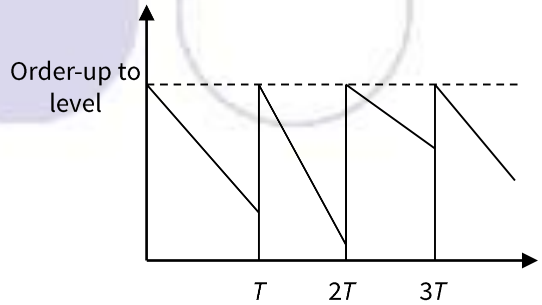
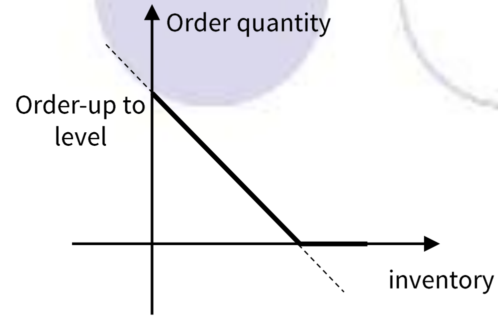
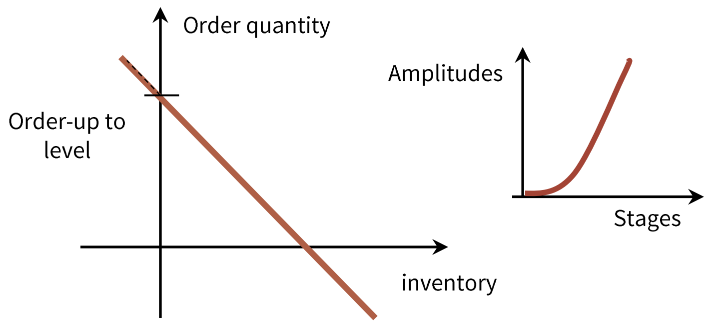
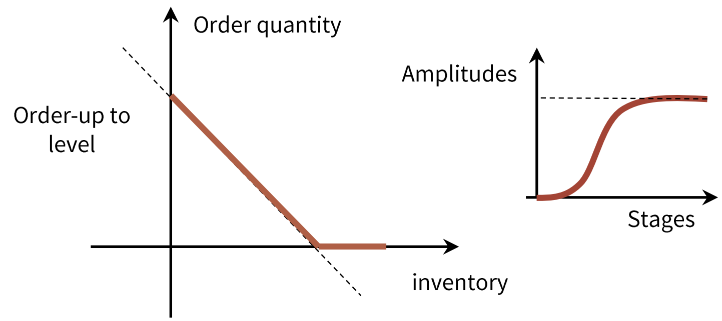
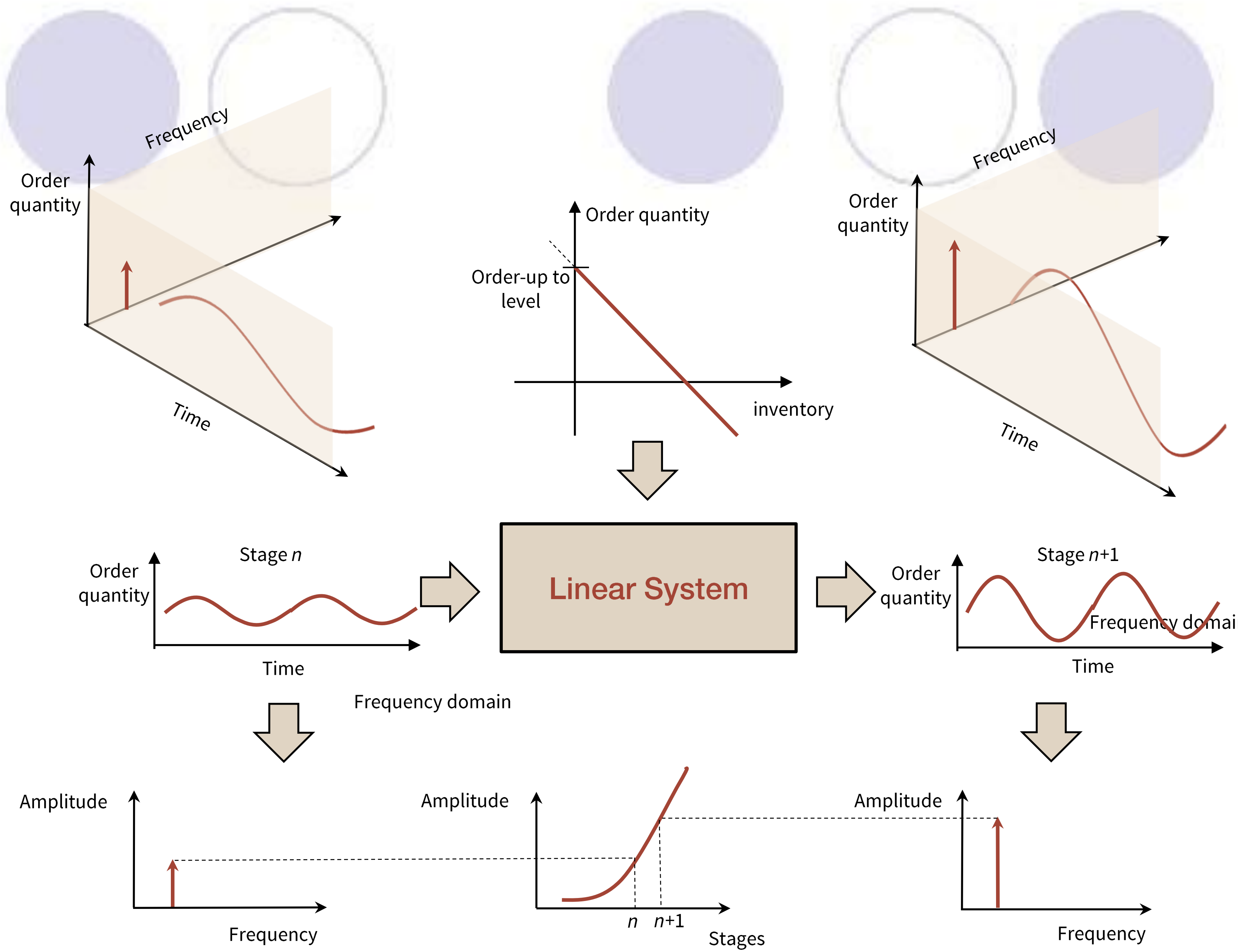
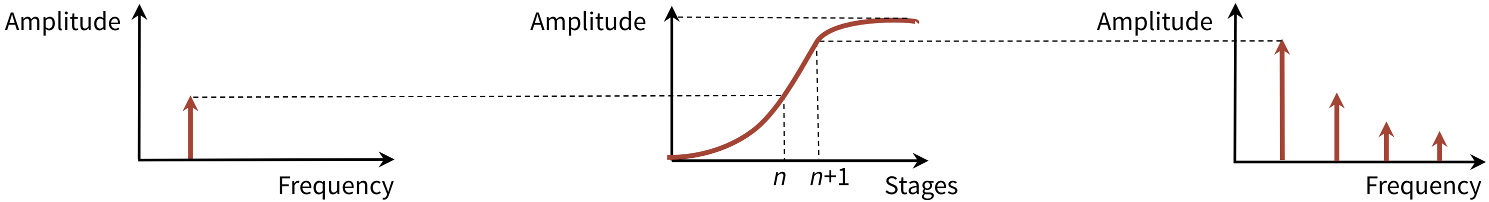

