not image features)

Small Motion

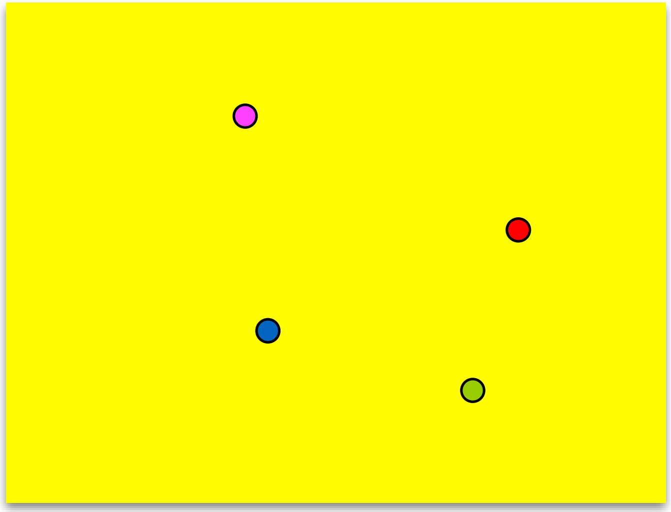
(pixels only move a little bit)

Implication: linearization of the brightness
constancy constraint

Approach



$I(x, y, t)$



$I(x, y, t')$

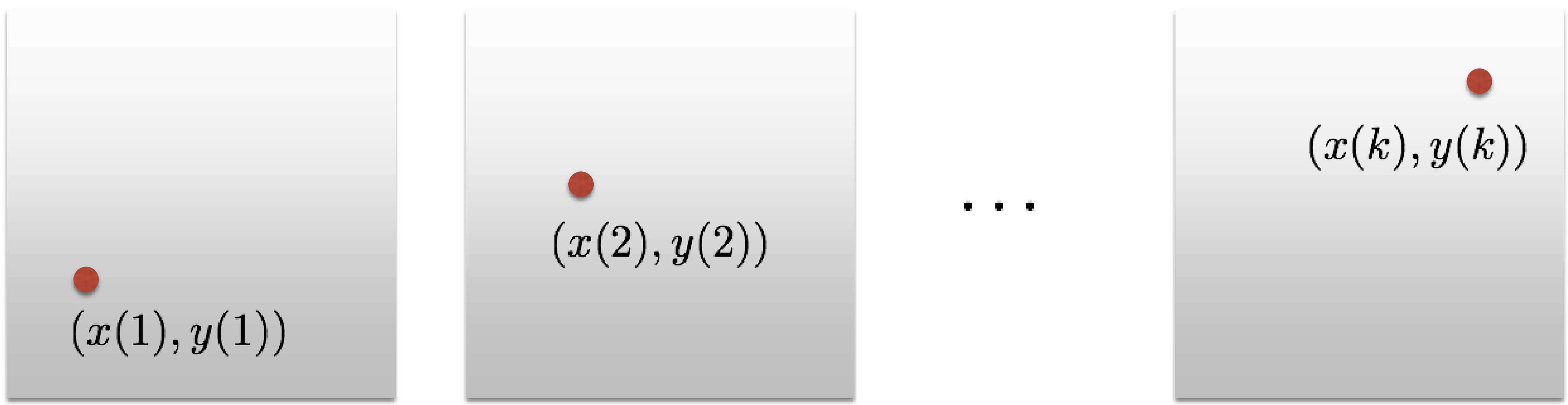
Look for nearby pixels with the same color

(small motion) (color constancy)

Assumption 1

Brightness constancy

Scene point moving through image sequence



Assumption 1

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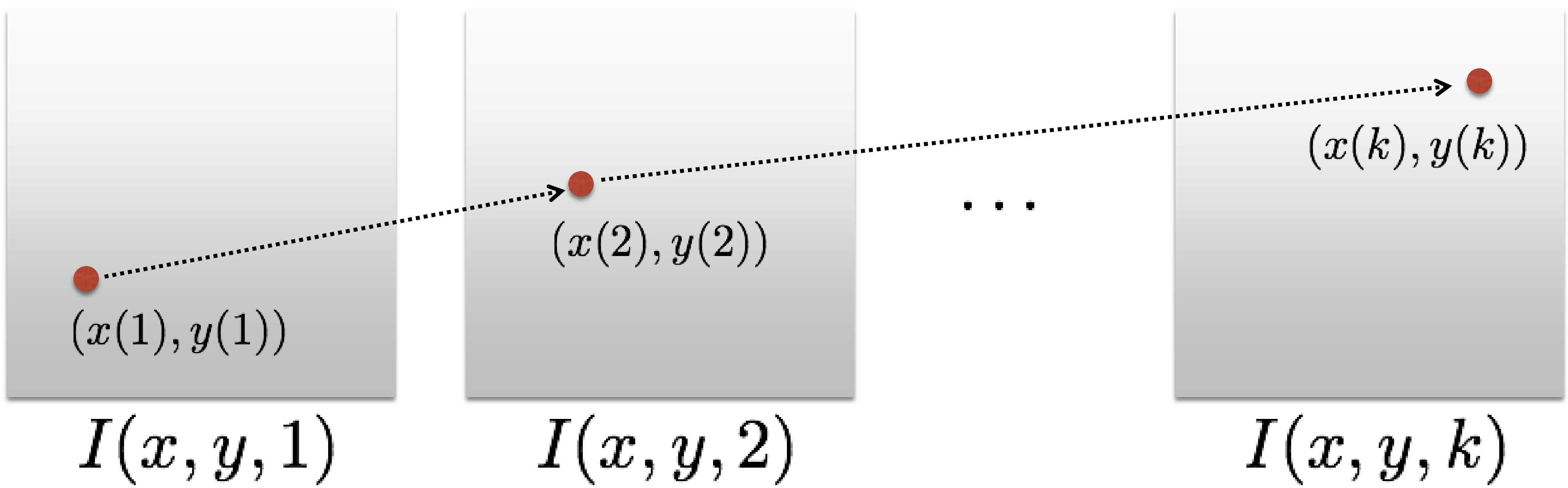
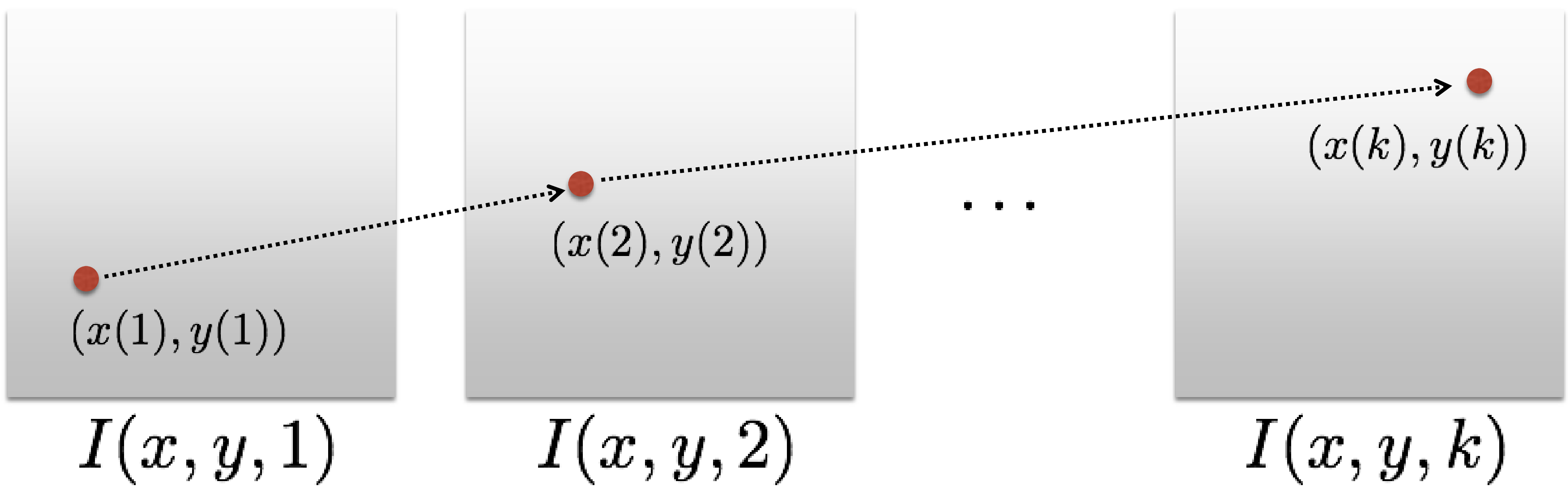


Assumption 1

Brightness constancy

Scene point moving through image sequence

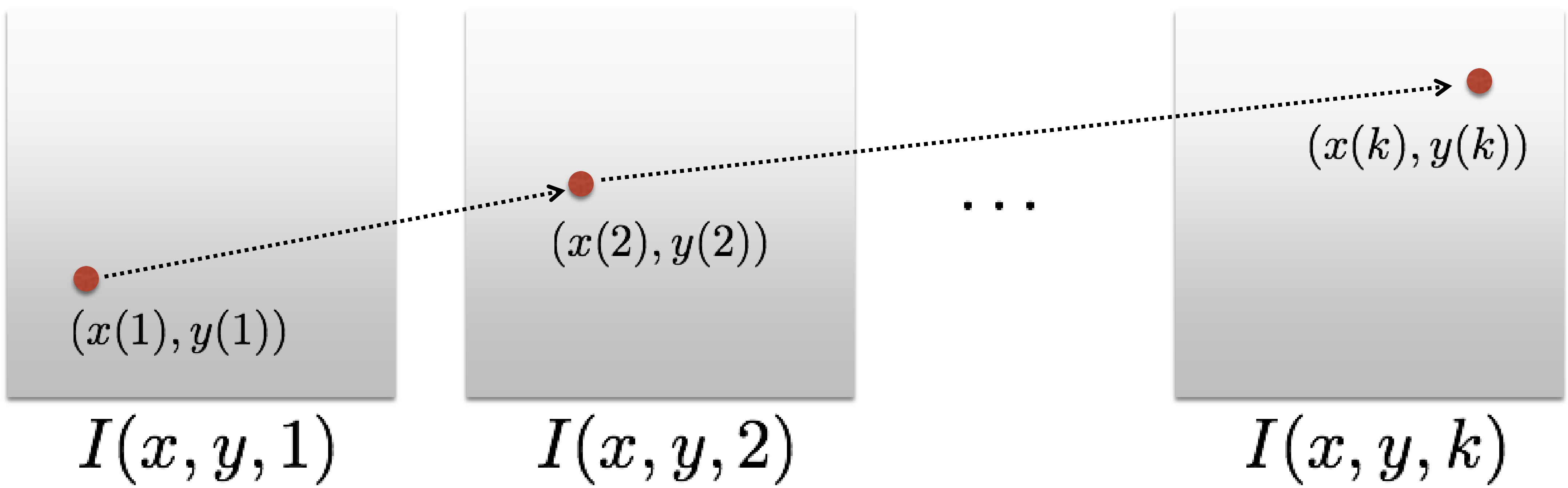


Assumption:Brightness of the point will remain the same

Assumption 1

Brightness constancy

Scene point moving through image sequence



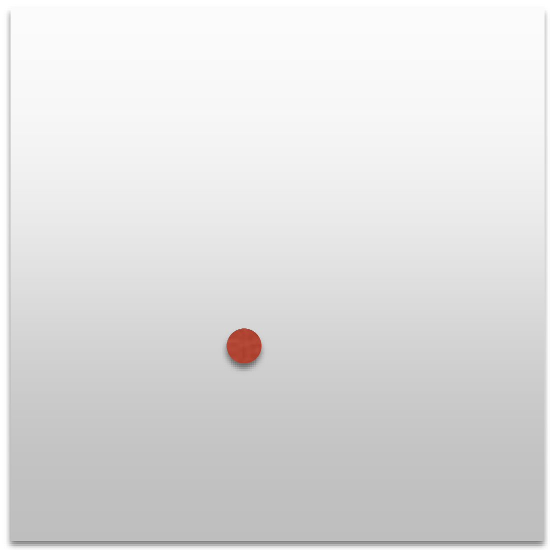
Assumption:Brightness of the point will remain the same

