Honours Degree of Bachelor of Science in Artificial Intelligence

Batch 22 - Level 2 (Semester 2)

CM 2320: Mathematical Methods

Chapter 1: Special Functions

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Special Functions

Learning Outcomes

By the end of this chapter, students will be able to;

1. recognize different type of special functions.
2. identify the properties of special functions.

Contents

1 Introduction

Most of the functions encountered in introductory analysis belong to the class of *elementary functions*. This class is composed of polynomials, rational functions, transcendental functions (trigonometric, exponential, logarithmic, and so on), and functions constructed by combining two or more of these functions through addition, subtraction, multiplication, division, or composition. Beyond these functions lies a class of special functions which are important in a variety of engineering and physics applications.

Real wold problems solutions are heavily interlaced with special functions like the gamma function, error function, Bessel functions, and so forth. Also, functions such as the Heaviside unit function and the impulse function, which are employed in a variety of engineering applications, are briefly discussed. Hence, a brief introduction of some of these special functions can be quite useful before discussing integral transforms themselves.

2 The Gamma Function

Definition 1. *The Gamma function is defined by the integral formula*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

The integral converges absolutely for $Re(z) > 0$.

The variable z may be real or complex. Although the integral is improper, it has been shown that it converges uniformly for all values of z for which $Re(z) > 0$. The function $\Gamma(z)$ is bounded and differentiable and, in fact, an analytic function throughout this domain.

2.1 Properties of the Gamma Function

The gamma function $\Gamma(z)$ has the following properties:

(1) $\Gamma(z+1) = z\Gamma(z)$

(2) $\Gamma(1) = 1$

(3) $\Gamma(n+1) = n!$, where $n = 0, 1, 2, \ldots$

2.2 Analytic Continuation

An analytic continuation of $\Gamma(z)$ to the left of the imaginary axis can be accomplished through repeated use of the recurrence formula $\Gamma(z+1) = z\Gamma(z)$ expressed in the form

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad z \neq 0. \tag{1}$$

The right-hand side of (1) is defined for all $Re(z) > -1$, $z \neq 0$, and thus this expression defines $\Gamma(z)$ in this domain. Replacing z by $z+1$ in (1) yields

$$\Gamma(z+1) = \frac{\Gamma(z+2)}{z+1}, \qquad z \neq -1.$$

and when substituted into (1), we obtain

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)}, \qquad z \neq 0, -1. \tag{2}$$

which now defines $\Gamma(z)$ for all $Re(z) > -2$, $z \neq 0, -1$. Continuing this process, we deduce that

$$\Gamma(z) = \frac{\Gamma(z+k)}{z(z+1)\dots(z+k-1)}, \quad z \neq 0, -1, \dots, -k+1, \tag{3}$$

where k is a positive integer. Equation (3) can be used to define the gamma function at every z with a negative real part except at negative integers and zero. The values $z = 0, -1, -2, \dots,$ are actually first-order poles of the function and thus

$$|\Gamma(-n)| = \infty, \qquad n = 0, 1, 2, \dots \tag{4}$$

The graph of the gamma function for $z = x$, a real variable, is sketched in the Figure 1.