Order-up to
level
T
2T
3T

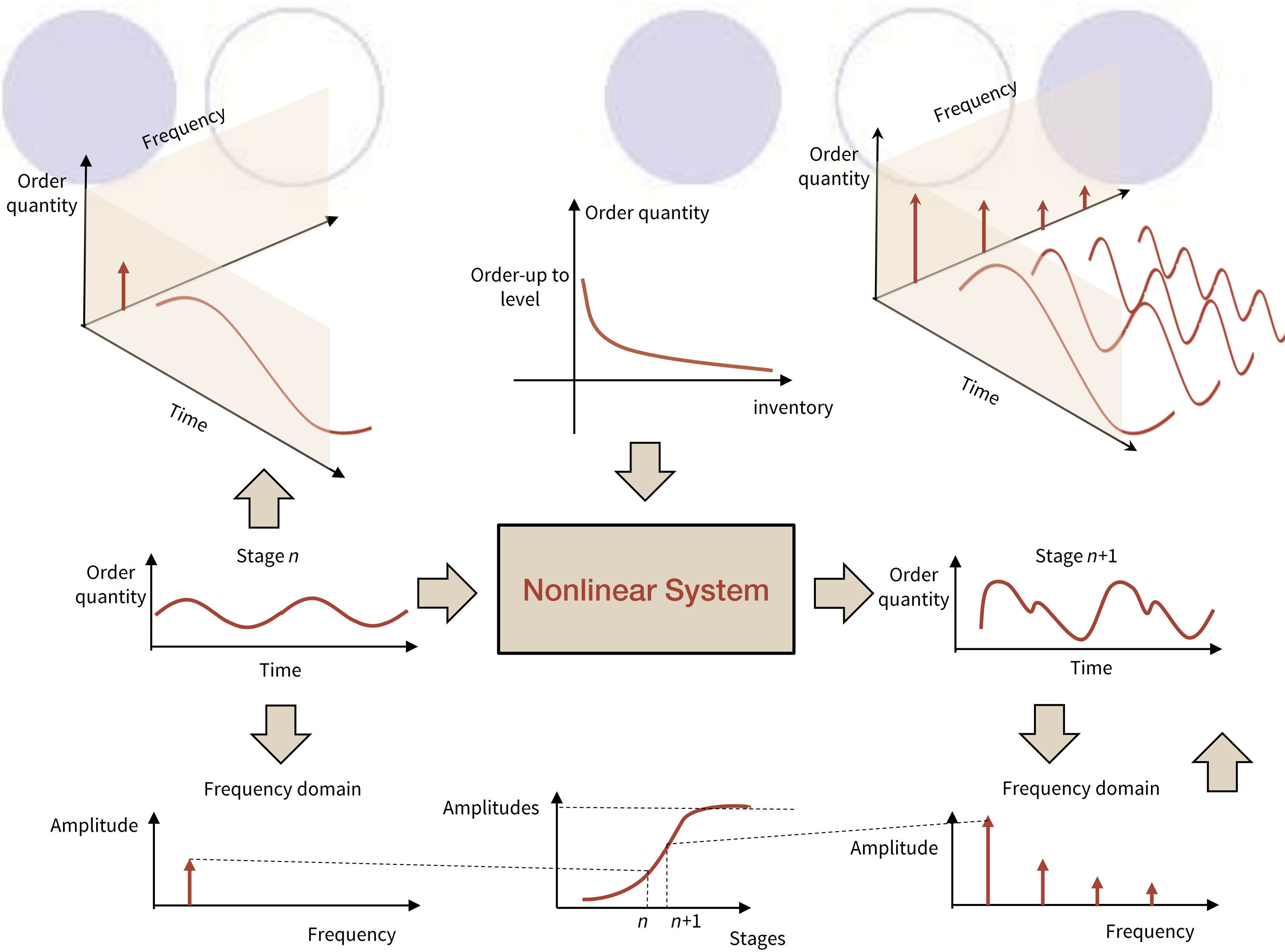

Order quantity
Order-up to
level
inventory

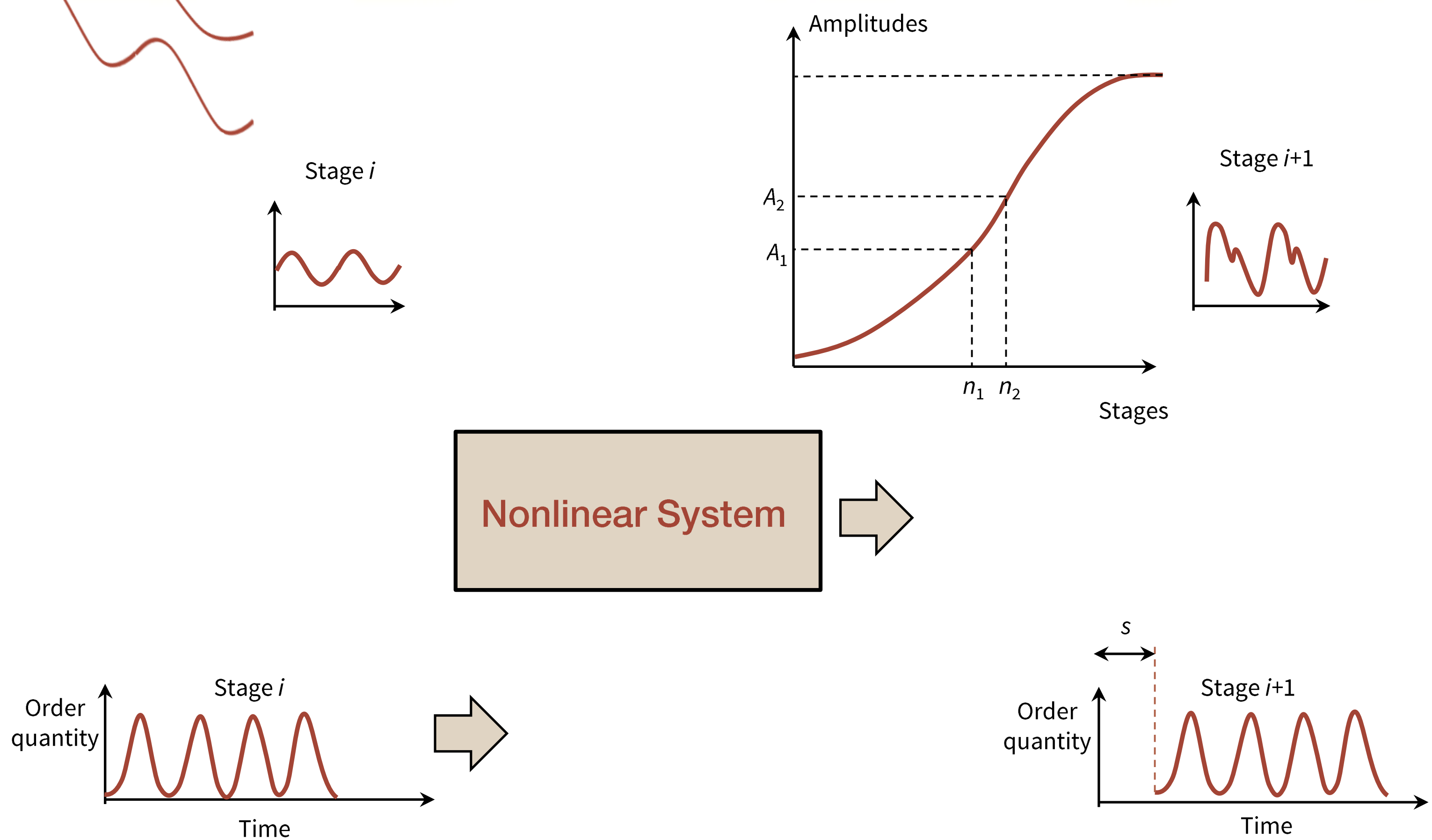

Order quantity
Order-up to
level
inventory
Amplitudes
Stages


Order quantity
Order-up to
level
inventory
Amplitudes
Stages

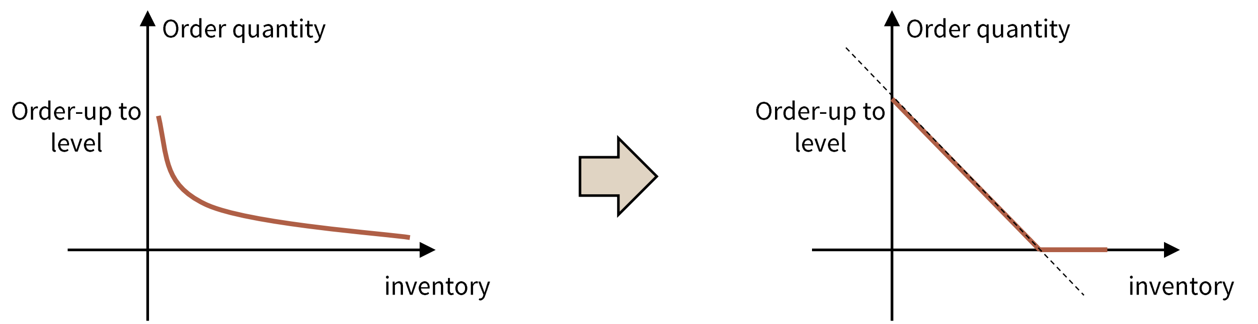

Frequency
Order quantity
Time
Order quantity
Order-up to level
inventory
Frequency
Order quantity
Time
Stage n
Order quantity
Time
Linear System
Stage n+1
Order quantity
Frequency domain
Time
Frequency domain
Amplitude
Frequency
Amplitude
n
n+1
Stages
Amplitude
Frequency

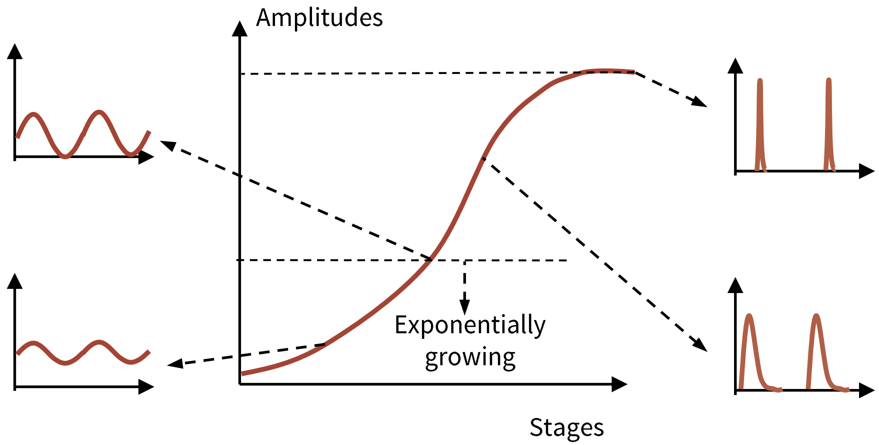

Amplitude
Frequency
Amplitude
n
n+1
Stages
Amplitude
Frequency

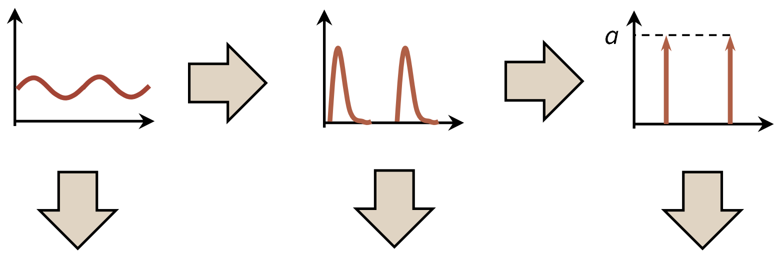

Frequency
Order quantity
Time
Order quantity
Order-up to level
inventory
Frequency
Order quantity
Time
Stage n
Order quantity
Time
Nonlinear System
Stage n+1
Order quantity
Time
Frequency domain
Amplitude
Frequency
Amplitudes
n
n+1
Stages
Frequency domain
Amplitude
Frequency

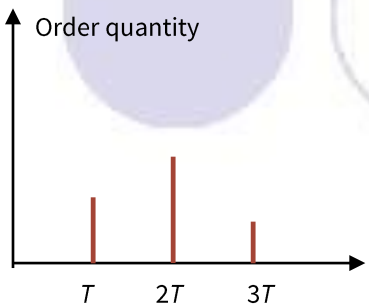


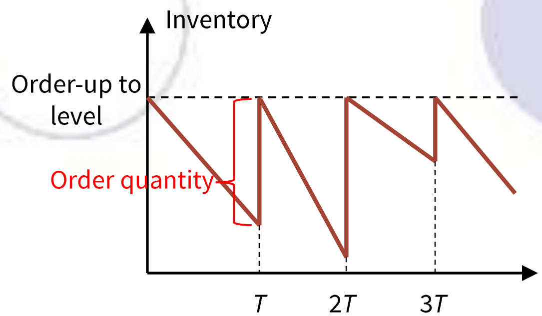
Stage i
Amplitudes
A_2
A_1
n_1
n_2
Stages
Stage i+1
Nonlinear System
s
Order quantity
Stage i
Time
Order quantity
Stage i+1
Time

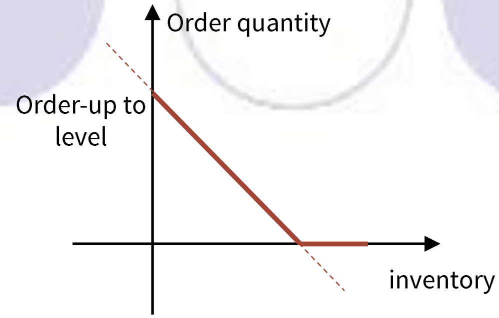

Order quantity
Order-up to level
inventory
Order quantity
Order-up to level
inventory

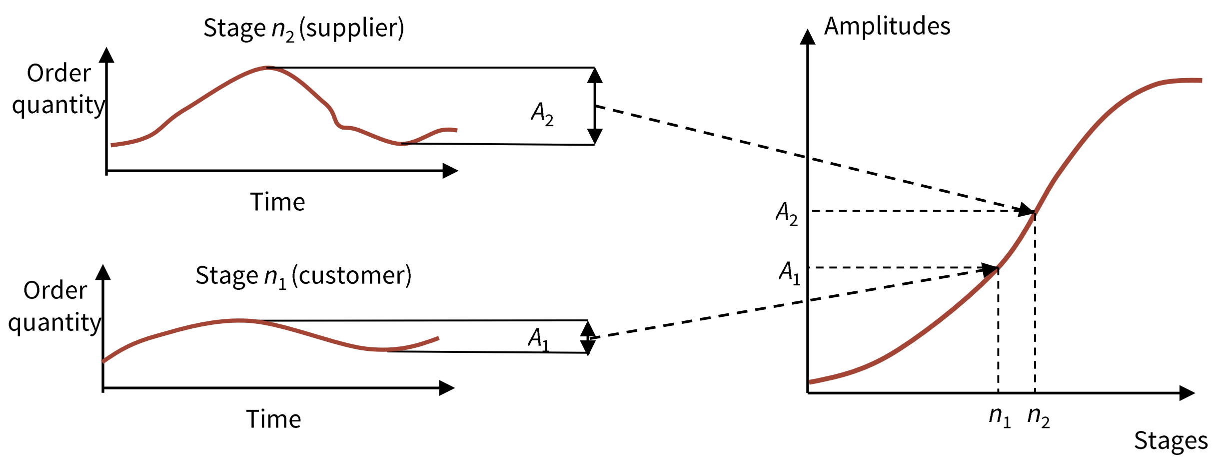

Amplitudes
Exponentially growing
Stages


a


Order quantity
T
2T
3T


Inventory
Order-up to level
Order quantity
T
2T
3T


Order quantity
Order-up to level
inventory


Stage n_2 (supplier)
Order quantity
A_2
Time
Stage n_1 (customer)
Order quantity
A_1
Time
Amplitudes
A_2
A_1
n_1
n_2
Stages

Decentralized Systems

- System-level instability
  - disturbances amplified across system
  - compromise system performance

- Can we achieve centralized performance with decentralized control?
  - stabilize
  - optimize

