Honours Degree of Bachelor of Science in Artificial Intelligence

Batch 22 - Level 2 (Semester 2)

CM 2320: Mathematical Methods

Chapter 1: Special Functions

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Special Functions

Learning Outcomes

By the end of this chapter, students will be able to;

1. identify Bessel functions, heaviside functions and related functions.
2. identify the properties of Bessel functions, heaviside functions and related functions.

Contents

1 Bessel Functions

1.1 Introduction

Definition 1. *Bessel functions of the first kind are defined by the series*

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)},$$

where the parameter ν denotes the order of the given Bessel function.

When $\nu = n$, $(n = 0, 1, 2, \dots)$, defines the Bessel function of integer order

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!}. \qquad n = 0, 1, 2, \dots. \tag{1}$$

The simplest representative of which is

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2}.$$

The graph of the $J_n(x)$, $n = 0, 1, 2, \dots$. are sketched in the Figure 1.

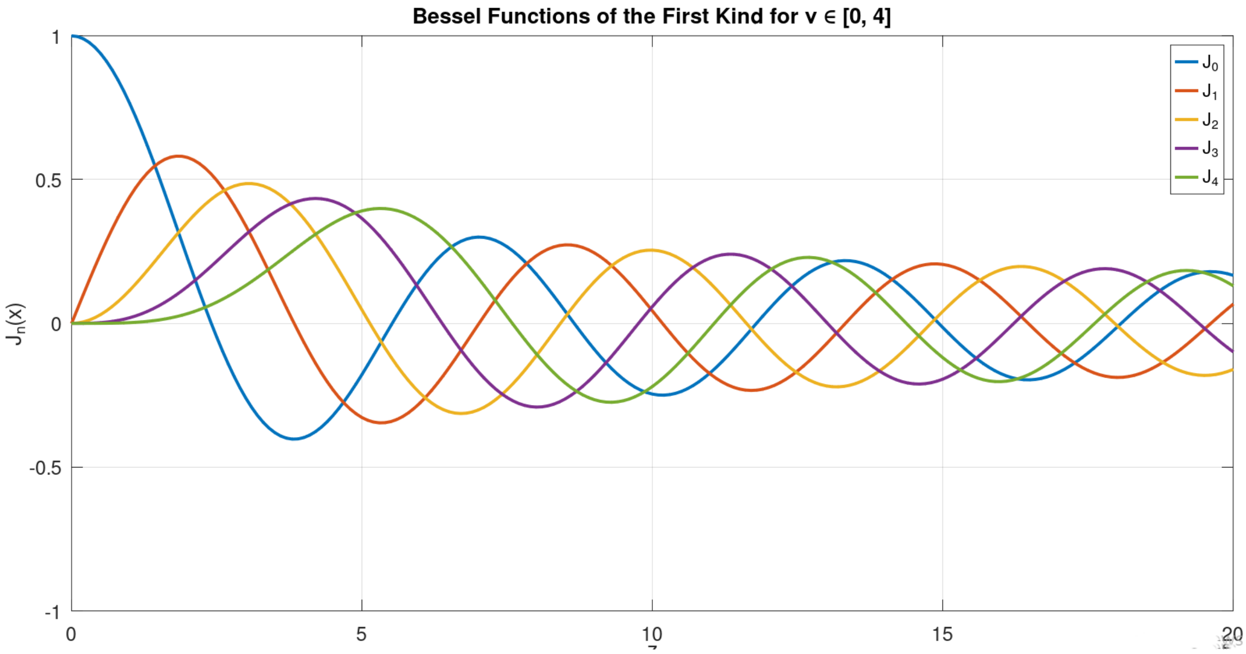

Figure 1: Graphs of Bessel functions $n = 0, 1, 2, 3, 4$.

1.2 Bessel Functions for Negative ν

The parameter ν in (1) may also take on negative values. For example, when $\nu = -n(n = 0, 1, 2, \dots)$, we get

$$\begin{aligned} J_{-n}(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k-n}}{k!(k-n)!} \\ &= \sum_{k=n}^{\infty} \frac{(-1)^k (z/2)^{2k-n}}{k!(k-n)!}, \end{aligned}$$

where we have used the fact that $1/(k-n)! = 0$, $(k = 0, 1, \dots, n-1)$. Finally the change of index $k = m + n$ yields

$$J_{-n}(z) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (z/2)^{2m+n}}{m!(m+n)!},$$

from which we deduce

$$J_{-n}(z) = (-1)^n J_n(z). \qquad n = 0, 1, 2, \dots.$$

1.3 Properties of Bessel Function

The Bessel function $J_{-n}(z)$ has the following properties:

(1) $\dfrac{d}{dz}\left[z^{\nu} J_{\nu}(z)\right] = z^{\nu} J_{\nu-1}(z)$

(2) $\dfrac{d}{dz}\left[z^{-\nu} J_{\nu}(z)\right] = -z^{-\nu} J_{\nu+1}(z)$

2 Heaviside Unit Function

Definition 2. *The Heaviside Unit functions $H(t)$ can be defined by*

$$H(t-a) = \begin{cases} 0, & t < a, \\ 1, & t > a. \end{cases}$$

This function has a jump discontinuity at $t = a$ of unit magnitude. The graph of the $H(t-1)$ is sketched in Figure 2.

