E1 244: Detection & Estimation Theory **Jan-May 2017**

Lecture 12 — February 14

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PURELY STOCHASTIC SIGNALS

Recap: Signal Detection in noise
- Deterministic signals
- Partly stochastic signals(Signals with Random parameters)
- Purely Stochastic signals(This Lecture)

12.1 Detection of Stochastic Signals in Noise

The basic problem that we are dealing is the following hypothesis testing problem:

$$H_0 : \underline{Y} = \underline{N}$$

$$H_1 : \underline{Y} = \underline{S} + \underline{N}$$

Where $\underline{N}$ is iid noise with zero mean and $\underline{S}$ is the purely stochastic signal with covariance matrix Σ_s that we are dealing with. And also $\underline{S}$ and $\underline{N}$ are independent.

$$\underline{N} \sim \mathcal{N}(\underline{0}, \sigma^2 I)\ ,\ \underline{S} \sim (\underline{0}, \Sigma_s) \text{ and } \underline{S} \perp\!\!\!\perp \underline{N}$$

Motivation: Signals in turbulent/highly random environments. e.g. radio astronomy,sonar,underwater signaling etc. In a more general scenario we have :

$$H_0 : \underline{Y} \sim \mathcal{N}(\mu_0, \Sigma_0)$$
vs
$$H_1 : \underline{Y} \sim \mathcal{N}(\mu_1, \Sigma_1)$$

The Log likelihood ratio of the observation $\underline{y} \epsilon R^n$ is

$$log(L(\underline{y})) = log(\frac{P_1(\underline{y})}{P_0(\underline{y})})$$

$$log(L(\underline{y})) = log(\frac{\frac{1}{(2\pi)^{0.5}|\Sigma_1|^{0.5}} exp(\frac{-(\underline{y}-\underline{u}_1)^T\Sigma_1^{-1}(\underline{y}-\underline{u}_1)}{2})}{\frac{1}{(2\pi)^{0.5}|\Sigma_0|^{0.5}} exp(\frac{-(\underline{y}-\underline{u}_0)^T\Sigma_0^{-1}(\underline{y}-\underline{u}_0)}{2})})$$

$$log(L(\underline{y})) = (1/2)\underline{y}^T(\Sigma_0^{-1} - \Sigma_1^{-1})\underline{y} + \underline{y}^T(-\Sigma_0^{-1}\mu_0 + \Sigma_1^{-1}\mu_1) + C$$

Note : C is a constant that does not depend on $\underline{y}$

Case-(i): When $\Sigma_0 = \Sigma_1$. Here the optimum detector is Linear detector and it is given by thresholding a Linear function of $\underline{y}$,ie.

$$y^T \Sigma_0^{-1}(\mu_1 - \mu_0) \gtreqless .....$$

Case-(ii): When $\mu_0 = \mu_1 = 0$
In this we get a quadratic detector which is of the following form

$$. \; y^T(\Sigma_0^{-1} - \Sigma_1^{-1})y \gtreqless .....$$

Since $\underline{y}^T\underline{y}$ gives energy, in general we can say that it is a " Energy Detector". More Specifically we can say that it is a " Weighted Energy Detector" . For the following case of hypothesis

$$H_0 : \underline{Y} = \underline{N}$$
$$\text{vs}$$
$$H_1 : \underline{Y} = \underline{S} + \underline{N}$$

The optimum detector is : (For the above case $\Sigma_0 = \sigma^2 I, \Sigma_1 = \sigma^2 I + \Sigma_s$)

$$\delta_0(\underline{y}) = \begin{cases} 1, & \underline{y}^T Q \underline{y} > \tau^1 \\ \gamma, & \underline{y}^T Q \underline{y} = \tau^1 \\ 0, & \underline{y}^T Q \underline{y} < \tau^1 \end{cases} \tag{12.1}$$

Where,

$$Q = \tfrac{1}{\sigma^2} I - (\sigma^2 I + \Sigma_s)^{-1}$$
$$Q = (I + \tfrac{1}{\sigma^2}\Sigma_s - I)(\sigma^2 I + \Sigma_s)^{-1}$$
$$Q = (\sigma^2 \Sigma_s)(\sigma^2 I + \Sigma_s)^{-1}$$

$\underline{y}^T Q \underline{y} \gtreqless \ldots$ is called a "Weighted energy detector" or a "Radiometer".

