Probability Basics

Computer Vision

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Random Variable

What is it?

Is it 'random'?

Is it a 'variable'?

Random Variable

What is it?

Is it ‘random’?

not in the traditional sense

Is it a ‘variable’?

not in the traditional sense

Random Variable:

a variable whose possible values are numerical outcomes of a random phenomenon

http://www.stat.yale.edu/Courses/1997-98/101/ranvar.htm

Random variable:

a measurable function from a probability space into a measurable space known as the state space (Doob 1996)

http://mathworld.wolfram.com/RandomVariable.html

Random variable:

a function that associates a unique numerical value with every outcome of an experiment

http://www.stats.gla.ac.uk/steps/glossary/probability_distributions.html

$$\text{outcome} \xrightarrow{\text{random variable}} \underset{\text{(index)}}{\text{value}}$$

outcome
(face of a penny)

random variable

value
(heads or tails)

0: heads

1: tails

What kind of random variable is this?

outcome
(face of a penny)

random variable

value
(heads or tails)

0: heads

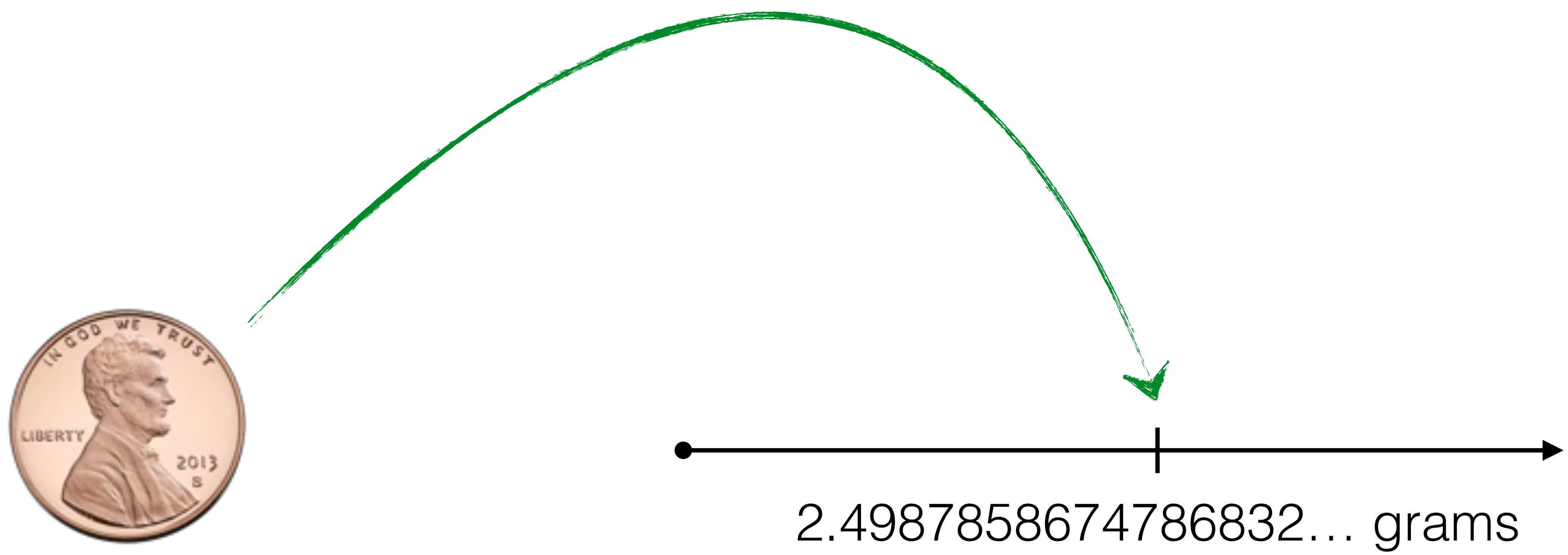
1: tails

Discrete.
Can enumerate all possible outcomes

outcome
(mass of a penny)

