Calculus for Team-Based Inquiry Learning

2024 Edition PREVIEW

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[1] calculus.tbil.org

[2] github.com/TeamBasedInquiryLearning/calculus/blob/main/LICENSE.md

Contents

Chapter 1

Limits (LT)

Learning Outcomes

How do we measure "close-by" values?
By the end of this chapter, you should be able to...

1. Find limits from the graph of a function.
2. Infer the value of a limit based on nearby values of the function.
3. Compute limits of functions given algebraically, using proper limit properties.
4. Determine where a function is and is not continuous.
5. Determine limits of functions at infinity.
6. Determine limits of functions approaching vertical asymptotes.

1.1 Limits graphically (LT1)

Learning Outcomes

- Find limits from the graph of a function.

Activity 1.1.1 In Figure 1 the graph of a function is given, but something is wrong. The graphic card failed and one portion did not render properly. We can't see what is happening in the neighborhood of $x = 2$.

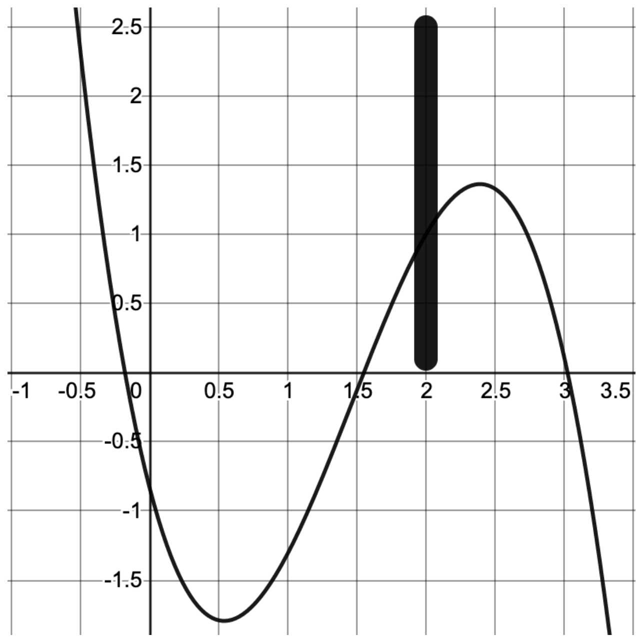


Figure 1 A graph of a function that has not been rendered properly.

(a) Imagine moving along the graph toward the missing portion from the left, so that you are climbing up and to the right toward the obscured area of the graph. What y-value are you approaching?

A. 0.5
B. 1
C. 1.5
D. 2
E. 2.5

(b) Think of the same process, but this time from the right. You're falling down and to the left this time as you come close to the missing portion. What y-value are you approaching?

A. 0.5
B. 1
C. 1.5
D. 2
E. 2.5

Activity 1.1.2 In Figure 2 the graphic card is working again and we can see more clearly what is happening in the neighborhood of $x = 2$.

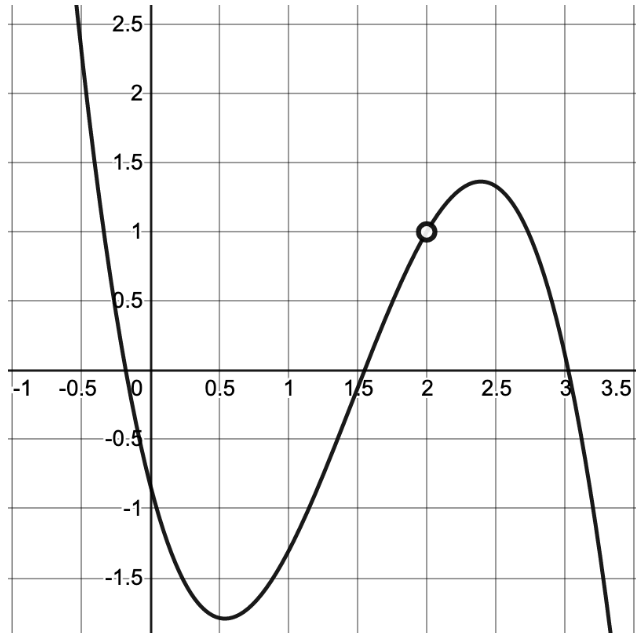


Figure 2 A graph of a function that has rendered properly

(a) What is the value of $f(2)$?

(b) What is the y-value that is approached as we move toward $x = 2$ from the left?

A. 0.5
B. 1
C. 1.5
D. 2
E. 2.5

(c) What is the y-value that is approached as we move toward $x = 2$ from the right?

A. 0.5
B. 1
C. 1.5
D. 2
E. 2.5

Remark 1.1.3 When studying functions in algebra, we often focused on the *value* of a function given a specific x-value. For instance, finding $f(2)$ for some function $f(x)$. In calculus, and here in Activity 1.1.1 and Activity 1.1.2, we have instead been exploring what is happening as we *approach* a certain value on a graph. This concept in mathematics is known as finding a limit.

Activity 1.1.4 Based on Activity 1.1.1 and Activity 1.1.2, write your first draft of the definition of a limit. What is important to include? (You can use concepts of limits from your daily life to motivate or define what a limit is.)

Definition 1.1.5 Given a function f, a fixed input $x = a$, and a real number L, we say that f **has limit** L **as** x **approaches** a, and write

$$\lim_{x \to a} f(x) = L$$

provided that we can make $f(x)$ as close to L as we like by taking x sufficiently close (but not equal) to a. If we cannot make $f(x)$ as close to a single value as we would like as x approaches a, then we say that f **does not have a limit as** x **approaches** a**.** $\Diamond$

Activity 1.1.6

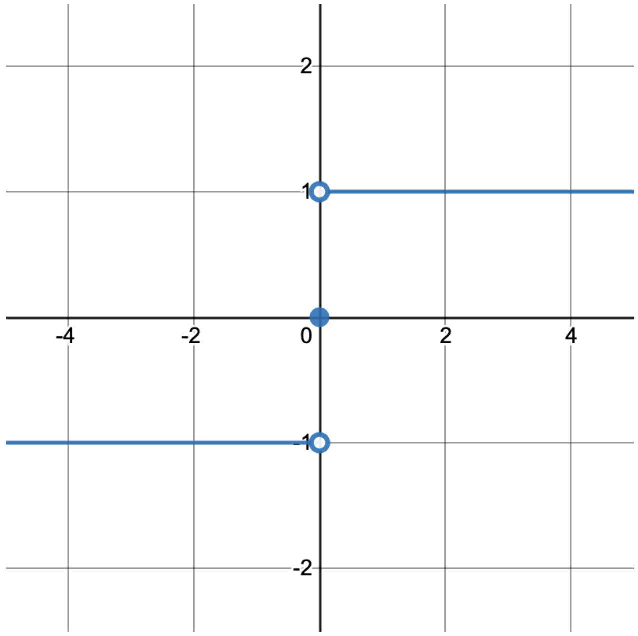


Figure 3 A piecewise-defined function

What is the limit as x approaches 0 in Figure 3?

A. The limit is 1

B. The limit is -1

C. The limit is 0

D. The limit is not defined

Definition 1.1.7 We say that f *has limit* L_1 *as* x *approaches* a *from the left* and write

$$\lim_{x\to a^-} f(x) = L_1$$

provided that we can make the value of $f(x)$ as close to L_1 as we like by taking x sufficiently close to a while always having $x < a$. We call L_1 the left-hand limit of f as x approaches a. Similarly, we say L_2 *is the right-hand limit of* f *as* x *approaches* a and write

$$\lim_{x\to a^+} f(x) = L_2$$

provided that we can make the value of $f(x)$ as close to L_2 as we like by taking x sufficiently close to a while always having $x > a$. $\Diamond$

Activity 1.1.8 Refer again to Figure 3 from Activity 1.1.6.

(a) Which of the following best matches the definition of right and left limits? (Note that DNE is short for ”does not exist.”)

A. The left limit is -1. The right limit is 1.

B. The left limit is 1. The right limit is -1.

C. The left limit DNE. The right limit is 1.

D. The left limit is -1. The right limit DNE.

E. The left limit DNE. The right limit DNE.

(b) What do you think the overall limit equals?

A. The limit is 1

B. The limit is -1

C. The limit is 0

D. The limit is not defined

Limits graphically (LT1)

Activity 1.1.9 Consider the following graph:

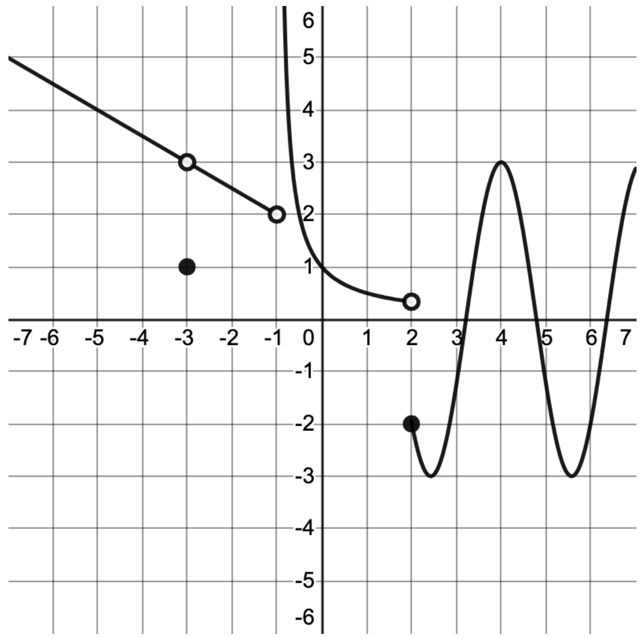


Figure 4 Another piecewise-defined function

(a) Find $\lim\limits_{x \to -3^-} f(x)$ and $\lim\limits_{x \to -3^+} f(x)$.

(b) Find $\lim\limits_{x \to -1^-} f(x)$ and $\lim\limits_{x \to -1^+} f(x)$.

(c) Find $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$.

(d) Find $\lim\limits_{x \to 4^-} f(x)$ and $\lim\limits_{x \to 4^+} f(x)$.

(e) For which x-values does the *overall* limit exist? Select all. If the limit exists, find it. If it does not, explain why.

A. -3

B. -1

C. 2

D. 4

Activity 1.1.10 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

1. $\lim_{x\to 5^-} f(x)$ is finite, but $\lim_{x\to 5^+} f(x)$ is infinite.
2. $\lim_{x\to -3} f(x) = -4$, but $f(-3) = 0$.
3. $\lim_{x\to -1^-} f(x) = -1$ but $\lim_{x\to -1^+} f(x) \neq -1$.

Activity 1.1.11 In this activity we will explore a mathematical theorem, the Squeeze Theorem Theorem 1.1.11.

(a) The part of the theorem that starts with "Suppose..." forms the assumptions of the theorem, while the part of the theorem that starts with "Then..." is the conclusion of the theorem. What are the assumptions of the Squeeze Theorem? What is the conclusion?

(b) The assumptions of the Squeeze Theorem can be restated informally as "the function g is squeezed between the functions f and h around a." Explain in your own words how the two assumptions result into a "squeezing effect."

(c) Let's see an example of the application of this theorem. First examine the following picture. Explain why, from the picture, it seems that both assumptions of the theorem hold.

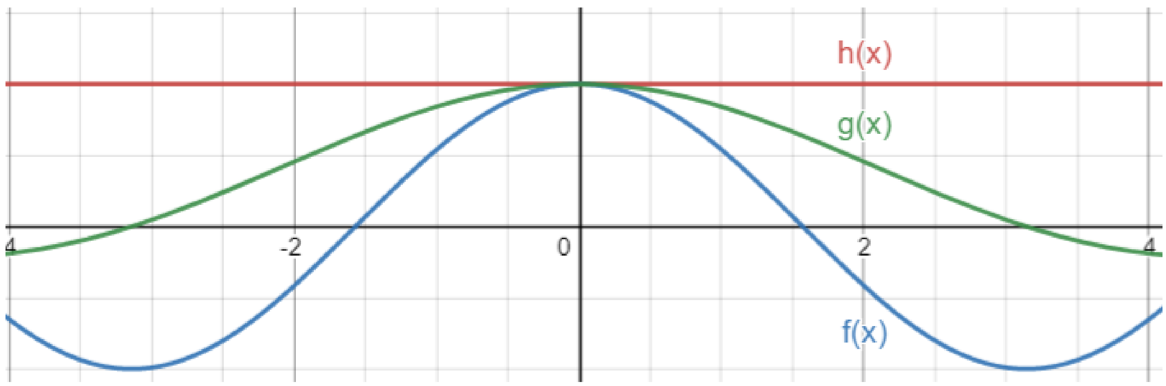


Figure 5 A pictorial example of the Squeeze Theorem.

(d) Match the functions $f(x), g(x), h(x)$ in the picture to the functions $\cos(x), 1, \frac{\sin(x)}{x}$.

(e) Using trigonometry, one can show algebraically that $\cos(x) \leq \frac{\sin(x)}{x} \leq 1$ for x values close to zero. Moreover, $\lim_{x\to 0} \cos(x) = \cos(0) = 1$ (we say that cosine is a continuous function). Use these facts and the Squeeze Theorem, to find the limit $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$.

1.2 Limits numerically (LT2)

Learning Outcomes

- Infer the value of a limit based on nearby values of the function.

Activity 1.2.1

Table 6

x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	0.1695	0.1699	0.1667	?	0.1667	0.1664	0.1639

Based on the values of Table 7, what is the best approximation for $\lim_{x \to 7} f(x)$?

A. the limit is approximately 7

B. the limit is approximately 0.17

C. the limit is approximately 0.16

D. the limit is approximately 0.1667

E. the limit is approximately 6.9999

Remark 1.2.2 Notice that the value we obtained in Activity 1.2.1 is only an approximation, based on the trends that we have seen within the table.

Activity 1.2.3

Table 7

x	1.25	1.5	1.75	2	2.25	2.5	2.75
$f(x)$	-0.7606	-0.13	0.4881	?	1.3119	1.33	0.9606

In Activity 1.1.1's Figure 1 we found an approximation to the limit of the function as x tends to 2. Now let us say you are also given a table of numerical values (Table 8) for the function. Given this new information which of the choices below best describes the limit of the function as x tends to 2?

A. There is not enough information because we do not know the value of the function at $x = 2$.

B. The limit can be approximated to be 1 because the data in the table and the graph show that from the left and the right the function approaches 1 as x goes to 2.

C. The limit can be approximated to be 1 because the values appear to approach 1 and the graph appears to approach 1, but we should zoom in on the graph to be sure.

D. The limit cannot be approximated because the function might not exist at $x = 2$.

Activity 1.2.4

Table 8

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	-0.4	-0.49	-0.499	?	0.499	0.49	0.4

Based on Table 9, what information can be inferred about $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, and $\lim\limits_{x\to 1} f(x)$?

A. $\lim\limits_{x\to 1^-} f(x) = -0.5$, $\lim\limits_{x\to 1^+} f(x) = 0.5$, and $\lim\limits_{x\to 1} f(x) = 0$

B. $\lim\limits_{x\to 1^-} f(x) = -0.5$, $\lim\limits_{x\to 1^+} f(x) = 0.5$, and $\lim\limits_{x\to 1} f(x)$ does not exist

C. $\lim\limits_{x\to 1^-} f(x) = 0.5$, $\lim\limits_{x\to 1^+} f(x) = -0.5$, and $\lim\limits_{x\to 1} f(x)$ does not exist

D. $\lim\limits_{x\to 1^-} f(x) = 0.5$, $\lim\limits_{x\to 1^+} f(x) = -0.5$, and $\lim\limits_{x\to 1} f(x) = 0$

Limits numerically (LT2)

Activity 1.2.5 Consider the following function $f(x) = 3x^3 + 2x^2 - 5x + 20$.

(a) Of the following options, at which values given would you evaluate $f(x)$ to best determine $\lim_{x \to 2} f(x)$ numerically?

A. 1.9, 1.99, 2.0, 2.01, 2.1

B. 1.98, 1.99, 2.0, 2.01, 2.02

C. 1.8, 1.9, 2.0, 2.1, 2.2

D. 1.0, 1.5, 2.0, 2.5, 3.0

(b) Use the values that you chose in part (a) to calculate an approximation for $\lim_{x \to 2} f(x)$.

(c) Which value best describes the limit that you obtained in part (b)?

A. The approximate value is 41.25

B. The approximate value is 41.5

C. The approximate value is 41.75

D. The approximate value is 42

Activity 1.2.6 In Figure 10 is the graph for $f(x) = \sin\left(\frac{1}{x}\right)$. Several values for $f(x)$ in the neighborhood of $x = 0$ are approximated in Table 11.

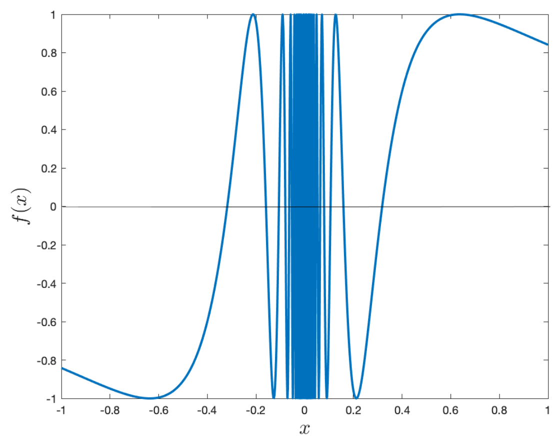


Figure 9 Graph of $f(x) = \sin(1/x)$.

Table 10

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.54402	0.50637	-0.82688	?	0.82688	-0.50637	-0.54402

(a) Based on the graph and table what is the best explanation for the limit as x tends to zero?

A. The limit does not exist because the left and right limits have opposite values.

B. The limit does not exist because we do not have enough information to answer the question.

C. The limit does not exist because the function is oscillating between -1 and 1.

D. The limit does not exist because you are dividing by zero when $x = 0$ for $f(x)$.

(b) Would your conclusion that resulted from Activity 1.2.6 change if the function was $f(x) = \cos(1/x)$ or $f(x) = \tan(1/x)$?

Limits numerically (LT2)

Activity 1.2.7 Use technology to complete the following table of values.

$$f(x) = \frac{x^2 - x - 12}{x^2 + 16\,x + 39}$$

x	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$							

Then explain how to use it to make an educated guess as to the value of the limit

$$\lim_{x \to -3} \frac{x^2 - x - 12}{x^2 + 16\,x + 39}$$

Activity 1.2.8 In this activity you will study the velocity of Usain Bolt in his Beijing 100 meters dash. He completed 100 meters in 9.69 seconds for an overall average speed of $100/9.69 = 10.32$ meters per second (about 23 miles per hour). But this is the average velocity on the whole interval. How fast was he at different instances? What was his maximum velocity? Let's explore this. The table Table 12 shows his split times recorded every 10 meters.

Table 11

t (seconds)	1.85	2.87	3.78	4.65	5.5	6.32	7.14	7.96	8.79	9.69
d (meters)	10	20	30	40	50	60	70	80	90	100

(a) What was the average velocity on the first 50 meters? On the second 50 meters?

(b) What was the average velocity between 30 and 50 meters? Between 50 and 70 meters?

(c) What was the average velocity between 40 and 50 meters? Between 50 and 60 meters?

(d) What is your best estimate for the Usain's velocity at the instant when he passed the 50 meters mark? This is your estimate for the instantaneous velocity.

(e) Using the table of values, explain why 50 meters is NOT the best guess for when the instantaneous velocity was the largest. What other point would be more reasonable?

1.3 Limits analytically (LT3)

Learning Outcomes

- Compute limits of functions given algebraically, using proper limit properties.

Limits analytically (LT3)

Remark 1.3.1 Recall that in Activity 1.2.5 we used numerical methods and table of values to find the limit of a relatively simple degree three polynomial at a point. This was inefficient, “there’s gotta be a better way!”

Activity 1.3.2 Given $f(x) = 3x^2 - \frac{1}{2}x + 4$, evaluate $f(2)$ and approximate $\lim_{x\to 2} f(x)$ numerically (or graphically). What do you think is more likely?

A. $\lim_{x\to 2} f(x) = f(2)$

B. $\lim_{x\to 2} f(x) \approx f(2)$

C. $\lim_{x\to 2} f(x) \neq f(2)$

Activity 1.3.3 The table below gives values of a few different functions.

Table 12

x	6.99	6.999	7.001	7.01
f(x)	13.99	13.999	14.001	14.01
g(x)	22.97	22.997	23.003	23.03
3f(x)	41.97	41.997	42.003	42.03
f(x)+g(x)	36.96	36.996	37.004	37.04
f(x)g(x)	321.350	321.935	322.065	322.650

Using the table above, which of the following is *least* likely to be true?

A. $\lim_{x\to 7} f(x) = 14$ and $\lim_{x\to 7} g(x) = 23$

B. $\lim_{x\to 7} 3f(x) = 3\lim_{x\to 7} f(x)$

C. $\lim_{x\to 7} (f(x) + g(x)) = \lim_{x\to 7} f(x) + \lim_{x\to 7} g(x)$

D. $\lim_{x\to 7} (f(x)g(x)) = f(7)\left(\lim_{x\to 7} g(x)\right)$

Remark 1.3.4 In Activity 1.3.3 we observed that limits seem to be "well-behaved" when combined with standard operations on functions. The next theorems, known as **Limit Laws**, tell us how limits interact with combinations of functions.

Activity 1.3.5 If $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = -3$, which of the following statements are true? Select all that apply!

A. $\lim_{x\to 2}(f(x)\cdot g(x)) = -6$

B. $\lim_{x\to 2}(f(x)+g(x)) = -1$

C. $\lim_{x\to 2}(f(x)-g(x)) = -2$

D. $\lim_{x\to 2}(f(x)/g(x)) = -2/3$

Activity 1.3.6 Below you are given the graphs of two functions. Compute the limits below (if possible).

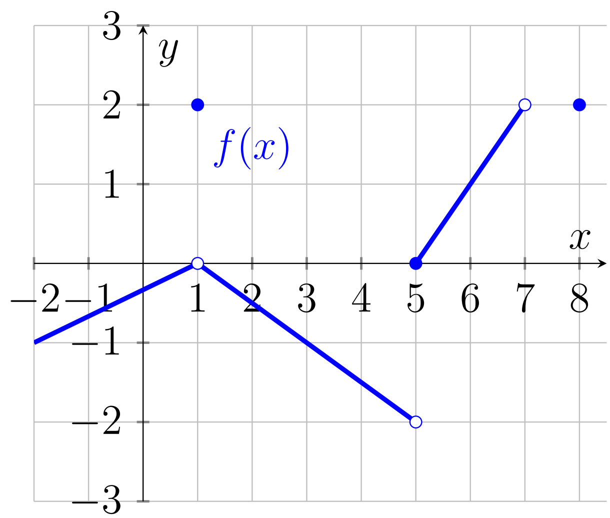


Figure 13 The graph of $f(x)$.

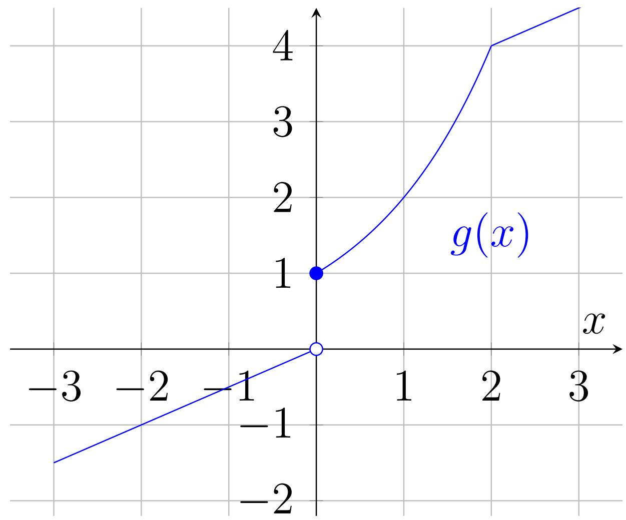


Figure 14 The graph of $g(x)$.

(a) $\lim_{x\to 1} f(x) + g(x)$.

(b) $\lim_{x\to 5^+} 3f(x)$.

(c) $\lim_{x\to 0^+} f(x)g(x)$.

(d) (Challenge) $\lim_{x\to 1} g(x)/f(x)$.

(e) (Challenge) $\lim_{x\to 0^+} f(g(x))$.

Activity 1.3.7 Given $p(x) = -3x^2 - 5x + 7$, which of the following limit laws would use to determine $\lim_{x \to 2} p(x)$? Choose all that apply.

A. Sums/Difference Law

B. Scalar Multiple Law

C. Product Law

D. Identity Law

E. Power Law

F. Constant Law

Limits analytically (LT3)

Activity 1.3.8 Given $p(x) = -3x^2 - 5x + 7$ and $q(x) = x^4 - x^2 + 3$, which of the following describes the most efficient way to determine $\lim_{x \to -1} \frac{p(x)}{q(x)}$?

A. Sums/difference, scalar multiple, and product laws

B. Theorem 1.3.10 and the quotient law

C. Power, sums/difference, scalar multiple, and constant laws

D. Quotient and root law

Activity 1.3.9 Consider taking the limit of a rational function $\frac{p(x)}{q(x)}$ as $x \to c$. If $q(c) = 0$, is it possible for $\lim_{x \to c} \dfrac{p(x)}{q(x)}$ to equal a number?

A. No, because $\frac{p(x)}{q(x)}$ is not defined at $x = c$ since $q(c) = 0$.

B. Yes, because if you graph $f(x) = \frac{x^2-1}{x-1}$, the value $f(1)$ is not defined, but the graph shows that the limit of $f(x)$ does exist as $x \to 1$.

C. No, because if you graph $g(x) = \frac{x^2+1}{x-1}$, the value $g(1)$ is not defined and the graph shows that the limit of $\lim_{x \to c} g(x)$ does not exist.

D. Yes, because we can use Theorem 1.3.12.

Limits analytically (LT3)

Activity 1.3.10 Let $f(x) = 2x$ and $g(x) = x$, which of the following statements is true?

A. $\lim_{x\to 0}(f(x)/g(x)) = 0$

B. $\lim_{x\to 0}(f(x)/g(x)) = 2$

C. $\lim_{x\to 0}(f(x)/g(x))$ cannot be determined

D. $\lim_{x\to 0}(f(x)/g(x))$ does not exist

Remark 1.3.11 When we compute the limit of a ratio where both the numerator and denominator have limit equal to zero, we have to compute the value of a $\frac{0}{0}$ **indeterminate form**. The value of an indeteminate form can be any real number or even infinity or not existent, we just do not know yet! We can usually determine the value of an indeterminate form using some algebraic manipulations of the expression given.

Definition 1.3.12 A function $f(x)$ has a **hole** at $x = c$ if $f(c)$ does not exist but $\lim_{x \to c} f(x)$ does exist and is equal to a real number. $\Diamond$

Example 1.3.13 The function $f(x) = \frac{x^2-1}{x-1}$ has a hole at $x = 1$ because $f(1)$ is not defined but

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2,$$

so the limit exists and is equal to a real number. Notice that $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$ is also an example of a limit giving an indeterminate form $\frac{0}{0}$ which we could then compute using an algebraic manipulation of the function given. $\square$

Activity 1.3.14 Determine the following limits and explain your reasoning.

$$\lim_{x \to -6} \frac{x^2 - 6\,x + 5}{x^2 - 3\,x - 18}$$

$$\lim_{x \to -1} \frac{x^2 - 1}{x^2 + 3\,x + 2}$$

$$\lim_{x \to 5} \frac{x - 5}{\sqrt{x + 31} - 6}$$

Activity 1.3.15 In activity Activity 1.2.8 you studied the velocity of Usain Bolt in his Beijing 100 meters dash. We will now study this situation analytically. To make our computations simpler, we will approximate that he could run 100 meters in 10 seconds and we will consider the model $d = f(t) = t^2$, where d is the distance in meters and t is the time in seconds.

Note 1.3.16 The average velocity is the ratio distance covered over time elapsed. If we consider the interval that starts at $t = a$ and has width h, written $[a, a+h]$, the average velocity on this interval is $\dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$. The instantaneous velocity at time $t = a$ is given by:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(a) Compute the average velocity on the interval $[5, 6]$. We think of this interval as $[5, 5+h]$ for the value of $h = 1$.

(b) Compute the average velocity starting at 5 seconds, but now with $h = 0.5$ seconds.

(c) We want to study the instantaneous velocity at $a = 5$ seconds. Find an expression for the average velocity on the interval $[5, 5+h]$, where h is an unspecified value.

(d) Expand your expression. When $h \neq 0$, you can simplify it!

(e) Recall that the instantaneous velocity is the limit of your expression as $h \to 0$. Find the instantaneous velocity given by this model at $t = 5$ seconds.

(f) The model $d = f(t) = t^2$ does not really capture the real-world situation. Think of at least one reason why this model does not fit the scenario of Usain Bolt's 100 meters dash.

1.4 Continuity (LT4)

Learning Outcomes

- Determine where a function is and is not continuous.

Continuity (LT4)

Remark 1.4.1 A continuous function is one whose values change smoothly, with no jumps or gaps in the graph. We'll explore the idea first, and arrive at a mathematical definition soon.

Activity 1.4.2 Which of the following scenarios best describes a continuous function?

A. The age of a person reported in years

B. The price of postage for a parcel depending on its weight

C. The volume of water in a tank that is gradually filled over time

D. The number of likes on my latest TikTok depending on the time since I posted it

Remark 1.4.3 How would you use the language of limits to clarify the definition of continuity?

Activity 1.4.4 A function f defined on $-4 < x < 4$ has the graph pictured below. Use the graph to answer each of the following questions.

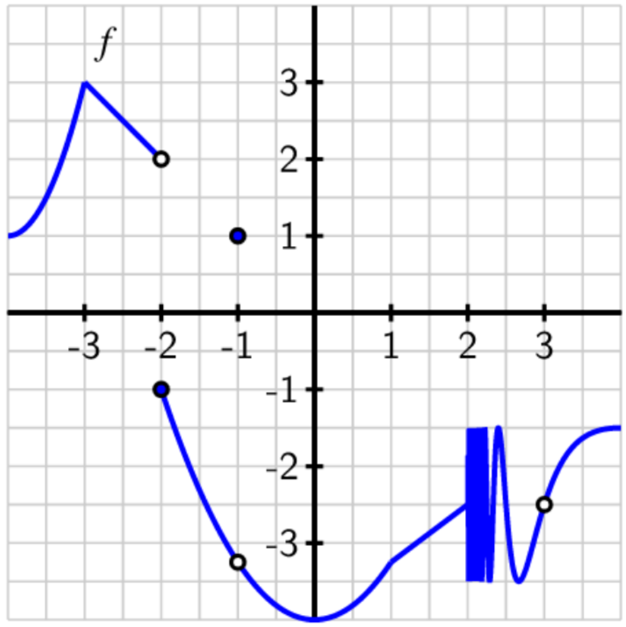


Figure 15

(a) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether the limit $\lim_{x\to a} f(x)$ exists. If the limit does not exist, be ready to explain why not.

(b) For each of the values of a where the limit of f exists, determine the value of $f(a)$ at each such point.

(c) For each such a value, is $f(a)$ equal to $\lim_{x\to a} f(x)$?

(d) Use your understanding of continuity to determine whether f is continuous at each value of a.

(e) Any revisions would you want to make to your definition of continuity that you arrived at toward the end of Remark 1.4.3?

Continuity (LT4)

Definition 1.4.5 A function f is **continuous** at $x = a$ provided that

1 f has a limit as $x \to a$

2 f is defined at $x = a$ (equivalently, a is in the domain of f), and

3 $\lim\limits_{x \to a} f(x) = f(a)$.

$\Diamond$

Activity 1.4.6 Suppose that some function $h(x)$ is continuous at $x = -3$. Use Definition 1.4.5 to decide which of the following quantities are equal to each other.

A. $\lim_{x \to -3^+} h(x)$

B. $\lim_{x \to -3^-} h(x)$

C. $\lim_{x \to -3} h(x)$

D. $h(-3)$

Activity 1.4.7 Consider the function f whose graph is pictured below (it's the same graph from Activity 1.4.4). In the questions below, consider the values $a = -3, -2, -1, 0, 1, 2, 3$.

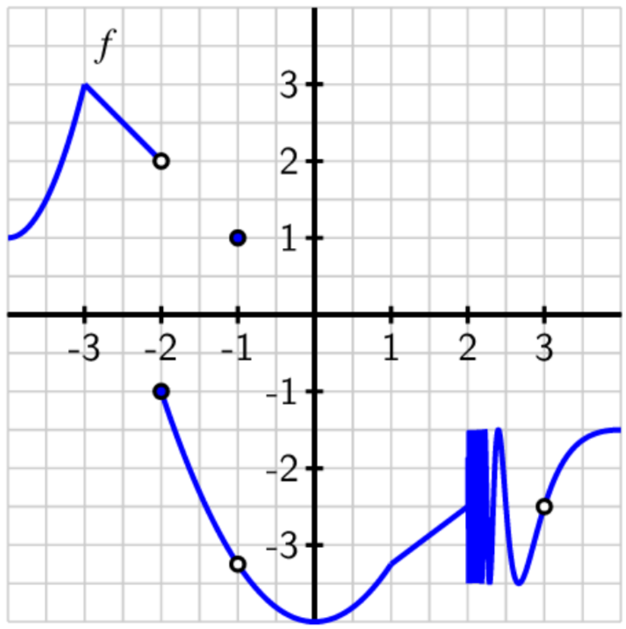


Figure 16

(a) For which values of a do we have $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$?

(b) For which values of a is $f(a)$ not defined?

(c) For which values of a does f have a limit at a, yet $f(a) \neq \lim_{x \to a} f(x)$?

(d) For which values of a does f fail to be continuous? Give a complete list of intervals on which f is continuous.

Activity 1.4.8 Which condition is *stronger*, meaning it implies the other?

A. f has a limit at $x = a$

B. f is continuous at $x = a$

Continuity (LT4)

Activity 1.4.9 Previously, you have used graphs, tables, and formulas to answer questions about limits. Which of those are suitable for answering questions about continuity?

A. Graphs only

B. Formulas only

C. Graphs and formulas only

D. Tables and formulas only

Activity 1.4.10 Consider the function f whose graph is pictured below.

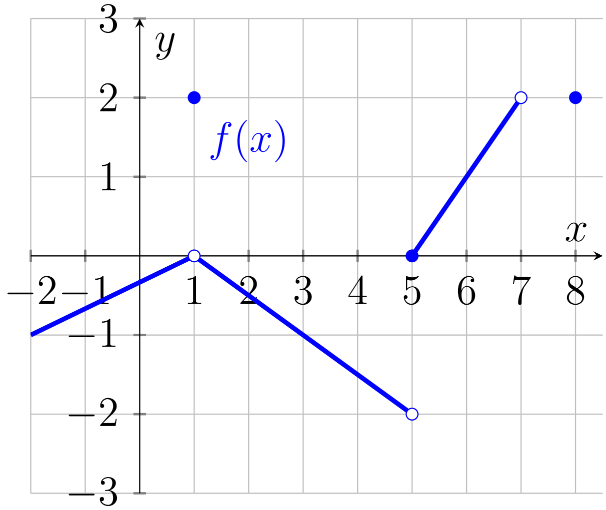


Figure 17 The graph of $f(x)$.

Give a list of x-values where $f(x)$ is not continuous. Be prepared to defend your answer based on Definition 1.4.5.

Remark 1.4.11 When $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, we say that f has a **removable discontinuity** at $x = a$. This is because if $f(a)$ were redefined to be equal to $\lim_{x \to a} f(x)$, the redefined function would be continuous at $x = a$, thus "removing" the discontinuity.

When the left and right limit exist separately, but are not equal, the discontinuity is not removable and is called a **jump discontinuity**.

Activity 1.4.12

(a) Determine the value of b to make $h(x)$ continuous at $x = 5$.

$$h(x) = \begin{cases} b - x, & x < 5 \\ -x^2 + 6\,x - 6, & x \geq 5 \end{cases}$$

(b) Classify the type of discontinuity present at $x = -6$ for the function $f(x)$.

$$f(x) = \begin{cases} -8\,x - 46, & x < -6 \\ 6, & x = -6 \\ 4\,x + 30, & x > -6 \end{cases}$$

Activity 1.4.13 Answer the questions below about piecewise functions. It may be helpful to look at some graphs.

(a) Which values of c, if any, could make the following function continuous on the real line?

$$g(x) = \begin{cases} x + c & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

(b) Which values of c, if any, could make the following function continuous on the real line?

$$h(x) = \begin{cases} 4 & x \leq c \\ x^2 & x > c \end{cases}$$

(c) Which values of c, if any, could make the following function continuous on the real line?

$$k(x) = \begin{cases} x & x \leq c \\ x^2 & x > c \end{cases}$$

Activity 1.4.14 In this activity we will explore a mathematical theorem, the Intermediate Value Theorem.

(a) To get an idea for the theorem, draw a continuous function $f(x)$ on the interval $[0, 10]$ such that $f(0) = 8$ and $f(10) = 2$. Find an input c where $f(c) = 5$.

(b) Now try to draw a graph similar to the previous one, but that does not have any input corresponding to the output 5. Then, find where your graph violates these conditions: $f(x)$ is continuous on $[0, 10]$, $f(0) = 8$, and $f(10) = 2$.

(c) The part of the theorem that starts with "Suppose…" forms the assumptions of the theorem, while the part of the theorem that starts with "Then…" is the conclusion of the theorem. What are the assumptions of the Intermediate Value Theorem? What is the conclusion?

(d) Apply the Intermediate Value Theorem to show that the function $f(x) = x^3 + x - 3$ has a zero (so crosses the x-axis) at some point between $x = -1$ and $x = 2$. (Hint: What interval of x values is being considered here? What is N? Why is N between $f(a)$ and $f(b)$?)

1.5 Limits with infinite inputs (LT5)

Learning Outcomes

- Determine limits of functions at infinity.

Activity 1.5.1 Consider the graph of the polynomial function $f(x) = x^3$. We want to think about what the long term behavior of this function might be. Which of the following best describes its behavior?

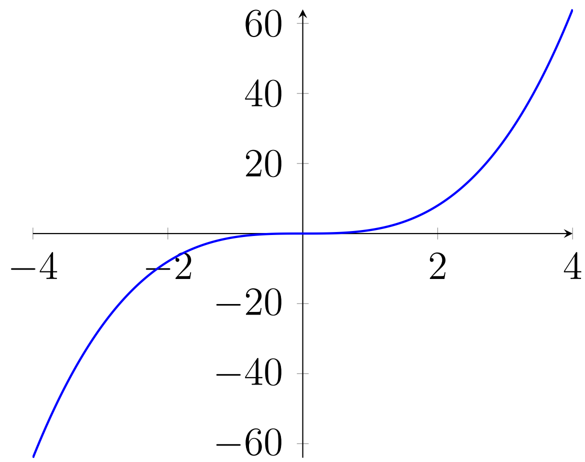


Figure 18 The graph of x^3.

A. As x gets larger, the function x^3 gets smaller and smaller.

B. As x gets more and more negative, the function x^3 gets more and more negative.

C. As x gets more and more positive, the function x^3 gets more and more negative.

D. As x gets smaller, the function x^3 gets smaller and smaller.

Remark 1.5.2 We say that "the limit as x tends to negative infinity of x^3 is negative infinity" and that "the limit as x tends to positive infinity of x^3 is positive infinity." In symbols, we write

$$\lim_{x\to+\infty} x^3 = +\infty\,,\ \lim_{x\to-\infty} x^3 = -\infty.$$

Activity 1.5.3 Consider the graph of the rational function $f(x) = 1/x^3$. We want to think about what the long term behavior of this function might be. Which of the following best describes its behavior?

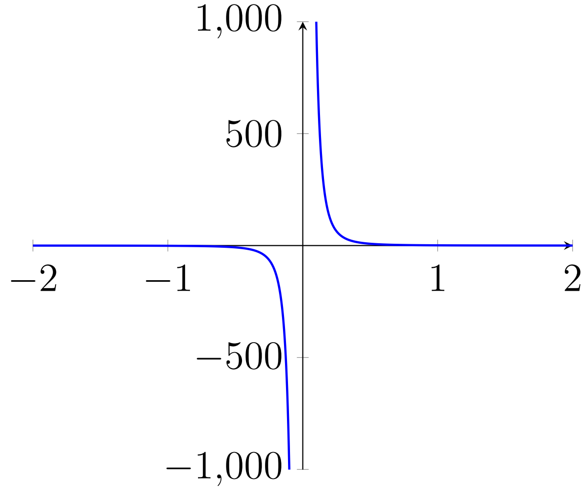


Figure 19 The graph of $1/x^3$.

A. As x tends to positive infinity, the function $1/x^3$ tends to positive infinity

B. As x tends to negative infinity, the function $1/x^3$ tends to 0

C. As x tends to positive infinity, the function $1/x^3$ tends to negative infinity

D. As x tends to 0, the function $1/x^3$ tends to 0

Definition 1.5.4 A function has a **horizontal asymptote** at $y = b$ when

$$\lim_{x \to +\infty} f(x) = b$$

or

$$\lim_{x \to -\infty} f(x) = b$$

This means that we can make the output of $f(x)$ as close as we want to b, as long as we take x a large enough positive number ($x \to \infty$) or a large enough negative number ($x \to -\infty$). $\Diamond$

Remark 1.5.5 We say that the function $1/x^3$ has horizontal asymptote y=0 because the limit as x tends to positive infinity of $1/x^3$ is 0. Alternatively, we could also justify it by saying that the limit as x goes to negative infinity is 0.

Activity 1.5.6 Which of the following functions have horizontal asymptotes? Select all!

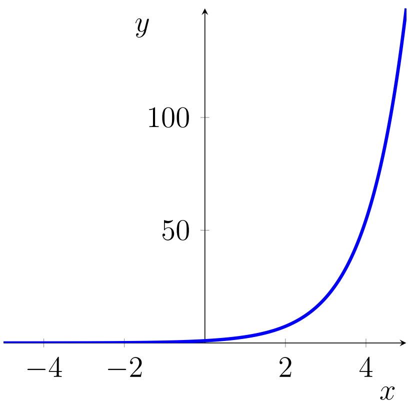


Figure 20 A

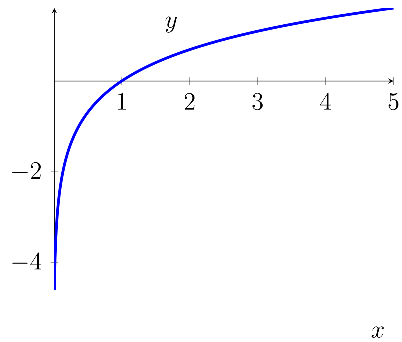


Figure 21 B

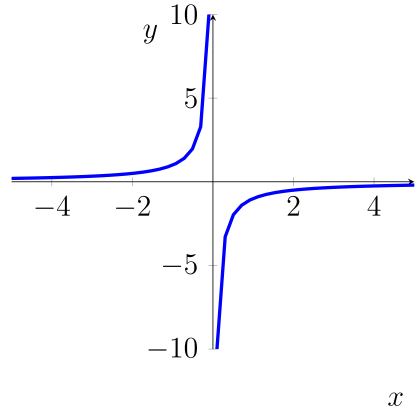


Figure 22 C

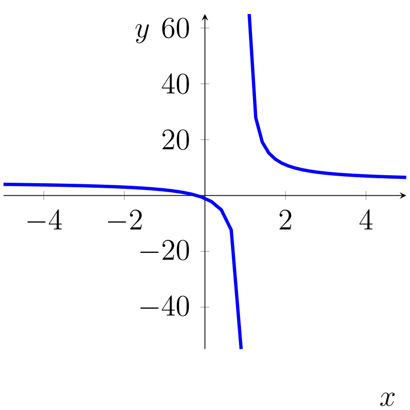


Figure 23 D

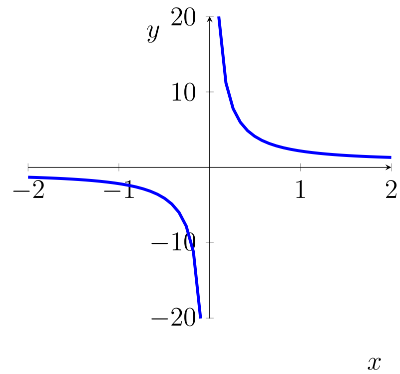


Figure 24 E

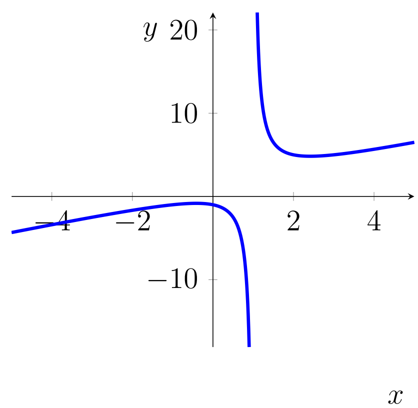


Figure 25 F

Limits with infinite inputs (LT5)

Activity 1.5.7 Recall that a rational function is a ratio of two polynomials. For any given rational function, what are all the possible behaviors as x tends to + or - infinity?

A. The only possible limit is 0

B. The only possible limits are 0 or $\pm\infty$

C. The only possible limits are 0, 1 or $\pm\infty$

D. The only possible limits are any constant number or $\pm\infty$

Limits with infinite inputs (LT5)

Activity 1.5.8 In this activity we will examine functions whose limit as x approaches positive and negative infinity is a nonzero constant.

(a) Graph the following functions and consider their limits as x approaches positive and negative infinity. Which function(s) have a limit that is nonzero and constant? Find each of these limits.

A. $f(x) = \dfrac{x^3 - x + 3}{2x^3 - 6x + 1}$

B. $f(x) = \dfrac{x^2 - 3}{5x^3 - 2x^2 + 5}$

C. $f(x) = \dfrac{x^4 - 3x - 2}{3x^3 - 5x + 1}$

D. $f(x) = \dfrac{10x^5 - 3x + 2}{5x^5 - 3x^2 + 1}$

E. $f(x) = \dfrac{-8x^2 - 5x + 1}{2x^2 - 2x + 3}$

(b) Conjecture a rule for how to determine that a rational function has a nonzero constant limit as x approaches positive and negative infinity. Test your rule by creating a rational function whose limit as $x \to \infty$ equals 3 and then check it graphically.

Activity 1.5.9 What about when the limit is not a nonzero constant? How do we recognize those? In this activity you will first conjecture the general behavior of rational functions and then test your conjectures.

(a) Consider a rational function $r(x) = \frac{p(x)}{q(x)}$. Looking at the numerator $p(x)$ and the denominator $q(x)$, when does the function $r(x)$ have limit equal to 0 as $x \to \infty$?

A. When the ratio of the leading terms is a constant.

B. When the degree of the numerator is greater than the degree of the denominator.

C. When the degree of the numerator is less than the degree of the denominator.

D. When the degree of the numerator is equal to the degree of the denominator.

(b) Consider a rational function $r(x) = \frac{p(x)}{q(x)}$. Looking at the numerator $p(x)$ and the denominator $q(x)$, when does the function $r(x)$ have limit approaching infinity as $x \to \infty$?

A. When the ratio of the leading terms is a constant.

B. When the degree of the numerator is greater than the degree of the denominator.

C. When the degree of the numerator is less than the degree of the denominator.

D. When the degree of the numerator is equal to the degree of the denominator.

(c) Conjecture a rule for the each of the previous two parts of the activity. Test your rules by creating a rational function whose limit as $x \to \infty$ equals 0 and another whose limit as $x \to \infty$ is infinite. Then check them graphically.

Activity 1.5.10 Explain how to find the value of each limit.

(a)

$$\lim_{x\to-\infty} -\frac{6\,x^4+7\,x^3-7}{6\,x-x^4+9} \quad\text{and}\quad \lim_{x\to+\infty} -\frac{6\,x^4+7\,x^3-7}{6\,x-x^4+9}$$

(b)

$$\lim_{x\to-\infty} -\frac{7\,x^4-5\,x^3+8}{3\,(2\,x^5+3\,x^2-3)} \quad\text{and}\quad \lim_{x\to+\infty} -\frac{7\,x^4-5\,x^3+8}{3\,(2\,x^5+3\,x^2-3)}$$

(c)

$$\lim_{x\to-\infty} \frac{3\,x^6+x^3-8}{7\,x-6\,x^5+7} \quad\text{and}\quad \lim_{x\to+\infty} \frac{3\,x^6+x^3-8}{7\,x-6\,x^5+7}$$

Activity 1.5.11 What is your best guess for the limit as x goes to $+\infty$ of the function graphed below?

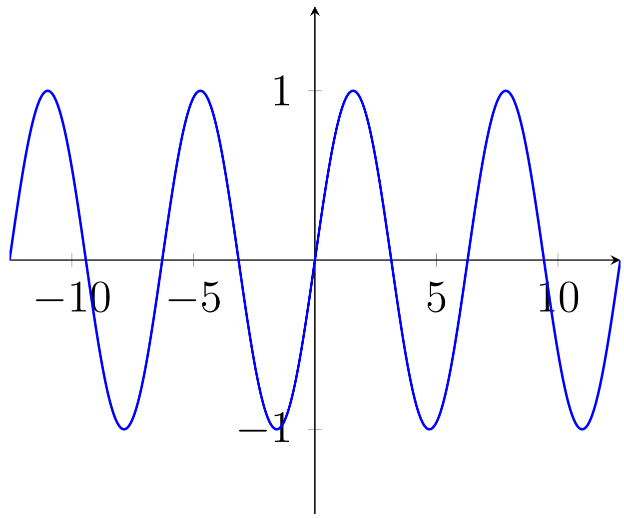


Figure 26 A mysterious periodic function.

A. The limit is 0

B. The limit is 1

C. The limit is -1

D. The limit is $+\infty$

E. The limit DNE

Activity 1.5.12 Compute the following limits.

(a) $\displaystyle\lim_{x \to -\infty} \frac{x^3 - x + 83}{1}$

(b) $\displaystyle\lim_{x \to -\infty} \frac{1}{x^3 - x + 83}$

(c) $\displaystyle\lim_{x \to +\infty} \frac{x + 3}{2 - x}$

(d) $\displaystyle\lim_{x \to -\infty} \frac{\pi - 3x}{\pi x - 3}$

(e) (Challenge) $\displaystyle\lim_{x \to +\infty} \frac{3e^x + 2}{2e^x + 3}$

(f) (Challenge) $\displaystyle\lim_{x \to -\infty} \frac{3e^x + 2}{2e^x + 3}$

Activity 1.5.13 The graph below represents the function $f(x) = \dfrac{2(x+3)(x+1)}{x^2-2x-3}$.

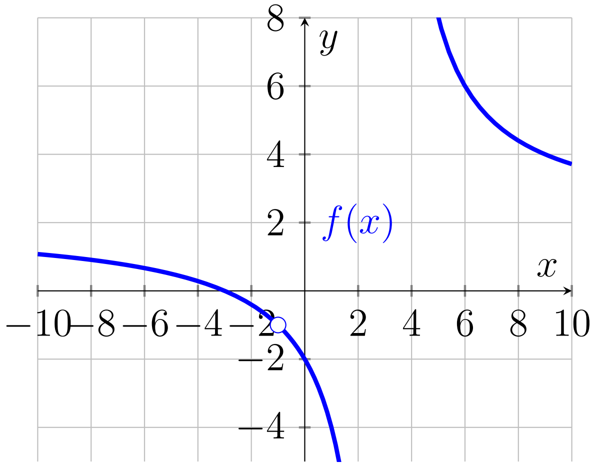


Figure 27 The graph of $f(x)$

(a) Find the horizontal asymptote of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.

(b) Use limit notation to describe the behavior of $f(x)$ at its horizontal asymptotes.

(c) Come up with the formula of a rational function that has horizontal asymptote $y = 3$.

(d) What do you think is happening at $x = -1$? We will come back to this in the next section!

Note 1.5.14 An exponential function $P(t) = a\,b^t$ exhibiting exponential decay will have the long term behavior $P(t) \to 0$ as $t \to \infty$. If we shift the graph up by c units, we obtain the new function $Q(t) = a\,b^t + c$, with the long term behavior $\lim_{t\to\infty} Q(t) = c$. A cooling object can be represented by the exponential decay model $Q(t) = a\,b^t + c$.

Activity 1.5.15 In this activity you will explore an exponential model for a cooling object.

Consider a cup of coffee initially at 100 degrees Fahrenheit. The said cup of coffee was forgotten this morning on the kitchen counter where the thermostat is set at 72 degrees Fahrenheit. From previous observations, we can assume that a cup of coffee looses 10 percent of its temperature each minute.

(a) In the long run, what temperature do you expect the coffee to tend to? Write your observation with limit notation.

(b) In the model $Q(t) = a\,b^t + c$, your previous answer gives you the value of one of the parameters in this model. Which one?

(c) From the information given, we notice that the cup of coffee has decay rate of 10% or $r = -0.1$. When an exponential model has decay rate r, its exponential base b has value $b = 1 + r$. Use this to find the value of b for the exponential model described in this scenario.

(d) Assume that the initial temperature corresponds to input $t = 0$. Use the data about the initial temperature to find the value of the parameter a in the model $Q(t) = a\,b^t + c$.

(e) You should have found that this scenario has exponential model $Q(t) = 28\,(0.9)^t + 72$. If you go back to drink the cup of coffee 30 minutes after it was left on the counter, what temperature will the coffee have reached?

1.6 Limits with infinite outputs (LT6)

Learning Outcomes

- Determine limits of functions approaching vertical asymptotes.

Activity 1.6.1 Consider the graph in Figure 33.

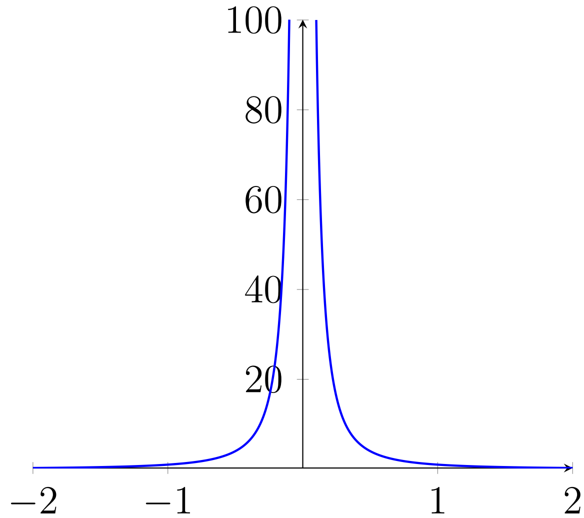


Figure 28 The graph of $1/x^2$.

(a) Which of the following best describes the limit as x approaches zero in the graph?

A. The limit is 0

B. The limit is positive infinity

C. The limit does not exist

D. This limit is negative infinity

(b) Which of the following best describes the relationship between the line $x = 0$ and the graph of the function?

A. The line $x = 0$ is a horizontal asymptote for the function

B. The function is not continuous at the point $x = 0$

C. The function is moving away from the line $x = 0$

D. The function is getting closer and closer to the line $x = 0$

E. The function has a jump in outputs around $x = 0$

Definition 1.6.2 A function has a **vertical asymptote** at $x = a$ when

$$\lim_{x \to a} f(x) = +\infty$$

or

$$\lim_{x \to a} f(x) = -\infty$$

The limit being equal to positive infinity means that we can make the output of $f(x)$ as large a positive number as we want as long as we are sufficiently close to $x = a$. Similarly, the limit being equal to negative infinity means that we can make the output of $f(x)$ as large a negative number as we want as long as we are sufficiently close to $x = a$. $\diamondsuit$

Limits with infinite outputs (LT6)

Activity 1.6.3 Select all of the following graphs which illustrate functions with vertical asymptotes.

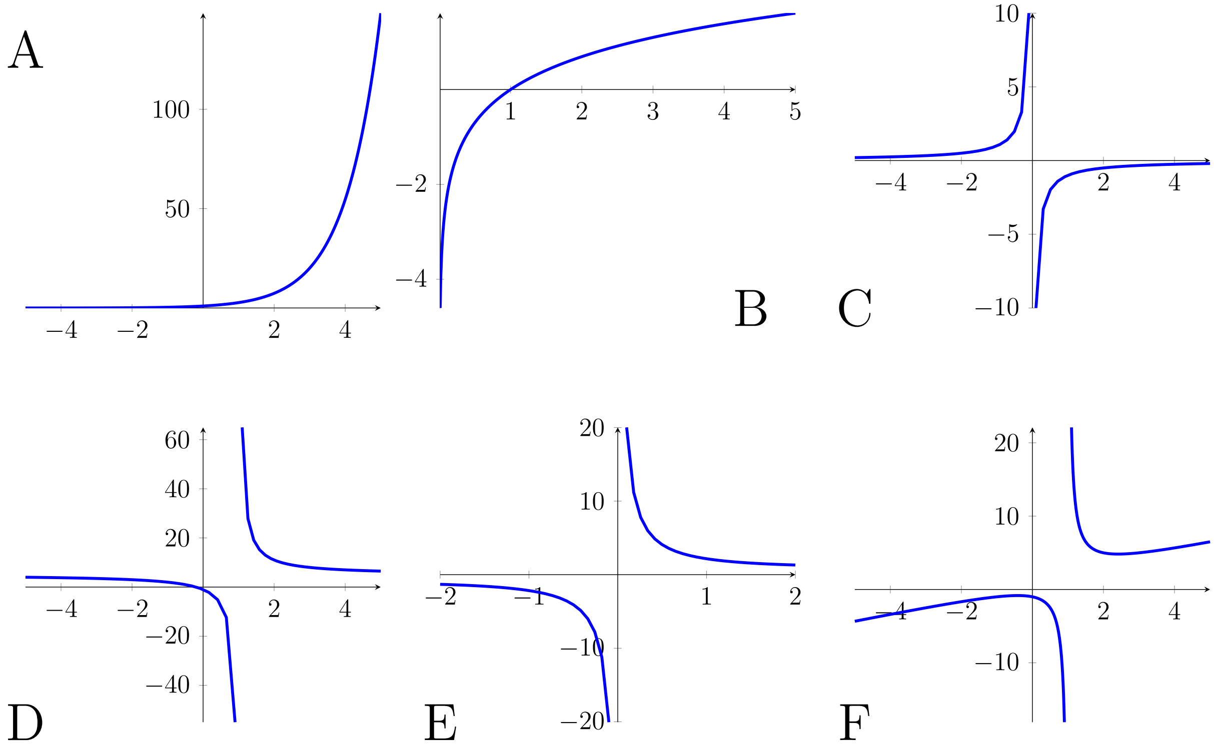


Figure 29 Choices for vertical asymptotes

Remark 1.6.4 If $x = a$ is a vertical asymptote for the function $f(x)$, the function $f(x)$ is not defined at $x = a$. As $f(a)$ does not exist, the function is NOT continuous at $x = a$. Moreover, the function's output tends to plus or minus infinity and so the limit is not equal to a number.

