Problem 5. Prove that for any integer n, at least one of the integers n, $n+2$, $n+4$ is divisible by 3.

Proof. By induction and cases.

Base case

When $n = 1$, the integers n, $n+2$, and $n+4$ are 1, 3, and 5 respectively. 3 is divisible by 3. Hence, the statement holds.

When $n = 2$, the integers n, $n+2$, and $n+4$ are 2, 4, and 6 respectively. Here, 6 is divisible by 3. The statement holds yet again.

Induction step

We assume that the statement holds for $n = k$.

We need to prove that the statement holds for the case when $n = k+1$.

When $n = k$, either

- $3|k$
  i.e. $(q \in \mathbb{N})(k = 3q)$
  $\therefore k = 3q$ or,
- $3|(k+2)$
  i.e. $(q \in \mathbb{N})(k+2 = 3q)$
  $\implies k = 3q - 2$ or,
- $3|(k+4)$
  i.e. $(q \in \mathbb{N})(k+4 = 3q)$
  $\implies k = 3q - 4$

For $n = k+1$, the integers are $k+1$, $k+3$, and $k+5$.

There can be 3 cases,

- If $3|k$, i.e. $k = 3q$, then the integers are $3q+1$, $3q+3$, and $3q+5$. (Substituting $k = 3q$ in $k+1$, $k+3$, and $k+5$)

  In this case, the second integer, $3q+3 \implies 3(q+1)$ is divisible by 3.
  Hence the statement is **TRUE**.

- If $3|k+2$, i.e. $k = 3q-2$, then the integers are $3q-1$, $3q+1$, and $3q+3$. (Substituting $k = 3q-2$ in $k+1$, $k+3$, and $k+5$)

  In this case, the third integer, $3q+3 \implies 3(q+1)$ is divisible by 3.
  Hence the statement is **TRUE**.

- If $3|k+4$, i.e. $k = 3q-4$, then the integers are $3q-3$, $3q-1$, and $3q+1$. (Substituting $k = 3q-4$ in $k+1$, $k+3$, and $k+5$)

  In this case, the first integer, $3q-3 \implies 3(q-1)$ is divisible by 3.
  Hence the statement is **TRUE**.

Since the statement is true for all the three cases, by the principle of mathematical induction, the theorem is proved.

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