Problem 7. Prove that for any natural number n,

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$$

Proof. By induction.

The expression in the left hand side can be written as $\sum_{i=1}^{n} 2^i$

Base case When $n = 1$, the left hand side is 2. Simplifying the right hand side,

$$\begin{aligned} 2^{(1)+1} - 2 &= 2^2 - 2 \\ &= 2 \end{aligned}$$

So, the identity is valid for $n = 1$.

Induction step Assume that the identity holds for $n = k$. Then,

$$\sum_{i=1}^{k} 2^i = 2^{k+1} - 2$$

Then, for $n = k + 1$

$$\begin{aligned} \sum_{i=1}^{k+1} 2^i &= \sum_{i=1}^{k} 2^i + 2^{k+1} \\ &= (2^{k+1} - 2) + 2^{k+1} && \text{[Using the induction step]} \\ &= 2(2^{k+1}) - 2 \\ &= 2^{k+2} - 2 \end{aligned}$$

Which is the expression to be expected if $n = k + 1$.

Hence, by the principle of mathematical induction, the theorem is proved.

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