Limits with infinite outputs (LT6)

Activity 1.6.5 Notice that as x goes to 0, the value of x^2 goes to 0 but the value of $1/x^2$ goes to infinity. What is the best explanation for this behavior?

A. When dividing by an increasingly small number we get an increasing big number

B. When dividing by an increasingly large number we get an increasing small number

C. A rational function always has a vertical asymptote

D. A rational function always has a horizontal asymptote

Remark 1.6.6 Informally, we say that the limit of "$\frac{1}{0}$" is infinite. Notice that this could be either positive or negative infinity, depending on how whether the outputs are becoming more and more positive or more and more negative as we approach zero.

Limits with infinite outputs (LT6)

Activity 1.6.7 Consider the rational function $f(x) = \frac{2}{x-3}$. Which of the following options best describes the limits as x approaches 3 from the right and from the left?

A. As $x \to 3^+$, the limit DNE, but as $x \to 3^-$ the limit is $-\infty$.

B. As $x \to 3^+$, the limit is $+\infty$, but as $x \to 3^-$ the limit is $-\infty$.

C. As $x \to 3^+$, the limit is $+\infty$, but as $x \to 3^-$ the limit is $+\infty$.

D. As $x \to 3^+$, the limit is $-\infty$, but as $x \to 3^-$ the limit is $-\infty$.

E. As $x \to 3^+$, the limit DNE and as $x \to 3^-$ the limit DNE.

Remark 1.6.8 When considering a ratio of functions $f(x)/g(x)$, the inputs a where $g(a) = 0$ are not in the domain of the ratio. If $g(a) = 0$ but $f(a)$ is not equal to 0, then $x = a$ is a vertical asymptote.

Activity 1.6.9 Consider the function $f(x) = \frac{x^2-1}{x-1}$. The line $x = 1$ is NOT a vertical asymptote for $f(x)$. Why?

A. When x is not equal to 1, we can simplify the fraction to $x - 1$, so the limit is 1.

B. When x is not equal to 1, we can simplify the fraction to $x + 1$, so the limit is 2.

C. The function is always equal to $x + 1$.

D. The function is always equal to $x - 1$.

Remark 1.6.10 Recall the definition of a hole from Definition 1.3.16. In Activity 1.6.9 we have a hole at $x = 1$.

Activity 1.6.11 Find all the vertical asymptotes of the following rational functions.

(a) $y = \frac{3x-4}{7x+1}$

(b) $y = \frac{x^2+10x+24}{x^2-2x+1}$

(c) $y = \frac{(x^2-4)(x^2+1)}{x^6}$

(d) $y = \frac{2x+1}{2x^2+8x-10}$

Activity 1.6.12 Explain and demonstrate how to find the value of each limit.

(a)

$$\lim_{x \to -3^-} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

(b)

$$\lim_{x \to -3^+} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

(c)

$$\lim_{x \to -3} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

Activity 1.6.13 The graph below represents the function $f(x) = \dfrac{(x+2)(x+4)}{x^2+3x-4}$.

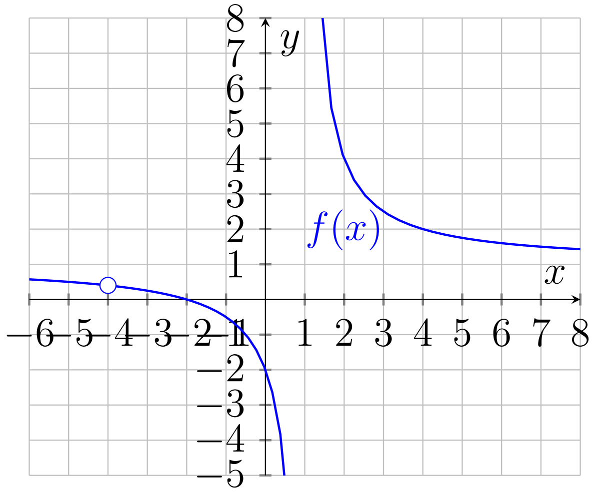


Figure 30 The graph of $f(x)$

(a) Explain the behavior of $f(x)$ at $x = -4$.

(b) Find the vertical asymptote(s) of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.

(c) Find the horizontal asymptote(s) of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.

(d) Use limit notation to describe the behavior of $f(x)$ at its asymptotes.

Activity 1.6.14 Consider the following rational function.

$$r(x) = \frac{5\,(x-3)(x-6)^3}{6\,(x+2)^3(x-3)}$$

(a) Explain how to find the horizontal asymptote(s) of $r(x)$, if there are any. Then express your findings using limit notation.

(b) Explain how to find the hole(s) of $r(x)$, if there are any. Then express your findings using limit notation.

(c) Explain how to find the vertical asymptote(s) of $r(x)$, if there are any. Then express your findings using limit notation.

(d) Draw a rough sketch of $r(x)$ that showcases all the limits that you have found above.

Activity 1.6.15 You want to draw a function with all these properties.

- $\lim_{x \to 3} f(x) = 5$
- $f(3) = 0$
- $\lim_{x \to 0^-} f(x) = -\infty$
- $\lim_{x \to 0^+} f(x) = 0$
- $\lim_{x \to +\infty} f(x) = 2$

Before you start drawing, consider the following guiding questions.

(a) At which x values will the limit not exist?

(b) What are the asymptotes of this function?

(c) At which x values will the function be discontinuous?

(d) Draw the graph of one function with all the properties above. Make sure that your graph is a function! You only need to draw a graph, writing a formula would be very challenging!

Chapter 2

Derivatives (DF)

Learning Outcomes

How can we measure the instantaneous rate of change of a function?
By the end of this chapter, you should be able to...

1. Estimate the value of a derivative using difference quotients, and draw corresponding secant and tangent lines on the graph of a function.
2. Find derivatives using the definition of derivative as a limit.
3. Compute basic derivatives using algebraic rules.
4. Compute derivatives using the Product and Quotient Rules.
5. Compute derivatives using the Chain Rule.
6. Compute derivatives using a combination of algebraic derivative rules.
7. Compute derivatives of implicitly-defined functions.
8. Compute derivatives of inverse functions.

2.1 Derivatives graphically and numerically (DF1)

Learning Outcomes

- Estimate the value of a derivative using difference quotients, and draw corresponding secant and tangent lines on the graph of a function.

Activity 2.1.1 In this activity you will study the velocity of a ball falling under gravity. The height of the ball (in feet) is given by the formula $f(t) = 64 - 16(t-1)^2$, where t is measured in seconds. We want to study the velocity at the instant $t = 2$, so we will look at smaller and smaller intervals around $t = 2$. For your convenience, below you will find a table of values for $f(t)$. Recall that the average velocity is given by the change in height over the change in time.

Table 31

t (seconds)	1	1.5	1.75	2	2.25	2.5	3
$f(t)$ (feet)	64	60	55	48	39	28	0

(a) To start we will look at an interval of length one before $t = 2$ and after $t = 2$, so we consider the intervals $[1, 2]$ and $[2, 3]$. What was the average velocity on the interval $[1, 2]$? What about on the interval $[2, 3]$?

(b) Now let's consider smaller intervals of length 0.5. What was the average velocity on the interval $[1.5, 2]$? What about on the interval $[2, 2.5]$?

(c) What was the average velocity on the interval $[1.75, 2]$? What about on the interval $[2, 2.25]$?

(d) If we wanted to approximate the velocity at the instant $t = 2$, what would be your best estimate for this instantaneous velocity?

Observation 2.1.2 If we want to study the velocity at the instant $t = 2$, it is helpful to study the average velocity on small intervals around $t = 2$. If we consider the interval $[2, 2+h]$, where h is the width of the interval, the average velocity is given by the difference quotient

$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}.$$

Observation 2.1.3 We want to be able to consider intervals before and after $t = 2$. A positive value of h will give an interval after $t = 2$. For example, the interval $[2, 3]$ for $h = 1$. A negative value of h will give an interval before $t = 2$. For example, the interval $[1, 2]$ corresponds $h = -1$. In the formula above, it looks like the interval would be $[2, 1]$, but the standard notation in an interval is to write the smallest number first. This does not change the difference quotient because

$$\frac{f(2+h)-f(2)}{(2+h)-2} = \frac{f(2)-f(2+h)}{2-(2+h)}.$$

Activity 2.1.4 Consider the height of the ball falling under gravity as in Table 37 .

(a) What was the average velocity on the interval $[2, 2+h]$ for $h = 1$ and $h = -1$?

(b) What was the average velocity on the interval $[2, 2+h]$ for $h = 0.5$ and $h = -0.5$?

(c) What was the average velocity on the interval $[2, 2+h]$ for $h = 0.25$ and $h = -0.25$?

(d) What is your best estimate for the limiting value of these velocities as $h \to 0$? Notice that this is your estimate for the instantaneous velocity at $t = 2$!

Definition 2.1.5 The instanteous velocity at $t = a$ is the limit as $h \to 0$ of the difference quotient $\frac{f(a+h)-f(a)}{h}$. In the activity above the instantaneous velocity at $t = 2$ is given by the limit

$$v(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$\Diamond$

Definition 2.1.6 The slope of the secant line to $f(x)$ through the points $x = a$ and $x = b$ is given by the difference quotient

$$\frac{f(b) - f(a)}{b - a}.$$

$\Diamond$

Activity 2.1.7 In this activity you will study the slope of a graph at a point. The graph of the function $g(x)$ is given below. For your convenience, below you will find a table of values for $g(x)$.

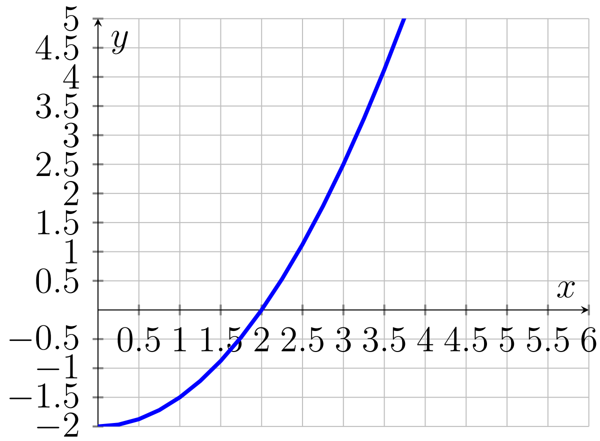


Figure 32 The graph of $g(x)$

Table 33

x	1	1.5	2	2.5	3
$g(x)$	-1.5	-0.875	0	1.125	2.5

(a) What is the slope of the line through $(1, g(1))$ and $(2, g(2))$? Draw this line on the graph of $g(x)$.

(b) What is the slope of the line through $(1.5, g(1.5))$ and $(2, g(2))$? Draw this line on the graph of $g(x)$.

(c) Draw the line tangent to $g(x)$ at $x = 2$. What would be your best estimate for the slope of this tangent line?

(d) Notice that the slope of the tangent line at $x = 2$ is positive. What feature of the graph of $f(x)$ around $x = 2$ do you think causes the tangent line to have positive slope?

A. The function $f(x)$ is concave up

B. The function $f(x)$ is increasing

C. The function $f(x)$ is concave down

D. The function $f(x)$ is decreasing

Observation 2.1.8 The slope of the secant line to $f(x)$ through the points $x = a$ and $x = b$ is given by the difference quotient $\frac{f(b)-f(a)}{b-a}$. As the point $x = b$ gets closer to $x = a$, the slope of the secant line tends to the slope of the tangent line. In symbols, the slope at $x = a$ is given by the *limit*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Letting $b = a + h$, we can also say that the slope of the tangent line at $x = a$ is given by the limit

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Definition 2.1.9 The derivative of $f(x)$ at $x = a$, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$\diamondsuit$

Observation 2.1.10 In Activity 2.1.1 and Activity 2.1.4 you studied a ball falling under gravity and estimated the instantaneous velocity as a limiting value of average velocities on smaller and smaller intervals. Drawing the corresponding secant lines, we see how the secant lines approximate better the tangent line, showing graphically what we previouly saw numericaly. Here is a Desmos animation showing the secant lines approaching the tangent line `https://www.desmos.com/calculator/bzs1bxz7fa`.

Activity 2.1.11 Suppose that the function $f(x)$ gives the position of an object at time x. Which of the following quantities are the same? Select all that apply!

A. The value of the derivative of $f(x)$ at $x = a$

B. The slope of the tangent line to $f(x)$ at $x = a$

C. The instantaneous velocity of the object at $x = a$

D. The difference quotient $\frac{f(a+h)-f(a)}{h}$

E. The limit $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

Observation 2.1.12 We can use the difference quotient $\frac{f(a+h)-f(a)}{h}$ for small values of h to *estimate* $f'(a)$, the value of the derivative at $x = a$.

Activity 2.1.13 Suppose that you know that the function $g(x)$ has values $g(-0.5) = 7$, $g(0) = 4$, and $g(0.5) = 2$. What is your best estimate for $g'(0)$?

A. $g'(0) \approx -3$

B. $g'(0) \approx -2$

C. $g'(0) \approx -6$

D. $g'(0) \approx -4$

E. $g'(0) \approx -5$

Activity 2.1.14 Suppose that you know that the function $f(x)$ has value $f(1) = 3$ and has derivative at $x = 1$ given by $f'(1) = 2$. Which of the following scenarios is most likely?

A. $f(2) = 3$ because the function is constant

B. $f(2) = 2$ because the derivative is constant

C. $f(2) \approx 1$ because the function's output decreases by about 2 units for each increase by 1 unit in the input

D. $f(2) \approx 5$ because the function's output increases by about 2 units for each increase by 1 unit in the input

Observation 2.1.15 We can use the derivative at $x = a$ to estimate the increase/decrease of the function $f(x)$ close to $x = a$. A positive derivative at $x = a$ suggests that the output values are increasing around $x = a$ approximately at a rate given by the value of the derivative. A negative derivative at $x = a$ suggests that the output values are decreasing around $x = a$ approximately at a rate given by the value of the derivative.

Activity 2.1.16 In this activity you will study the abolute value function $f(x) = |x|$. The absolute value function is a piecewise defined function which outputs x when x is positive (or zero) and outputs $-x$ when x is negative. So the absolute value always outputs a number which is positive (or zero). Here is the graph of this function.

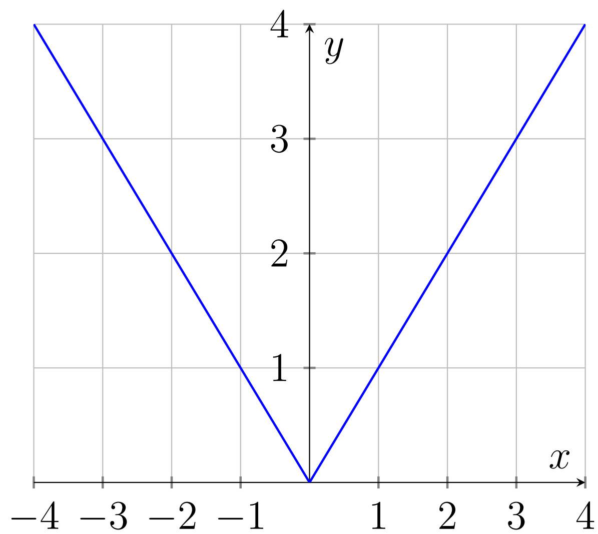


Figure 34 The graph of $|x|$

(a) What do you think is the slope of the function for any x value smaller than zero?

A. 0

B. 1

C. -1

D. DNE

(b) What do you think is the slope of the function for any x value greater than zero?

A. 0

B. 1

C. -1

D. DNE

(c) What do you think is the slope of the function at zero?

A. 0

B. 1

C. -1

D. DNE

Observation 2.1.17 Because the derivative at a point is defined in terms of a limit, the quantity $f'(a)$ *might not exist*! In that case we say that $f(x)$ is not differentiable at $x = a$. This might happen when the slope on the left of the point is different from the slope on the right, like in the case of the absolute value function. We call this behavior a corner in the graph.

Activity 2.1.18 Consider the graph of function $h(x)$.

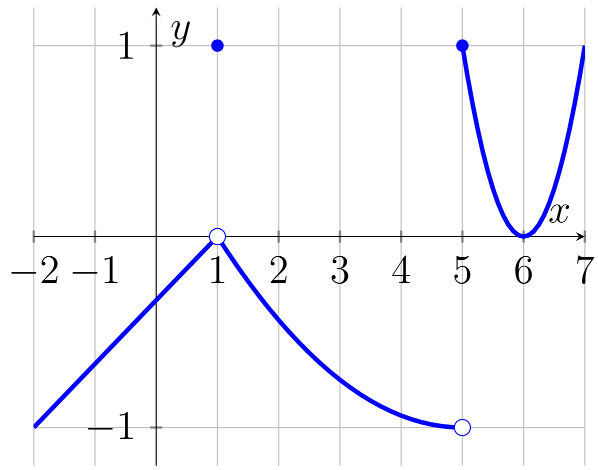


Figure 35 The graph of $h(x)$.

(a) For which of the following points a is $h'(a)$ positive? Select all that apply!

A. -1
B. 1
C. 2
D. 5
E. 6

(b) For which of the following points a is $h'(a)$ negative? Select all that apply!

A. -1
B. 1
C. 2
D. 5
E. 6

(c) For which of the following points a is $h'(a)$ zero? Select all that apply!

A. -1
B. 1
C. 2
D. 5
E. 6

(d) For which of the following points a the quantity $h'(a)$ does NOT exist? Select all that apply!

A. -1

B. 1

C. 2

D. 5

E. 6

Activity 2.1.19 Sketch the graph of a function $f(x)$ that satisfies the following criteria. (You do not need to define the function algebraically.)

- Defined and continuous on the interval $[-5, 5]$.
- $f'(x)$ does not exist at $x = 0$
- $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} < 0$
- The slope tangent to the graph of $f(x)$ at $x = 3$ is zero
- The rate of change of $f(x)$ when $x = -1$ is positive

Activity 2.1.20 You are given the graph of the function $f(x)$.

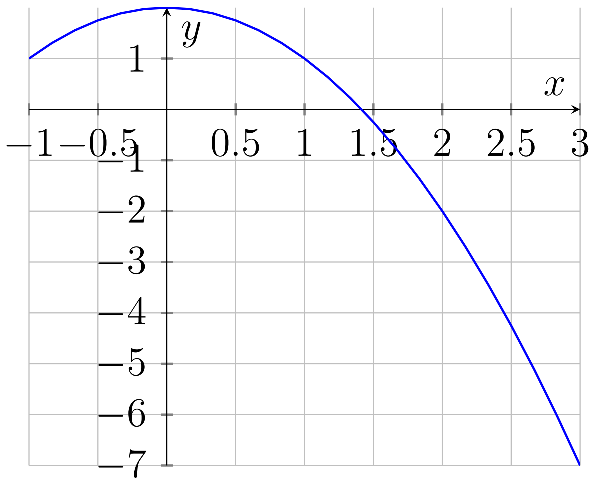


Figure 36 The graph of $f(x)$

(a) Using the graph, estimate the slope of the tangent line at $x = 2$. Make sure you can carefully describe your process for obtaining this estimate!

(b) If you call your approximation for the slope m, which one of the following expression gives you the equation of the tangent line at $x = 2$?

A. $y - 2 = m(x - 2)$

B. $y + 2 = m(x - 2)$

C. $y - 2 = m(x + 2)$

D. $y + 2 = m(x + 2)$

(c) Find the equation of the tangent line at $x = 2$.

2.2 Derivatives analytically (DF2)

Learning Outcomes

- Find derivatives using the definition of derivative as a limit.

Observation 2.2.1 Recall that $f'(a)$, the derivative of $f(x)$ at $x = a$, was defined as the limit as $h \to 0$ of the difference quotient on the interval $[a, a+h]$ as in Definition 2.1.9. If $f'(a)$ exists, then say that $f(x)$ is differentiable at a. If for some open interval (a, b), we have that $f'(x)$ exists for every point x in (a, b), then we say that $f(x)$ is differentiable on (a, b).

Activity 2.2.2 For the function $f(x) = x - x^2$ use the limit definition of the derivative at a point to compute $f'(2)$.

A. $f'(2) = \lim_{h \to 0} \dfrac{(2+h) - (2+h)^2 - 2 + 4}{h} = -3$

B. The limit $f'(2) = \lim_{h \to 0} \dfrac{(2+h) - (2+h)^2 - 2}{h}$ simplifies algebraically to $\lim_{h \to 0} \dfrac{-3h - h^2}{h}$ which does not exist, thus $f'(2)$ is not defined.

C. The limit $f'(2) = \lim_{h \to 0} \dfrac{(2+h) - (2+h)^2 - 2}{h}$ simplifies algebraically to $\lim_{h \to 0} \dfrac{h - h^2}{h}$ which does not exist, thus $f'(2)$ is not defined.

D. $f'(2) = \lim_{h \to 0} \dfrac{(2+h) - (2^2 + h^2) - 2 + 4}{h} = 1$

Activity 2.2.3 Consider the function $f(x) = 3 - 2x$. Which of the following best summarizes the average rates of changes of on f on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$?

A. The average rate of change on the intervals $[1, 4]$ and $[3, 7]$ are equal to the slope of $f(x)$, but the average rate of change of f cannot be determined on $[5, 5 + h]$ without a specific value of h.

B. The average rate of change on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$ are all different values.

C. The average rate of change on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$ are all equal to -2.

Activity 2.2.4 Can you find $f'(\pi)$ when $f(x) = 3 - 2x$ without doing any computations?

A. No, because we cannot compute the value $f(\pi)$.

B. No, because we cannot compute the average rate of change on the interval $[\pi, \pi + h]$.

C. Yes, $f'(\pi) = 3$ because the intercept of the tangent line at any point is equal to the constant intercept of $f(x)$.

D. Yes, $f'(\pi) = -2$ because the slope of the tangent line at any point is equal to the constant slope of $f(x)$.

Definition 2.2.5 Let $f(x)$ be function that is differentiable on an open interval (a,b). The derivative function of $f(x)$, denoted $f'(x)$, is given by the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

At any particular input $x = a$, the derivative function outputs $f'(a)$, the value of derivative at the point $x = a$. $\Diamond$

Remark 2.2.6 To specify the indendent variable of our function, we say that $f'(x)$ is the derivative of $f(x)$ with respect to x. For the derivative function of $y = f(x)$ we also use the notation:

$$f'(x) = y'(x) = \frac{dy}{dx} = \frac{df}{dx}.$$

The last type of notation is known as *differential* (or Leibniz) notation for the derivative.

Remark 2.2.7 Notice that our notation for the derivative function is based on the name that we assign to the function along with our choice of notation for indendent and dependent variables. For example, if we have a differentiable function $y = v(t)$, the derivative function of $v(t)$ with respect to t can be written as $v'(t) = y'(t) = \frac{dy}{dt} = \frac{dv}{dt}$.

Activity 2.2.8 In this activity you will consider $f(x) = -x^2 + 4$ and compute its derivative function $f'(x)$ using the limit definition of the derivative function Definition 2.2.5.

(a) What expression do you get when you simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(-(x+h)^2 + 4) - (-x^2 + 4)}{h}?$$

A. $\dfrac{x^2 + h^2 + 4 - x^2 - 4}{h} = \dfrac{h^2}{h}$

B. $\dfrac{-x^2 - h^2 + 4 + x^2 - 4}{h} = \dfrac{-h^2}{h}$

C. $\dfrac{-x^2 - 2xh - h^2 + 4 + x^2 - 4}{h} = \dfrac{-2xh - h^2}{h}$

D. $\dfrac{x^2 + 2xh + h^2 + 4 - x^2 - 4}{h} = \dfrac{2xh + h^2}{h}$

(b) After taking the limit as $h \to 0$, which of the following is your result for the derivative function $f'(x)$?

A. $f'(x) = x$

B. $f'(x) = -x$

C. $f'(x) = 2x$

D. $f'(x) = -2x$

Activity 2.2.9 Using the limit definition of the derivative, find $f'(x)$ for $f(x) = -x^2 + 2x - 4$. Which of the following is an accurate expression for $f'(x)$?

A. $f'(x) = 2x + 2$

B. $f'(x) = -2x$

C. $f'(x) = -2x + 2$

D. $f'(x) = -2x - 2$

Activity 2.2.10 Using the limit definition of the derivative, you want to find $f'(x)$ for $f(x) = \frac{1}{x}$. We will do this by first simplifying the difference quotient and then taking the limit as $h \to 0$.

(a) What expression do you get when you simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}?$$

A. $\dfrac{\frac{1}{x+h}}{h} = \dfrac{1}{(x+h)h}$

B. $\dfrac{\frac{h}{x+h}}{h} = \dfrac{h}{h(x+h)}$

C. $\dfrac{\frac{x-(x+h)}{(x+h)x}}{h} = \dfrac{-h}{h(x+h)x}$

D. $\dfrac{\frac{x-(x+h)}{(x+h)x}}{h} = \dfrac{-h^2}{(x+h)x}$

E. $\dfrac{\frac{h}{(x+h)x}}{h} = \dfrac{h}{h(x+h)x}$

(b) After taking the limit as $h \to 0$, which of the following is your result for the derivative function $f'(x)$?

A. $f'(x) = 0$

B. $f'(x) = 1/x$

C. $f'(x) = -1/x$

D. $f'(x) = 1/x^2$

E. $f'(x) = -1/x^2$

Activity 2.2.11 Find $f'(x)$ using the limit definition of the derivative. Then evaluate at $x = 8$.

$$f(x) = x^2 - 5\,x - 5$$

Definition 2.2.12 Once we have computed the first derivative $f'(x)$, the **second derivative** of $f(x)$ is the first derivative of $f'(x)$ or

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}.$$

$\Diamond$

Activity 2.2.13 Consider the function $f(x) = -x^2 + 2x - 4$. Earlier you saw that $f'(x) = -2x + 2$. What is the second derivative of $f(x)$?

A. $f''(x) = 2$

B. $f''(x) = -2$

C. $f''(x) = 2x$

D. $f''(x) = -2x$

Remark 2.2.14 The first derivative encodes information about the rate of change of the original function. In particular,

- If $f' > 0$, then f is increasing;
- If $f' < 0$, then f is decreasing;
- If $f' = 0$, then f has a horizontal tangent line (and it might have a max or min or it might just be changing pace).

The second derivative is the derivative of the derivative. It encodes information about the rate of change of the rate of change of the original function. In particular,

- If $f'' > 0$, then f' is increasing;
- If $f'' < 0$, then f' is decreasing;
- If $f'' = 0$, then f' has a horizontal tangent line (and it might have a max or min or it might just be changing pace).

Activity 2.2.15 Consider the function $f(x) = -x^2 + 2x - 4$. Earlier you saw that $f'(x) = -2x + 2$ and $f''(x) = -2$. What does this tell you about the graph of $f(x)$ for $x > 1$?

A. The graph is increasing and concave up

B. The graph is increasing and concave down

C. The graph is decreasing and concave up

D. The graph is decreasing and concave down

Observation 2.2.16 We have two ways to compute analytically the derivative at a point. For example, to compute $f'(1)$, the derivative of $f(x)$ at $x = 1$, we have two methods

1. We can directly compute $f'(1)$ by finding the difference quotient on the interval $[1, 1 + h]$ and then taking the limit as $h \to 0$.

2. We can first find the derivative function $f'(x)$ by computing the difference quotient on the interval $[x, x + h]$, then taking the limit as $h \to 0$, and finally evaluating the expression for $f'(x)$ at the input $x = 1$.

The latter approach is more convenient when you want to consider the value of the derivative function at multiple points!

Activity 2.2.17 Consider the function $f(x) = \frac{1}{x^2}$. You will find $f'(1)$ in two ways!

(a) Using the limit definition of the derivative at a point, compute the difference quotient on the interval $[1, 1+h]$ and then take the limit as $h \to 0$. What do you get?

A. -1

B. 1

C. 2

D. -2

(b) Now, using the limit definition of the derivative function, find $f'(x)$. Which of the following is your result for the derivative function $f'(x)$?

A. $f'(x) = -1/x^3$

B. $f'(x) = 1/x^3$

C. $f'(x) = -2/x^3$

D. $f'(x) = 2/x^3$

(c) Make sure that your answers match! So if you plug in $x = 1$ in $f'(x)$, you should get the same number you got when you computed $f'(1)$.

Activity 2.2.18 In this activity you will study (again!) the velocity of a ball falling under gravity. A ball is tossed vertically in the air from a window. The height of the ball (in feet) is given by the formula $f(t) = 64 - 16(t-1)^2$, where t is the seconds after the ball is launched. Recall that in Activity 2.1.1, you used numerical methods to approxmiate the instantaneous velocity of $f(t)$ to calculate $v(2)$!

(a) Using the limit definition of the derivative function, find the velocity function $v(t) = f'(t)$.

(b) Using the velocity function $v(t)$, what is $v'(1)$, the instantaneous velocity at $t = 1$?

A. -32 feet per second
B. 32 feet per second
C. 0 feet per second
D. -16 feet per second
E. 16 feet per second

(c) What behavior would explain your finding?

A. After 1 second the ball is falling at a speed of 32 meters per second.
B. After 1 second the ball is moving upwards at a speed of 32 meters per second.
C. After 1 second the ball reaches its highest point and it stops for an instant.
D. After 1 second the ball is falling at a speed of 16 meters per second.
E. After 1 second the ball is moving upwards at a speed of 16 meters per second.

Observation 2.2.19 A function can only be differentiable at $x = a$ if it is also continuous at $x = a$. But not all continuous functions are differentiable: when we have a corner in the graph of a the function, the function is continuous at the corner point, but it is not differentiable at that point!

Activity 2.2.20 In Observation 2.1.17, we said that a function is not differentiable when the limit that defines it does not exist. In this activity we will study differentiability analytically.

(a) Consider the following continuous function

$$g(x) = \begin{cases} x+2 & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

Consider the interval $[2, 2+h]$. When $h < 0$, the interval falls under the first definition of $g(x)$ and the derivative is always equal to 1. What is the derivative function for x values greater than 2? Show that at $x = 2$ the value of this derivative is not equal to 1 and so $g(x)$ is not differentiable at $x = 2$.

(b) Consider the following discontinuous function

$$g(x) = \begin{cases} x+2 & x \leq 2 \\ x & x > 2 \end{cases}$$

On both sides of $x = 2$ it seems that the slope is the same, but this function is still not differentiable at $x = 2$. Notice that $g(2) = 4$. When $h > 0$, the interval $[2, 2+h]$falls under the second definition of $g(x)$, but $g(2)$ is always fixed at 4. Compute the difference quotient $\frac{g(2+h)-g(2)}{h}$ assuming that $h > 0$ and notice that this does not simplify as expected! Moreover, if you take the limit as $h \to 0$, you will get infinity and not the expected slope of 1!

(c) Consider the following function

$$g(x) = \begin{cases} ax+2 & x \leq 2 \\ bx^2 & x > 2 \end{cases}$$

where a, b are some nonzero parameters you will find. Find an equation in a, b that needs to be true if we want the function to be continuous at $x = 2$. Also, find an equation in a, b that needs to be true if we want the function to be differentiable at $x = 2$. Solve the system of two linear equations... you should find that $a = -2$ and $b = -1/2$ are the only values that make the function differentiable (and continuous!).

2.3 Elementary derivative rules (DF3)

Learning Outcomes

- Compute basic derivatives using algebraic rules.

Observation 2.3.1 We know how to find the derivative function using the limit definition of the derivative. From the activities in the previous section, we have seen that this process gets cumbersome when the functions are more complicated. In this section we will discuss shortcuts to calculate derivatives, known as "differentiation rules".

Activity 2.3.2 In this activity we will try to deduce a rule for finding the derivative of a power function. Note, a power function is a function of the form $f(x) = x^n$ where n is any real number.

(a) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x$?

A. -1

B. 1

C. 0

D. Does not exist

(b) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^2$?

A. 0

B. $-2x$

C. $2x$

D. $2x + 1$

(c) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^3$?

A. $3x^2$

B. $-3x^2$

C. $3x^2 - 3x$

D. $-3x^2 + 3x$

(d) WITHOUT using the limit definition of the derivative, what is your best guess for $f'(x)$ when $f(x) = x^4$? (See if you can find a pattern from the first three tasks of this activity.)

A. $3x^2$

B. $3x^3$

C. $4x^2$

D. $4x^3$

Observation 2.3.3 We have been using $f'(x)$, read "f prime", to denote a derivative of the function $f(x)$. There are other ways to denote the derivative of $y = f(x)$: y' or $\frac{df}{dx}$, pronounced "dee-f dee-x". If you want to take the derivative of $f'(x)$, y', or $\frac{df}{dx}$ to get the second derivative of $f(x)$, the notation is $f''(x)$, y'', or $\frac{d^2f}{dx^2}$.

Activity 2.3.4 Using Theorem 2.3.3, which of the following statement(s) are true? For those statements that are wrong, give the correct derivative.

A. The derivative of $y = x^{10}$ is $y' = 10x^{11}$.

B. The derivative of $y = x^{-8}$ is $y' = -8x^{-9}$.

C. The derivative of $y = x^{100}$ is $y' = 100x^{99}$.

D. The derivative of $y = x^{-17}$ is $y' = -17x^{-16}$.

Activity 2.3.5 Using Theorem 2.3.6, which of the following statement(s) are true? Note: Pay attention to the independent variable (the input) of the function.

A. The derivative of $y(x) = 10$ is $y'(x) = 9$.

B. The derivative of $y(t) = x$ is $y'(t) = 0$.

C. The derivative of $y(a) = x^2$ is $y'(a) = 2x$.

D. The derivative of $y(x) = -5$ is $y'(x) = -4$.

Activity 2.3.6 What is the derivative of the function $y(x) = 12x^{2/3}$?

A. $y'(x) = 8x^{5/3}$.

B. $y'(x) = 18x^{-1/3}$.

C. $y'(x) = 8x^{-1/3}$.

D. $y'(x) = 18x^{5/3}$.

Activity 2.3.7 What are the first and second derivatives for the arbitrary quadratic function given by $f(x) = ax^2 + bx + c$, where a, b, c are any real numbers?

A. $f'(x) = 2ax + bx + c$, $f''(x) = 2a + b$.

B. $f'(x) = 2x + 1$, $f''(x) = 2$.

C. $f'(x) = 2ax + b$, $f''(x) = 2a$.

D. $f'(x) = ax + b$, $f''(x) = a$.

Elementary derivative rules (DF3)

Activity 2.3.8 We can look at power functions with fractional exponents like $f(x) = x^{\frac{1}{4}} = \sqrt[4]{x}$ or with negative exponents like $g(x) = x^{-4} = \frac{1}{x^4}$. What is the derivative of these two functions?

A. $f'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$, $g'(x) = \dfrac{-4}{x^3}$.

B. $f'(x) = \dfrac{1}{4}\sqrt[4]{x^3}$, $g'(x) = \dfrac{-4}{x^5}$.

C. $f'(x) = \dfrac{1}{4}\sqrt[4]{x^3}$, $g'(x) = \dfrac{-4}{x^3}$.

D. $f'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$, $g'(x) = \dfrac{-4}{x^5}$.

Observation 2.3.9 A special case of Theorem 2.3.13 is when $b = e$, where e is the base of the natural logarithm function. In this case let $f(x) = e^x$. Then

$$f'(x) = \ln(e)\, e^x = e^x.$$

So $f(x) = e^x$ is a special function for which $f'(x) = f(x)$.

Activity 2.3.10 The first derivative of the function $g(x) = x^e + e^x$ is given by $g'(x) = ex^{e-1} + e^x$. What is the second derivative of $g(x)$?

A. $g''(x) = x^e + e^x$.

B. $g''(x) = e(e-1)x^{e-2} + e^x$.

C. $g''(x) = ex^{e-1} + e^x$.

D. $g''(x) = e^x$.

Activity 2.3.11 The derivative of $f(x) = 7\sin(x) + 2e^x + 3x^{1/3} - 2$ is,

A. $f'(x) = 7\cos(x) + 2e^x + x^{-2/3} - 2x$.

B. $f'(x) = 7\cos(x) + 2e^x + -2x^{-2/3} - 2$.

C. $f'(x) = -7\sin(x) + e^x + x^{-2/3}$.

D. $f'(x) = -7\cos(x) + 2e^x\ln(x) + x^{-2/3}$.

E. $f'(x) = 7\cos(x) + 2e^x + x^{-2/3}$.

Activity 2.3.12 Which of the following statements is NOT true?

A. The derivative of $y = 2\ln(x)$ is $y' = \frac{2}{x}$.

B. The derivative of $y = \frac{\ln(x)}{2}$ is $y' = \frac{1}{2x}$.

C. The derivative of $y = \frac{2}{3}\ln(x)$ is $y' = \frac{3}{2x}$.

D. The derivative of $y = \ln(x^2)$ is $y' = \frac{2}{x}$.

Activity 2.3.13 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (scalar multiple, sum/difference, etc.) you are using in your work.

(a)

$$g(x) = 2\,\cos(x) - 3\,e^{x}$$

(b)

$$h(w) = \sqrt[5]{w^{7}} + \frac{6}{w^{5}}$$

(c)

$$f(t) = -4\,t^{5} + 5\,t^{3} + t - 8$$

Activity 2.3.14 Suppose that the temperature (in degrees Fahrenheit) of a cup of coffee, t minutes after forgetting it on a bench outside, is given by the function

$$f(t) = 40\,(0.5)^t + 50$$

Find $f(1)$ and $f'(1)$ and try to interpret your result in the context of this problem.

Activity 2.3.15 In this activity you will use our first derivative rules to study the slope of tangent lines.

(a) The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of these points.

(b) Find the equations of the two lines tangent to the parabola $y = (x-2)^2$ which pass through the origin. You will want to think about slope in two ways: as the derivative at $x = a$ and the rise over the run in a linear function through the origin and the point $(a, f(a))$. Use a graph to check your work and sketch the tangent lines on your graph.

Activity 2.3.16 Find the values of the parameters a, b, c for the quadratic polynomial $q(x) = ax^2 + bx + c$ that best approximates the graph of $f(x) = e^x$ at $x = 0$. This means choosing a, b, c such that

- $q(0) = f(0)$
- $q'(0) = f'(0)$
- $q''(0) = f''(0)$

Hint: find the values of $f(0), f'(0), f''(0)$. The values of $q(0), q'(0), q''(0)$ at zero will involve some parameters. You can solve for these parameters using the equations above.

2.4 The product and quotient rules (DF4)

Learning Outcomes

- Compute derivatives using the Product and Quotient Rules.

The product and quotient rules (DF4)

Activity 2.4.1 Let f and g be the functions defined by

$$f(t) = 2t^2\,,\; g(t) = t^3 + 4t.$$

(a) Find $f'(t)$ and $g'(t)$.

(b) Let $P(t) = 2t^2\,(t^3 + 4t)$ and observe that $P(t) = f(t) \cdot g(t)$. Rewrite the formula for P by distributing the $2t^2$ term. Then, compute $P'(t)$ using the power, sum, and scalar multiple rules.

(c) True or false: $P'(t) = f'(t) \cdot g'(t)$.

Activity 2.4.2 The product rule is a powerful tool, but sometimes it isn't necessary; a more elementary rule may suffice. For which of the following functions can you find the derivative without using the product rule? Select all that apply.

A. $f(x) = e^x \sin x$

B. $f(x) = \sqrt{x}(x^3 + 3x - 3)$

C. $f(x) = (4)(x^5)$

D. $f(x) = x \ln x$

Activity 2.4.3 Find the derivative of the following functions using the product rule.

(a) $f(x) = (x^2 + 3x)\sin x$

(b) $f(x) = e^x \cos x$

(c) $f(x) = x^2 \ln x$

Activity 2.4.4 Let f and g be the functions defined by

$$f(t) = 2t^2\,,\; g(t) = t^3 + 4t.$$

(a) Determine $f'(t)$ and $g'(t)$. (You found these previously in Activity 2.4.1.)

(b) Let $Q(t) = \frac{t^3+4t}{2t^2}$ and observe that $Q(t) = \frac{g(t)}{f(t)}$. Rewrite the formula for Q by dividing each term in the numerator by the denominator and use rules of exponents to write Q as a sum of scalar multiples of power functions. Then, compute $Q'(t)$ using the sum and scalar multiple rules.

(c) True or false: $Q'(t) = \frac{g'(t)}{f'(t)}$.

Activity 2.4.5 Just like with the product rule, there are times when we can find the derivative of a quotient using elementary rules rather than the quotient rule. For which of the following functions can you find the derivative without using the quotient rule? Select all that apply.

A. $f(x) = \dfrac{6}{x^3}$

B. $f(x) = \dfrac{2}{\ln x}$

C. $f(x) = \dfrac{e^x}{\sin x}$

D. $f(x) = \dfrac{x^3 + 3x}{x}$

Activity 2.4.6 Find the derivative of the following functions using the quotient rule (or, if applicable, an elementary rule).

(a) $f(x) = \frac{6}{x^3}$

(b) $f(x) = \frac{2}{\ln x}$

(c) $f(x) = \frac{e^x}{\sin x}$

(d) $f(x) = \frac{x^3+3x}{x}$

Activity 2.4.7 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (product, quotient, sum and difference, etc.) you are using in your work.

(a)

$$f(w) = -\frac{3\,w^2 + 5\,w - 2}{\sin(w)}$$

(b)

$$g(t) = \frac{t^2 + 6\,t + 1}{t^2}$$

(c)

$$h(t) = -2\left(t^2 + 3\,t + 3\right)\cos(t)$$

Note 2.4.8 We have found the derivatives of $\sin x$ and $\cos x$, but what about the other trigonometric functions? It turns out that the quotient rule along with some trig identities can help us! (See Khan Academy[1] for a reminder of trig identities.)

[1] `KhanAcademy.org`

Activity 2.4.9 Consider the function $f(x) = \tan x$, and remember that $\tan x = \frac{\sin x}{\cos x}$.

(a) What is the domain of f?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{(\cos x)(\cos x) + (\sin x)(\sin x)}{(\cos x)^2}.$$

(c) Which trig identity might be useful here to simplify this expression? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec x = \frac{1}{\cos x}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of x is $f'(x)$ defined? How does this domain compare to the domain of f?

Activity 2.4.10 Let $g(x) = \cot x$, and recall that $\cot x = \frac{\cos x}{\sin x}$.

(a) What is the domain of $g(x)$?

(b) Use the quotient rule to develop a formula for $g'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $g'(x)$ only in terms of the cosecant function.

(d) What is the domain of $g'(x)$? How does this domain compare to the domain of $g'(x)$?

Activity 2.4.11 Let $h(x) = \sec x$, and recall that $\sec x = \frac{1}{\cos x}$.

(a) What is the domain of $h(x)$?

(b) Use the quotient rule to develop a formula for $h'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $h'(x)$ only in terms of the the tangent and secant functions.

(d) What is the domain of $h'(x)$? How does this domain compare to the domain of $h'(x)$?

The product and quotient rules (DF4)

Activity 2.4.12 Let $p(x) = \csc x$, and recall that $\csc x = \frac{1}{\sin x}$.

(a) What is the domain of $p(x)$?

(b) Use the quotient rule to develop a formula for $p'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $h'(x)$ only in terms of the the cotangent and cosecant functions.

(d) What is the domain of $p'(x)$? How does this domain compare to the domain of $p'(x)$?

The product and quotient rules (DF4)

Activity 2.4.13 Consider the functions

$$f(x) = 3\cos(x),\ \ g(x) = x^2 + 3e^x$$

and the function $h(x)$ for which a table of values is given.

x	-1	0	2
$h(x)$	-4	-1	3
$h'(x)$	0	-1	1

In answering the following questions, be sure to explicitly denote which derivative rules (product, quotient, sum/difference, etc.) you are using in your work.

(a) Find the derivative of $f(x) \cdot g(x)$.

(b) Find the derivative of $\dfrac{f(x)}{g(x)}$.

(c) Find the value of the derivative of $f(x) \cdot h(x)$ at $x = -1$.

(d) Find the value of the derivative of $\dfrac{g(x)}{h(x)}$ at $x = 0$.

(e) Consider the function

$$r(x) = 3\cos(x) \cdot x.$$

Find $r'(x)$, $r''(x)$, $r'''(x)$, and $r^{(4)}(x)$ so the first, second, third, and fourth derivative of $r(x)$. What pattern do you notice? What do you expect the twelfth derivative of $r(x)$ to be?

Activity 2.4.14

(a) Differentiate $y = \dfrac{e^x}{x}, y = \dfrac{e^x}{x^2}, y = \dfrac{e^x}{x^3}$. Simplify your answers as much as possible.

(b) What do you expect the derivative of $y = \dfrac{e^x}{x^n}$ to be? Prove your guess!

(c) What do your answers above tell you above the shape of the graph of $y = \dfrac{e^x}{x^n}$? Study how the sign of the numerator and the denominator change in the first derivative to determine when the behavior changes!

Activity 2.4.15 The quantity q of skateboards sold depends on the selling price p of a skateboard, so we write $q = f(p)$. You are given that

$$f(140) = 15000, \quad f'(140) = -100$$

(a) What does the data provided tell you about the sales of skateboards?

(b) The total revenue, R, earned by the sale of skateboards is given by $R = q \cdot p = f(p) \cdot p$. Explain why.

(c) Find the derivative of the revenue when $p = 140$, so find the value of

$$\left.\frac{dR}{dp}\right|_{p=140}.$$

(d) What is the sign of the quantity above? What do you think would happen to the revenue if the price was changed from \$140 to \$141?

Activity 2.4.16 Let $f(v)$ be the gas consumption in liters per kilometer (l/km) of a car going at velocity v kilometers per hour (km/hr). So if the car is going at velocity v, then $f(v)$ tells you how many liters of gas the car uses to go one kilometer. You are given the following data

$$f(50) = 0.04, \quad f'(50) = 0.0004$$

(a) Let $g(v)$ be the distance (in kilometers) that the same car covers per liter of gas at velocity v. What are the units of the output of $g(v)$? Use these units to infer how to write $g(v)$ in terms of $f(v)$, then find $g(50)$ and $g'(50)$.

(b) Let $h(v)$ be the gas consumption over time, so the liters of gas consumed per hour by the same car going at velocity v. What are the units of the output of $h(v)$? Use these units to infer how to write $h(v)$ in terms of $f(v)$, then find $h(50)$ and $h'(50)$.

(c) How would you explain the practical meaning of your findings to a driver who knows no calculus?

2.5 The chain rule (DF5)

Learning Outcomes

- Compute derivatives using the Chain Rule.

Note 2.5.1 When we consider the consider the composition $f \circ g$ of the function f with the function g, we mean the composite function $f(g(x))$, where the function g is applied first and then f is applied to the output of g. We also call f the outside function whilst g is the inside function.

Activity 2.5.2

(a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Which of the following is a formula for $f(g(x))$?

A. $-4x^2 + 4x + 4$

B. $4x^2 - 4x + 5$

C. $-2x^2 + 9$

D. $-2x^2 + 4$

(b) One of the options above is a formula for $g(f(x))$. Which one?

Activity 2.5.3

(a) Consider the composite function $f(g(x)) = \sqrt{e^x}$. Which function is the outside function $f(x)$ and which one is the inside function $g(x)$?

A. $f(x) = x^2$, $g(x) = e^x$

B. $f(x) = \sqrt{x}$, $g(x) = e^x$

C. $f(x) = e^x$, $g(x) = \sqrt{x}$

D. $f(x) = e^x$, $g(x) = x^2$

(b) Using properties of exponents, we can rewrite the original function as $e^{\frac{x}{2}}$. Using this new expression, what is your new inside function and your new outside function?

(c) Consider the function $e^{\sqrt{x}}$. In this case, what are the inside and outside functions?

The chain rule (DF5)

Activity 2.5.4 In this activity we will build the intuition for the chain rule using a real-world scenario and differential notation for derivatives. Consider the following scenario.

My neighborhood is being invaded! The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. How fast is the squirrel population growing per week?

(a) The scenario gives you information regarding the rate of growth of $s(a)$, the squirrel population as a function of acorn availability (measured in bushels). What is the current value of $\frac{ds}{da}$?

A. 2

B. 100

C. 200

D. 50

(b) The scenario gives you information regarding the rate of growth of $a(t)$, the acorn availability as a function of time (measured in weeks). What is the current value of $\frac{da}{dt}$?

A. 2

B. 100

C. 200

D. 50

(c) Given all the information provided, what is your best guess for the value of $\frac{ds}{dt}$, the rate at which the squirrel population is growing per week?

A. 2

B. 100

C. 200

D. 50

(d) Given your answers above, what is the relationship between $\frac{ds}{da}$, $\frac{da}{dt}$, $\frac{ds}{dt}$?

Activity 2.5.5

(a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Notice that $f(g(x)) = -4x^2 + 4x + 4$. Which of the following is the derivative function of the composite function $f(g(x))$?

A. $-8x + 4$

B. $-4x$

C. $-2x$

D. 2

(b) One of the options above is a formula for $f'(x) \cdot g'(x)$. Which one? Notice that this is not the same as the derivative of $f(g(x))$!

The chain rule (DF5)

Activity 2.5.6 Consider the composite function $h(x) = \sqrt{e^x} = e^{\frac{x}{2}}$. For each of the two expressions, find the derivative using the chain rule. Which of the following expressions are equal to $h'(x)$? Select all!

A. $\dfrac{1}{2}\left(e^x\right)^{\frac{-1}{2}} \cdot e^x$

B. $\dfrac{1}{2}\left(e^x\right)^{\frac{3}{2}} \cdot e^x$

C. $\dfrac{1}{2}e^{\frac{-x}{2}}$

D. $e^{\frac{x}{2}} \cdot \dfrac{1}{2}$

E. $\dfrac{1}{2}\sqrt{e^x}$

F. $\sqrt{e^x} \cdot e^x$

Activity 2.5.7 Below you are given the graphs of two functions: $a(x)$ and $b(x)$. Use the graphs to compute vaules of composite functions and of their derivatives, when possible (there are points where the derivative of these functions is not defined!). Notice that to compute the derivative at a point, you first want to find the derivative as a function of x and then plug in the input you want to study.

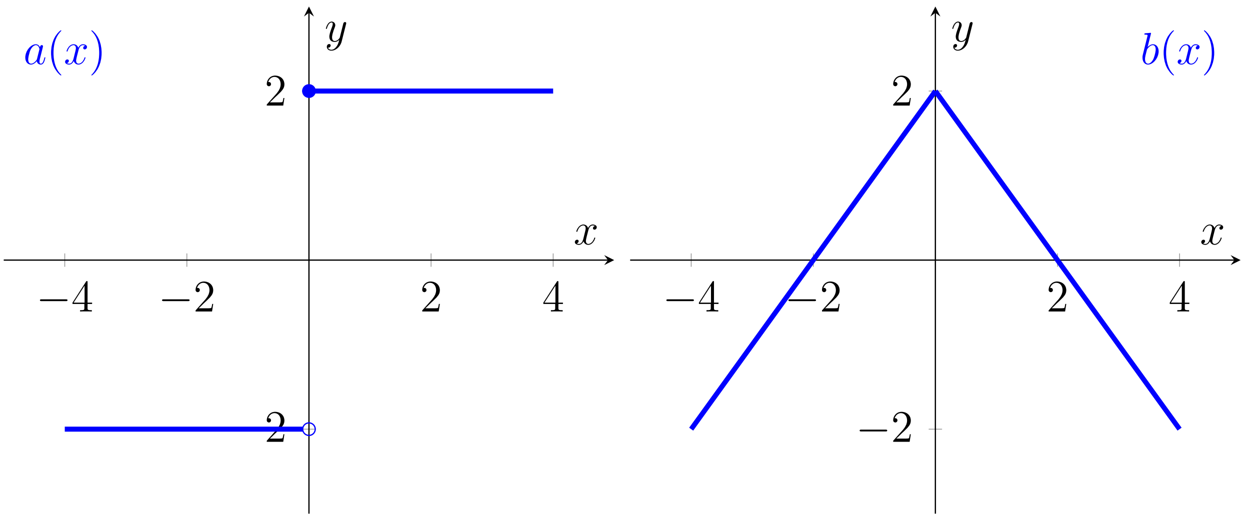


Figure 37 The graphs of $a(x)$ and $b(x)$

(a) Notice that the derivative of $a \circ b$ is given by $a'(b(x)) \cdot b'(x)$, so the derivative of $a \circ b$ at $x = 4$ is given by the quantity $a'(b(4)) \cdot b'(4) = a'(-2) \cdot b'(4)$, because $b(4) = -2$. Using the graphs to compute slopes, what is the derivative of $a \circ b$ at $x = 4$?

A. 0
B. -1
C. 1
D. -2
E. 2
F. The derivative does not exist at this point.

(b) Which of the following values is the derivative of $a \circ b$ at $x = 2$?

A. 0
B. -1
C. 1
D. -2
E. 2
F. The derivative does not exist at this point.

The chain rule (DF5)

(c) Which of the following values is the derivative of $b \circ a$ (different order!) at $x = -2$?

A. 0

B. -1

C. 1

D. -2

E. 2

F. The derivative does not exist at this point.

The chain rule (DF5)

Activity 2.5.8 In this activity you will study the derivative of $\cos^n(x)$ for different powers n.

(a) Consider the function $\cos^2(x) = (\cos(x))^2$. Combining power and chain rule, what do you get if you differentiate $\cos^2(x)$?

A. $-\cos^2(x)\sin(x)$

B. $-\cos^2(x)\sin(x)$

C. $2\cos(x)\sin(x)$

D. $-2\cos(x)\sin(x)$

(b) Consider the function $\cos^3(x)$. Find its derivative.

(c) Consider the function $\cos^n(x)$, for n any number. Find the general formula for its derivative.

The chain rule (DF5)

Activity 2.5.9 In this activity you will study the derivative of $b^{\cos(x)}$ for different bases b.

(a) Consider the function $e^{\cos(x)}$. Combining exponential and chain rule, what do you get if you differentiate $e^{\cos(x)}$?

A. $e^{\cos(x)}$

B. $-e^{\cos(x)}\sin(x)$

C. $e^{-\sin(x)}$

D. $e^{\cos(x)}\sin(x)$

(b) Consider the function $2^{\cos(x)}$. Find its derivative.

(c) Consider the function $b^{\cos(x)}$, for b any positive number. Find the general formula for its derivative.

Remark 2.5.10 Remember that exponential and power functions obey very different differentiation rules. This behavior continues when we consider composite function. The composite power function $f(x)^3$ has derivative

$$3[f(x)]^2 \cdot f'(x)$$

but the composite exponential function $3^{f(x)}$ has derivative

$$\ln(3)\, 3^{f(x)} \cdot f'(x)$$

The chain rule (DF5)

Activity 2.5.11 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (chain, product, quotient, sum/difference, etc.) you are using in your work.

1. $$f(x) = -(4\,x - 3\,e^{x} + 4)^{3}$$

2. $$k(w) = 9\,\cos\left(w^{\frac{7}{5}}\right)$$

3. $$h(y) = -3\,\sin\left(-5\,y^{2} + 2\,y - 5\right)$$

4. $$g(t) = 9\,\cos(t)^{\frac{7}{5}}$$

Answer.

1. $$f'(x) = 3\,(4\,x - 3\,e^{x} + 4)^{2}(3\,e^{x} - 4)$$

2. $$k'(w) = -\frac{63}{5}\,w^{\frac{2}{5}}\sin\left(w^{\frac{7}{5}}\right)$$

3. $$h'(y) = 6\,(5\,y - 1)\cos\left(-5\,y^{2} + 2\,y - 5\right)$$

4. $$g'(t) = -\frac{63}{5}\,\cos(t)^{\frac{2}{5}}\sin(t)$$

Activity 2.5.12 Notice that

$$\left(\frac{f(x)}{g(x)}\right) = \left(f(x) \cdot g(x)^{-1}\right)$$

Use this observation, the chain rule, the product rule, and the power rule (plus some fraction algebra) to deduce the quotient rule in a new way!

