Dispersion relations for spring-mass lattices

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1 Simple mass-spring lattice

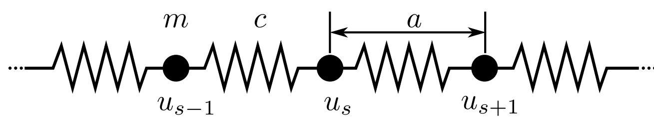


Figure 1. Simple mass-spring lattice.

1.1 Direct formulation

The force in the plane s cause by the displacement of the plane $s+p$ is proportional to the difference $u_{s+p} - u_s$ of the displacements. For brevity we will consider only nearest-neighbor interactions, so $p = \pm 1$. The total force on s comes from planes $s = \pm 1$:

$$F_s = c(u_{s+1} - u_s) + c(u_{s-1} - u_s). \tag{1}$$

The constant c is the stiffness between nearest-neighbour planes and will differ for longitudinal and transverse waves.

The equation of motion of the plane s is

$$m\ddot{u} = c(u_{s+1} + u_{s-1} - 2u_s),$$

assuming a harmonic time dependence $\exp(-i\omega t)$

$$-m\omega^2 u_s = c(u_{s+1} + u_{s-1} - 2u_s)\ . \tag{2}$$

Due to the Bloch-periodicity condition

$$u_{s\pm1} = u_s e^{\pm ika}.$$

So (2) is now

$$-m\omega^2 u_s = c(u_s \exp(ika) + u_s \exp(-ika) - 2u_s)$$

and canceling u_s from both sides, we have

$$\omega^2 m = -c[\exp(ika) + \exp(-ika) - 2]\ .$$

Using the identity $2\cos ka = \exp(ika) + \exp(-ika)$, we have the dispersion relation

$$\omega^2 = (2c/m)(1 - \cos ka) \ . \tag{3}$$

The boundary of the first Brillouin zone lies at $k = \pm\pi/a$. We show from (3) that the slope of ω versus k is zero at the zone boundary

$$\frac{d\omega^2}{dk} = (2ca/m)\sin ka = 0$$

at $k = \pm\pi/a$, $\sin ka = 0$.

By a trigonometric identity (2) may be written as

$$\omega^2 = (4c/m)\sin^2\frac{1}{2}ka, \qquad \omega = (4c/m)^{1/2}\left|\sin\frac{1}{2}ka\right| \ . \tag{4}$$

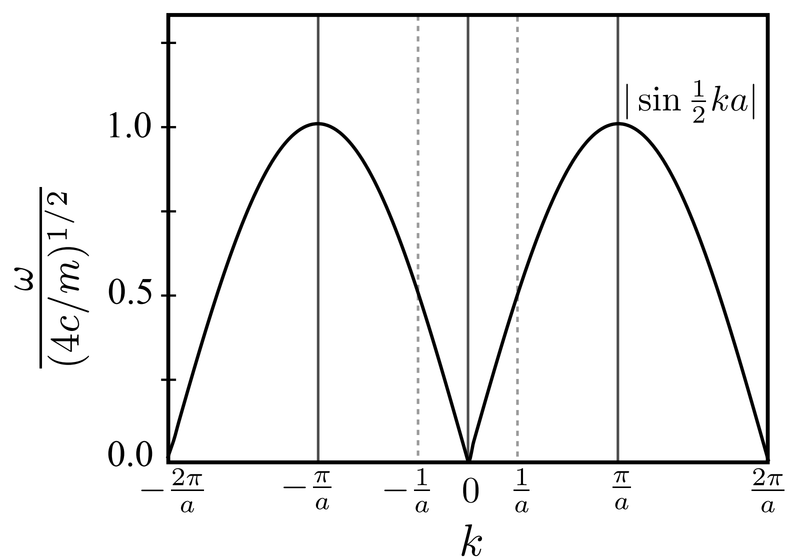


Figure 2. Plot of ω versus k. The region of $k << 1/a$ or $\lambda >> a$ corresponds to the continuum approximation; where ω is directly proportional to k. The First Brillouin Zone is placed between $-\pi/a$ and π/a.

1.2 Unit cell formulation

This problem could also be solved taking a single cell and imposing the Bloch conditions after the assemblage process for the matrices. The force balance (in frequency domain) is

$$c(u_2 - u_1) = -\frac{m}{2}\omega^2 u_1 \ ,$$
$$c(u_1 - u_2) = -\frac{m}{2}\omega^2 u_2 \ ,$$

we should take care since the total amount of mass in the cell should be consistent. That's why we choose each particle to have $m/2$. And this system could be expressed as

$$c\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\{\mathbf{u}\} = -\frac{m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\{\mathbf{u}\} \ , \tag{5}$$

multiplying the second row by $\exp(-ika)$ and the second column by its complex conjugate $\exp(ika)$ we get

$$c\begin{bmatrix} -1 & e^{ika} \\ e^{-ika} & -1 \end{bmatrix}\{\mathbf{u}\} = -\frac{m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\{\mathbf{u}\} \quad ,$$

adding the second row to first one, and then adding the second column to first one yields

$$c\begin{bmatrix} e^{ika}+e^{-ika}-2 & e^{ika}-1 \\ e^{-ika}-1 & -1 \end{bmatrix}\{\mathbf{u}\} = -\frac{m}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\{\mathbf{u}\} \quad .$$

Now, were interested just in the reduced system, we delete the second row and column and get

$$\frac{c}{m}\left[e^{ika}+e^{-ika}-2\right] = \omega^2 \quad , \tag{6}$$

and this could be rewritten as

$$\omega = (4c/m)^{1/2}\left|\sin\frac{1}{2}ka\right| \quad , \tag{7}$$

which is the same result obtained before.