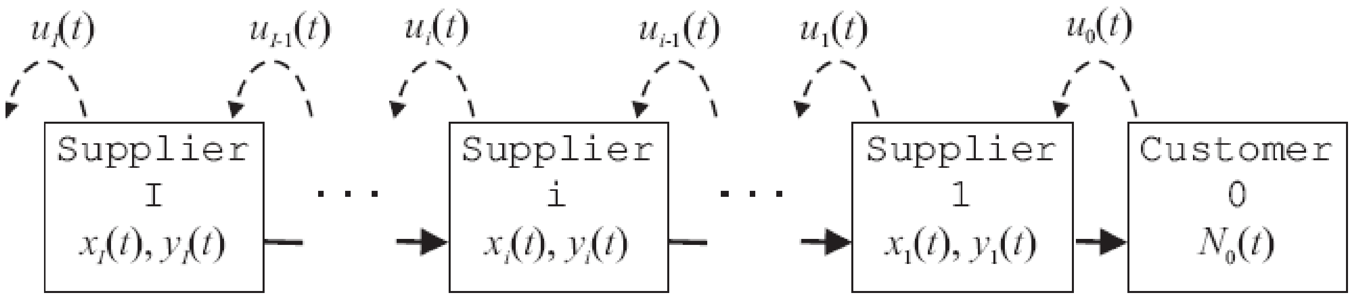
Outline

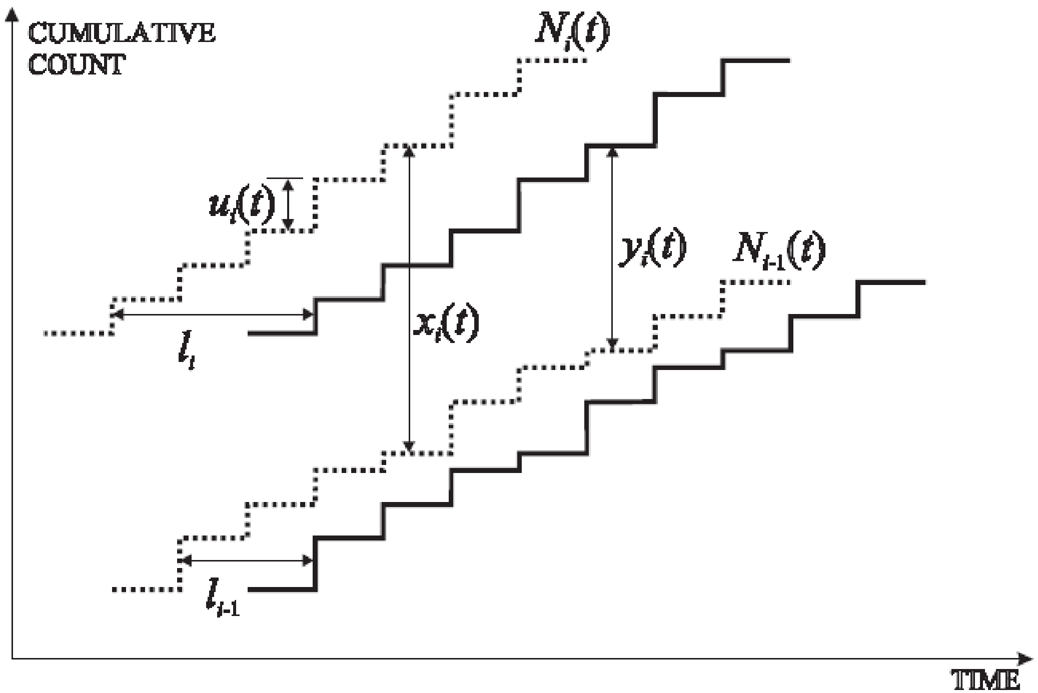
- Basics
  - Background
  - The bullwhip effect
- Deterministic chain stability
  - System formulation
  - Analytical results
- Stochastic chain stability
  - System formulation
  - Analytical results
- Toward optimality
  - Decentralized negotiations
  - Advance order commitment

Goal

- Establish a system-control framework
  - Generalize past work on deterministic and homogeneous system (Daganzo, 2001, 2003a, b; Dejonckheere *et al*., 2003a, b)
  - Further understand the bullwhip effect
- Develop <u>analytical methods</u> to examine the existence of the bullwhip effect

Supply Chain Representation





Definitions

$N_i(t)$ = cumulative orders placed by supplier i by period t

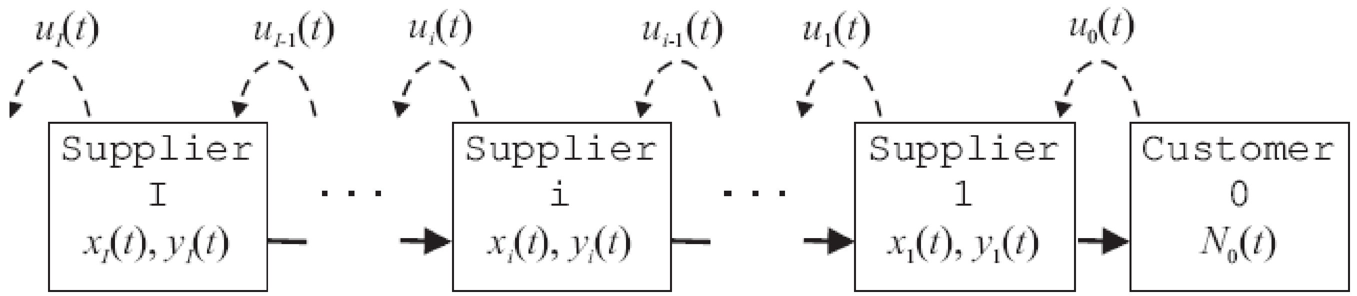
$u_i(t)$ = order placed by supplier i in period t

$y_i(t)$ = in-stock inventory

$x_i(t)$ = inventory position

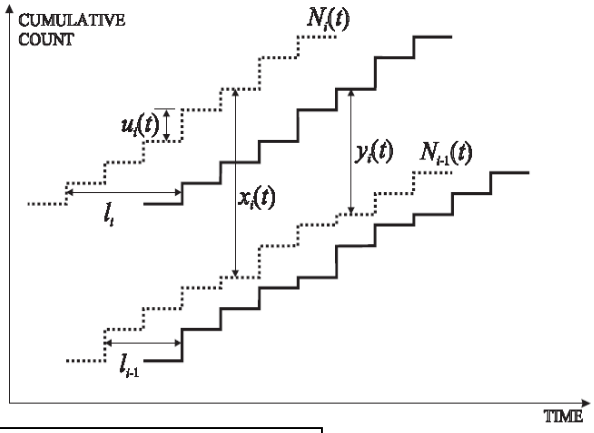
l_i = lead time

Assumptions



- Decentralized decision-making
- Rational ordering decisions
- Linear ordering policy
- Deterministic system operations

System Dynamics



Inventory dynamics:

$$x_i(t+1) = x_i(t) + u_i(t) - u_{i-1}(t), \forall i = 0, 1, ...,$$
$$y_i(t+1) = y_i(t) + u_i(t - l_i) - u_{i-1}(t), \forall i = 1, 2, ...$$

Ordering policy:

$$u_i(t) = \gamma_i + A_i(P)x_i(t) + B_i(P)y_i(t) + C_i(P)u_{i-1}(t-1), i = 1, 2, ...$$

P: shift operator; i.e. $P^k x(t) = x(t-k)$, $\forall$ integer $k \geq 0$

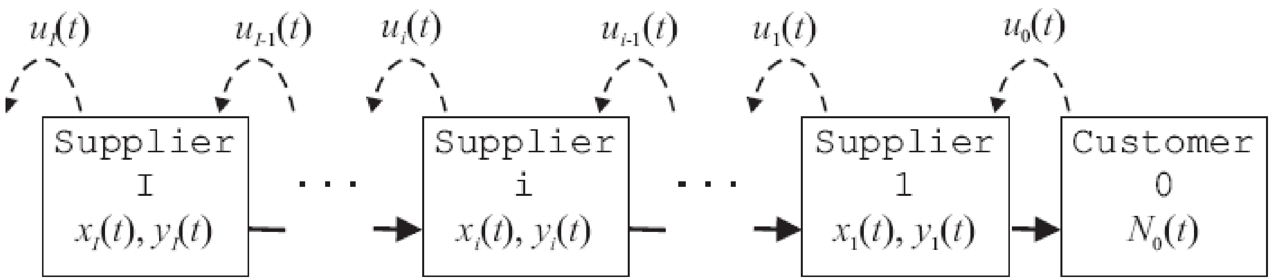
$A_i(\cdot)$, $B_i(\cdot)$, $C_i(\cdot)$: polynomials with real coefficients;

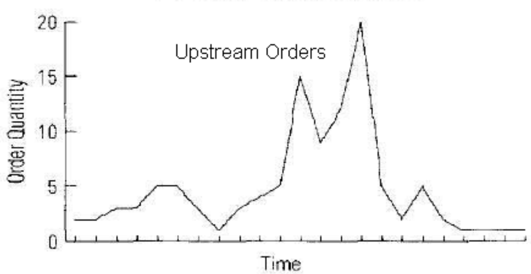
γ_i: constant

Equilibrium state:

$$u^\infty = \gamma_i + A_i(1)x_i^\infty + B_i(1)y_i^\infty + C_i(1)u^\infty,$$

The Bullwhip Effect Metric

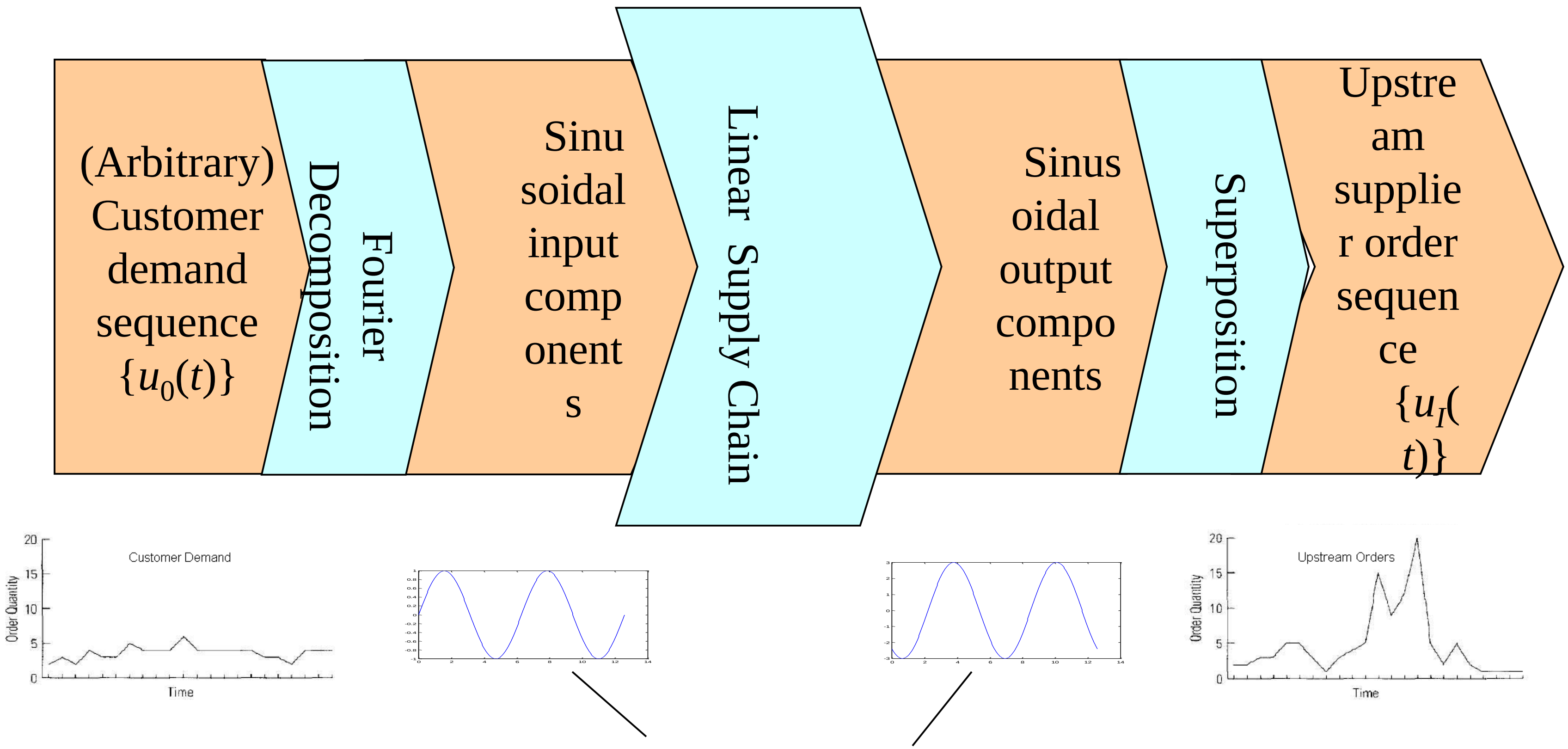




Worst-case RMSE amplification (L_2 norm gain)

$$\max_{\forall\{\bar{u}_0\}\neq 0} \frac{\left(\sum_{t=0}^{\infty} \bar{u}_I^2(t)\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \bar{u}_0^2(t)\right)^{\frac{1}{2}}}, \text{ where } \bar{u}_i(t) := u_i(t) - u^{\infty},\ i = 0, I\ .$$

Frequency Response Analysis



Frequency preserves; the amplitude and phase angle changes according to the transfer function.

The Bullwhip Effect Metrics

Time domain

- ✓ Worst-case RMSE amplification

  (L_2 norm gain)

$$\max_{\forall\{\bar{u}_0\}\neq 0} \frac{\left(\sum_{t=0}^{\infty} \bar{u}_I^2(t)\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \bar{u}_0^2(t)\right)^{\frac{1}{2}}}$$

Frequency domain

- ✓ The peak value on the Bode plot

  (H_∞ norm)

$$\max_{\forall w\in[0,2\pi)} \left|T_I(e^{jw})\right|$$

where T_I is the transfer function from customer demand to supplier I orders.

