Problem 8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Proof. Using ϵ definition of limits.

Writing the theorem formally,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_n - L| < \epsilon]$$
$$\implies (\forall M > 0)(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|Ma_n - ML| < \epsilon]$$

Let $\epsilon > 0$ be given. Using the initial assumption, for all $n \geq N$, we can find an n such that

$$|a_n - L| < \frac{\epsilon}{M}$$

Hence, for all $n \geq N$,

$$|Ma_n - ML| = M|a_n - L| < M \times \frac{\epsilon}{M} = \epsilon$$

By the definition of limits, this proves that $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

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