1.3 Finite case

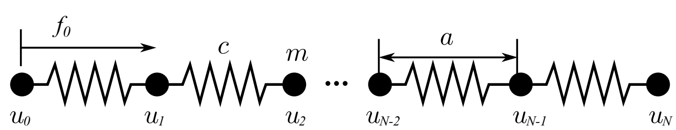


Figure 3. Finite simple mass-spring lattice.

In practice, we cannot have infinite (periodic) repetition of unit cells, and, at some point, we need to truncate them to make them finite (Figure 3). When the number of unit cells increase, the behavior should get closer to the infinite case, and the presence of bandgaps could be inferred from the analysis of the response in frequency.

If we start from a single unit cell the equations of motion for harmonic force of frequency ω are given by

$$\begin{aligned} &-c(u_0-u_1)+f_0 = -\omega^2 m u_0 \\ &-c(u_1-u_0) = -\omega^2 m u_1 \quad , \end{aligned}$$

or

$$\begin{bmatrix} -1+\Omega^2 & 1 \\ 1 & -1+\Omega^2 \end{bmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -\hat{f}_0 \\ 0 \end{pmatrix} \quad ,$$

where we normalized the equation. $\Omega^2 = \omega^2/\omega_0^2 = \omega^2/(k/m)$ and $\hat{f}_0 = f_0/c$. And the solutions for this system of equations is

$$\begin{aligned} u_0 &= \frac{\hat{f}_0(1-\Omega^2)}{\Omega^2(\Omega^2-2)} \\ u_1 &= \frac{\hat{f}_0}{\Omega^2(\Omega^2-2)} \quad . \end{aligned}$$

We are interested in the response of the last mass with respect to the force impinged in the first one, i.e.

$$\frac{u_1}{\hat{f}_0} = \frac{1}{\Omega^2(\Omega^2 - 2)} \quad .$$

If we take two unit cells the (dimensionless) system of equations read

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} + \Omega^2 \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\hat{f}_0 \\ 0 \\ 0 \end{pmatrix} \quad ,$$

where the solutions are

$$\begin{aligned} u_0 &= \frac{\hat{f}_0(\Omega^2 - \Omega - 1)(\Omega^2 + \Omega - 1)}{\Omega^2(\Omega - 1)(\Omega + 1)(\Omega^2 - 3)} \\ u_1 &= \frac{-\hat{f}_0}{\Omega^2(\Omega^2 - 3)} \\ u_2 &= \frac{\hat{f}_0}{\Omega^2(\Omega - 1)(\Omega + 1)(\Omega^2 - 3)} \quad . \end{aligned}$$

And the response in this case is given by

$$\frac{u_2}{\hat{f}_0} = \frac{1}{\Omega^2(\Omega - 1)(\Omega + 1)(\Omega^2 - 3)}$$

In the case of N unit cells the system of equations is

$$\begin{bmatrix} -1+\Omega^2 & 1 & \cdots & 0 & 0 \\ 1 & -2+\Omega^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2+\Omega^2 & 1 \\ 0 & 0 & \cdots & 1 & -1+\Omega^2 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} -\hat{f}_0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad .$$

Given the structure of the matrix it is possible that the system has a closed solution. Here, we solved the system numerically and plotted the ratio $u_N/\hat{f}_0$ for different N values (Figure 4).

2 Diatomic crystal

We write the equations of motion under the assumption that each plane interacts only with its nearest-neighbour planes and that the force constants are identical between all pairs of nearest-neighbour planes.

$$m_1 \frac{d^2 u_s}{dt^2} = c(v_s + v_{s-1} - 2u_s); \tag{8a}$$

$$m_2 \frac{d^2 v_s}{dt^2} = c(u_{s+1} + u_s - 2v_s). \tag{8b}$$

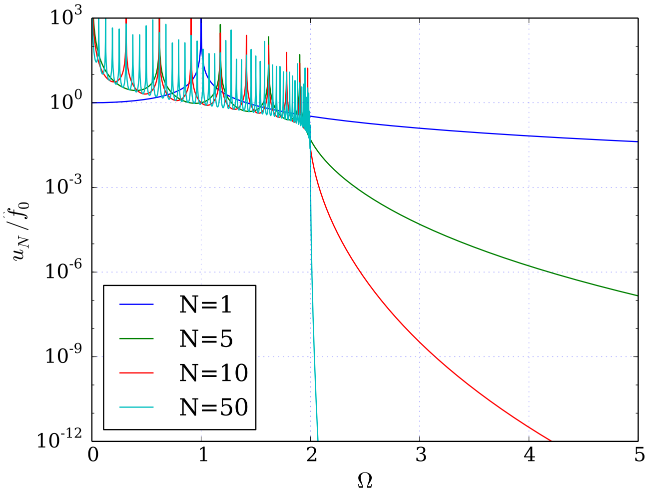


Figure 4. Frequency response for a finite mass-spring lattice.

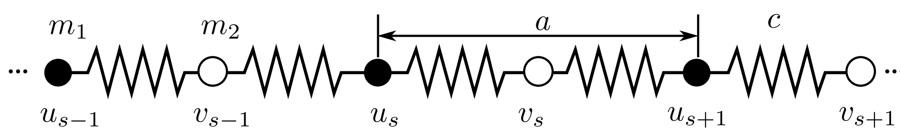


Figure 5. Lattice made with spring and two species of masses.