$$I(x(t), y(t), t) = C$$

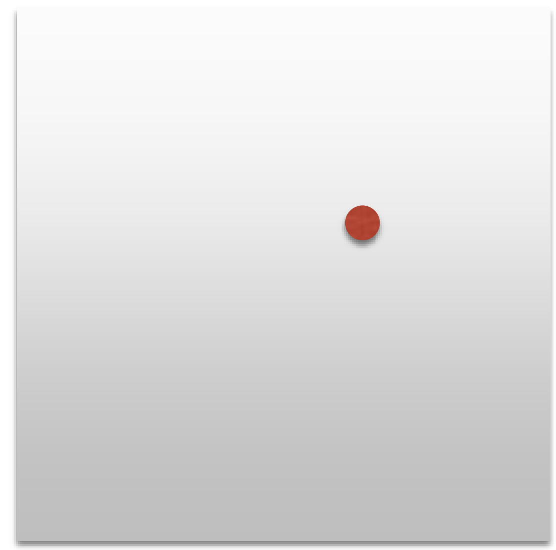
constant

Assumption 2

Small motion



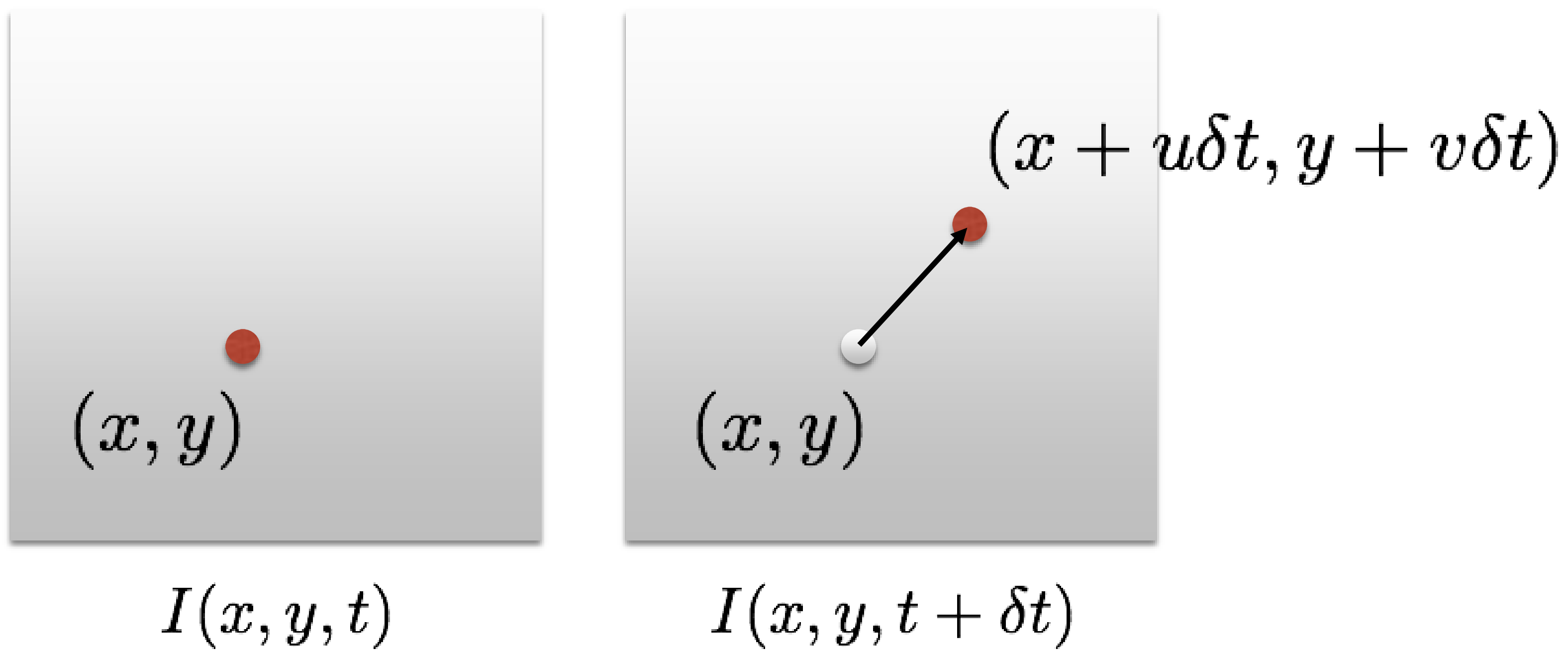
$I(x, y, t)$



$I(x, y, t + \delta t)$

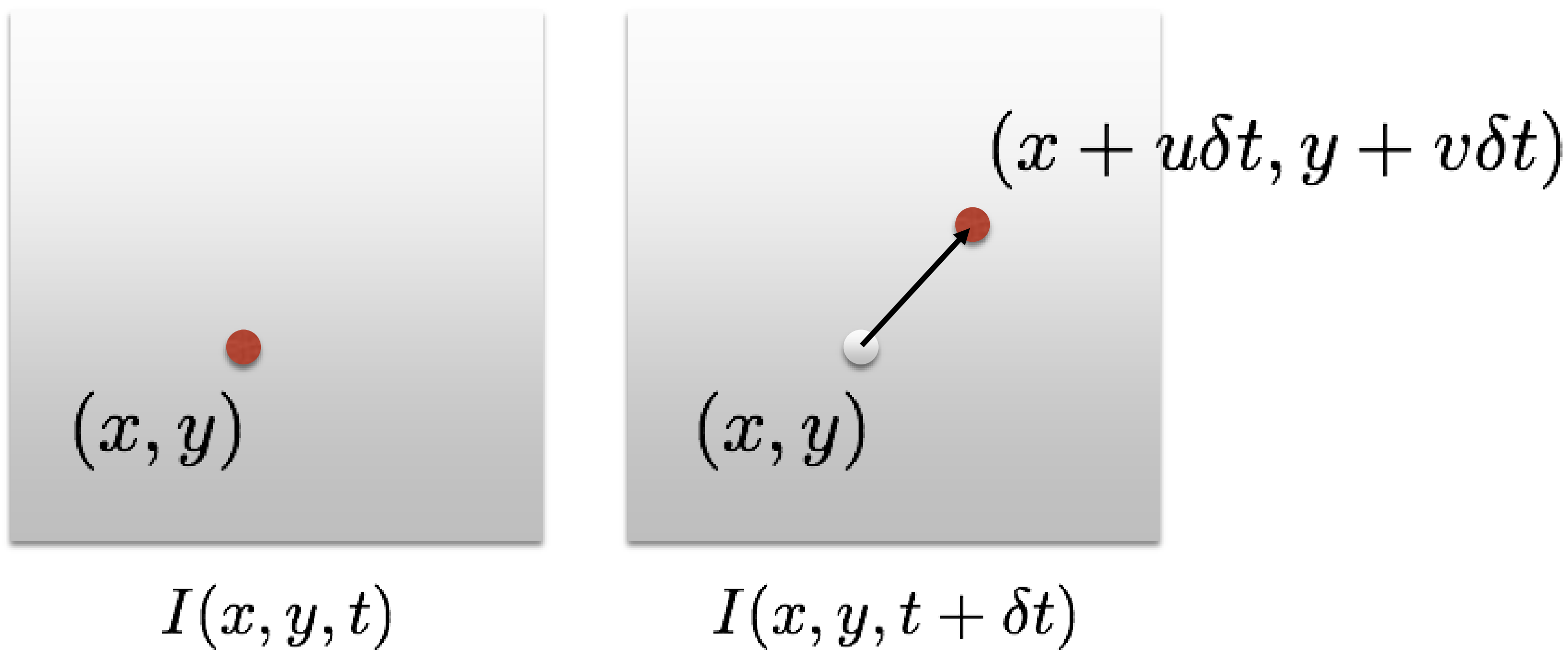
Assumption 2

Small motion



Assumption 2

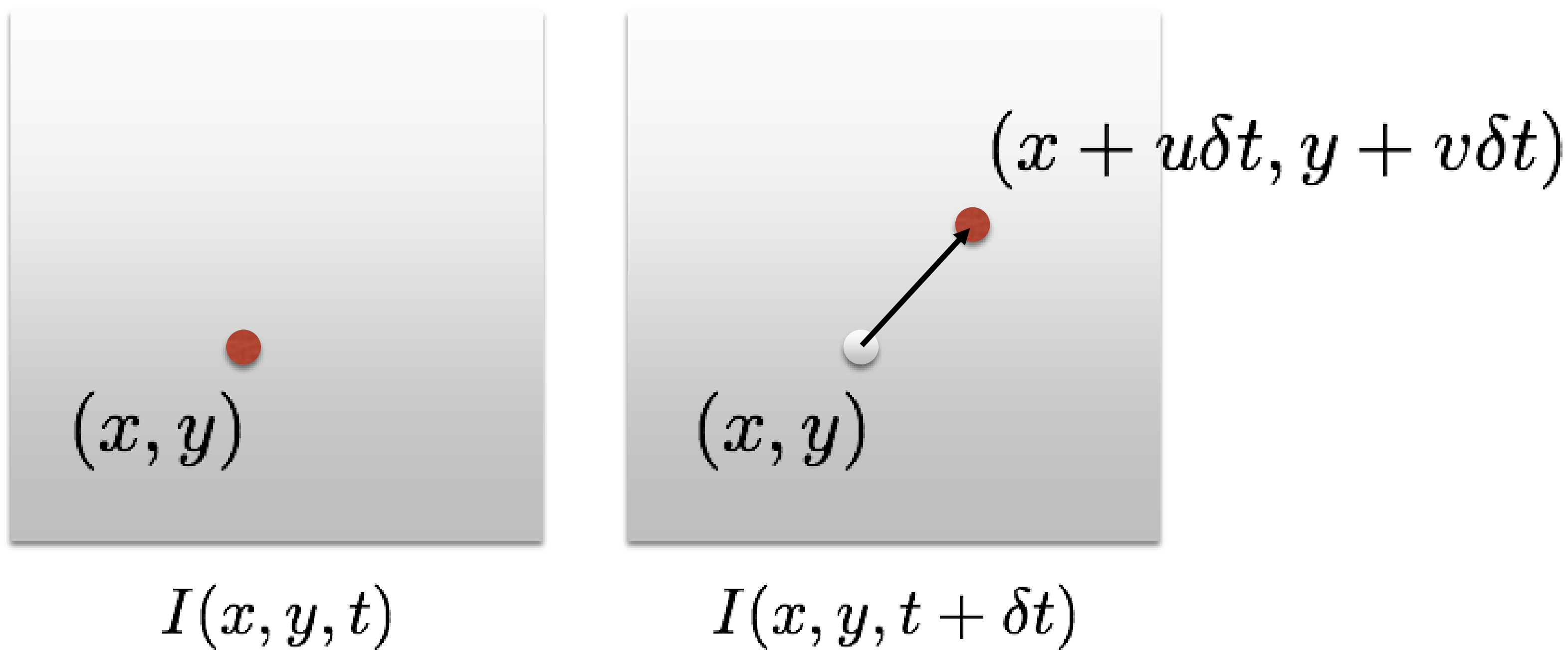
Small motion



Optical flow (velocities): (u, v) Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

Assumption 2

Small motion



Optical flow (velocities): (u, v) Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

For a ***really small space-time step***…

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

… the brightness between two consecutive image frames is the same

These assumptions yield the …

Brightness Constancy Equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative

partial derivative

Equation is not obvious. Where does this come from?