Activity 2.5.13 Remember my neighborhood squirrel invasion? The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. Considering this information as pertaining to the moment $t = 0$, you are given the following possible model for the squirrel:

$$s(a(t)) = 2a(t) + 10 = 2\left(50\sin(2t) + 60\right) + 10.$$

(a) Check that the model satisfies the data $\frac{ds}{da} = 2$ and $\frac{da}{dt}\Big|_{t=0} = 100$

(b) Find the derivative function $\frac{ds}{dt}$ and check that $\frac{ds}{dt}|_{t=0} = 200$.

(c) According to this model, what is the maximum and minimum squirrel population? What is the fastest rate of increase and decrease of the squirrel population? When will these extremal scenarions occur?

Activity 2.5.14 Suppose that a fish population at t months is approximated by

$$P(t) = 100 \cdot 4^{0.05t}$$

(a) Find $P(10)$ and use units to explain what this value tells us about the population.

(b) Find $P'(10)$ and use units to explain what this value tells us about the population. (If you want to avoid using a calculator, you can use the approximation $\ln(4) = 1.4$.)

2.6 Differentiation strategy (DF6)

Learning Outcomes

- Compute derivatives using a combination of algebraic derivative rules.

Activity 2.6.1 Consider the functions defined below:

$$f(x) = \sin((x^2 + 3x)\cos(2x))$$

$$g(x) = \sin(x^2 + 3x)\cos(2x)$$

(a) What do you notice that is similar about these two functions?

(b) What do you notice that is different about these two functions?

(c) Imagine that you are sorting functions into different categories based on how you would differentiate them. In what category (or categories) might these functions fall?

Remark 2.6.2 To take a derivative, we need to examine how the function is built and then proceed accordingly. Below are some questions you might ask yourself as you take the derivative of a function, especially one where multiple rules might need to be used:

1. How is this function built algebraically? What kind of function is this? What is the big picture?
2. Where do you start?
3. Is there an easier or more convenient way to write the function?
4. Are there products or quotients involved?
5. Is this function a composition of two (or more) elementary functions? If so, what are the outside and inside functions?
6. What derivative rules will be needed along the way?

Activity 2.6.3 Consider the function $f(x) = x^3\sqrt{3 - 8x^2}$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A. Chain rule
B. Power rule
C. Product rule
D. Quotient rule
E. Sum/difference rule

(b) What other rules would be needed along the way? Select all that apply.

A. Chain rule
B. Power rule
C. Product rule
D. Quotient rule
E. Sum/difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.4 Consider the function $f(x) = \left(\frac{\ln x}{(3x-4)^3}\right)^5$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A. Chain rule

B. Power rule

C. Product rule

D. Quotient rule

E. Sum/difference rule

(b) What other rules would be needed along the way? Select all that apply.

A. Chain rule

B. Power rule

C. Product rule

D. Quotient rule

E. Sum/difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.5 Consider the function $f(x) = \sin(\cos(\tan(2x^3 - 1)))$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A. Chain rule
B. Power rule
C. Product rule
D. Quotient rule
E. Sum/difference rule

(b) What other rules would be needed along the way? Select all that apply.

A. Chain rule
B. Power rule
C. Product rule
D. Quotient rule
E. Sum/difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.6 Consider the function $f(x) = \frac{x^2 e^x}{2x^3 - 5x + \sqrt{x}}$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A. Chain rule
B. Power rule
C. Product rule
D. Quotient rule
E. Sum/difference rule

(b) What other rules would be needed along the way? Select all that apply.

A. Chain rule
B. Power rule
C. Product rule
D. Quotient rule
E. Sum/difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.7 Find the derivative of the following functions. For each, include an explanation of the steps involved that references the algebraic structure of the function.

(a) $f(x) = e^{5x}(x^2 + 7^x)^3$

(b) $f(x) = \left(\frac{3x+1}{2x^6-1}\right)^5$

(c) $f(x) = \sqrt{\cos\left(2\,x^2 + x\right)}$

(d) $f(x) = \tan(xe^x)$

Activity 2.6.8 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (constant multiple, sum/difference, etc.) you are using in your work.

(a)

$$f(y) = \sqrt{\cos\left(6\,y^4 - 6\,y\right)}$$

(b)

$$g(t) = \left(\frac{5\,t^3 + 2}{4\,t^4 - 3}\right)^4$$

(c)

$$h(x) = -\left(5\,x^4 - 7\,x^3\right)^5 x^{\frac{1}{4}}$$

2.7 Differentiating implicitly defined functions (DF7)

Learning Outcomes

- Compute derivatives of implicitly-defined functions.

Differentiating implicitly defined functions (DF7)

Observation 2.7.1 Many of the equations that has been discussed so far fall under the category of an explicit equation. An explicit equation is one in which the relationship between x and y is given explicitly, such as $y = f(x)$. In this section we will examine when the relationship between x and y is given implicity. An implicit equation looks like $f(x, y) = g(x, y)$ where both sides of the equation may depend on both x and y.

Observation 2.7.2 Note that if we are taking the derivative of $f(x)$ with respect to x, then

$$\frac{d}{dx}(f(x)) = f'(x).$$

However, if we are taking the derivative of $g(y(x))$ with respect to x, then

$$\frac{d}{dx}(g(y)) = g'(y) \cdot \frac{dy}{dx}.$$

Activity 2.7.3 For this activity we want to find the equation of a tangent line for a circle with radius 5 centered at the origin, $x^2 + y^2 = 25$, at the point $(-3, -4)$.

(a) The derivative with respect to x for the equation of the circle is given by which expression.

A. $2x + 2y\dfrac{dy}{dx} = 25$

B. $2x + y\dfrac{dy}{dx} = 0$

C. $2x + 2y\dfrac{dy}{dx} = 0$

D. $2x + 2\dfrac{dy}{dx} = 25$

(b) Solving for $\frac{dy}{dx}$ gives?

A. $\dfrac{dy}{dx} = \dfrac{25 - 2x}{2y}$

B. $\dfrac{dy}{dx} = -\dfrac{2x}{y}$

C. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

D. $\dfrac{dy}{dx} = \dfrac{25 - 2x}{2}$

(c) Plug the point $(-3, -4)$ into the expression found above for the derivative to get the slope of the tangent line.

(d) Use the value for the slope of the tangent line to obtain the equation of the tangent line.

Activity 2.7.4 The curve given in Figure 50 is an example of an astroid. The equation of this astroid is $x^{2/3} + y^{2/3} = 3^{2/3}$. What is the derivative with respect x for this astroid? (Solve for $\frac{dy}{dx}$).

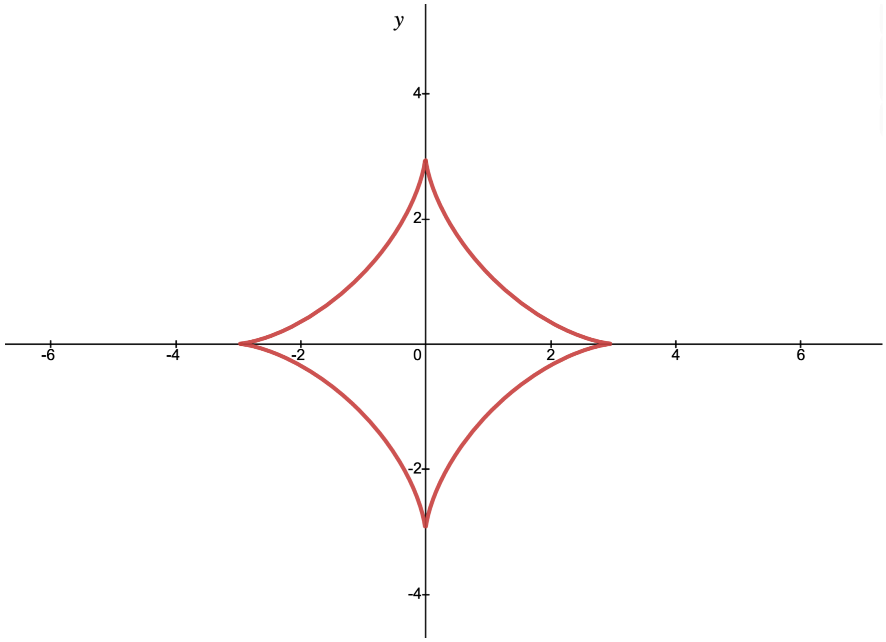


Figure 38 Graph of $x^{2/3} + y^{2/3} = 3^{2/3}$.

A. $\dfrac{dy}{dx} = \dfrac{x^{-1/3}}{y^{-1/3}}$

B. $\dfrac{dy}{dx} = \dfrac{y^{-1/3}}{x^{-1/3}}$

C. $\dfrac{dy}{dx} = \dfrac{3^{-1/3} - x^{-1/3}}{y^{-1/3}}$

D. $\dfrac{dy}{dx} = -\dfrac{x^{-1/3}}{y^{-1/3}}$

Activity 2.7.5 An example of a lemniscate is given in Figure 51. The equation of this lemniscate is $(x^2+y^2)^2 = x^2 - y^2$. What is the derivative with respect x for this lemniscate? (Solve for $\frac{dy}{dx}$).

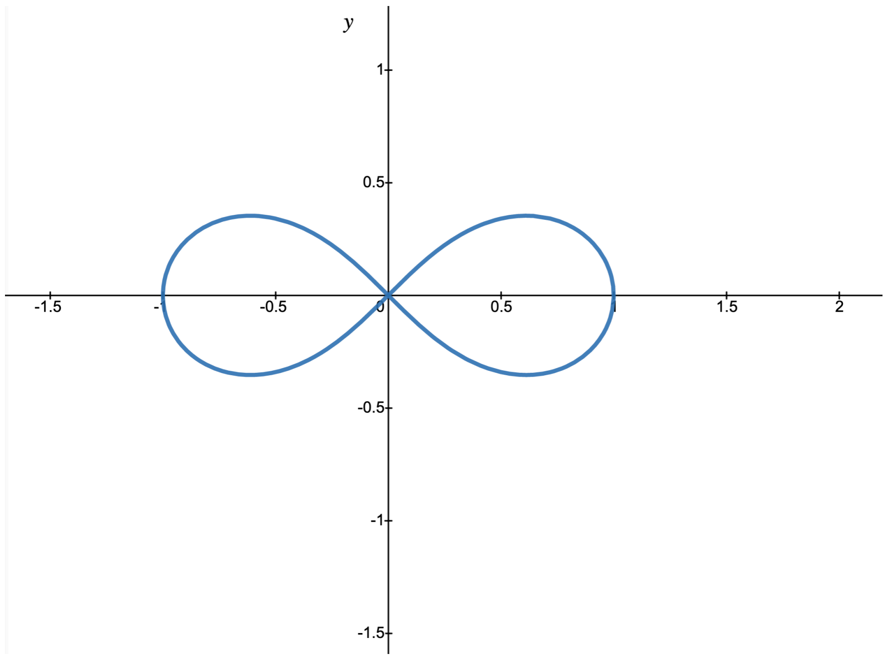


Figure 39 Graph of $(x^2+y^2)^2 = x^2 - y^2$.

A. $\dfrac{dy}{dx} = \dfrac{x(1-2(x^2+y^2))}{y+2(x^2+y^2)}$

B. $\dfrac{dy}{dx} = \dfrac{x(1-2(x^2+y^2))}{y(1+2(x^2+y^2))}$

C. $\dfrac{dy}{dx} = \dfrac{y(1+2(x^2+y^2))}{x(1-2(x^2+y^2))}$

D. $\dfrac{dy}{dx} = \dfrac{y+2(x^2+y^2)}{x(1-2(x^2+y^2))}$

Differentiating implicitly defined functions (DF7)

Activity 2.7.6 Explain how to use implicit differentiation to find $\frac{dy}{dx}$ for each of the following equations.

(a)

$$-5\,x^{5} - 5\,\cos\left(y\right) = 3\,y^{4} + 2$$

(b)

$$-5\,ye^{x} + 5\,\sin\left(x\right) = 0$$

Activity 2.7.7 To take the derivative of some explicit equations you might need to make it an implicit equation. For this activity we will find the derivative of $y = x^x$. Make the equation an implicit equation by taking natural logarithm of both sides, this gives $\ln(y) = x\ln(x)$. Knowing this, what is $\frac{dy}{dx}$? This process to find a derivative is known as logarithmic differentiation.

A. $\dfrac{dy}{dx} = x^x(\ln(x) + 1)$

B. $\dfrac{dy}{dx} = \dfrac{(\ln(x) + 1)}{x^x}$

C. $\dfrac{dy}{dx} = x^x(\ln(x) + x)$

D. $\dfrac{dy}{dx} = \dfrac{(\ln(x) + x)}{x^x}$

Activity 2.7.8 Valerie is building a square chicken coop with side length x. Because she needs a separate place for the rooster, she needs to put fence around the square and also along the diagonal line shown. The fence costs \$20 per linear meter, and she has a budget of \$900.

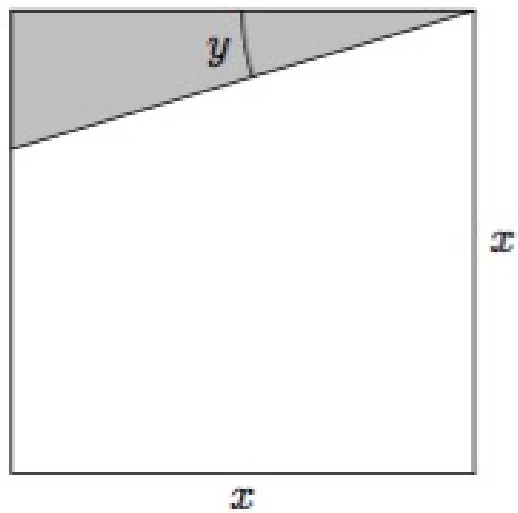


Figure 40 A diagram of the chicken coop.

(a) Which of the following equations gives the relationship between x and y? Make sure you can explain why!

A. $20x + \dfrac{80x}{\cos(y)} = 900$

B. $80x + \dfrac{20x}{\cos(y)} = 900$

C. $80x + \dfrac{20x}{\sin(y)} = 900$

D. $20x + \dfrac{80x}{\sin(y)} = 900$

(b) If Valerie builds the coop with $y = \pi/3$ (and wants to use her whole budget), find what side length x she needs to use.

(c) Find the slope of the curve at this point and interpret what this value tells Valerie.

2.8 Differentiating inverse functions (DF8)

Learning Outcomes

- Compute derivatives of inverse functions.

Differentiating inverse functions (DF8)

Remark 2.8.1 Let f^{-1} be the inverse function of f. The relationship between a function and its inverse can be expressed with the identity

$$f(f^{-1}(x)) = x.$$

Activity 2.8.2 In this activity you will use implicit differentiation and the inverse function identity in Remark 2.8.1 to find the derivative of $y = \ln(x)$.

(a) Suppose that $y = \ln(x)$. Then we have that

$$e^y = x.$$

Using implicit implicit differentiation, what do you get?

A. $\dfrac{dy}{dx} = \dfrac{x}{y}$

B. $\dfrac{dy}{dx} = \dfrac{1}{e^x}$

C. $\dfrac{dy}{dx} = \dfrac{x}{e^y}$

D. $\dfrac{dy}{dx} = \dfrac{1}{e^y}$

(b) Notice that we started with the relationship $e^y = x$. Use this to simplify $\frac{dy}{dx}$. You should get that when $y = \ln(x)$ we have that $\frac{dy}{dx} = \frac{1}{x}$... as expected!

Activity 2.8.3 In this activity we will try to find a general formula for the derivative of the inverse function. Let g be the inverse function of f. We have also used the notation f^{-1} before, but for the purpose of this problem, let us use g to avoid too many exponents. We can express the relationship "g is the inverse of f" with the equation from Remark 2.8.1

$$f(g(x)) = x.$$

(a) Looking at the equation $f(g(x)) = x$, what is the derivative with respect to x of the right hand side of the equation?

A. x

B. 1

C. 0

D. x^2

(b) Looking at the equation $f(g(x)) = x$, what is the derivative with respect to x of the left hand side of the equation?

A. $f'(g(x))$

B. $f'(g'(x))$

C. $f(g(x))\,g'(x)$

D. $f'(g(x))\,g'(x)$

(c) Setting the two sides of the equation equal after differentiating, we can solve for $g'(x)$. What do you get?

A. $g'(x) = \dfrac{x}{f(g(x))}$

B. $g'(x) = \dfrac{x}{f'(g(x))}$

C. $g'(x) = \dfrac{1}{f(g(x))}$

D. $g'(x) = \dfrac{1}{f'(g(x))}$

Remark 2.8.4 In the above activity you should have found that the derivative of $g = f^{-1}$, the inverse function of f, is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Notice that because of the chain rule, the derivative of f has to be evaluated at $\boldsymbol{f^{-1}(x)}$

Activity 2.8.5 In this problem you will apply the general formula for the derivative of the inverse function to find the values of some derivatives graphically.

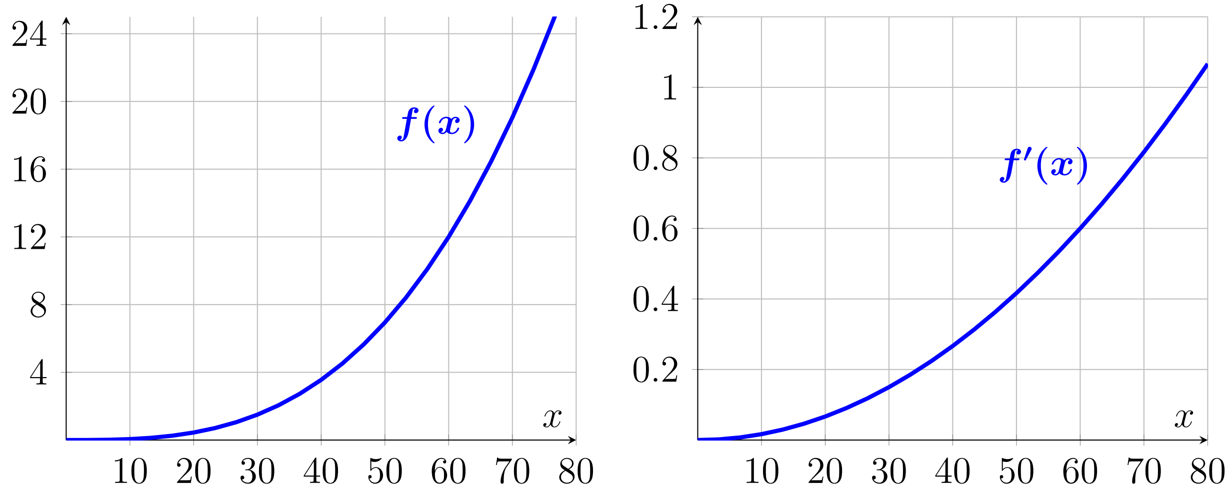


Figure 41 The graphs of $f(x)$ and $f'(x)$.

(a) The derivative of the inverse function at $x = 12$ given by $(f^{-1})'(12) = \frac{1}{f'(f^{-1}(12))}$. Using the graphs, what is your best approximation for this quantity?

A. $(f^{-1})'(12) \approx \dfrac{1}{0.2} = 5$

B. $(f^{-1})'(12) \approx \dfrac{1}{0.6} \approx 1.67$

C. $(f^{-1})'(12) \approx \dfrac{1}{0.4} = 2.5$

D. $(f^{-1})'(12) \approx \dfrac{1}{0.1} = 10$

(b) What is your best estimate for $(f^{-1})'(6)$?

A. $(f^{-1})'(6) \approx \dfrac{1}{0.2} = 5$

B. $(f^{-1})'(6) \approx \dfrac{1}{0.6} \approx 1.67$

C. $(f^{-1})'(6) \approx \dfrac{1}{0.4} = 2.5$

D. $(f^{-1})'(6) \approx \dfrac{1}{0.1} = 10$

Activity 2.8.6 Use the general formula for the derivative of the inverse function from Remark 2.8.4 to find...

(a) The derivative of the inverse function of $f(x) = e^x$... This should match the result of Activity 2.8.2!

(b) The derivative of the inverse function of $f(x) = \frac{1}{x}$... This should match a derative that you have seen before! See if you can explain why.

Definition 2.8.7 We can only invert the function $y = \sin(x)$ on the restricted domain $[-\pi/2, \pi/2]$ (Why?). On this domain we define arcsine by the condition

$$x = \sin^{-1}(y) \quad \text{when} \quad y = \sin(x).$$

$\diamond$

Activity 2.8.8 In this activity you will study the arcsine function.

(a) Consider the values of $y = \sin(x)$ given in the table below for an angle x between $-\pi/2$ and $\pi/2$. Fill in the corresponding values for the inverse function arcsine $x = \sin^{-1}(y)$. In other words, you need to provide the angle in $[-\pi/2, \pi/2]$ whose sine value is given. You can use the unit circle to help you remember which angles yield the given values of sine. The first entry is provided: a sine value of -1 corresponds to the angle $-\pi/2$.

Table 42

$y = \sin(x)$	-1	$-\sqrt{3}/2$	$-1/2$	0	$1/2$	$\sqrt{3}/2$	1
$x = \sin^{-1}(y)$	$-\pi/2$						

(b) From the graph of $y = \sin(x)$ and your table above, graph the arcsine function $y = \sin^{-1}(x)$

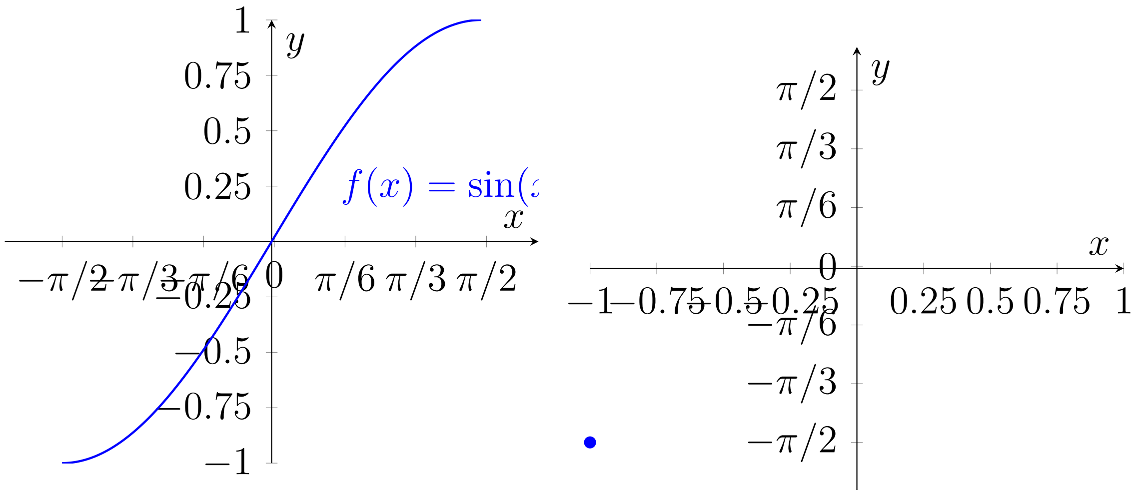


Figure 43 The graphs of $\sin(x)$ and one point on $\sin^{-1}(x)$.

(c) Let's now work with the function arccosine. Again, we need to restrict the domain of cosine to be able to invert the function (Why?). The convention is to restrict cosine to the domain $[0, \pi]$ in order to define arccosine. Given this restriction, what are the domain and range of arccosine? Create a table of values and graph the function arccosine.

(d) Let's now work with the function arctangent. Again, we need to restrict the domain of tangent to be able to invert the function (Why?). The convention is to restrict tangent to the domain $(-\pi/2, \pi/2)$ in order to define arctangent. Given this restriction, what are the domain and range of arctangent? Create a table of values and graph the function arctangent.

Activity 2.8.9 In this activity you will find a formula for the derivative of arctangent.

(a) Differentiate the implicit equation $\tan(y) = x$, what do you get for $\frac{dy}{dx}$?

A. $\dfrac{dy}{dx} = \dfrac{x}{\tan(y)}$

B. $\dfrac{dy}{dx} = \dfrac{1}{\tan(y)}$

C. $\dfrac{dy}{dx} = \dfrac{x}{\sec^2(y)}$

D. $\dfrac{dy}{dx} = \dfrac{1}{\sec^2(y)}$

(b) For what function $y = g(x)$ have you found the derivative $\frac{dy}{dx}$?

(c) We want to rewrite $\frac{dy}{dx}$ only in terms of x. Notice that

$$\tan^2(y) = \frac{\sin^2(y)}{\cos^2(y)} = \frac{1-\cos^2(y)}{\cos^2(y)}.$$

Multiplying out by the denominator, isolating, and solving for $\cos^2(y)$, we get that

A. $\cos^2(y) = \dfrac{\tan^2(y)}{\cos^2(y)}$

B. $\cos^2(y) = \dfrac{1}{\tan^2(y)+1}$

C. $\cos^2(y) = \dfrac{1-\cos^2(y)}{\tan^2(y)}$

D. $\cos^2(y) = \dfrac{1}{\tan^2(y)-1}$

(d) Finally, rewrite $\frac{dy}{dx}$ as $\frac{dy}{dx} = \cos^2(y)$ and use the fact that $\tan(y) = x$ to get a nice formula for the derivative of the arctangent function of x.

Remark 2.8.10 Consider the functions $y = \tan^{-1}(x)$. Using your algebra above, you should have found that

$$\frac{d}{dx}\Big(\tan^{-1}(x)\Big) = \frac{1}{1+x^2}.$$

In a similar fashion, one can find that

$$\frac{d}{dx}\Big(\sin^{-1}(x)\Big) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\Big(\cos^{-1}(x)\Big) = -\frac{1}{\sqrt{1-x^2}}.$$

Activity 2.8.11 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (product, quotient, sum and difference, etc.) you are using in your work.

(a)

$$k(t) = \frac{\arctan(-4\,t)}{\ln(-4\,t)}$$

(b)

$$j(u) = -5\,\arcsin(u)\log\left(u^6 + 2\right)$$

(c)

$$n(x) = \ln\left(-\arcsin(x) + 4\,\arctan(x)\right)$$

Answer.

1.

$$k'(t) = -\frac{\arctan(-4\,t)}{t\log(-4\,t)^2} - \frac{4}{(16\,t^2 + 1)\log(-4\,t)}$$

2.

$$j'(u) = -\frac{30\,u^5\arcsin(u)}{u^6 + 2} - \frac{5\,\log\left(u^6 + 2\right)}{\sqrt{-u^2 + 1}}$$

3.

$$n'(x) = \frac{\frac{1}{\sqrt{-x^2+1}} - \frac{4}{x^2+1}}{\arcsin(x) - 4\,\arctan(x)}$$

Activity 2.8.12

(a) Find the equation of the tangent line to $y = \tan^{-1}(x)$ at $x = 0$. Draw the function and the tangent on a graphing calculator to check your work!

(b) Find the equation of the tangent line to $y = \sin^{-1}(x)$ at $x = 0.5$. Draw the function and the tangent on a graphing calculator to check your work!

(c) Find the equation of the tangent line to $y = \cos^{-1}(x)$ at $x = -0.5$. Draw the function and the tangent on a graphing calculator to check your work!

Activity 2.8.13 Let $y = f(v)$ be the gas consumption (in ml/km) of a car at velocity v (in km/hr). We use the notation: ml for milliliters, km for kilometers, and hr for hours. Also consider the function $g(y)$, where $v = g(y)$ is the function that gives the velocity v (in km/hr) when the gas consumption is y (in ml/km). You are given the graphs of $f(v)$, $f'(v)$ below.

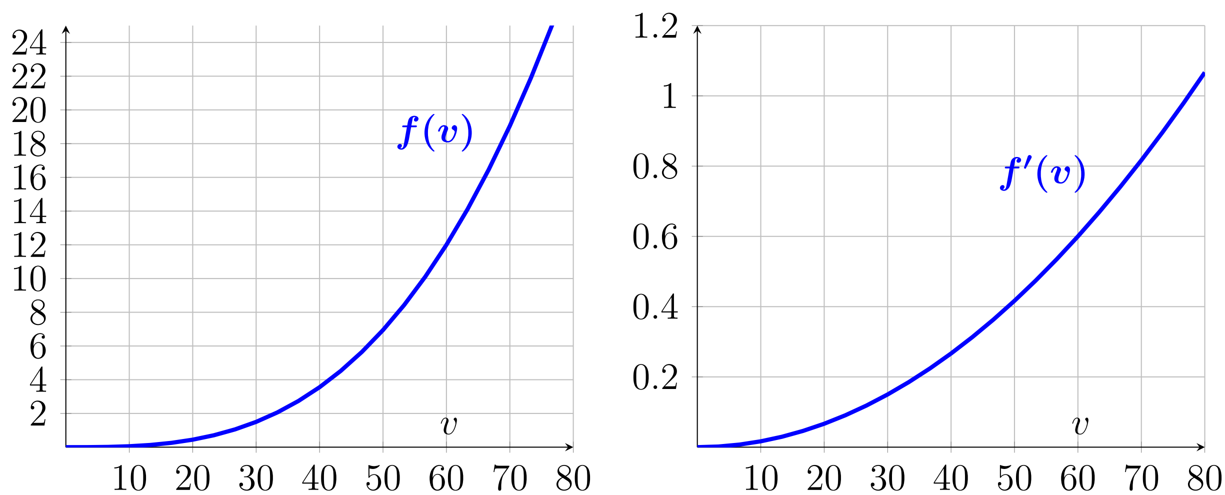


Figure 44 The graphs of $f(v)$, $f'(v)$.

(a) Estimate $f^{-1}(6)$. What does this value mean in the context of the problem?

(b) Using your answer from part (a), estimate the derivative of the inverse function of $f(x)$ at $x = 6$ i.e., compute $(f^{-1})'(6)$.

(c) What is the relationship between the functions f and g?

(d) Use the relationship between the functions f and g to estimate $g(12)$ and $g'(12)$. What do these values mean in the context of the problem?

Chapter 3

Applications of Derivatives (AD)

Learning Outcomes

How can we use derivatives to solve application questions?
By the end of this chapter, you should be able to...

1. Use derivatives to answer questions about rates of change and equations of tangents.
2. Use tangent lines to approximate functions.
3. Model and analyze scenarios using related rates.
4. Use the Extreme Value Theorem to find the global maximum and minimum values of a continuous function on a closed interval.
5. Determine where a differentiable function is increasing and decreasing and classify the critical points as local extrema.
6. Determine the intervals of concavity of a twice differentiable function and find all of its points of inflection.
7. Sketch the graph of a differentiable function whose derivatives satisfy given criteria.
8. Apply optimization techniques to solve various problems.
9. Compute the values of indeterminate limits using L'Hôpital's Rule.

3.1 Tangents, motion, and marginals (AD1)

Learning Outcomes

- Use derivatives to answer questions about rates of change and equations of tangents.

Definition 3.1.1 The *tangent line* of a function $f(x)$ at $x = a$ is the linear function $L(x)$

$$L(x) = f'(a)(x - a) + f(a).$$

Notice that this is the linear function with slope $f'(a)$ and passing through $(a, f(a))$ in point-slope form. $\Diamond$

Activity 3.1.2 For the following functions, find the required tangent line.

(a) Find the tangent line to $f(x) = \ln(x)$ at $x = 1$

A. $L(x) = x$

B. $L(x) = x + 1$

C. $L(x) = x - 1$

D. $L(x) = -x + 1$

(b) Find the tangent line to $f(x) = e^x$ at $x = 0$

A. $L(x) = x$

B. $L(x) = x + 1$

C. $L(x) = x - 1$

D. $L(x) = -x + 1$

Activity 3.1.3 Let $f(x) = -2\,x^4 + 4\,x^2 - x + 5$. Find an equation of the line tangent to the graph at the point $(-2, -9)$.

Definition 3.1.4 If a particle has position function $s = f(t)$, where t is measured in seconds and s is measured in meters, then the derivative of the position function tells us how the position is changing over time, so $f'(t)$ gives us the (instantenous) velocity in meters per second. Also, the derivative of the velocity gives us the change in velocity over time, so so $f''(t)$ gives us the (instantenous) acceleration in meters per second squared. Summarizing,

- $v(t) = f'(t)$ is the velocity of the particle in m/s.
- $a(t) = f''(t)$ is the acceleration of the particle in m/s^2.

$\diamondsuit$

Activity 3.1.5 A particle moves on a vertical line so that its y coordinate at time t is

$$y = t^3 - 9t^2 + 24t + 3$$

for $t \geq 0$. Here t is measured in seconds and y is measured in feet.

(a) Find the velocity and acceleration functions.

(b) Sketch graphs of the position, velocity and acceleration functions for $0 \leq t \leq 5$.

(c) When is the particle moving upward and when is it moving downward?

(d) When is the particle's velocity increasing?

(e) Find the total distance that the particle travels in the time interval $0 \leq t \leq 5$. Careful: the total distance is not the same as the displacement (the change in position)! Compute how much the particle moves up and add it to how much the particle moves down.

Activity 3.1.6 Suppose the position of an object in miles is modeled by the following function:

$$s(t) = -t^3 - 3\,t^2 - 5\,t + 8.$$

Explain and demonstrate how to find the object's position, velocity, and acceleration at 2 seconds. Use appropriate units for each.

Observation 3.1.7 In some cases, we want to also consider the speed of a particle, which is the absolute value of the velocity. In symbols $|v(t)| = |f'(t)|$ is the speed of the particle. A particle is speeding up when the speed is increasing.

Activity 3.1.8 Consider the speed of a particle. What is the behavior of the speed in relation to velocity and acceleration?

A. The speed is always positive and it is increasing when the velocity and the acceleration have the same sign.

B. The speed is positive when the velocity is positive and negative when the velocity is negative.

C. The speed is positive when the acceleration is positive and negative when the acceleration is negative.

D. The speed is always positive and it is increasing when the velocity and the acceleration have opposite signs.

Definition 3.1.9 In a parametric motion on a curve C given by $x = f(t)$ and $y = g(t)$ we have that

- $\frac{dx}{dt} = f'(t)$ is the rate of change of $f(t)$, one component of the slope (or velocity)
- $\frac{dy}{dt} = g'(t)$ is the rate of change of $g(t)$, one component of the slope (or velocity)
- $\frac{dy}{dx}$ is the actual slope (or velocity) of the object and by the chain rule $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

$\lozenge$

Activity 3.1.10 An airplane is cruising at a fixed height and traveling in a pattern described by the parametric equations

$$x = 4t, \quad y = -t^4 + 4t - 1,$$

where x, y have units of miles, and t is in hours.

(a) Find the slope of the curve.

(b) What is the slope of the curve at $(0, -1)$.

(c) Write the equation of the tangent line to the curve at $(0, -1)$.

Definition 3.1.11 If $C(x)$ is the cost of producing x items and $R(x)$ is the revenue from selling x items, then $P(x) = R(x) - C(x)$ is the profit. We can study their derivatives, the marginals

- $C'(x)$ is the marginal cost, the rate of change of the cost per unit change in production;
- $R'(x)$ is the marginal revenue, the rate of change of the revenue per unit change in sales;
- $P'(x) = R'(x) - C'(x)$ is the marginal profit, the rate of change of the profit per unit change in sales (assuming we are selling all the items produced).

$\diamondsuit$

Activity 3.1.12 The manager of a computer shop has to decide how many computers to store in the back of the shop. If she stores a large number, she has to pay extra in storage costs. If she stores only a small number, she will have to reorder more often, which will involve additional handling costs. She has found that if she stores x computers, the storage and handling costs will be C dollars, where

$$C(x) = 10x^3 - 900x^2 + 16000x + 210000$$

(a) What is the fixed cost of the computer shop, the cost when no computers are in storage? In practical terms this may account for rent and utilities expenses.

(b) Find the marginal cost

(c) Now suppose that x computers give revenue $R(x) = 1000x$. What is the marginal revenue? What is the real world interpretation of your finding?

(d) Find a formula for the profit function $P(x)$ and find the marginal profit using the marginal revenue and the marginal cost (assuming the number of items produced and sold is equal and given by x).

Activity 3.1.13 A gizmo is sold for \$63 per item. Suppose that the number of items produced is equal to the number of items sold and that the cost (in dollars) of producing x gizmos is given by the following function:

$$C(x) = 4\,x^3 + 10\,x^2 + 7\,x + 4.$$

Explain and demonstrate how to find the marginal revenue, the marginal cost, and the marginal profit in this situation.

Definition 3.1.14 A cooling object has temperature modelled by

$$y = ae^{-kt} + c,$$

where a, c, k are positive constants determined by the local conditions. $\Diamond$

Activity 3.1.15 Consider a cup of coffee initially at 100°F. The said cup of coffee was forgotten this morning in my living room where the thermostat is set at 72°F. I also observed that when I initially prepared the coffee, the temperature was decreasing at a rate of 3.8 degrees per minute.

(a) In the long run, what temperature do you expect the coffee to tend to? Use this information in the model $y = ae^{-kt} + c$ to determine the value of c.

(b) Using the initial temperature of the coffee and your value of c, find the value of a in the model $y = ae^{-kt}t + c$.

(c) The scenario also gives you information about the value of the rate of change at $t = 0$. Use this additional information to determine the model $y = ae^{-kt}t + c$ completely.

(d) You should find that the temperature model for this coffee cup is $y = 72 + 38e^{-0.1t}$. Explain how the values of each parameter connects to the information given.

3.2 Linear approximation (AD2)

Learning Outcomes

- Use tangent lines to approximate functions.

Linear approximation (AD2)

Definition 3.2.1 The *linear approximation* (or tangent line approximation or linearization) of a function $f(x)$ at $x = a$ is the tangent line $L(x)$ at $x = a$. In formulas, $L(x)$ is the linear function

$$L(x) = f'(a)(x - a) + f(a).$$

Notice that this is obtained by writing the tangent line to $f(x)$ at $(a, f(a))$ in point-slope form and calling the resulting linear function $L(x)$. The linear approximation $L(x)$ is a linear function that looks like $f(x)$ when we zoom in near $x = a$. $\diamondsuit$

Activity 3.2.2 Without using a calculator, we will use calculus to approximate $\ln(1.1)$.

(a) Find the equation of the tangent line to $\ln(x)$ at $x = 1$. This will be your linear approximation $L(x)$. What do you get for $L(x)$?

A. $L(x) = x$

B. $L(x) = x + 1$

C. $L(x) = x - 1$

D. $L(x) = -x + 1$

(b) As 1.1 is close to 1, we can use $L(1.1)$ to approximate $\ln(1.1)$. What approximation do you get?

A. $\ln(1.1) \approx 1.1$

B. $\ln(1.1) \approx 2.1$

C. $\ln(1.1) \approx 0.1$

D. $\ln(1.1) \approx -0.1$

(c) Sketch the tangent line $L(x)$ on the same plane as the graph of $\ln(x)$. What do you notice?

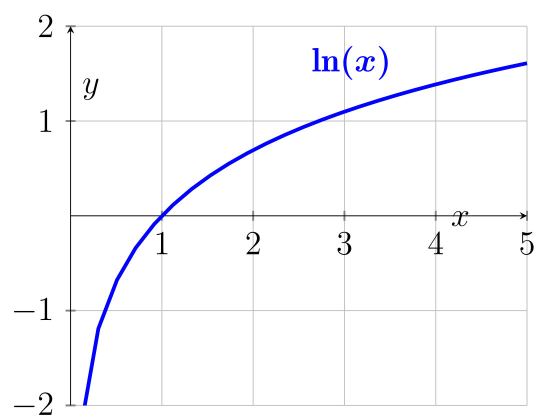


Figure 45 The graph of $\ln(x)$

Activity 3.2.3 Using the equation of the tangent line to the graph of $\ln(x)$ at $x = 1$ and the shape of this graph, you can show that for all values of x, we have that $\ln(x) \le x - 1$.

(a) Compute the second derivative of $\ln(x)$. What do you notice about the sign of the second derivative of $\ln(x)$? What does this tell you about the shape of the graph?

(b) Conclude that because the graph of $\ln(x)$ has a certain shape, the graph will bend below the tangent line and so that $\ln(x)$ will always be smaller than the tangent line approximation $L(x) = x - 1$.

Activity 3.2.4 In this activity you will approximate power functions near $x = 1$.

(a) Find the tangent line approximation to x^2 at $x = 1$.

A. $L(x) = 2x$

B. $L(x) = 2x + 1$

C. $L(x) = 2x - 1$

D. $L(x) = -2x + 1$

(b) Show that for any constant k, the tangent line approximation to x^k at $x = 1$ is $L(x) = k(x - 1) + 1$.

(c) Someone claims that the square root of 1.1 is about 1.05. Use the linear approximation to check this estimate. Do you think this estimate is about right? Why or why not?

(d) Is the actual value $\sqrt{1.1}$ above or below 1.05? What feature of the graph of $\sqrt{x}$ makes this an over or under estimate?

Remark 3.2.5 If a function $f(x)$ is *concave up* around $x = a$, then the function is turning upwards from its tangent line. So when we use a linear approximation, the value of the approximation will be below the actual value of the function and the approximation is an underestimate. If a function $f(x)$ is *concave down* around $x = a$, then the function is turning downwards from its tangent line. So when we use a linear approximation, the value of the approximation will be above the actual value of the function and the approximation is an overestimate.

Linear approximation (AD2)

Activity 3.2.6 Suppose f has a continuous positive second derivative and Δx is a small increment in x (like h in the limit definition of the derivative). Which one is larger...

$$f(1 + \Delta x) \quad \text{or} \quad f'(1)\Delta x + f(1) \quad ?$$

Activity 3.2.7 A certain function $p(x)$ satisfies $p(7) = 49$ and $p'(7) = 8$.

1. Explain how to find the local linearization $L(x)$ of $p(x)$ at 7.
2. Explain how to estimate the value of $p(6.951)$.
3. Suppose that $p'(7) = 0$ and you know that $p''(x) < 0$ for $x < 7$. Explain how to determine if your estimate of $p(6.951)$ is too large or too small.
4. Suppose that $p''(x) > 0$ for $x > 7$. Use this fact and the additional information above to sketch an accurate graph of $y = p(x)$ near $x = 7$.

Answer.

1. $L(x) = 8\,x - 7$
2. $p(6.951) \approx 48.6064$
3. The estimate is too large.

Activity 3.2.8 Let's find the quadratic polynomial

$$q(x) = ax^2 + bx + c$$

where a, b, c are parameters to be determined so that $q(x)$ best approximates the graph of $f(x) = \ln(x)$ at $x = 1$.

(a) We want to choose a, b, c such that our quadratic polynomial resembles $f(x)$ at $x = 1$. First thing, we want $f(1) = q(1)$. What equation in a, b, c does this condition give you?

A. $a + b + c = 1$

B. $a + b + c = 0$

C. $c = 0$

D. $c = 1$

(b) We also want $f'(1) = q'(1)$. What equation in a, b, c does this condition give you?

(c) Finally, we want $f''(1) = q''(1)$. What equation in a, b, c does this condition give you?

(d) Find a solution to this system of linear equations! Your answer will give you values of a, b, c that can be used to draw a quadratic approximating the natural logarithm. You can check your answer on Desmos https://www.desmos.com/calculator/bad2xrwmvl

Observation 3.2.9 A linear approximation $L(x)$ to $f(x)$ at $x = a$ is a linear function with

$$L(a) = f(a), \quad L'(a) = f'(a).$$

A quadratic approximation $Q(x)$ to $f(x)$ at $x = a$ is a quadratic function with

$$Q(a) = f(a), \quad Q'(a) = f'(a), \quad Q''(a) = f''(a).$$

Activity 3.2.10 Find the linear approximation $L(x)$ of $\cos(x)$ at $x = 0$. Then find the quadratic approximation $Q(x)$ of $\cos(x)$ at $x = 0$. Graph both and compare the two approximations!

Linear approximation (AD2)

Activity 3.2.11 Suppose the function $p(x)$ satisfies $p(-2) = 5$, $p'(-2) = 1$, and $p''(x) < 0$ for x values nearby -2.

(a) Explain and demonstrate how to find the linearization $L(x)$ of $p(x)$ at $x = -2$.

(b) Explain and demonstrate how to estimate the value of $p(-2.03)$ using this linearization.

(c) Explain why your estimate of $p(-2.03)$ is greater than or less than the actual value.

(d) Sketch a possible graph of $p(x)$ and its linearization $L(x)$ nearby $x = -2$ to illustrate your findings.

3.3 Related rates (AD3)

Learning Outcomes

- Model and analyze scenarios using related rates.

Remark 3.3.1 So far we have been interested in the instantaneous rate at which one variable, say y, changes with respect to another, say x, leading us to compute and interpret $\frac{dy}{dx}$. We also have situations where several variables change together and often each quantity is a function of time, represented by the variable t. Knowing how the quantities are related, we will determine how their rates of change with respect to time are related.

Example 3.3.2 In a sense, the chain rule is our first example of related rates: recall that when y is a function of x, which in turn is a function of t, we are considering the composite function $y(x(t))$, and we learned that by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Notice that the chain rule gives a relationship between three rates: $\frac{dy}{dt}, \frac{dy}{dx}, \frac{dx}{dt}$. $\square$

Related rates (AD3)

Activity 3.3.3 Remember the squirrels taking over my neighborhood? The population s grows based on acorn availability a, at a rate of 2 squirrels per bushel. The acorn availability a is currently growing at a rate of 100 bushels per week. What is $\frac{ds}{dt}$ in this situation?

A. 2

B. 100

C. 200

D. Not enough information

Example 3.3.4 In a more serious example, suppose that air is being pumped into a spherical balloon so that its volume increases at a constant rate of 20 cubic inches per second. Since the balloon's volume and radius are related, by knowing how fast the volume is changing, we ought to be able to discover how fast the radius is changing. Can we determine how fast is the radius of the balloon increasing when the balloon's diameter is 12 inches? $\square$

Activity 3.3.5 A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why? Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note as well that in the setting of this problem, *both V and r* are changing with time t. Hence both V and r may be viewed as *implicit* functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$. Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.At this point in the problem, by differentiating we have "related the rates" of change of V and r. Recall that we are given in the problem that the balloon is being inflated at a constant *rate* of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?From part (c), we know the value of $\frac{dV}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

Remark 3.3.6 In problems where two or more quantities are related to one another, like in the case that all of the variables involved are functions of time t, we are interested in finding out how their rates of change are related; we call these *related rates* problems. Once we have an equation establishing the relationship among the variables, we differentiate the equation, usually implicitly with respect to time, to find connections among the rates of change.

Remark 3.3.7 A guide to solving related rated problems.

1. *Picture it!* Draw a diagram to represent the situation.

2. *What do we know?* Make a list of all quantities you are given in the problem, choosing clearly defined variable names for them. If a quantity is changing (a rate), then it should be labeled as a derivative.

3. *What do we want to know?* Make a list of all quantities to be determined. Again, choose clearly defined variable names.

4. *How are the variables related to each other?* Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.

5. *How are the rates related?* Differentiate implicitly with respect to time. This will give an equation that relates the rates together.

6. *Time to evaluate!* Evaluate the derivatives and variables at the information relevant to the instant at which a certain rate of change is sought.

Remark 3.3.8 Volume formulas.

- A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$
- A vertical cylinder of radius r and height h has volume $V = \pi r^2 h$
- A cone of radius r and height h has volume $V = \frac{\pi}{3} r^2 h$

Activity 3.3.9 A vertical cylindrical water tank has a radius of 1 meter. If water is pumped out at a rate of 3 cubic meters per minute, at what rate will the water level drop?

(a) Draw a figure to represent the situation. Introduce variables that measure the radius of the water's surface, the water's depth in the tank, and the volume of the water. Label your diagram.

(b) What information about rates of changes does the problem give you?

(c) Recall that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$. What is the related rates equation in the context of the vertical cylindrical tank? What derivative rules did you use to find this equation?

A. $\frac{dV}{dt} = \pi 2r\frac{dh}{dt}$

B. $\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$

C. $\frac{dV}{dt} = \pi\frac{dr}{dt}h$

D. $\frac{dV}{dt} = \pi 2r\frac{dr}{dt}h + \pi r^2\frac{dh}{dt}$

E. $\frac{dV}{dt} = \pi 2rh + \pi r^2$

(d) Which variable(s) have a constant value in this situation? Why?

A. The variable measuring the radius of the water's surface

B. The variable measuring the depth of the water

C. The variable measuring the volume of the water

(e) Which variable(s) have a constant rate of change in this situation? Why?

A. The variable measuring the radius of the water's surface

B. The variable measuring the depth of the water

C. The variable measuring the volume of the water

(f) Using your finding above, find at what rate the water level is dropping.

(g) If the full tank contains 12 cubic meters of water, how long does it take to empty the tank?

(h) Confirm your finding in the previous part by finding the initial water level for 12 cubic centimeters of water and determine how long it takes for the water level to reach 0.

Activity 3.3.10 A water tank has the shape of an inverted circular cone (the cone points downwards) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.

(b) Say that r is the radius and h the depth of the water at a given time, t. Notice that at any point of time there is a fixed proportion between the depth and the radius of the volume of water, forced by the shape of the tank. What proportional equation relates the radius and height of the water, and why?

(c) Determine an equation that gives the volume of water in the tank as a function of only the depth h of the water (so eliminate the radius from the volume equation using the previous part).

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time t.

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

(f) When is the water rising most rapidly?

A. $h = 3$

B. $h = 4$

C. $h = 5$

D. The water level rises at a constant rate

Related rates (AD3)

Activity 3.3.11 Recall that in a right triangle with sides a, b and hypotenuse c we have the relationship

$$a^2 + b^2 = c^2,$$

also known in the western world as the Pythagorean theorem (even though this result was well know well before his time by other civilizations).

(a) Notice that by differentiating the equation above with respect to t we get a relationship between $a, b, c, \frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}$. Find this related rates equation.

(b) A rectangle has one side of 8 cm. How fast is the diagonal of the rectangle changing at the instant when the other side is 6 cm and increasing at a rate of 3 cm per minute?

(c) A 10 m ladder leans against a vertical wall and the bottom of the ladder slides away at a rate of 0.5 m/sec. When is the top of the ladder sliding the fastest down the wall?

A. When the bottom of the ladder is 4 meters from the wall.

B. When the bottom of the ladder is 8 meters from the wall.

C. The top of the ladder is sliding down at a constant rate.

Activity 3.3.12 Suppose a car was 75 miles east of a town, traveling west at 75 mph. A second car was 120 miles north of the same town, traveling south at 70 mph. At this exact moment, how fast is the distance between the cars changing?

3.4 Extreme values (AD4)

Learning Outcomes

- Use the Extreme Value Theorem to find the global maximum and minimum values of a continuous function on a closed interval.

Extreme values (AD4)

Remark 3.4.1 In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications—say to identify a point at which maximum profit or minimum cost occurs—or in theory to characterize the behavior of a function or a family of related functions.

Example 3.4.2 Consider the familiar example of a parabolic function such as $s(t) = -16t^2+32t+48$. This function represents the height of an object tossed vertically straight up: its maximum value occurs at the vertex of the parabola and represents the greatest height the object reaches. This maximum value is an especially important point on the graph and we can notice that the function changes from increasing to decreasing at this point.

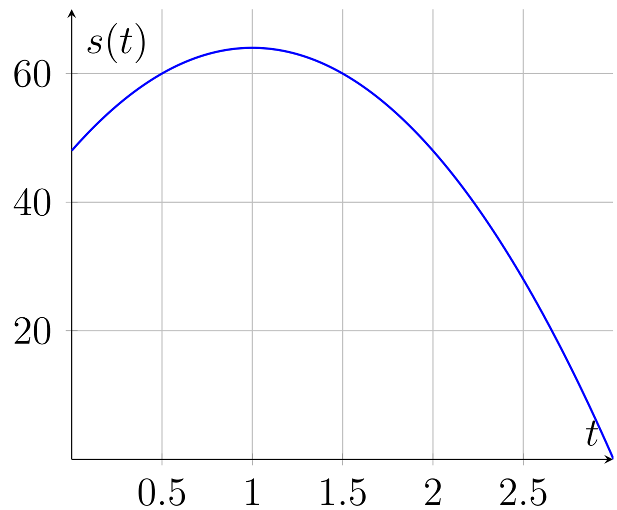


Figure 46 The graph of $s(t) = -16t^2 + 32t + 48$

$\square$

Extreme values (AD4)

Definition 3.4.3 We say that $f(x)$ has a **global maximum** at $x = c$ provided that $f(c) \geq f(x)$ for all x in the domain of the function. We also say that $f(c)$ is a global maximum value for the function. On the other hand, we say that $f(x)$ has a **global minimum** at $x = c$ provided that $f(c) \leq f(x)$ for all x in the domain of the function. We also say that $f(c)$ is a global minimum value for the function. The global maxima and minima are also known as the **global extrema** (or extreme values or absolute extrema) of the function. $\Diamond$

Activity 3.4.4 According to Definition 3.4.3, which of the following statements best describes the global extrema of the function in Figure 64?

A. The global maximum is $t = 1$, because this is where the function goes from increasing to decreasing.

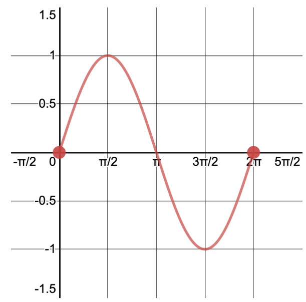
B. The global maximum is $s(1) = 64$, because $s(t) \leq 64$ for every other input t.

C. The graph has two global minima at the endpoints because the endpoints must be global extrema.

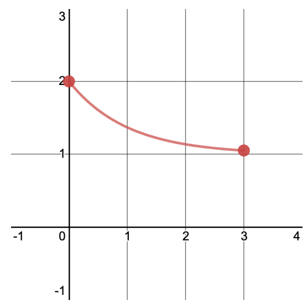
D. The graph has no global minimum.

Observation 3.4.5 There can be some issues when trying to determine the global mimimum and maximum values of a function only using its graph. The Extreme Value Theorem will guarantee the existence of global extrema on a closed interval. Then we will see how to use derivatives to find algebraically the extrema of a function.

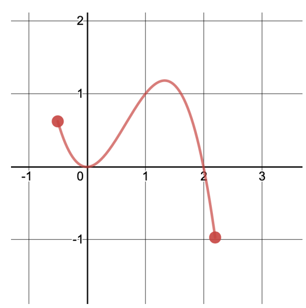
Activity 3.4.6 For each of the following figures, decide where the global extrema are located.



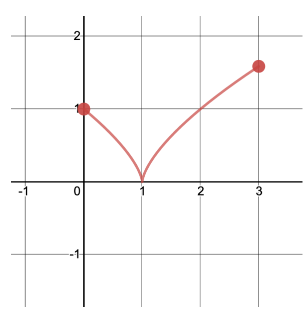
(a) **Figure 47**



(b) **Figure 48**



(c) **Figure 49**



(d) **Figure 50**

Activity 3.4.7 The Extreme Value Theorem (EVT) guarantees a global maximum and a global minumum for which of the following?

A. $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[-5, 0]$.

B. $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[0, 4]$.

C. $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[4, 6]$.

D. $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[6, 10]$.

Activity 3.4.8 For the following activity, draw a sketch of a function that has the following properties.

(a) The function is continuous and has an global minimum but no global maximum.

(b) The function is continuous and has an global maximum but no global minimum.

Definition 3.4.9 We say that $x = c$ is a **critical point** (or critical number) of $f(x)$ if $x = c$ is in the domain of $f(x)$ and either $f'(c) = 0$ or $f'(c)$ does not exist. $\diamondsuit$

Activity 3.4.10 Which of the following are critical numbers for $f(x) = \frac{1}{3}x^3 - 2x + 2$?

A. $x = \sqrt{2}$ and $x = -\sqrt{2}$.

B. $x = \sqrt{2}$.

C. $x = 2$ and $x = 0$.

D. $x = 2$.

Remark 3.4.11 The Closed Interval Method. The following is a way of finding the global extrema of a continuous function f on a closed interval $[a, b]$.

1 Make a list of all critical points of f in (a, b). (Do not include any critical points outside of the interval).

2 Add the endpoints a and b to the list.

3 Evaluate f at all points on your list.

4 The smallest output occurs at the global minimum. The largest output occurs at the global maximum.

Activity 3.4.12 What are the global extrema for $f(x) = 3x^4 - 4x^3$ on $[-1, 2]$.

A. Global maximum is when $x = 0$ and global minimum when $x = 1$.

B. Global maximum is when $x = 2$ and global minimum when $x = -1$.

C. Global maximum is when $x = 2$ and global minimum when $x = 1$.

D. Global maximum is when $x = 0$ and global minimum when $x = -1$.

Activity 3.4.13 What are the global extrema for $f(x) = x\sqrt{4-x}$ on $[-2, 4]$.

A. Global maximum is when $x = -2$ and global minimum when $x = \frac{8}{3}$.

B. Global maximum is when $x = 4$ and global minimum when $x = \frac{8}{3}$.

C. Global maximum is when $x = \frac{8}{3}$ and global minimum when $x = -2$.

D. Global maximum is when $x = 4$ and global minimum when $x = -2$.

Activity 3.4.14 Explain how to find the global minimum and global maximum values of the function $f(x) = -2\,x^3 + 18\,x^2 + 42\,x + 33$ on the interval $[-2, 2]$.

Activity 3.4.15 In this problem you will consider the function $g(x)$.

$$g(x) = \begin{cases} x^3 - 3x & x < 0 \\ x^2 - 4x + 2 & x \geq 0 \end{cases}$$

(a) What can you say about the point $x = 0$?