We look for a solution in the form of a traveling wave, now with different amplitudes u, v on alternate phases:

$$u_s = u\exp(ika)\exp(-i\omega t); \tag{9a}$$
$$v_s = v\exp(ika)\exp(-i\omega t). \tag{9b}$$

Replacing (9) in (8) we have

$$-\omega^2 m_1 u = cv[1+\exp(-ika)] - 2cu;$$
$$-\omega^2 m_2 v = cu[1+\exp(ika)] - 2cv.$$

It has no trivial solution only if the determinant vanishes, i.e.

$$\begin{vmatrix} 2c - m_1\omega^2 & -c[1+\exp(-ika)] \\ -c[1+\exp(ika)] & 2c - m_2\omega^2 \end{vmatrix} = 0,$$

or

$$m_1 m_2 \omega^4 - 2c(m_1+m_2)\omega^2 + 2c^2(1-\cos ka) = 0 \ . \tag{10}$$

Solving this biquadratic equation and doing some algebraic manipulations, we get

$$\omega^2 = \frac{c}{m_1 m_2}\left[m_1 + m_2 \pm \sqrt{(m_1+m_2)^2 - 2m_1m_2(1-\cos ka)}\right] \ . \tag{11}$$

Let's examine the limiting cases $ka << 1$ and $ka = \pm\pi$ at the boundary zone. For small ka we have $\cos ka \approx 1 - \frac{1}{2}k^2a^2 + \cdots$, and the two roots are

$$\omega^2 \approx 2c\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{(optical branch)} \tag{12}$$

$$\omega^2 \approx \frac{\frac{1}{2}c}{m_1+m_2}k^2a^2 \quad \text{(acoustical branch)} \ . \tag{13}$$

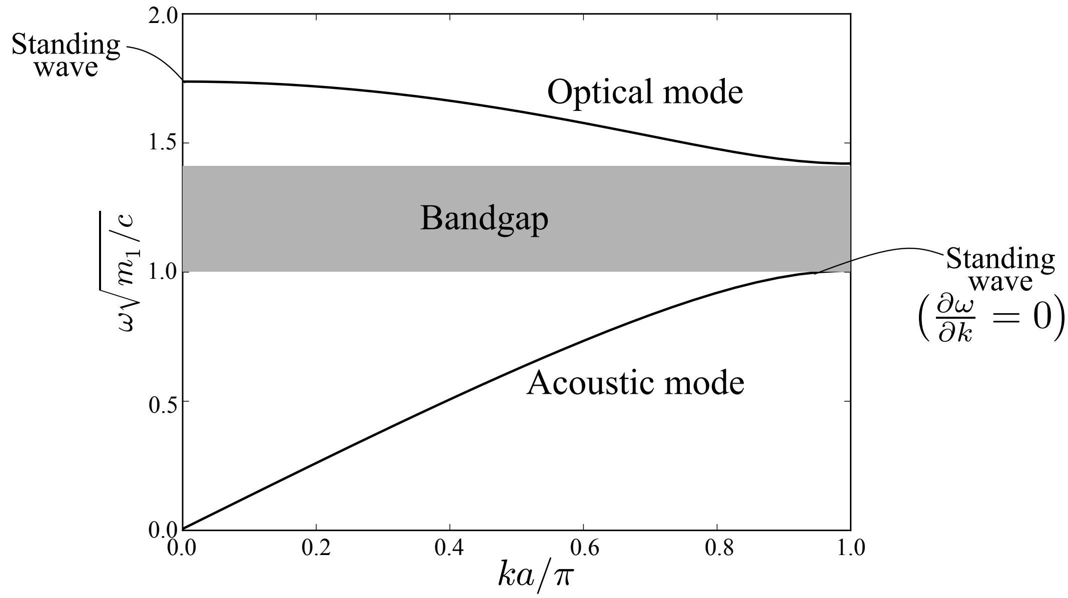


Figure 6. Optical and acoustical branches of the dispersion relation for a diatomic linear lattice. The values are computed for a ratio $m_2/m_1 = 2$.

At $k_{max} = \pm\pi/a$ the roots are

$$\omega^2 = 2c/m_1, \quad \omega^2 = 2c/m_2 \ .$$

3 Three masses lattice

The equations are

$$m_1 \frac{d^2 u_s}{dt^2} = c(v_s + w_{s-1} - 2u_s); \tag{14a}$$

$$m_2 \frac{d^2 v_s}{dt^2} = c(w_s + u_s - 2v_s); \tag{14b}$$

$$m_3 \frac{d^2 w_s}{dt^2} = c(u_{s+1} + v_s - 2w_s). \tag{14c}$$

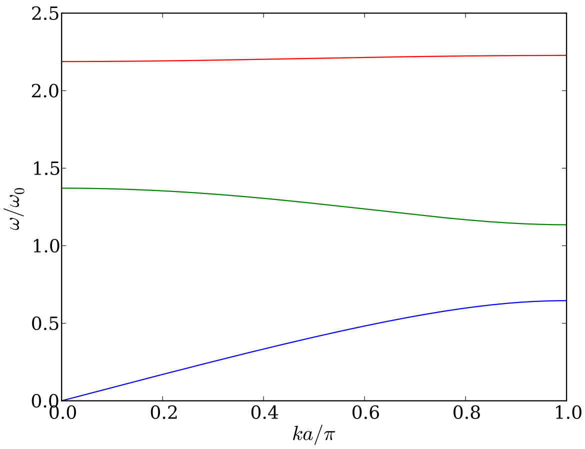
After assume an harmonic solution and apply Bloch conditions the following system is found

$$\begin{bmatrix} -2 & 1 & \exp(-ika) \\ 1 & -2 & 1 \\ \exp(ika) & 1 & -2 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = -\frac{\omega^2}{\omega_0^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \ , \tag{15}$$

