3. Kinematic Modeling

This section of the paper deals with the geometric kinematic modeling of a triple singularity drive mobile platform. This modeling is split into two parts: inverse kinematics, which can be used to solve for the steering angles of the singularity drive modules given a desired robot trajectory, and the forward kinematics, which can be used to estimate the position and orientation of the robot using sensor data.

To solve for these equations, we break the desired motion of the robot into two components: translation of the center of mass and rotation about the center of mass. By superposition, any motion of the robot can be decoupled into these two components. In order to simplify the kinematics, we make the following assumptions:

- There is no slip between the drive wheels and the ground.
- The robot operates on a smooth, hard ground plane.
- The center of mass is perfectly balanced between each of the singularity drive modules, and each point of contact with the ground supports equal weight.
- The motion of the gimbals does not significantly change the location of the center of mass in the mobile reference frame.

3.1 Nomenclature

$O\text{-}XY$:	the global coordinate system
$O\text{-}X_RY_R$:	the robot's inertial coordinate system, centered about the center of mass

$O\text{-}X_W Y_W$: the singularity drive wheel coordinate system, centered on the pivot point of the singularity drive module

V_x, V_y, ω: the linear and angular velocity components of the robot in the global reference frame

v_{Rx}, v_{Ry}, ω_R: the linear and angular velocity components of the robot in the inertial reference frame

v_{Wx}, v_{Wy}, ω_W: the linear and angular velocity components of the robot in the wheel reference frame

v_w: the wheel velocity vectors in the inertial reference frame

v_{wx}, v_{wy}: the components of the wheel velocity vector in the $O\text{-}X_R Y_R$ coordinate system

v_{wr}, v_{wt}: the rotational and translational components of the wheel velocity vectors

R: the distance from the center of mass of the robot to the ground contact point of the singularity drive module

θ_w: the wheel angle of each singularity drive

ω_d: the angular velocity of the drive wheels

r_{eff}: the effective radius of the drive wheel in contact with the ground

γ: the tilt angle of the drive shaft, measured from the ZW axis

θ_t: the gimbal steer angle

α, β: the axial and lateral gimbal rotation angles

l: the unit vector along the drive axis of the singularity drive

x_1, y_1, z_1: the components of the unit vector l in the $O\text{-}X_W Y_W$ coordinate system

d_1, h: the projections of unit vector l in the $Z_W X_W$ *and* $X_W Y_W$ planes respectively

3.2 Inverse Kinematics

We begin the kinematic modeling of the singularity drive with the inverse kinematics. These relationships govern the motion of the robot in response to steering servo inputs and drive wheel angular velocity. Shown in Figure 11 below is the notation used in this derivation.