12.2 Performance analysis of a quadratic detector:

Here want to find : $P_j[\underline{Y}^T Q \underline{Y} > \tau^1]$ where j ϵ {0,1}

So we have to Decompose Σ_s using Singular Value Decomposition(SVD)

$$\Sigma_s = \sum_{k=1}^{n} \lambda_k \underline{v}_k \underline{v}^T$$

where $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ are " Ortho Normal" and also we can express I as $I = \sum_{k=1}^{n} \underline{v}_k \underline{v}^T$

Using the expressions for Σ_s and I as given above, we can write the following

$$(\sigma^2 I + \Sigma_s)^{-1} = \sum_{k=1}^{n} (\lambda_k + \sigma^2)^{-1} \underline{v}_k \underline{v}^T$$

and hence we can write the following expression

$$(\sigma^2 \Sigma_s)(\sigma^2 I + \Sigma_s)^{-1} = \sum_{k=1}^{n} (\tfrac{\lambda_k}{(\lambda_k+\sigma^2)\sigma^2}) \underline{v}_k \underline{v}^T$$

So we finally obtain the following expression with the help of SVD

$$\underline{y}^T Q \underline{y} = \sum_{k=1}^{n} y_k^2$$

where

$$y_k = (\sqrt{(}\tfrac{\lambda_k}{(\lambda_k+\sigma^2)\sigma^2})) \underline{y}^T \underline{v}_k$$

Exercise : Suppose if Y is multivariate then show that

$$\bar{Y}_k = (\sqrt{(}\tfrac{\lambda_k}{(\lambda_k+\sigma^2)\sigma^2})) \underline{Y}^T \underline{v}_k$$

Where $\bar{Y}_k$ for $k = 1, \ldots, n$ are independent, Guassian random variables with mean zero

Let $\sigma_{jk}^2 = var(\bar{Y}_k | H_j)$ for j=0,1 and k=1,$\ldots, n$

$$\sigma_{jk}^2 = \begin{cases} \frac{\lambda_k}{(\lambda_k+\sigma^2)}, & j = 0 \\ \frac{\lambda_k}{\sigma^2}, & j = 1 \end{cases} \tag{12.2}$$

The pdfs of $T_k = \bar{Y}_k^2$ under H_j is :

$$P_{T_k}(t|H_j) = \begin{cases} \frac{1}{(\sqrt{2\pi t}\sigma_{jk}} e^{\frac{-t}{2\sigma_{jk}^2}}, & t >= 0 \\ 0, & t < 0 \end{cases} \tag{12.3}$$

This is a gamma Density with parameters $\frac{1}{2}, \frac{-1}{2\sigma_{jk}^2}$

For a Gamma Density with parameters a and b i.e GAMMA(a,b), the pdf is $\alpha\ x^{a-1} e^{-bx}$

Hence the pdf of $T = \sum_{k=1}^{n} T_k$ where $T_k = \underline{Y}^T Q \underline{Y}$ is given as

$$P_T = P_{T_1} * P_{T_2} * \dots P_{T_n} \text{ where } * \text{ denotes convolution}$$

$P_T = \mathcal{F}^{-1}[\prod_{k=1}^{n} \phi_{T_k}]$, where ϕ_{T_k} is the charecteristic function of T_k and is given as :

$$\phi_{T_k}(u) = \mathcal{E}[e^{iuT_k}] \text{ where } f_X(x) = \frac{1}{2}\int_{-\infty}^{\infty} \phi_X(t)e^{-itx}dt$$

$$P_T(t|H_j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu}\prod_{k=1}^{n} \phi_{T_k}(u)du$$

For the case of Gamma Distribution, the charecteristic function is given by $\phi_{T_k}(u) = [1-2iu\sigma_j k^2]^{-1}$

Generally $P_T(t|H_j)$ is intractable, but it is possible to find when $\sigma_{j1} = \sigma_{j2} = \dots = \sigma_{jk} = \sigma_j$ where j=0,1

In this Special case,

$$P_T(t|H_j) = GAMMA(\tfrac{n}{2}, \tfrac{-1}{2\sigma_j^2}) = \Gamma(\tfrac{n}{2}, \tfrac{-1}{2\sigma_j^2})$$

$$\sigma_{j1} = \sigma_{j2} = \dots = \sigma_{jk} = \sigma_j \text{ if and only if } \lambda_1 = \lambda_2 = \dots = \lambda_n = \sigma_n^2$$

$$\Sigma_s = \sigma_s^2 I \text{ i.e } \underline{S} \sim \mathcal{N}(\underline{0}, \sigma_s^2 I)$$