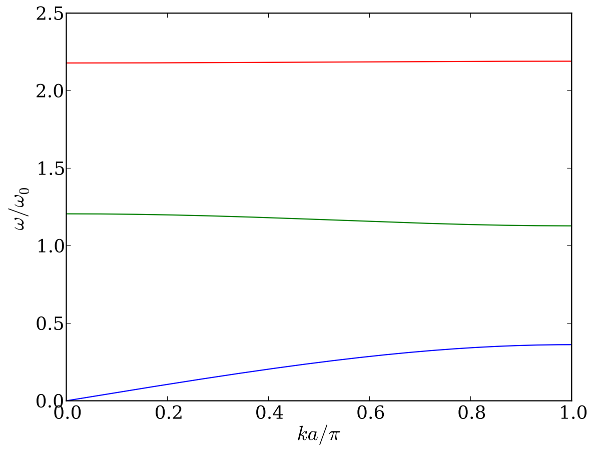
where $\omega_0^2 = c/m_1$, $\mu_1 = m_2/m_1$ and $\mu_2 = m_3/m_1$. The characteristic polynomial for this system is

$$\mu_1\mu_2 x^3 - 2[\mu_1\mu_2 - \mu_1 - \mu_2]x^2 + 3[\mu_1 + \mu_2 + 1]x + 2[\cos ka - 1] = 0 \ . \tag{16}$$

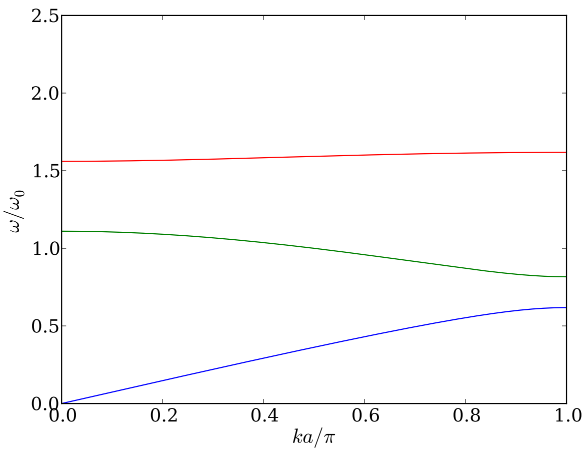
Figure 7 presents some dispersion curves for different mass ratios.



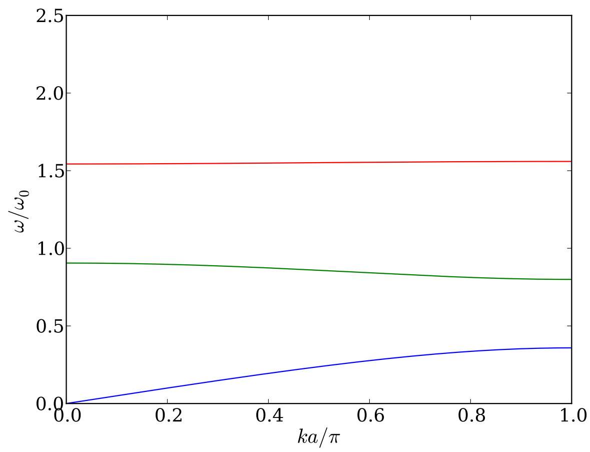
(a) $\mu_1 = 1/2,\ \mu_2 = 3.$



(b) $\mu_1 = 1/2,\ \mu_2 = 10.$



(c) $\mu_1 = 2,\ \mu_2 = 3.$



(d) $\mu_1 = 2,\ \mu_2 = 10.$

Figure 7. Dispersion curves for different values of mass ratios μ_1, μ_2.

4 2D square lattice

For the simple cases shown before is easy to formulate the complete balance of forces taking into account the first neighbours. In general is easier to take the unit cell for the lattice and find the resulting system through row and column operations, just as exemplified before. The system of equation in frequency domain is

$$c\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}\begin{Bmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{Bmatrix} = -\frac{\omega}{4}m\mathbb{I}_8\begin{Bmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{Bmatrix}\ , \tag{17}$$

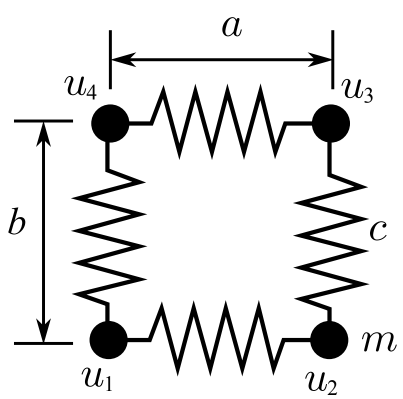


Figure 8. Unit cell for a two dimensional square lattice with a unique species.

and after applying the row operations for the Bloch-conditions imposition, we get

$$\frac{4c}{m}\begin{bmatrix} 1-\cos(k_x a) & 0 \\ 0 & 1-\cos(k_y b)\end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \omega^2 \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} . \tag{18}$$

Or, equivalently

$$\omega_x^2 = \frac{8c}{m}\left(\sin\frac{1}{2}k_x a\right)^2 , \qquad \omega_y^2 = \frac{8c}{m}\left(\sin\frac{1}{2}k_y b\right)^2 \quad . \tag{19}$$

Figure 9 shows the dispersion relation for this problem.

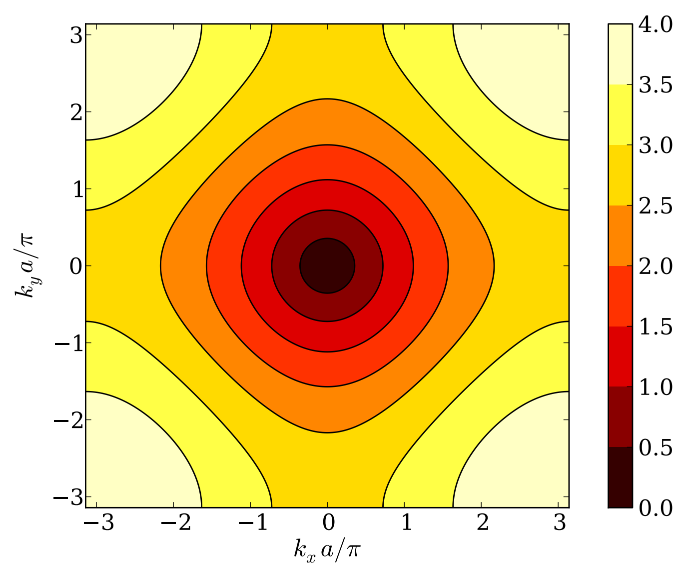


Figure 9. Dispersion relations–shown as isofrequency contours.

