Honours Degree of Bachelor of Science in Artificial Intelligence

Batch 22 - Level 2 (Semester 2)

CM 2320: Mathematical Methods

Chapter 4: Partial Differential Equations

Dr. Thilini Piyatilake
Senior Lecturer
Department of Computational Mathematics
University of Moratuwa

Partial Differential Equations

Learning Outcomes

By the end of this chapter, students will be able to;

1. identify the necessity of partial differential equations to describe real world problem.
2. classify the partial differential equations.
3. distinguish between the 3 classes of 2^{nd} order, linear PDE's. Know the physical problems each class represents and the physical/mathematical characteristics of each.

Contents

1 Introduction

1.1 Definition

Definition 1. *A **Partial Differential Equation (PDE)** is one or more equations connecting partial derivatives of one or more unknown functions (or dependent variables). In* *addition the equation(s) can contain known functions. All functions are functions of two or more independent variables.*

Examples:

- $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0 \Rightarrow$ 2D Laplace equation
- $\dfrac{\partial f}{\partial t} = \alpha \dfrac{\partial^2 f}{\partial x^2} \Rightarrow$ 1D Diffusion equation
- $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2} \Rightarrow$ 1D Wave equation
- $\dfrac{\partial^2 f}{\partial x^2} + f \dfrac{\partial^2 f}{\partial y^2} + 2 = 0 \Rightarrow$ 2D equation

1.2 Difference Between ODE and PDE

In mathematics, an ordinary differential equation of *ODE* is a differential equation containing a *function or function of one independent variable and its derivatives.* The term "ordinary" is used in contrast with the term "partial" differential equation or PDE which may be with respect to more than one independent variable. The general form of ODE is

$$F(x, y, y^{'}, y^{''}, \ldots, y^{n}) = 0,$$

where y is a function of x, $y^{'} = \dfrac{dy}{dx}$ is the first derivative with respect to x, and $y^n = \dfrac{d^n y}{dx^n}$ is the n^{th} derivative with respect to x

1.3 General Features of PDE

A PDE is an equation stating a relationship between function of *two or more independent variables* and the partial derivatives of this function with respect to these independent variables. The dependent variable f is used as a generic dependent variable. In most problems in engineering and science, the dependent variables are either *space* (x, y, z) or *space and time* (x, y, z, t). The dependent variable f depends on the physical problem being modeled.

2 Properties of PDEs

There are a number of properties by which PDEs can be separated into families of similar equations. The three main properties are

1. order,
2. homogeneity and
3. linearity.

2.1 Order

The *magenta*order of a partial differential equation is the *order of highest derivative* engage in the equation.

Examples:

- $c^2 \dfrac{\partial^4 u}{\partial x^4} + 2\dfrac{\partial^2 u}{\partial t^2} = 0 \;\; \Rightarrow$ Fourth order
- $\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \;\; \Rightarrow$ Second order
- $\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial^2 u}{\partial x \partial y}\right)^2 + \dfrac{\partial^2 u}{\partial y^2} = x^2 + y^2 \;\; \Rightarrow$ Second order
- $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 \;\; \Rightarrow$ First order

2.1.1 Classification of Linear 2$^{\text{nd}}$ Order PDE

The general form

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} + D\frac{\partial \phi}{\partial x} + E\frac{\partial \phi}{\partial y} + F\phi + G = 0.$$

We can classify the 2$^{\text{nd}}$ order PDE as follows:

1. Elliptic if $(B^2 - 4AC) < 0$
2. Parabolic if $(B^2 - 4AC) = 0$
3. Hyperbolic if $(B^2 - 4AC) > 0$

Exercise 1. *Classify the following partial differential equations.*

1. $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + g = 0$
2. $\dfrac{\partial^2 U}{\partial t^2} = c^2 \dfrac{\partial^2 U}{\partial x^2}$
3. $U_t = kU_{xx}$

2.1.2 Well Known 2nd Order PDEs

- Elliptic
  - Poisson equation
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + g = 0$$
  - Laplace equation
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$
    We can use Poisson and Laplace equations to describe steady state problems.

- Parabolic
  - Heat or diffusion equation
$$\frac{\partial U}{\partial t} = K\frac{\partial^2 U}{\partial x^2}$$
    To model time dependent temperature distribution along a heated 1D bar or the concentration of a certain chemical in a thin tube.
  - Black-Scholes equation
$$\frac{\partial v}{\partial t} + rs\frac{\partial v}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} = rv,$$
    where $v(s, t)$ is the value of a share option, s is the share price, r is the interest rate, and s is the share "volatility".

- Hyperbolic
  - Wave equation
$$\frac{\partial^2 U}{\partial t^2} = c\frac{\partial^2 U}{\partial x^2}$$
    To model a vibration of a guitar string or 1D supersonic flow.

2.2 Homogeneity

In the partial differential equation $U(f) = C$ where f is the dependent variable of all independent variables (x, y, z) and U involves.

1. If $C = 0$ then PDE is a Homogeneous equation.
2. If $C \neq 0$ then PDE is a Non Homogeneous equation.

Examples

- $u_t = u_x \ \Rightarrow$ Homogeneous
- $u_{xx} + u_{yy} = 0 \ \Rightarrow$ Homogeneous
- $u_{xx} + u_{yy} = x^2 + y^2 \ \Rightarrow$ Non Homogeneous
- $u_x u_{xx} + (u_y)^2 = 0 \ \Rightarrow$ Homogeneous
- $a(x, y)u_{xx} + 2u_{xy} + 3x^2 u_{yy} = 4e^x \ \Rightarrow$ Non Homogeneous

2.3 Linearity

If f satisfies the following properties:

1. $U(f+g) = U(f) + U(g)$
2. $U(kf) = kU(f)$, for any scalar k

then the equation is called a *linear partial differential* equation.

Examples

- Heat Equation: $f_t - f_{xx} = 0 \;\; \Rightarrow$ Linear
- Inviscid Burger's Equation: $f_t + ff_x = 0 \;\; \Rightarrow$ Non Linear