Problem 10. Give an example of a family of intervals A_n, $n = 1, 2, ...,$ such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Proof. Let the set A_n be defined as $[0, \frac{1}{n}]$

For $n = 1$,

$$A_1 = [0, 1]$$

For $n + 1$ i.e. $n = 2$,

$$A_2 = [0, \frac{1}{2}]$$

Since $[0, \frac{1}{2}] \subset [0, 1]$, i.e. $A_2 \subset A_1$, $A_{n+1} \subset A_n$ holds.

As $n \to \infty$, $\frac{1}{n} \to 0$. So $\bigcap_{n=1}^{\infty} A_n = [0, 0]$

Since, the interval is inclusive, $0 \in A_n$. The expression $\bigcap_{n=1}^{\infty} A_n$ will hence equate to 0. $0 \in \mathbb{R}$

Therefore, the example given has the required stated properties.

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