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

Insight:

If the time step is really small,

we can *linearize* the intensity function

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

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$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

partial derivative

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

fixed point

assuming small motion

cancel terms

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

$$I(x,y,t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x,y,t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

cancel terms

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

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(First order approximation, two variables)

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assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by δt

take limit $\delta t \to 0$

$$I(x+u\delta t, y+v\delta t, t+\delta t) = I(x,y,t)$$

Multivariable Taylor Series Expansion
(First order approximation, two variables)

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$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$$I(x+u\delta t, y+v\delta t, t+\delta t) = I(x,y,t)$$

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$$I(x,y,t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x,y,t)$$

assuming small motion

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take limit $\delta t \to 0$

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Brightness Constancy Equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Brightness Constancy Equation

$$I_x u + I_y v + I_t = 0$$

(x-flow) (y-flow)

shorthand notation

$$\nabla I^\top \boldsymbol{v} + I_t = 0$$

(1 x 2) (2 x 1)

vector form

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

temporal gradient

How do you compute these terms?

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

Forward difference
Sobel filter
Scharr filter
…

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

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spatial derivative

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

frame differencing

Frame differencing

t

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

-

$t+1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

=

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	9	9	9	9
0	9	0	0	0
0	9	0	0	0
0	9	0	0	0

(example of a forward difference)

Example:

t

3 x 3 patch

1	1	1	1	1
1	1	1	1	1
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**

$t + 1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	**10**	**10**	**10**
1	1	**10**	**10**	**10**
1	1	**10**	**10**	**10**

$$I_x = \frac{\partial I}{\partial x}$$

-	0	0	0	-
-	0	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-

-1 0 1

$$I_y = \frac{\partial I}{\partial y}$$

-	-	-	-	-
0	0	0	0	0
0	**9**	**9**	**9**	**9**
-	0	0	0	0
0	0	0	0	0
-	-	-	-	-

-1
0
1

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	**9**	**9**	**9**	**9**
0	**9**	0	0	0
0	**9**	0	0	0
0	**9**	0	0	0

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

How do you compute this?

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$	$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$	$I_t = \frac{\partial I}{\partial t}$
spatial derivative	**optical flow**	**temporal derivative**
Forward difference Sobel filter Scharr filter …	**We need to solve for this!** (this is the unknown in the optical flow problem)	frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

spatial derivative	optical flow	temporal derivative
$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$	$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$	$I_t = \frac{\partial I}{\partial t}$

Forward difference
Sobel filter
Scharr filter
…

(u, v)

Solution lies on a line

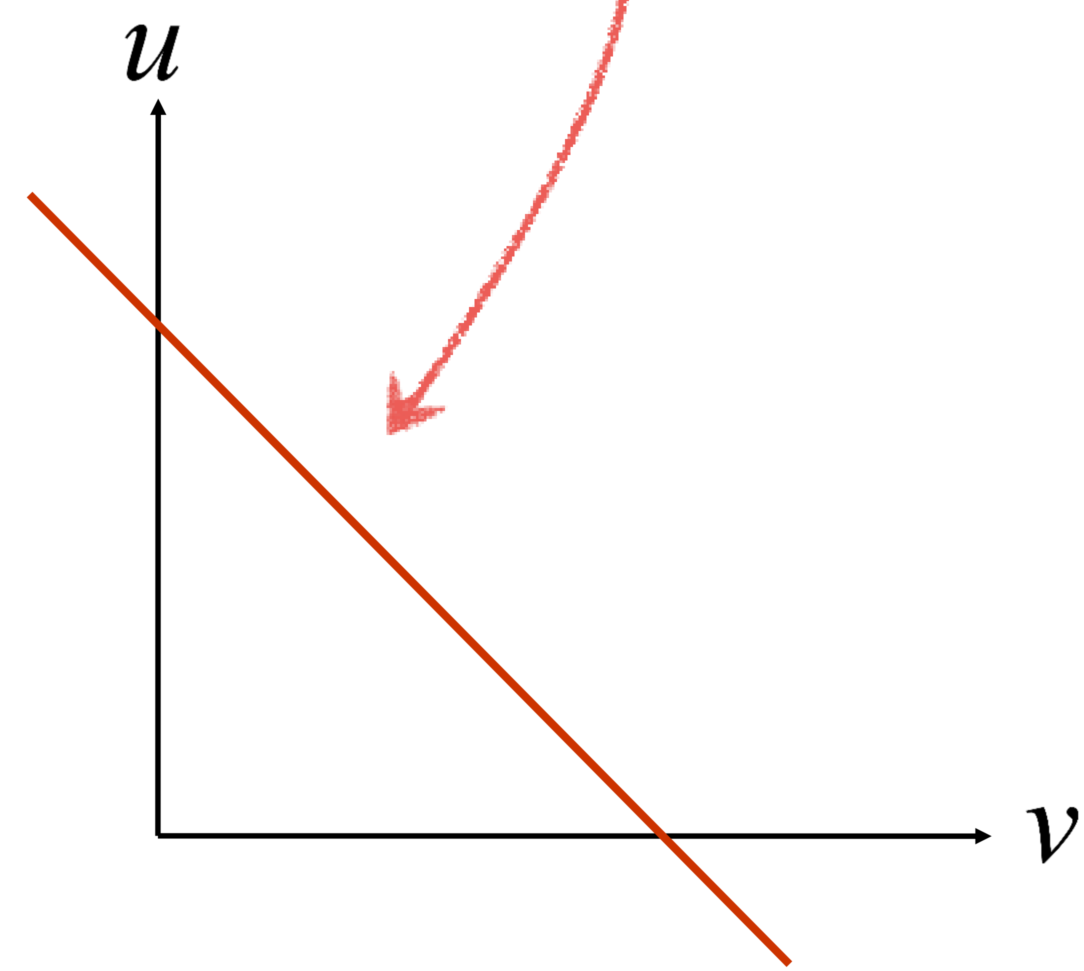
Cannot be found uniquely
with a single constraint

frame differencing

Solution lies on a straight line

$$\boxed{I_x u + I_y v + I_t = 0}$$

many combinations of u and v will satisfy the equality



The solution cannot be determined uniquely with a single constraint (a single pixel)

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$\boxed{u = \frac{dx}{dt} \quad v = \frac{dy}{dt}}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

How can we use the brightness constancy equation to estimate the optical flow?

unknown

$$I_x u + I_y v + I_t = 0$$

known

We need at least ____ equations to solve for 2 unknowns.

unknown

$$I_x u + I_y v + I_t = 0$$

known

Where do we get more equations (constraints)?

Horn-Schunck Optical Flow (1981)

brightness constancy

small motion

'smooth' flow

(flow can vary from pixel to pixel)

global method
(dense)

Lucas-Kanade Optical Flow (1981)

method of differences

'constant' flow

(flow is constant for all pixels)

local method
(sparse)

Constant flow

Where do we get more equations (constraints)?

$$I_x u + I_y v + I_t = 0$$

Assume that the surrounding patch (say 5x5) has
'**constant flow**'

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us equations

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$
\begin{gathered}
I_x(\boldsymbol{p}_1)u + I_y(\boldsymbol{p}_1)v = -I_t(\boldsymbol{p}_1) \\
I_x(\boldsymbol{p}_2)u + I_y(\boldsymbol{p}_2)v = -I_t(\boldsymbol{p}_2) \\
\vdots \\
I_x(\boldsymbol{p}_{25})u + I_y(\boldsymbol{p}_{25})v = -I_t(\boldsymbol{p}_{25})
\end{gathered}
$$

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}$$

Matrix form

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}$$

$$\underset{25 \times 2}{A} \qquad \underset{2 \times 1}{x} \qquad \underset{25 \times 1}{b}$$

How many equations? How many unknowns? How do we solve this?

Least squares approximation

$\hat{x} = \arg\min_{x} ||Ax - b||^2$ is equivalent to solving $A^\top A\hat{x} = A^\top b$

Least squares approximation

$$\hat{x} = \arg\min_x ||Ax - b||^2$$ is equivalent to solving $$A^\top A\hat{x} = A^\top b$$

To obtain the least squares solution solve:

$$\begin{array}{ccc} A^\top A & \hat{x} & A^\top b \end{array}$$

$$\left[\begin{array}{cc} \sum\limits_{p\in P} I_x I_x & \sum\limits_{p\in P} I_x I_y \\ \sum\limits_{p\in P} I_y I_x & \sum\limits_{p\in P} I_y I_y \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right] = - \left[\begin{array}{c} \sum\limits_{p\in P} I_x I_t \\ \sum\limits_{p\in P} I_y I_t \end{array}\right]$$

where the summation is over each pixel $\boldsymbol{p}$ in patch $\boldsymbol{P}$

$$x = (A^\top A)^{-1} A^\top b$$

Least squares approximation

$$\hat{x} = \arg\min_x ||Ax - b||^2$$ is equivalent to solving $$A^\top A\hat{x} = A^\top b$$

To obtain the least squares solution solve:

$$\begin{array}{ccc} A^\top A & \hat{x} & A^\top b \\ \left[\begin{array}{cc} \sum_{p\in P} I_x I_x & \sum_{p\in P} I_x I_y \\ \sum_{p\in P} I_y I_x & \sum_{p\in P} I_y I_y \end{array}\right] & \left[\begin{array}{c} u \\ v \end{array}\right] = - & \left[\begin{array}{c} \sum_{p\in P} I_x I_t \\ \sum_{p\in P} I_y I_t \end{array}\right] \end{array}$$

where the summation is over each pixel $\boldsymbol{p}$ in patch $\boldsymbol{P}$

Sometimes called 'Lucas-Kanade Optical Flow'
(can be interpreted to be a special case of the LK method with a translational warp model)

When is this solvable?

$$A^\top A\hat{x} = A^\top b$$

$A^\mathsf{T}A$ should be invertible

$A^\mathsf{T}A$ should not be too small

λ_1 and λ_2 should not be too small

$A^\mathsf{T}A$ should be well conditioned

λ_1/λ_2 should not be too large (λ_1=larger eigenvalue)

Where have you seen this before?

$$A^\top A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Where have you seen this before?

$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!