(b) In addition to $x = 0$, find the other two critical points. What are the critical points of $g(x)$?

A. $x = 0$, $x = 1$, $x = 2$

B. $x = 0$, $x = -1$, $x = 2$

C. $x = 0$, $x = -1$, $x = -2$

D. $x = 0$, $x = 1$, $x = -2$

(c) Can you use the Closed Interval Method on $[-4, -1]$? If you can, find the global max and min. If you can't, explain why.

(d) Can you use the Closed Interval Method on $[1, 4]$? If you can, find the global max and min. If you can't, explain why.

(e) Can you use the Closed Interval Method on $[-1, 1]$? If you can, find the global max and min. If you can't, explain why.

3.5 Derivative tests (AD5)

Learning Outcomes

- Determine where a differentiable function is increasing and decreasing and classify the critical points as local extrema.

Definition 3.5.1 We say that $f(x)$ has a **local maximum** at $x = c$ provided that $f(c) \geq f(x)$ for all x near c. We also say that $f(c)$ is a local maximum value for the function. On the other hand, we say that $f(x)$ has a **local minimum** at $x = c$ provided that $f(c) \leq f(x)$ for all x near c. We also say that $f(c)$ is a local minimum value for the function. The local maxima and minima are also known as the *local extrema* (or relative extrema) of the function. $\diamondsuit$

Observation 3.5.2 To find the extreme values of a function we can consider all its *local extrema* (local maxima and minima) and study them to find which one(s) give the largest and smallest values on the function. But how do you find the local/relative extrema? We will see that we can detect local extrema by computing the first derivative and finding the critical points of the function. By finding the critical points, we will produce a list of candidates for the extrema of the function.

Activity 3.5.3 We have encountered several terms recently, so we should make sure that we understand how they are related. Which of the following statements are true?

A. In a closed interval an endpoint is always a local extrema but it might or might not be a global extrema.

B. In a closed interval an endpoint is always a global extrema.

C. A crtical point is always a local extrema but it might or might not be a global extrema.

D. A local extrema only occurs where the first derivative is equal to zero.

E. A local extrema always occurs at a critical point.

F. A local extrema might occur at a critical point or at an endpoint of a closed interval.

Activity 3.5.4

(a) Sketch the graph of a continuous function that is increasing on $(-\infty, -2)$, constant on the interval $(3, 5)$, and decreasing on the interval $(-2, 3)$.

(b) How would you describe the derivative of the function on each interval?

A. For $x < -2$ we have $f'(x) < 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x) > 0$.

B. For $x < -2$ we have $f'(x) > 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x)$ is undefined.

C. For $x < -2$ we have $f'(x) > 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x) = 0$.

D. For $x < -2$ we have $f'(x) < 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x)$ is constant.

Activity 3.5.5 Look back at the graph you made for Activity 3.5.4.

Which of the following best describes what is occurring when graph changes behavior?

A. There is a critical point.

B. There is a local maximum or minimum.

C. The derivative is undefined.

D. The derivative is equal to zero.

Observation 3.5.6 Critical points detect changes in the behavior of a function. We will use critical points as ”break points” in studying the behavior of a function. To understand what happens at the critical points we use the Derivative Tests.

Activity 3.5.7 Let $f(x) = x^4 - 4x^3 + 4x^2$

(a) Find all critical points of $f(x)$. Draw them on the same number line.

(b) What intervals have been created by subdividing the number line at the critical points?

(c) Pick an x-value that lies in each interval. Determine whether $f'(x)$ is positive or negative at each point.

(d) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?

(e) List all local extrema.

Activity 3.5.8 Consider the function $f(x) = -x^3 + 3\,x + 4$.

(a) Find the open intervals where $f(x)$ is increasing or decreasing.

(b) Find the local extrema of $f(x)$.

Remark 3.5.9 Dealing with discontinuities. Our previous activity dealt with a function that was continuous for all real numbers. Because of that, we could trust our chart to point out local extrema. Let's now consider what might happen if a function has any discontinuities.

Activity 3.5.10 Draw a function that is increasing on the left of $x = 1$, discontinuous at $x = 1$, such that $f(1) = \lim_{x \to 1^+} f(x)$, and decreasing to the right of $x = 1$. Does the derivative of $f(x)$ exist at $x = 1$? Does your graph have a local maximum or minimum at $x = 1$?

Activity 3.5.11 Let $f(x) = \frac{x}{(x-2)^2}$.

(a) Note that $f(x)$ is not defined for $x = 2$. But the function may be increasing on one side of $x = 2$ and decreasing on the other! So we include $x = 2$ on your number line.

(b) Find all critical points of $f(x)$. Plot them and any discontinuities for $f(x)$ on the same number line.

(c) What intervals have been created by subdividing the number line at the critical points and at the discontinuities?

(d) Pick an x-value that lies in each interval. Determine whether $f'(x)$ is positive or negative each point.

(e) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?

(f) List all local maxima and local minima.

Derivative tests (AD5)

Activity 3.5.12 For each of the following functions, find the intervals on which $f(x)$ is increasing or decreasing. Then identify any local extrema using either the First or Second Derivative Test.

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = \frac{1}{2}x + \cos x$ on $(0, 2\pi)$

(c) $f(x) = (x^2 - 9)^{2/3}$

(d) $f(x) = \ln(2x - 1)$. (Hint: think about the domain of this one before you get started!)

(e) $f(x) = \frac{x^2}{x^2-4}$

Activity 3.5.13

(a) Suppose f is continuous and differentiable on $[a, b]$ and also suppose that $f(a) = f(b)$. What is the average rate of change of $f(x)$ on $[a, b]$? What does the MVT (Mean Value Theorem) tell you?

(b) Use part (a) to show with the MVT that $f(x) = (x-1)^2 + 3$ has a critical point on $[0, 2]$.

3.6 Concavity and inflection (AD6)

Learning Outcomes

- Determine the intervals of concavity of a twice differentiable function and find all of its points of inflection.

Concavity and inflection (AD6)

Observation 3.6.1 In addition to asking *whether* a function is increasing or decreasing, it is also natural to inquire *how* a function is increasing or decreasing. Activity 3.6.2 describes three basic behaviors that an increasing function can demonstrate on an interval, as pictured in Figure 71

Activity 3.6.2 Sketch a sequence of tangent lines at various points to each of the following curves in Figure 71.

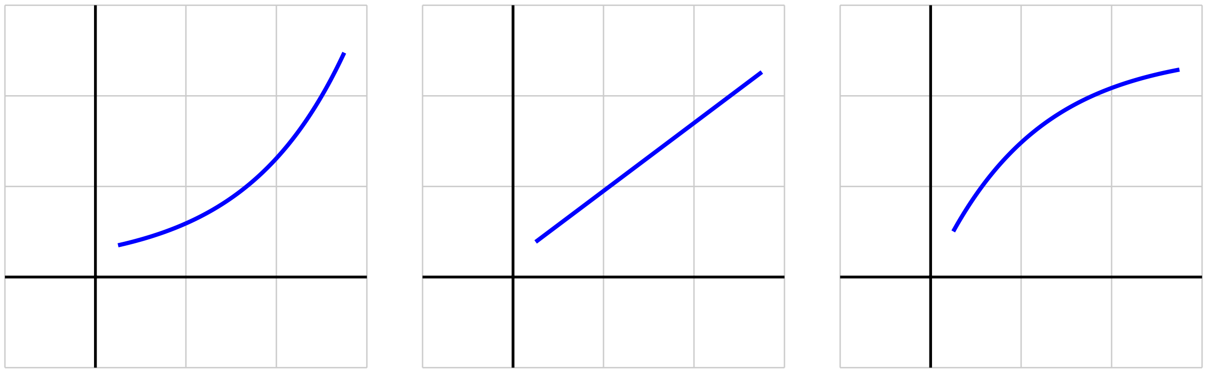


Figure 51 Three increasing functions

(a) Look at the curve pictured on the left of Figure 71. How would you describe the slopes of the tangent lines as you move from left to right?

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines remain constant as you move from left to right.

C. The slopes of the tangent lines increase as you move from left to right.

(b) Look at the curve pictured in the middle of Figure 71. How would you describe the slopes of the tangent lines as you move from left to right?

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines remain constant as you move from left to right.

C. The slopes of the tangent lines increase as you move from left to right.

(c) Look at the curve pictured on the right of Figure 71. How would you describe the slopes of the tangent lines as you move from left to right?

Concavity and inflection (AD6)

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines remain constant as you move from left to right.

C. The slopes of the tangent lines increase as you move from left to right.

Remark 3.6.3 On the leftmost curve in Figure 71, as we move from left to right, the slopes of the tangent lines will increase. Therefore, the rate of change of the pictured function is increasing, and this explains why we say this function is *increasing at an increasing rate.*

Observation 3.6.4 We must be extra careful with our language when dealing with negative numbers. For example, it can be tempting to say that "-100 is bigger than -2." But we must remember that "greater than" describes how numbers lie on a number line: -100 is less than -2 becomes it comes earlier on the number line. It might be helpful to say that "-100 is "more negative" than -2."

Activity 3.6.5 Sketch a sequence of tangent lines at various points to each of the following curves in Figure 72.

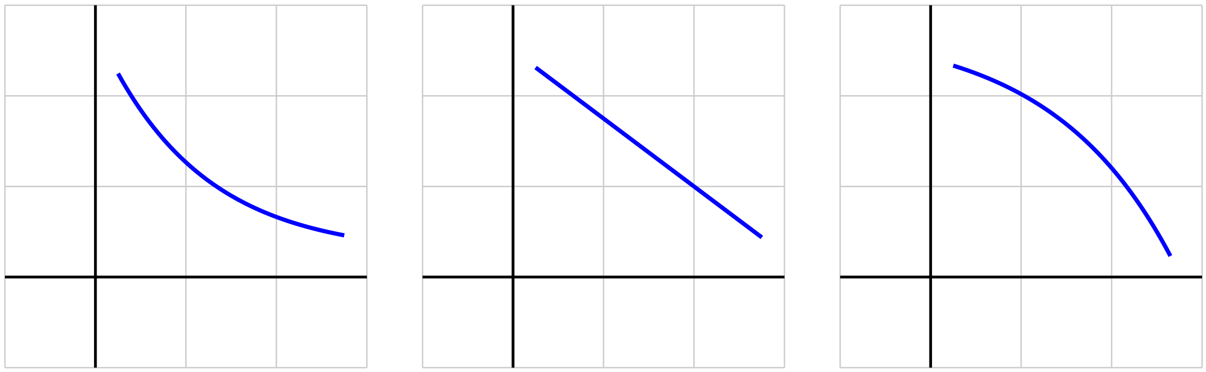


Figure 52 From left to right, three functions that are all decreasing.

(a) Look at the curve pictured on the left in Figure 72. How would you describe the slopes of the tangent lines as you move from left to right?

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines remain constant as you move from left to right.

C. The slopes of the tangent lines increase as you move from left to right.

(b) Look at the curve pictured in the middle in Figure 72. How would you describe the slopes of the tangent lines as you move from left to right?

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines remain constant as you move from left to right.

C. The slopes of the tangent lines increase as you move from left to right.

(c) Look at the curve pictured on the right in Figure 72. How would you describe the slopes of the tangent lines as you move from left to right?

Concavity and inflection (AD6)

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines remain constant as you move from left to right.

C. The slopes of the tangent lines increase as you move from left to right.

Concavity and inflection (AD6)

Remark 3.6.6 Recall the terminology of concavity: when a curve bends upward, we say its shape is concave up. When a curve bends downwards, we say its shape is concave down.

Activity 3.6.7 Look at in Figure 73. Which curve is concave up? Which one is concave down? Why? Try to explain using the graph!

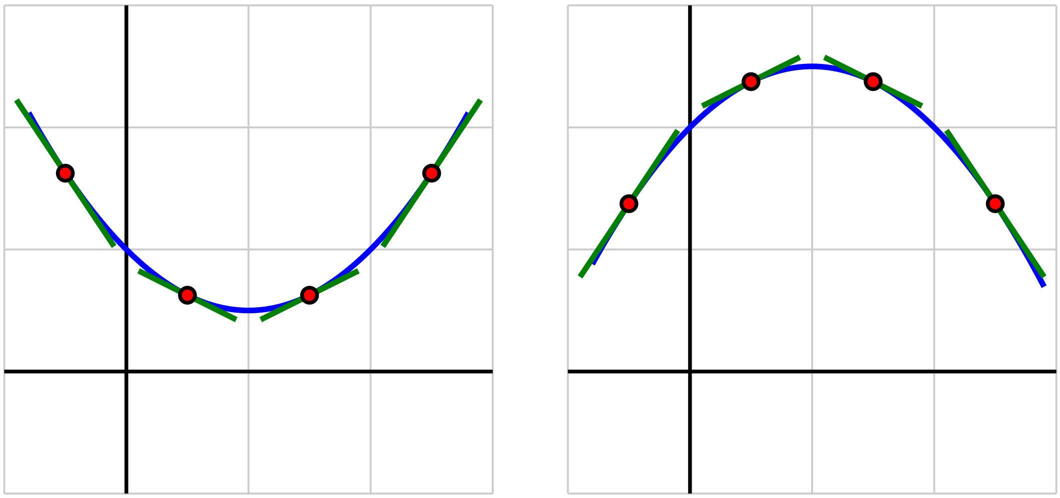


Figure 53 Two concavity, which is which?

Definition 3.6.8 Let f be a differentiable function on some interval (a, b). Then f is **concave up** on (a, b) if and only if f' is increasing on (a, b); f is **concave down** on (a, b) if and only if f' is decreasing on (a, b). $\diamondsuit$

Activity 3.6.9 Look at how the slopes of the tangent lines change from left to right for each of the two graphs in Figure 73

(a) Look at the curve pictured on the left in Figure 73. How would you describe the slopes of the tangent lines as you move from left to right?

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines increase as you move from left to right.

C. The slopes of the tangent lines go from increasing to decreasing as you move from right to left.

D. The slopes of the tangent lines go from decreasing to increasing as you move from right to left.

(b) Which of the following statements is true about the function on the left in Figure 73?

A. $f'(x) > 0$ on the entire interval shown.

B. $f'(x) < 0$ on the entire interval shown.

C. $f''(x) > 0$ on the entire interval shown.

D. $f''(x) < 0$ on the entire interval shown.

(c) Look at the curve pictured on the right in Figure 73. How would you describe the slopes of the tangent lines as you move from left to right?

A. The slopes of the tangent lines decrease as you move from left to right.

B. The slopes of the tangent lines increase as you move from left to right.

C. The slopes of the tangent lines go from increasing to decreasing as you move from right to left.

D. The slopes of the tangent lines go from decreasing to increasing as you move from right to left.

(d) Which of the following statements is true about the function on the right in Figure 73?

A. $f'(x) > 0$ on the entire interval shown.

B. $f'(x) < 0$ on the entire interval shown.

C. $f''(x) > 0$ on the entire interval shown.

D. $f''(x) < 0$ on the entire interval shown.

Observation 3.6.10 In the previous section, we saw in Activity 3.5.8 how to use critical points of the function and the sign of the first derivative to identify intervals of increase/decrease of a function. The next activity Activity 3.6.12 uses the critical points of the first derivative function and the sign of the second derivative (accordingly to Theorem 3.6.10) to identify where the original function is concave up/down.

Activity 3.6.11 Let $f(x) = x^4 - 54x^2$.

(a) Find all the zeros of $f''(x)$.

(b) What intervals have been created by subdividing the number line at zeros of $f''(x)$?

(c) Pick an x-value that lies in each interval. Determine whether $f''(x)$ is positive or negative at each point.

(d) On which intervals is $f'(x)$ increasing? On which intervals is $f'(x)$ decreasing?

(e) List all the intervals where $f(x)$ is concave up and all the intervals where $f(x)$ is concave down.

Definition 3.6.12 If $x = c$ is a point where $f''(x)$ changes sign, then the concavity of graph of $f(x)$ changes at this point and we call $x = c$ an **inflection point** of $f(x)$. $\Diamond$

Activity 3.6.13 Use the results from Activity 3.6.12 to identify all of the inflection points of $f(x) = x^4 - 4x^3 + 4x^2$.

Activity 3.6.14 For each of the following functions, describe the open intervals where it is concave up or concave down, and any inflection points.

(a) $f(x) = -\frac{1}{4}x^5 - \frac{5}{2}x^4 - \frac{15}{2}x^3$

(b) $f(x) = \frac{3}{20}x^5 + x^4 - \frac{5}{2}x^3$

Activity 3.6.15 Consider the following table. The values of the first and second derivatives of $f(x)$ are given on the domain $[0, 7]$. The function $f(x)$ does not suddenly change behavior between the points given, so the table gives you enough information to completely determine where $f(x)$ is increasing, decreasing, concave up, and concave down.

x	0	1	2	3	4	5	6	7
$f'(x)$	2	0	-2	0	2	1	0	-1
$f''(x)$	-2	-1	0	1	0	-1	0	3

(a) List all the critical points of $f(x)$ that you can find using the table above.

(b) Use the First Derivative Test to classify the critical numbers (decide if they are a max or min). Write full sentence stating the conclusion of the test for each critical number.

(c) On which interval(s) is $f(x)$ increasing? On which interval(s) is $f(x)$ decreasing? List all the critical points of $f(x)$ that you can find using the table above.

(d) There is one critical number for which the Second Derivative Test is inconclusive. Which one? You can still determine if it is a max or min using the First Derivative Test!

(e) List all the critical points of $f'(x)$ that you can find using the table above.

(f) On which intervals is $f(x)$ concave up? On which intervals is $f(x)$ concave down?

(g) List all the inflection points of $f(x)$ that you can find using the table above.

3.7 Graphing with derivatives (AD7)

Learning Outcomes

- Sketch the graph of a differentiable function whose derivatives satisfy given criteria.

Remark 3.7.1 In Section 3.5 and Section 3.6 we learned how the first and second derivatives give us information about the graph of a function. Specifically, we can determine the intervals where a function is increasing, decreasing, concave up, or concave down as well as any local extrema or inflection points. Now we will put that information together to sketch the graph of a function.

Activity 3.7.2 Which of the following features best describe the curve graphed below?

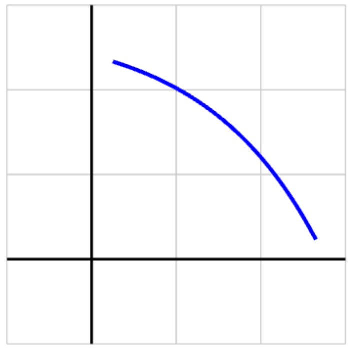


Figure 54

A. Increasing and concave up

B. Increasing and concave down

C. Decreasing and concave up

D. Decreasing and concave down

Activity 3.7.3

(a) Which of the following features best describe the curve graphed below?

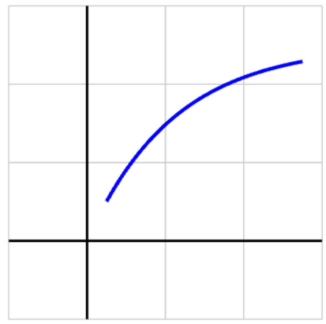


Figure 55

A. $f' > 0$ and $f'' > 0$

B. $f' > 0$ and $f'' < 0$

C. $f' < 0$ and $f'' > 0$

D. $f' < 0$ and $f'' < 0$

(b) For each of the *other* three answer choices, sketch a curve that matches that description.

Activity 3.7.4 For each prompt that follows, sketch a possible graph of a function on the interval $-3 < x < 3$ that satisfies the stated properties.

(a) A function $f(x)$ that is increasing on $-3 < x < 3$, concave up on $-3 < x < 0$, and concave down on $0 < x < 3$.

(b) A function $g(x)$ that is increasing on $-3 < x < 3$, concave down on $-3 < x < 0$, and concave up on $0 < x < 3$.

(c) A function $h(x)$ thatis decreasing on $-3 < x < 3$, concave up on $-3 < x < -1$, neither concave up nor concave down on $-1 < x < 1$, and concave down on $1 < x < 3$.

(d) A function $p(x)$ that is decreasing and concave down on $-3 < x < 0$ and is increasing and concave down on $0 < x < 3$.

Observation 3.7.5 To draw an accurate sketch, we must keep in mind additional characteristics of a function, such as the domain and the horizontal and vertical asymptotes (when they exist). The next problem Activity 3.7.6 includes those aspects in addition to increasing, decreasing, and concavity features.

Activity 3.7.6 The following chart describes the values of $f(x)$ and its first and second derivatives at or between a few given values of x, where $\nexists$ denotes that $f(x)$ does not exist at that value of x.

x		-8		-6		-3		0		2		5		8		11		13	
$f(x)$		3		5		$\nexists$		-5		$\nexists$		4		$\nexists$		-5		-3	
$f'(x)$	$+$		$+$		$-$		$-$		$-$		$-$		$+$		$+$		$+$		$+$
$f''(x)$	$+$		$-$		$-$		$+$		$-$		$+$		$+$		$-$		$-$		$-$

Assume that $f(x)$ has vertical asymptotes at each x-value where $f(x)$ does not exist, that $\lim_{x\to-\infty} f(x) = 1$, and that $\lim_{x\to\infty} f(x) = -1$.

(a) List all the asymptotes of $f(x)$ and mark them on the graph.

(b) Does $f(x)$ have any local maxima or local minima? If so, at what point(s)?

(c) Does $f(x)$ have any inflection points? If so, at what point(s)?

(d) Use the information provided to sketch a reasonable graph of $f(x)$. Watch changes in behavior due to changes in the sign of each derivative.

Remark 3.7.7 A guide to curve sketching.

1. Identify the domain of the function.
2. Identify any vertical or horizontal asymptotes, if they exist.
3. Find $f'(x)$. Then use it to determine the intervals where the function is increasing and the intervals where the function is decreasing. State any local extrema.
4. Find $f''(x)$. Then use it to determine the intervals where the function is concave up and the intervals where the function is concave down. State any inflection points.
5. Put everything together and draw sketch.

Activity 3.7.8 Sketch the graph of each of the following functions using the guide to curve sketching found in Remark 3.7.7

(a) $f(x) = x^4 - 4x^3 + 10$

(b) $f(x) = \frac{x^2-4}{x^2-9}$

(c) $f(x) = x + 2\cos x$ on the interval $[0, 2\pi]$

(d) $f(x) = \frac{x^2+x-2}{x+3}$

(e) $f(x) = \frac{x}{\sqrt{x^2+2}}$

(f) $f(x) = x^6 + \frac{12}{5}x^5 - 12x^4 + 10$

3.8 Applied optimization (AD8)

Learning Outcomes

- Apply optimization techniques to solve various problems.

Activity 3.8.1 The box. Help your company design an open box (no lid) with maximum volume given the following constraints:

- The box must be made from the following material: an 8 by 8 inches piece of cardboard.
- To create the box, you are asked to cut out a square from each corner of the 8 by 8 inches piece of cardboard and to fold up the flaps to create the sides.

(a) Draw a diagram illustrating how the box is created.

(b) Explain why the volume of the box is a function of the side length x of the cutout squares.

(c) Express the volume of the box V as a function of the length of the cuts x.

(d) What is a realistic domain of the function $V(x)$?

(e) What cut length x maximizes the volume of the box?

Remark 3.8.2 A guide for optimization problems.

1. Draw a diagram and introduce variables.
2. Determine a function of a single variable that models the quantity to be optimized.
3. Decide the domain on which to consider the function being optimized.
4. Use calculus to identify the global maximum and/or minimum of the quantity being optimized.
5. Conclusion: what are the optimal points and what optimal values do we obtain at these points?

Activity 3.8.3 According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where the "girth" is the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

(a) Let x represent the length of one side of the square end and y the length of the longer side. Label these quantities appropriately on the image shown in Figure 76.

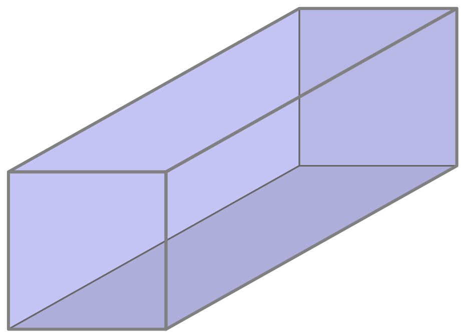


Figure 56 A rectangular parcel with a square end.

(b) What is the quantity to be optimized in this problem?

A. maximize volume (call this V)

B. maximize the girth plus length (call this P)

C. minimize volume (call this V)

D. minimize the girth plus length (call this P)

(c) Which formula below represents the quantity you want to optimize in terms of x and y?

A. $V = x^2y$

B. $V = xy^2$

C. $P = 2x + y$

D. $P = 4x + y$

(d) The problem statement tells us that the parcel's girth plus length (P) may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length P to equal 108 inches. What equation does this constraint give us involving x and y?

A. $108 = 4x + y$

B. $108 = 2x + y$

C. $108 = x^2 + y$

D. $108 = xy^2$

(e) The equation above gives the relationship between x and y. For ease of notation, solve this equation for y as a function on x and then find a formula for the volume of the parcel as a function of the single variable x. What is the formula for $V(x)$?

A. $V(x) = x^2(108 - 4x)$

B. $V(x) = x(108 - 4x)^2$

C. $V(x) = x^2(108 - 2x)$

D. $V(x) = x(108 - 2x)^2$

(f) Over what domain should we consider this function? To answer this question, notice that the problem gives us the constraint that P (girth plus length) is 108 inches. This constraint produces intervals of possible values for x and y.

A. $0 \leq x \leq 108$

B. $0 \leq y \leq 108$

C. $0 \leq x \leq 27$

D. $0 \leq y \leq 27$

(g) Use *calculus* to find the global maximum of the volume of the parcel on the domain you just determined. Justify that you have found the global maximum using either the Closed Interval Method, the First Derivative Test, or the Second Derivative Test!

Remark 3.8.4 Notice that a critical point might or might not be an global maximum or minimum, so just finding the critical points is not enough to answer an optimization problem. Moreover, some of the critical points might be outside of the domain imposed by the context and thus they cannot be feasible optimal points.

Activity 3.8.5 Revenue = Number of tickets × Price of ticket. Waterford movie theater currently charges \$8 for a ticket. At this price, the theater sells 200 tickets daily. The general manager wonders if they can generate more revenue by increasing the price of a tickets. A survey shows that they will lose 20 customers for every dollar increase in the ticket price.

(a) If the price of a movie ticket is increased by d dollars, write a formula for the price P in terms of d.

(b) If the price of a ticket is increased by one dollar, how many many customers will the theater lose?

(c) Write a formula for the number of tickets sold T as a function of a price increase of d dollars.

(d) Consider the new price of a ticket $P(d)$ and the new number of tickets sold $T(d)$. Write a formula for the revenue earned by ticket sales $R(d)$ as a function of a price increase of d dollars.

(e) What is a realistic domain for the function $R(d)$?

(f) What increase in price d should the general manager choose to maximize the revenue? What price would a movie ticket cost then and what would the revenue be at that price?

(g) Suppose now that the cost of running the business when the price is increased by d dollars is given by $C(d) = 10d^3 - 40d^2 + 40d + 600$. If the manager decides that they will definitely increase the price, what price increase d maximizes the profit? (Recall that Profit = Revenue - Cost).

Activity 3.8.6 Modeling given a geometric shape. The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter.

(a) What choice of dimensions will make the rectangular area in the center as large as possible?

(b) What should the dimensions so the total area enclosed by the running track is maximized?

Activity 3.8.7 Modeling in algebraic situations.

(a) Find the coordinates of the point on the curve $y = \sqrt{x}$ closest to the point $(1, 0)$.

(b) The sum of two positive numbers is 48. What is the smallest possible value of the sum of their squares?

Activity 3.8.8 Suppose that if a widget is priced at $176, then you are able to sell 672 units each day. According to a survey of customers, increasing this price by $1 will result in losing 4 daily sales; decreasing by $1 will gain 4 daily sales. Your manager asks you how to adjust the price of a widget to maximize the revenue (widgets sold times price). Write an explanation of what this change in price should be and why.

3.9 Limits and Derivatives (AD9)

Learning Outcomes

- Compute the values of indeterminate limits using L'Hôpital's Rule.

Remark 3.9.1 When we compute a limit algebraically, we often encounter the indeterminate form

$$\frac{0}{0}$$

but this means that limit can equal any number, infinity, or it might not exist. When we encounter an indeterminate form, we just do not know (yet) what the value of the limit is.

Activity 3.9.2 We can compute limits that give indeterminate forms via algebraic manipulations. Consider

$$\lim_{x\to 1} \frac{4x-4}{x^2-1}.$$

(a) Verify that this limit gives an indeterminate form of the type $\frac{0}{0}$.

(b) As you are computing a limit, you can cancel common factors. After you simplify the fraction, what is the limit?

A. 4

B. 2

C. $\dfrac{1}{2}$

D. The limit does not exist.

Remark 3.9.3 Consider the limits

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

Notice that these limits give indeterminate forms of the type $\frac{0}{0}$. However, these limits are equal to $f'(a)$, the derivative of $f(x)$ at $x = a$. If you can compute $f'(a)$, then you have computed the value of the limit!

Activity 3.9.4 Use the limit definition of the derivative to compute the following limits. Each limit is $f'(a)$, the derivative of some function $f(x)$ at some point $x = a$. You need to determine the function and the point to find the value of the limit: $f'(a)$.

(a) Notice that $\lim\limits_{x \to 0} \dfrac{e^{2+x} - e^2}{x}$ is the derivative of e^x at $x = 2$ (where x was used for h). Given this observation, what is this limit equal to?

A. 2

B. e

C. e^2

D. The limit does not exist.

(b) Consider $\lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x}$. This limit is also the limit definition of some derivative at some point. What is the value of this limit?

A. 1

B. 0

C. $\ln(2)$

D. The limit does not exist.

Activity 3.9.5 Compute the following limits using the limit defintion of the derivative at a point.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x}$

(b) $\displaystyle\lim_{x\to 0}\frac{\tan(x)}{x}$

(c) $\displaystyle\lim_{x\to 0}\frac{\cos(\frac{\pi}{3}+x)-\frac{1}{2}}{x}$

Remark 3.9.6 When we compute a limit algebraically, we might encounter the indeterminate form

$$\frac{\infty}{\infty}$$

but this means that limit can equal any number, infinity, or it might not exist. When we encounter an indeterminate form, we just do not know (yet) what the value of the limit is.

Activity 3.9.7 We can compute limits that give indeterminate forms via algebraic manipulations. Consider

$$\lim_{x \to +\infty} \frac{2x^2+1}{x^2-1}.$$

(a) Verify that this limit gives an indeterminate form of the type $\frac{\infty}{\infty}$.

(b) You can manipulate this fraction algebraically by dividing numerator and denominator by x^2. Then, notice that $\pm\frac{1}{x^2} \to 0$ as $x \to \infty$. Given these observations, what is the given limit equal to?

A. 2

B. 1

C. $\dfrac{1}{2}$

D. The limit does not exist.

Activity 3.9.8 Look back at some limits that gave you an indeterminate form. Can you use L'Hôpital's Rule to find the limit? If using the L'Hôpital's Rule is appropriate, then try to compute the limit this way. It should give you the same result.

Activity 3.9.9 In Activity 1.1.12, when we started to study limits, we encountered the Squeeze Theorem and computed the limit $\lim_{x\to 0} \frac{\sin(x)}{x}$ using this theorem. Let's find new ways to compute this limit.

(a) Thinking about x as the length of an interval h, this limit is actually equal to the value of some derivative, so $f'(a) = \lim_{h\to 0} \frac{\sin(h)}{h}$. What function $f(x)$ and what point $x = a$ would lead to this limit? Use these to find $f'(a)$, the value of this limit (in a new way!).

(b) Verify, one more time, that this limit is indeed an indeterminate form. Then use L'Hôpital's Rule to find this limit (again, in another way!).

Activity 3.9.10 For the following limits, check if they give an indeterminate form. If they do, try to use L'Hôpital's Rule. Does it help? It may or may not, or you may just need to use the rule repeatedly. Either way, try to compute the value of the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

(c) $\displaystyle\lim_{x\to \infty} \frac{3x^2 + 3}{x^2 + 2x}$

(d) $\displaystyle\lim_{x\to 0^+} \frac{\ln(x)}{-x}$

(e) $\displaystyle\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$

(f) $\displaystyle\lim_{x\to 0} \frac{\sin^2(3x)}{5x^3 - 3x^2}$

Activity 3.9.11 For each limit, explain if L'Hôpital's Rule may be applied. If it can, explain how to use this rule to find the limit.

(a)

$$\lim_{x\to\infty} \frac{-8\,x + 3\,e^{x}}{7\,x - 3\,e^{x}}$$

(b)

$$\lim_{x\to 0} \frac{6\,\cos(8\,x)}{4\,x - 7}$$

(c)

$$\lim_{x\to 0} \frac{-9\,\cos(3\,x) + 9}{-3\,x}$$

(d)

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x^2 - 13\,x + 36}$$

Activity 3.9.12 There are situations in which using L'Hôpital's Rule does not help and you do need some algebra skills! Consider the function $r(x) = \frac{x}{\sqrt{x^2+2}}$ and suppose that we want to find the limits as x tends to $\pm\infty$.

(a) Check that the limit as $x \to +\infty$ gives an indeterminate form $\frac{\infty}{\infty}$. Then try to use L'Hôpital's Rule... what happens? What if you use it again?

(b) We need to use algebra to handle this limit. Informally, we would like to cancel the highest powers at the numerator and denominator. Look at the denominator, $\sqrt{x^2+2}$. We want to factor out an x^2 under the square root. What do you get?

A. $\sqrt{x^2\left(1+\frac{2}{x}\right)}$

B. $\sqrt{x^2\left(1+\frac{2}{x^2}\right)}$

C. $\sqrt{x^2\,(1+x)}$

D. $\sqrt{x^2\,(1+x^2)}$

(c) Now we need to be careful when computing $\sqrt{x^2}$ as $\sqrt{x^2} = |x|$. The absolute value function $|x|$ equals $+x$ when we have a positive input and $-x$ when we have a negative output. So we have the two limits.

$$\lim_{x\to+\infty} \frac{x}{|x|\sqrt{(1+\frac{2}{x^2})}}$$

$$\lim_{x\to-\infty} \frac{x}{|x|\sqrt{(1+\frac{2}{x^2})}}$$

Thinking about what happens to the absolute values as you go towards positive or negative infinity, find the values of these two limits... The two limits have different values!

Chapter 4

Definite and Indefinite Integrals (IN)

Learning Outcomes

By the end of this chapter, you should be able to...

1. Use geometric formulas to compute definite integrals.
2. Approximate definite integrals using Riemann sums.
3. Determine basic antiderivatives.
4. Solve basic initial value problems.
5. Evaluate a definite integral using the Fundamental Theorem of Calculus.
6. Find the derivative of an integral using the Fundamental Theorem of Calculus.
7. Use definite integrals to find area under a curve.
8. Use definite integral(s) to compute the area bounded by several curves.

4.1 Geometry of definite integrals (IN1)

Learning Outcomes

- Use geometric formulas to compute definite integrals.

Definition 4.1.1 The **definite integral** for a positive function $f(x) \geq 0$ between the points $x = a$ and $x = b$ is the area between the function and the x-axis. We denote this quantity as $\displaystyle\int_a^b f(x)\,dx$ $\Diamond$

Remark 4.1.2 For some functions which have known geometric shapes (like pieces of lines or circles) we can already compute these area exactly and we will do so in this section. But for most functions we do not know quite yet how to compute these areas. In the next section, we will see that because we can compute the areas of rectangles quite easily, we can always try to approximate a shape with rectangles, even if this could be a very coarse approximation.

Activity 4.1.3 Consider the linear function $f(x) = 2x$. Sketch a graph of this function. Consider the area between the x-axis and the function on the interval $[0, 1]$. What is $\int_0^1 f(x)\,dx$?

A. 1

B. 2

C. 3

D. 4

Activity 4.1.4 Consider the linear function $f(x) = 4x$. What is $\int_0^1 f(x)\,dx$?

A. 1

B. 2

C. 3

D. 4

Activity 4.1.5 Consider the linear function $f(x) = 2x + 2$. Notice that on the interval $[0, 1]$, the shape formed between the graph and the x-axis is a trapezoid. What is $\int_0^1 f(x)\,dx$?

A. 1

B. 2

C. 3

D. 4

Activity 4.1.6 Consider the function $f(x) = \sqrt{4 - x^2}$. Notice that on the domain $[-2, 2]$, the shape formed between the graph and the x-axis is a semicircle. What is $\int_{-2}^{2} f(x)\, dx$?

A. π

B. 2π

C. 3π

D. 4π

Definition 4.1.7 If a function $f(x) \leq 0$ on $[a, b]$, then we define the integral between a and b to be

$$\int_a^b f(x)\,dx \;=\; (-1)\times\text{area between the graph of and the axis on the interval }.$$

So the definite integral for a negative function is the "negative" of the area between the graph and the x-axis. ◊

Geometry of definite integrals (IN1)

Activity 4.1.8 Explain how to use geometric formulas for area to compute the following definite integrals. For each part, sketch the function to support your explanation.

1. $$\int_1^6 (-3\,x + 6)\,dx$$

2. $$\int_2^6 (-3\,x + 6)\,dx$$

3. $$\int_1^5 \left(-\sqrt{-(x-1)^2 + 16}\right)\,dx$$

Activity 4.1.9 The graph of $g(t)$ and the areas A_1, A_2, A_3 are given below.

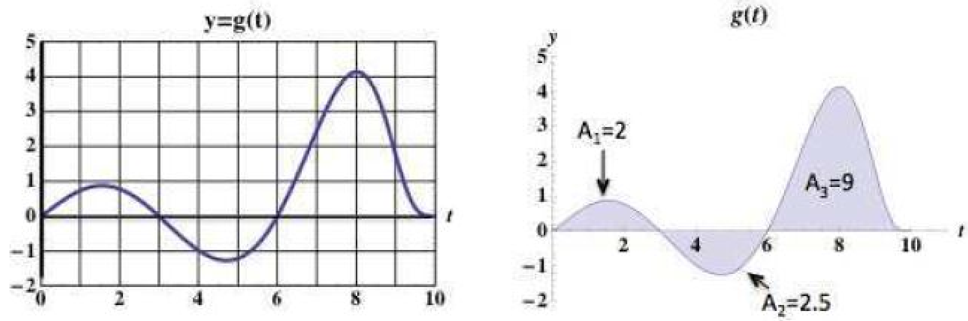


Figure 57

(a) Find $\int_3^3 g(t)\,dt$

(b) Find $\int_3^6 g(t)\,dt$

(c) Find $\int_0^{10} g(t)\,dt$

(d) Suppose that $g(t)$ gives the velocity in fps at time t (in seconds) of a particle moving in the vertical direction. A positive velocity indicates that the particle is moving up, a negative velocity indicates that the particle is moving down. If the particle started at a height of 3ft, at what height would it been after 3 seconds? After 6 seconds? After 10 seconds? At what time does the particle reach the highest point in this time interval?

4.2 Approximating definite integrals (IN2)

Learning Outcomes

- Approximate definite integrals using Riemann sums.

Activity 4.2.1 Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

(a) On the left-hand axes provided in Figure 82, sketch a labeled graph of the velocity function $v(t) = 3$.

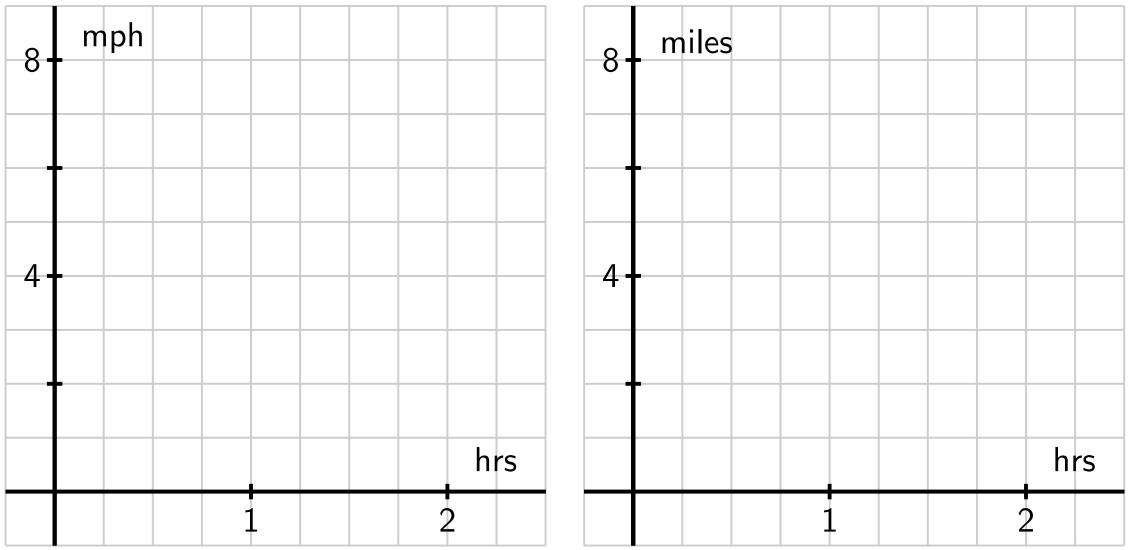


Figure 58 At left, axes for plotting $y = v(t)$; at right, for plotting $y = s(t)$.

Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question (d).

(b) How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?

(c) Find an algebraic formula, $s(t)$, for the position of the person at time t, assuming that $s(0) = 0$. Explain your thinking.

(d) On the right-hand axes provided in Figure 82, sketch a labeled graph of the position function $y = s(t)$.

(e) For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v.

Activity 4.2.2 Suppose that a person is walking in such a way that her velocity varies slightly according to the information given in Table 83 and graph given in Figure 84.

t	$v(t)$
0.00	1.500
0.25	1.789
0.50	1.938
0.75	1.992
1.00	2.000
1.25	2.008
1.50	2.063
1.75	2.211
2.00	2.500

Table 59 Velocity data for the person walking.

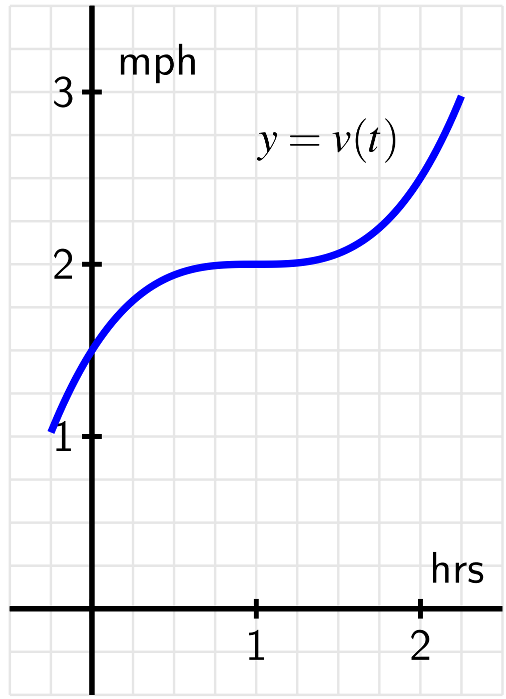


Figure 60 The graph of $y = v(t)$.

(a) Using the grid, graph, and given data appropriately, estimate the distance traveled by the walker during the two hour interval from $t = 0$ to $t = 2$. You should use time intervals of width $\Delta t = 0.5$, choosing a way to use the function consistently to determine the height of each rectangle in order to approximate distance traveled.

(b) How could you get a better approximation of the distance traveled on $[0, 2]$? Explain, and then find this new estimate.

(c) Now suppose that you know that v is given by $v(t) = 0.5t^3 - 1.5t^2 + 1.5t + 1.5$. Remember that v is the derivative of the walker's position function, s. Find a formula for s so that $s' = v$.

(d) Based on your work in (c), what is the value of $s(2) - s(0)$? What is the meaning of this quantity?

Activity 4.2.3 Explain how to approximate the area under the curve $f(x) = -9\,x^3 + 3\,x - 9$ on the interval $[4, 10]$ using a right Riemann sum with 3 rectangles of uniform width.

4.3 Elementary antiderivatives (IN3)

Learning Outcomes

- Determine basic antiderivatives.

Definition 4.3.1 If g and G are functions such that $G' = g$, we say that G is an **antiderivative** of g.

The collection of all antiderivatives of g is called the **general antiderivative** or **indefinite integral**, denoted by $\int g(x)\,dx$. All antiderivatives differ by a constant C (since $\frac{d}{dx}[C] = 0$), so we may write:

$$\int g(x)\,dx = G(x) + C.$$

$\diamondsuit$

Activity 4.3.2 Consider the function $f(x) = \cos x$. Which of the following could be $F(x)$, an antiderivative of $f(x)$?

A. $\sin x$

B. $\cos x$

C. $\tan x$

D. $\sec x$

Activity 4.3.3 Consider the function $f(x) = x^2$. Which of the following could be $F(x)$, an antiderivative of $f(x)$?

A. $2x$

B. $\frac{1}{3}x^3$

C. x^3

D. $\frac{2}{3}x^3$

Remark 4.3.4 We now note that whenever we know the derivative of a function, we have a *function-derivative pair*, so we also know the antiderivative of a function. For instance, in Activity 4.3.2 we could use our prior knowledge that

$$\frac{d}{dx}[\sin(x)] = \cos(x),$$

to determine that $F(x) = \sin(x)$ is an antiderivative of $f(x) = \cos(x)$. F and f together form a function-derivative pair. Every elementary derivative rule leads us to such a pair, and thus to a known antiderivative.

In the following activity, we work to build a list of basic functions whose antiderivatives we already know.

Activity 4.3.5 Use your knowledge of derivatives of basic functions to complete Table 86 of antiderivatives. For each entry, your task is to find a function F whose derivative is the given function f.

Table 61 Familiar basic functions and their antiderivatives.

given function, $f(x)$	antiderivative, $F(x)$
k, (k is constant)	
x^n, $n \neq -1$	
$\frac{1}{x}$, $x > 0$	
$\sin(x)$	
$\cos(x)$	
$\sec(x)\tan(x)$	
$\csc(x)\cot(x)$	
$\sec^2(x)$	
$\csc^2(x)$	
e^x	
a^x $(a > 1)$	
$\frac{1}{1+x^2}$	
$\frac{1}{\sqrt{1-x^2}}$	

Activity 4.3.6 Using this information, which of the following is an antiderivative for $f(x) = 5\sin(x) - 4x^2$?

A. $F(x) = -5\cos(x) + \frac{4}{3}x^3$.

B. $F(x) = 5\cos(x) + \frac{4}{3}x^3$.

C. $F(x) = -5\cos(x) - \frac{4}{3}x^3$.

D. $F(x) = 5\cos(x) - \frac{4}{3}x^3$.

Activity 4.3.7 Find the general antiderivative for each function.

(a)

$$f(x) = -4\sec^2(x)$$

(b)

$$f(x) = \frac{8}{\sqrt{x}}$$

Activity 4.3.8 Find each indefinite integral.

(a)

$$\int (-9\,x^4 - 7\,x^2 + 4)\,dx$$

(b)

$$\int 3\,e^x\,dx$$

4.4 Initial Value Problems (IN4)

Learning Outcomes

- Solve basic initial value problems.

Initial Value Problems (IN4)

Note 4.4.1 In this section we will discuss the relationship between antiderivatives and solving simple differential equations. A differential equation is an equation that has a derivative. For this section we will focus on differential equations of the form

$$\frac{dy}{dx} = f(x).$$

Our goal is to find a relationship of $y(x)$ that satisfies the differential equation. We can solve for $y(x)$ by finding the antiderivative of $f(x)$.

Activity 4.4.2 Which of the following equations for $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^2 + 2x.$$

A. $y(x) = \dfrac{x^3}{3} + x^2 + 4$

B. $y(x) = 2x + 2$

C. $y(x) = \dfrac{x^3}{3} + x^2 + 10$

D. $y(x) = \dfrac{x^3}{3} + x^2$

E. $y(x) = 2x$

Remark 4.4.3 In Activity 4.4.2 there are more than one solution that satisfies the differential equation. In fact their is a family of functions that satisfies the differential equation, that is

$$f(x) = \frac{x^3}{3} + x^2 + c_1,$$

where c_1 is an arbitrary constant yet to be defined. To find c_1 we have to have some initial value for the differential equation, $y(x_0) = y_0$, where the point (x_0, y_0) is the starting point for the differential equation. In general this section we will focus on solving initial value problems (differential equation with an initial condition) of the form,

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

Activity 4.4.4 Which of the following equations for $y(x)$ satisfies the differential equation and initial condition,

$$\frac{dy}{dx} = x^2 + 2x, \quad y(3) = 16.$$

A. $y(x) = \frac{x^3}{3} + x^2 - 4$

B. $y(x) = \frac{x^3}{3} + x^2 + 2$

C. $y(x) = \frac{x^3}{3} + x^2 - 2$

D. $y(x) = \frac{x^3}{3} + x^2 + 16$

Activity 4.4.5 Which of the following functions satisfies the initial value problem,

$$\frac{dy}{dx} = \sin(x), \quad y(0) = 1.$$

A. $y(x) = \cos(x)$

B. $y(x) = \cos(x) + 2$

C. $y(x) = \cos(x) + 1$

D. $y(x) = -\cos(x)$

E. $y(x) = -\cos(x) + 2$

Activity 4.4.6 One of the applications of initial value problems is calculating the distance traveled from a point based on the velocity of the object. Given that the velocity of the of an object in km/hr is approximated by $v(t) = \cos(t) + 1$, what is the approximate distance travelled by the object after 1 hour?

A. $s(1) \approx 1$ km

B. $s(1) \approx 0.1585$ km

C. $s(1) \approx 1.8415$ km

D. $s(1) \approx 2.3415$ km

Activity 4.4.7 So far we have only been going from velocity to position of an object. Recall that to find the acceleration of an object, you can take the derivative of the velocity of an object. Let use say we have the acceleration of a falling object in $\mathrm{m/s^2}$ given by $a(t) = -9.8$.What is the velocity of the falling object, if the initial velocity is given by $v(0) = 0$ m/s.

A. $v(t) = -9.8t$ m

B. $v(t) = -9.8t$ m/s

C. $v(t) = 9.8t$ m/s

D. $v(t) = 9.8t + 1$ m

What is the position of the object, if the initial position is given by $s(0) =$ 10 m.

A. $s(t) = 4.9t + 10$ m

B. $s(t) = -4.9t^2 + 14.9$ m

C. $s(t) = -4.9t^2 + 10$ m

D. $s(t) = 4.9t + 5.1$ m

Activity 4.4.8 Let $f'(x) = -12\,x - 6$. Find $f(x)$ such that $f(5) = -179$.

4.5 FTC for definite integrals (IN5)

Learning Outcomes

- Evaluate a definite integral using the Fundamental Theorem of Calculus.

Activity 4.5.1 Find the area beteween $f(x) = \frac{1}{2}x + 2$ and the x-axis from $x = 2$ to $x = 6$.

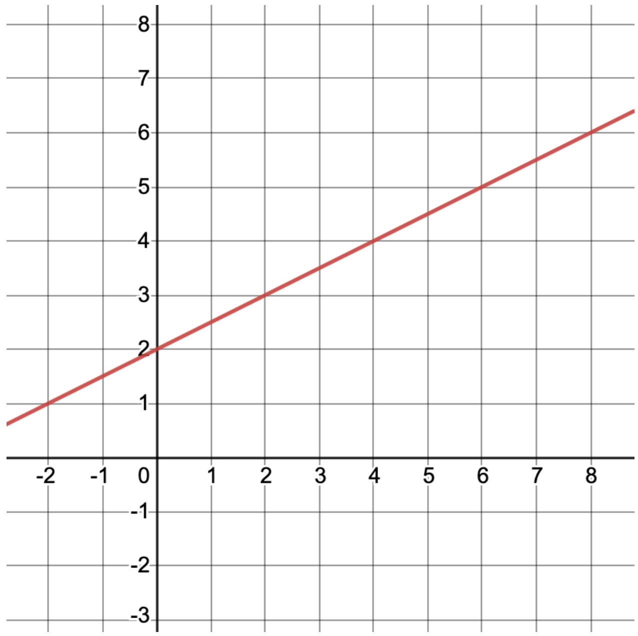


Figure 62

Activity 4.5.2 Approximate the area under the curve $f(x) = (x-1)^2 + 2$ on the interval $[1, 5]$ using a left Riemann sum with four uniform subdivisions. Draw your rectangles on the graph.

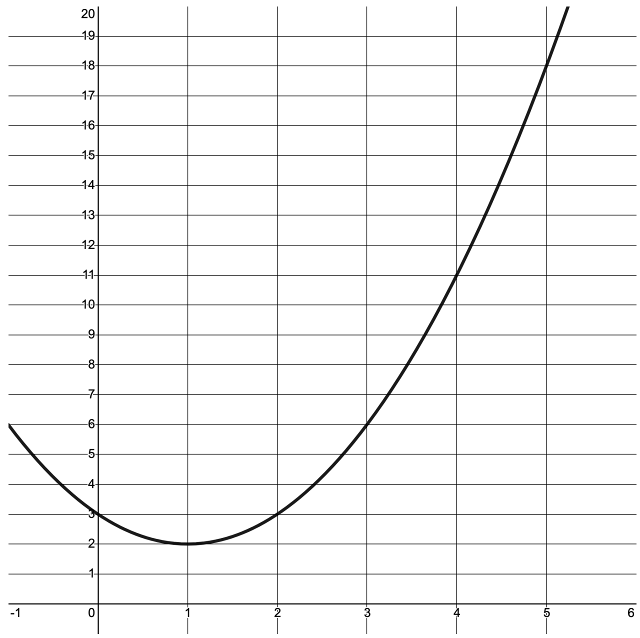


Figure 63

FTC for definite integrals (IN5)

Definition 4.5.3 Let $f(x)$ be a continuous function on the interval $[a, b]$. Divide the interval into n subdivisions of equal width, Δx, and choose a point x_i in each interval. Then, the definite integral of $f(x)$ from a to b is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

$\Diamond$

Activity 4.5.4 How does $\int_2^6 \left(\frac{1}{2}x+2\right)\,dx$ relate to Activity 4.5.1? Could you use Activity 4.5.1 to find $\int_0^4 \left(\frac{1}{2}x+2\right)\,dx$? What about $\int_1^7 \left(\frac{1}{2}x+2\right)\,dx$?

Remark 4.5.5 Properties of Definite Integrals.

1. If f is defined at $x = a$, then $\displaystyle\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
3. If f is integrable on $[a, b]$ and c is in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
4. If f is integrable on $[a, b]$ and k is a constant, then kf is integrable on $[a, b]$ and $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.
5. If f and g are integrable on $[a, b]$, then $f \pm g$ are integrable on $[a, b]$ and $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

FTC for definite integrals (IN5)

Activity 4.5.6 Suppose that $\int_1^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 4$.Find each of the following.

(a) $\int_1^7 f(x)\,dx$

(b) $\int_5^1 f(x)\,dx$

(c) $\int_7^7 f(x)\,dx$

(d) $3\int_5^7 f(x)\,dx$

Observation 4.5.7 We've been looking at two big things in this chapter: antiderivatives and the area under a curve. In the early days of the development of calculus, they were not known to be connected to one another. The integral sign wasn't originally used in both instances. (Gottfried Leibniz introduced it as an elongated S to represent the sum when finding the area.) Connecting these two seemingly separate problems is done by the Fundamental Theorem of Calculus

Activity 4.5.8 Evaluate the following definite integrals. Include a sketch of the graph with the area you've found shaded in. Approximate the area to check to see if your definite integral answer makes sense. (Note: Just a guess, you don't have to use Riemann sums. Use the grid to help.)