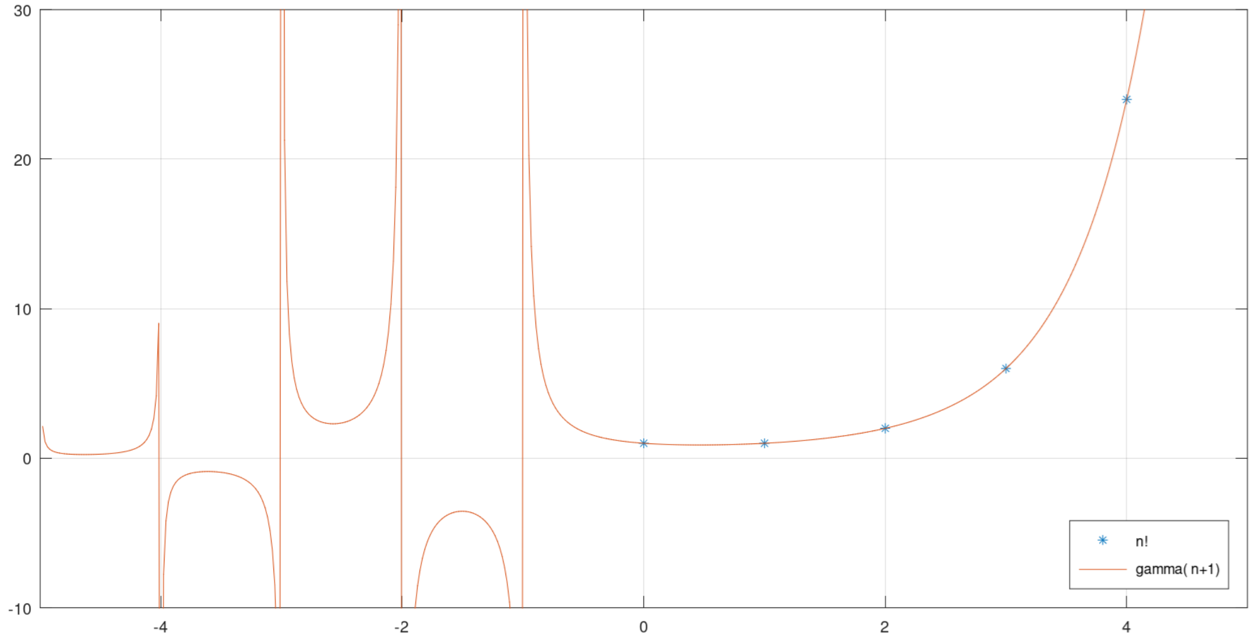

Figure 1: Graph of the gamma function

Exercise 1. *Evaluate the following expressions.*

(1) $\Gamma(-1/2)$

(2) $\Gamma(-3/2)$

Exercise 2. *Show that* $\Gamma(1/2) = \sqrt{\pi}$.

3 The Error Function and Related Functions

3.1 The Error Function

Definition 2. *The error function is defined by the integral*

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt,$$

where the variable z may be real or complex.

3.1.1 Taylor Series Expansion for Exponential Function

A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

Taylor series of exponential functions about a point $x = 0$ is

$$\begin{aligned} e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

3.1.2 Series Representation of the Error Function

By representing the exponential function in terms of its power series expansion, we have

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} dt,$$

from which we deduce (term-wise integration of power series is permitted)

$$erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

This series converges everywhere in the finite complex plane; therefore, $erf(z)$ is an entire function. Examination of the series reveals that the error function is an **odd function**, i.e.,

$$\boldsymbol{erf(-z) = -erf(z)}.$$

3.1.3 Properties of Error Function

The error function $erf(z)$ has the following properties:

- $erf(0) = 0$
- $erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = \frac{\Gamma(1/2)}{\sqrt{\pi}} = 1$

The graph of $erf(x)$, where x is real, is shown in Figure 2.

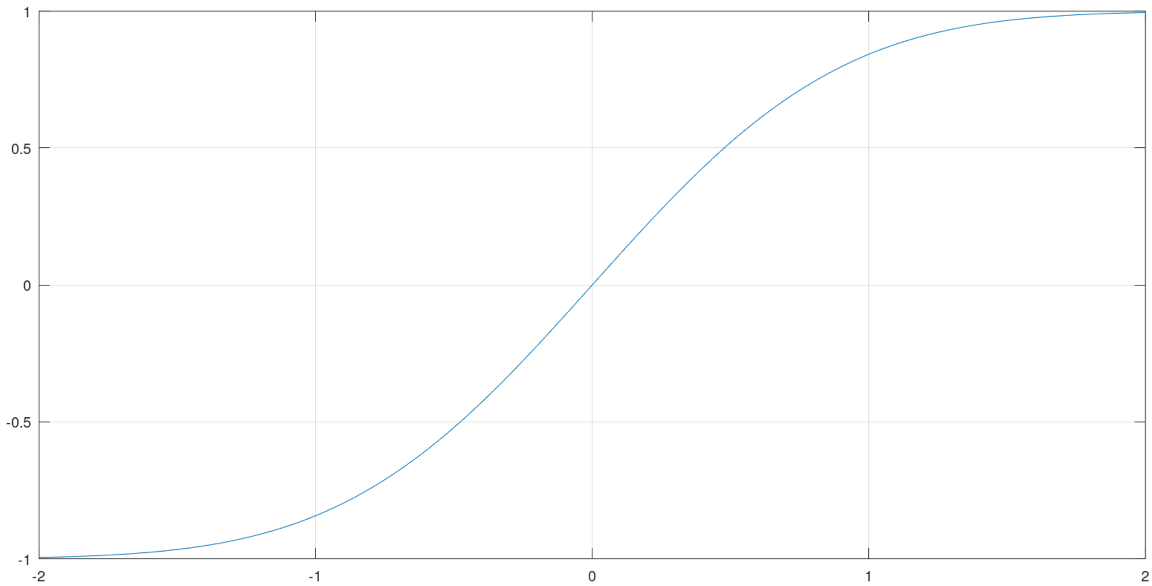

Figure 2: Graph of the error function

3.2 Complementary Error Function

In some applications it is useful to introduce the *complementary error* function

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.$$

Using properties of integrals, it follows that

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt,$$

from which we deduce

$$erfc(z) = 1 - erf(z)$$

Hence, all properties of $erfc(z)$ are easily derived from those of $erf(z)$.