Implications

- Corners are when $\lambda 1$, $\lambda 2$ are big; this is also when Lucas-Kanade optical flow works best

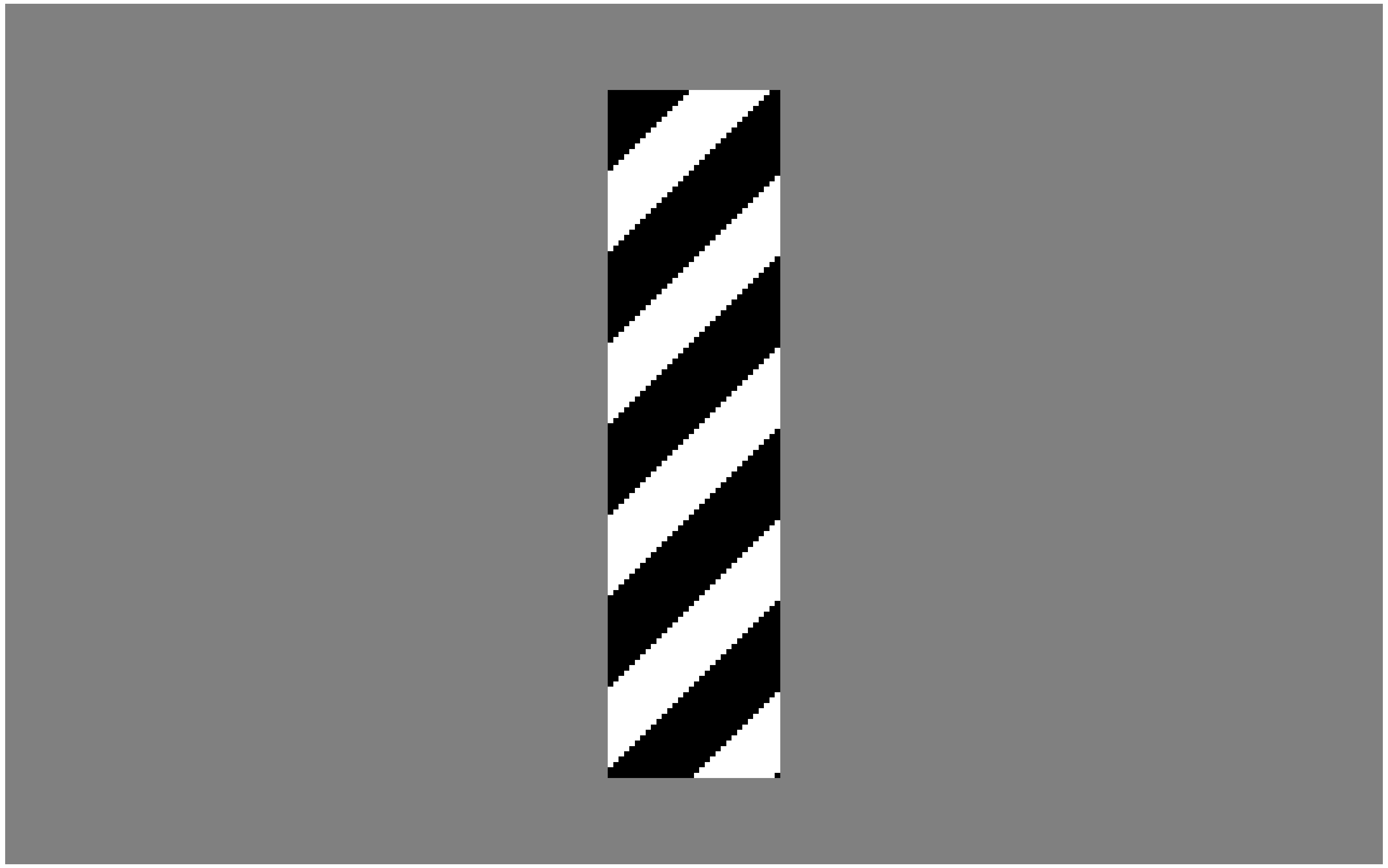
- Corners are regions with two different directions of gradient (at least)

- Corners are good places to compute flow!

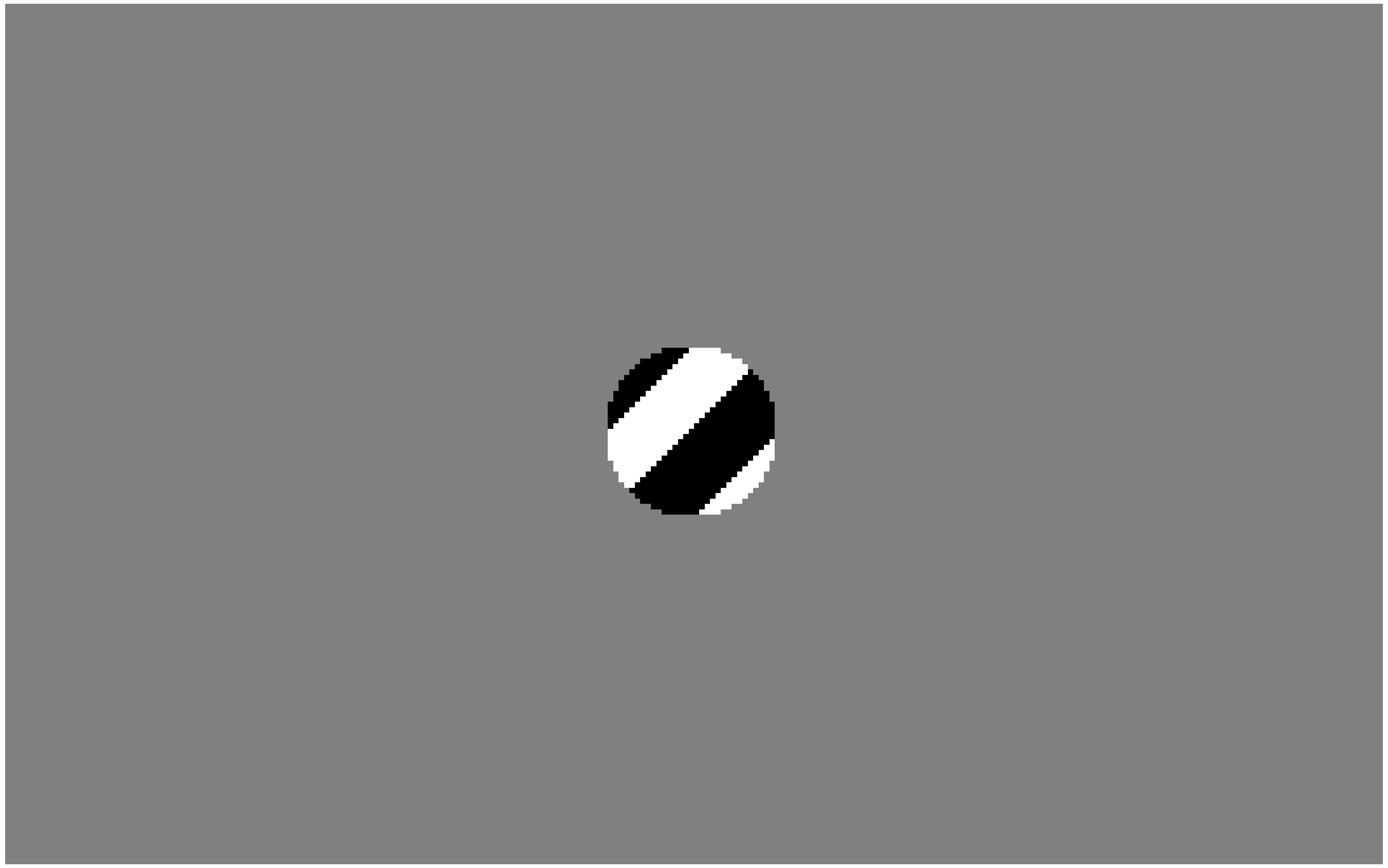
What happens when you have no 'corners'?

You want to compute optical flow.
What happens if the image patch contains only a line?