(a) $\displaystyle\int_0^2 \left(x^2+3\right)\, dx$

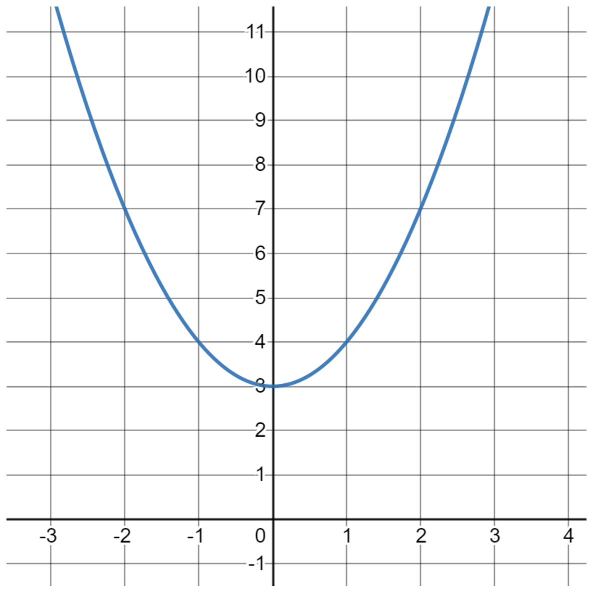


Figure 64

(b) $\displaystyle\int_1^4 \left(\sqrt{x}\right)\, dx$

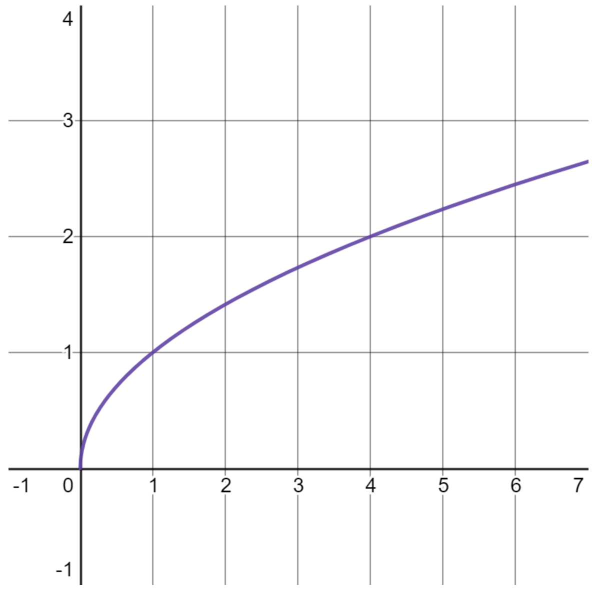


Figure 65

(c) $\displaystyle\int_{-\pi/4}^{\pi/2} (\cos x)\ dx$

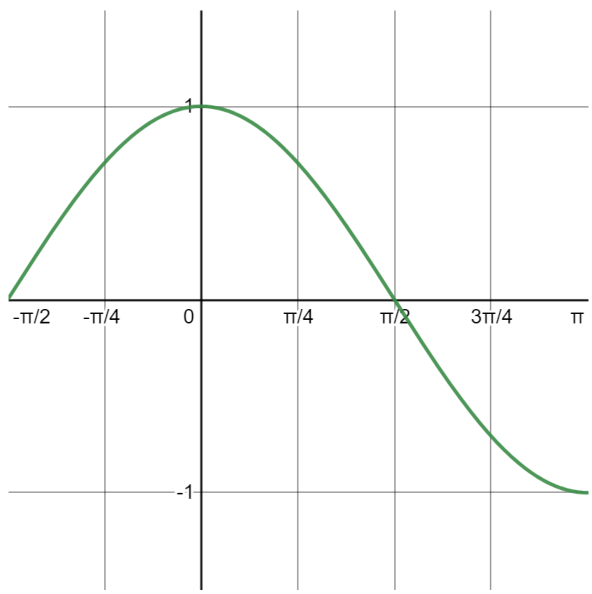


Figure 66

Activity 4.5.9 Find the area between $f(x) = 2x - 6$ on the interval $[0, 8]$ using

1. geometry
2. the definite integral

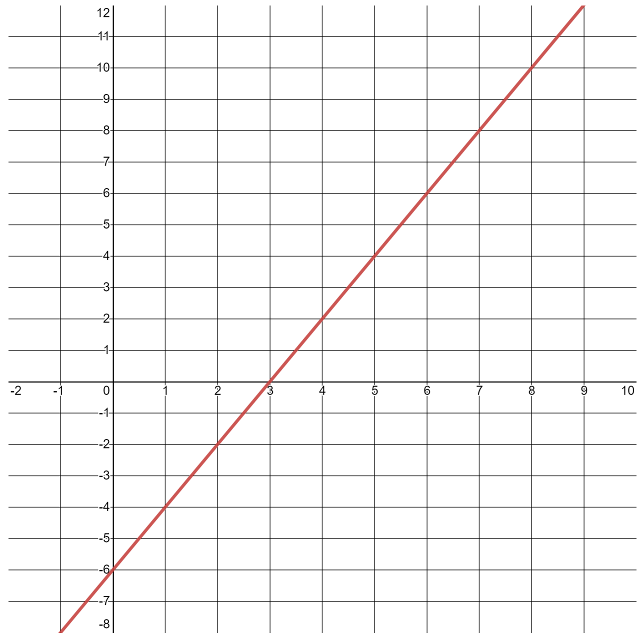


Figure 67

What do you notice?

Activity 4.5.10 Find the area bounded by the curves $f(x) = e^x - 2$, the x-axis, $x = 0$, and $x = 1$.

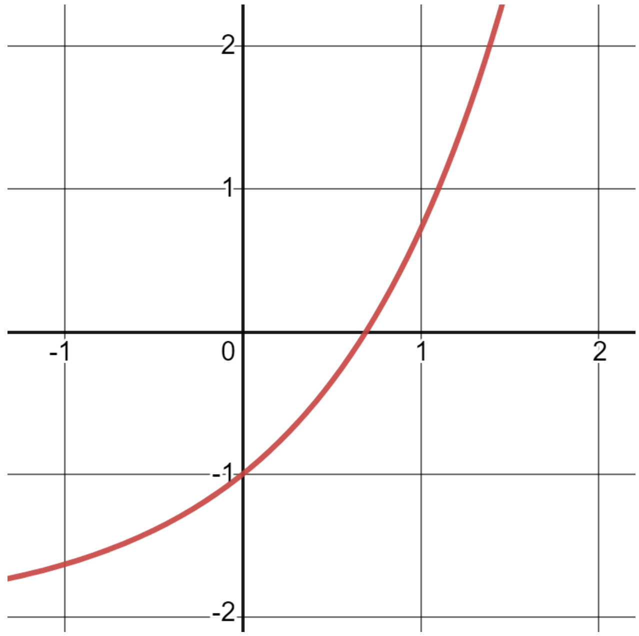


Figure 68

Activity 4.5.11 Set up a definite integral that represents the shaded area. Then find the area of the given region using the definite integral.

(a) $y = \frac{1}{x^2}$

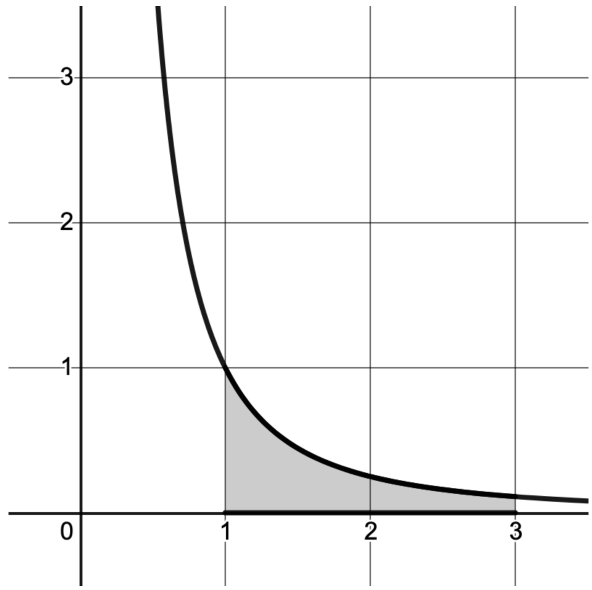


Figure 69

(b) $y = 3x^2 - x^3$

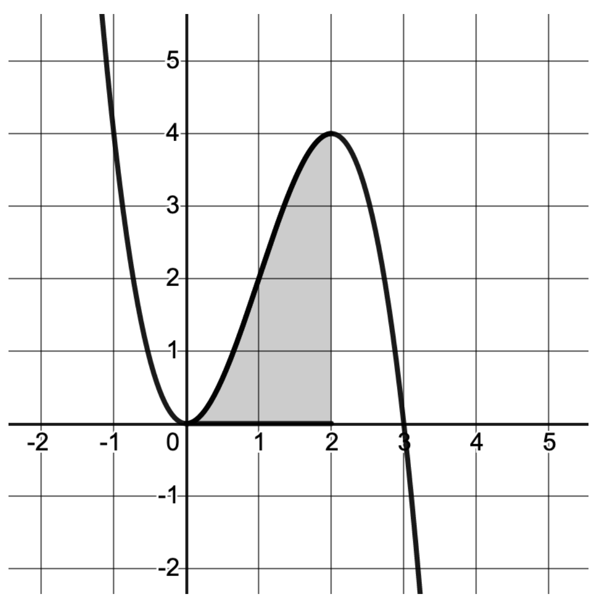


Figure 70

Activity 4.5.12 Explain how to compute the exact value of each of the following definite integrals using the Fundamental Theorem of Calculus. Leave all answers in exact form, with no decimal approximations.

(a)

$$\int_{-3}^{-2} \left(-9\,x^3 - 9\,x^2 + 1\right)\,dx$$

(b)

$$\int_{\frac{7}{6}\,\pi}^{\frac{5}{4}\,\pi} \left(-3\,\sin\left(x\right)\right)\,dx$$

(c)

$$\int_{2}^{6} \left(3\,e^{x}\right)\,dx$$

4.6 FTC for derivatives of integrals (IN6)

Learning Outcomes

- Find the derivative of an integral using the Fundamental Theorem of Calculus.

Note 4.6.1 In this section we extend the Fundamental Theorem of Calculus discussed in Section 4.5 to include taking the derivatives of integrals. We will call this addition to the Fundamental Theorem of Calculus (FTC) part II. First we will introduce part II and then discuss the implications of this addition.

Activity 4.6.2 For the following activity we will explore the Fundamental Theorem of Calculus Part II.

(a) Given that $A(x) = \int_a^x t^3\,dt$, then by the Fundamental Theorem of Calculus Part I,

A. $A(x) = x^3 - a^3$

B. $A(x) = a^4 - x^4$

C. $A(x) = \frac{1}{4}(x^4 - a^4)$

D. $A(x) = 3x^2$

(b) Using what you found for $A(x)$, what is $A'(x)$

A. $A'(x) = 3x^2$

B. $A'(x) = 4a^3 - 4x^3$

C. $A'(x) = x^3$

D. $A'(x) = 6x$

(c) Use the Fundamental Theorem of Calculus Part II to find $A'(x)$. What do you notice between what you got above and using FTC Part II? Which method do you prefer?

A. $A'(x) = 3x^2$

B. $A'(x) = 4a^3 - 4x^3$

C. $A'(x) = x^3$

D. $A'(x) = 6x$

Activity 4.6.3 Given $A(x) = \int_x^b e^t \, dt$, what is $A'(x)$?

A. $A'(x) = -e^x$

B. $A'(x) = e^x$

C. $A'(x) = e^b - e^x$

D. $A'(x) = e^x - e^b$

Observation 4.6.4 For the first two activities we have only explored when the function of the limits of the integrand are x. Now we want to see what happens when the limits are more complicated. To do this we will follow a similar procedure as that done in activity 1.

FTC for derivatives of integrals (IN6)

Activity 4.6.5 Recall that by the Fundamental Theorem of Calculus Part I, $\int_a^b f(t)\,dt = F(b) - F(a)$.

(a) Let $A(x) = \int_x^{x^2} f(t)\,dt$ and re-write using FTC Part I.

(b) Using what you got find $A'(x)$. Explain what derivative rule(s) you used.

(c) Using what you found what is the derivative of $A(x) = \int_x^{x^2} (t+2)\,dt$?

A. $A'(x) = 2x(x+2) - (x+2)$

B. $A'(x) = (x+2) - 2x(x^2+2)$

C. $A'(x) = (x^2+2) - (x+2)$

D. $A'(x) = 2x(x^2+2) - (x+2)$

Remark 4.6.6 Now we have some thoughts of how to generalize the FTC Part II when the limts are more complicated.

Activity 4.6.7 Given $A(x) = \int_{x^3}^{x^5} (\sin(t) - 2)\, dt$, what is $A'(x)$?

4.7 Area under curves (IN7)

Learning Outcomes

- Use definite integrals to find area under a curve.

Remark 4.7.1 A geometrical interpretation of

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

(Definition 4.5.3) defines $\int_a^b f(x)dx$ as the **net area** between the graph of $y = f(x)$ and the x-axis. By net area, we mean the area above the x-axis (when $f(x)$ is positive) minus the area below the x-axis (when $f(x)$ is negative).

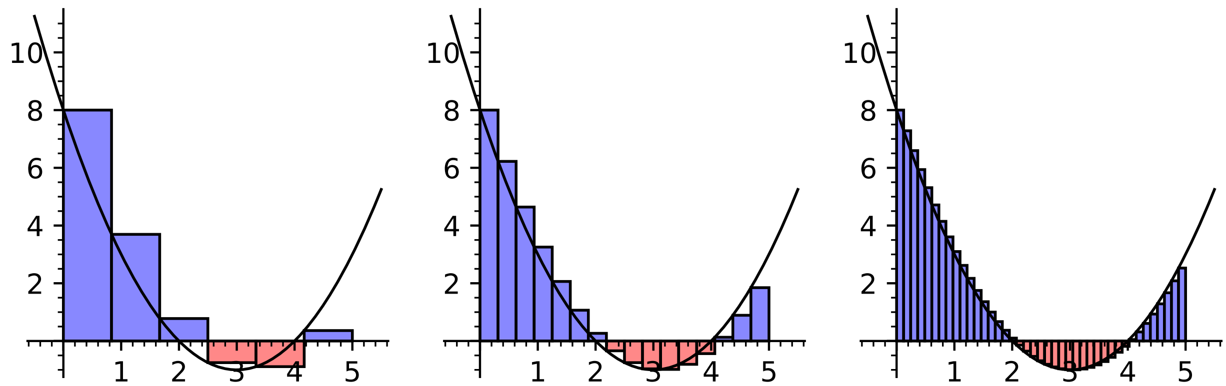


Figure 71 Improving approximations of $\int_0^5 (x-2)(x-4)dx$

Activity 4.7.2

(a) Write the net area between $f(x) = 6\,x^2 - 18\,x$ and the x-axis from $x = 2$ to $x = 7$ as a definite integral.

(b) Evaluate this definite integral to verify the net area is equal to 265 square units.

Observation 4.7.3 In order to find the total area between a curve and the x-axis, one must break up the definite integral at points where $f(x) = 0$, that is, wherever $f(x)$ may change from positive to negative, or vice versa.

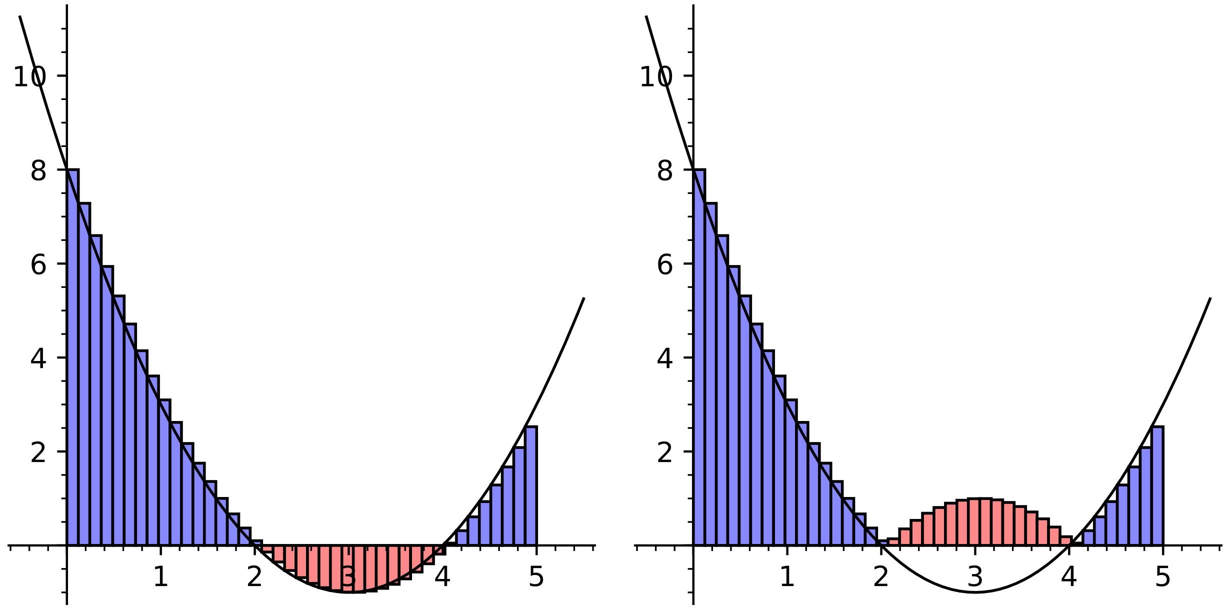


Figure 72 Partitioning $\int_0^5 (x-2)(x-4)dx$ at $x = 2$ and $x = 4$.

Since $f(x) = (x-2)(x-4)$ is zero when $x = 2$ and $x = 4$, we may compute the total area between $y = (x-2)(x-4)$ and the x-axis using absolute values as follows:

$$\text{Area} = \left|\int_0^2 (x-2)(x-4)dx\right| + \left|\int_2^4 (x-2)(x-4)dx\right| + \left|\int_4^5 (x-2)(x-4)dx\right|$$

Activity 4.7.4 Follow these steps to find the total area between $f(x) = 6\,x^2 - 18\,x$ and the x-axis from $x = 2$ to $x = 7$.

(a) Find all values for x where $f(x) = 6\,x^2 - 18\,x$ is equal to 0.

(b) Only one such value is between $x = 2$ and $x = 7$. Use this value to fill in the ? below, then verify that its value is 279 square units.

$$\text{Area} = \left| \int_2^{?} \left(6\,x^2 - 18\,x\right)\,dx \right| + \left| \int_{?}^{7} \left(6\,x^2 - 18\,x\right)\,dx \right|$$

Activity 4.7.5 Answer the following questions concerning $f(x) = 6\,x^2 - 96$.

(a) What is the total area between $f(x) = 6\,x^2 - 96$ and the x-axis from $x = -1$ to $x = 9$?

(b) What is the net area between $f(x) = 6\,x^2 - 96$ and the x-axis from $x = -1$ to $x = 9$?

4.8 Area between curves (IN8)

Learning Outcomes

- Use definite integral(s) to compute the area bounded by several curves.

Remark 4.8.1 In Section 4.7, we learned how to find the area between a curve and the x-axis ($f(x) = 0$) using a definite integral. What if we want the area between any two functions? What if the x-axis is not one of the boundaries?

In this section, we'll investigate how a definite integral may be used to represent the area between two curves.

Activity 4.8.2 Consider the functions given by $f(x) = 5 - (x-1)^2$ and $g(x) = 4 - x$.

(a) Use algebra to find the points where the graphs of f and g intersect.

(b) Sketch an accurate graph of f and g on the xy plane, labeling the curves by name and the intersection points with ordered pairs.

(c) Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the x-axis on the interval between the intersection points of f and g. Shade this area in your sketch.

(d) Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the x-axis on the interval between the intersection points of f and g. Shade this area in your sketch.

(e) Let's denote the area between $y = f(x)$ and the x-axis as A_f and the area between $y = g(x)$ and the x-axis as A_g. How could we use A_f and A_g to find exact area between f and g between their intersection points?

A. We could find $A_f + A_g$ to find the area between the curves.

B. We could find $A_f - A_g$ to find the area between the curves.

C. We could find $A_g - A_f$ to find the area between the curves.

Note 4.8.3 We've seen from Activity 4.8.2 that a natural way to think about the area between two curve is as the area beneath the upper curve minus the area beneath the lower curve.

Activity 4.8.4 We now look for a general way of writing definite integrals for the area between two given curves, $f(x)$ and $g(x)$. Consider this area, illustrated in Figure 103.

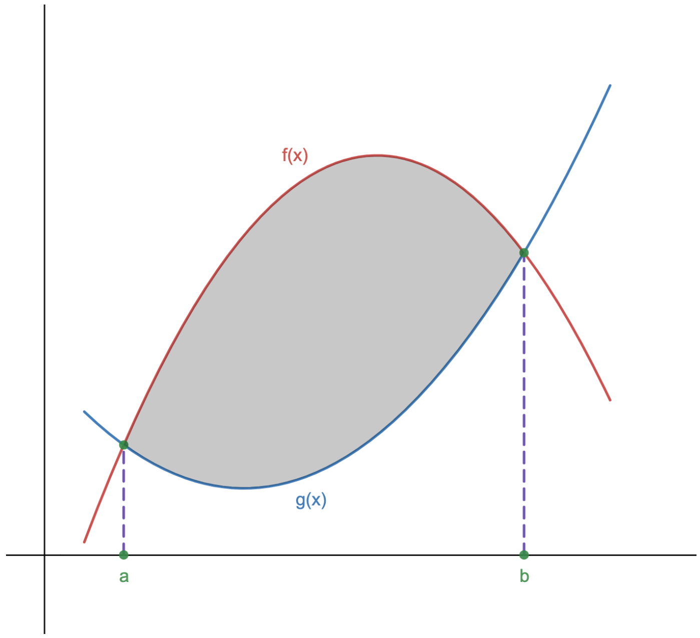


Figure 73 Area between $f(x)$ and $g(x)$.

(a) How could we represent the shaded area in Figure 103?

A. $\displaystyle\int_b^a f(x)\,dx - \int_b^a g(x)\,dx$

B. $\displaystyle\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

C. $\displaystyle\int_b^a g(x)\,dx - \int_b^a f(x)\,dx$

D. $\displaystyle\int_a^b g(x)\,dx - \int_a^b f(x)\,dx$

(b) The two definite integrals above can be rewritten as one definite integral using the sum and difference property of definite integrals:

If f and g are continuous functions, then

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Use the property above to represent the shaded area in Figure 103 using one definite integral.

Area between curves (IN8)

A. $\displaystyle\int_b^a (f(x) - g(x))\,dx$

B. $\displaystyle\int_a^b (f(x) - g(x))\,dx$

C. $\displaystyle\int_b^a (g(x) - f(x))\,dx$

D. $\displaystyle\int_a^b (g(x) - f(x))\,dx$

Area between curves (IN8)

Fact 4.8.5 *If two curves $y = f(x)$ and $y = g(x)$ intersect at $(a, g(a))$ and $(b, g(b))$, and for all x such that $a \leq x \leq b$, $f(x) \geq g(x)$, then the area between the curves is $A = \int_a^b (f(x) - g(x))\, dx$.*

Activity 4.8.6 In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

(a) The finite region bounded by $y = \sqrt{x}$ and $y = \frac{1}{4}x$.

(b) The finite region bounded by $y = 12 - 4x^2$ and $y = x^2 - 8$.

(c) The area bounded by the y-axis, $f(x) = \cos(x)$, and $g(x) = \sin(x)$, where we consider the region formed by the first positive value of x for which f and g intersect.

(d) The finite regions between the curves $y = x^3 - 2x$ and $y = x^2$.

Activity 4.8.7 Let **R** be the finite region bounded by the graphs of $y = (x+5)^2 - 1$ and $y = 7\,x + 34$.

Sketch an illustration of **R**, and then explain how to express the area of **R** in the following two ways:(Do not evaluate either definite integral.)

1. As a definite integral with respect to x.
2. As a definite integral with respect to y.

Chapter 5

Techniques of Integration (TI)

Learning Outcomes

How do we use various techniques to integrate less simple functions?
By the end of this chapter, you should be able to...

1. Evaluate various integrals via the substitution method.
2. Compute integrals using integration by parts.
3. Compute integrals involving products of trigonometric functions.
4. Use trigonometric substitution to compute indefinite integrals.
5. I can integrate functions using a table of integrals.
6. I can integrate functions using the method of partial fractions.
7. I can select appropriate strategies for integration.
8. I can compute improper integrals.

5.1 Substitution method (TI1)

Learning Outcomes

- Evaluate various integrals via the substitution method.

Activity 5.1.1 Answer the following.

(a) Using the chain rule, which of these is the derivative of e^{x^3} with respect to x?

A. e^{3x^2}

B. $x^3e^{x^3-1}$

C. $3x^2e^{x^3}$

D. $\frac{1}{4}e^{x^4}$

(b) Based on this result, which of these would you suspect to equal $\int x^2e^{x^3}\,dx$?

A. $e^{x^3+1}+C$

B. $\frac{1}{3x}e^{x^3+1}+C$

C. $3e^{x^3}+C$

D. $\frac{1}{3}e^{x^3}+C$

Substitution method (TI1)

Activity 5.1.2 Recall that if u is a function of x, then $\frac{d}{dx}[u^7] = 7u^6u'$ by the Chain Rule.

For each question, choose from the following.

A. $\dfrac{1}{7}u^7 + C$ B. $u^7 + C$ C. $7u^7 + C$ D. $\dfrac{6}{7}u^7 + C$

(a) What is $\int 7u^6u'\,dx$?

(b) What is $\int u^6u'\,dx$?

(c) What is $\int 6u^6u'\,dx$?

Activity 5.1.3 Based on these activities, which of these choices seems to be a viable strategy for integration?

A. Memorize an integration formula for every possible function.

B. Attempt to rewrite the integral in the form $\int g'(u)u'dx = g(u) + C$.

C. Keep differentiating functions until you come across the function you want to integrate.

Fact 5.1.4 *By the chain rule,*

$$\frac{d}{dx}[g(u) + C] = g'(u)u'.$$

There is a dual integration technique reversing this process, known as the ***substitution method****.*

This technique involves choosing an appropriate function u in terms of x to rewrite the integral as follows:

$$\int f(x)\,dx = \cdots = \int g'(u)u'dx = g(u) + C.$$

Observation 5.1.5 Recall that $\frac{du}{dx} = u'$, and so $du = u'\,dx$. This allows for the following common notation:

$$\int f(x)\,dx = \cdots = \int g'(u)du = g(u) + C.$$

Therefore, rather than dealing with equations like $u' = \frac{du}{dx} = x^2$, we will prefer to write $du = x^2\,dx$.

Substitution method (TI1)

Activity 5.1.6 Consider $\int x^2 e^{x^3}\,dx$, which we conjectured earlier to be $\frac{1}{3}e^{x^3}+C$.

Suppose we decided to let $u = x^3$.

(a) Compute $\frac{du}{dx} = ?$, and rewrite it as $du = ?\,dx$.

(b) This $?\,dx$ doesn't appear in $\int x^2 e^{x^3}\,dx$ exactly, so use algebra to solve for $x^2\,dx$ in terms of du.

(c) Replace $x^2 dx$ and x^3 with u, du terms to rewrite $\int x^2 e^{x^3}\,dx$ as $\int \frac{1}{3}e^u\,du$.

(d) Solve $\int \frac{1}{3}e^u\,du$ in terms of u, then replace u with x^3 to confirm our original conjecture.

Example 5.1.7 Here is how one might write out the explanation of how to find $\int x^2 e^{x^3}\,dx$ from start to finish:

$$\int x^2 e^{x^3}\,dx \qquad\qquad \text{Let } u = x^3$$

$$du = 3x^2\,dx$$

$$\frac{1}{3}du = x^2\,dx$$

$$\begin{aligned}\int x^2 e^{x^3}\,dx &= \int e^{(x^3)}(x^2\,dx)\\ &= \int e^u \frac{1}{3}du\\ &= \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3} + C\end{aligned}$$

$\square$

Activity 5.1.8 Which step of the previous example do you think was the most important?

A. Choosing $u = x^3$.

B. Finding $du = 3x^2\,dx$ and $\frac{1}{3}du = x^2\,dx$.

C. Substituting $\int x^2 e^{x^3}\,dx$ with $\int \frac{1}{3}e^u\,du$.

D. Integrating $\int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C$.

E. Unsubstituting $\frac{1}{3}e^u + C$ to get $\frac{1}{3}e^{x^3} + C$.

Activity 5.1.9 Below are two correct solutions to the same integral, using two different choices for u. Which method would you prefer to use yourself?

$$\int x\sqrt{4x+4}\,dx \quad \text{Let } u = x+1$$

$$4u = 4x+4$$
$$x = u-1$$
$$du = dx$$

$$\begin{aligned}\int x\sqrt{4x+4}\,dx &= \int (u-1)\sqrt{4u}\,du \\ &= \int (2u^{3/2} - 2u^{1/2})\,du \\ &= \frac{4}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\ &= \frac{4}{5}(x+1)^{5/2} \\ &\quad - \frac{4}{3}(x+1)^{3/2} + C\end{aligned}$$

$$\int x\sqrt{4x+4}\,dx \quad \text{Let } u = \sqrt{4x+4}$$

$$u^2 = 4x+4$$
$$x = \frac{1}{4}u^2 - 1$$
$$dx = \frac{1}{2}u\,du$$

$$\begin{aligned}\int x\sqrt{4x+4}\,dx &= \int \left(\frac{1}{4}u^2 - 1\right)(u)\left(\frac{1}{2}u\,du\right) \\ &= \int \left(\frac{1}{8}u^4 - \frac{1}{2}u^2\right)\,du \\ &= \frac{1}{40}u^5 - \frac{1}{6}u^3 + C \\ &= \frac{1}{40}(4x+4)^{5/2} \\ &\quad - \frac{1}{6}(4x+4)^{3/2} + C\end{aligned}$$

Substitution method (TI1)

Activity 5.1.10 Suppose we wanted to try the substitution method to find $\int e^x \cos(e^x + 3)\, dx$. Which of these choices for u appears to be most useful?

A. $u = x$, so $du = dx$

B. $u = e^x$, so $du = e^x\, dx$

C. $u = e^x + 3$, so $du = e^x\, dx$

D. $u = \cos(x)$, so $du = -\sin(x)\, dx$

E. $u = \cos(e^x + 3)$, so $du = e^x \sin(e^x + 3)\, dx$

Activity 5.1.11 Complete the following solution using your choice from the previous activity to find $\int e^x \cos(e^x+3)\,dx$.

$$\int e^x \cos(e^x+3)\,dx \qquad \text{Let } u = ?$$

$$du = ?\,dx$$

$$\begin{aligned}\int e^x \cos(e^x+3)\,dx &= \int ?\,du\\ &= \cdots\\ &= \sin(e^x+3)+C\end{aligned}$$

Activity 5.1.12 Complete the following integration by substitution to find $\int \frac{x^3}{x^4+4}\, dx$.

$$\int \frac{x^3}{x^4+4}\, dx \qquad\qquad \text{Let } u = ?$$

$$\begin{aligned} du &= ?\, dx \\ ?du &= ?\, dx \end{aligned}$$

$$\begin{aligned} \int \frac{x^3}{x^4+4}\, dx &= \int \frac{?}{?}\, du \\ &= \cdots \\ &= \frac{1}{4}\ln|x^4+4| + C \end{aligned}$$

Substitution method (TI1)

Activity 5.1.13 Given that $\int \frac{x^3}{x^4+4}\,dx = \frac{1}{4}\ln|x^4+4| + C$, what is the value of $\int_0^2 \frac{x^3}{x^4+4}\,dx$?

A. $\dfrac{8}{20}$

B. $-\dfrac{8}{20}$

C. $\dfrac{1}{4}\ln(20) - \dfrac{1}{4}\ln(4)$

D. $\dfrac{1}{4}\ln(4) - \dfrac{1}{4}\ln(20)$

Activity 5.1.14 What's wrong with the following computation?

$$\int_0^2 \frac{x^3}{x^4+4}\,dx \qquad\qquad \text{Let } u = x^4+4$$

$$du = 4x^3\,dx$$

$$\frac{1}{4}du = x^3\,dx$$

$$\begin{aligned}\int_0^2 \frac{x^3}{x^4+4}\,dx &= \int_0^2 \frac{1/4}{u}\,du \\ &= \left[\frac{1}{4}\ln|u|\right]_0^2 \\ &= \frac{1}{4}\ln 2 - \frac{1}{4}\ln 0\end{aligned}$$

A. The wrong u substitution was made.

B. The antiderivative of $\frac{1/4}{u}$ was wrong.

C. The x values $0, 2$ were plugged in for the variable u.

Example 5.1.15 Here's one way to show the computation of this definite integral by tracking x values in the bounds.

$$\int_0^2 \frac{x^3}{x^4+4}\,dx \qquad\qquad \text{Let } u = x^4+4$$

$$du = 4x^3\,dx$$

$$\frac{1}{4}du = x^3\,dx$$

$$\begin{aligned}\int_{x=0}^{x=2} \frac{x^3}{x^4+4}\,dx &= \int_{x=0}^{x=2} \frac{1/4}{u}\,du \\ &= \left[\frac{1}{4}\ln|u|\right]_{x=0}^{x=2} \\ &= \left[\frac{1}{4}\ln|x^4+4|\right]_{x=0}^{x=2} \\ &= \frac{1}{4}\ln(20) - \frac{1}{4}\ln(4)\end{aligned}$$

□

Example 5.1.16 Instead of unsubstituting u values for x values, definite intergrals may be computed by also substituting x values in the bounds with u values. Use this idea to complete the following solution:

$$\int_1^3 x^2 e^{x^3}\,dx$$

Let $u = ?$

$$du = 3x^2\,dx$$

$$\frac{1}{3}du = x^2\,dx$$

$$\begin{aligned}\int_1^3 x^2 e^{x^3}\,dx &= \int_{x=1}^{x=3} e^{(x^3)}(x^2\,dx)\\ &= \int_{u=?}^{u=?} e^u \frac{1}{3}du\\ &= \left[\frac{1}{3}e^u\right]_?^?\\ &= ?\end{aligned}$$

$\square$

Example 5.1.17 Here is how one might write out the explanation of how to find $\int_1^3 x^2 e^{x^3}\,dx$ from start to finish by leaving bounds in terms of x instead:

$$\int_1^3 x^2 e^{x^3}\,dx \qquad \text{Let } u = x^3$$

$$du = 3x^2\,dx$$

$$\frac{1}{3}du = x^2\,dx$$

$$\begin{aligned}
\int_1^3 x^2 e^{x^3}\,dx &= \int_{x=1}^{x=3} e^{(x^3)}(x^2\,dx)\\
&= \int_{x=1}^{x=3} e^u \frac{1}{3}du\\
&= \left[\frac{1}{3}e^u\right]_{x=1}^{x=3}\\
&= \left[\frac{1}{3}e^{x^3}\right]_{x=1}^{x=3}\\
&= \frac{1}{3}e^{3^3} - \frac{1}{3}e^{1^3}\\
&= \frac{1}{3}e^{27} - \frac{1}{3}e
\end{aligned}$$

$\square$

Activity 5.1.18 Use substitution to show that

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2e^2 - 2e.$$

Activity 5.1.19 Use substitution to show that

$$\int_0^{\pi/4} \sin(2\theta)\, d\theta = \frac{1}{2}.$$

Activity 5.1.20 Use substitution to show that

$$\int u^5(u^3+1)^{1/3}\,du = \frac{1}{7}(u^3+1)^{7/3} - \frac{1}{4}(u^3+1)^{4/3} + C.$$

Substitution method (TI1)

Activity 5.1.21 Consider $\int (3x-5)^2\,dx$.

(a) Solve this integral using substitution.

(b) Replace $(3x-5)^2$ with $(9x^2-30x+25)$ in the original integral, the solve using the reverse power rule.

(c) Which method did you prefer?

Substitution method (TI1)

Activity 5.1.22 Consider $\int \tan(x)\, dx$.

(a) Replace $\tan(x)$ in the integral with a fraction involving sine and cosine.

(b) Use substitution to solve the integral.

5.2 Integration by Parts (TI2)

Learning Outcomes

- Compute integrals using integration by parts.

Activity 5.2.1 Answer the following.

(a) Using the product rule, which of these is derivative of x^3e^x with respect to x?

A. $3x^2e^x$

B. $3x^2e^x + x^3e^x$

C. $3x^2e^{x-1}$

D. $\frac{1}{4}x^4e^x$

(b) Based on this result, which of these would you suspect to equal $\int 3x^2e^x + x^3e^x\,dx$?

A. $x^3e^x + C$

B. $x^3e^x + \frac{1}{4}x^4e^x + C$

C. $6xe^x + 3x^2e^x + C$

D. $6xe^x + 3x^2e^x + 3x^2e^x + x^3e^x + C$

Activity 5.2.2 Answer the following.

(a) Which differentiation rule is easier to implement?

A. Product Rule

B. Chain Rule

(b) Which differentiation strategy do expect to be easier to reverse?

A. Product Rule

B. Chain Rule

Activity 5.2.3 Answer the following.

(a) Which of the following equations is equivalent to the formula $\frac{d}{dx}[uv] = u'v + uv'$?

A. $uv' = -\dfrac{d}{dx}(uv) - vu'$

B. $uv' = -\dfrac{d}{dx}(uv) + vu'$

C. $uv' = \dfrac{d}{dx}(uv) + vu'$

D. $uv' = \dfrac{d}{dx}(uv) - vu'$

(b) Which of these is the most concise result of integrating both sides with respect to x?

A. $\displaystyle\int (uv')\,dx = uv - \int (vu')\,dx$

B. $\displaystyle\int (u)\,dv = uv - \int (v)\,du$

C. $\displaystyle\int (uv')\,dx = uv - \int (vu')\,dx + C$

D. $\displaystyle\int (u)\,dv = uv - \int (v)\,du + C$

Integration by Parts (TI2)

Fact 5.2.4 *By the product rule, $\frac{d}{dx}[uv] = u'v + uv'$ and, subsequently, $uv' = \frac{d}{dx}[uv] - u'v$. There is a dual integration technique reversing this process, known as **integration by parts**.*

This technique involves using algebra to rewrite an integral of a product of functions in the form $\int (u)\, dv$ and then using the equality

$$\int (u)\, dv = uv - \int (v)\, du.$$

Integration by Parts (TI2)

Activity 5.2.5 Consider $\int xe^x\,dx$. Suppose we decided to let $u = x$.

(a) Compute $\frac{du}{dx} = ?$, and rewrite it as $du = ?\,dx$.

(b) What is the best candidate for dv?

A. $dv = x\,dx$

B. $dv = e^x$

C. $dv = x$

D. $dv = e^x\,dx$

(c) Given that $dv = e^x\,dx$, find $v = ?$.

(d) Show why $\int xe^x\,dx$ may now be rewritten as $xe^x - \int e^x\,dx$.

(e) Solve $\int e^x\,dx$, and then give the most general antiderivative of $\int xe^x\,dx$.

Example 5.2.6 Here is how one might write out the explanation of how to find $\int xe^x\,dx$ from start to finish:

$$\int xe^x\,dx$$

$$\begin{aligned} u &= x & dv &= e^x\,dx \\ du &= 1\cdot\,dx & v &= e^x \end{aligned}$$

$$\begin{aligned} \int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \end{aligned}$$

$\square$

Activity 5.2.7 Which step of the previous example do you think was the most important?

A. Choosing $u = x$ and $dv = e^x\,dx$.

B. Finding $du = 1\,dx$ and $v = e^x\,dx$.

C. Applying integration by parts to rewrite $\int xe^x\,dx$ as $xe^x - \int e^x\,dx$.

D. Integrating $\int e^x\,dx$ to get $xe^x - e^x + C$.

Integration by Parts (TI2)

Activity 5.2.8 Consider the integral $\int x^9 \ln(x)\, dx$. Suppose we proceed using integration by parts. We choose $u = \ln(x)$ and $dv = x^9\, dx$.What is du?What is v?What do you get when plugging these pieces into integration by parts?Does the new integral $\int v\, du$ seem easier or harder to compute than the original integral $\int x^9 \ln(x)\, dx$?

A. The original integral is easier to compute.

B. The new integral is easier to compute.

C. Neither integral seems harder than the other one.

Integration by Parts (TI2)

Activity 5.2.9 Consider the integral $\int x^9 \ln(x)\,dx$ once more. Suppose we still proceed using integration by parts. However, this time we choose $u = x^9$ and $dv = \ln(x)\,dx$. Do you prefer this choice or the choice we made in Activity 5.2.8?

A. We prefer the substitution choice of $u = \ln(x)$ and $dv = x^9\,dx$.

B. We prefer the substitution choice of $u = x^9$ and $dv = \ln(x)\,dx$.

C. We do not have a strong preference, since these choices are of the same difficulty.

Integration by Parts (TI2)

Activity 5.2.10 Consider the integral $\int x\cos(x)\,dx$. Suppose we proceed using integration by parts. Which of the following candidates for u and dv would best allow you to evaluate this integral?

A. $u = \cos(x)$, $dv = xdx$

B. $u = \cos(x)\,dx$, $dv = x$

C. $u = x\,dx$, $dv = \cos(x)$

D. $u = x$, $dv = \cos(x)\,dx$

Integration by Parts (TI2)

Activity 5.2.11 Evaluate the integral $\int x\cos(x)\,dx$ using integration by parts.

Activity 5.2.12 Now use integration by parts to evaluate the integral $\int_{\frac{\pi}{6}}^{\pi} x\cos(x)\,dx$.

Integration by Parts (TI2)

Activity 5.2.13 Consider the integral $\int x \arctan(x)\, dx$. Suppose we proceed using integration by parts. Which of the following candidates for u and dv would best allow you to evaluate this integral?

A. $u = x\, dx$, $dv = \arctan(x)$

B. $u = \arctan(x)$, $dv = x\, dx$

C. $u = x \arctan(x)$, $dv = \, dx$

D. $u = x$, $dv = \arctan(x)\, dx$

Integration by Parts (TI2)

Activity 5.2.14 Consider the integral $\int e^x \cos(x)\, dx$. Suppose we proceed using integration by parts. Which of the following candidates for u and dv would best allow you to evaluate this integral?

A. $u = e^x$, $dv = \cos(x)\, dx$

B. $u = \cos(x)$, $dv = e^x\, dx$

C. $u = e^x\, dx$, $dv = \cos(x)$

D. $u = \cos(x)\, dx$, $dv = e^x$

Activity 5.2.15 Suppose we started using integration by parts to solve the integral $\int e^x \cos(x)\,dx$ as follows:

$$\int e^x \cos(x)\,dx$$

$$\begin{aligned} u &= \cos(x) & dv &= e^x\,dx \\ du &= -\sin(x)\,dx & v &= e^x \end{aligned}$$

$$\begin{aligned} \int e^x \cos(x)\,dx &= \cos(x)e^x - \int e^x(-\sin(x)\,dx) \\ &= \cos(x)e^x + \int e^x \sin(x)\,dx \end{aligned}$$

We will have to use integration by parts a second time to evaluate the integral $\int e^x \sin(x)\,dx$. Which of the following candidates for u and dv would best allow you to continue evaluating the original integral $\int e^x \cos(x)\,dx$?

A. $u = e^x$, $dv = \sin(x)\,dx$

B. $u = \sin(x)$, $dv = e^x\,dx$

C. $u = e^x\,dx$, $dv = \sin(x)$

D. $u = \sin(x)\,dx$, $dv = e^x$

Integration by Parts (TI2)

Activity 5.2.16 Use integration by parts to show that $\displaystyle\int_0^{\frac{\pi}{4}} x\sin(2x)\,dx = \frac{1}{4}$.

Integration by Parts (TI2)

Activity 5.2.17 Consider the integral $\int t^5 \sin(t^3)\, dt$.

(a) Use the substitution $x = t^3$ to rewrite the integral in terms of x.

(b) Use integration by parts to evaluate the integral in terms of x.

(c) Replace x with t^3 to finish evaluating the original integral.

Integration by Parts (TI2)

Activity 5.2.18 Use integration by parts to show that $\int \ln(z)\,dz = z\ln(z) - z + C$.

Integration by Parts (TI2)

Activity 5.2.19 Given that that $\displaystyle\int \ln(z)\,dz = z\ln(z) - z + C$, evaluate $\displaystyle\int (\ln(z))^2\,dz$.

Integration by Parts (TI2)

Activity 5.2.20 Consider the antiderivative $\int (\sin(x))^2 dx$.

(a) Noting that $\int (\sin(x))^2 dx = \int (\sin(x))(\sin(x))dx$ and letting $u = \sin(x)$, $dv = \sin(x)dx$, what equality does integration by parts yield?

A $\int (\sin(x))^2 dx = \sin(x)\cos(x) + \int (\cos(x))^2 dx$.

B $\int (\sin(x))^2 dx = -\sin(x)\cos(x) + \int (\cos(x))^2 dx$.

C $\int (\sin(x))^2 dx = \sin(x)\cos(x) - \int (\cos(x))^2 dx$.

D $\int (\sin(x))^2 dx = -\sin(x)\cos(x) - \int (\cos(x))^2 dx$.

(b) Using the fact that $(\cos(x))^2 = 1 - (\sin(x))^2$ to rewrite the above equality.

(c) Solve algebraically for $\int (\sin(x))^2 dx$.

Activity 5.2.21 Modifying the approach from Activity 5.2.20, use parts to find $\displaystyle\int (\cos(x))^2 dx$.

5.3 Integration of trigonometry (TI3)

Learning Outcomes

- Compute integrals involving products of trigonometric functions.

Activity 5.3.1 Consider $\displaystyle\int \sin(x)\cos(x)\,dx$. Which substitution would you choose to evaluate this integral?

A. $u = \sin(x)$

B. $u = \cos(x)$

C. $u = \sin(x)\cos(x)$

D. Substitution is not effective

Activity 5.3.2 Consider $\int \sin^4(x)\cos(x)\,dx$. Which substitution would you choose to evaluate this integral?

A. $u = \sin(x)$

B. $u = \sin^4(x)$

C. $u = \cos(x)$

D. Substitution is not effective

Integration of trigonometry (TI3)

Activity 5.3.3 Consider $\int \sin^4(x)\cos^3(x)\,dx$. Which substitution would you choose to evaluate this integral?

A. $u = \sin(x)$

B. $u = \cos^3(x)$

C. $u = \cos(x)$

D. Substitution is not effective

Integration of trigonometry (TI3)

Activity 5.3.4 It's possible to use subtitution to evaluate $\int \sin^4(x)\cos^3(x)\,dx$, by taking advantage of the trigonometric identity $\sin^2(x)+\cos^2(x)=1$.

Complete the following substitution of $u=\sin(x), du=\cos(x)\,dx$ by filling in the missing ?s.

$$\begin{aligned}
\int \sin^4(x)\cos^3(x)\,dx &= \int \sin^4(x)(\ ?\ )\cos(x)\,dx \\
&= \int \sin^4(x)(1-?)\cos(x)\,dx \\
&= \int ?(1-?)\,du \\
&= \int (u^4-u^6)\,du \\
&= \frac{1}{5}u^5-\frac{1}{7}u^7+C \\
&= ?
\end{aligned}$$

Activity 5.3.5 Trying to substitute $u = \cos(x), du = -\sin(x)\,dx$ in the previous example is less successful.

$$\begin{aligned}\int \sin^4(x)\cos^3(x)\,dx &= -\int \sin^3(x)\cos^3(x)(-\sin(x)\,dx)\\ &= -\int \sin^3(x)u^3\,du\\ &= \cdots ?\end{aligned}$$

Which feature of $\sin^4(x)\cos^3(x)$ made $u = \sin(x)$ the better choice?

A. The even power of $\sin^4(x)$

B. The odd power of $\cos^3(x)$

Activity 5.3.6 Try to show

$$\int \sin^5(x)\cos^2(x)\,dx = -\frac{1}{7}\cos^7(x) + \frac{2}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$$

by first trying $u = \sin(x)$, and then trying $u = \cos(x)$ instead.

Which substitution worked better and why?

A. $u = \sin(x)$ due to $\sin^5(x)$'s odd power.

B. $u = \sin(x)$ due to $\cos^2(x)$'s even power.

C. $u = \cos(x)$ due to $\sin^5(x)$'s odd power.

D. $u = \cos(x)$ due to $\cos^2(x)$'s even power.

Observation 5.3.7 When integrating the form $\int \sin^m(x)\cos^n(x)\,dx$:

- If sin's power is odd, rewrite the integral as $\int g(\cos(x))\sin(x)\,dx$ and use $u = \cos(x)$.
- If cos's power is odd, rewrite the integral as $\int h(\sin(x))\cos(x)\,dx$ and use $u = \sin(x)$.

Integration of trigonometry (TI3)

Activity 5.3.8 Let's consider $\displaystyle\int \sin^2(x)\, dx$.

(a) Use the fact that $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ to rewrite the integrand using the above identities as an integral involving $\cos(2x)$.

(b) Show that the integral evaluates to $\frac{1}{2}\,x - \frac{1}{4}\,\sin\left(2\,x\right) + C$.

Activity 5.3.9 Let's consider $\int \sin^2(x)\cos^2(x)\,dx$.

(a) Use the fact that $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ and $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ to rewrite the integrand using the above identities as an integral involving $\cos^2(2x)$.

(b) Use the above identities to rewrite this new integrand as one involving $\cos(4x)$.

(c) Show that integral evaluates to $\frac{1}{8}\,x - \frac{1}{32}\,\sin(4\,x) + C$.

Integration of trigonometry (TI3)

Activity 5.3.10 Consider $\int \sin^4(x)\cos^4(x)\,dx$. Which would be the most useful way to rewrite the integral?

A. $\displaystyle\int (1-\cos^2(x))^2\cos^4(x)\,dx$

B. $\displaystyle\int \sin^4(x)(1-\sin^2(x))^2\,dx$

C. $\displaystyle\int \left(\frac{1-\cos(2x)}{2}\right)^2\left(\frac{1+\cos(2x)}{2}\right)^2\,dx$

Activity 5.3.11 Consider $\int \sin^3(x)\cos^5(x)\,dx$. Which would be the most useful way to rewrite the integral?

A. $\int (1-\cos^2(x))\cos^5(x)\sin(x)\,dx$

B. $\int \sin^3(x)\left(\frac{1+\cos(2x)}{2}\right)^2\cos(x)\,dx$

C. $\int \sin^3(x)(1-\sin^2(x))^2\cos(x)\,dx$

Remark 5.3.12 We might also use some other trigonometric identities to manipulate our integrands, listed in Appendix B.

Activity 5.3.13 Consider $\displaystyle\int \sin(\theta)\sin(3\theta)\,d\theta$.

(a) Find an identity from Appendix B which could be used to transform our integrand.

(b) Rewrite the integrand using the selected identity.

(c) Evaluate the integral.

5.4 Trigonometric Substitution (TI4)

Learning Outcomes

- Use trigonometric substitution to compute indefinite integrals.

Trigonometric Substitution (TI4)

Activity 5.4.1 Consider $\int \sqrt{9-4x^2}\,dx$. Which substitution would you choose to evaluate this integral?

A. $u = 9 - 4x^2$

B. $u = \sqrt{9-4x^2}$

C. $u = 3 - 2x$

D. Substitution is not effective

Activity 5.4.2 To find $\int \sqrt{9-4x^2}\,dx$, we will need a more advanced substitution. Which of these candidates is most reasonable?

A. Let v satisfy $9-4x^2 = 9-9e^{2v} = 9e^{-2v}$.

B. Let θ satisfy $9-4x^2 = 9-9\sin^2\theta = 9\cos^2\theta$.

C. Let w satisfy $9-4x^2 = 4-8\ln|w| = 4\ln|2w|$.

D. Let ϕ satisfy $9-4x^2 = 4-4\cos^2\phi = 4\sin^2\phi$.

Activity 5.4.3 Fill in the missing ?s for the following calculation.

$$\begin{aligned}
\text{Let } 9 - 4x^2 &= 9 - 9\sin^2\theta = 9\cos^2\theta \\
4x^2 &= ? \\
x &= ? \\
dx &= ?\, d\theta
\end{aligned}$$

$$\begin{aligned}
\int \sqrt{9 - 4x^2}\, dx &= \int \sqrt{?}\, (?\, d\theta) \\
&= \int \frac{9}{2}\cos^2\theta\, d\theta
\end{aligned}$$

Activity 5.4.4 From Section 5.3 we may find $\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$.

Use this to continue your work in the previous activity and complete the integration by trigonometric substitution.

$$\begin{aligned}\sin(\theta) &= ?\\ \theta &= \arcsin(?)\\ \cos(\theta) &= ?\sqrt{?}\end{aligned}$$

$$\begin{aligned}\int \sqrt{9-4x^2}\,dx = \cdots &= \int \frac{9}{2}\cos^2\theta\, d\theta\\ &= \frac{9}{2}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C\\ &= \frac{9}{4}(?) + \frac{9}{4}(?)(?) + C\end{aligned}$$

Activity 5.4.5 Use similar reasoning to complete the following proof that $\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$.

$$\begin{aligned}
\text{Let } 1 - x^2 = 1 - {?}\theta &= {?}\theta \\
x^2 &= {?} \\
x &= {?} \\
dx &= {?}\, d\theta \\
\theta &= {?}
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\, dx &= \int \frac{1}{\sqrt{?}}\, ({?}\, d\theta) \\
&= \int d\theta \\
&= {?} + C \\
&= \arcsin(x) + C
\end{aligned}$$

Activity 5.4.6 Substitutions of the form

$$16 - 25x^2 = 16 - 16\sin^2 x = 16\cos^2 x$$

are made possible due to the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$.

Which two of these four identities can be obtained from dividing both sides of $\sin^2(x) + \cos^2(x) = 1$ by $\cos^2(x)$ and rearranging?

A. $\tan^2(x) - 1 = \sec^2(x)$

B. $\tan^2(x) + 1 = \sec^2(x)$

C. $\sec^2(x) - 1 = \tan^2(x)$

D. $\sec^2(x) + 1 = \tan^2(x)$

Observation 5.4.7 In summary, certain quadratic expressions inside an integral may be substituted with trigonometric functions to take advantage of trigonometric identities and simplify the integrand:

$$\text{Let } b^2 - a^2x^2 = b^2 - b^2\sin^2(\theta) = b^2\cos^2(\theta)$$
$$\text{So } x = \frac{b}{a}\sin(\theta)$$

$$\text{Let } b^2 + a^2x^2 = b^2 + b^2\tan^2(\theta) = b^2\sec^2(\theta)$$
$$\text{So } x = \frac{b}{a}\tan(\theta)$$

$$\text{Let } a^2x^2 - b^2 = b^2\sec^2(\theta) - b^2 = b^2\tan^2(\theta)$$
$$\text{So } x = \frac{b}{a}\sec(\theta)$$

Activity 5.4.8 Complete the following trignometric substitution to find $\displaystyle\int \frac{3}{4+25x^2}\,dx$.

$$\begin{aligned} \text{Let } 4+25x^2 &= 2+?\theta = ?\theta \\ 25x^2 &= ? \\ x &= ? \\ dx &= ?\,d\theta \\ \theta &= ? \end{aligned}$$

$$\begin{aligned} \int \frac{3}{4+25x^2}\,dx &= \int \frac{3}{?}\,(?\,d\theta) \\ &= \int ?d\theta \\ &= ?+C \\ &= \frac{3}{10}\arctan(\frac{5}{2}x)+C \end{aligned}$$

Activity 5.4.9 Complete the following trignometric substitution to find $\int \frac{7}{x\sqrt{9x^2-16}}\,dx$.

$$\begin{aligned}
\text{Let } 9x^2-16 &= ?\theta - 16 = ?\theta \\
9x^2 &= ? \\
x &= ? \\
dx &= ?\,d\theta \\
\theta &= ?
\end{aligned}$$

$$\begin{aligned}
\int \frac{7}{x\sqrt{9x^2-16}}\,dx &= \int \frac{7}{?\sqrt{?}}\,(?\,d\theta) \\
&= \int ?d\theta \\
&= ? + C \\
&= \frac{7}{4}\,\mathrm{arcsec}(\frac{3}{4}x) + C
\end{aligned}$$

Trigonometric Substitution (TI4)

Activity 5.4.10 Use appropriate trignometric substitutions and the given trigonometric integrals to find each of the following.

(a)

$$\begin{aligned}\int \frac{\sqrt{-9\,x^2+16}}{x^2}\,dx &= \cdots \\ &= \int \frac{3\cos^2\theta}{\sin^2\theta}\,d\theta \\ &= -3\theta - 3\frac{\cos\theta}{\sin\theta} + C \\ &= -3\ \arcsin\left(?\right) - \frac{\sqrt{?}}{?} + C\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2\sqrt{9\,x^2-16}}{x}\,dx &= \cdots \\ &= \int 8\tan^2\theta\,d\theta \\ &= 8\tan\theta - 8\theta + C \\ &= ?\sqrt{?} - 8\ \operatorname{arcsec}\left(?\right) + C\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{1}{\sqrt{81\,x^2+4}}\,dx &= \cdots \\ &= \int \frac{1}{9}\sec\theta\,d\theta \\ &= \frac{1}{9}\log|\sec\theta + \tan\theta| + C \\ &= \frac{1}{9}\ \log\left|? + \frac{1}{2}\sqrt{?}\right| + C\end{aligned}$$

Activity 5.4.11 Consider the unit circle $x^2 + y^2 = 1$. Find a function $f(x)$ so that $y = f(x)$ is the graph of the upper-half semicircle of the unit circle.

Activity 5.4.12

(a) Find the area under the curve $y = f(x)$ from Activity 5.4.11.

(b) How does this value compare to what we know about areas of circles?

5.5 Tables of Integrals (TI5)

Learning Outcomes

- I can integrate functions using a table of integrals.

Tables of Integrals (TI5)

Activity 5.5.1 Consider the integral $\int \sqrt{16 - 9x^2}\, dx$. Which of the following substitutions appears most promising to find an antiderivaitve for this integral?

A. $u = 16 - 9x^2$

B. $u = 9x^2$

C. $u = 3x$

D. $u = x$

Activity 5.5.2 The form of which entry from Appendix A best matches the form of the integral $\displaystyle\int \sqrt{16-9x^2}\,dx$?

A. b.
B. c.
C. g.
D. h.

Activity 5.5.3 For each of the following integrals, identify which entry from Appendix A best matches the form of that integral.

