Honours Degree of Bachelor of Science in Artificial Intelligence

Batch 22 - Level 2 (Semester 2)

CM 2320: Mathematical Methods

Chapter 2: Laplace Transform

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Laplace Transform

Learning Outcomes

By the end of this chapter, students will be able to;

1. define a Laplace transform.
2. derive Laplace transforms of elementary functions.
3. use a standard list of Laplace transforms to determine the transform of common functions.
4. prove the properties of Laplace transform.

Contents

1 Introduction

The Laplace transform is used to solve linear constant coefficient differential equations. This is achieved by transforming them to algebraic equations. The algebraic equations are solved, then the inverse Laplace transform is used to obtain a solution in terms of the original variables. This technique can be applied to both single and simultaneous differential equations. The Laplace transform is also used to produce transfer functions for the elements of an engineering system. These are represented in diagrammatic form as blocks. The various blocks of the system, corresponding to the system elements, are connected together and the result is a block diagram for the whole system. By breaking down a system in this way it is much easier to visualize how the various parts of the system interact and so a transfer function model is complementary to a time domain model and is a valuable way of viewing an engineering system.

1.1 Definition of the Laplace Transform

Let $f(t)$ be a function of time t. In many real problems only values of $t \geq 0$ are of interest. Hence $f(t)$ is given for $t \geq 0$, and for all $t < 0$, $f(t)$ is taken to be 0.

Definition 1. *The Laplace transform of $f(t)$ is $F(s)$, defined by*

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

To denote the Laplace transform of $f(t)$ we write $\mathcal{L}\{f(t)\}$ or $\mathcal{L}(f)$. We use a lower case letter to represent the time domain function and an upper case letter to represent the s domain function. The variable s may be real or complex.

Exercise 1. *Find the Laplace transforms of the following functions.*

(a) 1

(b) e^{at}

1.2 Laplace Transform of Some Common Functions

Determining the Laplace transform of a given function, $f(t)$, is essentially an exercise in integration. In order to save effort a look-up table is often used. Figure 1 lists some common functions and their corresponding Laplace transforms.

Function, $f(t)$	*Laplace transform, $F(s)$*	*Function, $f(t)$*	*Laplace transform, $F(s)$*
1	$\frac{1}{s}$	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
t	$\frac{1}{s^2}$	$\sinh bt$	$\frac{b}{s^2-b^2}$
t^2	$\frac{2}{s^3}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{-at}\sinh bt$	$\frac{b}{(s+a)^2-b^2}$
e^{at}	$\frac{1}{s-a}$	$e^{-at}\cosh bt$	$\frac{s+a}{(s+a)^2-b^2}$
e^{-at}	$\frac{1}{s+a}$	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$u(t)$ unit step	$\frac{1}{s}$
$\cos bt$	$\frac{s}{s^2+b^2}$	$u(t-d)$	$\frac{e^{-sd}}{s}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\delta(t)$	1
		$\delta(t-d)$	e^{-sd}

Figure 1: The Laplace transforms of some common functions.

Exercise 2. *Use standard list to determine the Laplace transform of each of the following functions:*

(a) t^3

(b) $\sin(4t)$

(c) e^{-2t}

(d) $\cos\left(\frac{t}{2}\right)$

2 Properties of the Laplace Transform

There are some useful properties of the Laplace transform that can be exploited. They allow us to find the Laplace transforms of more difficult functions. The properties we shall examine are:

1. linearity;
2. shift theorems;
3. final value theorem.

2.1 Linearity

Let f and g be two functions of t and let k be a constant which may be negative. Then

$$\mathcal{L}\left\{f+g\right\} = \mathcal{L}\left\{f\right\} + \mathcal{L}\left\{g\right\},$$
$$\mathcal{L}\left\{kf\right\} = k\mathcal{L}\left\{f\right\}.$$

Exercise 3. *Find the Laplace transforms of the following functions:*

(a) $3+2t$

(b) $5t^2 - 2e^t$

2.2 Shift Theorem

2.2.1 First Shift Theorem

Theorem 1. *If* $\mathcal{L}\left\{f(t)\right\} = F(s)$ *then*

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a),$$

where a is a constant.

We obtain $F(s+a)$ by replacing every s in $F(s)$ by $s+a$. The variable s has been shifted by an amount a.

Exercise 4. *(a) Use standard list to find the Laplace transform of*

$$f(t) = t\sin(5t).$$

(b) Use the first shift theorem to write down

$$\mathcal{L}\left\{e^{-3t}t\sin(5t)\right\}.$$

2.2.2 Second Shift Theorem

Theorem 2. *If* $\mathcal{L}\{f(t)\} = F(s)$ *then*

$$\mathcal{L}\{H(t-d)f(t-d)\} = e^{-sd}F(s), \qquad d > 0.$$

The function, $H(t-d)f(t-d)$, is obtained by moving $H(t)f(t)$ to the right by an amount d. This is illustrated in below Figure.

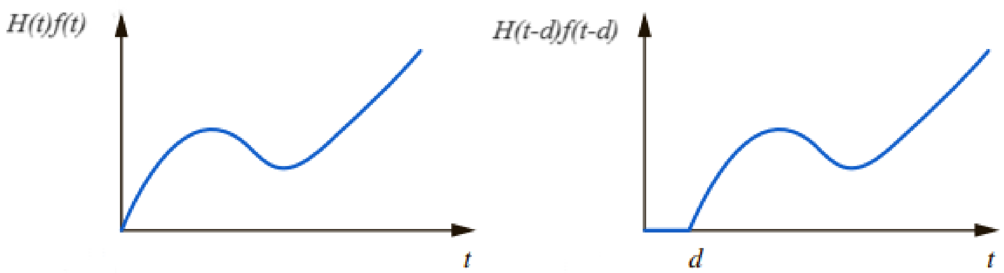

Figure 2: Shifting the function $H(t)f(t)$ to the right by an amount d yields the function $H(t-d)f(t-d)$.

Note that because $f(t)$ is defined to be 0 for $t < 0$, then $f(t-d) = 0$ for $t < d$.

Exercise 5. *Given* $\mathcal{L}\{f(t)\} = \dfrac{2s}{s+9}$, *find* $\mathcal{L}\{H(t-2)f(t-2)\}$.

2.3 Final Value Theorem

This theorem applies only to real values of s and for functions, $f(t)$, which possess a limit as $t \to \infty$.

Theorem 3. *The final value theorem states:*

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t).$$

Some care is needed when applying the theorem. The Laplace transform of some functions exists only for $Re(s) > 0$ and for these functions taking the limit as $s \to 0$ is not sensible.

Exercise 6. *Verify the final value theorem for* $f(t) = e^{-2t}$.