$\xrightarrow{\text{random variable}}$

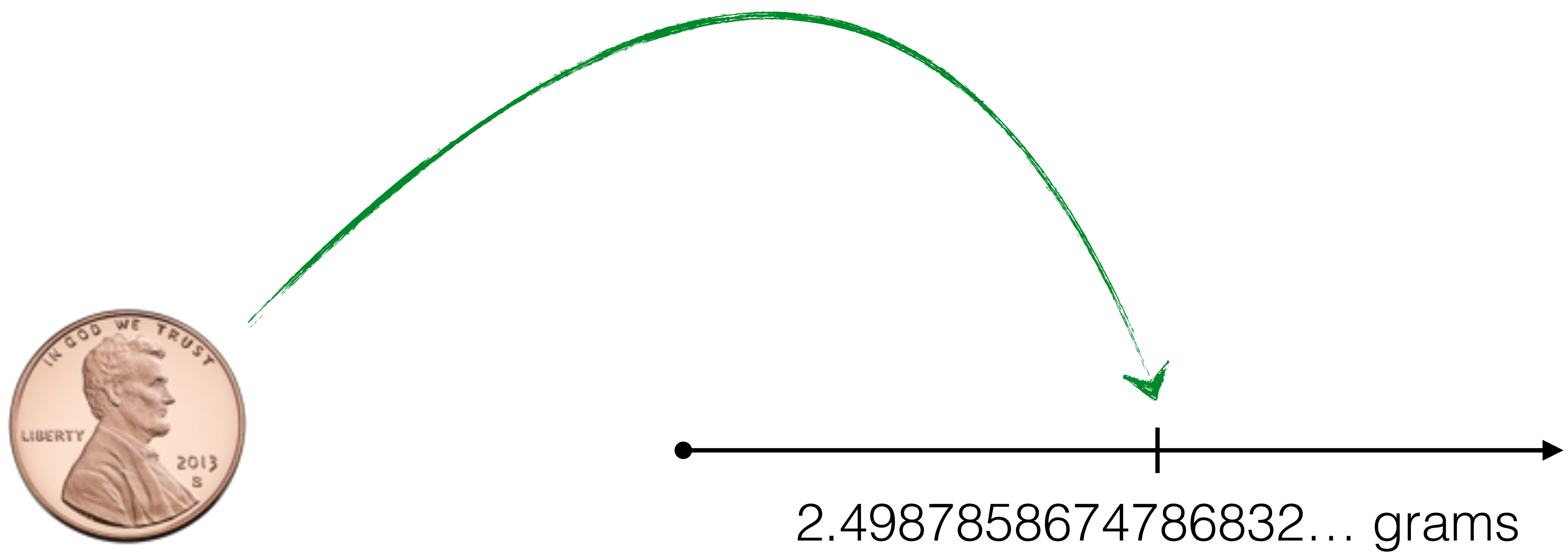
value
(a number)



outcome
(mass of a penny)