(a) $\displaystyle\int \frac{25x^2}{\sqrt{25x^2-9}}\,dx$

(b) $\displaystyle\int \frac{81x^2}{\sqrt{16-x^2}}\,dx$

(c) $\displaystyle\int \frac{1}{10x\sqrt{100-x^2}}\,dx$

(d) $\displaystyle\int \frac{7}{\sqrt{25x^2-9}}\,dx$

(e) $\displaystyle\int \frac{1}{\sqrt{25x^2+16}}\,dx$

Example 5.5.4 Here is how one might write out the explanation of how to find $\displaystyle\int \frac{3}{x\sqrt{49x^2-4}}\,dx$ from start to finish:

$$\int \frac{3}{x\sqrt{49x^2-4}}\,dx \qquad\qquad \begin{aligned} &\text{Let } u^2 = 49x^2 \\ &\text{Let } a^2 = 4 \\ &\quad u = 7x \\ &\quad du = 7\,dx \\ &\quad \frac{1}{7}\,du = dx \\ &\quad a = 2 \end{aligned}$$

$$\begin{aligned} \int \frac{3}{x\sqrt{49x^2-4}}\,dx &= 3\int \frac{1}{x\sqrt{49x^2-4}}(\,dx) \\ &= 3\int \frac{1}{\frac{u}{7}\sqrt{u^2-a^2}}\left(\frac{1}{7}\,du\right) \\ &= 3\int \frac{1}{u\sqrt{u^2-a^2}}\,du \qquad \text{which best matches f.} \\ &= 3\left(\frac{1}{a}\operatorname{arcsec}\left(\frac{u}{a}\right)\right)+C \\ &= \frac{3}{2}\operatorname{arcsec}\left(\frac{7x}{2}\right)+C \end{aligned}$$

$\square$

Activity 5.5.5 Which step of the previous example do you think was the most important?

A. Choosing $u^2 = 49x^2$ and $a^2 = 4$.

B. Finding $u = 7x$, $du = 7\,dx$, $\frac{1}{7}\,du = \,dx$, and $a = 2$.

C. Substituting $\frac{3}{x\sqrt{49x^2 - 4}}\,dx$ with $3\int \frac{1}{u\sqrt{u^2 - a^2}}\,du$ and finding the best match of f from Appendix A.

D. Integrating $3\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = 3(\frac{1}{a}\operatorname{arcsec}(\frac{u}{a})) + C$.

E. Unsubstituting $3(\frac{1}{a}\operatorname{arcsec}(\frac{u}{a})) + C$ to get $\frac{3}{2}\operatorname{arcsec}(\frac{7x}{2}) + C$.

Activity 5.5.6 Consider the integral $\int \frac{1}{\sqrt{64 - 9x^2}}\, dx$. Suppose we proceed using Appendix A. We choose $u^2 = 9x^2$ and $a^2 = 64$.

(a) What is u?

(b) What is du?

(c) What is a?

(d) What do you get when plugging these pieces into the integral $\int \frac{1}{\sqrt{64 - 9x^2}}\, dx$?

(e) Is this a good substitution choice or a bad substitution choice?

Activity 5.5.7 Consider the integral $\int \frac{1}{\sqrt{64 - 9x^2}}\, dx$ once more. Suppose we still proceed using Appendix A. However, this time we choose $u^2 = x^2$ and $a^2 = 64$. Do you prefer this choice of substitution or the choice we made in Activity 5.5.6?

A. We prefer the substitution choice of $u^2 = x^2$ and $a^2 = 64$.

B. We prefer the substitution choice of $u^2 = 9x^2$ and $a^2 = 64$.

C. We do not have a strong preference, since these substitution choices are of the same difficulty.

Tables of Integrals (TI5)

Activity 5.5.8 Use the appropriate substitution and entry from Appendix A to show that $\displaystyle\int \frac{7}{x\sqrt{4+49x^2}}\,dx = -\frac{7}{2}\ln\left|\frac{2+\sqrt{49x^2+4}}{7x}\right| + C$.

Activity 5.5.9 Use the appropriate substitution and entry from Appendix A to show that $\displaystyle\int \frac{3}{5x^2\sqrt{36-49x^2}}\,dx = -\frac{\sqrt{36-49x^2}}{60x} + C.$

Tables of Integrals (TI5)

Activity 5.5.10 Evaluate the integral $\int 8\sqrt{4x^2-81}\,dx$. Be sure to specify which entry is used from Appendix A at the corresponding step.

5.6 Partial fractions (TI6)

Learning Outcomes

- I can integrate functions using the method of partial fractions.

Activity 5.6.1 Consider $\int \frac{x^2+x+1}{x^3+x}\,dx$. Which substitution would you choose to evaluate this integral?

A. $u = x^3$

B. $u = x^3 + x$

C. $u = x^2 + x + 1$

D. Substitution is not effective

Activity 5.6.2 Using the method of substitution, which of these is equal to $\int \frac{5}{x+7} dx$?

A. $5 \ln |x+7| + C$

B. $\frac{5}{7} \ln |x+7| + C$

C. $5 \ln |x| + 5 \ln |7| + C$

D. $\frac{5}{7} \ln |x| + C$

Observation 5.6.3 To avoid repetitive substitution, the following integral formulas will be useful.

$$\int \frac{1}{x+b}dx = \ln|x+b| + C$$

$$\int \frac{1}{(x+b)^2}dx = -\frac{1}{x+b} + C$$

$$\int \frac{1}{x^2+b^2}dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right) + C$$

Partial fractions (TI6)

Activity 5.6.4 Which of the following is equal to $\dfrac{1}{x} + \dfrac{1}{x^2+1}$?

A. $\dfrac{2x}{x^2+x+1}$

B. $\dfrac{x^3+x}{x^2+x+1}$

C. $\dfrac{2x}{x^3+x}$

D. $\dfrac{x^2+x+1}{x^3+x}$

Activity 5.6.5 Based on the previous activities, which of these is equal to $\displaystyle\int \frac{x^2+x+1}{x^3+x}dx$?

A. $\ln|x| + \arctan(x) + C$

B. $\ln|x^2+x+1| + C$

C. $\ln|x^3+x| + C$

D. $\arctan(x^3+x) + C$

Activity 5.6.6 Suppose we know

$$\frac{10x-11}{x^2+x-2} = \frac{7}{x-1} + \frac{3}{x+2}.$$

Which of these is equal to $\int \frac{10x-11}{x^2+x-2} dx$?

A. $7 \ln |x-1| + 3 \arctan(x+2) + C$

B. $7 \ln |x-1| + 3 \ln |x+2| + C$

C. $7 \arctan(x-1) + 3 \arctan(x+2) + C$

D. $7 \arctan(x-1) + 3 \ln |x+2| + C$

Partial fractions (TI6)

Observation 5.6.7 To find integrals like $\int \frac{x^2+x+1}{x^3+x}dx$ and $\int \frac{10x-11}{x^2+x-2}dx$, we'd like to **decompose** the fractions into simpler **partial fractions** that may be integrated with these formulas

$$\int \frac{1}{x+b}dx = \ln|x+b| + C$$

$$\int \frac{1}{(x+b)^2}dx = -\frac{1}{x+b} + C$$

$$\int \frac{1}{x^2+b^2}dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right) + C$$

Fact 5.6.8 Partial Fraction Decomposition. *Let* $\dfrac{p(x)}{q(x)}$ *be a rational function, where the degree of* p *is less than the degree of* q*.*

1. Linear Terms: *Let* $(x-a)^n$ *divide* $q(x)$*, Then the decomposition of* $\frac{p(x)}{q(x)}$ *will contain the terms*

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$

2. Quadratic Terms: *Let* $(x^2+bx+c)^n$ *divide* $q(x)$*, where* x^2+bx+c *is irreducable. Then the decomposition of* $\frac{p(x)}{q(x)}$ *will contain the terms*

$$\frac{B_1x + C_1}{x^2+bx+c} + \frac{B_2x + C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx + C_n}{(x^2+bx+c)^n}.$$

Example 5.6.9 Following is an example of a rather involved partial fraction decomposition.

$$\begin{aligned}
&\frac{7\,x^6 - 4\,x^5 + 41\,x^4 - 20\,x^3 + 24\,x^2 + 11\,x + 16}{x(x-1)^2(x^2+4)^2} \\
=&\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}
\end{aligned}$$

Using some algebra, it's possible to find values for A through G to determine

$$\begin{aligned}
&\frac{7\,x^6 - 4\,x^5 + 41\,x^4 - 20\,x^3 + 24\,x^2 + 11\,x + 16}{x(x-1)^2(x^2+4)^2} \\
=&\frac{1}{x} + \frac{2}{x-1} + \frac{3}{(x-1)^2} + \frac{4x+5}{x^2+4} + \frac{6x+7}{(x^2+4)^2}.
\end{aligned}$$

□

Partial fractions (TI6)

Activity 5.6.10 Which of the following is the form of the partial fraction decomposition of $\dfrac{x^3-7x^2-7x+15}{x^3(x+5)}$?

A. $\dfrac{A}{x}+\dfrac{B}{x+5}$

B. $\dfrac{A}{x^3}+\dfrac{B}{x+5}$

C. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x+5}$

D. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{Dx+E}{x+5}$

Partial fractions (TI6)

Activity 5.6.11 Which of the following is the form of the partial fraction decomposition of $\dfrac{x^2+1}{(x-3)^2(x^2+4)^2}$?

A. $\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{C}{x^2+4}+\dfrac{D}{(x^2+4)^2}$

B. $\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{Cx+D}{(x^2+4)^2}$

C. $\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{C}{x^2+4}+\dfrac{Dx+E}{(x^2+4)^2}$

D. $\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{Cx+D}{x^2+4}+\dfrac{Ex+F}{(x^2+4)^2}$

Partial fractions (TI6)

Activity 5.6.12 Consider that the partial decomposition of $\dfrac{x^2+5x+3}{(x+1)^2x}$ is

$$\frac{x^2+5x+3}{(x+1)^2x} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x}.$$

What equality do we obtain if we multiply both sides of the above equation by $(x+1)^2x$?

A. $x^2+5x+3 = Ax(x+1) + Bx + C(x+1)^2$

B. $x^2+5x+3 = A(x+1) + B(x+1)^2 + Cx$

C. $x^2+5x+3 = Ax(x+1) + Bx + C(x+1)$

D. $x^2+5x+3 = Ax(x+1) + Bx^2 + C(x+1)^2$

Activity 5.6.13 Use your choice in Activity 5.6.12 (which must hold for any x value) to answer the following.

(a) By substituting $x = 0$ into the equation, we may find:

A. $A = 1$ B. $B = -2$ C. $C = 3$

(b) By substituting $x = -1$ into the equation, we may find:

A. $A = -4$ B. $B = 1$ C. $C = 5$

Partial fractions (TI6)

Activity 5.6.14 Using the results of Activity 5.6.13, show how to rewrite our choice from Activity 5.6.12

$$?x^2 + ?x = Ax^2 + Ax.$$

What value of A satisfies this equation?

A. -2 B. 3 C. 4 D. -5

Activity 5.6.15 By using the form of the decomposition $\dfrac{x^2+5x+3}{(x+1)^2x} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x}$ and the coefficients found in Activity 5.6.13 and Activity 5.6.14, evaluate $\displaystyle\int \frac{x^2+5x+3}{(x+1)^2x}dx$.

Activity 5.6.16 Given that $\frac{x^3 - 7x^2 - 7x + 15}{x^3(x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+5}$ do the following to find A, B, C, and D.

(a) Eliminate the fractions to obtain

$$x^3 - 7x^2 - 7x + 15 = A(?)(?) + B(?)(?) + C(?) + D(?).$$

(b) Plug in an x value that lets you find the value of C.

(c) Plug in an x value that lets you find the value of D.

(d) Use other algebra techniques to find the values of A and B.

Partial fractions (TI6)

Activity 5.6.17 Given your choice in Activity 5.6.16 Find $\int \frac{x^3 - 7x^2 - 7x + 15}{x^3(x+5)} dx$.

Partial fractions (TI6)

Activity 5.6.18 Consider the rational expression $\dfrac{2x^3+2x+4}{x^4+2x^3+4x^2}$. Which of the following is the partial fraction decomposition of this rational expression?

A. $\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{2x-1}{x^2+2x+4}$

B. $\dfrac{2}{x}+\dfrac{0}{x^2}+\dfrac{-1}{x^2+2x+4}$

C. $\dfrac{0}{x}+\dfrac{1}{x^2}+\dfrac{-1}{x^2+2x+4}$

D. $\dfrac{0}{x}+\dfrac{1}{x^2}+\dfrac{2x-1}{x^2+2x+4}$

Partial fractions (TI6)

Activity 5.6.19 Given your choice in Activity 5.6.18 Find $\int \frac{2x^3+2x+4}{x^4+2x^3+4x^2}dx$.

Partial fractions (TI6)

Activity 5.6.20 Given that $\dfrac{2x+5}{x^2+3x+2} = \dfrac{-1}{x+2} + \dfrac{3}{x+1}$, find $\displaystyle\int_0^3 \frac{2x+5}{x^2+3x+2}dx$.

Partial fractions (TI6)

Activity 5.6.21 Evaluate $\displaystyle\int \frac{4x^2-3x+1}{(2x+1)(x+2)(x-3)}dx.$

5.7 Integration strategy (TI7)

Learning Outcomes

- I can select appropriate strategies for integration.

Integration strategy (TI7)

Activity 5.7.1 Consider the integral $\int e^t \tan(e^t) \sec^2(e^t)\, dt$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

A. The method of substitution

B. The method of integration by parts

C. Trigonometric substitution

D. Using a table of integrals

E. The method of partial fractions

Integration strategy (TI7)

Activity 5.7.2 Consider the integral $\int \frac{2x+3}{1+x^2}\,dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

A. The method of substitution

B. The method of integration by parts

C. Trigonometric substitution

D. Using a table of integrals

E. The method of partial fractions

Integration strategy (TI7)

Activity 5.7.3 Consider the integral $\int \frac{x}{\sqrt[3]{1-x^2}}\,dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

A. The method of substitution

B. The method of integration by parts

C. Trigonometric substitution

D. Using a table of integrals

E. The method of partial fractions

Integration strategy (TI7)

Activity 5.7.4 Consider the integral $\int \frac{1}{2x\sqrt{1-36x^2}}\,dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

A. The method of substitution

B. The method of integration by parts

C. Trigonometric substitution

D. Using a table of integrals

E. The method of partial fractions

Integration strategy (TI7)

Activity 5.7.5 Consider the integral $\int t^5 \cos(t^3)\, dt$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

A. The method of substitution

B. The method of integration by parts

C. Trigonometric substitution

D. Using a table of integrals

E. The method of partial fractions

Integration strategy (TI7)

Activity 5.7.6 Consider the integral $\int \frac{1}{1+e^x}\,dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

A. The method of substitution

B. The method of integration by parts

C. Trigonometric substitution

D. Using a table of integrals

E. The method of partial fractions

5.8 Improper integrals (TI8)

Learning Outcomes

- I can compute improper integrals.

Improper integrals (TI8)

Activity 5.8.1 Recall $\int \frac{1}{x^2} dx = -\frac{1}{x} + C$. Compute the following definite integrals.

(a) $\displaystyle\int_{1/100}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{1/100}^{1}$

(b) $\displaystyle\int_{1/10000}^{1} \frac{1}{x^2} dx$

(c) $\displaystyle\int_{1/1000000}^{1} \frac{1}{x^2} dx$

Improper integrals (TI8)

Activity 5.8.2 What do you notice about $\int_a^1 \frac{1}{x^2}dx$ as a approached 0 in Activity 5.8.1?

A. $\int_a^1 \frac{1}{x^2}dx$ approaches 0.

B. $\int_a^1 \frac{1}{x^2}dx$ approaches a finite constant greater than 0.

C. $\int_a^1 \frac{1}{x^2}dx$ approaches ∞.

D. There is not enough information.

Activity 5.8.3 Compute the following definite integrals, again using $\displaystyle\int \frac{1}{x^2}dx = -\frac{1}{x} + C$.

(a) $\displaystyle\int_1^{100} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^{100}$

(b) $\displaystyle\int_1^{10000} \frac{1}{x^2}dx$

(c) $\displaystyle\int_1^{1000000} \frac{1}{x^2}dx$

Activity 5.8.4 What do you notice about $\int_1^b \frac{1}{x^2}dx$ as b approached ∞ in Activity 5.8.3?

A. $\int_1^b \frac{1}{x^2}dx$ approaches 0.

B. $\int_1^b \frac{1}{x^2}dx$ approaches a finite constant greater than 0.

C. $\int_1^b \frac{1}{x^2}dx$ approaches ∞.

D. There is not enough information.

Improper integrals (TI8)

Activity 5.8.5 Recall $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$. Compute the following definite integrals.

(a) $\displaystyle\int_{1/100}^{1} \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_{1/100}^{1}$

(b) $\displaystyle\int_{1/10000}^{1} \frac{1}{\sqrt{x}} dx$

(c) $\displaystyle\int_{1/1000000}^{1} \frac{1}{\sqrt{x}} dx$

Activity 5.8.6

(a) What do you notice about the integral $\int_a^1 \frac{1}{\sqrt{x}} dx$ as a approached 0 in Activity 5.8.5?

A. $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches 0.

B. $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches a finite constant greater than 0.

C. $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches ∞.

D. There is not enough information.

(b) How does this compare to what you found in Activity 5.8.1?

Activity 5.8.7 Compute the following definite integrals using $\displaystyle\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} + C$.

(a) $\displaystyle\int_1^{100} \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_1^{100}$

(b) $\displaystyle\int_1^{10000} \frac{1}{\sqrt{x}}dx$

(c) $\displaystyle\int_1^{1000000} \frac{1}{\sqrt{x}}dx$

Activity 5.8.8

(a) What do you notice the integral $\int_1^b \frac{1}{\sqrt{x}} dx$ as b approached ∞ in Activity 5.8.7?

A. $\int_1^b \frac{1}{\sqrt{x}} dx$ approaches 0.

B. $\int_1^b \frac{1}{\sqrt{x}} dx$ approaches a finite constant greater than 0.

C. $\int_1^b \frac{1}{\sqrt{x}} dx$ approaches ∞.

D. There is not enough information.

(b) How does this compare to what you found in Activity 5.8.3?

Definition 5.8.9 For a function $f(x)$ and a constant a, we let $\displaystyle\int_a^\infty f(x)dx$ denote

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\left(\int_a^b f(x)dx\right).$$

If this limit is a defined real number, then we say $\displaystyle\int_a^\infty f(x)dx$ is **convergent**. Otherwise, it is **divergent**.

Similarly,

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\left(\int_a^b f(x)dx\right).$$

$\diamondsuit$

Activity 5.8.10 Which of these limits is equal to $\displaystyle\int_1^\infty \frac{1}{x^2}dx$?

A. $\displaystyle\lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx$

B. $\displaystyle\lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$

C. $\displaystyle\lim_{b\to\infty} \left[-\frac{1}{b}+1\right]$

D. All of these.

Activity 5.8.11 Given the result of Activity 5.8.10, what is $\int_1^\infty \frac{1}{x^2}dx$?

A. 0

B. 1

C. ∞

D. $-\infty$

Activity 5.8.12 Does $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ converge or diverge?

A. Converges because $\lim_{b\to 0^+} \left[2\sqrt{b} - 2\right]$ converges.

B. Diverges because $\lim_{b\to 0^+} \left[2\sqrt{b} - 2\right]$ diverges.

C. Converges because $\lim_{b\to \infty} \left[2\sqrt{b} - 2\right]$ converges.

D. Diverges because $\lim_{b\to \infty} \left[2\sqrt{b} - 2\right]$ diverges.

Definition 5.8.13 For a function $f(x)$ with a vertical asymptote at $x = c > a$, we let $\displaystyle\int_a^c f(x)dx$ denote

$$\int_a^c f(x)dx = \lim_{b \to c^-} \left(\int_a^b f(x)dx \right).$$

For a function $f(x)$ with a vertical asymptote at $x = c < b$, we let $\displaystyle\int_c^b f(x)dx$ denote

$$\int_c^b f(x)dx = \lim_{a \to c^+} \left(\int_a^b f(x)dx \right).$$

$\diamondsuit$

Activity 5.8.14 Which of these limits is equal to $\int_0^1 \frac{1}{\sqrt{x}} dx$?

A. $\lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx$

B. $\lim_{a \to 0^+} \left[2\sqrt{x}\right]_a^1$

C. $\lim_{a \to 0^+} \left[2 - 2\sqrt{a}\right]$

D. All of these.

Activity 5.8.15 Given the this result, what is $\int_0^1 \frac{1}{\sqrt{x}} dx$?

A. 0

B. 1

C. 2

D. ∞

Improper integrals (TI8)

Activity 5.8.16 Does $\int_0^1 \frac{1}{x^2}dx$ converge or diverge?

A. Converges because $\lim_{a\to 0^+}\left[-1+\frac{1}{a}\right]$ converges.

B. Diverges because $\lim_{a\to 0^+}\left[-1+\frac{1}{a}\right]$ diverges.

C. Converges because $\lim_{a\to 1^-}\left[-1+\frac{1}{a}\right]$ converges.

D. Diverges because $\lim_{a\to 1^-}\left[-1+\frac{1}{a}\right]$ diverges.

Improper integrals (TI8)

Activity 5.8.17 Explain and demonstrate how to write each of the following improper integrals as a limit, and why this limit converges or diverges.

(a) $\displaystyle\int_{-2}^{+\infty} \frac{1}{\sqrt{x+6}} dx.$

(b) $\displaystyle\int_{-4}^{-2} \frac{1}{(x+4)^{\frac{4}{3}}} dx.$

(c) $\displaystyle\int_{-5}^{0} \frac{1}{(x+5)^{\frac{5}{9}}} dx.$

(d) $\displaystyle\int_{10}^{+\infty} \frac{1}{(x-8)^{\frac{4}{3}}} dx.$

Fact 5.8.18 *Suppose that* $0 < p$ *and* $p \neq 1$. *Applying the integration power rule gives us the indefinite integral* $\int \frac{1}{x^p} dx = \frac{1}{(1-p)} x^{1-p} + C$.

Improper integrals (TI8)

Activity 5.8.19

(a) If $0 < p < 1$, which of the following statements must be true? Select all that apply.

A. $1 - p < 0$

B. $1 - p > 0$

C. $1 - p < 1$

D. $\int_1^\infty \frac{1}{x^p} dx$ converges.

E. $\int_1^\infty \frac{1}{x^p} dx$ diverges.

(b) If $p > 1$, which of the following statements must be true? Select all that apply.

A. $1 - p < 0$

B. $1 - p > 0$

C. $1 - p < 1$

D. $\int_1^\infty \frac{1}{x^p} dx$ converges.

E. $\int_1^\infty \frac{1}{x^p} dx$ diverges.

Activity 5.8.20

(a) If $0 < p < 1$, which of the following statements must be true?

A. $\displaystyle\int_0^1 \frac{1}{x^p} dx$ converges.

B. $\displaystyle\int_0^1 \frac{1}{x^p} dx$ diverges.

(b) If $p > 1$, which of the following statements must be true?

A. $\displaystyle\int_0^1 \frac{1}{x^p} dx$ converges.

B. $\displaystyle\int_0^1 \frac{1}{x^p} dx$ diverges.

Activity 5.8.21 Consider when $p = 1$. Then $\frac{1}{x^p} = \frac{1}{x}$ and $\displaystyle\int \frac{1}{x^p}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$.

(a) What can we conclude about $\displaystyle\int_1^\infty \frac{1}{x} dx$?

A. $\displaystyle\int_1^\infty \frac{1}{x} dx$ converges.

B. $\displaystyle\int_1^\infty \frac{1}{x} dx$ diverges.

C. There is not enough information to determine whether this integral converges or diverges.

(b) What can we conclude about $\displaystyle\int_0^1 \frac{1}{x} dx$?

A. $\displaystyle\int_0^1 \frac{1}{x} dx$ converges.

B. $\displaystyle\int_0^1 \frac{1}{x} dx$ diverges.

C. There is not enough information to determine whether this integral converges or diverges.

Improper integrals (TI8)

Fact 5.8.22 *Let $c, p > 0$.*

- $\displaystyle\int_0^c \frac{1}{x^p}dx$ *converges if and only if* $p < 1$.
- $\displaystyle\int_c^\infty \frac{1}{x^p}dx$ *converges if and only if* $p > 1$.

Activity 5.8.23 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

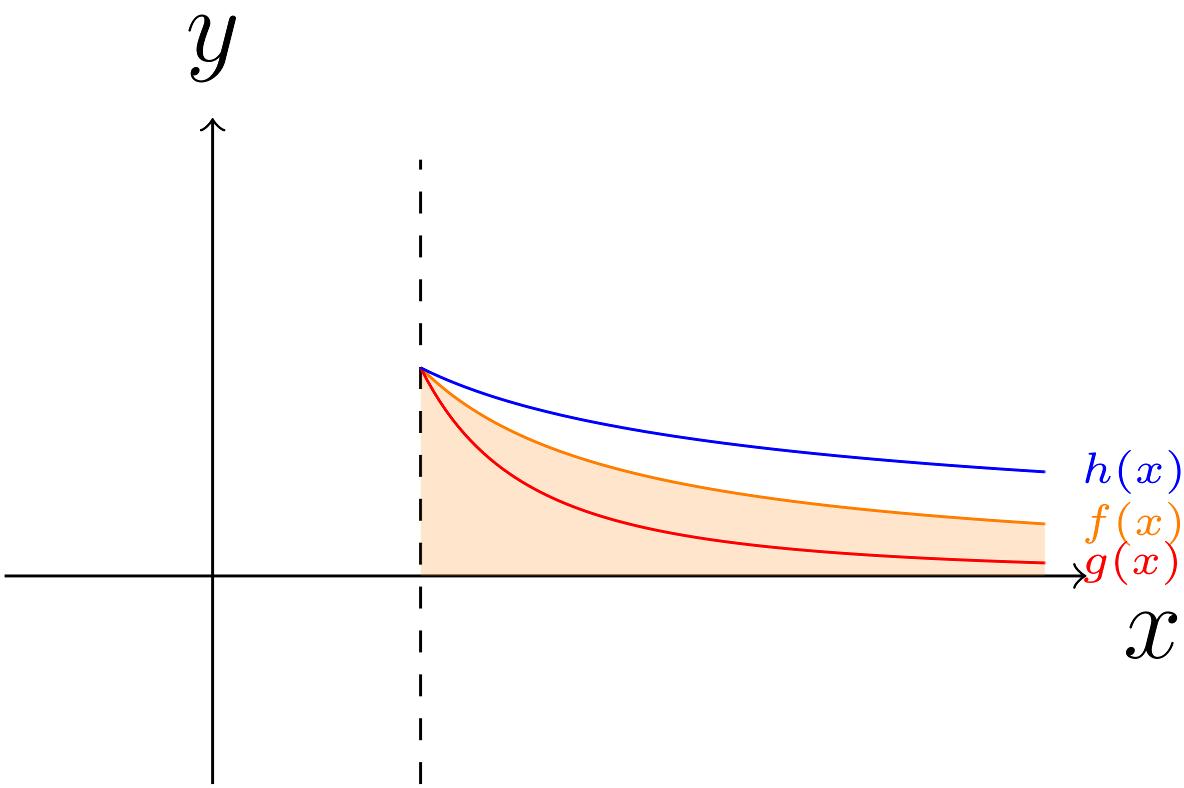


Figure 74 Plots of $f(x), g(x), h(x)$

If $\int_1^\infty f(x)dx$ is convergent, what can we say about $g(x), h(x)$?

A. $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are both convergent.

B. $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are both divergent.

C. Whether or not $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are convergent or divergent cannot be determined.

D. $\int_1^\infty g(x)dx$ is convergent and $\int_1^\infty h(x)dx$ is divergent.

E. $\int_1^\infty g(x)dx$ is convergent and $\int_1^\infty h(x)dx$ could be either convergent or divergent.

Activity 5.8.24 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

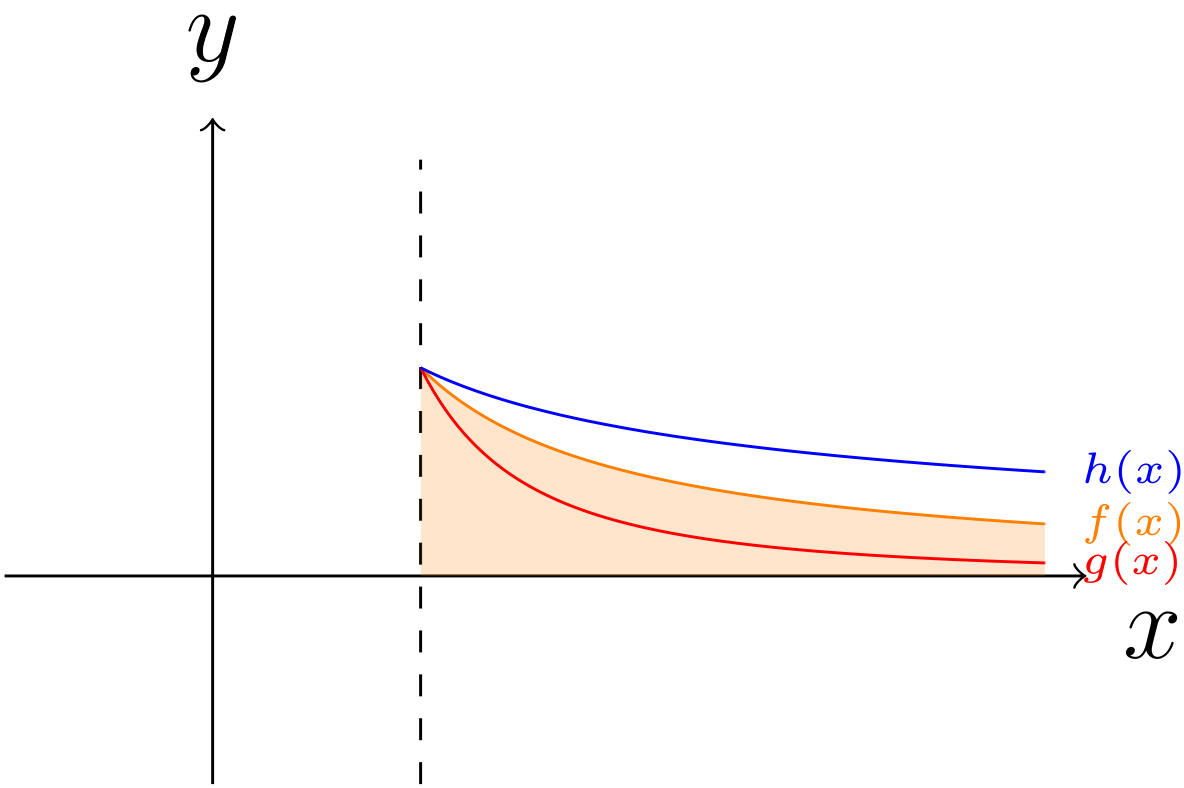


Figure 75 Plots of $f(x), g(x), h(x)$

If $\int_1^\infty f(x)dx$ is divergent, what can we say about $g(x), h(x)$?

A. $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are both convergent.

B. $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are both divergent.

C. Whether or not $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are convergent or divergent cannot be determined.

D. $\int_1^\infty g(x)dx$ could be either convergent or dicergent and $\int_1^\infty h(x)dx$ is divergent.

Improper integrals (TI8)

E. $\int_1^\infty g(x)dx$ is convergent and $\int_1^\infty h(x)dx$ is divergent.

Activity 5.8.25 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

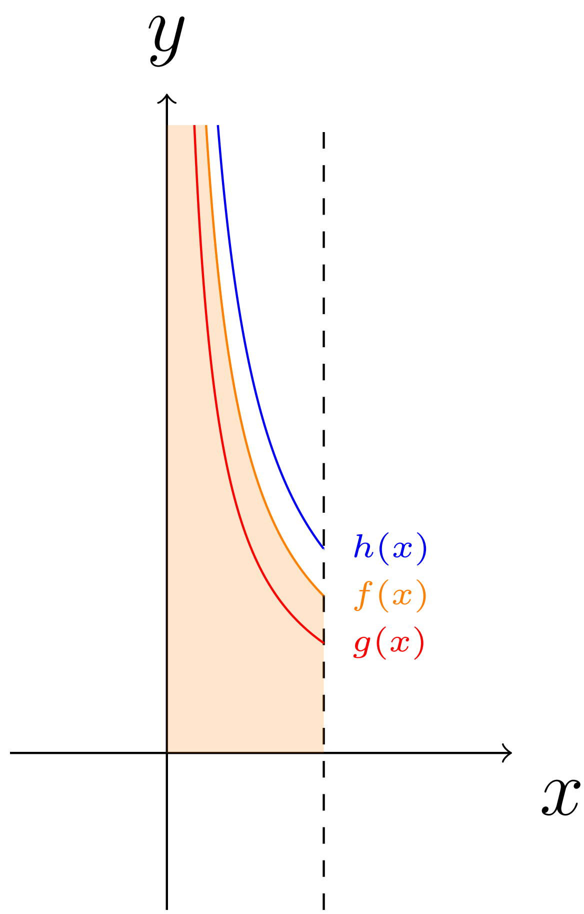


Figure 76 Plots of $f(x), g(x), h(x)$

If $\displaystyle\int_0^1 f(x)dx$ is convergent, what can we say about $g(x)$ and $h(x)$?

A. $\displaystyle\int_0^1 g(x)dx$ and $\displaystyle\int_0^1 h(x)dx$ are both convergent.

B. $\displaystyle\int_0^1 g(x)dx$ and $\displaystyle\int_0^1 h(x)dx$ are both divergent.

C. Whether or not $\displaystyle\int_0^1 g(x)dx$ and $\displaystyle\int_0^1 h(x)dx$ are convergent or divergent cannot be determined.

D. $\displaystyle\int_0^1 g(x)dx$ is convergent and $\displaystyle\int_0^1 h(x)dx$ is divergent.

Improper integrals (TI8)

E. $\int_0^1 g(x)dx$ is convergent and $\int_0^1 h(x)dx$ can either be convergent or divergent.

Activity 5.8.26 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

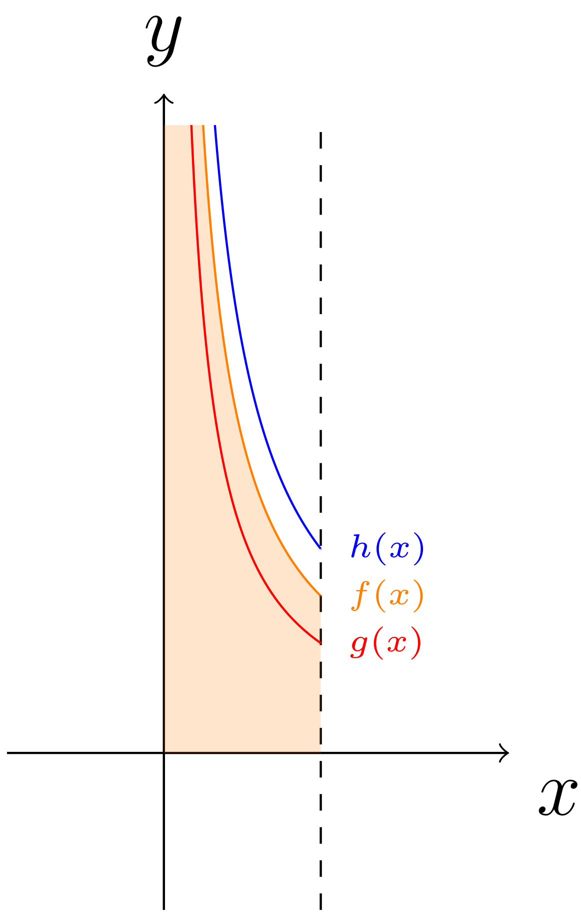


Figure 77 Plots of $f(x), g(x), h(x)$

If $\int_0^1 f(x)dx$ is dinvergent, what can we say about $g(x)$ and $h(x)$?

A. $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are both convergent.

B. $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are both divergent.

C. Whether or not $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are convergent or divergent cannot be determined.

D. $\int_0^1 g(x)dx$ can be either convergent or divergent and $\int_0^1 h(x)dx$ is divergent.

E. $\int_0^1 g(x)dx$ is convergent and $\int_0^1 h(x)dx$ is divergent.

Fact 5.8.27 *Let $f(x), g(x)$ be functions such that for $a < x < b$, $0 \leq f(x) \leq g(x)$. Then*

$$0 \leq \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

In particular:

- *If $\int_a^b g(x)dx$ converges, so does the smaller $\int_a^b f(x)dx$.*
- *If $\int_a^b f(x)dx$ diverges, so does the bigger $\int_a^b g(x)dx$.*

Activity 5.8.28 Compare $\frac{1}{x^3+1}$ to one of the following functions where $x > 2$ and use this to determine if $\int_2^{\infty} \frac{1}{x^3+1} dx$ is convergent or divergent.

A. $\dfrac{1}{x}$

B. $\dfrac{1}{\sqrt{x}}$

C. $\dfrac{1}{x^2}$

D. $\dfrac{1}{x^3}$

Activity 5.8.29 Comparing $\frac{1}{x^3-4}$ to which of the following functions where $x > 3$ allows you to determine that $\int_3^\infty \frac{1}{x^3-4}\,dx$ converges?

A. $\dfrac{1}{x^3+x}$

B. $\dfrac{1}{4x^3}$

C. $\dfrac{1}{x^3}$

D. $\dfrac{1}{x^3 - x^3/2}$

Activity 5.8.30

(a) Find $\displaystyle\int_{\pi/2}^{a} \cos(x)dx$.

(b) Which of the following is true about $\int_{\pi/2}^{\infty} \cos(x)dx$?

A. $\displaystyle\int_{\pi/2}^{\infty} \cos(x)dx$ is convergent.

B. $\displaystyle\int_{\pi/2}^{\infty} \cos(x)dx$ is divergent.

C. More information is needed.

Chapter 6

Applications of Integration (AI)

Learning Outcomes

How can we use integrals to solve application questions?
By the end of this chapter, you should be able to...

1. Compute the average value of a function on an interval.
2. Estimate the arclength of a curve with Riemann sums and find an integral which computes the arclength.
3. Compute volumes of solids of revolution.
4. Compute surface areas of surfaces of revolution.
5. Set up integrals to solve problems involving density, mass, and center of mass.
6. Set up integrals to solve problems involving work.
7. Set up integrals to solve problems involving force and/or pressure.

6.1 Average Value (AI1)

Learning Outcomes

- Compute the average value of a function on an interval.

Average Value (AI1)

Activity 6.1.1 Suppose a car drives due east at 70 miles per hour for 2 hours, and then slows down to 40 miles per hour for an additional hour.

(a) How far did the car travel in these 3 hours?

A. 110 miles

B. 150 miles

C. 180 miles

D. 220 miles

(b) What was its average velocity over these 3 hours?

A. 55 miles per hour

B. 60 miles per hour

C. 70 miles per hour

D. 75 miles per hour

Activity 6.1.2 Suppose instead the car starts with a velocity of 30 miles per hour, and increases velocity linearly according to the function $v(t) = 30 + 20t$ so its velocity after three hours is 90 miles per hour.

(a) How can we model the car's distance traveled using calculus?

A. Integrate velocity, because position is the rate of change of velocity.

B. Integrate velocity, because velocity is the rate of change of position.

C. Differentiate velocity, because position is the rate of change of velocity.

D. Differentiate velocity, because velocity is the rate of change of position.

(b) Then, which of these expressions is a mathematical model for the car's distance traveled after 3 hours?

A. $\displaystyle\int (30 + 20t)\, dt$

B. $\displaystyle\int (30t + 10t^2)\, dt$

C. $\displaystyle\int_0^3 (30 + 20t)\, dt$

D. $\displaystyle\int_0^3 (30t + 10t^2)\, dt$

(c) How far did the car travel in these 3 hours?

A. 110 miles

B. 150 miles

C. 180 miles

D. 220 miles

(d) Thus, what was its average velocity over three hours?

A. 55 miles per hour

B. 60 miles per hour

C. 70 miles per hour

D. 75 miles per hour

Observation 6.1.3 To obtain the average velocity of an object traveling with velocity $v(t)$ for $a \leq t \leq b$, we may find its distance traveled by calculating $\int_a^b v(t)$. Thus, the average velocity is obtained by dividing by the time $b-a$ elapsed:

$$\frac{1}{b-a}\int_a^b v(t)\,dt.$$

For example, the following calculuation confirms the previous activity:

$$\frac{1}{3-0}\int_0^3 (30+20t)\,dt.$$

Average Value (AI1)

Definition 6.1.4 Given a function $f(x)$ defined on $[a, b]$, it's average value is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

$\Diamond$

Activity 6.1.5

(a) Which of the following expressions represent the average value of $f(x) = -12\,x^2 + 8\,x + 4$ over the interval $[-1, 2]$?

A. $\frac{1}{3} \int_{-1}^{2} \left(-12\,x^2 + 8\,x + 4\right)\,dx$

B. $\frac{-1}{1} \int_{1}^{2} \left(-12\,x^2 + 8\,x + 4\right)\,dx$

C. $\frac{1}{2} \int_{1}^{2} \left(-12\,x^2 + 8\,x + 4\right)\,dx$

D. $\frac{-1}{4} \int_{-1}^{2} \left(-12\,x^2 + 8\,x + 4\right)\,dx$

(b) Show that the average value of $f(x) = -12\,x^2 + 8\,x + 4$ over the interval $[-1, 2]$ is -4.

Activity 6.1.6

(a) Which of the following expressions represent the average value of $f(x) = x\cos(x^2) + x$ on the interval $[\pi, 4\pi]$?

A. $\displaystyle \frac{1}{3\pi}\int_0^{4\pi} \left(x\cos(x^2) + x\right)dx$

B. $\displaystyle \frac{1}{4\pi}\int_0^{4\pi} \left(x\cos(x^2) + x\right)dx$

C. $\displaystyle \frac{1}{3\pi}\int_{\pi}^{4\pi} \left(x\cos(x^2) + x\right)dx$

D. $\displaystyle \frac{1}{4\pi}\int_{\pi}^{4\pi} \left(x\cos(x^2) + x\right)dx$

(b) Find the average value of $f(x) = x\cos(x^2) + x$ on the interval $[\pi, 4\pi]$ using the chosen expression.

Average Value (AI1)

Activity 6.1.7 Find the average value of $g(t) = \dfrac{t}{t^2+1}$ on the interval $[0, 4]$.

Average Value (AI1)

Activity 6.1.8 A shot of a drug is administered to a patient and the quantity of the drug in the bloodstream over time is $q(t) = 3te^{-0.25t}$, where t is measured in hours and q is measured in milligrams. What is the average quantity of this drug in the patient's bloodstream over the first 6 hours after injection?

Activity 6.1.9 Which of the following is the average value of $f(x)$ over the interval $[0, 8]$?

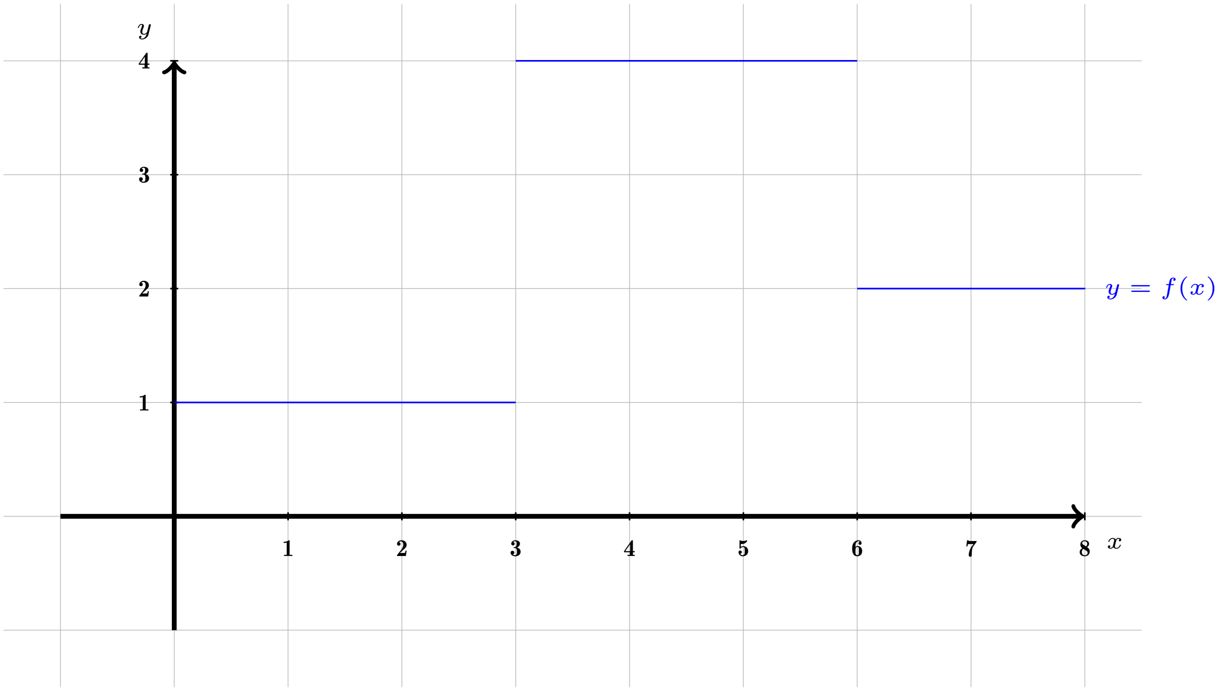


Figure 78 Plot of $f(x)$.

$$\text{Note } f(x) = \begin{cases} 1, & 0 \leq x \leq 3 \\ 4, & 3 < x \leq 6 \\ 2, & 6 < x \leq 8 \end{cases}.$$

A. 4

B. 2

C. $\frac{7}{3}$

D. 19

E. 2.375

6.2 Arclength (AI2)

Learning Outcomes

- Estimate the arclength of a curve with Riemann sums and find an integral which computes the arclength.

Activity 6.2.1 Suppose we wanted to find the arclength of the parabola $y = -x^2 + 6x$ over the interval $[0, 4]$.

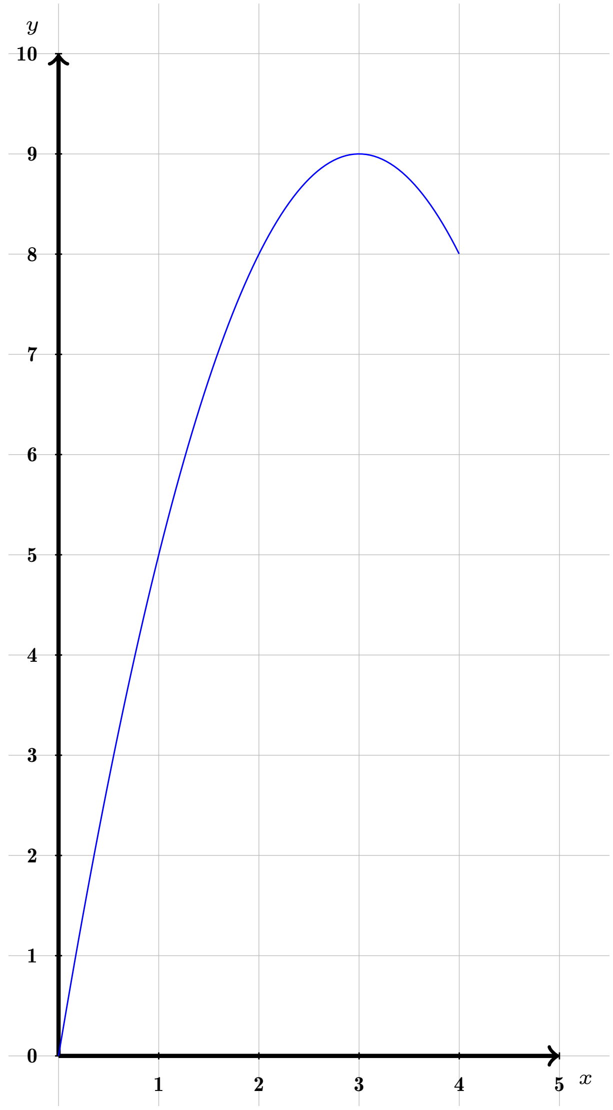


Figure 79 Plot of $y = -x^2 + 6x$ over $[0, 4]$.

(a) Suppose we wished to estimate this length with two line segments where $\Delta x = 2$.

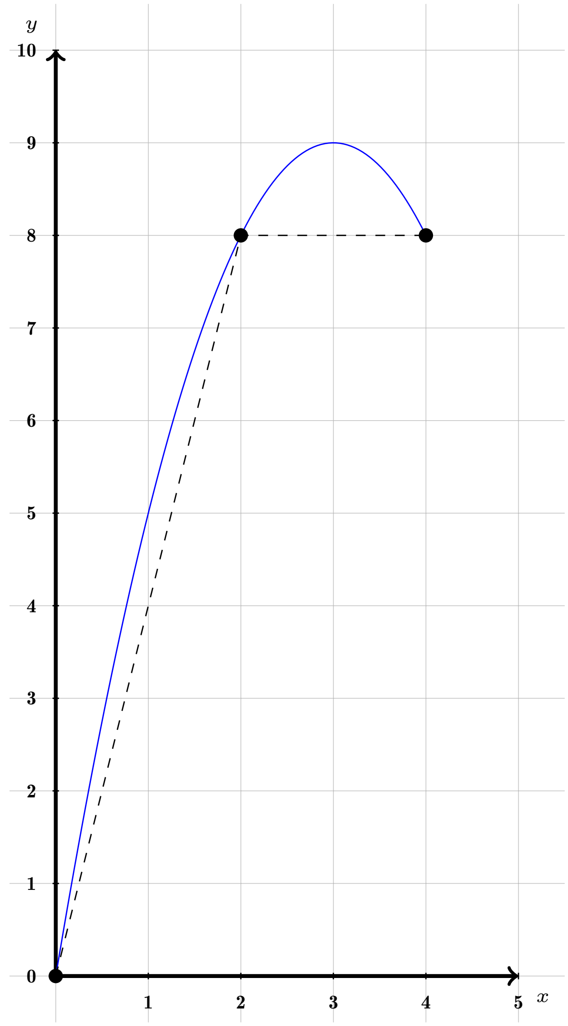


Figure 80 Plot of $y = -x^2 + 6x$ over $[0, 4]$ with two line segments where $\Delta x = 2$.

Which of the following expressions represents the sum of the lengths of the line segments with endpoints $(0, 0)$, $(2, 8)$ and $(4, 8)$?

A. $\sqrt{4+8}$

B. $\sqrt{2^2+8^2}+\sqrt{(4-2)^2+(8-8)^2}$

C. $\sqrt{4^2+8^2}$

D. $\sqrt{2^2+8^2}+\sqrt{4^2+8^2}$

(b) Suppose we wished to estimate this length with four line segments where $\Delta x = 1$.

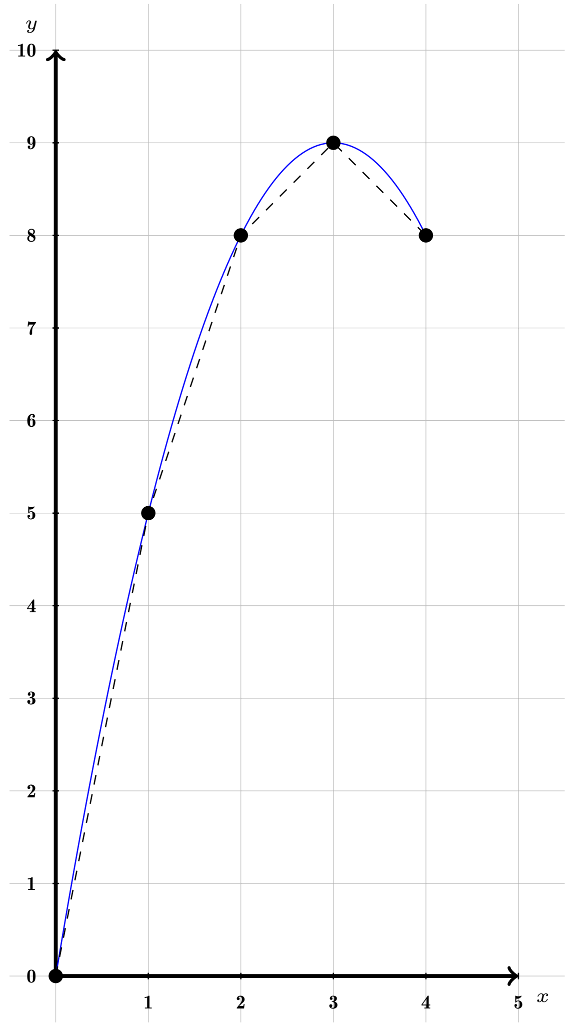


Figure 81 Plot of $y = -x^2 + 6x$ over $[0, 4]$ with four line segments where $\Delta x = 1$.

Which of the following expressions represents the sum of the lengths of the line segments with endpoints $(0, 0)$, $(1, 5)$, $(2, 8)$, $(3, 9)$ and $(4, 8)$?

A $\sqrt{4^2 + 8^2}$

B $\sqrt{1^2 + (5-0)^2} + \sqrt{1^2 + (8-5)^2} + \sqrt{1^2 + (9-8)^2} + \sqrt{1^2 + (8-9)^2}$

C $\sqrt{1^2 + 5^2} + \sqrt{2^2 + 8^2} + \sqrt{3^2 + 9^2} + \sqrt{4^2 + 8^2}$

(c) Suppose we wished to estimate this length with n line segments where $\Delta x = \dfrac{4}{n}$. Let $f(x) = -x^2 + 6x$.

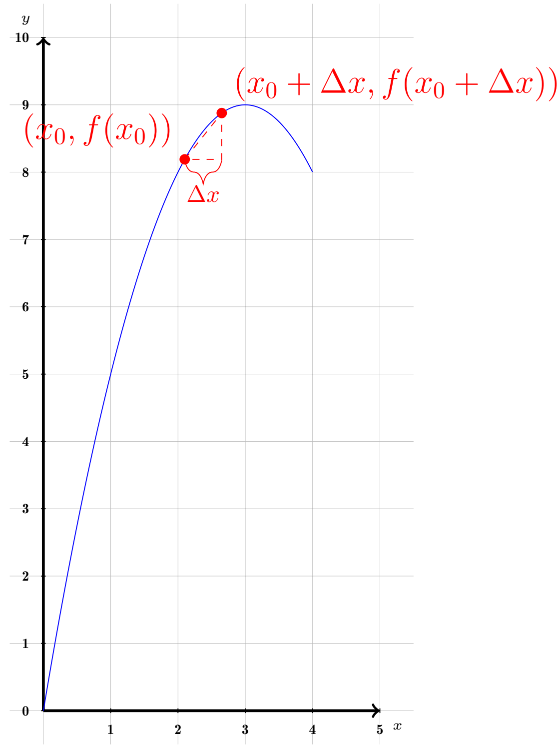


Figure 82 Plot of $y = -x^2 + 6x$ over $[0, 4]$ with n line segments where $\Delta x = \dfrac{4}{n}$.

Which of the following expressions represents the length of the line segment from $(x_0, f(x_0))$ to $(x_0 + \Delta x, f(x_0 + \Delta x))$?

A. $\sqrt{x_0^2 + f(x_0)^2}$

B. $\sqrt{(x_0 + \Delta x)^2 + f(x_0 + \Delta x)^2}$

C. $\sqrt{(\Delta x)^2 + f(\Delta x)^2}$

D. $\sqrt{(\Delta x)^2 + (f(x_0 + \Delta x) - f(x_0))^2}$

(d) Which of the following Riemann sums best estimates the arclength of the parabola $y = -x^2 + 6x$ over the interval $[0, 4]$? Let $f(x) = -x^2 + 6x$.

A. $\sum \sqrt{(\Delta x)^2 + f(\Delta x)^2}$

B. $\sum \sqrt{(x_i + \Delta x)^2 + f(x_i + \Delta x)^2}$

C. $\sum \sqrt{x_i^2 + f(x_i)^2}$

D. $\sum \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}$

(e) Note that

$$\sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2} = \sqrt{(\Delta x)^2 \left(1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2\right)}$$

Arclength (AI2)

$$= \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2}\,\Delta x.$$

Which of the following best describes $\lim_{\Delta x \to 0} \dfrac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$?

A. 0

B. 1

C. $f'(x_i)$

D. This limit is undefined.

Arclength (AI2)

Fact 6.2.2 *Given a differentiable function $f(x)$, the **arclength** of $y = f(x)$ defined on $[a, b]$ is computed by the integral*

$$\begin{aligned}\lim_{n\to\infty}\sum\sqrt{(\Delta x)^2+(f(x_i+\Delta)-f(x_i))^2} &= \lim_{n\to\infty}\sum\sqrt{1+\left(\frac{f(x_i+\Delta x)-f(x_i)}{\Delta x}\right)^2}\Delta x\\ &= \int_a^b\sqrt{1+(f'(x))^2}dx.\end{aligned}$$

Activity 6.2.3 Use Fact 6.2.2 to find an integral which measures the arclength of the parabola $y = -x^2 + 6x$ over the interval $[0, 4]$.

Activity 6.2.4 Consider the curve $y = 2^x - 1$ defined on $[1, 5]$.

(a) Estimate the arclength of this curve with two line segments where $\Delta x = 2$.

x_i	$(x_i, f(x_i))$	$(x_i + \Delta x, f(x_i + \Delta x))$	Length of segment
1			
3			

(b) Estimate the arclength of this curve with four line segments where $\Delta x = 1$.

x_i	$(x_i, f(x_i))$	$(x_i + \Delta x, f(x_i + \Delta x))$	Length of segment
1			
2			
3			
4			

(c) Find an integral which computes the arclength of the curve $y = 2^x - 1$ defined on $[1, 5]$.

Arclength (AI2)

Activity 6.2.5 Consider the curve $y = 5e^{-x^2}$ over the interval $[-1, 4]$.