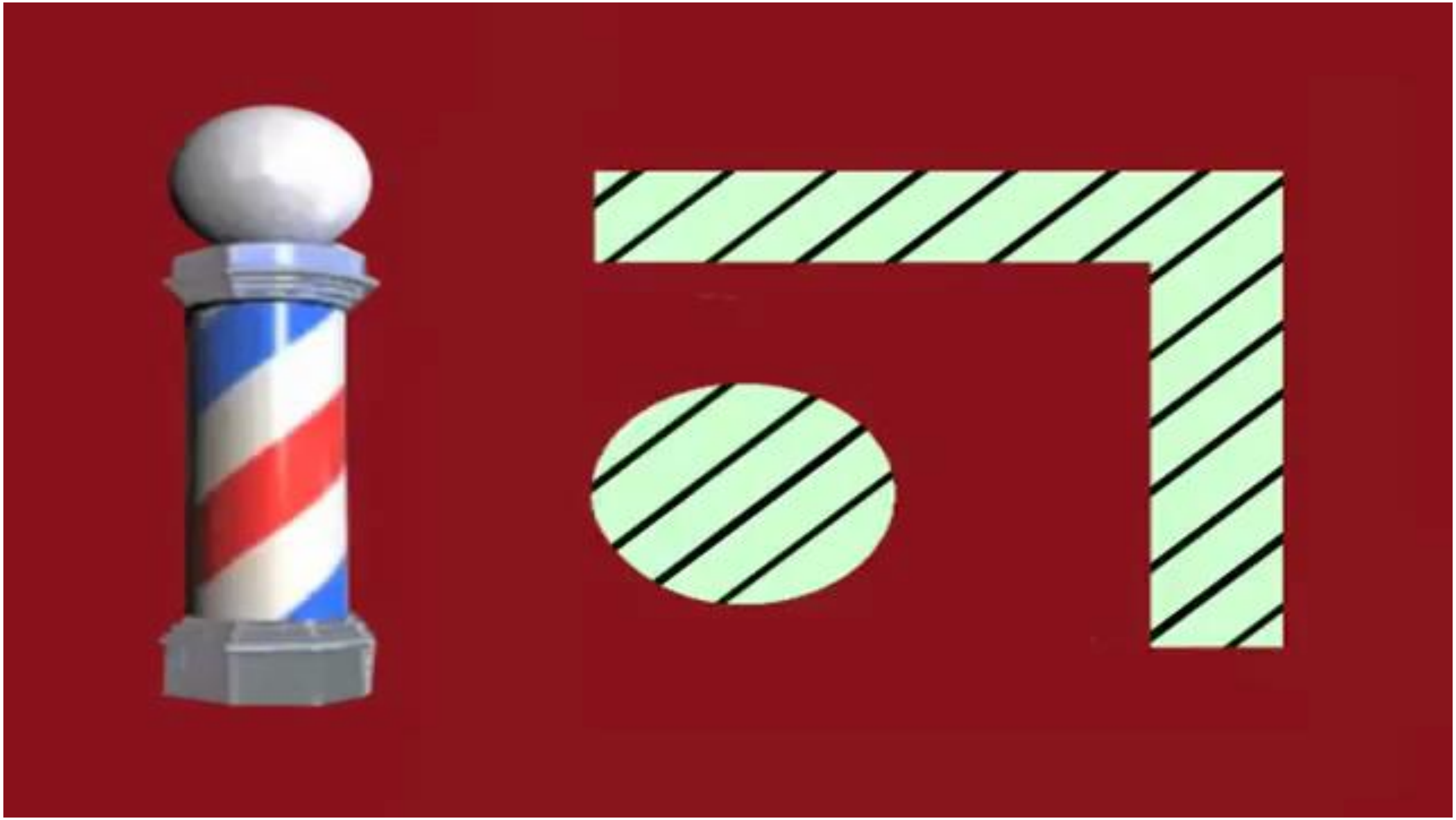
Barber’s pole illusion



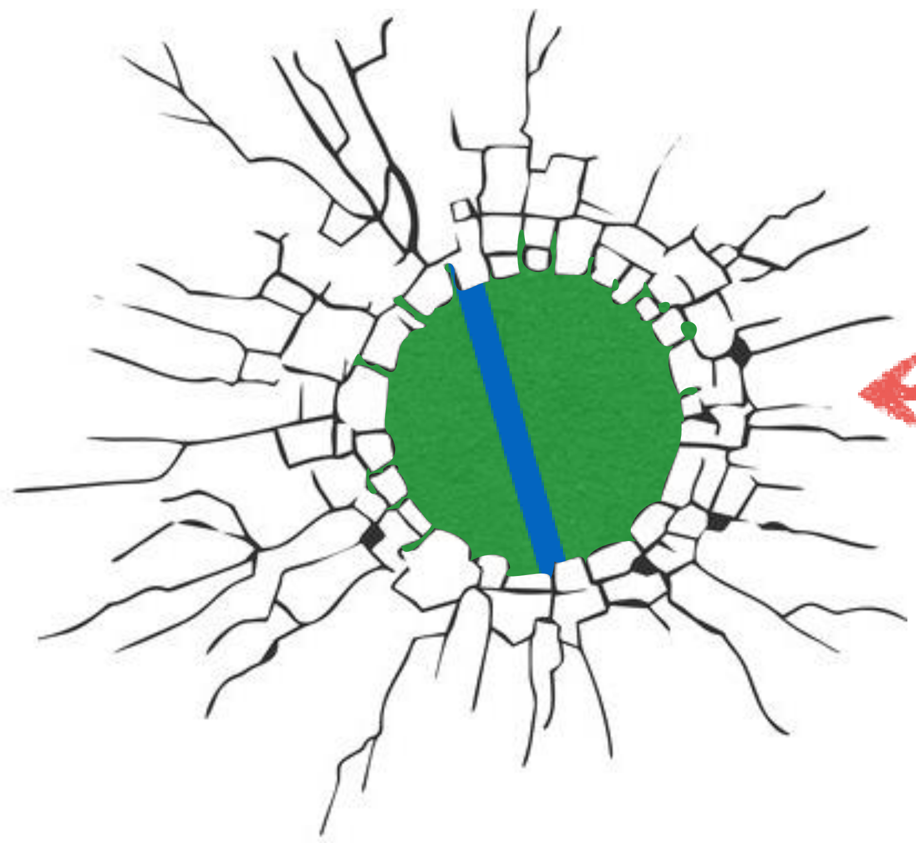
Barber's pole illusion



Barber's pole illusion



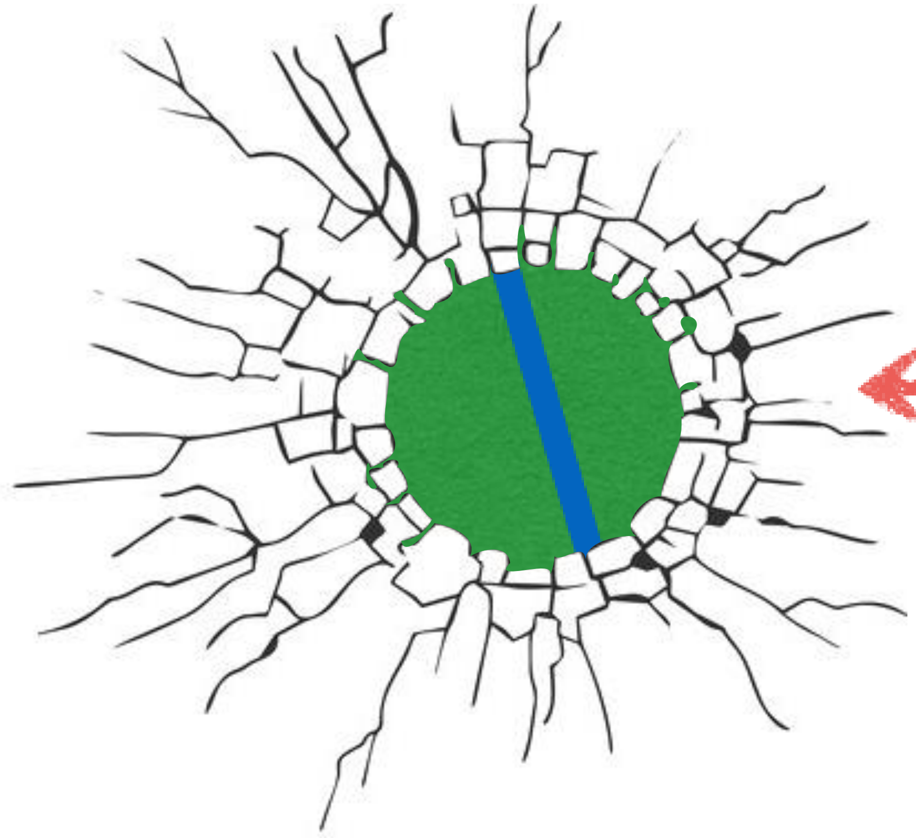
Aperture Problem



small visible image patch

In which direction is the line moving?

Aperture Problem



small visible image patch

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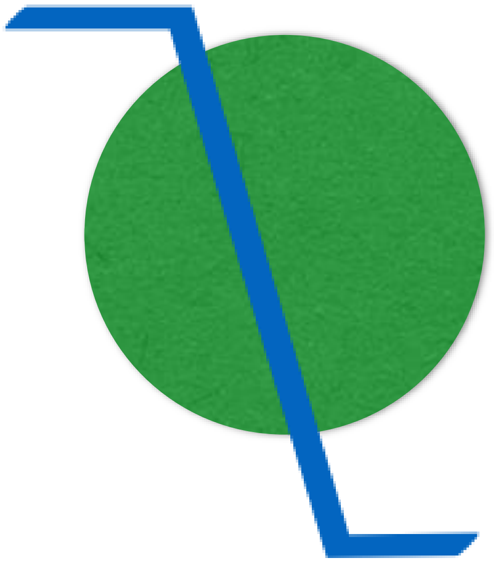
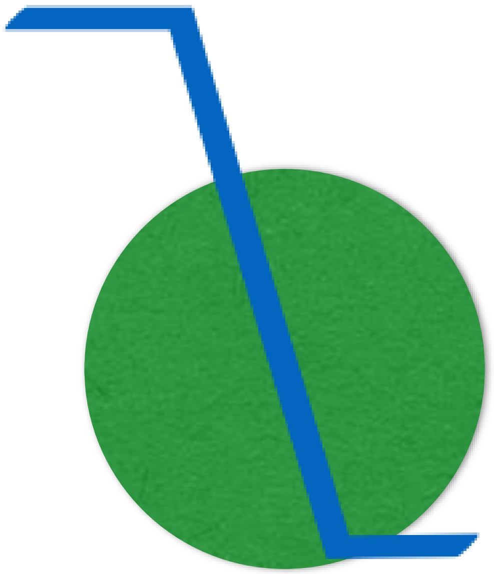
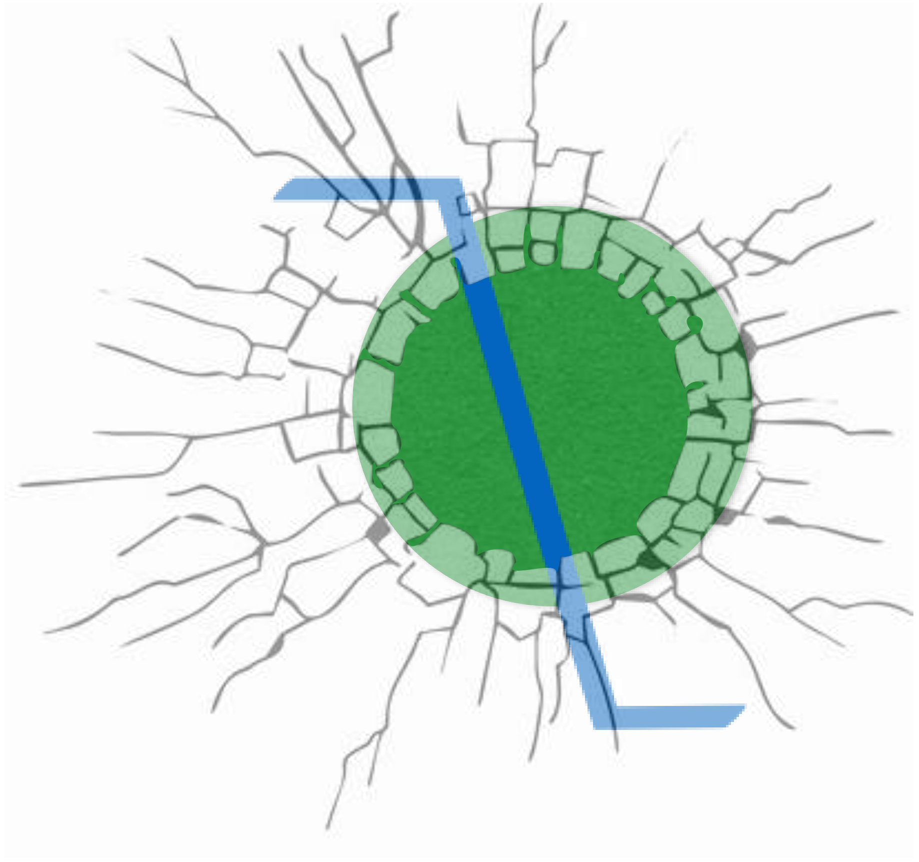
Aperture Problem