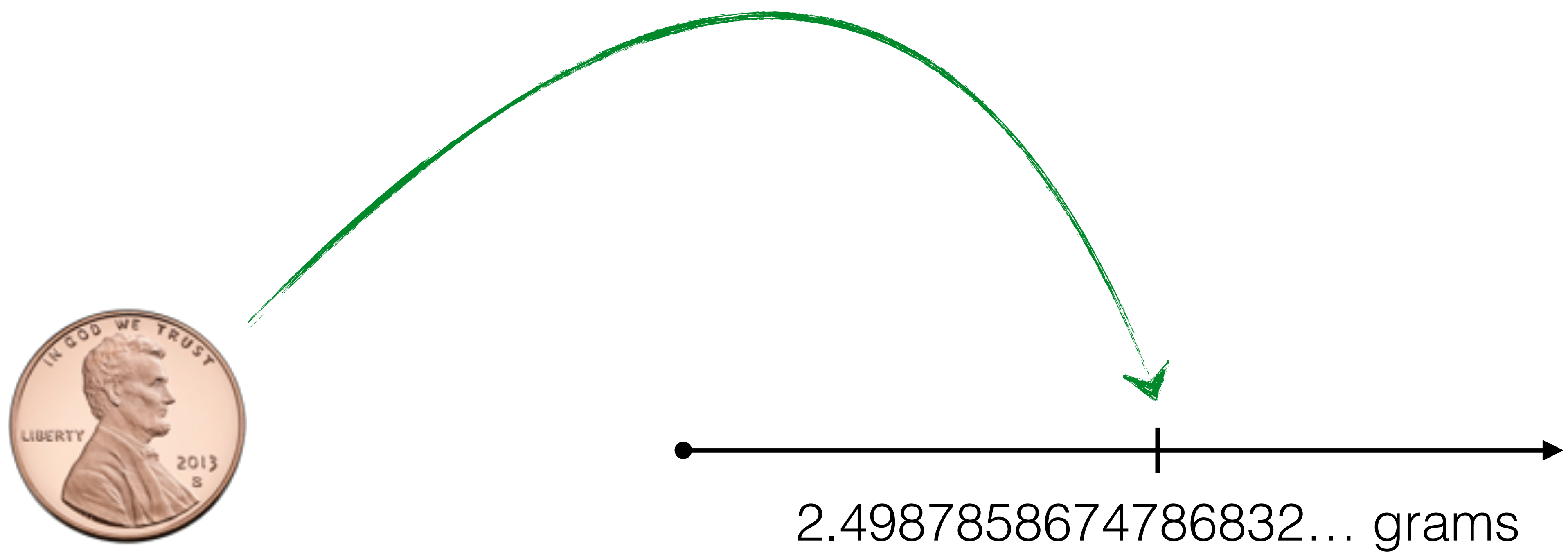
random variable

value
(a number)



What kind of random variable is this?

outcome (mass of a penny) $\xrightarrow{\text{random variable}}$ value (a number)



Continuous.
Cannot enumerate all possible outcomes

<u>Random Variables</u> are typically denoted with a capital letter

$$X, Y, A, \ldots$$

<u>Values</u> of an RV are typically denoted with lower case

$$X = x$$

"RV is equal to a certain value"

$$x$$

or just the value
(many times this is also used to mean the RV!
You'll have to figure it out from the context)

Probability:

the chance that a particular event (or set of events) will occur expressed on a linear scale from 0 (impossibility) to 1 (certainty)