System Dynamics

- For each variable, take its difference from the equilibrium-state value

$\forall\ i, t > 0$

$$\bar{x}_i(t+1) = \bar{x}_i(t) + \bar{u}_i(t) - \bar{u}_{i-1}(t)$$

$$\bar{y}_i(t+1) = \bar{y}_i(t) + \bar{u}_i(t-l_i) - \bar{u}_{i-1}(t)$$

$$\bar{u}_i(t) = A_i(P)\bar{x}_i(t) + B_i(P)\bar{y}_i(t) + C_i(P)\bar{u}_{i-1}(t-1)$$

- In the frequency domain (z-transform):

$$(z-1)X_i(z) = U_i(z) - U_{i-1}(z)$$

$$(z-1)Y_i(z) = z^{-l_i}U_i(z) - U_{i-1}(z)$$

$$U_i(z) = A_i(z^{-1})X_i(z) + B_i(z^{-1})Y_i(z) + z^{-1}C_i(z^{-1})U_{i-1}(z)$$

Eliminating $X_i(z)$ and $Y_i(z)$,

$$U_i(z) = \frac{z^{-1}C_i(z^{-1}) - (z-1)^{-1}\left[A_i(z^{-1}) + B_i(z^{-1})\right]}{1-(z-1)^{-1}\left[A_i(z^{-1}) + z^{-l_i}B_i(z^{-1})\right]}U_{i-1}(z),\ \ i = 1,2,\ldots$$

The Transfer Function

- The transfer function from the customer demand to the upstream orders of supplier *I* is

$$T_I(z) := \prod_{i=1}^{I} T_{i-1,i}(z)$$

where

$$T_{i-1,i}(z) := \frac{z^{-1}C_i(z^{-1}) - (z-1)^{-1}\left[A_i(z^{-1}) + B_i(z^{-1})\right]}{1-(z-1)^{-1}\left[A_i(z^{-1}) + z^{-l_i}B_i(z^{-1})\right]}$$

- To examine the bullwhip effect, check the H_∞ norm: whether $\exists w \in [0, 2\pi)$, such that

$$\left|T_I(e^{jw})\right| = \prod_{i=1}^{I}\left|T_{i-1,i}(e^{jw})\right| > 1.$$

Deterministic Chain Results

Theorem 1 (Sufficient condition for instability)

Supplier $I+1$ in the deterministic (LTI) supply chain experiences the bullwhip effect if

$$\sum_{i=1}^{I} \frac{1+B_i(1)l_i - C_i(1)}{A_i(1)+B_i(1)} > 0.$$

Corollary 1 (Homogeneous chain)

When $A_i = A$, $B_i = B$, $C_i = C$, and $l_i = l$, $\forall i$, all upstream suppliers experience the bullwhip effect if

$$\frac{1+B(1)l - C(1)}{A(1)+B(1)} > 0.$$

Example (Homogeneous Chain)

- Order-up-to policy
  - Variable “order-up-to level”: two-period moving-average demand forecasting
  - Lead time l = 2

$$A(P) = -1, B(P) = 0, C(P) = \tfrac{1}{2}(1+P)l = 1+P$$

- By Corollary 1:

$$\frac{1 + B(1)l - C(1)}{A(1) + B(1)} = \frac{1+0-2}{-1+0} = 1 > 0.$$

The bullwhip effect exists for sure!

Outline

- Basics
  - Background
  - The bullwhip effect
- Deterministic chain stability
  - System formulation
  - Analytical results
- Stochastic chain stability
  - System formulation
  - Analytical results
- Toward optimality
  - Decentralized negotiations
  - Advance order commitment

Motivation

- Stochastic Environment
  - Unreliable shipments, variable lead times, price fluctuations, etc.
  - Randomness may affect system stability (bullwhip effect)
- Develop analytical conditions to examine the existence of the bullwhip effect

System Dynamics

- Return to the time-domain formulation

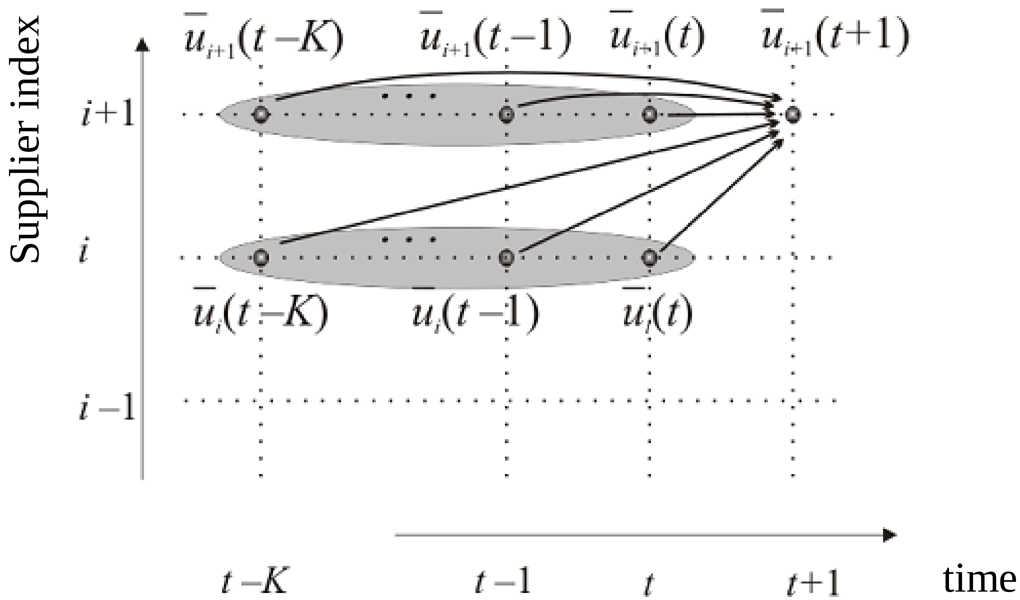
  $\forall\ i, t > 0$

$$\bar{x}_i(t+1) = \bar{x}_i(t) + \bar{u}_i(t) - \bar{u}_{i-1}(t)$$
$$\bar{y}_i(t+1) = \bar{y}_i(t) + \bar{u}_i(t-l_i) - \bar{u}_{i-1}(t)$$
$$\bar{u}_i(t) = A_i(P)\bar{x}_i(t) + B_i(P)\bar{y}_i(t) + C_i(P)\bar{u}_{i-1}(t-1)$$

- Simple algebra gives

$$\bar{u}_i(t+1) = [1 + A_i(P) + P^{l_i} B(P)]\bar{u}_i(t) + [(1-P)C_i(P) - B_i(P) - A_i(P)]\bar{u}_{i-1}(t)$$

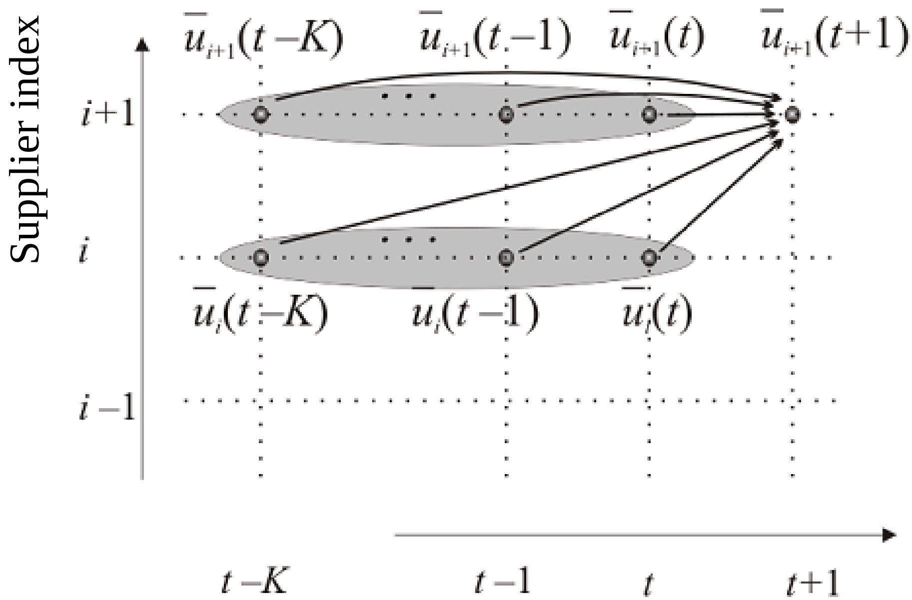
System Dynamics

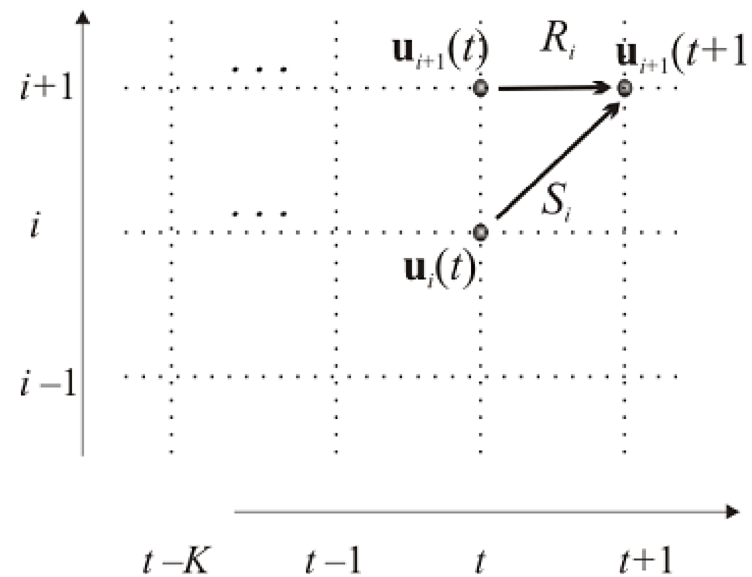


- Simple algebra gives

$$\bar{u}_i(t+1) = [1 + A_i(P) + P^{l_i} B(P)]\bar{u}_i(t) + [(1-P)C_i(P) - B_i(P) - A_i(P)]\bar{u}_{i-1}(t)$$

System Dynamics





- Let $\mathbf{u}_i(t) := [\bar{u}_i(t), \bar{u}_i(t-1), \ldots, \bar{u}_i(t-K)]^{\mathrm{T}}$, then

$$\mathbf{u}_i(t+1) = R_i \cdot \mathbf{u}_i(t) + S_i \cdot \mathbf{u}_{i-1}(t), \ \forall i, t > 0$$

Markovian Jump Linear System (MJLS)

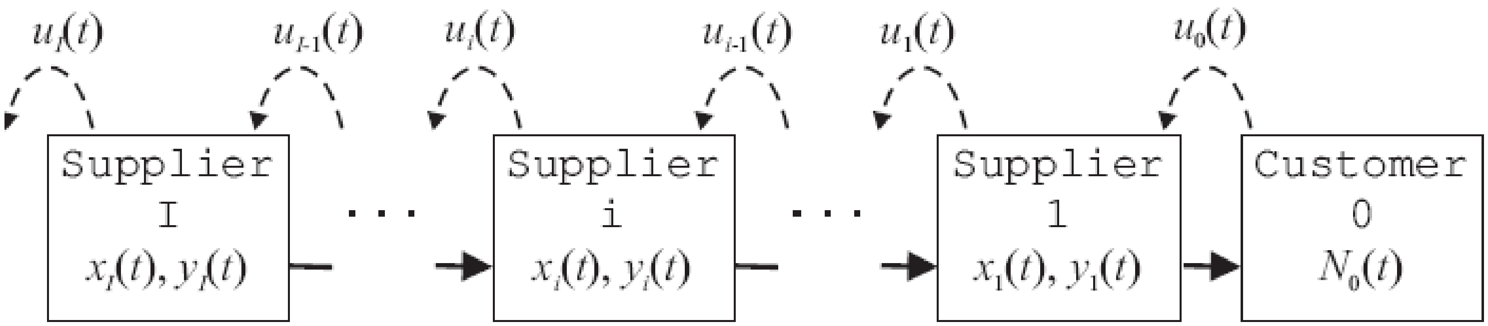
- Allow stochastic model parameters
- Consider

$$\mathbf{u}_i(t+1) = R_{\theta_i(t)} \cdot \mathbf{u}_i(t) + S_{\theta_i(t)} \cdot \mathbf{u}_{i-1}(t), \; \forall i, t > 0$$

where matrix pair $\left\{ R_{\theta_i(t)}, S_{\theta_i(t)} \right\}$ takes value from a finite set according to an exogenous Markov chain $\{\theta_i(t)\}$.

$\theta_i(t) \in \mathcal{M}_i = \{1, 2, \ldots, M_i\}$, with transition probability matrix $\mathcal{P}_i = [p^i{}_{mn}]_{M_i \times M_i}$.