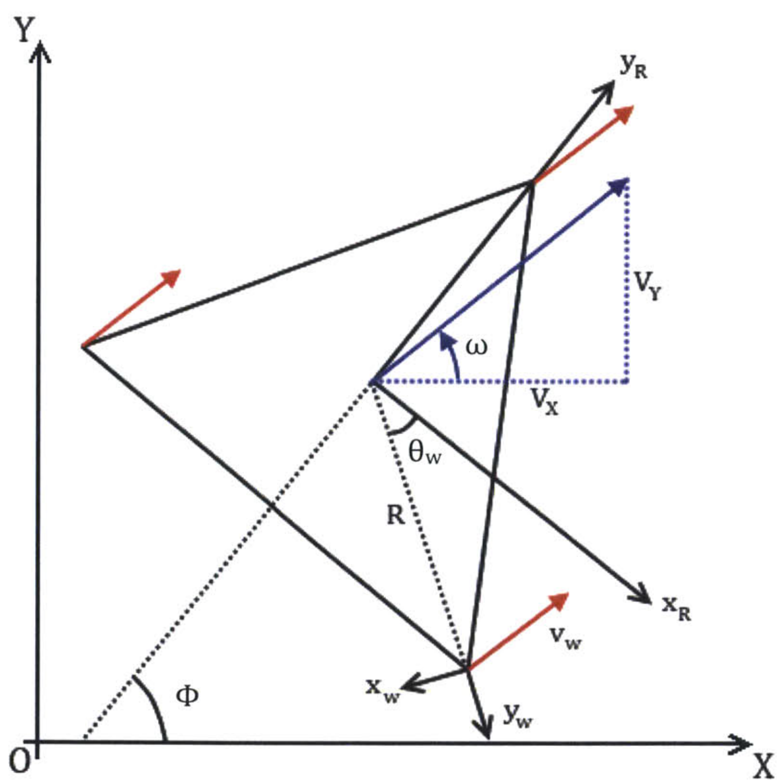

FIGURE 11. ROBOT PLATFORM NOTATION

There are three coordinate systems used in this derivation. First is the world (static) reference frame, labeled *O-XY*; second is the mobile (robot) reference frame, labeled $O\text{-}X_RY_R$ and located at the center of mass; and third is the wheel reference frame, labeled $O\text{-}X_WY_W$ and fixed to the center point of the gimbal/drive wheel assembly. The angle between the X and X_R axes, which defines the robot's global orientation, is Φ. Input parameters for the inverse kinematics are the desired components of velocity V_X and V_Y and the angular velocity ω of the robot with respect to the static reference frame *O-XY*. In order to express these desired movement vectors in the mobile reference frame, we use the following coordinate transformation:

$$\begin{bmatrix} v_{Rx} \\ v_{Ry} \\ \omega_R \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_X \\ V_Y \\ \omega \end{bmatrix} \tag{1}$$

The motion of the robot is broken down into two components: translational and rotational. Corresponding velocity vectors at each wheel (v_w) can be decoupled into these components as well. The magnitude of these wheel velocity vectors is given by the following equation, where R is the distance from the center of mass to the wheel:

$$|v_{wr}| = \frac{\phi_R}{R} \tag{2}$$

The angle between each wheel and the X_R-axis of the mobile coordinate system is denoted by θ_w. The rotational component of the i-th (i = 1, 2, 3) wheel velocity vector is given by:

$$\begin{bmatrix} v_{wrx} \\ v_{wry} \end{bmatrix}_i = \begin{bmatrix} 0 & 0 & \frac{\sin\theta_{wi}}{R} \\ 0 & 0 & \frac{\cos\theta_{wi}}{R} \end{bmatrix} \begin{bmatrix} v_{Rx} \\ v_{Ry} \\ \omega_R \end{bmatrix} \tag{3}$$

The translational component of the i-th wheel velocity vector is equivalent to the desired robot velocity $\boldsymbol{v_R}$:

$$\begin{bmatrix} v_{wtx} \\ v_{wty} \end{bmatrix}_i = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_{Rx} \\ v_{Ry} \\ \omega_R \end{bmatrix} \tag{4}$$

By superposition, we can sum Equations 3 and 4 in order to find the expression for the i-th wheel velocity vector:

$$v_w = v_{wr} + v_{wt} \tag{5}$$

$$\begin{bmatrix} v_{wx} \\ v_{wy} \end{bmatrix}_i = \begin{bmatrix} 1 & 1 & \frac{\sin \theta_{wi}}{R} \\ 1 & 1 & \frac{\cos \theta_{wi}}{R} \end{bmatrix} \begin{bmatrix} v_{Rx} \\ v_{Ry} \\ \omega_R \end{bmatrix} \tag{6}$$

Another coordinate transformation will allow these wheel velocity vectors to be expressed in their respective wheel's coordinate system:

$$\begin{bmatrix} v_{Wx} \\ v_{Wy} \end{bmatrix}_i = \begin{bmatrix} \cos \theta_{wi} & \sin \theta_{wi} \\ -\sin \theta_{wi} & \cos \theta_{wi} \end{bmatrix} \begin{bmatrix} v_{wx} \\ v_{wy} \end{bmatrix} \tag{7}$$

By combining Equations 1, 6, and 7, we can find the full kinematic relationship between the desired global velocity, orientation, and the wheel velocity vectors:

$$\begin{bmatrix} v_{Wx} \\ v_{Wy} \end{bmatrix}_i = \overline{H_i} \begin{bmatrix} V_X \\ V_Y \\ \omega \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} v_{Wx} \\ v_{Wy} \end{bmatrix}_i = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \end{bmatrix} \begin{bmatrix} V_X \\ V_Y \\ \omega \end{bmatrix} \tag{9}$$

