Honours Degree of Bachelor of Science in Artificial Intelligence

Batch 22 - Level 2 (Semester 2)

CM 2320: Mathematical Methods

Chapter 2: Laplace Transform

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Laplace Transform

Learning Outcomes

By the end of this chapter, students will be able to;

1. derive the Laplace transform of the derivative of an expression.
2. derive the Laplace transform of the integral of an expression.
3. define the inverse Laplace transform.
4. determine the inverse Laplace transforms of simple functions.
5. define the convolution.
6. calculate the convolution of functions.
7. apply the convolution theorem to obtain inverse Laplace transforms.
8. solve differential equations using Laplace transforms.

Contents

1 Laplace Transform of Derivatives and Integrals

In this section we shall use the Laplace transform to solve differential equations. In order to do this we need to be able to find the Laplace transform of derivatives of functions.

Let $f(t)$ be a function of t, and $f^{'}$ and $f^{''}$ are the first and second derivatives of f. The Laplace transform of $f(t)$ is $F(s)$. Then,

$$\mathcal{L}\left\{f^{'}\right\} = sF(s) - f(0),$$
$$\mathcal{L}\left\{f^{''}\right\} = s^2F(s) - sf(0) - f^{'}(0).$$

where $f(0)$ and $f'(0)$ are the initial values of f and f'. The general case for the Laplace transform of an n^{th} derivative is

$$\mathcal{L}\left\{f^{(n)}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \cdots - f^{(n-1)}(0).$$

Proof

Another useful result is

$$\mathcal{L}\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}F(s).$$

Proof

Exercise 1. *The Laplace transform of $x(t)$ is $X(s)$. Given $x(0) = 2$ and $x'(0) = -1$, write expressions for the Laplace transforms of following expressions.*

1. $2x^{''} - 3x^{'} + x$
2. $-x^{''} + 2x^{'} + x$

2 Inverse Laplace Transform

2.1 Definition

Definition 1. *If the* ***Laplace transform of a function $f(t)$ is $F(s)$****, i.e.* $\mathcal{L}\{f(t)\} = F(s)$, *then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is written as,*

$$f(t) = \mathcal{L}^{-1}\{F(s)\},$$

and call $\mathcal{L}^{-1}$ the inverse Laplace transform.

Example:
Since $\mathcal{L}\{1\} = \frac{1}{s}$ then $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$.
Like $\mathcal{L}$, $\mathcal{L}^{-1}$ can be shown to be a linear operator.

Exercise 2. *Find the inverse Laplace transforms of the following:*

1. $\dfrac{2}{s^3}$
2. $\dfrac{s}{s^2+1}$
3. $\dfrac{s+1}{s^2+1}$
4. $\dfrac{10}{(s+2)^4}$
5. $\dfrac{s+3}{s^2+6s+13}$

2.2 Using Partial Fractions to Find the Inverse Laplace Transform

The inverse Laplace transform of a fraction is often best found by expressing it as its partial fractions, and finding the inverse transform of these.

Exercise 3. *Find the inverse Laplace transforms of the following:*

1. $\dfrac{4s-1}{s^2-s}$

2. $\dfrac{6s+8}{s^2+3s+2}$

2.3 Finding the Inverse Laplace Transform Using Complex Numbers

We now look at a method of finding inverse Laplace transforms using complex numbers. Essentially the method is one using partial fractions, but where all the factors in the denominator are linear. That is, there are no quadratic factors.

Exercise 4. *Find the inverse Laplace transforms of the following functions:*

1. $\dfrac{s+3}{s^2+6s+13}$

2. $\dfrac{2s+3}{s^2+6s+13}$

3 The Convolution Theorem

3.1 The Convolution

Definition 2. *Let $f(t)$ and $g(t)$ be two piecewise continuous functions. The* ***convolution*** *of $f(t)$ and $g(t)$, denoted $(f * g)(t)$, is defined by*

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv.$$

Exercise 5. *Find the convolution of $2t$ and t^3.*

3.1.1 Commutative Property

The convolution is **commutative**. That is, for any functions $f(t)$ and $g(t)$. That is $f*g = g*f$.

Exercise 6. *Show that $f * g = g * f$, where $f(t) = 2t$ and $g(t) = t^3$.*

3.2 The Convolution Theorem

Theorem 1. *Let $f(t)$ and $g(t)$ be piecewise continuous functions, with $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. The inverse Laplace transform of a product of transforms, $F(s)G(s)$ is*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Proof

Exercise 7. *Use the convolution theorem to find the inverse Laplace transforms of the following function.*

$$\frac{1}{(s+2)(s+3)}$$

4 Solving Differential Equations

4.1 Solving Linear Constant Coefficient Differential Equations Using the Laplace Transform

We now apply the Laplace transform to finding the particular solution of differential equations. The initial conditions are automatically satisfied when solving an equation using the Laplace transform. They are contained in the transform of the derivative terms.
The Laplace transform of the equation is found. This transforms the differential equation into an algebraic equation. The transform of the dependent variable is found and then the inverse transform is calculated to yield the required particular solution.

Exercise 8. *Solve the differential equation*

$$\frac{dx}{dt} + x = 0, \qquad x(0) = 3,$$

using Laplace transforms.

Exercise 9. *Solve the differential equation*

$$y^{''} - y = -t^2, \qquad y(0) = 2, \;\; y^{'}(0) = 0,$$

using Laplace transforms.

4.2 Solve Simultaneous Differential Equations Using the Laplace Transform

Exercise 10. *Solve the simultaneous differential equations*

$$x' + x + \frac{y'}{2} = 1,$$
$$\frac{x'}{2} + y' + y = 0,$$

subjected to the initial conditions $x(0) = y(0) = 0$.