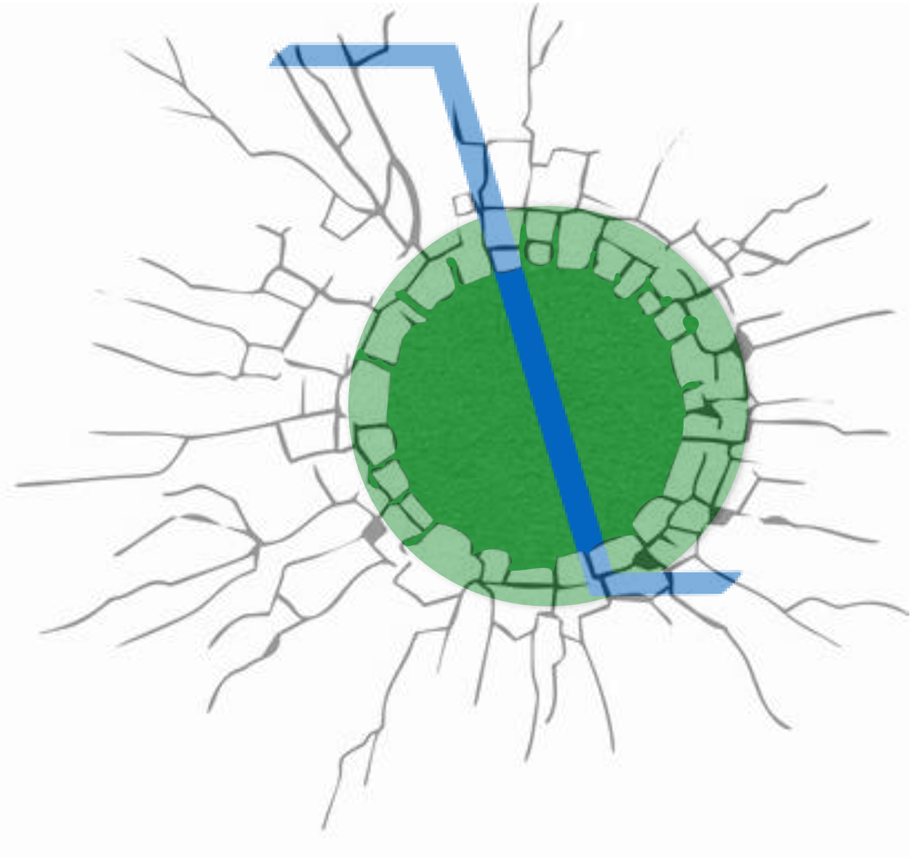
Aperture Problem

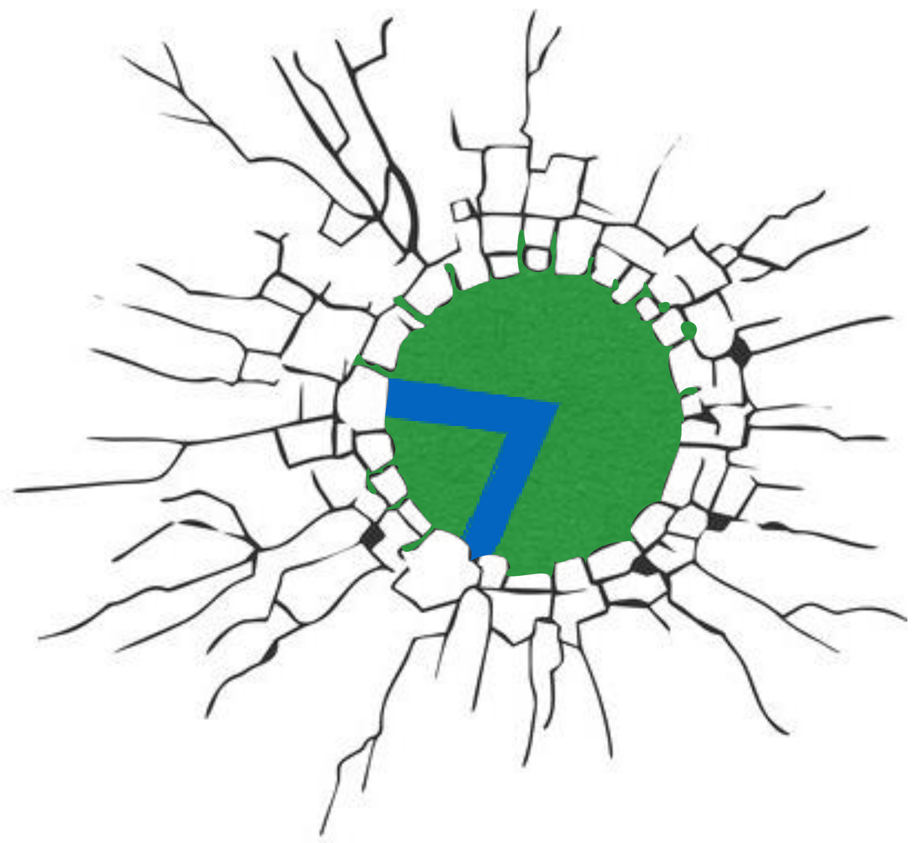


Aperture Problem

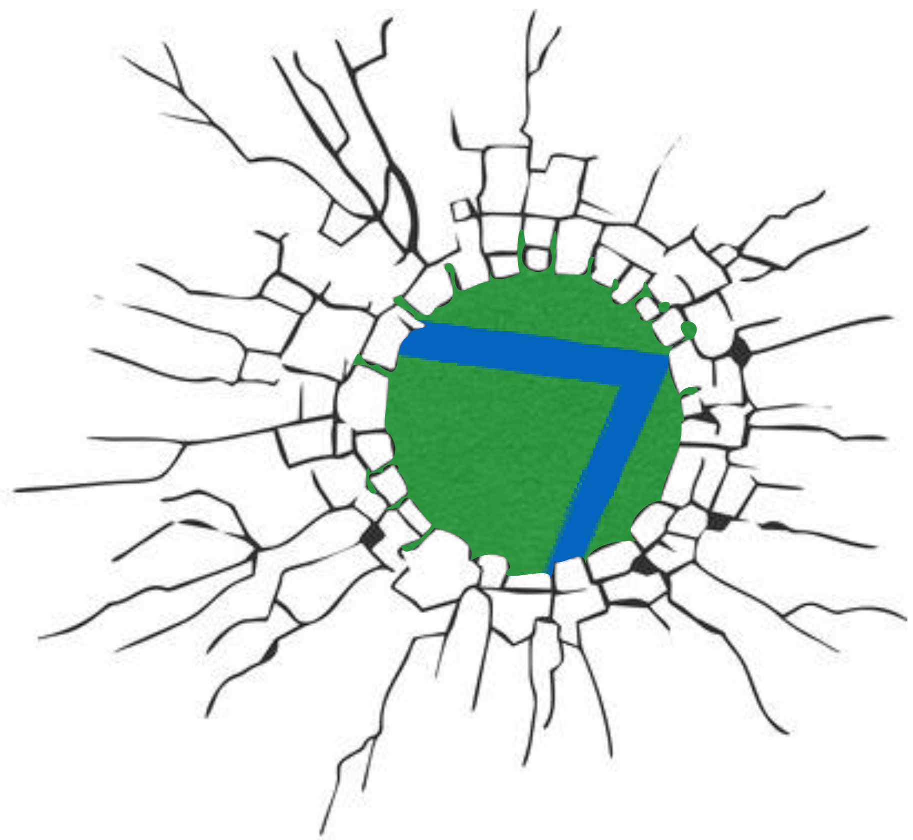


Aperture Problem





Want patches with different gradients to
the avoid aperture problem



Want patches with different gradients to the avoid aperture problem

H(x,y) = y

1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5

optical flow: (1,1)

I(x,y)

-	-	-	-
-	1	1	1
-	2	2	2
-	3	3	3
-	4	4	4

$$I_x u + I_y v + I_t = 0$$

Compute gradients

$$I_x(3,3) = 0$$
$$I_y(3,3) = 1$$
$$I_t(3,3) = I(3,3) - H(3,3) = -1$$

Solution:

$$v = 1$$

We recover the v of the optical flow but not the u.

This is the aperture problem.

Horn-Schunck optical flow

Horn-Schunck Optical Flow (1981)

brightness constancy

small motion

'smooth' flow
(flow can vary from pixel to pixel)

global method
(dense)

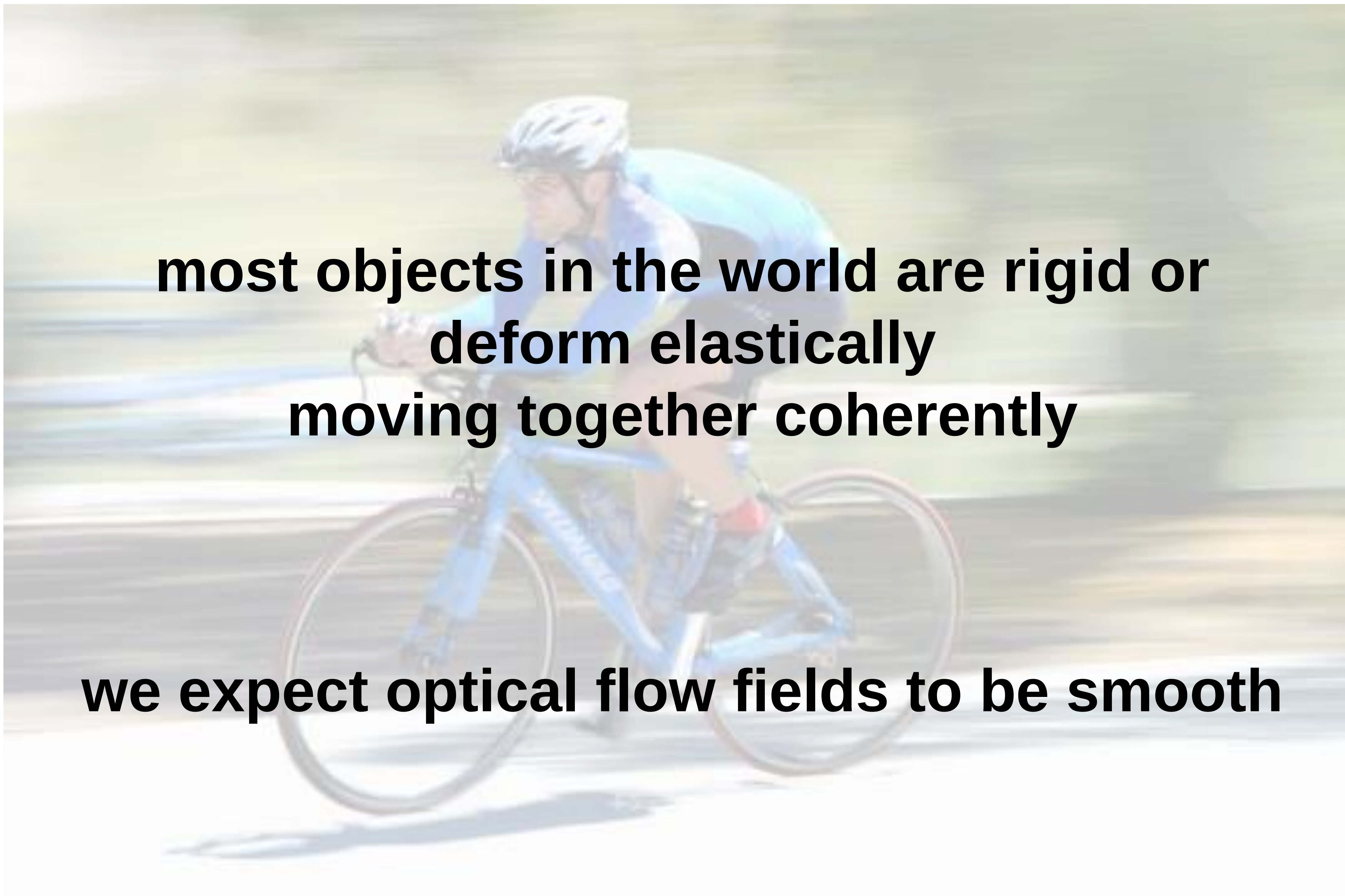
Lucas-Kanade Optical Flow (1981)

method of differences

'constant' flow
(flow is constant for all pixels)

local method
(sparse)

Smoothness



most objects in the world are rigid or deform elastically
moving together coherently

we expect optical flow fields to be smooth

Key idea

(of Horn-Schunck optical flow)

Enforce

brightness constancy

Enforce

smooth flow field

to compute optical flow

Key idea

(of Horn-Schunck optical flow)