5 2D rectangular lattice with diagonal springs

In this case we have a rectangular cell with springs with constant c in the sides. We added diagonal springs with constants c_d (see Figure 10). It should be noted that the value c_d is the *effective* spring

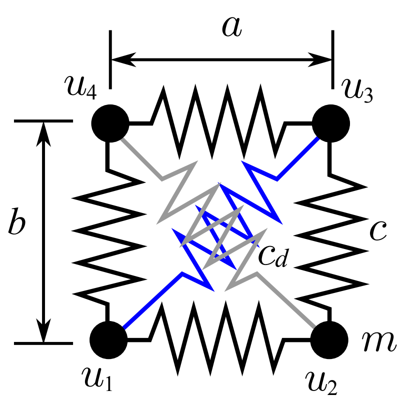


Figure 10. Unit cell for a two dimensional square lattice with a unique species and diagonal springs.

in the x and y directions –they are the same for a square lattice, in the case of $a \neq b$ the constants should vary.

The system of equation in frequency domain is

$$\mathbb{K}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = -\frac{\omega^2}{4} m \mathbb{I}_8 \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{20}$$

with

$$\mathbb{K} = \begin{bmatrix} -c_d - c_1 & 0 & c_1 & 0 & c_d & 0 & 0 & 0 \\ 0 & -c_d - c_2 & 0 & 0 & 0 & c_d & 0 & c_2 \\ c_1 & 0 & -c_d - c_1 & 0 & 0 & 0 & c_d & 0 \\ 0 & 0 & 0 & -c_d - c_2 & 0 & c_2 & 0 & c_d \\ c_d & 0 & 0 & 0 & -c_d - c_1 & 0 & c_1 & 0 \\ 0 & c_d & 0 & c_2 & 0 & -c_d - c_2 & 0 & 0 \\ 0 & 0 & c_d & 0 & c_1 & 0 & -c_d - c_1 & 0 \\ 0 & c_2 & 0 & c_d & 0 & 0 & 0 & -c_d - c_2 \end{bmatrix}$$

And the solutions are

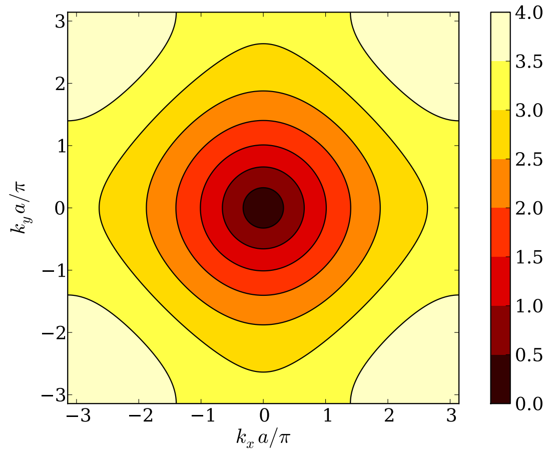
$$\begin{aligned} \omega_1^2 &= \frac{4c_1}{m}\left[1 - \cos k_x a\right] + \frac{4c_d}{m}\left[1 - \cos k_x a \cos k_y b\right] \\ \omega_2^2 &= \frac{4c_2}{m}\left[1 - \cos k_y a\right] + \frac{4c_d}{m}\left[1 - \cos k_x a \cos k_y b\right] \quad . \end{aligned}$$

equivalently (after some manipulations)

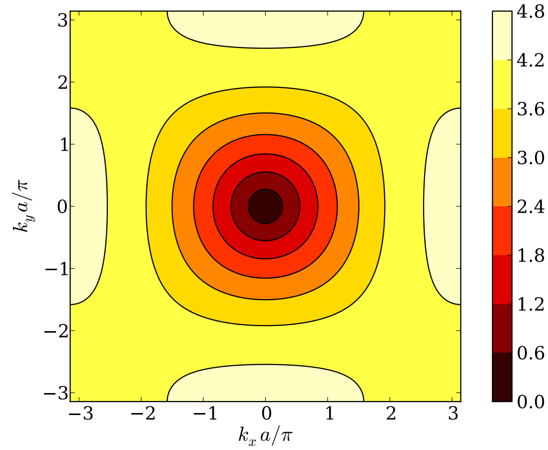
$$\omega_1^2 = \frac{8c_1}{m}\sin^2\left(\frac{1}{2}k_x a\right) + \frac{4c_d}{m}\left[\sin^2\frac{1}{2}\left(k_x a + k_y b\right) + \sin^2\frac{1}{2}\left(k_x a - k_y b\right)\right] \tag{21}$$

$$\omega_2^2 = \frac{8c_2}{m}\sin^2\left(\frac{1}{2}k_y b\right) + \frac{4c_d}{m}\left[\sin^2\frac{1}{2}\left(k_x a + k_y b\right) + \sin^2\frac{1}{2}\left(k_x a - k_y b\right)\right] \tag{22}$$

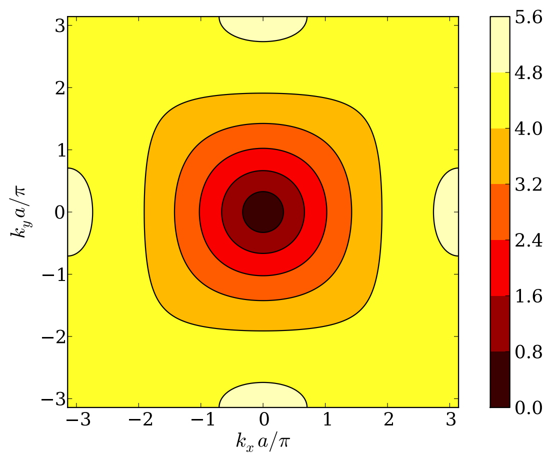
We recover the square lattice (without diagonals) making $c_d = 0$. The dispersion curves for this problem varying the ratio c_d/c are shown in Figure 11.



