Problem 4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

Proof. By using the definition that every odd number can be written in the form $2k + 1$, where k is an integer.

Trying out the expressions for $n = 0$

$$\begin{aligned} 4n + 1 &= 4(0) + 1 \\ &= 1 \\ 4n + 3 &= 4(0) + 3 \\ &= 3 \end{aligned}$$

1 and 3 are odd numbers. They can be expressed in the form of $4n + 1$ and $4n + 3$ so the statement seems true.

If $2k + 1$ can be written in the forms $4n + 1$ and $4n + 3$, the theorem will be proved.

There can be two cases. Either k is even or k is odd.

- **Case 1 (k is even)**

  Since k is even, it can written as $k = 2n$, where n is an integer.

  $$\begin{aligned} 2k + 1 &= 2(2n) + 1 && [\text{Substituting } k = 2n \text{ in } 2k + 1] \\ &= 4n + 1 \end{aligned}$$

  Hence, some of the odd numbers can be written in the form $4n + 1$.

- **Case 2 (k is odd)**

  Since k is odd, it can written as $k = 2n + 1$, where n is an integer.

  $$\begin{aligned} 2k + 1 &= 2(2n + 1) + 1 && [\text{Substituting } k = 2n + 1 \text{ in } 2k + 1] \\ &= 4n + 3 \end{aligned}$$

  Hence, the remaining odd numbers can be written in the form $4n + 3$.

$\therefore$ It is proved that every odd natural number is either of the forms $4n+1$ or $4n+3$, where n is an integer.

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