Enforce
brightness constancy

Enforce
smooth flow field

to compute optical flow

Enforce **brightness constancy**

$$\boxed{I_x u + I_y v + I_t = 0}$$

For every pixel,

$$\min_{u,v} \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

Enforce
brightness constancy

$$\boxed{I_x u + I_y v + I_t = 0}$$

For every pixel,

$$\min_{u,v} \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

lazy notation for $I_x(i,j)$

Key idea

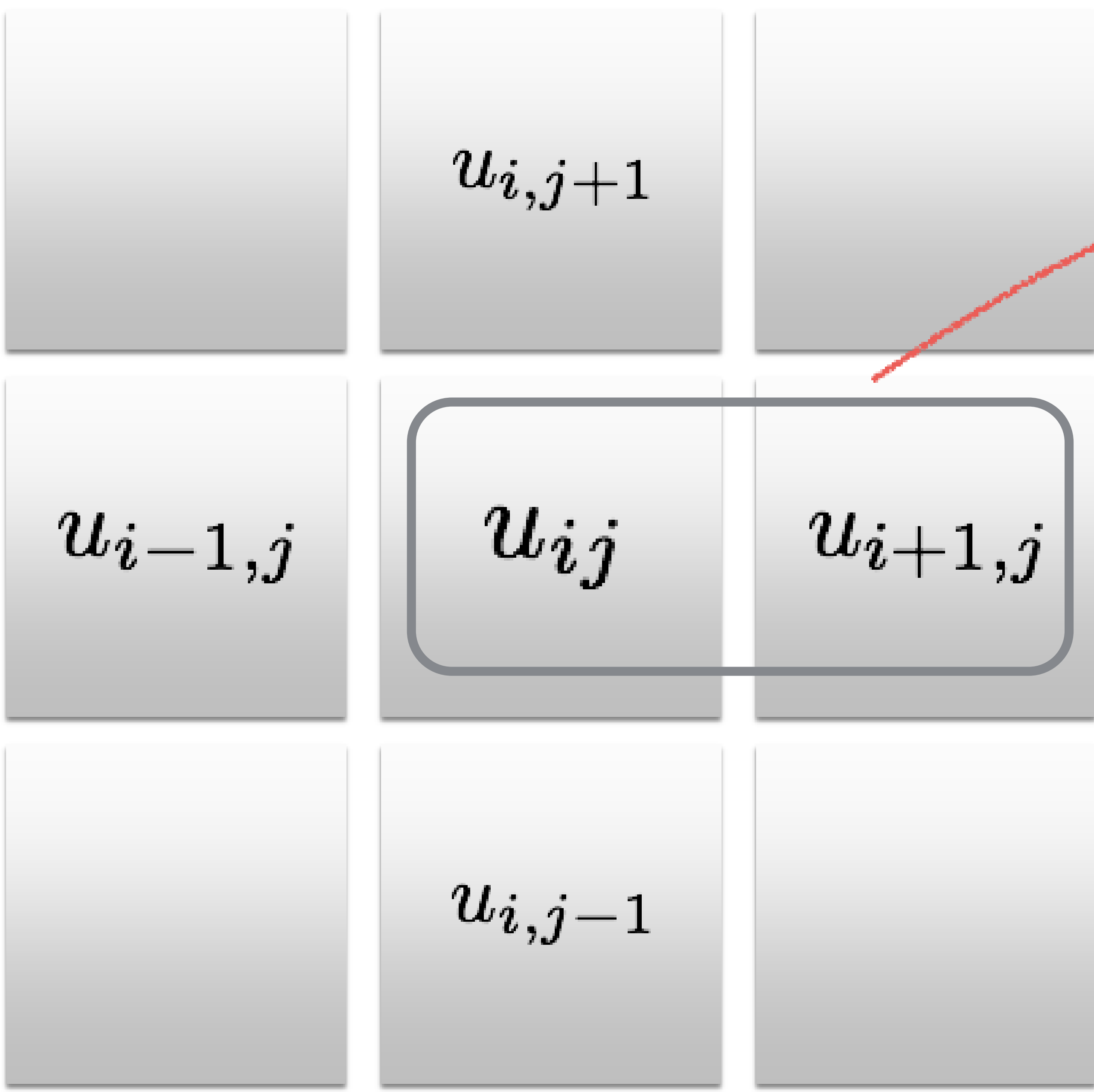
(of Horn-Schunck optical flow)

Enforce
brightness constancy

Enforce
smooth flow field

to compute optical flow

Enforce **smooth flow field**

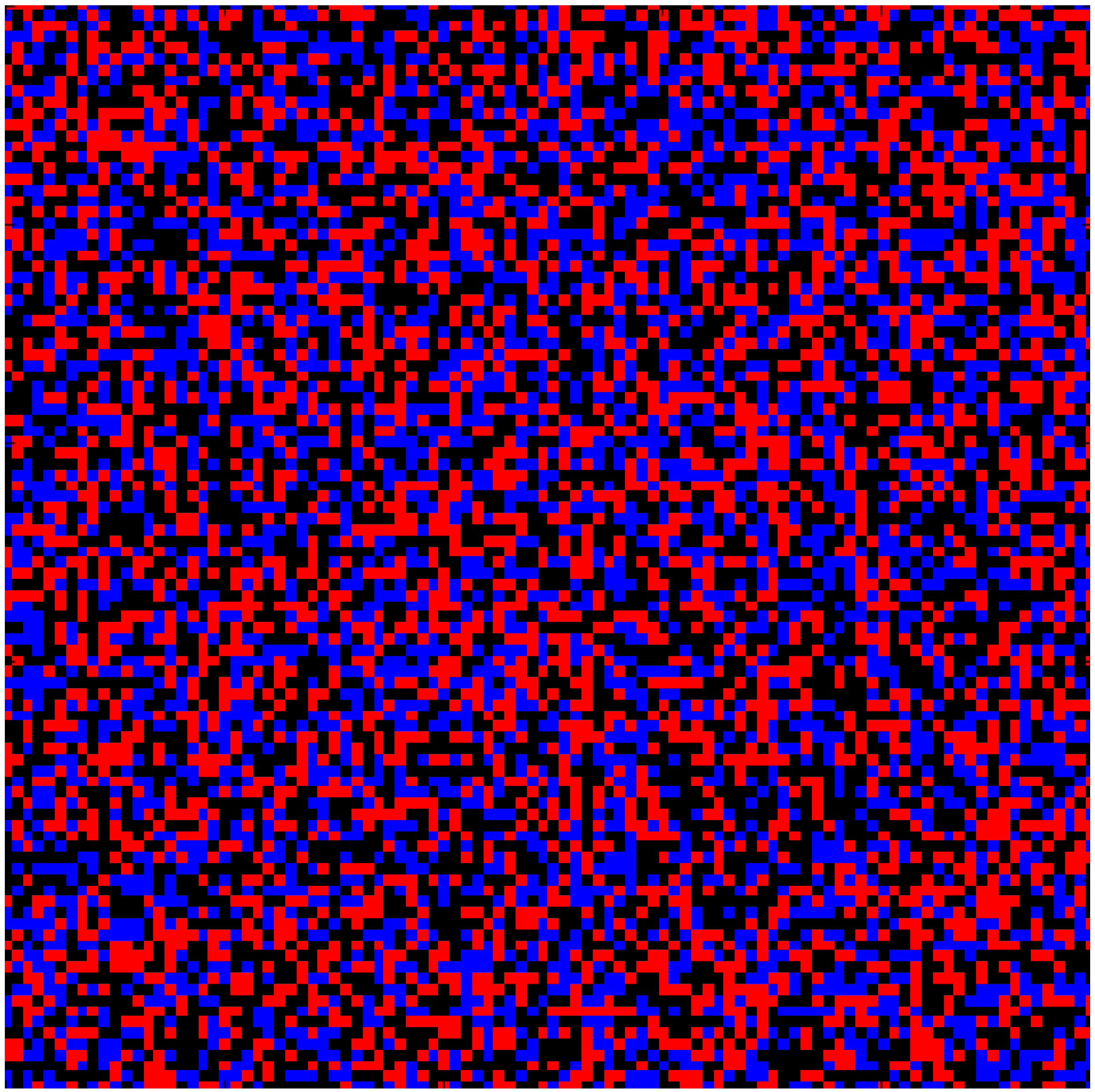


$$\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$$

u-component of flow

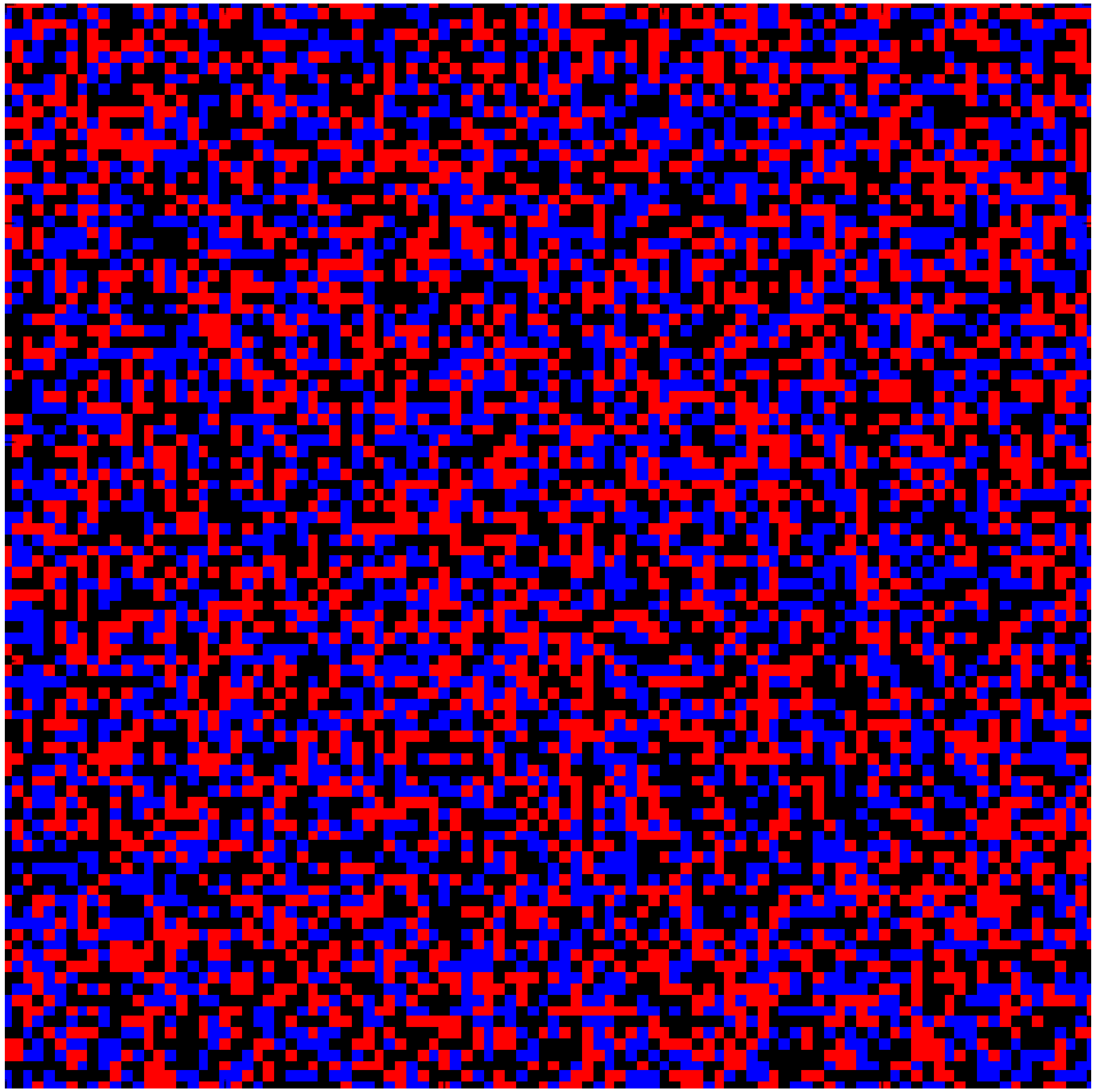
Which flow field optimizes the objective?