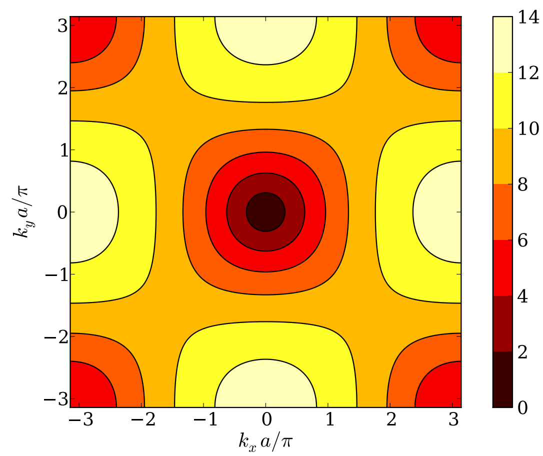
(a) $c_d/c = 0.1$ and $a = b$.



(b) $c_d/c = \sqrt{2}/2$ and $a = b$.



(c) $c_d/c = 1$ and $a = b$.



(d) $c_d/c = 10$ and $a = b$.

Figure 11. Dispersion curves for different values of spring ratios c_d/c.

6 2D square lattice with body mass and diagonal springs

In this case we have a rectangular cell with springs with constant c in the sides. We added a second mass m_2 to the center of the cell, and diagonal springs with constant c_d (see Figure 12). It should be noted that the value c_d is the *effective* spring in the x and y directions –they are the same for a square lattice, in the case of $a \neq b$ the constants should vary.

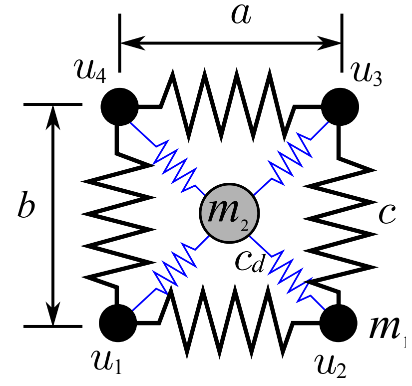


Figure 12. Unit cell for a two dimensional square lattice with a unique species in the corners and another one in the center that is linked by diagonal springs.

$$\mathbb{K}\begin{Bmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5 \end{Bmatrix} = -\omega^2\mathbb{M}\begin{Bmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 u_5\\ v_5 \end{Bmatrix} \tag{23}$$

with

$$\mathbb{K} = \begin{bmatrix}
-c_s - c_d & 0 & c_s & 0 & 0 & 0 & 0 & 0 & c_d & 0\\
0 & -c_s - c_d & 0 & 0 & 0 & 0 & 0 & c_s & 0 & c_d\\
c_s & 0 & -c_s - c_d & 0 & 0 & 0 & 0 & 0 & c_d & 0\\
0 & 0 & 0 & -c_s - c_d & 0 & c_s & 0 & 0 & 0 & c_d\\
0 & 0 & 0 & 0 & -c_s - c_d & 0 & c_s & 0 & c_d & 0\\
0 & 0 & 0 & c_s & 0 & -c_s - c_d & 0 & 0 & 0 & c_d\\
0 & 0 & 0 & 0 & c_s & 0 & -c_s - c_d & 0 & c_d & 0\\
0 & c_s & 0 & 0 & 0 & 0 & 0 & -c_s - c_d & 0 & c_d\\
c_d & 0 & c_d & 0 & c_d & 0 & c_d & 0 & -4\,c_d & 0\\
0 & c_d & 0 & c_d & 0 & c_d & 0 & c_d & 0 & -4\,c_d
\end{bmatrix},$$

and

$$\mathbb{M} = \begin{bmatrix} \frac{m_1}{4}\mathbb{I}_6 & 0\\ 0 & m_2\mathbb{I}_2 \end{bmatrix} .$$

After applying the Bloch theorem we obtain (showing just the lower diagonal due to the Hermiticity)

$$\mathbb{K}_R = \begin{bmatrix}
-4\left[c_d - c_s(\cos(k_y a) - 1)\right] & \bullet & \bullet & \bullet\\
0 & -4\left[c_d - c_s(\cos(k_y b) - 1)\right] & \bullet & \bullet\\
c_d\left(e^{i\,k_x a} + 1\right)\left(e^{i\,k_y b} + 1\right) & 0 & -4c_d & \bullet\\
0 & c_d\left(e^{i\,k_x a} + 1\right)\left(e^{i\,k_y b} + 1\right) & 0 & -4c_d
\end{bmatrix}$$

and

$$\mathbb{M}_R = \begin{bmatrix} m_1\mathbb{I}_2 & 0 \\ 0 & m_2\mathbb{I}_2 \end{bmatrix} \; .$$