The Bullwhip Effect Metric



(Expected L_2 norm gain)

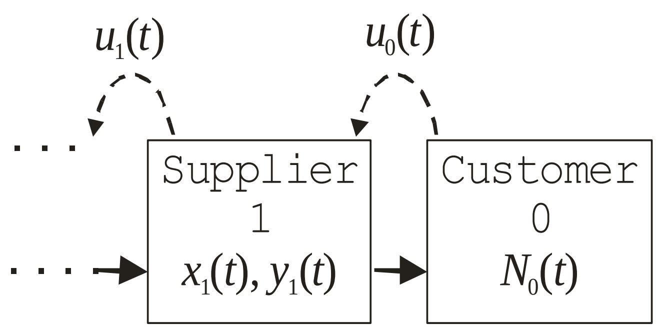
There is no bullwhip effect if

$$\max_{\forall\{\bar{u}_0\}\neq 0} \frac{\mathrm{E}\left(\sum_{t=0}^{\infty} \bar{u}_I^2(t)\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \bar{u}_0^2(t)\right)^{\frac{1}{2}}} \leq 1,$$

where the expectation is taken across Markov chain realizations.

Simplifying Assumption

- Homogeneous supply chain
  - Only need to consider one supplier, i.e., $i = 1$



  - Drop subscript i; e.g., $\theta(t) := \theta_i(t)$
  - When $\theta(t) = m$, let $R_m := R_{\theta(t)}, S_m := S_{\theta(t)}$

Stability Results

Theorem 2 (Sufficient condition for stability)

The bullwhip effect is avoided if there exists non-zero, positive semi-definite matrices $G \geq 0$ and $H := \mathrm{diag}(h_0, h_1, \ldots, h_K) \geq 0$ such that

$$\begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} - \sum_{m=1}^{M} p_{nm} \begin{bmatrix} R_m & S_m \\ E & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} R_m & S_m \\ E & 0 \end{bmatrix} \geq 0, \forall n \in \mathcal{M},$$

where and E is the identity matrix.

Theorem 3 (Necessary condition for stability)

The condition in Theorem 2 is also necessary if:

a) the system is “weakly controllable” (Ji and Chizeck, 1988)

b) the transition probabilities satisfy $p_{nm} \equiv p_m, \forall n \in \mathcal{M}$.

Stability Results

Corollary 2 (Deterministic chains)

The bullwhip effect is avoided in deterministic LTI chains **if and only if** there exists non-zero matrices $G \geq 0$ and $H := \mathrm{diag}(h_0, h_1, \ldots, h_K) \geq 0$ such that

$$\begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} - \begin{bmatrix} R & S \\ E & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} R & S \\ E & 0 \end{bmatrix} \geq 0 ,$$

where and E is the identity matrix.

Example (Deterministic Chain)

A family of "order-based" policies with advance demand information

$$\bar{u}_1(t+1) = \alpha \cdot \bar{u}_1(t) + [\beta_0 + \beta_1 P + \cdots + \beta_K P^K]\bar{u}_0(t)$$

where

$$|\alpha| < 1, \alpha + \beta_0 + \beta_1 + \cdots \beta_K = 1 \qquad \text{(properness)}$$

$$\alpha, \beta_0, \beta_1, \cdots \beta_K \geq 0 \qquad \text{(with advance demand information)}$$

Note:

$$R=\begin{bmatrix}\alpha & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix},\ S=\begin{bmatrix}\beta_0 & \beta_1 & \cdots & \beta_K\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}.$$

When G and H are as follows:

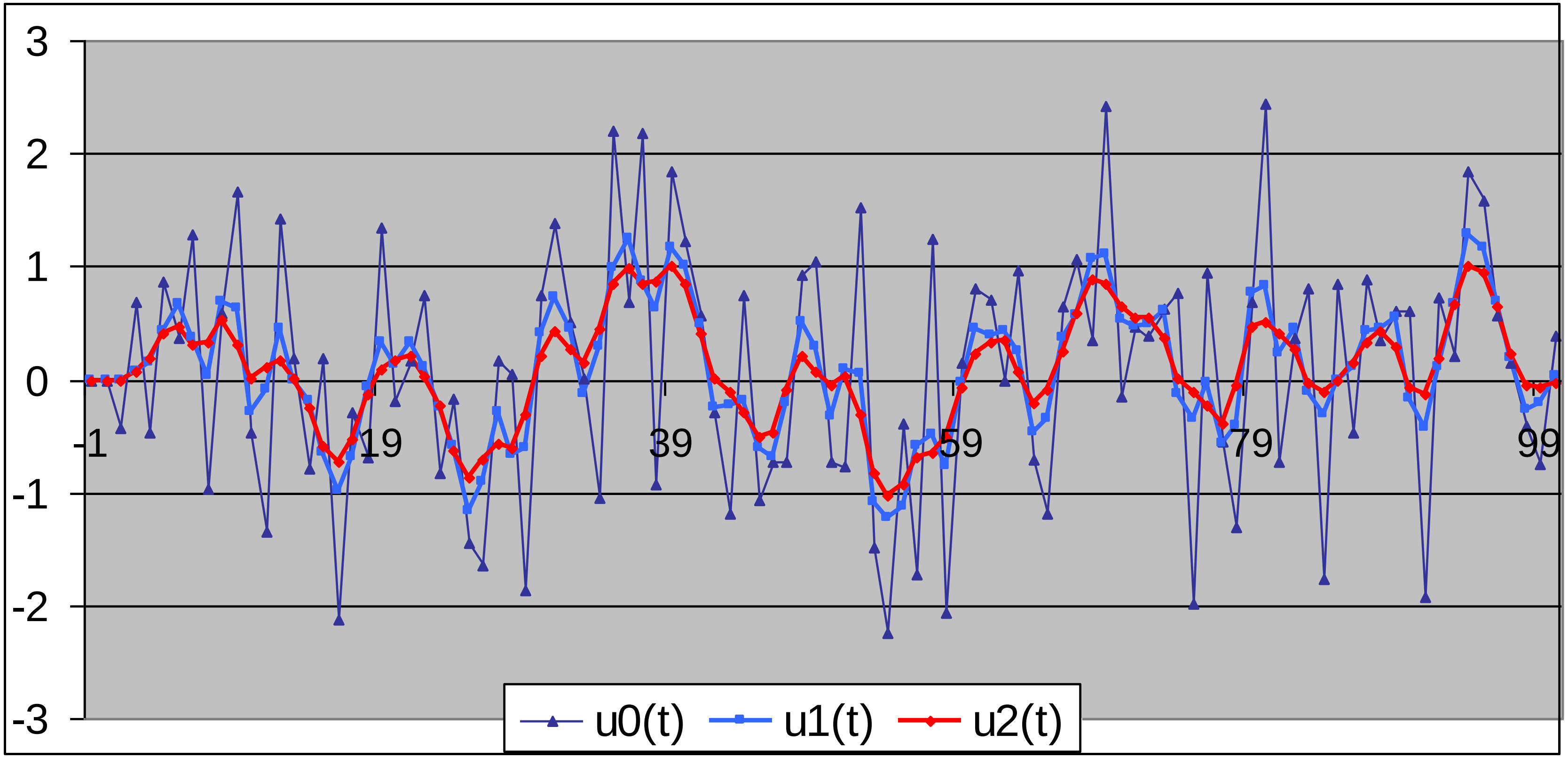
$$G=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & \sum_{k=1}^{K}\beta_k & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \beta_{K-1}+\beta_K & 0\\ 0 & 0 & \cdots & 0 & \beta_K\end{bmatrix},\ H=\begin{bmatrix}\beta_0 & 0 & \cdots & 0\\ 0 & \beta_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta_K\end{bmatrix},$$

the stability condition in Corollary 2:

$$\begin{bmatrix}G & 0\\ 0 & H\end{bmatrix}-\begin{bmatrix}R & S\\ E & 0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}G & 0\\ 0 & H\end{bmatrix}\begin{bmatrix}R & S\\ E & 0\end{bmatrix}\geq 0$$

is satisfied. There is no bullwhip effect.

Simulation



$\bar{u}_0(t)$ ~ i.i.d. Gaussian (0, 1), $\alpha = 0.4$, $\beta_0 = \beta_1 = 0.3$ (K=1)

Example (Stochastic Chain)

- Shipments lost with probability *p*
  - Transition probability matrix ($|\mathcal{M}| = 2$)

$$\mathcal{P} = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}$$

- The order-up-to policy
  - Safe mode

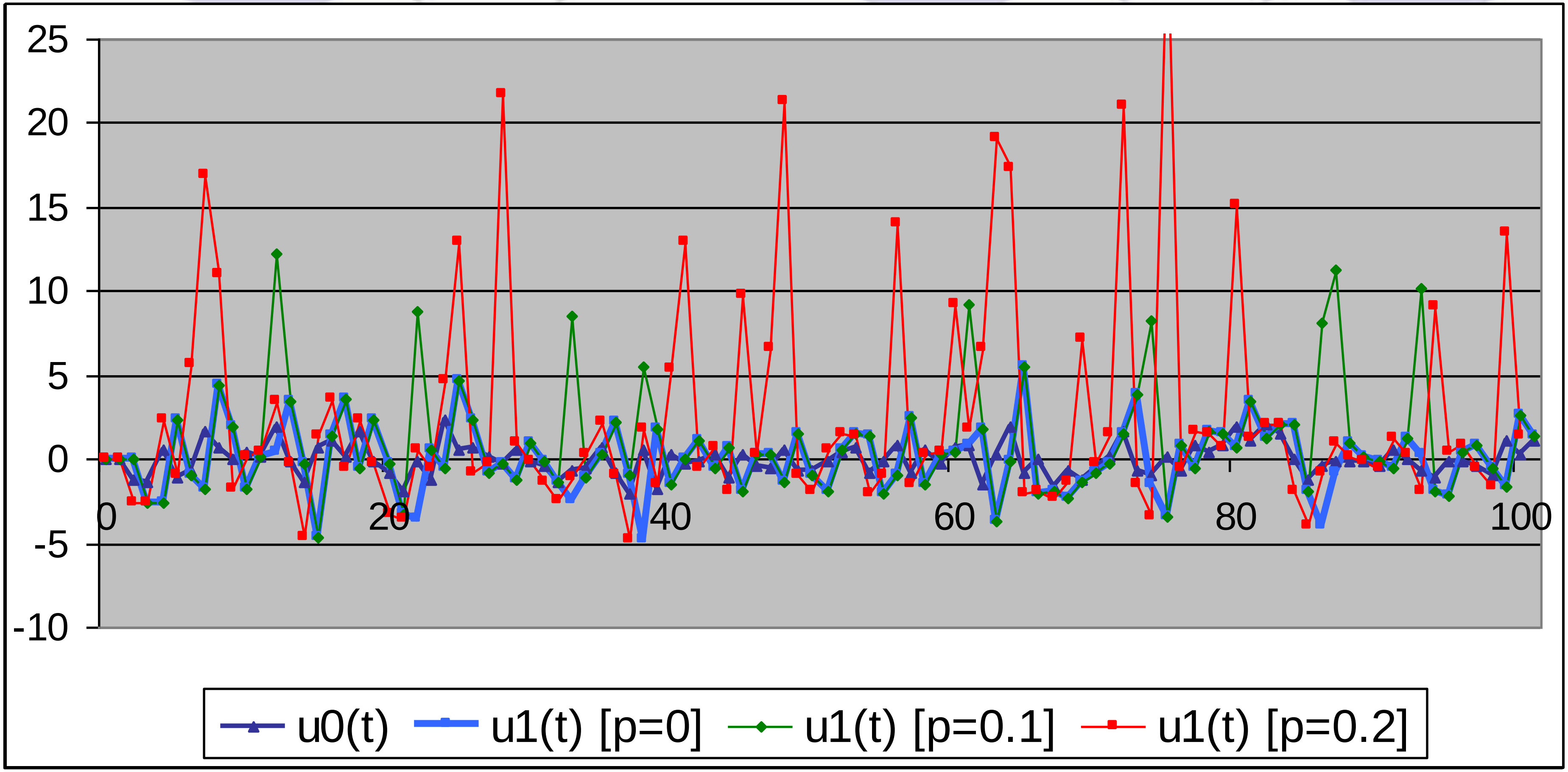
$$\bar{u}_1(t+1) = 2\bar{u}_0(t) - \bar{u}_0(t-2)$$

  - Loss mode

$$\bar{u}_1(t+1) = \bar{u}_1(t-2) + 2\bar{u}_0(t) - \bar{u}_0(t-2) + u^{\infty}$$