$$\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$$

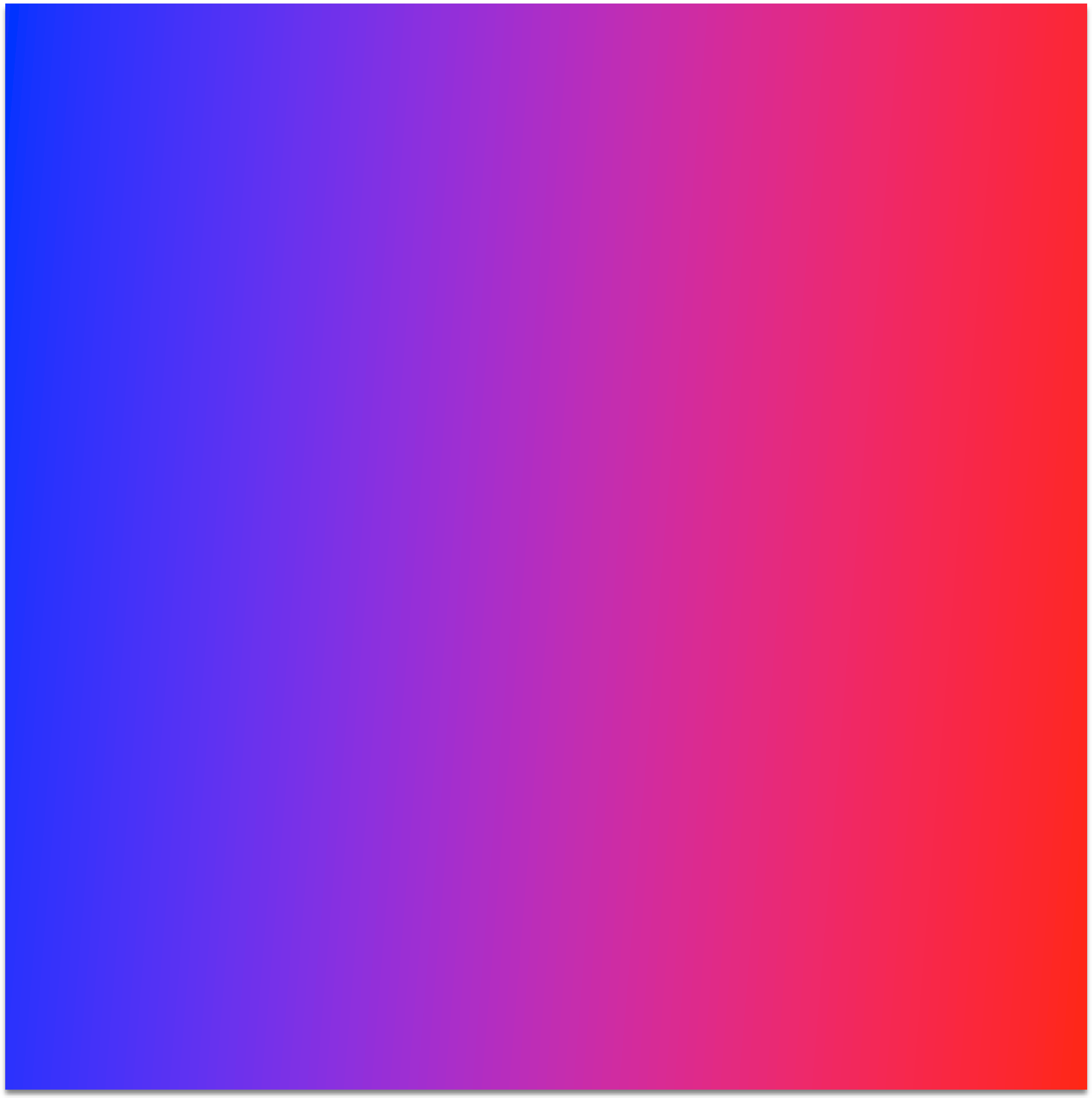


$$\sum_{ij} (u_{ij} - u_{i+1,j})^2 \quad ? \quad \sum_{ij} (u_{ij} - u_{i+1,j})^2$$

Which flow field optimizes the objective? $\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$



big



small

Key idea

(of Horn-Schunck optical flow)

Enforce

brightness constancy

Enforce

smooth flow field

to compute optical flow

bringing it all together…

Horn-Schunck optical flow

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ \underbrace{E_s(i,j)}_{\text{smoothness}} + \underbrace{\lambda}_{\text{weight}} \underbrace{E_d(i,j)}_{\text{brightness constancy}} \right\}$$

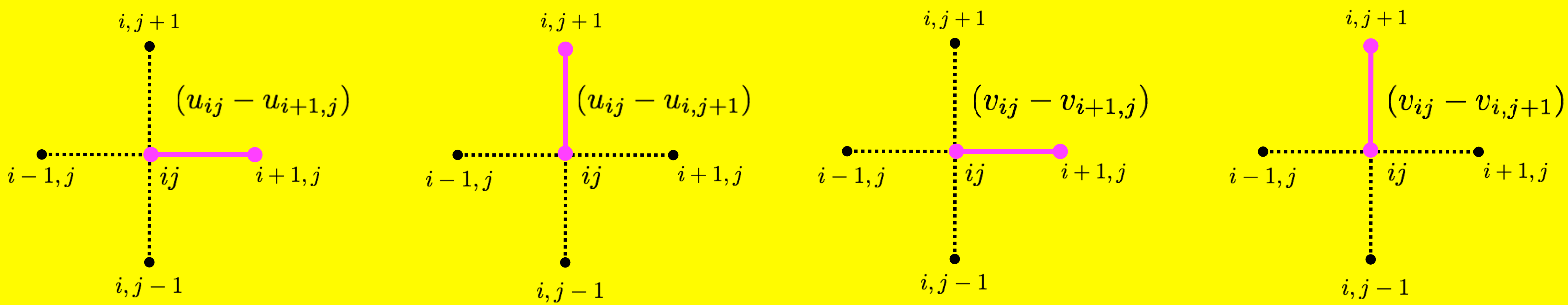
HS optical flow objective function

Brightness constancy

$$E_d(i,j) = \Big[I_x u_{ij} + I_y v_{ij} + I_t \Big]^2$$

Smoothness

$$E_s(i,j) = \frac{1}{4}\Big[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \Big]$$



How do we solve this minimization problem?

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

How do we solve this minimization problem?

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

Compute partial derivative, derive update equations

(gradient decent!)

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \underbrace{\frac{1}{4}\left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2\right]}_{\text{smoothness term}} + \lambda \underbrace{\left[I_x u_{ij} + I_y v_{ij} + I_t\right]^2}_{\text{brightness constancy}} \right\}$$

Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} =$$

how many u terms depend on k and l?

Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} =$$

how many u terms depend on k and l?

FOUR from smoothness

ONE from brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}}=2(u_{kl}-\bar{u}_{kl})+2\lambda(I_x u_{kl}+I_y v_{kl}+I_t)I_x$$

how many u terms depend on k and l?

FOUR from smoothness

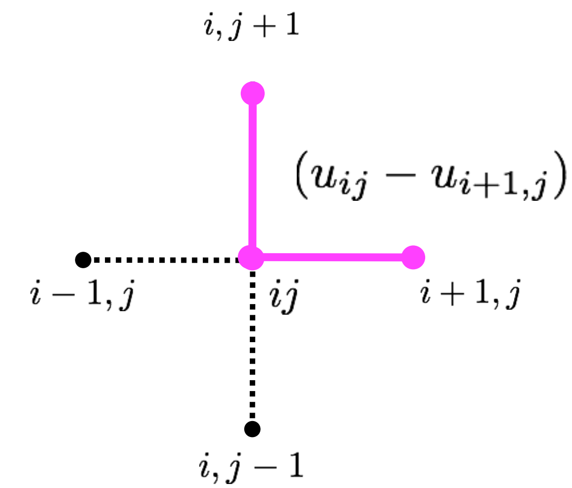
ONE from brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

$$(u_{ij}^2-2u_{ij}u_{i+1,j}+u_{i+1,j}^2) \qquad (u_{ij}^2-2u_{ij}u_{i,j+1}+u_{i,j+1}^2)$$

(variable will appear four times in sum)

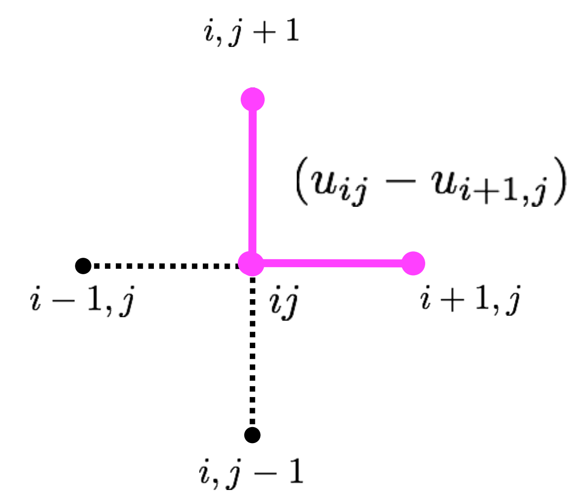


Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

$$(u_{ij}^2-2u_{ij}u_{i+1,j}+u_{i+1,j}^2) \qquad (u_{ij}^2-2u_{ij}u_{i,j+1}+u_{i,j+1}^2)$$

(variable will appear four times in sum)



$$\frac{\partial E}{\partial u_{kl}}=2(u_{kl}-\bar{u}_{kl})+2\lambda(I_x u_{kl}+I_y v_{kl}+I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}}=2(v_{kl}-\bar{v}_{kl})+2\lambda(I_x u_{kl}+I_y v_{kl}+I_t)I_y$$

short hand for local average

$$\bar{u}_{ij}=\frac{1}{4}\left\{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right\}$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

Where are the extrema of E?

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

Where are the extrema of E?

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}u_{kl} = (1 + \lambda I_x^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}v_{kl} = (1 + \lambda I_y^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

Rearrange to get update equations:

$$\boxed{\begin{aligned} \underbrace{\hat{u}_{kl}}_{\text{new value}} &= \underbrace{\bar{u}_{kl}}_{\text{old average}} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \\ \hat{v}_{kl} &= \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y \end{aligned}}$$

Recall: $$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

When lambda is small (lambda inverse is big)…

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value — old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

Recall: $\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$

When lambda is small (lambda inverse is big)...

$$\hat{u}_{kl} = \bar{u}_{kl} - \cancel{\frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2}} I_x$$

goes to zero

new value

old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \cancel{\frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2}} I_y$$

goes to zero

Recall: $$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

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goes to zero

new value

old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \cancel{\frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2}} I_y$$

goes to zero

…we only care about smoothness.

ok, take a step back, why did we do all this math?

We are solving for the optical flow (u,v) given two constraints

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

smoothness

brightness constancy

We needed the math to minimize this
(now to the algorithm)

Horn-Schunck
Optical Flow Algorithm

1. Precompute image gradients $I_y \quad I_x$

2. Precompute temporal gradients I_t

3. Initialize flow field $\boldsymbol{u} = \mathbf{0}$ $\boldsymbol{v} = \mathbf{0}$

4. While not converged

   Compute flow field updates for each pixel:

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \qquad \hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

Just 8 lines of code!

References

Basic reading:
- Szeliski, Section 8.4.