University of Moratuwa

Faculty of Information Technology
Department of Computational Mathematics

Honours Degree of Bachelor of Science in Artificial Intelligence

Level 2 - Semester 2 - Examination

CM 2320 - Mathematical Methods

Time Allowed: 3 Hours — August 2024

ADDITIONAL MATERIAL

1. Table of Laplace Transforms

INSTRUCTIONS TO CANDIDATES

1. This paper contains 4 questions on 6 pages (including this page).
2. The total marks obtainable for this examination is 100. The marks assigned for each question are included in square brackets.
3. This examination accounts for 70% of the module assessment.
4. This is a closed book examination.
5. Answer **ALL** questions.
6. Start to answer **a new question** on **new page**.
7. **All the necessary steps** for the answers should be **clearly indicated**.
8. Calculators are ALLOWED.
9. If a page is not printed, please inform the supervisor immediately.

END OF INSTRUCTIONS

Question 1 [**Total Marks Allocated: 25 Marks**]

a) By using techniques involving the *Gamma function*, find the value of

$$\int_0^\infty e^{-x^4}\,dx,$$

in the form $\Gamma(k)$, where k is a rational constant. [5 Marks]

b) If X is a normal random variable, its probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2},$$

where m is the mean value of X and σ^2 the variance. The probability that $X \leq y$ is defined by

$$P(X \leq y) = \int_{-\infty}^{y} p(x)dx.$$

(i) Show that

$$P(X \leq y) = \frac{1}{2}\left[1 + \; erf\left(\frac{y-m}{\sqrt{2}\sigma}\right)\right].$$

[5 Marks]

(ii) What is the probability $P(X \leq y)$ in the limit $y \to \infty$? [5 Marks]

c) The *Dirac delta* function is defined by the filter property

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a),$$

for a smooth function $f(t)$. Find the numerical value of following functions.

(i) $\displaystyle\int_{-\infty}^{\infty} \delta(3t-2)t^2 dt$ [5 Marks]

(ii) $\displaystyle\int_{-5}^{3} (2\delta(t) + 3\delta'(t))dt$ [5 Marks]

Question 2 [Total Marks Allocated: 25 Marks]

a) Define the Laplace transformation of the function $f(t)$. [2 Marks]

b) Find the Laplace transform of the function e^{-t}. [4 Marks]

c) Let $f'(t)$ denotes the first derivative of a function $f(t)$. If Laplace transformation of $f'(t)$ is given by

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

then show that

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

[4 Marks]

d) Use *convolution theorem* to find the inverse Laplace transforms of the following function

$$\frac{1}{(s^2+1)(s+1)}.$$

[6 Marks]

e) Use Laplace transformation and convolution theorem to solve the following differential equation

$$\frac{d^2x}{dt^2} + x = e^{-t},$$

subject to $x(0) = 0$ and $x'(0) = 0$. [4 Marks]

f) A system is modelled by the differential equations

$$\frac{dp}{dt} + p = k(t) \quad (1)$$

$$3\frac{dq}{dt} - q = p(t) \quad (2)$$

In (1) the input is $k(t)$ and the output is $p(t)$. In (2), $p(t)$ is the input and $q(t)$ is the final output of the system. Find the *overall* system transfer function assuming the initial conditions $p(0) = 0$ and $q(0) = 0$. [5 Marks]

Continued...

Question 3 [Total Marks Allocated: 25 Marks]

a) Let $S(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right]$ be the *Fourier series* for the function

$$f(t) = \begin{cases} 0, & -\pi < t < 0, \\ t, & 0 < t < \pi. \end{cases}$$

and $f(t+2\pi) = f(t)$.

(i) Sketch the graph of $f(t)$ on the interval $-3\pi < t < 3\pi$. [3 Marks]

(ii) State the precise numerical value of $S(t)$ for each t in the interval $-\pi \leq t \leq \pi$. [3 Marks]

(iii) Compute the Fourier coefficients a_0, a_n and b_n for $f(t)$. [8 Marks]

(iv) Using Fourier series of $f(t)$ deduce the following formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$ [3 Marks]

b) Consider the function

$$f(t) = \begin{cases} 5, & -2 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that the *Fourier transform* of $f(t)$ is given by $F(\omega) = \dfrac{10\sin(2\omega)}{\omega}$. [5 Marks]

(ii) The *first shift theorem* of Fourier transform is given by

$$\mathcal{F}\left\{e^{iat}f(t)\right\} = F(\omega - a),$$

for the constant a. Find the Fourier transform of $e^{-it}f(t)$. [3 Marks]

Continued...

Question 4 [Total Marks Allocated: 25 Marks]

One-dimensional wave equation models the two-dimensional dynamics of a vibrating string which is stretched and clamped at its end points (say at $x = 0$ and $x = L$) is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2},$$

where x denotes the spatial variable, t denotes time, $u(x,t)$ measures the vertical displacement of string at point x at time t, L is the length of the string, and C is a constant.

a) What is the *order* of the equation? Explain your answer. [2 Marks]

b) Determine whether the equations is *parabolic, hyperbolic* or *elliptic*. [2 Marks]

c) Given $u(x,t) = u_0 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi C t}{L}\right)$, where u_0 is a constant. Is $u(x,t)$, a solution? Explain your answer. [4 Marks]

d) Verify the boundary conditions $u(0,t) = u(L,t) = 0$, for the solution given in part (c). [4 Marks]

e) Verify the initial condition $\frac{\partial u}{\partial t}(x,0) = 0$, for the solution given in part (c). [4 Marks]

f) Verify that the principle of superposition holds for any two solutions, u_1 and u_2, of the equation. [4 Marks]

g) Separate the equation into a system of ordinary differential equations. [5 Marks]

END OF PAPER

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Table of Laplace Transforms

Casual Function $f(t)$	Laplace Transform $F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$\cos(bt)$	$\frac{s}{s^2-b^2}$