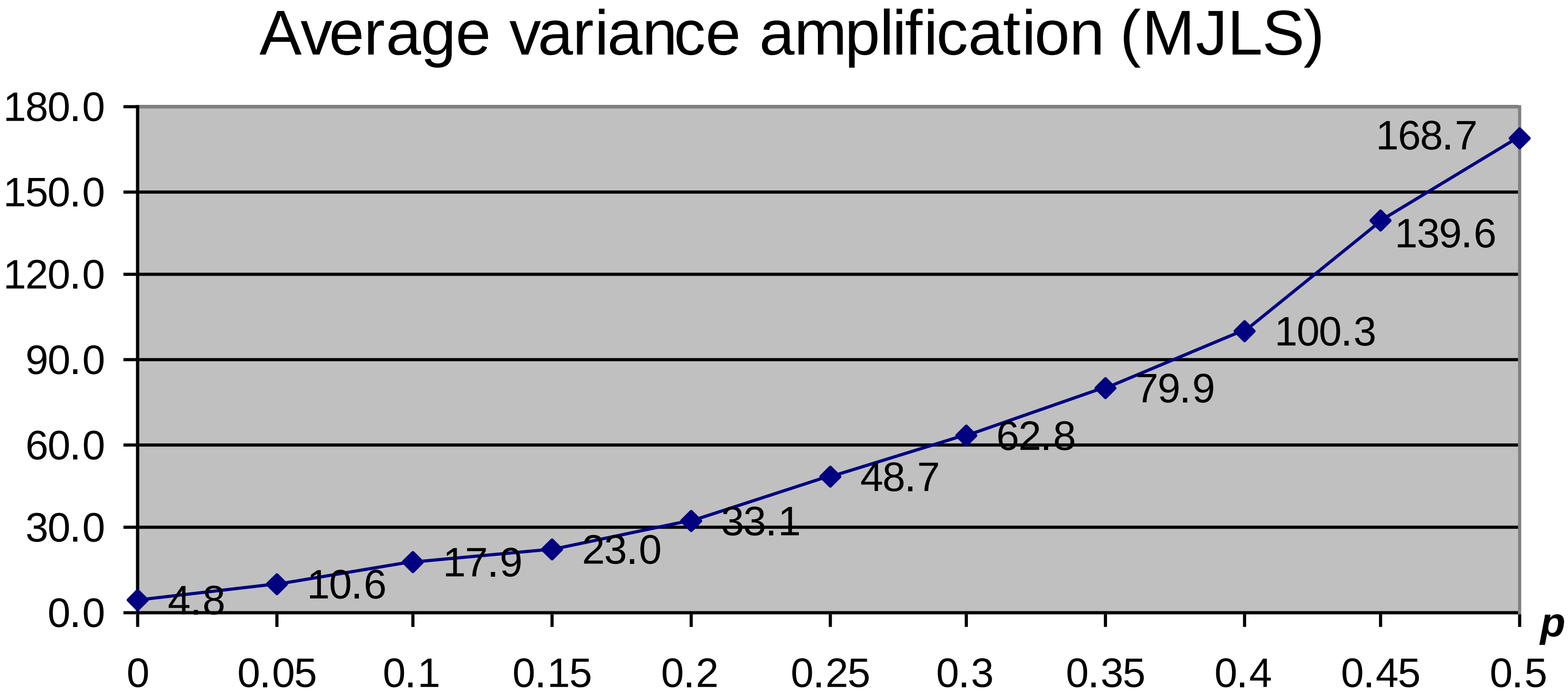
Example (Stochastic Chain)

- Let $\mathbf{u}_1(t) := [\bar{u}_1(t), \bar{u}_1(t-1), \bar{u}_1(t-2), u^\infty]^{\mathrm{T}}, \mathbf{u}_0(t) := [\bar{u}_0(t), \bar{u}_0(t-1), \bar{u}_0(t-2), u^\infty]^{\mathrm{T}}$.
  - Safe mode (m = 1) $R_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_1 = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$
  - Loss mode (m = 2) $R_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$
- Result
  - A numerical search reveals that matrices G and H satisfying Theorem 2 do not exist, $\forall p \in [0,1]$
  - The bullwhip effect exists



$\bar{u}_0(t)$ ~ i.i.d. Gaussian (0, 1)

Plot $\bar{u}_0(t), \bar{u}_1(t)$ under both deterministic condition (p = 0) and stochastic conditions (p = 0.1, p = 0.2)

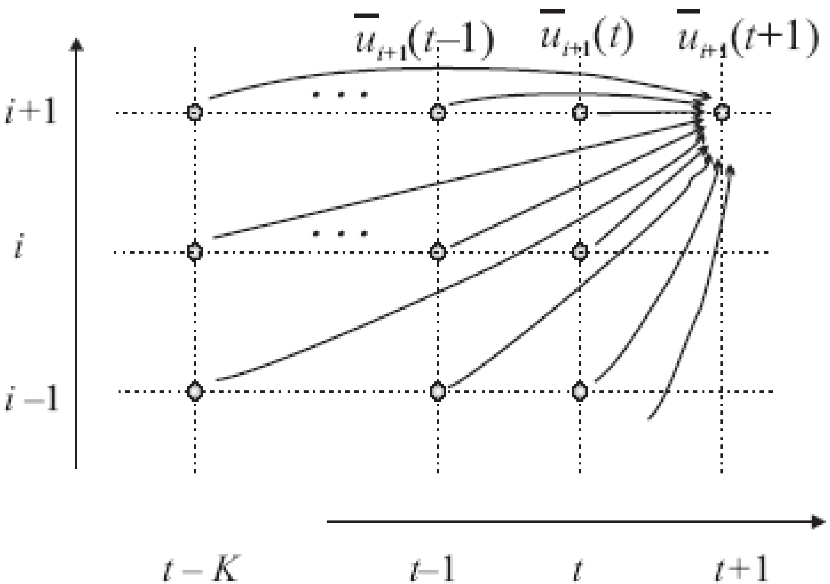

Average variance amplification (MJLS)
180.0
150.0
120.0
90.0
60.0
30.0
0.0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
p
4.8
10.6
17.9
23.0
33.1
48.7
62.8
79.9
100.3
139.6
168.7

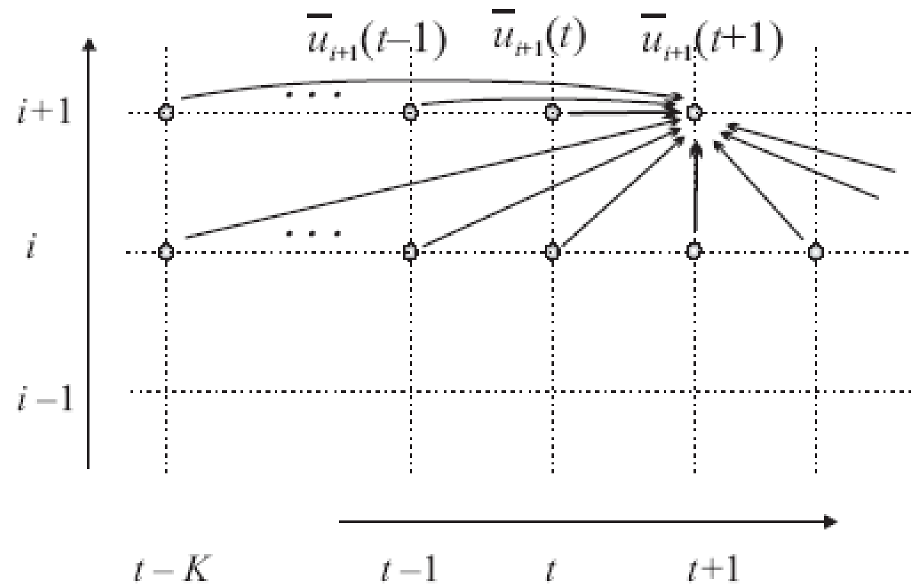
Outline

- Basics
  - Background
  - The bullwhip effect
- Deterministic chain stability
  - System formulation
  - Analytical results
- Stochastic chain stability
  - System formulation
  - Analytical results
- Toward optimality
  - Decentralized negotiations
  - Advance order commitment

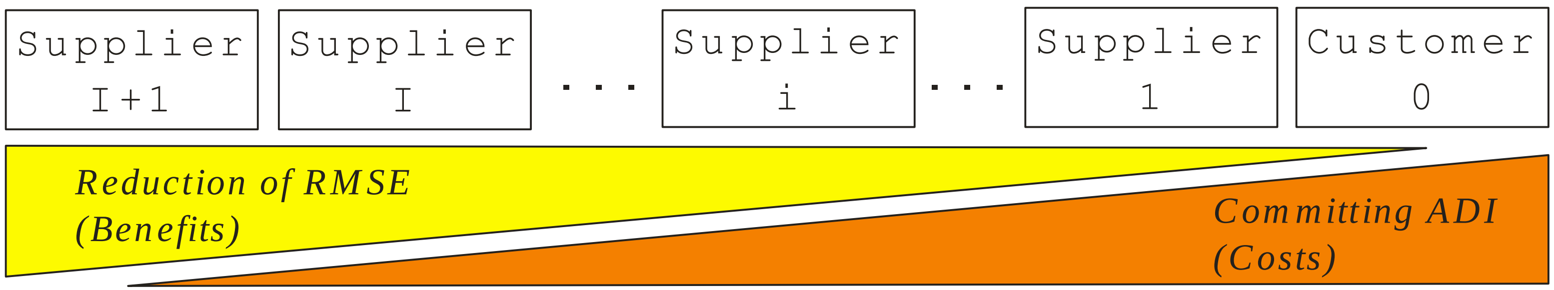
Achieving Stability / Optimality

- Sharing information among suppliers (Lee *et al*., 2000; Simchi-Levi and Zhao, 2003; etc.)
- Advance demand information (ADI) (Hariharan and Zipkin, 1995; Ouyang and Daganzo, 2005; etc.)





Advance Order Commitments



- Downstream suppliers committing to advance orders …
  - Reduces upstream order variations (introduces benefits)
  - Increases own costs
- Able to quantify benefits / costs for every supplier
- Idealized optimum
  - Coordination among suppliers
  - Total benefits exceeds total costs for sufficiently long chains

Decentralized Negotiations

- Negotiations
  - Neighboring suppliers negotiate discounts for advance order commitments and RMSE reductions
- If suppliers are not greedy
  - system reaches the same optimum as if there was a coordinating agent
- If suppliers are greedy and impatient
  - system may reach sub-optimum

Summary

- System-control framework
  - Supply chains
  - The bullwhip effect
- System-level stability
  - Deterministic inhomogeneous chain
  - Stochastic homogeneous chain
- System-level optimality
  - Advance order commitment
  - Decentralized negotiations

Questions?

Thank you!