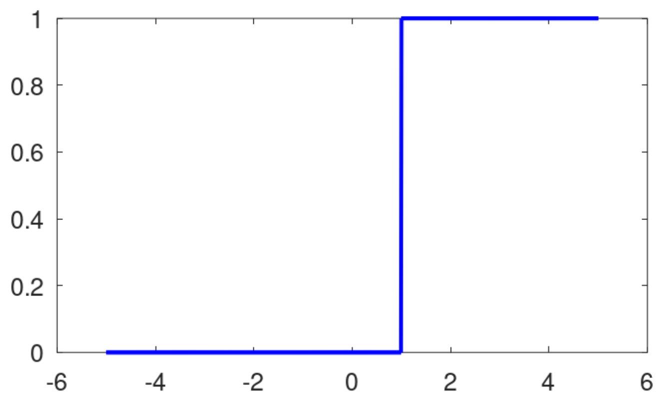

Figure 2: Graph of the Heaviside Unit function.

The main utility of the Heaviside unit function is that it acts like a "switch" to tum another function on or off at some time.

For instance, the function $f(t) = H(t-1)\cos(2\pi t)$, is clearly zero for $t < 1$ and assumes the graph of the cosine function for $t > 1$ as shown in Figure 3.

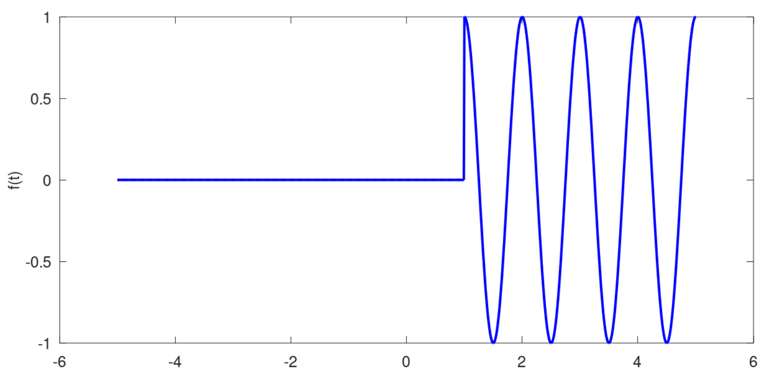

Figure 3: Graph of the function $f(t) = H(t-1)\cos(2\pi t)$.

3 Related Function

Definition 3. *The related functions is the rectangle function $f(t)$ defined by*

$$f(t) = \begin{cases} 1, & a < t < b, \\ 0, & \textit{otherwise.} \end{cases}$$

Related functions can be easily expressed in terms of the Heaviside unit function as

$$f(t) = H(t-a) - H(t-b).$$

The graph of related function is shown in Figure 4.

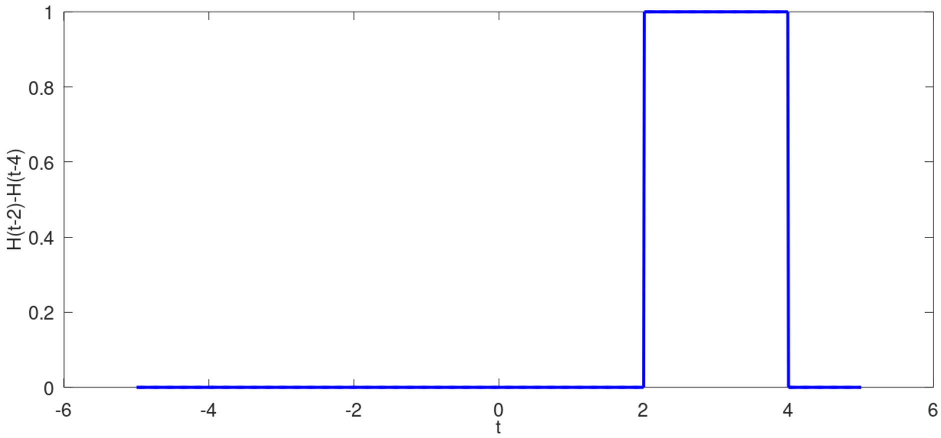

Figure 4: Graph of the related function.