The eigenvalues for this problem are lengthy, but a second order expansion around $(k_x, k_y) = (0, 0)$ (linear in the case o its square root) is shown

$$\begin{aligned}
\omega_1^2 =& \frac{2c_s(k_x a)^2 + c_d\left((k_x a)^2 + (k_y b)^2\right)}{m_1 + m_2} \\
\omega_2^2 =& \frac{2c_s(k_y b)^2 + c_d\left((k_x a)^2 + (k_y b)^2\right)}{m_1 + m_2} \\
\omega_3^2 =& \frac{2c_s(k_x a)^2 + 2c_d\left[2m_1^2 + 2m_2^2 + \left(4 - (k_x a)^2 + (k_y b)^2\right) m_1 m_2\right]}{m_1 m_2(m_2 + m_1)} \\
\omega_4^2 =& \frac{2c_s(k_y b)^2 + 2c_d\left[2m_1^2 + 2m_2^2 + \left(4 - (k_x a)^2 + (k_y b)^2\right) m_1 m_2\right]}{m_1 m_2(m_2 + m_1)} \; .
\end{aligned}$$

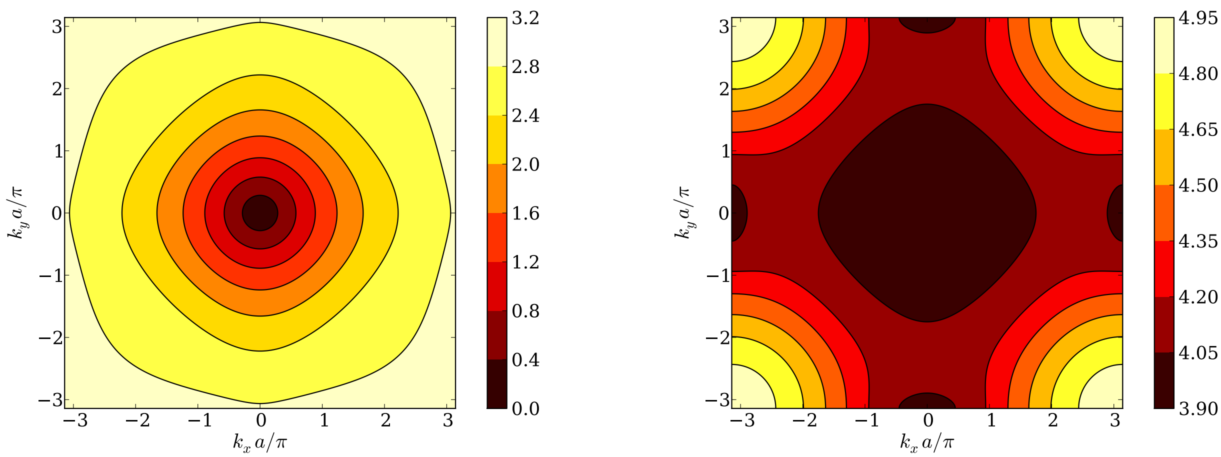
Assuming $a = b$, we can group the modes as degenerate ones and consider the most general propagation in the plane (due to the orthogonality)

$$\omega_{\text{low}}^2 = \frac{2(c_s + c_d)(k_x^2 + k_y^2)a^2}{m_1 + m_2} \equiv \frac{2(c_s + c_d)\mathbf{k}^2 a^2}{m_1 + m_2} \; ,$$

the same argument holds for the other two, giving

$$\omega_{\text{high}}^2 = \frac{2c_s\mathbf{k}^2 a^2 + 2c_d\left[4m_1^2 + 4m_2^2 + (8 - \mathbf{k}^2 a^2) m_1 m_2\right]}{m_1 m_2(m_2 + m_1)} \; .$$

Figure 13 shows the dispersion modes for ratios $m_2/m_1 = 1$ and $c_d/c_s = 1$. Figure 14 shows



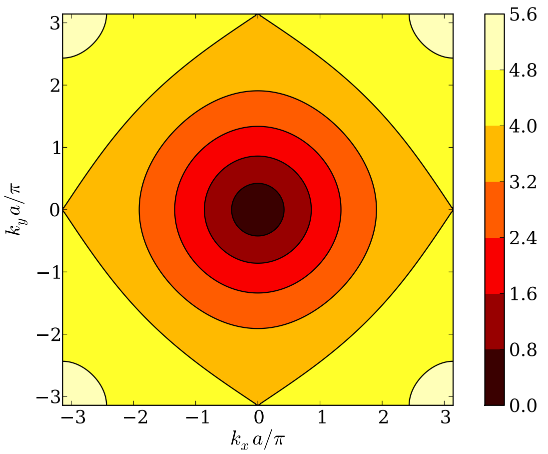
(a) First branch.

(b) Second branch.

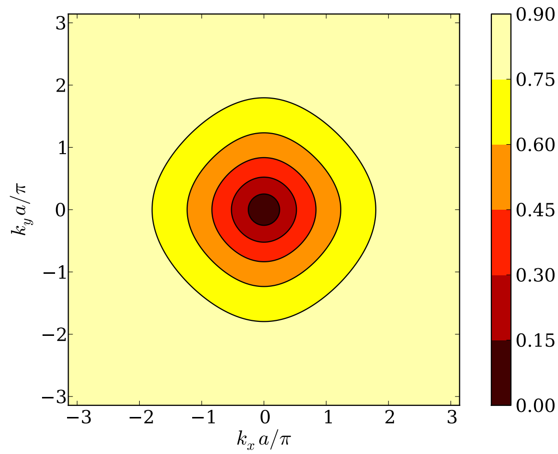
Figure 13. $c_d/c = 1$, $m_1/m_2 = 1$ and $a = b$.

the first branch of the dispersion curves for different ratios m_2/m_1 and c_d/c_s.