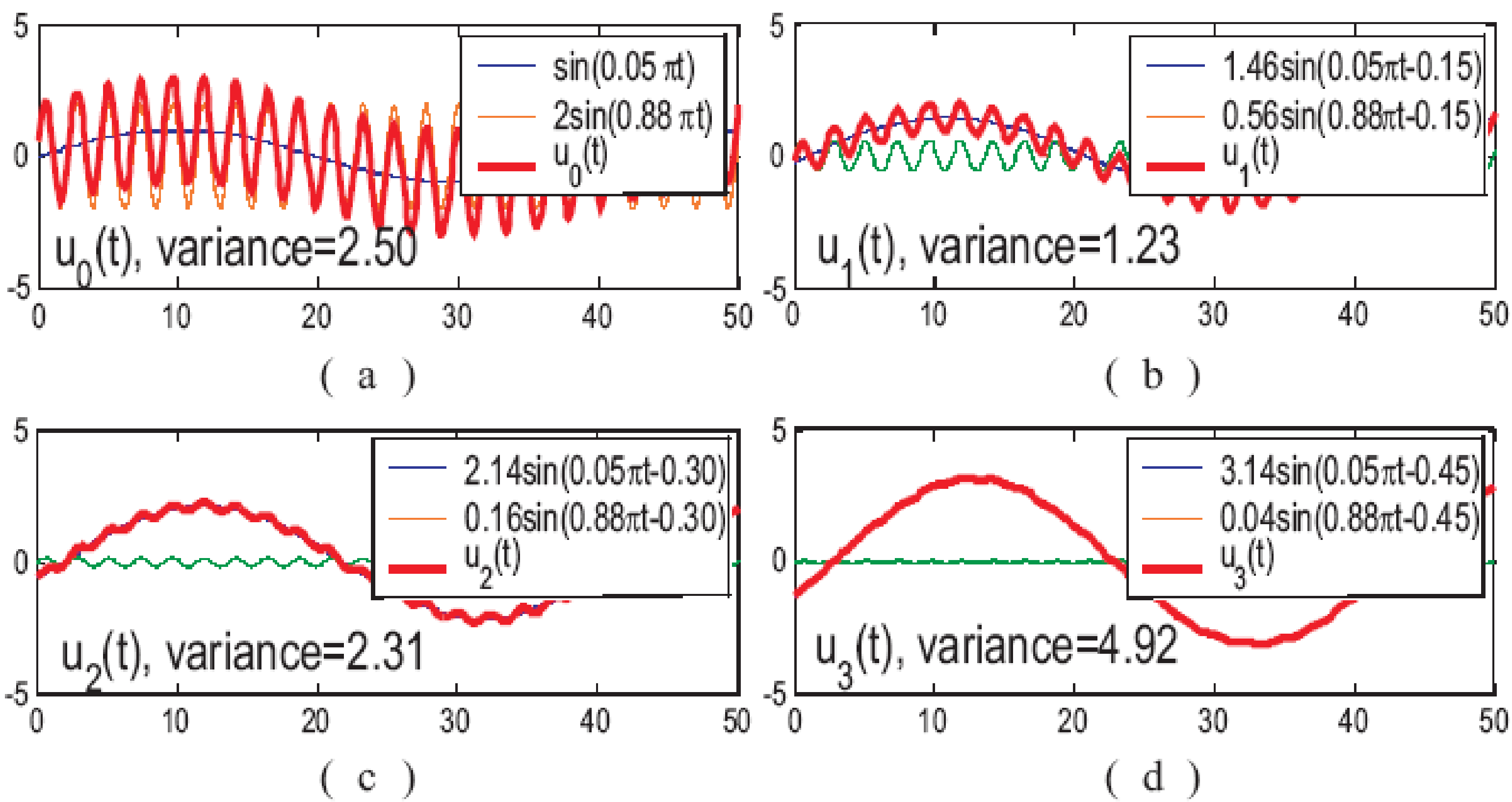
Back-up Slides

Bullwhip Effect Metrics

Time domain	Frequency domain
✓ Worst-case variance amplification (L_2 norm) $$\max_{\forall\{\bar{u}_0\}\neq 0} \frac{\left(\sum_{t=0}^{\infty} \bar{u}_I^2(t)\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \bar{u}_0^2(t)\right)^{\frac{1}{2}}}$$	✓ The peak value on the Bode plot (H_∞ norm) $$\max_{\forall w\in[0,2\pi)} \left|T_I(e^{jw})\right|$$
× Average-case variance amplification for white-noise input sequence × Variance amplification with certain demand process	× Noise-bandwidth (H_2 norm)

Bullwhip Effect Metrics

- The “general replenishment rule” proposed by Dejonckheere (2003a)
- Two sinusoidal input signals: $\sin(0.05\pi t) + 2\sin(0.88\pi t)$
- Amplification ratio: 1.464 and 0.282 through each stage; phase change: -0.1507 and -0.1515



Theorem 1 Proof (Deterministic Chains)

Boyd and Desoer (1985) showed that $\log \|T_I(e^\sigma)\|$ is subharmonic with regard to σ and satisfies the Poisson Inequality:

$$\log|T_I(e^y)| \le \frac{1}{\pi}\int_{-\infty}^{+\infty} \log|T_I(e^{jw})| \frac{y\,dw}{y^2+w^2}, \forall y \in (0,\infty)$$

Divide both sides by y, and let $y \to 0^+$,

$$\begin{aligned}\lim_{y\to 0+} \frac{1}{y}\log|T_I(e^y)| &\le \lim_{y\to 0+}\frac{1}{\pi}\int_{-\infty}^{+\infty} \log|T_I(e^{jw})| \frac{dw}{y^2+w^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \log|T_I(e^{jw})| \frac{dw}{w^2}\end{aligned}$$

Note that $T_I(z) = \prod_{i=1}^{I} T_{i-1,i}(z)$, therefore

$$\lim_{y\to 0+} \frac{1}{y}\log|T_I(e^y)| = \sum_{i=1}^{I} \lim_{y\to 0+} \frac{1}{y}\log|T_{i-1,i}(e^y)|.$$

By Taylor expansion at $y = 0$,

$$\begin{aligned}T_{i-1,i}(e^y) &= T_{i-1,i}(e^y)|_{y=0} + [T_{i-1,i}(e^y)]'_y|_{y=0}\cdot y + o(y) \\ &= 1 + \frac{1+B_i(1)l_i - C_i(1)}{A_i(1)+B_i(1)}\cdot y + o(y)\end{aligned}$$

Theorem 1 Proof (Deterministic Chains)

At the neighborhood of 0^+, $\frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} \cdot y + o(y) \ll 1$, therefore

$$\begin{aligned} |T_{i-1,i}(e^y)| &= \left|1 + \frac{1+B_i(1)l_i - C_i(1)}{A_i(1)+B_i(1)} \cdot y + o(y)\right| \\ &= 1 + \frac{1+B_i(1)l_i - C_i(1)}{A_i(1)+B_i(1)} \cdot y + o(y), \end{aligned}$$

By l'Hôpital's Rule,

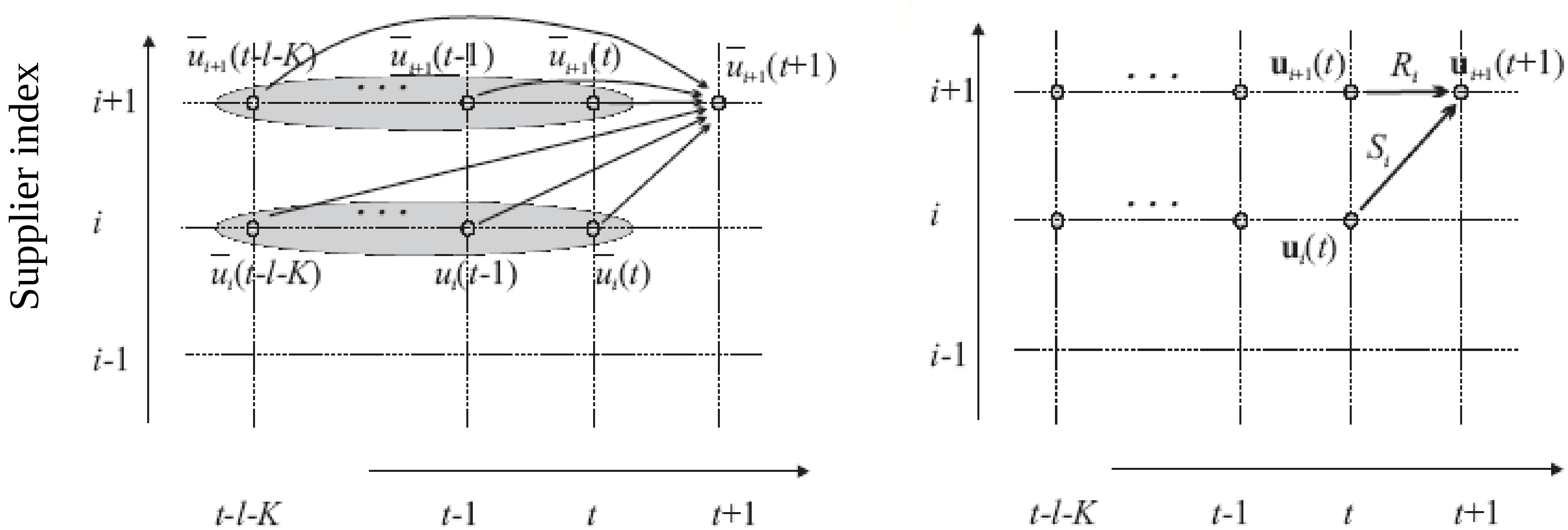
$$\begin{aligned} \lim_{y\to 0^+} \frac{1}{y} \log |T_{i-1,i}(e^y)| &= \lim_{y\to 0^+} \frac{|T_{i-1,i}(e^y)|'_y}{|T_{i-1,i}(e^y)|} \\ &= \lim_{y\to 0^+} \frac{\frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} + O(y)}{1 + \frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} \cdot y + o(y)} \\ &= \frac{1+B_i(1)l_i - C_i(1)}{A_i(1)+B_i(1)}. \end{aligned}$$

we have

$$\sum_{i=1}^{I} \frac{1+B_i(1)l_i - C_i(1)}{A_i(1)+B_i(1)} \le \frac{1}{\pi} \int_{-\infty}^{+\infty} \log |T_I(e^{jw})| \frac{dw}{w^2}.$$

Back

System Dynamics



- Let $\mathbf{u}_i(t) := [\bar{u}_i(t), \bar{u}_i(t-1), \ldots, \bar{u}_i(t-l_i-K)]'$, then

$$\mathbf{u}_i(t+1) = R_i \cdot \mathbf{u}_i(t) + S_i \cdot \mathbf{u}_{i-1}(t), \forall i, t > 0$$

where

$$R_i = \begin{bmatrix} \alpha_0^i & \alpha_1^i & \alpha_2^i & \ldots & \alpha_{K+l_i-1}^i & \alpha_{K+l_i}^i \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \quad S_i = \begin{bmatrix} \beta_0^i & \beta_1^i & \ldots & \beta_{K+1}^i & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Almost-surely stability (Mariton 1995)

Theorem 2 (Time stability). *The system described above is almost surely stable if for $w(t) \equiv 0$ and every initial state $(u(0), \theta(0))$,*

$$Pr\{\lim_{t \to \infty} \|u(t)\| = 0\} = 1.$$

With an ergodic Markov chain $\theta(t)$, almost sure stability is achieved when

$$\sum_{m=1}^{M} \pi_m |\sigma(R_m)| < 1$$

where π_m is the long-run probability of mode m, and $|\sigma(R_m)|$ is the spectral radius of matrix R_m.

Variance Amplification Bounds

Theorem 4 (Bounds for bullwhip effect metric)

The bullwhip effect metric (variance amplification) is bounded by μ ($\mu > 0$), if there exists non-zero matrices $G \geq 0$ and $H := \operatorname{diag}(h_0, h_1, \ldots, h_K) \geq 0$ such that

$$\begin{bmatrix} G & 0 \\ 0 & \mu \cdot H \end{bmatrix} - \sum_{m=1}^{M} p_{nm} \begin{bmatrix} R_m & S_m \\ E & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} R_m & S_m \\ E & 0 \end{bmatrix} \geq 0, \forall n \in \mathcal{M},$$

where and E is the identity matrix.

Theorem 2 Proof (Stochastic Chains)

Define a scalar function $V(u) := gu^2$. $g \geq 0$, we have

$$\mathrm{E}_{\theta(0),\ldots,\theta(T)} \sum_{t=0}^{T} [V(u(t+1)) - V(u(t))] = \mathrm{E}_{\theta(0),\ldots,\theta(T)} V(u(T+1)) \geq 0$$

we need to prove that

$$\mathrm{E}\left[\sum_{t=0}^{\infty} u^2(t)\right] \leq \sum_{t=0}^{\infty} w^2(t).$$

Define the expected L_2 norm of a truncated stochastic sequence $\{u(0), u(1), \ldots, u(T)\}$ as

$$\begin{aligned}
&\mathrm{E}_{\theta(0),\ldots,\theta(T-1)}\left[\sum_{t=0}^{T} u^2(t)\right] \\
\leq\ & \sum_{t=0}^{T} w^2(t) + \mathrm{E}_{\theta(0),\ldots,\theta(T)}\left[\sum_{t=0}^{T}\left(u^2(t) - w^2(t) + V(u(t+1)) - V(u(t))\right)\right] \\
=\ & \sum_{t=0}^{T} w^2(t) + \sum_{t=0}^{T} \mathrm{E}_{\theta(0),\ldots,\theta(T)}\left[u^2(t) - w^2(t) + gu^2(t+1) - gu^2(t)\right] \\
=\ & \sum_{t=0}^{T} w^2(t) - \sum_{t=0}^{T} \mathrm{E}_{\theta(0),\ldots,\theta(t)}\left\{[\mathbf{u}(t)'\mathbf{w}(t)'] F_{\theta(t)} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix}\right\}
\end{aligned}$$

Theorem 2 Proof (Stochastic Chains)

where

$$F_{\theta(t)} := \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} - \begin{bmatrix} R_{\theta(t)} & S_{\theta(t)} \\ I & 0 \end{bmatrix}' \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} R_{\theta(t)} & S_{\theta(t)} \\ I & 0 \end{bmatrix},$$

and

$$\mathbf{u}(t) = [u(t), u(t-1), ..., u(t-l-K)]', \ \mathbf{w}(t) = [w(t), w(t-1), ..., w(t-l-K)]'.$$