Where:

$$H_{11} = \cos\phi\,(\cos\theta_{wi} + \sin\theta_{wi}) - \sin\phi\,(\cos\theta_{wi} + \sin\theta_{wi})$$

$$H_{12} = \cos\phi\,(\cos\theta_{wi} + \sin\theta_{wi}) + \sin\phi\,(\cos\theta_{wi} + \sin\theta_{wi})$$

$$H_{13} = \frac{2\cos\theta_{wi}\sin\theta_{wi}}{R}$$

$$H_{21} = \cos\phi\,(\cos\theta_{wi} - \sin\theta_{wi}) - \sin\phi\,(\cos\theta_{wi} - \sin\theta_{wi})$$

$$H_{22} = \cos\phi\,(\cos\theta_{wi} - \sin\theta_{wi}) + \sin\phi\,(\cos\theta_{wi} - \sin\theta_{wi})$$

$$H_{23} = \frac{\cos^2\theta_{wi} - \sin^2\theta_{wi}}{R}$$

The magnitude of the wheel velocity is equal to:

$$|v_W| = \sqrt{v_{Wx}^2 + v_{Wy}^2} \tag{10}$$

Equation 9 dictates the wheel velocity vectors with respect to each wheel's own reference frame. Next, we need to determine the gimbal angles that correspond to these desired wheel velocities. First, we consider the tilt angle of the drive wheel assembly. Recall that this angle directly determines the effective gear ratio of the drive assembly by altering the effective radius of the wheel. The necessary effective radius to match the desired speed is given by the following equation, where ω_d is the angular velocity of the drive motor:

$$r_{eff} = \frac{|v_W|}{\omega_d} \tag{11}$$

In this case, because there is an infinitely variable gear ratio between the drive motor and the ground, the motor speed is held constant and can be an arbitrary value. The relationship between gimbal tilt angle γ and the effective radius of the drive wheel is given here, where r_w is the spherical radius of the drive wheel:

$$\sin\gamma = \frac{r_{eff}}{r_w} \tag{12}$$

Combining Equations 11 and 12 gives us an expression for the required tilt angle as a function of drive wheel radius and the desired wheel velocity:

$$\gamma = \sin^{-1}\left(\frac{|v_W|}{\omega_d \cdot r_w}\right) \tag{13}$$

In order to determine the steering angles for the singularity drive gimbals, we need to establish additional notation. Each Y_w-axis is aligned radially outward with the center of the mass of the robot. Let these Y_w-axes correspond to lateral gimbal rotation, while the X_w-axes correspond to axial gimbal rotation; β and α measure the angles of rotation in these respective axes.

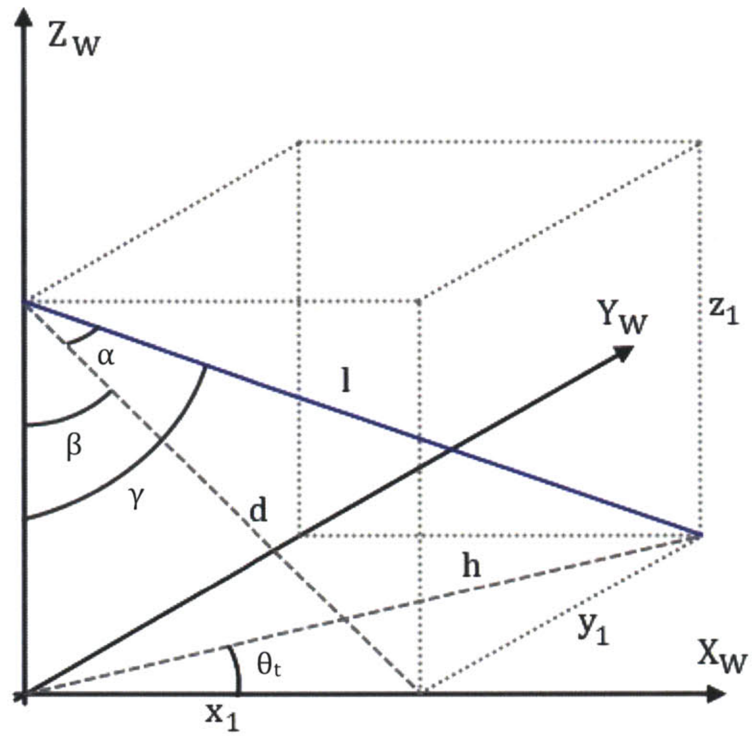

FIGURE 12. GIMBAL ANGLE NOTATION

Let unit vector ***l*** represent the direction of the drive axle in 3D space. Additionally, let x_1, y_1, and z_1 represent the three components of unit vector ***l*** in the $O\text{-}X_wY_wZ_w$ coordinate system. Tilt angle γ is measured between vector **l** and the Z_w-axis, while the steer angle θ_t measures the angle between the X_w-axis and the projection of ***l*** on the X_wY_x plane (itself denoted as length h). Note that θ_t is perpendicular to the wheel velocity vector v_W due to the spin of the motor shaft (which is positive in the counter clockwise direction from the top):