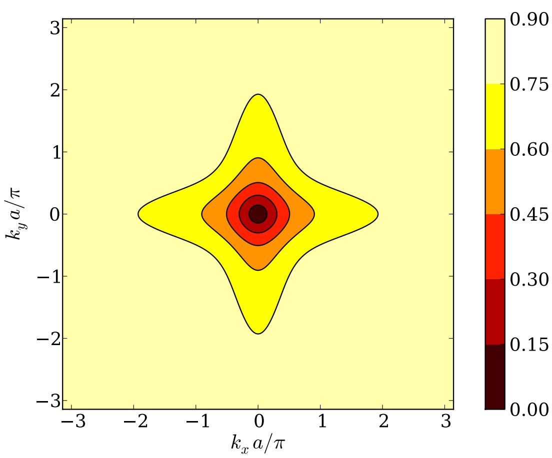
Figure 15 shows the second branch of the dispersion curves for different ratios m_2/m_1 and c_d/c_s.



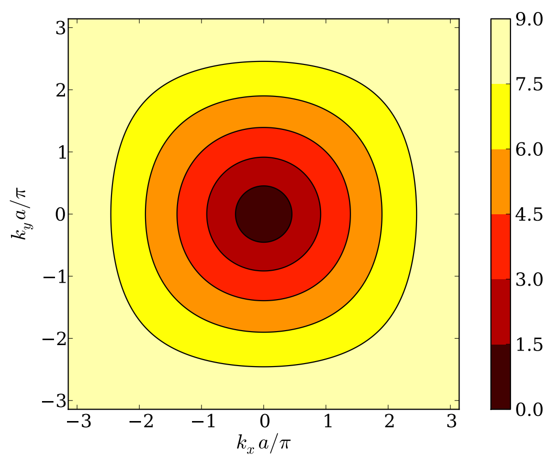
(a) $c_d/c = 0.1$, $m_2/m_1 = 1$ and $a = b$.



(b) $c_d/c = 10$, $m_2/m_1 = 1$ and $a = b$.

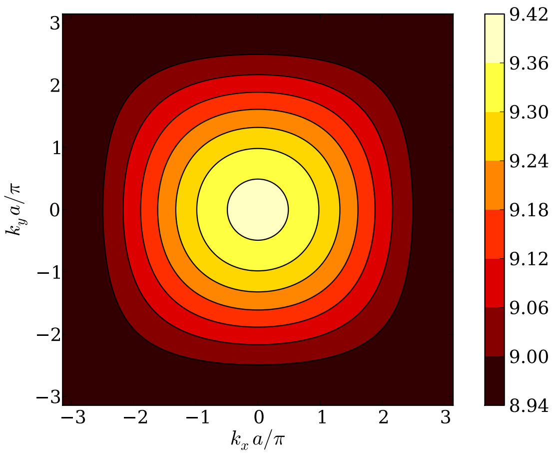


(c) $c_d/c = 1$, $m_2/m_1 = 0.1$ and $a = b$.

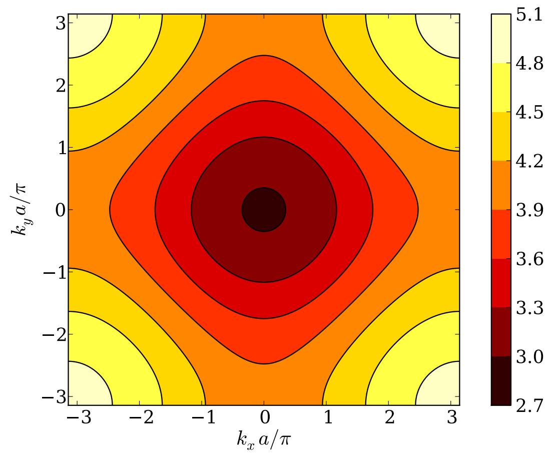


(d) $c_d/c = 0.1$, $m_2/m_1 = 10$ and $a = b$.

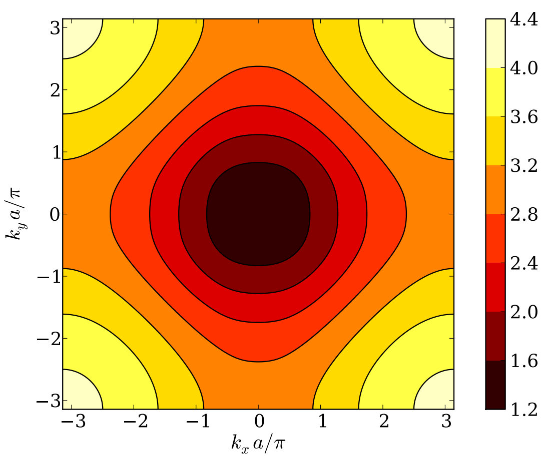
Figure 14. First branch of the dispersion curves.



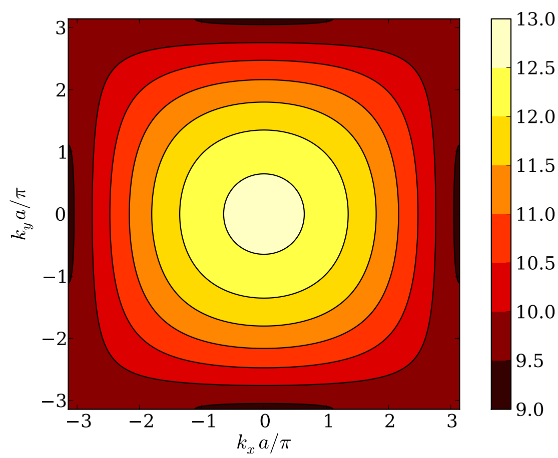
(a) $c_d/c = 0.1$, $m_2/m_1 = 1$ and $a = b$.



(b) $c_d/c = 10$, $m_2/m_1 = 1$ and $a = b$.



(c) $c_d/c = 1$, $m_2/m_1 = 0.1$ and $a = b$.



(d) $c_d/c = 0.1$, $m_2/m_1 = 10$ and $a = b$.

Figure 15. Second branch of the dispersion curves.