(a) Estimate this arclength with 5 line segments where $\Delta x = 1$.

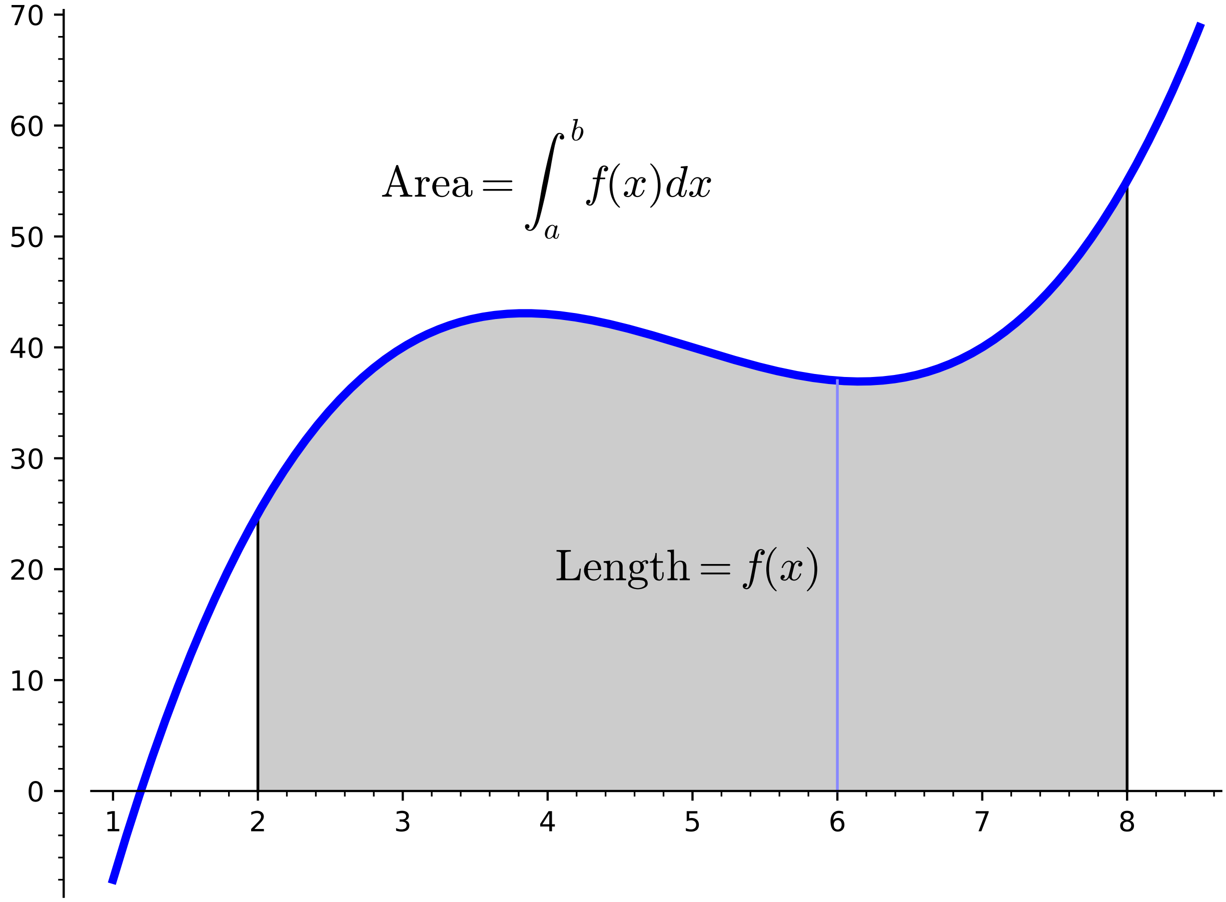
(b) Find an integral which computes this arclength.

6.3 Volumes of Revolution (AI3)

Learning Outcomes

- Compute volumes of solids of revolution.

Activity 6.3.1 Consider the following visualization to decide which of these statements is most appropriate for describing the relationship of lengths and areas.



A. Length is the integral of areas.

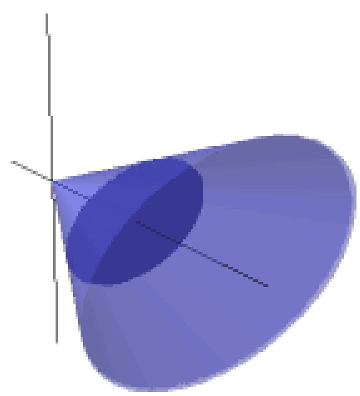
B. Area is the integral of lengths.

C. Length is the derivative of areas.

D. None of these.

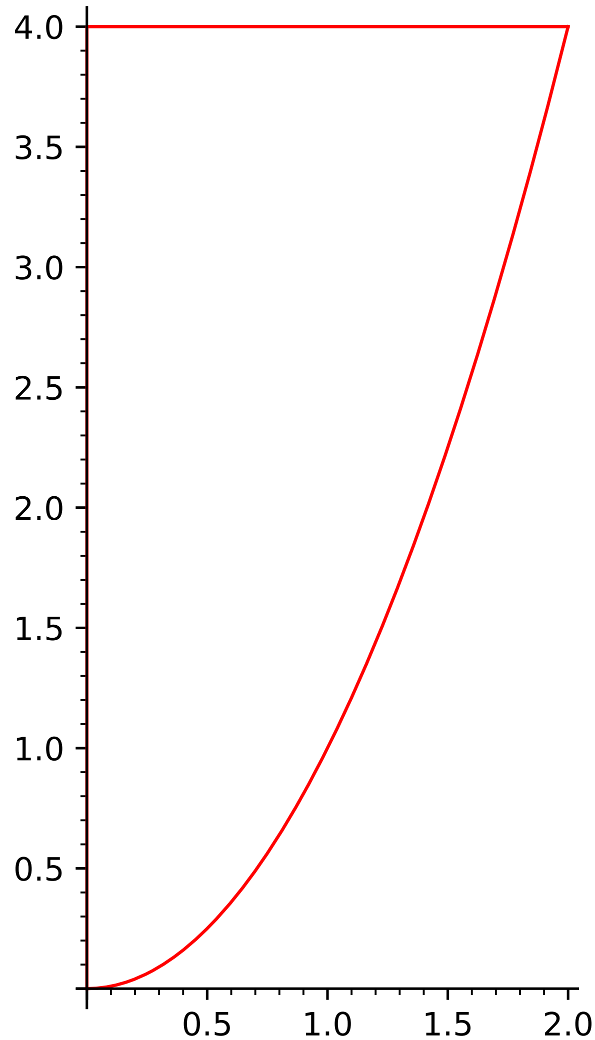
Definition 6.3.2 We define the **volume** of a solid with cross sectional area given by $A(x)$ laying between $a \leq x \leq b$ to be the definite integral

$$\text{Volume} = \int_a^b A(x)\,dx.$$



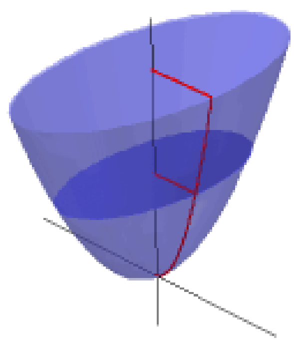
$\Diamond$

Activity 6.3.3 We will be focused on the volumes of solids obtained by revolving a region around an axis. Let's use the running example of the region bounded by the curves $x = 0, y = 4, y = x^2$.



(a) Consider the below illustrated revolution of this region, and the cross-section drawn from a horizontal line segment. Choose the most appropriate description of this illustration.

Volumes of Revolution (AI3)



A. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's x-value.

B. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's y-value.

C. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's x-value.

D. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's y-value.

(b) Which of these formulas is most appropriate to find this illustration's cross-sectional area?

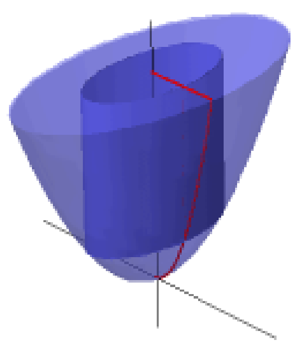
A. πr^2

B. $2\pi rh$

C. $\pi R^2 - \pi r^2$

D. $\dfrac{1}{2}bh$

(c) Consider the below illustrated revolution of this region, and the cross-section drawn from a vertical line segment. Choose the most appropriate description of this illustration.



A. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's x-value.

B. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's y-value.

C. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's x-value.

D. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's y-value.

(d) Which of these formulas is most appropriate to find this illustration's cross-sectional area?

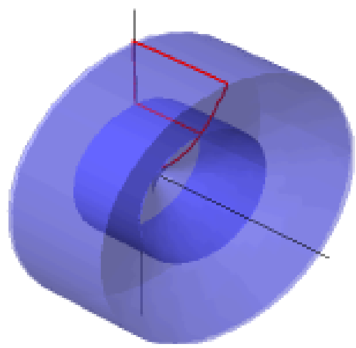
A. πr^2

B. $2\pi rh$

C. $\pi R^2 - \pi r^2$

D. $\frac{1}{2}bh$

(e) Consider the below illustrated revolution of this region, and the cross-section drawn from a horizontal line segment. Choose the most appropriate description of this illustration.



A. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's x-value.

B. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's y-value.

C. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's x-value.

D. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's y-value.

(f) Which of these formulas is most appropriate to find this illustration's cross-sectional area?

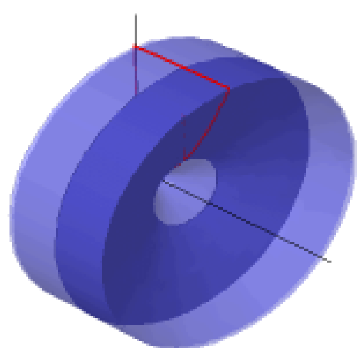
A. πr^2

B. $2\pi r h$

C. $\pi R^2 - \pi r^2$

D. $\dfrac{1}{2}bh$

(g) Consider the below illustrated revolution of this region, and the cross-section drawn from a vertical line segment. Choose the most appropriate description of this illustration.



A. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's x-value.

B. Region is rotated around the x-axis; the cross-sectional area is determined by the line segment's y-value.

C. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's x-value.

D. Region is rotated around the y-axis; the cross-sectional area is determined by the line segment's y-value.

(h) Which of these formulas is most appropriate to find this illustration's cross-sectional area?

A. πr^2

B. $2\pi rh$

C. $\pi R^2 - \pi r^2$

D. $\frac{1}{2}bh$

Remark 6.3.4 Generally when solving problems without the aid of technology, it's useful to draw your region in two dimensions, choose whether to use a horizontal or vertical line segment, and draw its rotation to determine the cross-sectional shape.

When the shape is a disk, this is called the **disk method** and we use one of these formulas depending on whether the cross-sectional area depends on x or y.

$$V = \int_a^b \pi r(x)^2\, dx, \qquad V = \int_a^b \pi r(y)^2\, dy.$$

When the shape is a washer, this is called the **washer method** and we use one of these formulas depending on whether the cross-sectional area depends on x or y.

$$V = \int_a^b \left(\pi R(x)^2 - \pi r(x)^2\right)\, dx, \qquad V = \int_a^b \left(\pi R(y)^2 - \pi r(y)^2\right)\, dy.$$

When the shape is a cylindrical shell, this is called the **shell method** and we use one of these formulas depending on whether the cross-sectional area depends on x or y.

$$V = \int_a^b 2\pi r(x)h(x)\, dx, \qquad V = \int_a^b 2\pi r(y)h(y)\, dy.$$

Activity 6.3.5 Let's now consider the region bounded by the curves $x = 0, x = 1, y = 0, y = 5e^x$, rotated about the x-axis.

(a) Sketch two copies of this region in the xy plane.

(b) Draw a vertical line segment in one region and its rotation around the x-axis. Draw a horizontal line segment in the other region and its rotation around the x-axis.

(c) Consider the method required for each cross-section drawn. Which would be the *easiest* strategy to proceed with?

A. The horizontal line segment, using the disk/washer method.

B. The horizontal line segment, using the shell method.

C. The vertical line segment, using the disk/washer method.

D. The vertical line segment, using the shell method.

(d) Let's proceed with the vertical segment. Which formula is most appropriate for the radius?

A. $r(x) = x$

B. $r(x) = 5e^x$

C. $r(x) = 5\ln(x)$

D. $r(x) = \frac{1}{5}\ln(x)$

(e) Which of these integrals is equal to the volume of the solid of revolution?

A. $\int_0^1 25\pi e^{2x}\,dx$

B. $\int_0^1 5\pi^2 e^x\,dx$

C. $\int_0^2 25\pi e^x\,dx$

D. $\int_0^2 5\pi^2 e^{2x}\,dx$

Activity 6.3.6 Let's now consider the same region, bounded by the curves $x = 0, x = 1, y = 0, y = 5e^x$, but this time rotated about the y-axis.

(a) Sketch two copies of this region in the xy plane.

(b) Draw a vertical line segment in one region and its rotation around the y-axis. Draw a horizontal line segment in the other region and its rotation around the y-axis.

(c) Consider the method required for each cross-section drawn. Which would be the *easiest* strategy to proceed with?

A. The horizontal line segment, using the disk/washer method.

B. The horizontal line segment, using the shell method.

C. The vertical line segment, using the disk/washer method.

D. The vertical line segment, using the shell method.

(d) Let's proceed with the vertical segment. Which formula is most appropriate for the radius?

A. $r(x) = x$

B. $r(x) = 5e^x$

C. $r(x) = 5\ln(x)$

D. $r(x) = \dfrac{1}{5}\ln(x)$

(e) Which formula is most appropriate for the height?

A. $h(x) = x$

B. $h(x) = 5e^x$

C. $h(x) = 5\ln(x)$

D. $h(x) = \dfrac{1}{5}\ln(x)$

(f) Which of these integrals is equal to the volume of the solid of revolution?

A. $\displaystyle\int_0^1 5\pi^2 x e^x \, dx$

B. $\displaystyle\int_0^1 10\pi x e^x \, dx$

C. $\displaystyle\int_0^2 5\pi x e^x \, dx$

D. $\displaystyle\int_0^2 10\pi x^2 e^x \, dx$

Activity 6.3.7 Consider the region bounded by $y = 2\,x + 3, y = 0, x = 4, x = 7$.

(a) Find an integral which computes the volume of the solid formed by rotating this region about the x-axis.

(b) Find an integral which computes the volume of the solid formed by rotating this region about the y-axis.

6.4 Surface Areas of Revolution (AI4)

Learning Outcomes

- Compute surface areas of surfaces of revolution.

Fact 6.4.1 *A* ***frustum*** *is the portion of a cone that lies between one or two parallel planes.*

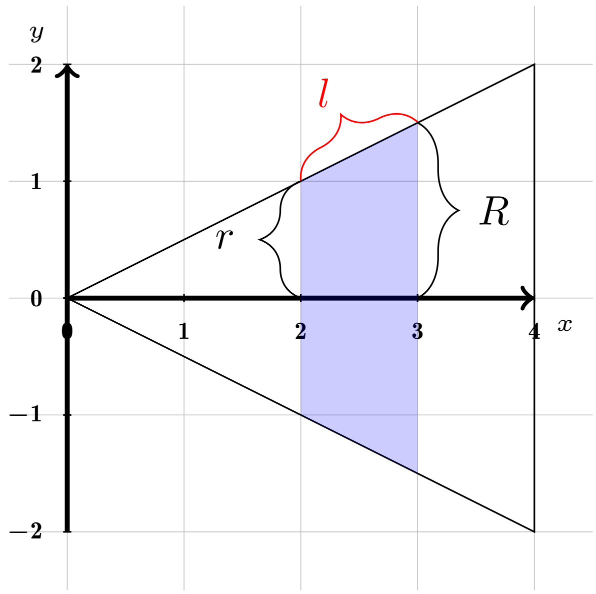


Figure 83 *Plot of a frustum.*

The surface area of the "side" of the frustum is:

$$2\pi\frac{r+R}{2}\cdot l$$

where r and R are the radii of the bases, and l is the length of the side.

Note that if $r = R$, this reduces to the surface area of a "side" of a cylinder.

Activity 6.4.2 Suppose we wanted to find the surface area of the the solid of revolution generated by rotating

$$y = \sqrt{x}, 0 \leq x \leq 4$$

about the y-axis.

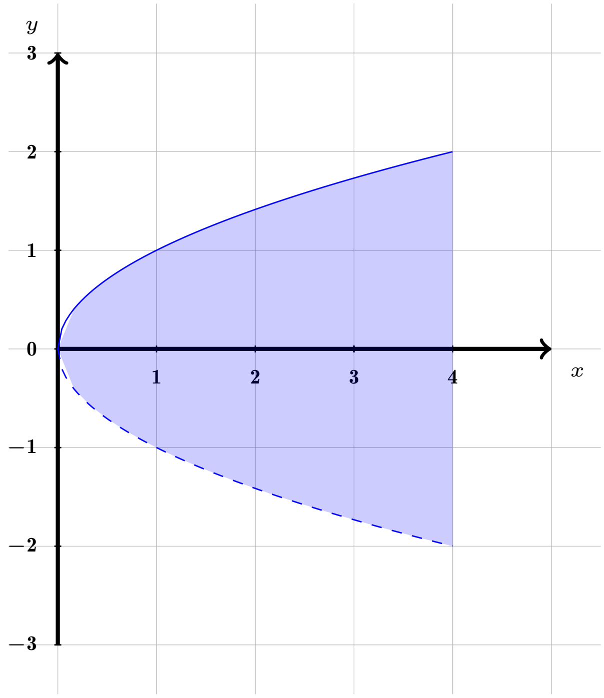


Figure 84 Plot of bounded region rotated about x-axis.

(a) Suppose we wanted to estimate the surface area with two frustums with $\Delta x = 2$.

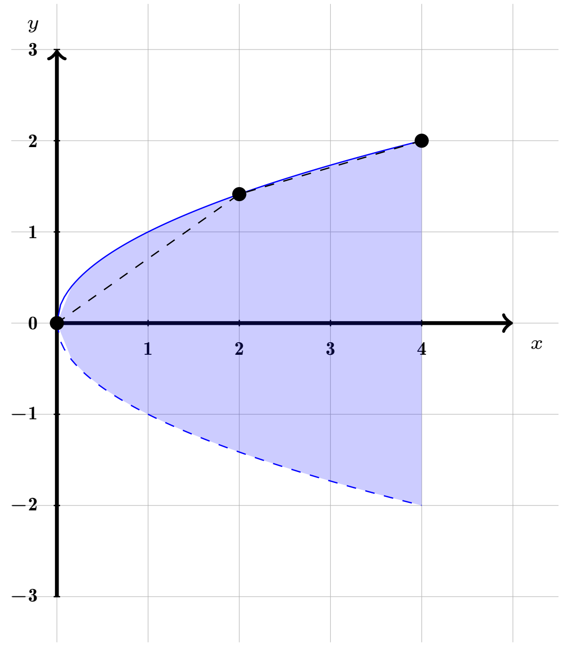


Figure 85 Plot of bounded region rotated about x-axis.

What is the surface area of the frustum formed by rotating the line segment from $(0,0)$ to $(2,\sqrt{2})$ about the x-axis?

A $2\pi\dfrac{0+\sqrt{2}}{2}\cdot 2$

B $2\pi\dfrac{0+\sqrt{2}}{2}\cdot\sqrt{2^2+\sqrt{2}^2}$

C $\pi\sqrt{2}^2\cdot 2$

D $\pi\sqrt{2}^2\cdot\sqrt{2^2+\sqrt{2}^2}$

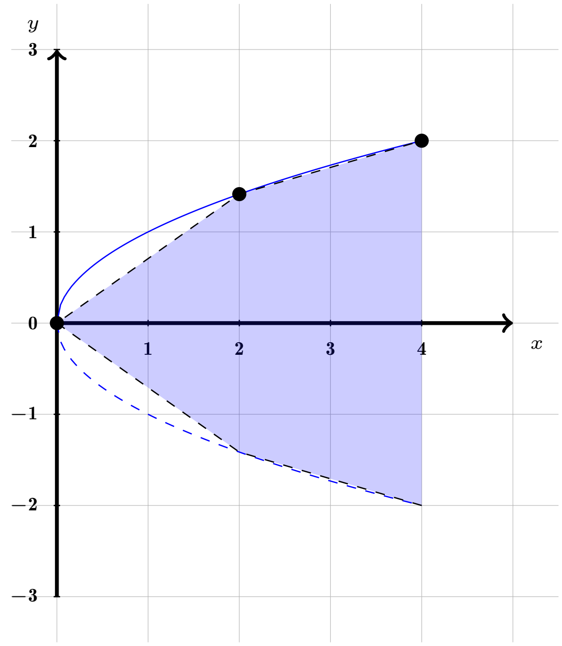


Figure 86 Plot of bounded region rotated about the x-axis.

(b) What is the surface area of the frustum formed by rotating the line segment from $(2, \sqrt{2})$ to $(4, 2)$ about the x-axis?

A $2\pi\dfrac{4+\sqrt{2}}{2}\cdot\sqrt{2}$

B $2\pi\dfrac{4+\sqrt{2}}{2}\cdot\sqrt{6}$

C $2\pi\dfrac{4+\sqrt{2}}{2}\cdot\sqrt{6-2\sqrt{2}}$

(c) Suppose we wanted to estimate the surface area with four frustums with $\Delta x = 1$.

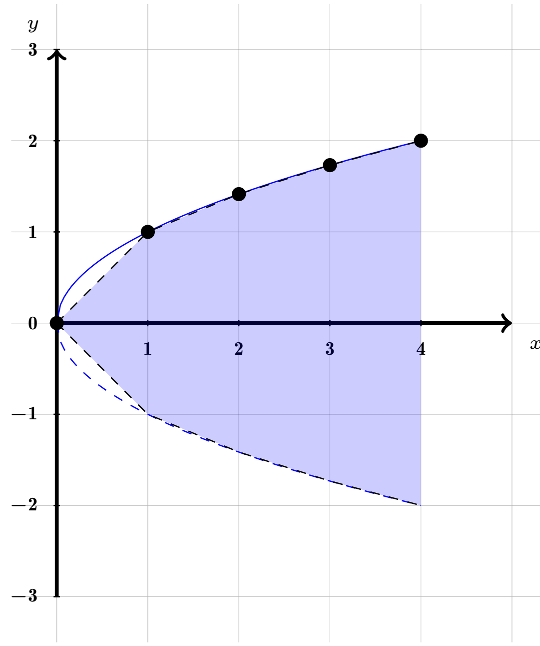


Figure 87 Plot of bounded region rotated about x-axis.

x_i	Δx	r_i	R_i	l	Estimated Surface Area
$x_1 = 0$	1	0	1	$\sqrt{1^2 + 1^2}$	
$x_2 = 1$	1	1	$\sqrt{2}$	$\sqrt{1^2 + (\sqrt{2} - 1)^2}$	
$x_3 = 2$	1	$\sqrt{2}$	$\sqrt{3}$		
$x_4 = 3$	1	3	2		

(d) Suppose we wanted to estimate the surface area with n frustums.

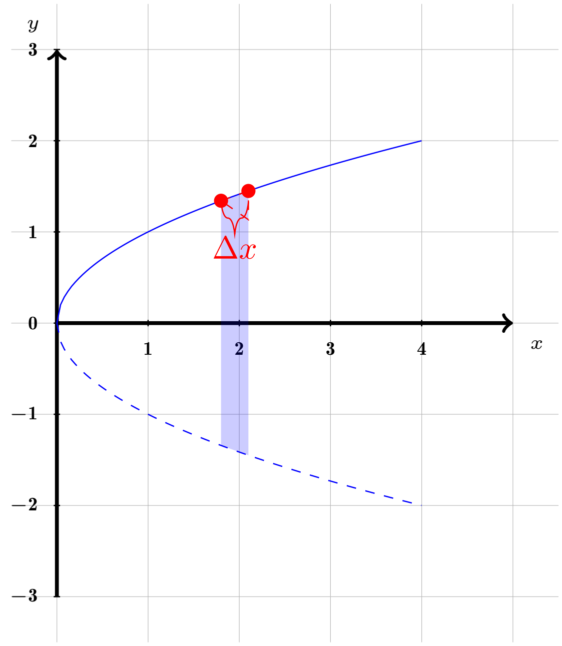


Figure 88 Plot of bounded region rotated about x-axis.

Let $f(x) = \sqrt{x}$. Which of the following expressions represents the surface area generated bo rotating the line segment from $(x_0, f(x_0))$ to $(\Delta x, f(x_0 + \Delta x))$ about the x-axis?

A $\pi \left(\dfrac{f(x_0) + f(x_0 + \Delta x)}{2}\right)^2 \sqrt{(\Delta x)^2 + (f(x_0 + \Delta x) - f(x_0))^2}.$

B $2\pi \dfrac{f(x_0) + f(x_0 + \Delta x)}{2} \sqrt{(\Delta x)^2 + (f(x_0 + \Delta x) - f(x_0))^2}.$

C $2\pi \dfrac{f(x_0) + f(x_0 + \Delta x)}{2} \Delta x.$

(e) Which of the following Riemann sums best estimates the surface area of the solid generated by rotating $y = \sqrt{x}$ over $[0, 4]$ about the x-axis ? Let $f(x) = \sqrt{x}$.

A $\sum \pi \left(\dfrac{f(x_i) + f(x_i + \Delta x)}{2}\right)^2 \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}.$

B $\sum 2\pi \dfrac{f(x_i) + f(x_i + \Delta x)}{2} \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}.$

C $\sum 2\pi \dfrac{f(x_i) + f(x_0 + \Delta x)}{2} \Delta x.$

Fact 6.4.3 *Recall from Fact 6.2.2 that*

$$\begin{aligned}
\lim_{\Delta x\to 0}\sqrt{(\Delta x)^2+(f(x_i+\Delta x)-f(x_i))^2} &= \lim_{\Delta x\to 0}\sqrt{(\Delta x)^2\left(1+\left(\frac{f(x_i+\Delta x)-f(x_i)}{\Delta x}\right)^2\right)}\\
&= \lim_{\Delta x\to 0}\sqrt{1+\left(\frac{f(x_i+\Delta x)-f(x_i)}{\Delta x}\right)^2}\Delta x\\
&= \sqrt{1+(f'(x))^2}dx,
\end{aligned}$$

and that

$$\lim_{\Delta x\to 0}\frac{f(x_i)+f(x_i+\Delta x)}{2}=f(x_i).$$

Thus given a function $f(x)\geq 0$ over $[a,b]$, the surface area of the solid generated by rotating this function about the x-axis is

$$SA=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}dx.$$

Surface Areas of Revolution (AI4)

Activity 6.4.4 Consider again the solid generated by rotating $y = \sqrt{x}$ over $[0, 4]$ about the x-axis.

(a) Find an integral which computes the surface area of this solid.

(b) If we instead rotate $y = \sqrt{x}$ over $[0, 4]$ about the y-axis, what is an integral which computes the surface area for this solid?

Activity 6.4.5 Consider again the function $f(x) = \ln(x) + 1$ over $[1, 5]$.

(a) Find an integral which computes the surface area of the solid generated by rotating the above curve about the x-axis.

(b) Find an integral which computes the surface area of the solid generated by rotating the above curve about the y-axis.

6.5 Density, Mass, and Center of Mass (AI5)

Learning Outcomes

- Set up integrals to solve problems involving density, mass, and center of mass.

Activity 6.5.1 Consider a rectangular prism with a 10 meters $\times$ 10 meters square base and height 20 meters. Suppose the density of the material in the prism increases with height, following the function $\delta(h) = 10 + h$ kg/m^3, where h is the height in meters.

(a) If one were to cut this prism, parallel to the base, into 4 pieces with height 5 meters, what would the volume of each piece be?

(b) Consider the piece sitting on top of the slice made at height $h = 5$. Using a density of $\delta(5) = 15$ kg/m^3, and the volume you found in (a), estimate the mass of this piece.

A. $500 \cdot 5 = 2500$ kg

B. $500 \cdot 15 = 7500$ kg

C. $500 \cdot 15 \cdot 5 = 37500$ kg

(c) Is this estimate the actual mass of this piece?

Activity 6.5.2 Consider all 4 slices from Activity 6.5.1.

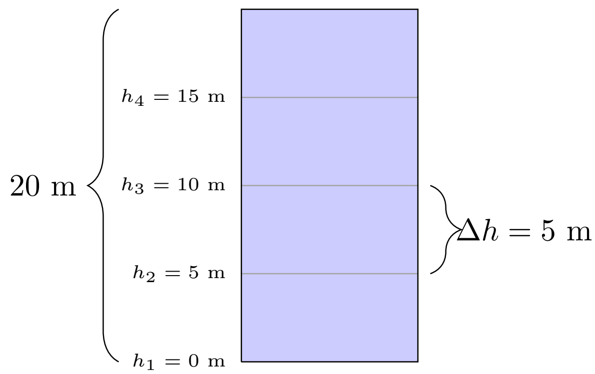


Figure 89 $10 \times 10 \times 20$ prism sliced into 4 pieces.

(a) Fill out the following table.

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_4 = 15$ m	$\delta(15) = 25$ kg/m^3	500 m^3	
$h_3 = 10$ m	$\delta(10) = 20$ kg/m^3	500 m^3	
$h_2 = 5$ m	$\delta(5) = 15$ kg/m^3	500 m^3	7500 kg
$h_1 = 0$ m	$\delta(0) = 10$ kg/m^3	500 m^3	

(b) What is the estimated mass of the rectangular prism?

Activity 6.5.3 Suppose instead that we sliced the prism from Activity 6.5.1 into 5 pices of height 4 meters.

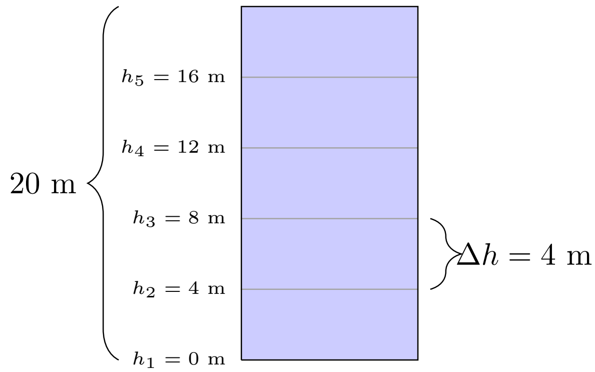


Figure 90 $10 \times 10 \times 20$ prism sliced into 5 pieces.

(a) Fill out the following table.

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_5 = 16$ m	$\delta(16) = 26$ kg/m^3	400 m^3	
$h_4 = 12$ m	$\delta(12) = 22$ kg/m^3	400 m^3	
$h_3 = 8$ m	$\delta(8) = 18$ kg/m^3	400 m^3	
$h_2 = 4$ m	$\delta(4) = 14$ kg/m^3	400 m^3	
$h_1 = 0$ m	$\delta(0) = 10$ kg/m^3	400 m^3	

(b) What is the estimated mass of the rectangular prism?

Activity 6.5.4 Which of the estimates computed in Activity 6.5.2 and Activity 6.5.3 is a better estimate of the mass of the prism?

A. Activity 6.5.2, 4 pieces is a better estimate.

B. Activity 6.5.3, 5 pieces is a better estimate.

Activity 6.5.5 Suppose now that we slice the prism from Activity 6.5.1 into slices of height Δh meters.

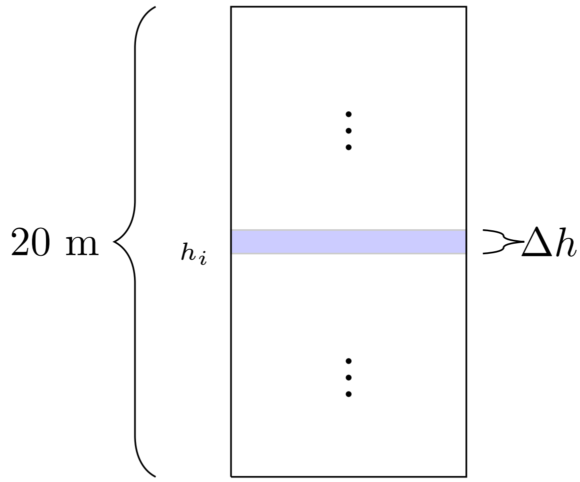


Figure 91 $10 \times 10 \times 20$ prism sliced into many pieces.

(a) Consider the piece sitting atop the slice made at height h_i. Using $\delta(h_i) = 10 + h_i$ as the estimate for the density of this piece, what is the mass of this piece?

A. $(10 + h)100 \cdot h_i$

B. $(10 + \Delta h)100 \cdot h_i$

C. $(10 + h_i)100 \cdot \Delta h$

D. $(10 + h_i)100 \cdot h$

Activity 6.5.6 Consider a cylindrical cone with a base radius of 15 inches and a height of 60 inches. Suppose the density of the cone is $\delta(h) = 15 + \sqrt{h}$ oz/in^3.

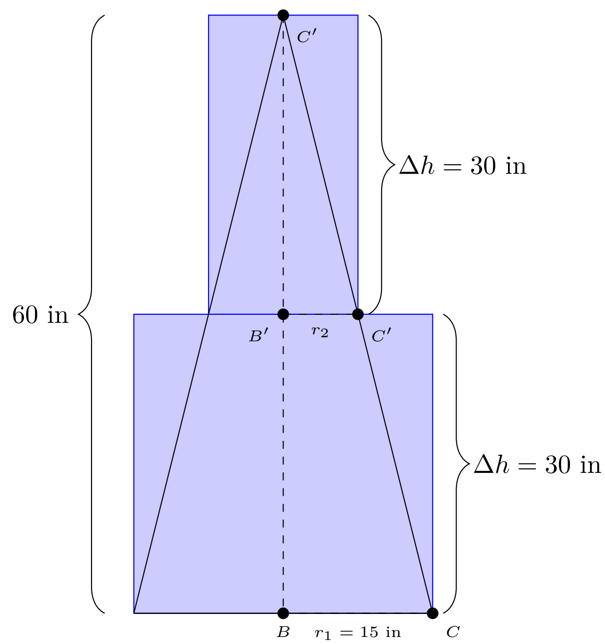


Figure 92 15× 60 cylindrical cone sliced into two pieces.

(a) Let r_2 be the radius of the circular cross section of the cone, made at height 30 inches. Recall that $\Delta ABC, \Delta AB'C'$ are similar triangles, what is r_2?

A. 15 inches.

B. 7.5 inches.

C. 30 inches.

D. 60 inches.

(b) What is the volume of a cylinder with radius $r_1 = 15$ inches and height 30 inches?

(c) What is the volume of a cylinder with radius r_2 inches and height 30 inches?

Activity 6.5.7 Suppose that we estimate the mass of the cone from Activity 6.5.6 with 2 cylinders of height 30 inches.

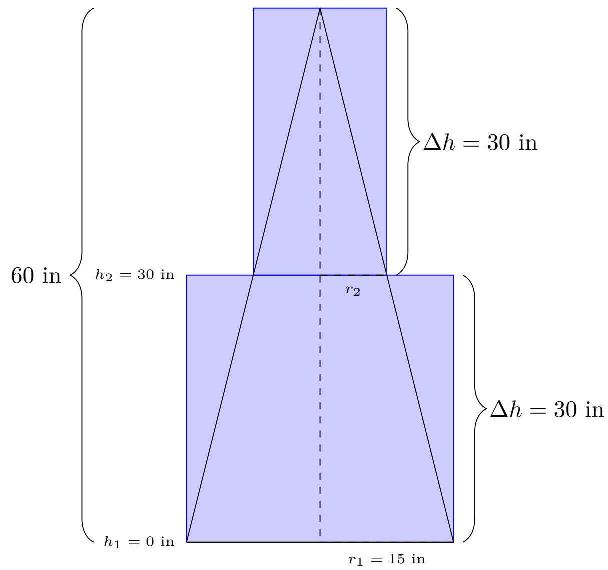


Figure 93 15× 60 cylindrical cone sliced into two pieces.

(a) Fill out the following table.

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_2 = 30$ m	$\delta(30) = 15 + \sqrt{30}$ oz/in^3	$\pi(7.5)^2 \cdot 30$ in^3	
$h_1 = 0$ m	$\delta(0) = 15$ oz/in^3	$\pi(15)^2 \cdot 30$ in^3	

(b) What is the estimated mass of the cone?

Activity 6.5.8 Suppose that we estimate the mass of the cone from Activity 6.5.6 with 3 cylinders of height 20 inches.

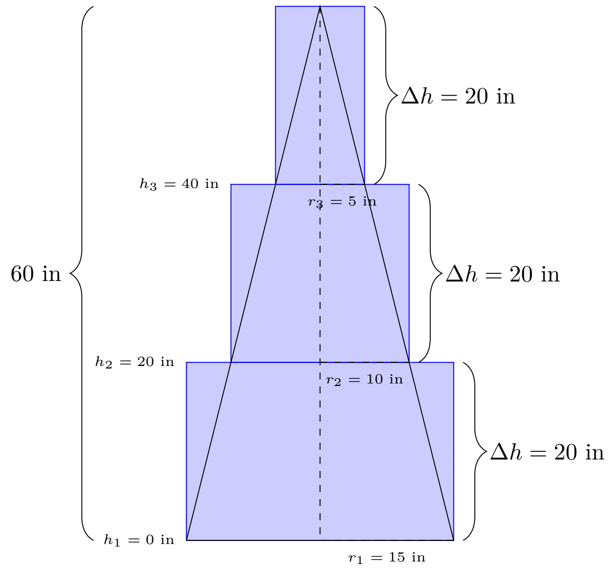


Figure 94 15× 60 cylindrical cone sliced into three pieces.

(a) Fill out the following table.

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_2 = 40$ m	$\delta(40) = 15 + \sqrt{40}$ oz/in^3	$\pi(5)^2 \cdot 20$ in^3	
$h_2 = 20$ m	$\delta(20) = 15 + \sqrt{20}$ oz/in^3	$\pi(10)^2 \cdot 20$ in^3	
$h_1 = 0$ m	$\delta(0) = 15$ oz/in^3	$\pi(15)^2 \cdot 20$ in^3	

(b) What is the estimated mass of the cone?

Activity 6.5.9 Suppose that we estimate the mass of the cone from Activity 6.5.6 with cylinders of height Δh.

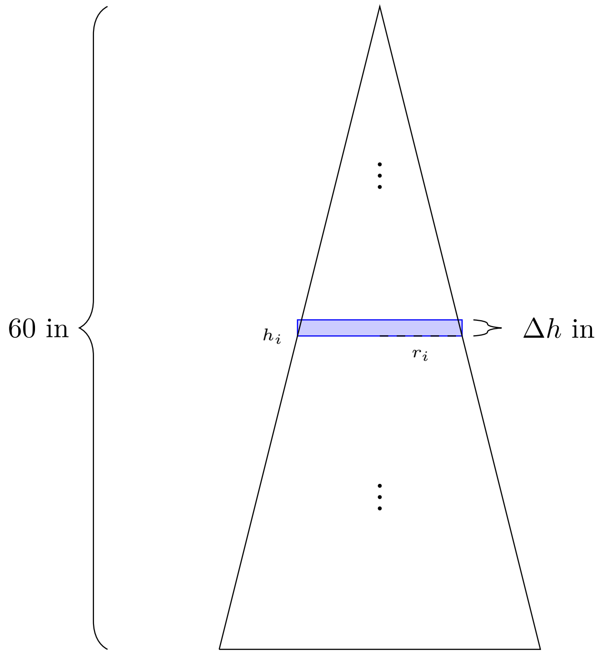


Figure 95 15× 60 cylindrical cone sliced into many pieces.

(a) Consider the piece sitting atop the slice made at height h_i. Using $\delta(h_i) = 15 + \sqrt{h_i}$ as the estimate for the density of this cylinder, what is the mass of this cylinder?

A. $(15 + \sqrt{h})\pi r_i^2 \cdot \Delta h$

B. $(15 + \sqrt{h_i})\pi r_i^2 \cdot \Delta h$

C. $(15 + \Delta h)\pi r_i^2 \cdot \Delta h_i$

D. $(15 + \sqrt{h_i})\pi r^2 \cdot \Delta h$

Activity 6.5.10 Consider a solid where the cross section of the solid at $x = x_i$ has area $A(x_i)$, and the density when $x = x_i$ is $\delta(x_i)$.

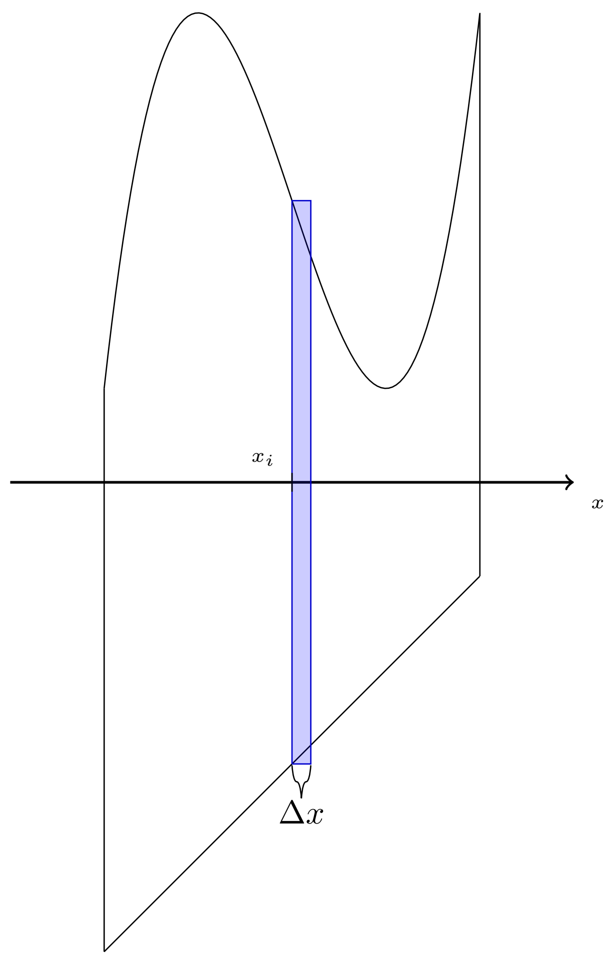


Figure 96 Solid approximated with prisms of width Δx.

(a) If we used prisms of width Δx to approximate this solid, what is the mass of the slice associated with x_i?

A. $A(x)\delta(x)\Delta x$

B. $\pi A(x)^2\delta(x_i)\Delta x$

C. $A(x_i)\delta(x_i)\Delta x$

D. $A(x_i)\delta(x_i)\Delta x_i$

Fact 6.5.11 *Consider a solid where the cross section of the solid at $x = x_i$ has area $A(x_i)$, and the density when $x = x_i$ is $\delta(x_i)$. Suppose the interval $[a, b]$ represents the x values of this solid. If one slices the solid into n pieces of width $\Delta x = \frac{b-a}{n}$, then one can approximate the mass of the solid by*

$$\sum_{i=1}^{n} \delta(x_i) A(x_i) \Delta x.$$

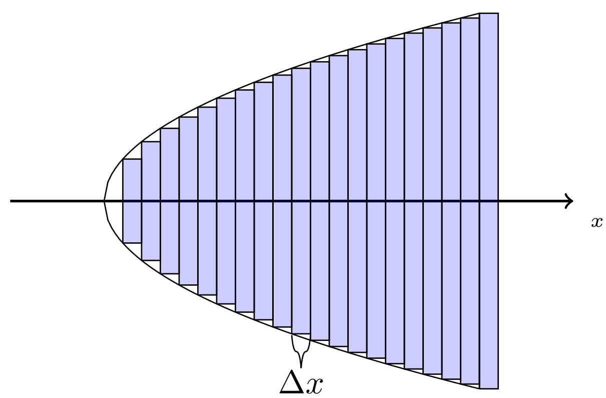


Figure 97 *Solid approximated with prisms of width Δx.*

We can then find actual mass by taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \delta(x_i) A(x_i) \Delta x \right) = \int_a^b \delta(x) A(x) dx.$$

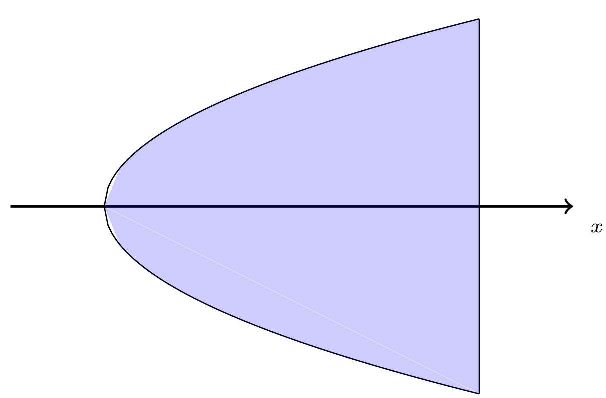


Figure 98 *Solid mass.*

Activity 6.5.12 Consider that for the prism from Activity 6.5.1, a cross section of height h is $A(h) = 10^2 = 100$ m^2. Also recall that the density of the prism is $\delta(h) = 10 + h$ kg/m^3, where h is the height in meters.

Use Fact 6.5.11 to find the mass of the prism.

Activity 6.5.13 Consider that for the cone from Activity 6.5.6, a cross section of height h is $A(h) = \pi r^2$ in^2, where r is the radius of the circular cross-section at height h inches. Also recall that the density of the cone is $\delta(h) = 15 + \sqrt{h}$ oz/in^3, where h is the height in inches.

(a) When the height is h inches, what is r?Use similar triangles:

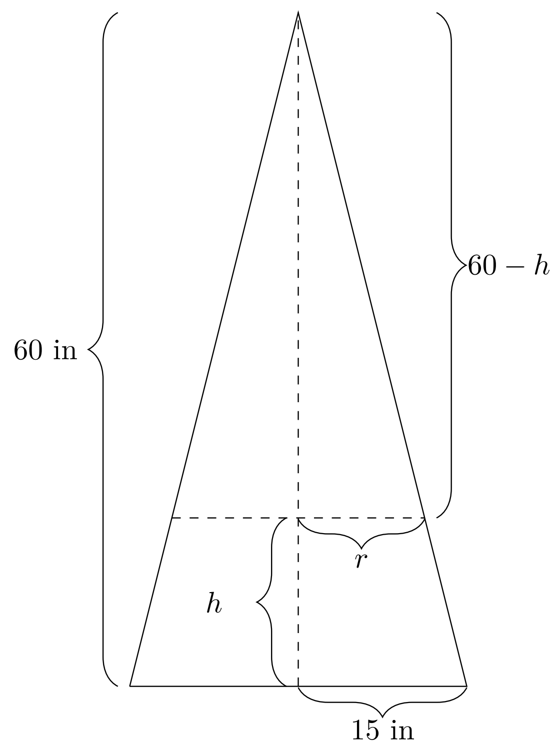


Figure 99 The right triangles in this figure are similar.

(b) Find $A(h)$ as a function of h using this information.

(c) Use Fact 6.5.11 to find the mass of the cone.

Activity 6.5.14 Consider a pyramid with a 8×8 ft square base and a height of 16 feet. Suppose the density of the pyramid is $\delta(h) = 10 + \cos(\pi h)$ lb/ft^3 where h is the height in feet.

(a) When the height is h feet, what is the area of the square cross section at that height, $A(h)$?Use similar triangles:

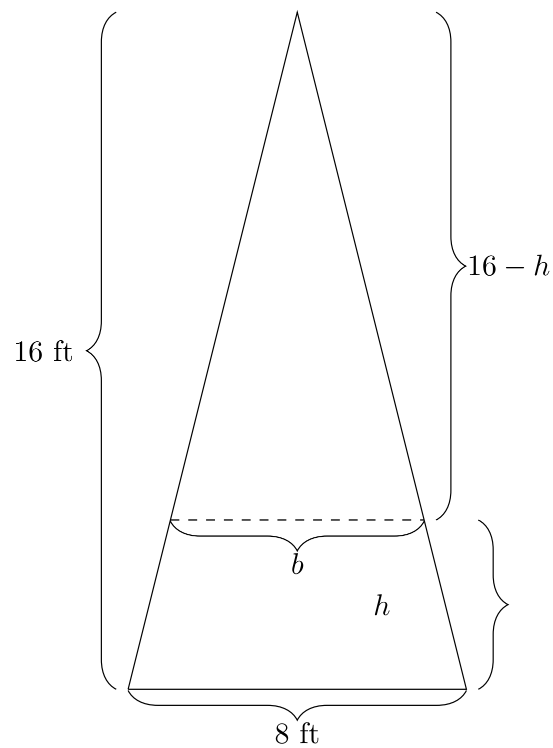


Figure 100 The triangles in this figure are similar.

(b) Use Fact 6.5.11 to find the mass of the pyramid.

Activity 6.5.15 Consider a board sitting atop the x-axis with six 1×1 blocks each weighing 1 kg placed upon it in the following way: two blocks are atop the 1, three blocks are atop the 2, and one block is atop the 6.

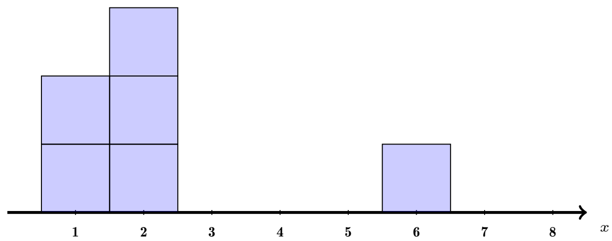


Figure 101 Six 1 kg blocks atop the x-axis.

Which of the following describes the x-value of the center of gravity of the board with the blocks?

A. $\dfrac{1+6}{2} = 3.5.$

B. $\dfrac{1+2+6}{3} = 3.$

C. $\dfrac{2\cdot 1 + 3\cdot 2 + 1\cdot 6}{6} \approx 2.3333.$

Activity 6.5.16 Consider a board sitting atop the x-axis with six 1×1 blocks each weighing 1 kg placed upon it in the following way: two blocks are atop the 1, three blocks are atop the 2, and one block is atop the 8.

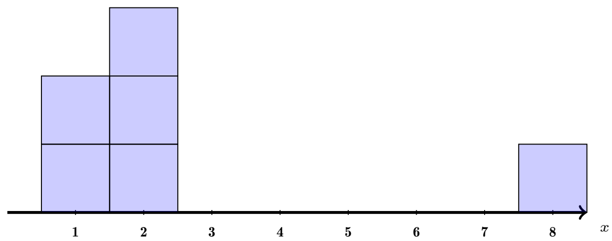


Figure 102 Six 1 kg blocks atop the x-axis.

Find the x-value of the center of gravity of the board with the blocks.

Fact 6.5.17 *Consider a solid where the cross section of the solid at $x = x_i$ has area $A(x_i)$, and the density when $x = x_i$ is $\delta(x_i)$. Suppose the interval $[a, b]$ represents the x values of this solid. Since each slice has approximate mass $\delta(x_i)A(x_i)\delta(x_i)$, we can approximate the center of mass by taking the weighted "average" of the x_i-values weighted by the associated mass:*

$$\frac{\sum_{i=1}^{n} x_i\delta(x_i)A(x_i)\Delta x}{\sum_{i=1}^{n} \delta(x_i)A(x_i)\Delta x}.$$

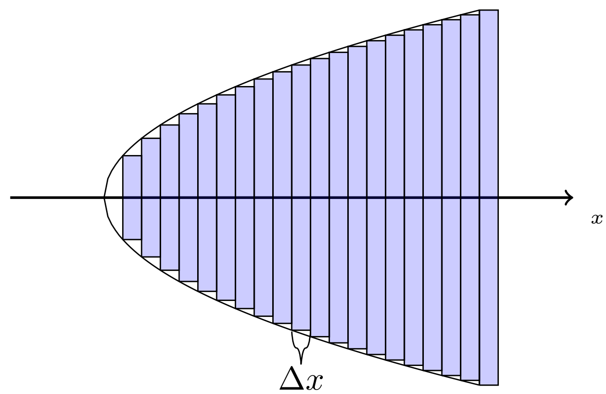


Figure 103 *Solid approximated with prisms of width Δx.*

We can then find actual center of mass by taking the limit as $n \to \infty$:

$$\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n} x_i\delta(x_i)A(x_i)\Delta x}{\sum_{i=1}^{n} \delta(x_i)A(x_i)\Delta x}\right) = \frac{\int_a^b x\delta(x)A(x)dx}{\int_a^b \delta(x)A(x)dx} = \frac{\int_a^b x\delta(x)A(x)dx}{\textit{The Total Mass}}.$$

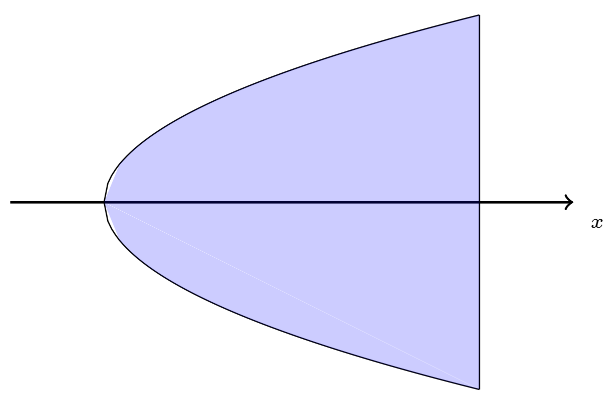


Figure 104 *Solid mass.*

Activity 6.5.18 Consider that for the prism from Activity 6.5.12, a cross section of height h is $A(h) = 10^2 = 100$ m^2. Also recall that the density of the prism is $\delta(h) = 10 + h$ kg/m^3, where h is the height in meters, and that we found the total mass to be 40000 kg.

Use Fact 6.5.17 to find the height where the center of mass occurs.

Activity 6.5.19 Consider that for the prism from Activity 6.5.13, a cross section of height h is $A(h) = \pi \cdot \left(\frac{60-h}{4}\right)^2$ in^2. Also recall that the density of the cone is $\delta(h) = 15 + \sqrt{h}$ oz/in^3, where h is the height in inches, and that we found the total mass to be about 142492.6 oz.

Use Fact 6.5.17 to find the height where the center of mass occurs.

Activity 6.5.20 Consider that for the pyramid from Activity 6.5.14, a cross section of height h is $A(h) = \pi \cdot \left(\frac{16-h}{2}\right)^2$ ft^2. Also recall that the density of the pyramid is $\delta(h) = 10 + \cos \pi h$ lb/feet^3, where h is the height in feet, and that we found the total mass to be about 3414.14.6 lbs.

Use Fact 6.5.17 to find the height where the center of mass occurs.

6.6 Work (AI6)

Learning Outcomes

- Set up integrals to solve problems involving work.

Fact 6.6.1 *Given a physical object m, the* ***work*** *done on that object is*

$$W = Fd = mad,$$

where F is the force applied to the object over a distance of d. Recall that force $F = ma$, where m is the mass of the object, and a is the acceleration applied to it.

Work (AI6)

Activity 6.6.2 Consider a bucket with 10 kg of water being pulled against the acceleration of gravity, $g = 9.8$ m/s^2, at a constant speed for 20 meters. Using Fact 6.6.1, what is the work needed to pull this bucket up 20 meters in kgm$^2/s^2$ (or Nm)?

A. 10 kgm$^2/s^2$

B. 20 kgm$^2/s^2$

C. 98 kgm$^2/s^2$

D. 200 kgm$^2/s^2$

E. 1960 kgm$^2/s^2$

Activity 6.6.3 Consider the bucket from Activity 6.6.2 with 10 kg of water, being pulled against the acceleration of gravity, $g = 9.8$ m/s^2, at a constant speed for 20 meters. Suppose that halfway up at a height of 10m, 5kg of water spilled out, leaving 5kg left. How much total work does it take to get this bucket to a height of 20m?

A. 980 kgm$^2/s^2$ or Nm

B. 1470 kgm$^2/s^2$ or Nm

C. 1960 kgm$^2/s^2$ or Nm

Activity 6.6.4 Suppose a 10 kg bucket of water is constantly losing water as it's pulled up, so at a height of h meters, the mass of the bucket is $m(h) = 2 + 8e^{-0.2h}$ kg.

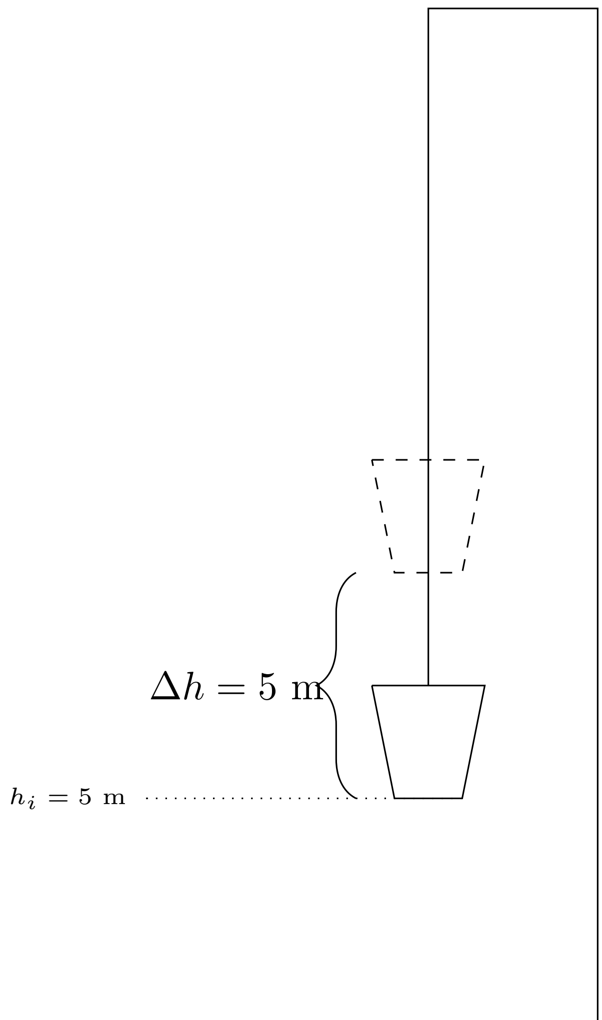


Figure 105 Bucket 5 m in the air, to be hoisted by another 5 meters.