$$\theta_t = \angle\overrightarrow{v_W} + \frac{\pi}{2} \tag{14}$$

Length d measures the length of vector **l**'s projection onto the Z_wX_w plane. The values of z_1 and h are equal to the following:

$$z_1 = l \cos \gamma \tag{15}$$

$$h = l \sin \gamma \tag{16}$$

The lengths of components x_1 and y_1 are equal to:

$$x_1 = l \sin \gamma \cos \theta_t \tag{17}$$

$$y_1 = l \sin \gamma \sin \theta_t \tag{18}$$

Similarly, the length of d is found to be:

$$d = \sqrt{l \cos \gamma + l^2 \sin^2 \gamma \cos^2 \theta_t} \tag{19}$$

Using these values, the values of gimbal angles α and β can be found:

$$\beta = \sin^{-1}\left(\frac{x_1}{d_1}\right) \tag{20}$$

$$\beta = \sin^{-1}\left(\frac{\sin \gamma \cos \theta_t}{\sqrt{\cos \gamma + \sin^2 \gamma \cos^2 \theta_t}}\right) \tag{21}$$

$$\alpha = \sin^{-1}\left(\frac{y_1}{l}\right) \tag{22}$$

$$\alpha = \sin^{-1}(\sin \gamma \sin \theta_t) \tag{23}$$

3.3 Forward Kinematics

The forward kinematics of this system can be used to estimate the global velocity of the robot using angle measurements at each of the singularity drive gimbals. Inputs for the forward kinematics are the gimbal angles α and β. In order to derive the proper relationships, we begin with the same gimbal vector diagram shown in Figure 12. We can solve for x_1, y_1, d_1, and h in terms of the gimbal angles:

$$x_1 = \cos\alpha \sin\beta \tag{24}$$

$$y_1 = \sin\alpha \tag{25}$$

$$h = \sqrt{x_1^2 + y_1^2} \tag{26}$$

We can use these values to solve for the tilt angle γ and the steer angle θ_t:

$$\gamma = \sin^{-1} h \tag{27}$$

$$\theta_t = \tan^{-1}\left(\frac{y_1}{x_1}\right) \tag{28}$$

By combining Equations 12 and 27, we can calculate the effective radius of the drive wheel on the ground:

$$r_{eff} = h \cdot r_w \tag{29}$$

We can rearrange Equation 11 to get an expression for the magnitude of the wheel velocity vector using this effective radius and the angular speed of the drive wheel:

$$|v_w| = w_d \cdot r_{eff} \tag{30}$$

Using this magnitude, Equation 29, and some basic trigonometry, we can find the following expressions for the components of the wheel velocity vectors at each wheel:

$$v_{wxi} = |v_w| \cos\left(\theta_t - \frac{\pi}{2}\right) \tag{31}$$

$$v_{wyi} = |v_w| \sin\left(\theta_t - \frac{\pi}{2}\right) \tag{32}$$

In a traditional system, we could rearrange the inverse kinematic equations and solve for the robot's global velocity and orientation. However, the singularity drive system is overdetermined, which means that we cannot invert the (non-square)H matrix in Equation 9 to do so. Therefore, we rely on a best-fit solution to find the most accurate approximation of the robot's velocity and orientation.

The following equation takes all of the gimbal angle measurements for three singularity drive modules and produces the most consistent linear and angular velocities of the robot with those measurements- even if those measurements do not agree [2]:

$$\begin{bmatrix} V_X \\ V_Y \\ \omega \end{bmatrix} = \left[\overline{H}_s{}^T + \overline{H}_s\right]^{-1} \overline{H}_s{}^T \begin{bmatrix} v_{wx1} \\ v_{wy1} \\ v_{wx2} \\ v_{wy2} \\ v_{wx3} \\ v_{wy3} \end{bmatrix} \tag{33}$$

Where:

$$\overline{H}_s = \begin{bmatrix} \overline{H}_1 \\ \overline{H}_2 \\ \overline{H}_3 \end{bmatrix} \tag{34}$$