Problem 9. Given an infinite collection A_n , $n = 1, 2, ...$ of intervals of the real line, their *intersection* is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals A_n, $n = 1, 2, ...$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

Proof. Let the set A_n be defined as $(0, \frac{1}{n})$

For $n = 1$,

$$A_1 = (0, 1)$$

For $n + 1$ i.e. $n = 2$,

$$A_2 = (0, \frac{1}{2})$$

Since $(0, \frac{1}{2}) \subset (0, 1)$, i.e. $A_2 \subset A_1$, $A_{n+1} \subset A_n$ holds.

As $n \to \infty, \frac{1}{n} \to 0$. So $\bigcap_{n=1}^{\infty} A_n = (0, 0)$

Since, the interval is exclusive, $0 \notin A_n$. The expression $\bigcap_{n=1}^{\infty} A_n$ will hence equate to $\emptyset$.

Therefore, the example given has the required stated properties.

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