http://mathworld.wolfram.com/Probability.html

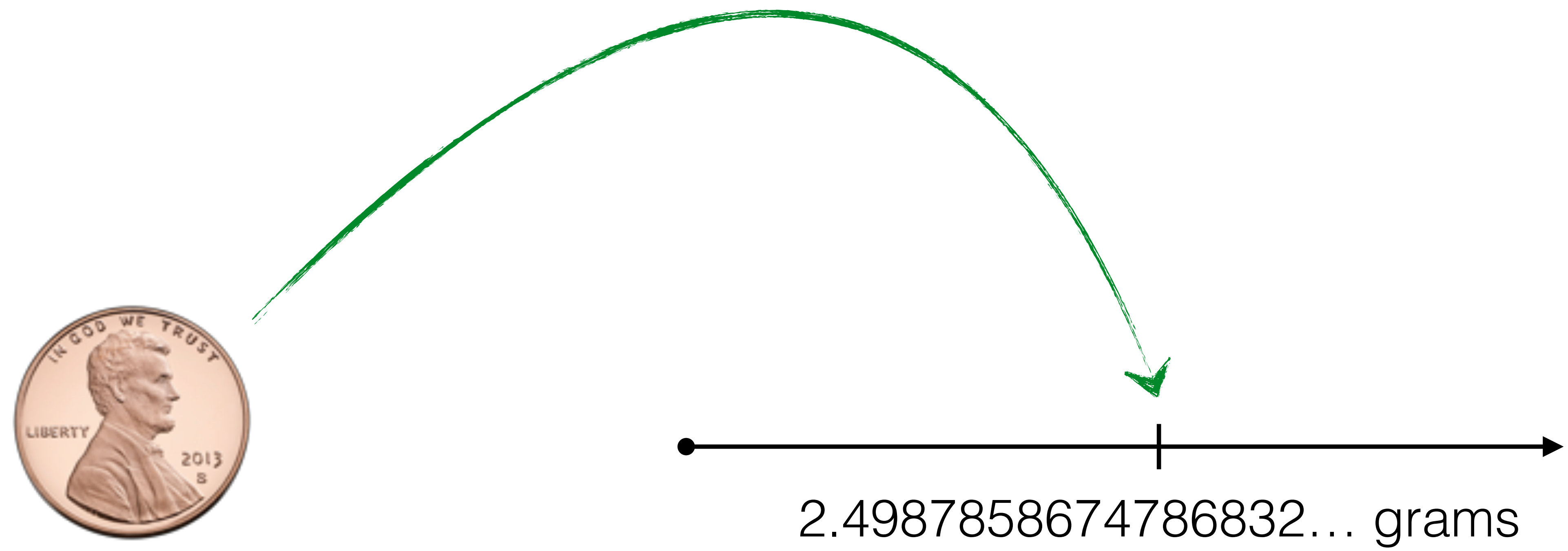
0: heads

1: tails

$$p(X = 0) = 0.5$$

$$p(X = 1) = 0.5$$

$$p(X = 0) + p(X = 1) = 1.0$$



$$\int p(x)dx = 1$$

Probability Axioms:

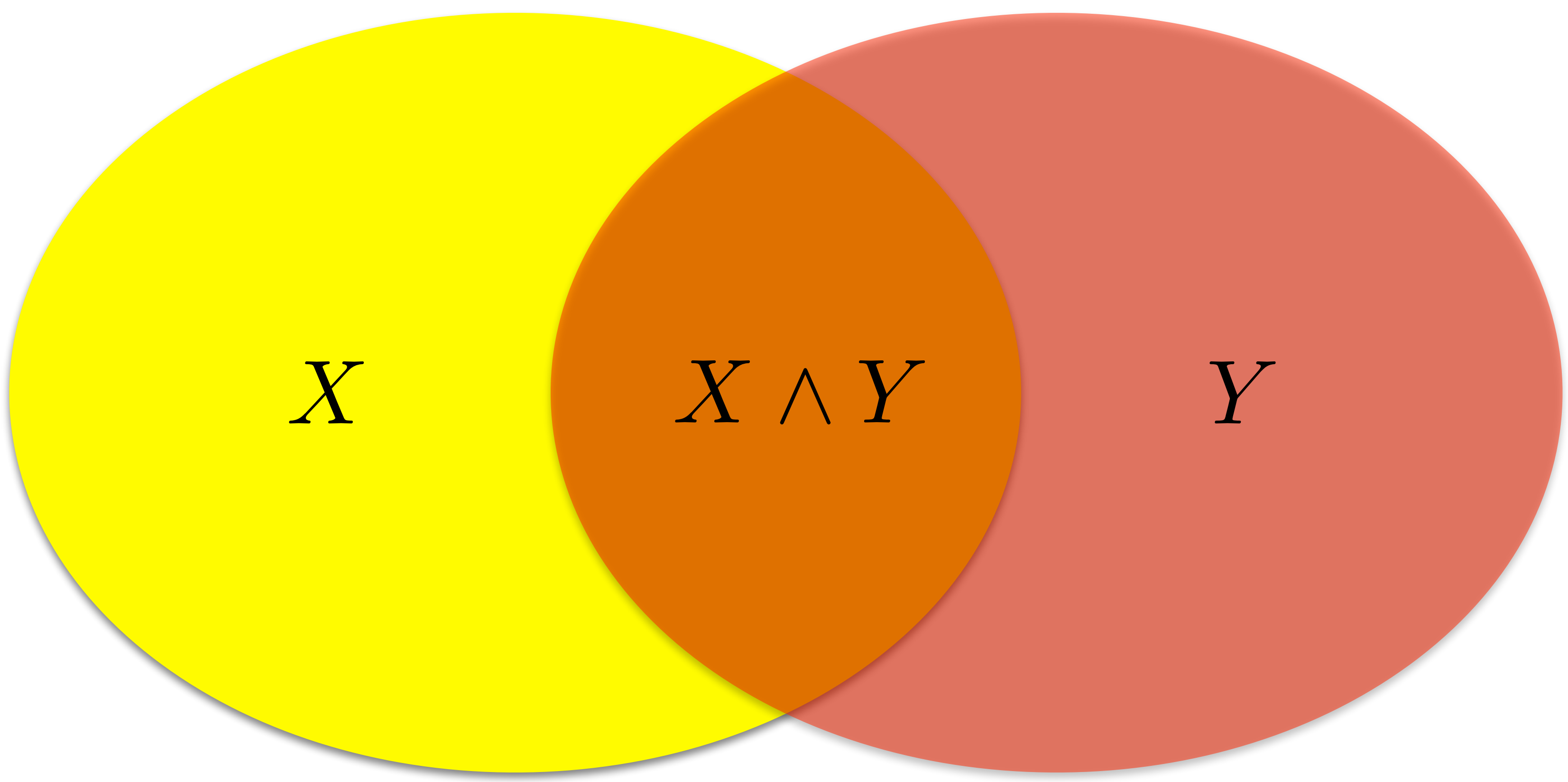
$$0 \leq p(x) \leq 1$$

$$p(\text{true}) = 1$$

$$p(\text{false}) = 0$$

$$p(X \vee Y) = p(X) + p(Y) - P(X \wedge Y)$$

$$p(X \vee Y) = p(X) + p(Y) - P(X \wedge Y)$$



Joint Probability

$$p(x, y)$$

When random variables are **independent**
(a sequence of coin tosses)

$$p(x, y) = p(x)p(y)$$

When random variables are **dependent**

$$p(x, y) = p(x|y)p(y)$$

this is a conditional probability defined next …

Conditional Probability

$$p(x|y)$$

Conditional probability of x given y

$p(x|y)$ is the short hand for $?$

in terms of the random variables **X** and **Y**

Conditional Probability

$$p(x|y)$$

Conditional probability of x given y

$p(x|y)$ is the short hand for $p(X = x|Y = y)$

How is it related to the joint probability?

$$p(x|y) = \frac{p(x, y)}{?}$$

Conditional Probability

$$p(x|y)$$

Conditional probability of x given y

$p(x|y)$ is the short hand for $p(X = x|Y = y)$

$$p(x|y) = \frac{p(x, y)}{p(y)}$$

Conditional probability is the probability of the union of the events x and y divided by the probability of event y

Bayes' Rule

likelihood

$$p(x|y) = \frac{p(y|x)\ ?}{?}$$

posterior

What's the relationship between the posterior and the likelihood?

Bayes' Rule

likelihood prior

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

posterior

evidence (observation prior)

How do you compute the evidence (observation prior)?

Bayes' Rule

likelihood prior

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

posterior

evidence (observation prior)

How do you compute the evidence (observation prior)?

$$p(x|y) = \frac{p(y|x)p(x)}{\sum_{x'} p(y|x')p(x')}$$

evidence (expanded)

Bayes' Rule

likelihood prior

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

posterior

evidence

Evidence (observation prior) is also called the **normalization factor**

$$p(x|y) = \eta p(y|x)p(x)$$

$$p(x|y) = \frac{1}{Z}p(y|x)p(x)$$

Bayes' Rule with 'evidence'

$$p(x|y,e) = \frac{p(y|x,e)p(x|e)}{p(y|e)}$$

Marginalization

$$p(x) = \sum_{y} p(x, y)$$

Marginalize out y

Conditioning

$$p(x) = \sum_{y} p(x|y)p(y)$$

Conditioned on y

Example: A visit to the Dentist

- **Toothache:** boolean variable indicating whether the patient has a toothache
- **Cavity:** boolean variable indicating whether the patient has a cavity
- **Catch:** whether the dentist's probe catches in the cavity

'not'

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

A table of joint probabilities

Joint probability over three (dependent) variables

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

$$p(cavity) = ?$$

Recall: $p(x) = \sum_y p(x, y)$ Marginalization

Joint probability over three (dependent) variables

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

$$p(cavity) = ?$$

$$p(cavity) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$$

Joint probability over three (dependent) variables

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

$$p(cavity|toothache) = ?$$

Joint probability over three (dependent) variables

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
$\neg$ *cavity*	**.016**	**.064**	**.144**	**.576**

$$
\begin{aligned}
p(cavity|toothache) &= \frac{p(cavity, toothache)}{p(toothache)} \\
&= \frac{0.108 + 0.012}{0.108 + 0.012 + 0.016 + 0.064} \\
&= 0.6
\end{aligned}
$$