Hence,

$$\mathcal{P}_j[\underline{Y}^T Q\underline{Y} > \tau^1] = 1 - \Gamma(\tfrac{n}{2}, \tfrac{\tau^1}{2\sigma_j^2})$$

where $\Gamma(\frac{n}{2}, \frac{\tau^1}{2\sigma_j^2})$ is the CDF of $\underline{Y}^T Q\underline{Y}$ at τ^1. This is an "Incomplete Gamma Function"

eg: For False alarm probability α,

$$\tau^1 = 2\sigma_0^2\Gamma^{-1}(\tfrac{n}{2}, 1-\alpha)$$

Correspondingly,

$$P_D = 1 - \Gamma(\tfrac{n}{2}, \tfrac{\sigma_0^2}{\sigma_1^2}\Gamma^{-1}(\tfrac{n}{2}, 1-\alpha))$$

Note:

Performance of this detector is a function of

1) n (i.e size of the vector)

2) $\frac{\sigma_0^2}{\sigma_1^2} = \frac{\frac{\sigma_s^2}{\sigma_s^2+\sigma^2}}{\frac{\sigma_s^2}{\sigma^2}} = \frac{1}{1+\frac{\sigma_s^2}{\sigma^2}}$

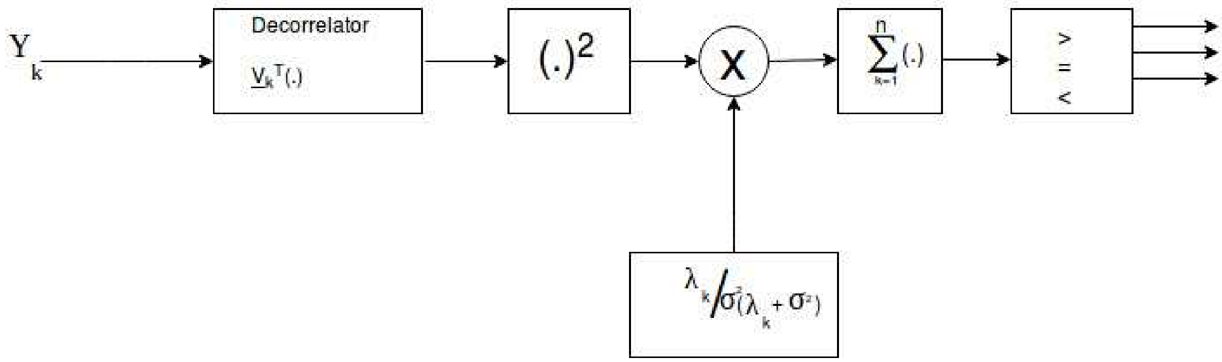

Figure 12.1: Decorrelator-Energy Detector

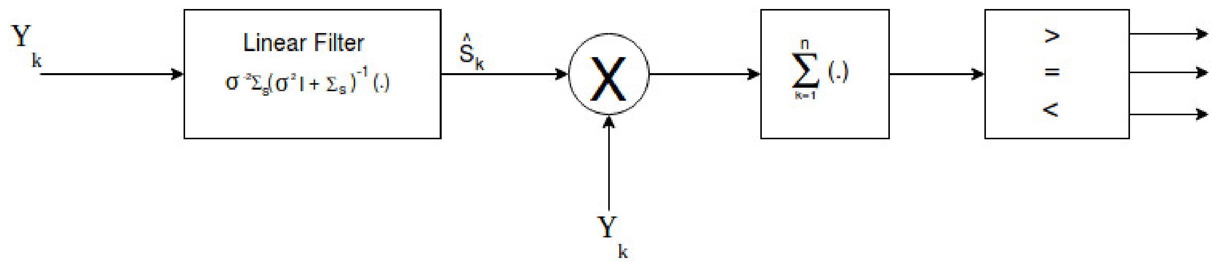

Figure 12.2: Estimator Correlator

12.3 Detector Structure

$\underline{y}^T Q \underline{y} \gtreqless \dots$ where Q=$(\sigma^2\Sigma_s)(\sigma^2 I + \Sigma_s)^{-1}$

(i)Decorrelator-Energy Detector:
The block diagram of this type of detector is as shown in the Figure 12.1.Where $\{v_1, v_2, v_3..........v_n\}$ is the orthonormal basis w.r.t Σ_s

(ii)Estimator-Correlator Detector:
The block diagram of this type of detector is as shown in the Figure12.2