For $\forall t, 0 \leq t \leq T$,

$$\begin{aligned}
& \mathrm{E}_{\theta(0),\ldots,\theta(t)} \left\{ [\mathbf{u}(t)'\mathbf{w}(t)'] F_{\theta(t)} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \right\} \\
\overset{(a)}{=} \; & \mathrm{E}_{\theta(0),\ldots,\theta(t-1)} \left\{ \mathrm{E}_{\theta(t)} \left[[\mathbf{u}(t)'\mathbf{w}(t)'] F_{\theta(t)} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \middle| \theta(0), \ldots, \theta(t-1) \right] \right\} \\
\overset{(b)}{=} \; & \mathrm{E}_{\theta(0),\ldots,\theta(t-1)} \left\{ [\mathbf{u}(t)'\mathbf{w}(t)'] \, \mathrm{E}_{\theta(t)} \left[F_{\theta(t)} | \theta(0), \ldots, \theta(t-1) \right] \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \right\} \\
\overset{(c)}{=} \; & \mathrm{E}_{\theta(0),\ldots,\theta(t-1)} \left\{ [\mathbf{u}(t)'\mathbf{w}(t)'] \left(\sum_{m=1}^{M} p_{\theta(t-1)m} F_m \right) \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \right\} \\
\overset{(d)}{\geq} \; & 0.
\end{aligned}$$

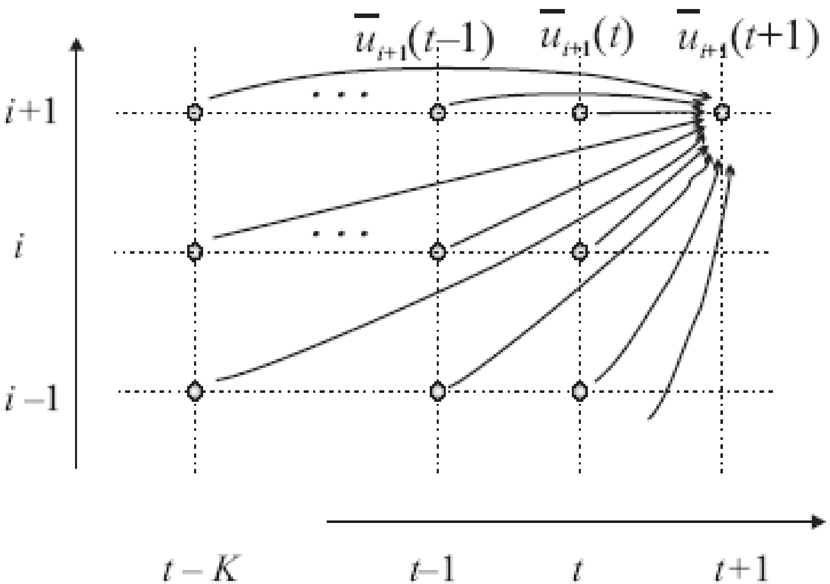
……

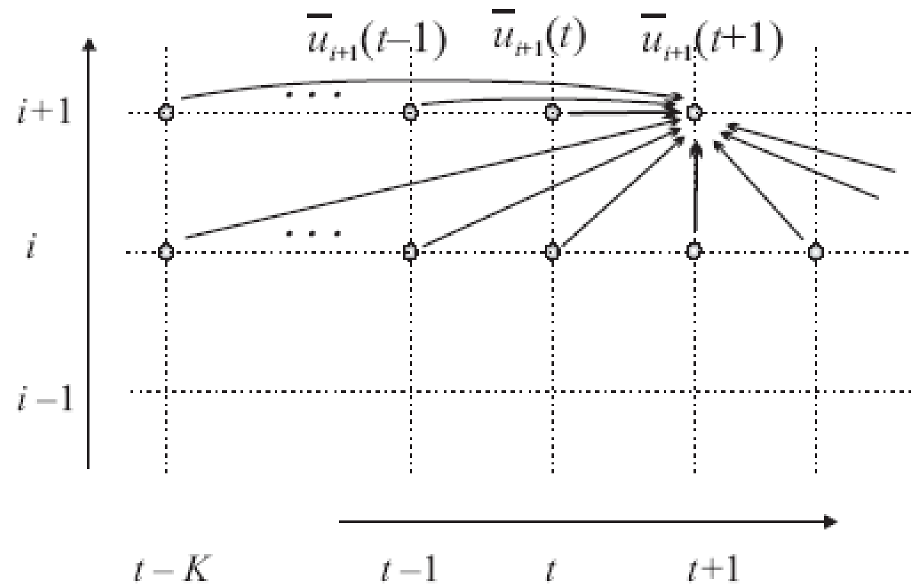
Outline

- Basics
  - Background
  - The bullwhip effect
- Deterministic inhomogeneous chain
  - System formulation
  - Analytical results
- Stochastic homogeneous chain
  - System formulation
  - Analytical results
- A decentralized contracting scheme
  - Advance order commitment

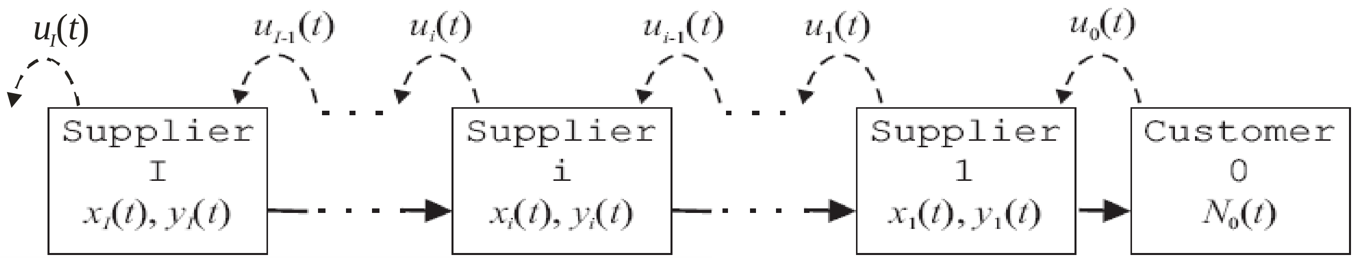
Achieving System-level Stability

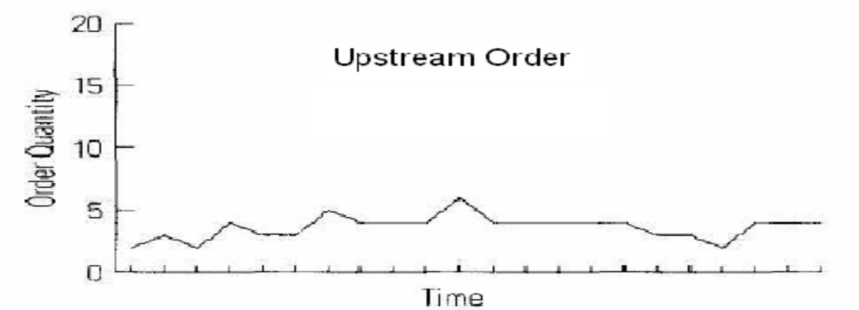
- Sharing information among suppliers (Lee *et al*. 2000; Simchi-Levi and Zhao, 2003; etc.)
- Advance demand information (Hariharan and Zipkin 1995; Ouyang and Daganzo, 2005; etc.)





Advance Order Commitments





- Downstream suppliers placing advance orders …
  - Reduces upstream order variations (introduces benefits)
  - Increases own costs
- Able to quantify benefits / costs for every supplier
- Feasibility
  - Total benefits exceeds total costs (an "imaginary broker" can profit)

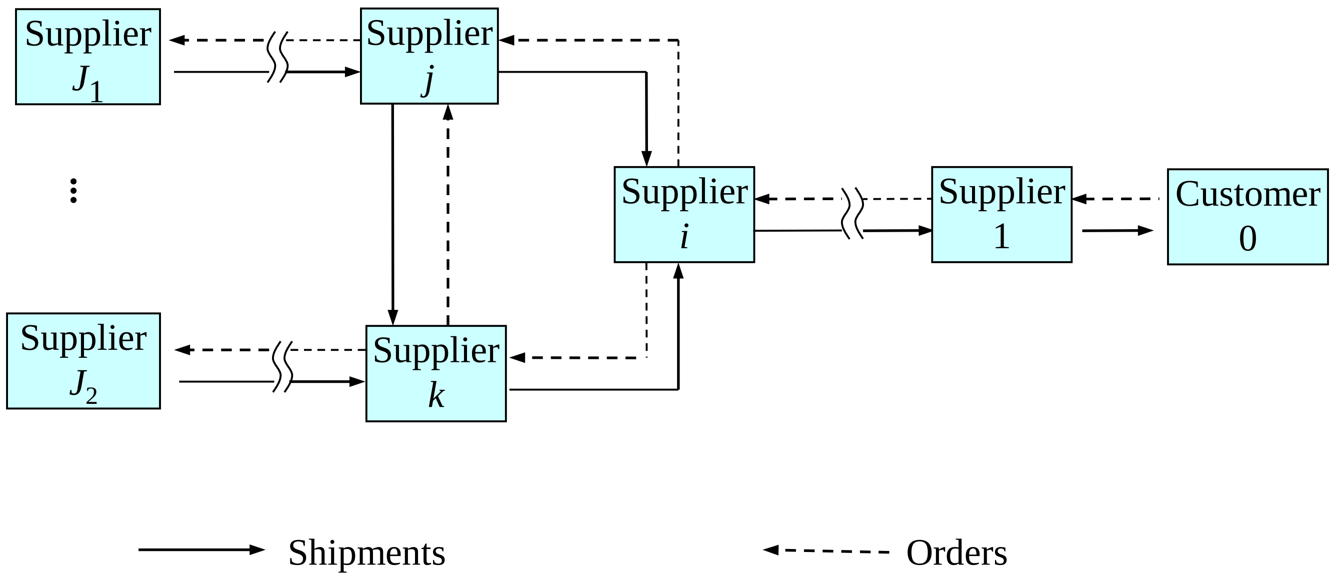
A Decentralized Option

- Contracting
  - Neighboring suppliers negotiate discounts for advance order commitments and variance reductions
  - System reaches the same equilibrium as if there was a "broker"
- No coordinating agent is necessary

Extensions

- Supply network
- Nonlinear system
  - ordering policy
  - operation (e.g., load-dependent lead time)
- Endogenous MJLS

Supply Networks



- $G = (V \cup W, E)$, V = {supplier node}, W = {customer node}, E={ordering arc};
- Inventory $\mathbf{x}(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\}$, orders $\mathbf{u}(t) = \{u_{ij}(t): (i, j) \in E\}$;
- lead time $\{l_{ij}\}$, shipment loss $\{\rho_{ij}\}$, $\forall (i, j) \in E$.

Supply Networks

System dynamics:

Inventory dynamics: $x_i(t+1) = x_i(t) + \sum_{j:(i,j)\in A} \left(\rho_{ij} \cdot u_{ij}(t - l_{ij})\right) - \sum_{k:(k,i)\in E} u_{ki}(t)$, $i = 1, \ldots, n$, $\mathbf{x}(0) = \mathbf{0}$;

Ordering policy: $u_{ij}(t) = \sum_{k:(i,k)\in E} A_{ik}(P) u_{ik}(t) + \frac{1}{\rho_{ij}} \left[\alpha_{ij}^0 x_i(t) + B_{ij}(P) \cdot \sum_{k:(k,i)\in E} u_{ki}(t) \right]$, $\forall (i, j) \in E$, $i > 0$

Supply Networks

Motion Equations:

$$X_i(z) = \frac{1}{z-1}\left[\sum_{j:(i,j)\in E}\left(\rho_{ij}\cdot U_{ij}(z)\cdot z^{-l_{ij}}\right) - \sum_{k:(k,i)\in E} U_{ki}(z)\right],\ i = 1, \ldots, n$$

$$U_{ij}(z) = \frac{\left(B_{ij}(z^{-1}) - \alpha_{ij}^{0}(z-1)^{-1}\right)}{\rho_{ij}\left(1 - A_{ij}(z^{-1}) - \alpha_{ij}^{0} z^{-l_{ij}}(z-1)^{-1}\right)} \cdot \sum_{k:(k,i)\in E} U_{ki}(z)$$
$$+ \frac{\sum_{k:(i,k)\in E, k\neq j}\left[U_{ik}(z)\cdot\left(\alpha_{ij}^{0}\rho_{ik} z^{-l_{ik}}(z-1)^{-1} + A_{ik}(z^{-1})\right)\right]}{\rho_{ij}\left(1 - A_{ij}(z^{-1}) - \alpha_{ij}^{0} z^{-l_{ij}}(z-1)^{-1}\right)},\ \forall (i,j)\in E,\ i>0$$

Transfer Function:

$$T_i(z) = \frac{\sum_{j:(i,j)\in E} U_{ij}(z)}{U_{01}(z)}$$