(a) What is the mass of the bucket at height $h_i = 5$ m?

(b) Assuming that the bucket does not lose water, estimate the amount of work needed to lift this bucket up $\Delta h = 5$ meters.

Activity 6.6.5 using the same the bucket from Activity 6.6.4, consider the bucket's mass at heights $h_i = 0, 5, 10, 15$ meters.

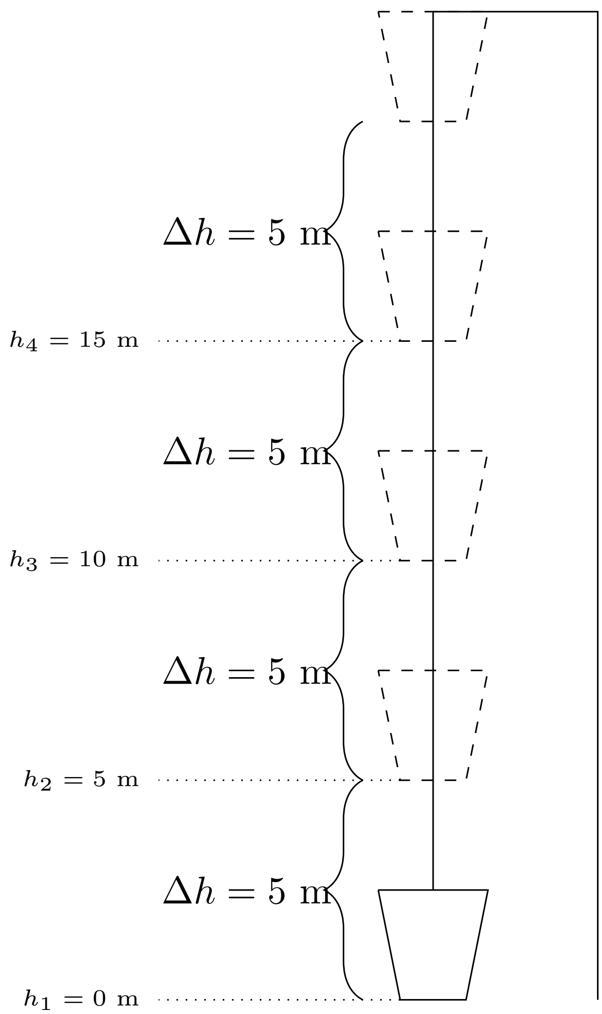


Figure 106 Bucket lifted 5 m at a time.

(a) Fill out the following table, estimating the work it would take to lift the bucket 20 meters.

h_i	Mass $m(h_i)$	Distance	Estimated Work
$h_4 = 15$ m	$m(15) = 2 + 8e^{-0.2 \cdot 15} \approx 2.398$ kg	5 m	
$h_3 = 10$ m	$m(10) = 2 + 8e^{-0.2 \cdot 10} \approx 3.083$ kg	5 m	
$h_2 = 5$ m	$m(5) = 2 + 8e^{-0.2 \cdot 5} \approx 4.943$ kg	5 m	242.207 Nm
$h_1 = 0$ m	$m(5) = 2 + 8e^{-0.2 \cdot 0} = 10$ kg	5 m	

(b) What is the total estimated work to lift this bucket 20 meters?

Activity 6.6.6 If we estimate the mass and work of the bucket from Activity 6.6.5 at height h_i with intervals of length Δh meters, which of the following best represents the Riemann sum of the work it would take to lift this bucket 20 meters?

A. $\sum h_i \cdot 9.8\Delta h.$ Nm

B. $\sum \left(2 + 8e^{-0.02h}\right) \cdot 9.8\Delta m$ Nm

C. $\sum \left(2 + 8e^{-0.02h_i}\right) \cdot 9.8\Delta h$ Nm

D. $\sum \left(2 + 8e^{-0.02h_i}\right) \cdot 9.8\Delta m$ Nm

Activity 6.6.7 Based on the Riemann sum chosen in Activity 6.6.6, which of the following integrals computes the work it would take to lift this bucket 20 meters?

A. $\displaystyle\int_0^{20} h_i \cdot 9.8dh.$ Nm

B. $\displaystyle\int_0^{20} \left(2+8e^{-0.02h}\right) \cdot 9.8dm$ Nm

C. $\displaystyle\int_0^{20} \left(2+8e^{-0.02h}\right) \cdot 9.8dh$ Nm

D. $\displaystyle\int_0^{20} \left(2+8e^{-0.02h_i}\right) \cdot 9.8dh$ Nm

Activity 6.6.8 Based on the integral chosen in Activity 6.6.7, compute the work it would take to lift this bucket 20 meters.

Observation 6.6.9 A "how to" for applying integrals to physics.

1. Estimate the value over a piece of the problem with x value x_i over interval of length Δx.

2. Find a Riemann sum using (1) which estimates the value in question.

3. Convert the Riemann sum to an integral and solve.

Activity 6.6.10 Consider a cylindrical tank filled with water, where the base of the cylinder has a radius of 3 meters and a height of 10 meters. Consider a 2 meter thick slice of water sitting 6 meters high in the tank. Using the fact that the mass of this water is $1000 \cdot \pi(3)^2 \cdot 2 = 18000\pi$ kg, estimate how much work is needed to lift this slice 4 more meters to the top of the tank.

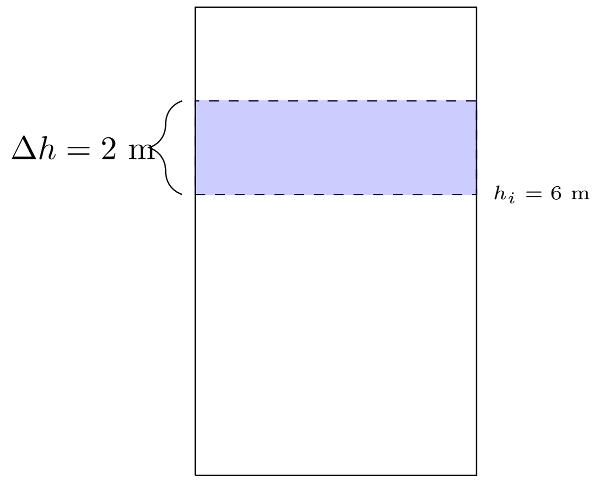


Figure 107 2m thick slice of water lifted 4m.

A. $18000\pi \cdot 4$ Nm

B. $18000\pi \cdot 9.8$ Nm

C. $18000\pi \cdot 4 \cdot 9.8$ Nm

D. $18000\pi \cdot 6$ Nm

E. $18000\pi \cdot 6 \cdot 9.8$ Nm

Activity 6.6.11 Consider the cylindrical tank filled with water from Activity 6.6.10. We wish to estimate the amount of work required to pump all the water out of the tank. Suppose we slice the water into 5 pieces and estimate the work it would take to lift each piece out of the tank.

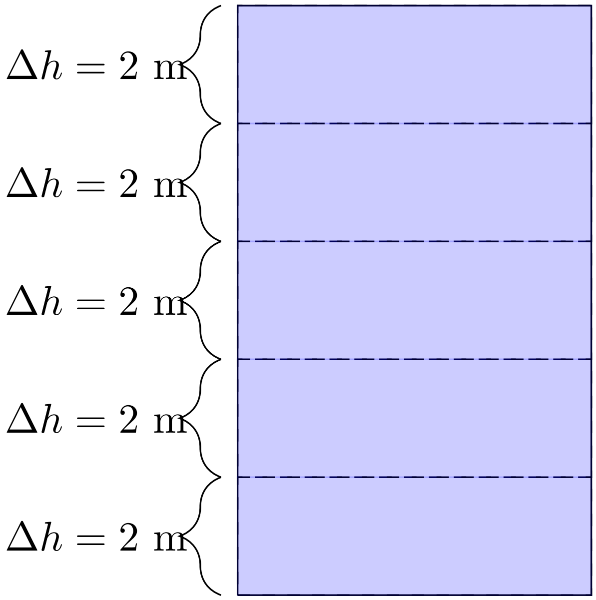


Figure 108 2m thick slices of water.

(a) Fill out the following table, estimating the work it would take to pump all the water out.

h_i	Mass	Distance	Estimated Work
$h_5 = 8$ m	18000π kg		
$h_4 = 6$ m	18000π kg	4 m	705600π Nm
$h_3 = 4$ m	18000π kg		
$h_2 = 2$ m	18000π kg		
$h_1 = 0$ m	18000π kg	10 m	

(b) What is the total estimated work to pump out all the water?

Activity 6.6.12 Recall Activity 6.6.11. If we estimate the work needed to lift slices of thickness Δh m at heights h_i m, which of the following Riemann sums best estimates the total work needed to pump all the water from the tank?

A. $\sum 1000 \cdot \pi 3^2 \cdot 9.8(10 - h)\Delta h$ Nm

B. $\sum 1000 \cdot \pi 3^2 \cdot 9.8(10 - h_i)\Delta h$ Nm

C. $\sum 1000 \cdot \pi (h_i)^2 \cdot 9.8(10 - h)\Delta h$ Nm

D. $\sum 1000 \cdot \pi (h_i)^2 \cdot 9.8(10 - h_i)\Delta h$ Nm

Activity 6.6.13 Based on the Riemann sum chosen in Activity 6.6.12, which of the following integrals computes the work it would take to pump all the water from the tank?

A. $\displaystyle\int_0^{10} 9000\pi \cdot 9.8(10-h)dh$ Nm

B. $\displaystyle\int_0^{10} 1000\pi \cdot 9.8h^2(10-h)dh$ Nm

Activity 6.6.14 Based on the integral chosen in Activity 6.6.13, compute the work it would take to pump all the water out of the tank.

Activity 6.6.15 Consider a cylindrical truncated-cone tank where the radius on the bottom of the tank is 10 m, the radius at the top of the tank is 100 m, and the height of the tank is 100m.

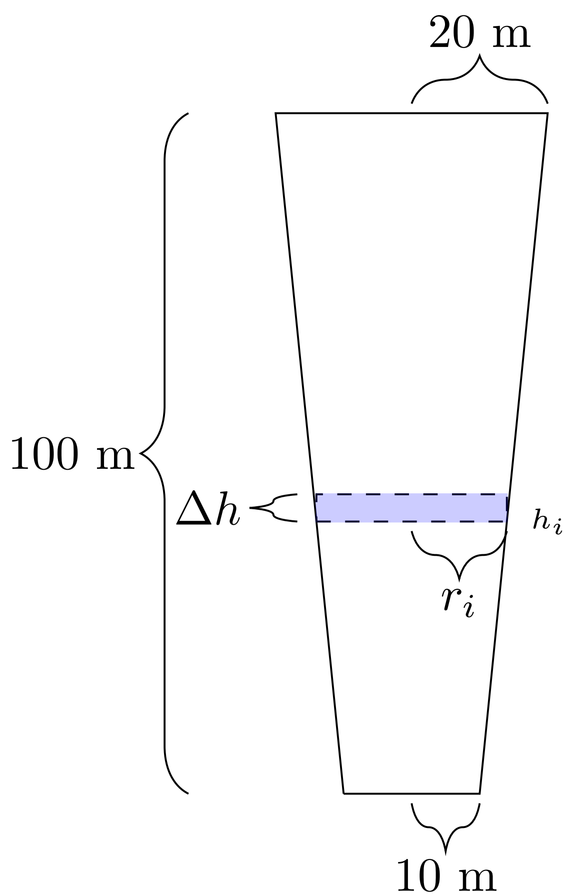


Figure 109 A slice at height h_i of width Δh.

(a) What is the radius r_i in meters of the cross section made at height h_i meters?

(b) What is the volume of a cylinder with radius r_i meters with width Δh meters?

(c) Using the fact that water has density 1000 kg/m^3, what is the mass of the volume of water you found in (b)?

(d) How far must this cylinder of water be lifted to be out of the tank?

Activity 6.6.16 Recall the computations done in Activity 6.6.15.

(a) Find a Riemann sum which estimates the total work needed to pump all the water out of this tank, using slices at heights h_i m, of width Δh m.

(b) Use (a) to find an integral expression which computes the amount of work needed to pump all the water out of this tank.

(c) Evaluate the integral found in (b).

6.7 Force and Pressure (AI7)

Learning Outcomes

- Set up integrals to solve problems involving force and/or pressure.

Force and Pressure (AI7)

Fact 6.7.1 *Recall that **pressure** is measured as force over area:*

$$P = F/A.$$

Rewriting this, we have that $F = PA$.

Activity 6.7.2 Consider a trapezoid-shaped dam that is 60 feet wide at its base and 90 feet wide at its top. Assume the dam is 20 feet tall with water that rises to its top. Water weighs 62.4 pounds per cubic foot and exerts $P = 62.4d$ lbs/ft^2 of pressure at depth d ft. Consider a rectangular slice of this dam at height h_i feet and width b_i.

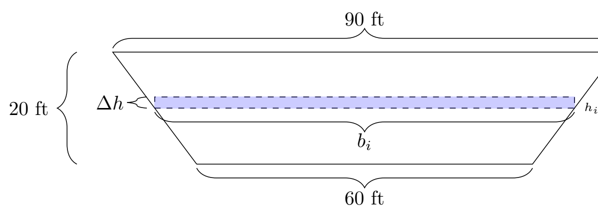


Figure 110 A slice at height h_i of width Δh.

(a) At a height of h_i feet, what is the base of the rectangle b_i?

(b) What is the area of a rectangle with base b_i feet and height Δh feet?

(c) Using a depth of $20 - h_i$ feet, how much pressure is exerted on this rectangle?

(d) Using the pressure found in (c), the area in (b), and Fact 6.7.1, how much force is exerted on this rectangle?

Activity 6.7.3 Recall the computations done in Activity 6.7.2.

(a) Find a Riemann sum which estimates the total force exerted on the dam, using slices at heights h_i m, of width Δh m.

(b) Use (a) to find an integral expression which computes the amount of force exerted on this dam.

(c) Evaluate the integral found in (b).

Chapter 7

Coordinates and Vectors (CO)

Learning Outcomes

How do we use alternative coordinates and vectors to describe points in the plane?

By the end of this chapter, you should be able to...

1. Sketch the graph of a two-dimensional parametric/vector equation, and convert such equations into equations of only x and y.
2. Compute derivatives and tangents related to two-dimensional parametric/vector equations.
3. Compute arclengths related to two-dimensional parametric/vector equations.
4. Convert points and equations between polar and Cartesian coordinates and equations.
5. Compute arclengths of curves given in polar coordinates.
6. Compute areas bounded by curves given in polar coordinates.

7.1 Parametric/vector equations (CO1)

Learning Outcomes

- Sketch the graph of a two-dimensional parametric/vector equation, and convert such equations into equations of only x and y.

Activity 7.1.1 Consider how we might graph the equation $y = 2 - x^2$ in the xy-plane.

(a) Complete the following chart of xy values by plugging each x value into the equation to produce its y value.

Table 111 Chart of x and y values to graph

x	y
-2	
-1	1
0	
1	
2	

(b) Plot each point (x, y) in your chart in the xy plane.

(c) Connect the dots to obtain a reasonable sketch of the equation's graph.

Activity 7.1.2 Suppose that we are told that at after t seconds, an object is located at the x-coordinate given by $x = t - 2$ and the y-coordinate given by $y = -t^2 + 4t - 2$.

(a) Complete the following chart of txy values by plugging each t value into the equations to produce its x and y values.

Table 112 Chart of x and y values for each t

t	x	y
0		
1	-1	1
2		
3		
4		

(b) Plot each point (x, y) in your chart in the xy plane, labeling it with its t value.

(c) Connect the dots to obtain a reasonable sketch of the equation's graph.

Definition 7.1.3 Graphs in the xy plane can be described by **parametric equations** $x = f(t)$ and $y = g(t)$, where plugging in different values of t into the functions f and g produces different points of the graph.

The t-values may be thought of representing the moment of *time* when an object is located at a particular position, and the graph may be thought of as the path the object travels throughout time. $\Diamond$

Activity 7.1.4 Earlier we obtained the same graphs for the xy equation $y = 2 - x^2$ and the parametric equations $x = t - 2$ and $y = -t^2 + 4t - 2$. Do the following steps to find out why.

(a) Which of the following equations describes t in terms of x?

A. $t = x - 2$

B. $t = x + 2$

C. $t = 2x$

D. $t = -2x$

(b) Which of these is the result of plugging this choice in for t in the parametric equation for y?

A. $y = -x + 2^2 + 4x + 2 - 2$

B. $y = -(x + 2)^2 + 4(x + 2) - 2$

C. $y = -x^2 + 2^2 + 4x + 4 \cdot 2 - 2$

(c) Show how to simplify this choice to obtain the equation $y = 2 - x^2$.

Fact 7.1.5 *One method of graphing parametric equations $x = f(t)$ and $y = g(t)$ is to combine them into a single equation only involving x and y, and using your usual graphing techniques.*

Activity 7.1.6 Parametric equations have the advantage of describing paths that cannot be described by a function $y = h(x)$. One such example is the graph of $x = 3\sin(\pi t)$ and $y = -3\cos(\pi t)$. (Use technology or the approximation $\sqrt{2} \approx 0.707$ to approximate coordinates as needed.)

(a) Complete the folowing table.

Table 113 Chart of approximate x and y values

t	x	y
0		
1/4		
1/2		
3/4	2.12	2.12
1		
5/4		
3/2		
7/4		
2		

(b) Plot these (x, y) points in the xy plane and connect the dots to draw a sketch of the graph.

(c) What do you obtain by plugging the parametric equations into the expression $x^2 + y^2$?

A. $x^2 + y^2 = -6\sin(\pi x)\cos(\pi x)$

B. $x^2 + y^2 = 9$

C. $x^2 + y^2 = 6\sin(\pi x)\cos(\pi x)$

D. $x^2 + y^2 = 0$

(d) Which of these describes the xy equation and graph given by these parametric equations?

A. a parabola

B. a line

C. a circle

D. a square

(e) The graph of these parametric equations cannot be described by a function. Why?

A. The graph fails the vertical line test.

B. The graph fails the horizontal line test.

C. The graph doesn't extend vertically to $+\infty$.

D. The graph doesn't extend horizontally to $-\infty$.

Definition 7.1.7 The parametric equations $x = f(t)$ and $y = g(t)$ are sometimes written in the form of the **vector equation** $\vec{r} = \langle f(t), g(t) \rangle$.

For example, the parametric equations $x = 3\sin(\pi t)$ and $y = -3\cos(\pi t)$ may be combined into the single vector equation $\vec{r} = \langle 3\sin(\pi t), -3\cos(\pi t) \rangle$.

$\Diamond$

Activity 7.1.8 Consider the vector equation $\vec{r} = \langle 2t - 3, -6t + 13 \rangle$.

(a) What are the corresponding parametric equations?

A. $x = 2t - 3$ and $y = -6t + 13$

B. $y = 2t - 3$ and $x = -6t + 13$

C. $xy = 2t - 3 - 6t + 13$

D. Vector equations cannot be converted into parametric equations.

(b) Draw a table of t, x, and y values with $t = 0, 1, 2, 3, 4$.

(c) Plot these (x, y) points in the plane and connect the dots to sketch the graph of this vector equation.

(d) Solve for t in terms of x and plug into the y parametric equation to show that this is the vector equation for the line $y = -3x + 4$.

7.2 Parametric/vector derivatives (CO2)

Learning Outcomes

- Compute derivatives and tangents related to two-dimensional parametric/vector equations.

Parametric/vector derivatives (CO2)

Activity 7.2.1 Consider the parametric equations $x = 2t - 1$ and $y = (2t-1)(2t-5)$. The coordinate on this graph at $t = 2$ is $(3, -3)$.

(a) Which of the following equations of x, y describes the graph of these paramteric equations?

A. $y = 2x(x+2) = 2x^2 + 2x$

B. $y = 2x(x-2) = 2x^2 - 2x$

C. $y = x(x+4) = x^2 + 4x$

D. $y = x(x-4) = x^2 - 4x$

(b) Which of the following describes the slope of the line tangent to the graph at the point $(3, -3)$?

A. $\frac{dy}{dx} = 2x+4$, which is 10 when $x = 3$.

B. $\frac{dy}{dx} = 2x + 4$, which is 8 when $t = 2$.

C. $\frac{dy}{dx} = 2x - 4$, which is 2 when $x = 3$.

D. $\frac{dy}{dx} = 2x - 4$, which is 0 when $t = 2$.

(c) Note that the parametric equation for y simplifies to $y = 4t^2 - 12t + 5$. What do we get for the derivatives $\frac{dx}{dt}$ of $x = 2t - 1$ and $\frac{dy}{dt}$ for $y = 4t^2 - 12t + 5$?

A. $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 8t - 12$.

B. $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 8t - 12$.

C. $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 6t + 5$.

D. $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 6t + 5$.

(d) It follows that when $t = 2$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 4$. Which of the following conjectures seems most likely?

A. The slope $\frac{dy}{dx}$ could also be found by computing $\frac{dx}{dt} + \frac{dy}{dt}$.

B. The slope $\frac{dy}{dx}$ could also be found by computing $\frac{dy/dt}{dx/dt}$.

C. The slope $\frac{dy}{dx}$ is always equal to $\frac{dx}{dt}$.

D. The slope $\frac{dy}{dx}$ is always equal to $\frac{dy}{dt}$.

Parametric/vector derivatives (CO2)

Fact 7.2.2 *Suppose x is a function of t, and y may be thought of as a function of either x or t. Then the Chain Rule requires that*

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

This provides the slope formula for parametric equations:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Activity 7.2.3 Let's draw the picture of the line tangent to the parametric equations $x = 2t - 1$ and $y = (2t - 1)(2t - 5)$ when $t = 2$.

(a) Use a t, x, y chart to sketch the parabola given by these parametric equations for $0 \leq t \leq 3$, including the point $(3, -3)$ when $t = 2$.

(b) Earlier we determined that the slope of the tangent line was 2. Draw a line with slope 2 passing through $(3, -3)$ and confirm that it appears to be tangent.

(c) Use the point-slope formula $y - y_0 = m(x - x_0)$ along with the slope 2 and point $(3, -3)$ to find the exact equation for this tangent line.

A. $y = 2x - 10$

B. $y = 2x - 9$

C. $y = 2x - 8$

D. $y = 2x - 7$

Activity 7.2.4 Consider the vector equation $\vec{r}(t) = \left\langle 3t^2 - 9, t^3 - 3t \right\rangle$.

(a) What are the corresponding parametric equations and their derivatives?

A. $y = 3t^2 - 9$ and $x = t^3 - 3t$; $\frac{dy}{dt} = 9t$ and $\frac{dx}{dt} = 3t - 6$

B. $x = 3t^2 - 9$ and $y = t^3 - 3t$; $\frac{dx}{dt} = 9t$ and $\frac{dy}{dt} = 3t - 6$

C. $y = 3t^2 - 9$ and $x = t^3 - 3t$; $\frac{dy}{dt} = 6t$ and $\frac{dx}{dt} = 3t^2 - 3$

D. $x = 3t^2 - 9$ and $y = t^3 - 3t$; $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 3t^2 - 3$

(b) The formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ allows us to compute slopes as which of the following functions of t?

A. $\dfrac{6t}{t^2 + 3}$

B. $\dfrac{6t}{t^2 + 1}$

C. $\dfrac{t^2 - 1}{2t}$

D. $\dfrac{2t}{3t^2 - 1}$

(c) Find the point, tangent slope, and tangent line equation (recall $y - y_0 = m(x - x_0)$) corresponding to the parameter $t = -3$.

A. Point $(-12, 9)$, slope $-\frac{4}{3}$, EQ $y = -\frac{4}{3}x - 7$

B. Point $(18, -18)$, slope $-\frac{4}{3}$, EQ $y = -\frac{4}{3}x + 6$

C. Point $(-12, 9)$, slope $\frac{3}{4}$, EQ $y = \frac{3}{4}x - 8$

D. Point $(18, -18)$, slope $\frac{3}{4}$, EQ $y = \frac{3}{4}x + 5$

7.3 Parametric/vector arclength (CO3)

Learning Outcomes

- Compute arclengths related to two-dimensional parametric/vector equations.

Example 7.3.1 In Figure 166, the blue curve is the graph of the parametric equations $x = t^2$ and $y = t^3$ for $1 \leq t \leq 2$. This curve connects the point $(1, 1)$ to the point $(4, 8)$. The red dashed line is the straight line segment connecting these points.

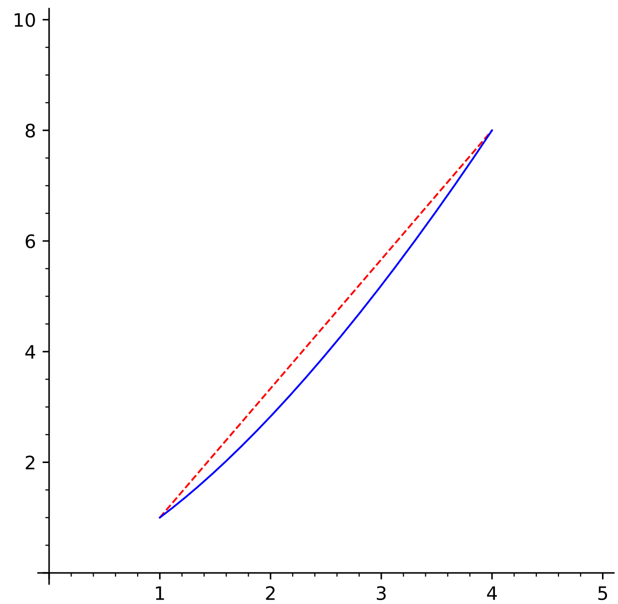


Figure 114 A parametric curve and segment from $(1, 1)$ to $(4, 8)$

□

Activity 7.3.2 Let's first investigate the length of the dashed red line segment in Figure 166.

(a) Draw a right triangle with the red dashed line segment as its hypotenuse, one leg parallel to the x-axis, and the other parallel to the y-axis.

How long are these legs?

A. 3 and 7.

B. 4 and 8.

C. 3 and 8.

D. 4 and 7.

(b) The Pythagorean theorem states that for a right triangle with leg lengths a, b and hypotenuse length c, we have...

A. $a = b = c$.

B. $a + b = c$.

C. $a^2 = b^2 = c^2$.

D. $a^2 + b^2 = c^2$.

(c) Using the leg lengths and Pythagorean theorem, how long must the red dashed hypotenuse be?

A. $\sqrt{20} \approx 4.47$.

B. $\sqrt{58} \approx 7.62$.

C. $\sqrt{67} \approx 8.19$.

D. $\sqrt{100} = 10$.

(d) Compared with the blue parametric curve connecting the same two points, is the red dashed line segement length an overestimate or underestimate?

A. Overestimate: the blue curve is shorter than the red line.

B. Underestimate: the blue curve is longer than the red line.

C. Exact: the blue curve is exactly as long as the red line.

Fact 7.3.3 *Recall that the linear distance between two points* (x_1, y_1) *and* (x_2, y_2) *may be computed by the distance formula*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note that $\Delta x = |x_2 - x_1|$ *and* $\Delta y = |y_2 - y_1|$ *measure leg lengths of a right triangle whose hypotenuse is the distance we want to measure, so we may rewrite this formula as*

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This formula will need to be modified to measure a curved path between two points.

Observation 7.3.4 By approximating the curve by several (say N) segements connecting points along the curve, we obtain a better approximation than a single line segment. For example, the illustration shown in Figure 167 gives three segments whose distances sum to about 7.6315, while the actual length of the curve turns out to be about 7.6337.

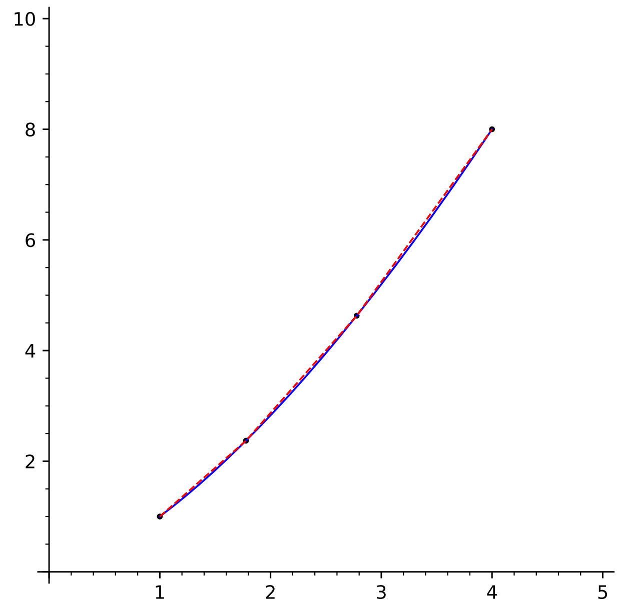


Figure 115 Subdividing a parametric curve where $N = 3$

Activity 7.3.5 How should we modify the distance formula $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ to measure arclength as illustrated in Figure 167?

(a) Let $\Delta L_1, \Delta L_2, \Delta L_3$ describe the lengths of each of the three segements. Which expression describes the total length of these segments?

A. $\Delta L_1 \times \Delta L_2 \times \Delta L_3$

B. $\Delta L_1 + 2\Delta L_2 + 3\Delta L_3$

C. $\displaystyle\sum_{i=1}^{3} \Delta L_i$

(b) We can let each $\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. But we will find it useful to involve the parameter t as well, or more accurately, the change Δt_i of t between each point of the subdivision.

Which of these is algebraically the same as the above formula for ΔL_i?

A. $\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}$

B. $\sqrt{\left[\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2\right] \Delta t_i}$

C. $\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\Delta t_i$

(c) Finally, we'll want to increase N from 3 so that it limits to ∞. What can we conclude when that happens?

A. Each segment is infintely small.

B. $\Delta x_i \to 0$

C. $\frac{\Delta x_i}{\Delta t_i} \to \frac{dx}{dt}$

D. All of the above.

Observation 7.3.6 Put together, and limiting the subdivisions of the curve $N \to \infty$, we obtain the Riemann sum

$$\lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} \Delta t_i.$$

Thus arclength along a parametric curve from $a \leq t \leq b$ may be calculated by using the corresponding definite integral

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Activity 7.3.7 Let's gain confidence in the arclength formula

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

by checking to make sure it matches the distance formula for line segments.

The parametric equations $x = 3t - 1$ and $y = 2 - 4t$ for $1 \leq t \leq 3$ represent the segment of the line $y = -\frac{4}{3}x - \frac{2}{3}$ connecting $(2, -2)$ to $(8, -10)$.

(a) Find dx/dt and dy/dt, and substitute them into the formula above along with $a = 1$ and $b = 3$.

(b) Show that the value of this formula is 10.

(c) Show that the length of the line segment connecting $(2, -2)$ to $(8, -10)$ is 10 by applying the distance formula directly instead.

Activity 7.3.8 For each of these parametric equations, use

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

to write a definite integral that computes the given length. (Do not evaluate the integral.)

(a) The portion of $x = \sin 3t, y = \cos 3t$ where $0 \leq t \leq \pi/6$.

(b) The portion of $x = e^t, y = \ln t$ where $1 \leq t \leq e$.

(c) The portion of $x = t + 1, y = t^2$ between the points $(3, 4)$ and $(5, 16)$.

Activity 7.3.9 Let's see how to modify $\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ to produce the arclength of the graph of a function $y = f(x)$.

(a) Let $x = t$. How can $\frac{dx}{dt}$ be simplified?

A. dx

B. dt

C. 1

D. 0

(b) Given $x = t$, how should $\frac{dy}{dt}$ and dt be rewritten?

A. $\frac{dy}{dt} = \frac{dy}{dx}$ and $dt = dx$.

B. $\frac{dy}{dt} = \frac{dx}{dt}$ and $dt = dx$.

C. $\frac{dy}{dt} = \frac{dy}{dx}$ and $dt = 1$.

D. $\frac{dy}{dt} = \frac{dy}{dt}$ and $dt = 1$.

(c) Write a modified, simplified formula for $\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ with t replaced with x.

7.4 Polar coordinates (CO4)

Learning Outcomes

- Convert points and equations between polar and Cartesian coordinates and equations.

Fact 7.4.1 *"As the crow flies" is an idiom used to describe the most direct path between two points. The* ***polar coordinate system*** *is a useful parametrization of the plane that, rather than describing horizontal and vertical position relative to the origin in the usual way, describes a point in terms of distance from the origin and direction. The origin is also known as the* ***pole*** *(hence polar coordinates).*

Let $\overline{OP}$ *be a line segment from the origin to a given point* P *in the plane. The length of* $\overline{OP}$ *is the distance (or* ***radius****)* r *from the origin to* P*. The* ***polar axis*** *is a ray starting at the origin.*

To define the "direction" of P*, we form an angle* θ *by letting the polar axis serve as the initial ray and* $\overrightarrow{OP}$ *as the terminal ray. We will set the positive* x*-axis as the polar axis and assume the movement in the positive direction is counter-clockwise (as in trigonometry). Notice that, unlike in the rectangular (or* Cartesian*) coordinate system, the polar coordinates* (r, θ) *for a point are not unique, as we could turn either way to face a given point (or even spin around a number of times before facing that direction).*

Furthermore, by allowing r *to be negative, we can also "walk backwards" to get to a point by facing in the opposite direction. Rather than the grid lines defined by specific values for* x *and* y *in the rectangular coordinate system, specific values of* r *correspond to circles of radius* r *centered about the origin, and specific values of* θ *correspond to lines going through the pole (called* ***radial lines****).*

Activity 7.4.2

(a) Plot the Cartesian point $P = (x, y) = (\sqrt{3}, -1)$ and draw line segments connecting the origin to P, the origin to $(x, y) = (\sqrt{3}, 0)$, and P to $(x, y) = (\sqrt{3}, 0)$.

(b) Solve the triangle formed by the line segments you just drew (i.e. find the lengths of all sides and the measures of each angle).

(c) Find all polar coordinates for the Cartesian point $(x, y) = (\sqrt{3}, -1)$.

(d) Find Cartesian coordinates for the polar point $(r, \theta) = \left(-\sqrt{2}, \frac{3\pi}{4}\right)$.

Activity 7.4.3 Graph each of the following.

(a) $r = 1$

(b) $r = -1$

(c) $\theta = \dfrac{\pi}{6}$

(d) $\theta = \dfrac{7\pi}{6}$

(e) $\theta = \dfrac{-5\pi}{6}$

(f) $1 \leq r < -1,\ 0 \leq \theta \leq \dfrac{\pi}{2}$

(g) $-3 \leq r \leq 2,\ \theta = \dfrac{\pi}{4}$

(h) $r \leq 0,\ \theta = \dfrac{-\pi}{2}$

(i) $\dfrac{2\pi}{3} \leq \theta \leq \dfrac{5\pi}{6}$

(j) $r = 3\sec(\theta)$

Fact 7.4.4 *If a polar graph is symmetric about the x-axis, then if the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ also lies on the graph.*

Fact 7.4.5 *If a polar graph is symmetric about the y-axis, then if the point* (r, θ) *lies on the graph, then the point* $(r, \pi - \theta)$ *or* $(-r, -\theta)$ *also lies on the graph.*

Fact 7.4.6 *If a polar graph is rotationally symmetric about the origin, then if the point* (r, θ) *lies on the graph, then the point* $(-r, \theta)$ *or* $(r, \pi + \theta)$ *also lies on the graph.*

Activity 7.4.7

(a) Find a polar form of the the Cartesian equation $x^2 + (y-3)^2 = 9$.

Activity 7.4.8 Find a Cartesian form of each of the given polar equations.

(a) $r^2 = 4r\cos(\theta)$

(b) $r = \dfrac{4}{2\cos(\theta) - \sin(\theta)}$

7.5 Polar Arclength (CO5)

Learning Outcomes

- Compute arclengths of curves given in polar coordinates.

Polar Arclength (CO5)

Activity 7.5.1 Recall that the length of a parametric curve is given by

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(a) Let $x(t) = r\cos(\theta)$ and $y(t) = r\sin(\theta)$ and show that the length of a polar curve $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by

$$\int_{\theta=\alpha}^{\theta=\beta} \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

(b) Find an integral computing the arclength of the polar curve defined by $r = 3\cos(\theta) - 2$ on $\pi/3 \le \theta \le \pi$.

(c) Find the length of the cardioid $r = 1 - \cos(\theta)$.

7.6 Polar area (CO6)

Learning Outcomes

- Compute areas bounded by curves given in polar coordinates.

Polar area (CO6)

Fact 7.6.1 *The area of the "fan-shaped" region between the pole and* $r = f(\theta)$ *as the angle* θ *ranges from* α *to* β *is given by*

$$\int_{\theta=\alpha}^{\theta=\beta} \frac{r^2}{2} d\theta.$$

Activity 7.6.2

(a) Find an integral computing the area of the region defined by $0 \leq r \leq -\cos(\theta) + 5$ and $\pi/2 \leq \theta \leq 3\pi/4$.

(b) Find the area enclosed by the cardioid $r = 2(1 + \cos(\theta)$.

(c) Find the area enclosed by one loop of the 4-petaled rose $r = \cos(2\theta)$.

Chapter 8

Sequences and Series (SQ)

Learning Outcomes

By the end of this chapter, you should be able to...

1. Define and use explicit and recursive formulas for sequences.
2. Determine if a sequence is convergent, divergent, monotonic, or bounded, and compute limits of convergent sequences.
3. Compute the first few terms of a telescoping or geometric partial sum sequence, and find a closed form for this sequence, and compute its limit.
4. Determine if a geometric series converges, and if so, the value it converges to.
5. Use the divergence, alternating series, and integral tests to determine if a series converges or diverges.
6. Use the direct comparison and limit comparison tests to determine if a series converges or diverges.
7. Use the ratio and root tests to determine if a series converges or diverges.
8. Determine if a series converges absolutely or conditionally.

8.1 Sequence Formulas (SQ1)

Learning Outcomes

- Define and use explicit and recursive formulas for sequences.

Sequence Formulas (SQ1)

Activity 8.1.1 Which of the following are sequences?

A. monthly gas bill

B. days in the year

C. how long you wash dishes

D. $1, 1, 2, 3, 5, 8, \ldots$

E. how much you spend on groceries

Activity 8.1.2 Consider the sequence $1, 2, 4, \ldots$.

(a) Which of the choices below reasonably continues this sequence of numbers?

A. $7, 12, 24, \ldots$

B. $7, 11, 16, \ldots$

C. $8, 16, 32, \ldots$

D. $1, 2, 4, \ldots$

E. $7, 12, 20, \ldots$

(b) Where possible, find a formula that allows us to move from one term to the next one.

Remark 8.1.3 As seen in the previous activity, having too few terms may prevent us from finding a unique way to continue creating a sequence of numbers. In fact, we need sufficiently many terms to uniquely continue a sequence of numbers (and how many terms is sufficient depends on which sequence of numbers you are trying to generate). Sometimes, we do not want to write out all of the terms needed to allow for this. Therefore, we will want to find short-hand notation that allows us to do so.

Definition 8.1.4 A **sequence** is a list of real numbers. Let a_n denote the nth term in a sequence. We will use the notation $\{a_n\}_{n=1}^{\infty} = a_1, a_2, \ldots, a_n, \ldots$. A general formula that indicates how to explicitly find the n-th term of a sequence is the **closed form** of the sequence. $\diamondsuit$

Sequence Formulas (SQ1)

Activity 8.1.5 Consider the sequence $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$. Which of the following choices gives a closed formula for this sequence? Select all that apply.

A. $\left\{\left(\frac{1}{3}\right)^{n-1}\right\}_{n=1}^{\infty}$

B. $\left\{\left(\frac{1}{3}\right)^{n}\right\}_{n=1}^{\infty}$

C. $\left\{\left(\frac{1}{3}\right)^{n-1}\right\}_{n=2}^{\infty}$

D. $\left\{\left(\frac{1}{3}\right)^{n+1}\right\}_{n=0}^{\infty}$

E. $\left\{\left(\frac{1}{3}\right)^{n}\right\}_{n=0}^{\infty}$

Sequence Formulas (SQ1)

Activity 8.1.6 Let a_n be the nth term in the sequence $\left\{\dfrac{n+1}{n}\right\}_{n=1}^{\infty}$. Which of the following terms corresponds to the 27^{th} term of this sequence?

A. $\dfrac{27}{26}$

B. $\dfrac{26}{27}$

C. $\dfrac{27}{28}$

D. $\dfrac{28}{27}$

E. $\dfrac{29}{28}$

Sequence Formulas (SQ1)

Activity 8.1.7 Let a_n be the nth term in the sequence $\left\{\frac{n+1}{n}\right\}_{n=2}^{\infty}$. Which of the following terms corresponds to the 27^{th} term of this sequence?

A. $\frac{27}{26}$

B. $\frac{26}{27}$

C. $\frac{27}{28}$

D. $\frac{28}{27}$

E. $\frac{29}{28}$

Activity 8.1.8 Let a_n be the nth term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Identify the 81st term of this sequence.

A. $\frac{1}{79}$

B. $\frac{1}{80}$

C. $\frac{1}{81}$

D. $\frac{1}{82}$

E. $\frac{1}{83}$

Sequence Formulas (SQ1)

Activity 8.1.9 Find a closed form for the sequence $0, 3, 8, 15, 24, \ldots$.

Activity 8.1.10 Find a closed form for the sequence $\frac{12}{1}, \frac{16}{2}, \frac{20}{3}, \frac{24}{4}, \frac{28}{5}, \ldots$.

Sequence Formulas (SQ1)

Activity 8.1.11 Let a_n be the nth term in the sequence $1, 1, 2, 3, 5, 8, \ldots$. Find a formula for a_n.

Sequence Formulas (SQ1)

Definition 8.1.12 A sequence is **recursive** if the terms are defined as a function of previous terms (with the necessary initial terms provided). $\diamond$

Sequence Formulas (SQ1)

Activity 8.1.13 Consider the sequence defined by $a_1 = 6$ and $a_{k+1} = 4a_k - 7$ for $k \geq 1$. What are the first four terms?

Activity 8.1.14 Consider the sequence $2, 7, 22, 67, 202, \ldots$. Which of the following offers the best recursive formula for this sequence?

A. $a_{n+1} = 3a_n + 1$

B. $a_1 = 2, a_k = 3a_{k-1} + 1$ for $k > 1$

C. $a_1 = 2, a_2 = 7, a_k = 3a_{k-1} + 1$ for $k > 2$

Sequence Formulas (SQ1)

Activity 8.1.15 Once more, consider the sequence $1, 1, 2, 3, 5, 8, \ldots$ from Activity 8.1.11. Suppose $a_1 = 1$ and $a_2 = 1$. Give a recursive formula for a_n for all $n \geq 3$.

Activity 8.1.16 Give a recursive formula that generates the sequence $1, 2, 4, 8, 16, 32, \ldots$.

Activity 8.1.17

(a) Find the first 5 terms of the following sequence:

- $a_n = 3 \cdot 2^n.$,

(b) Find a closed form for the following sequence:

- $4, 5, 8, 13, 20, \ldots,$

(c) Find a recursive form for the following sequence:

- $-3, 2, 7, 12, 17, \ldots,$

Activity 8.1.18

(a) Find the first 5 terms of the following sequence:

- $a_n = 5\,n + 4.$,

(b) Find a closed form for the following sequence:

- $0, 1, 4, 9, 16, \ldots$,

(c) Find a recursive form for the following sequence:

- $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$,

8.2 Sequence Properties and Limits (SQ2)

Learning Outcomes

- Determine if a sequence is convergent, divergent, monotonic, or bounded, and compute limits of convergent sequences.

Sequence Properties and Limits (SQ2)

Activity 8.2.1 We will consider the function $f(x) = \dfrac{4x+8}{x}$.

(a) Compute the limit $\displaystyle\lim_{x\to\infty} \frac{4x+8}{x}$.

A. 0

B. 8

C. 1

D. 4

(b) Determine on which intervals $f(x)$ is increasing and/or decreasing. (Hint: compute $f'(x)$ first.)

(c) Which statement best describes $f(x)$ for $x > 0$?

A. $f(x)$ is bounded above by 4

B. $f(x)$ is bounded below by 4

C. $f(x)$ is bounded above and below by 4

D. $f(x)$ is not bounded above

E. $f(x)$ is not bounded below

Definition 8.2.2 Given a sequence $\{x_n\}$:

- $\{x_n\}$ is *monotonically increasing* if $x_{n+1} > x_n$ for *every* choice of n.
- $\{x_n\}$ is *monotonically non-decreasing* if $x_{n+1} \geq x_n$ for *every* choice of n.
- $\{x_n\}$ is *monotonically decreasing* if $x_{n+1} < x_n$ for *every* choice of n.
- $\{x_n\}$ is *monotonically non-increasing* if $x_{n+1} \leq x_n$ for *every* choice of n.

All of these sequences would be *monotonic*. $\Diamond$

Activity 8.2.3 Consider the sequence $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

(a) Compute $x_{n+1} - x_n$.

(b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.

A. $x_{n+1} - x_n > 0$ for every choice of n.

B. $x_{n+1} - x_n \geq 0$ for every choice of n.

C. $x_{n+1} - x_n < 0$ for every choice of n.

D. $x_{n+1} - x_n \leq 0$ for every choice of n.

(c) Which of the following (if any) describe $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$?

A. Monotonically increasing.

B. Monotonically non-decreasing.

C. Monotonically decreasing.

D. Monotonically non-increasing.

Activity 8.2.4 Consider the sequence $\left\{\dfrac{n^2+1}{n}\right\}_{n=1}^{\infty}$.

(a) Compute $x_{n+1} - x_n$.

(b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.

A. $x_{n+1} - x_n > 0$ for every choice of n.

B. $x_{n+1} - x_n \geq 0$ for every choice of n.

C. $x_{n+1} - x_n < 0$ for every choice of n.

D. $x_{n+1} - x_n \leq 0$ for every choice of n.

(c) Which of the following (if any) describe $\left\{\dfrac{n^2+1}{n}\right\}_{n=1}^{\infty}$?

A. Monotonically increasing.

B. Monotonically non-decreasing.

C. Monotonically decreasing.

D. Monotonically non-increasing.

Activity 8.2.5 Consider the sequence $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$.

(a) Compute $x_{n+1} - x_n$.

(b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.

A. $x_{n+1} - x_n > 0$ for every choice of n.

B. $x_{n+1} - x_n \geq 0$ for every choice of n.

C. $x_{n+1} - x_n < 0$ for every choice of n.

D. $x_{n+1} - x_n \leq 0$ for every choice of n.

(c) Which of the following (if any) describe $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$?

A. Monotonically increasing.

B. Monotonically non-decreasing.

C. Monotonically decreasing.

D. Monotonically non-increasing.

Activity 8.2.6 Consider the sequence $\left\{ \dfrac{2}{3^n} \right\}_{n=0}^{\infty}$.

(a) Compute $x_{n+1} - x_n$.

(b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.

A. $x_{n+1} - x_n > 0$ for every choice of n.

B. $x_{n+1} - x_n \geq 0$ for every choice of n.

C. $x_{n+1} - x_n < 0$ for every choice of n.

D. $x_{n+1} - x_n \leq 0$ for every choice of n.

(c) Which of the following (if any) describe $\left\{ \dfrac{2}{3^n} \right\}_{n=0}^{\infty}$?

A. Monotonically increasing.

B. Monotonically non-decreasing.

C. Monotonically decreasing.

D. Monotonically non-increasing.

Definition 8.2.7 A sequence $\{x_n\}$ is *bounded* if there are real numbers b_u, b_ℓ such that

$$b_\ell \leq x_n \leq b_u$$

for every n. $\Diamond$

Activity 8.2.8 Consider the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ from Activity 8.2.3.

(a) Is there a b_u such that $x_n \leq b_u$ for every n? If so, what would be one such b_u?

(b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n? If so, what would be one such b_ℓ?

(c) Is $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ bounded?

Activity 8.2.9 Consider the sequence $\left\{\dfrac{n^2+1}{n}\right\}_{n=1}^{\infty}$ from Activity 8.2.4.

(a) Is there a b_u such that $x_n \leq b_u$ for every n? If so, what would be one such b_u?

(b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n? If so, what would be one such b_ℓ?

(c) Is $\left\{\dfrac{n^2+1}{n}\right\}_{n=1}^{\infty}$ bounded?

Activity 8.2.10 Consider the sequence $\left\{\dfrac{n+1}{n}\right\}_{n=1}^{\infty}$ from Activity 8.2.5.

(a) Is there a b_u such that $x_n \leq b_u$ for every n? If so, what would be one such b_u?

(b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n? If so, what would be one such b_ℓ?

(c) Is $\left\{\dfrac{n+1}{n}\right\}_{n=1}^{\infty}$ bounded?

Activity 8.2.11 Consider the sequence $\left\{\dfrac{2}{3^n}\right\}_{n=1}^{\infty}$ from Activity 8.2.6.

(a) Is there a b_u such that $x_n \leq b_u$ for every n? If so, what would be one such b_u?

(b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n? If so, what would be one such b_ℓ?

(c) Is $\left\{\dfrac{2}{3^n}\right\}_{n=1}^{\infty}$ bounded?

Definition 8.2.12 Given a sequence $\{x_n\}$, we say x_n has *limit* L, denoted

$$\lim_{n\to\infty} x_n = L$$

if we can make x_n as close to L as we like by making n sufficiently large. If such an L exists, we say $\{x_n\}$ *converges to* L. If no such L exists, we say $\{x_n\}$ *does not converge.* $\Diamond$

Activity 8.2.13

(a) For each of the following, determine if the sequence converges.

A. $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

B. $\left\{\dfrac{n^2+1}{n}\right\}_{n=1}^{\infty}$.

C. $\left\{\dfrac{n+1}{n}\right\}_{n=1}^{\infty}$.

D. $\left\{\dfrac{2}{3^n}\right\}_{n=0}^{\infty}$.

(b) Where possible, find the limit of the sequence.

Activity 8.2.14

(a) Determine to what value $\left\{\dfrac{4n}{n+1}\right\}_{n=0}^{\infty}$ converges.

(b) Which of the following ia most likely true about $\left\{\dfrac{4n(-1)^n}{n+1}\right\}_{n=0}^{\infty}$?

A. $\left\{\dfrac{4n(-1)^n}{n+1}\right\}_{n=0}^{\infty}$ converges to 4.

B. $\left\{\dfrac{4n(-1)^n}{n+1}\right\}_{n=0}^{\infty}$ converges to 0.

C. $\left\{\dfrac{4n(-1)^n}{n+1}\right\}_{n=0}^{\infty}$ converges to -4.

D. $\left\{\dfrac{4n(-1)^n}{n+1}\right\}_{n=0}^{\infty}$ does not converge.

Activity 8.2.15 For each of the following sequences, determine which of the properties: *monotonic*, *bounded* and *convergent*, the sequence satisfies. If a sequence is convergent, determine to what it converges. $\{3n\}_{n=0}^{\infty}$. $\left\{\frac{n^3}{3^n}\right\}_{n=0}^{\infty}$. $\left\{\frac{n}{n+3}\right\}_{n=1}^{\infty}$. $\left\{\frac{(-1)^n}{n+3}\right\}_{n=1}^{\infty}$.

Sequence Properties and Limits (SQ2)

Fact 8.2.16 *If a sequence is monotonic and bounded, then it is convergent.*

8.3 Partial Sum Sequence (SQ3)

Learning Outcomes

- Compute the first few terms of a telescoping or geometric partial sum sequence, and find a closed form for this sequence, and compute its limit.

Partial Sum Sequence (SQ3)

Activity 8.3.1 Consider the sequence $\{a_n\}_{n=0}^{\infty} = \left\{ \frac{1}{2^n} \right\}_{n=0}^{\infty}$.

(a) Find the first 5 terms of this sequence.

(b) Compute the following:

(a) a_0.

(b) $a_0 + a_1$.

(c) $a_0 + a_1 + a_2$.

(d) $a_0 + a_1 + a_2 + a_3$.

(e) $a_0 + a_1 + a_2 + a_3 + a_4$.

Partial Sum Sequence (SQ3)

Activity 8.3.2 Consider the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

(a) Find the first 5 terms of this sequence.

(b) Compute the following:

(a) a_1.

(b) $a_1 + a_2$.

(c) $a_1 + a_2 + a_3$.

(d) $a_1 + a_2 + a_3 + a_4$.

(e) $a_1 + a_2 + a_3 + a_4 + a_5$.

Partial Sum Sequence (SQ3)

Definition 8.3.3 Given a sequence $\{a_n\}_{n=0}^{\infty}$ define the $k^{\mathbf{th}}$ **partial sum** for this sequence to be

$$A_k = \sum_{i=0}^{k} a_i = a_0 + a_1 + a_2 + \cdots + a_k.$$

Note that $\{A_n\}_{n=0}^{\infty} = A_0, A_1, A_2, \ldots$ is itself a sequence called the *partial sum sequence.*

More generally, partial sums may be defined for any starting index. Given $\{a_n\}_{n=N}^{\infty}$, let

$$A_k = \sum_{i=N}^{k} a_i = a_N + a_{N+1} + a_{N+2} + \cdots + a_k.$$

$\Diamond$

Activity 8.3.4

(a) A_0

(b) A_1

(c) A_2

(d) A_3

(e) A_{100}

Partial Sum Sequence (SQ3)

Activity 8.3.5 Consider the sequence $a_n = \frac{2}{3^n}$. What is the best way to find the 100th partial sum A_{100}?

A. Sum the first 101 terms of the sequence $\{a_n\}$.

B. Find a closed form for the partial sum sequence $\{A_n\}$.

Activity 8.3.6 Expand the following polynomial products, and then reduce to as few summands as possible.

1. $(1-x)(1+x+x^2)$.
2. $(1-x)(1+x+x^2+x^3)$.
3. $(1-x)(1+x+x^2+x^3+x^4)$.
4. $(1-x)(1+x+x^2+\cdots+x^n)$, where n is any nonnegative integer.

Partial Sum Sequence (SQ3)

Activity 8.3.7 Suppose $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$. Without actually computing this sum, which of the following is equal to $\left(1 - \frac{1}{2}\right) S_5$?

A. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$.

B. $1 - \frac{1}{64}$.

C. $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32}$.

Partial Sum Sequence (SQ3)

Activity 8.3.8 Recall from Activity 8.3.4 that $A_{100} = 2 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \cdots + \frac{2}{3^{100}} = 2\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{100}}\right)$.

(a) Which of the following is equal to $\left(1 - \frac{1}{3}\right) A_{100}$?

A. $1 - \frac{1}{3^{101}}$.

B. $1 - \frac{1}{3^{100}}$.

C. $2\left(1 - \frac{1}{3^{101}}\right)$.

D. $2\left(1 - \frac{1}{3^{100}}\right)$.

(b) Based on your previous choice, write out an expression for A_{100}.

Partial Sum Sequence (SQ3)

Activity 8.3.9 Suppose that $\{b_n\}_{n=0}^{\infty} = \{(-2)^n\}_{n=0}^{\infty} = \{1, -2, 4, -8, \ldots\}$. Let $B_n = \sum_{i=0}^{n} b_i$ be the nth partial sum of $\{b_n\}$.

(a) Find simple expressions for the following:

(a) $(1-(-2))B_{10}$.

(b) $(1-(-2))B_{30}$.

(c) $(1-(-2))B_n$. Choose from the following:

A. $1+(-2)^n$.

B. $1-(-2)^n$.

C. $1+(-2)^{n+1}$.

D. $1-(-2)^{n+1}$.

E. $1-2^n$.

(b) Based on your previous answers, solve for the following:

(a) B_{10}.

(b) B_{30}.

(c) B_n. Choose from the following:

A. $\dfrac{1-(-2)^{n+1}}{1-(-2)}$

B. $\dfrac{1-(-2)^{n+1}}{1-2}$

C. $\dfrac{1-(-2)^{n+1}}{1+(-2)}$

D. $\dfrac{1-(-2)^{n}}{1-2}$

E. $\dfrac{1-(-2)^{n}}{1-(-2)}$

Partial Sum Sequence (SQ3)

Activity 8.3.10 Consider the following sequences:

1. $\{a_n\}_{n=0}^{\infty} = \left\{\left(-\frac{2}{3}\right)^n\right\}_{n=0}^{\infty}$.
2. $\{b_n\}_{n=0}^{\infty} = \{2 \cdot (-1)^n\}_{n=0}^{\infty}$.
3. $\{c_n\}_{n=0}^{\infty} = \{-3 \cdot (1.2)^n\}_{n=0}^{\infty}$.

(a) Find the closed form for the nth partial sum for the geometric sequence $A_n = \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} \left(-\frac{2}{3}\right)^n$.

A. $\frac{3}{5}\left(1 - \left(-\frac{2}{3}\right)^{n+1}\right)$.

B. $\frac{5}{3}\left(1 - \left(-\frac{2}{3}\right)^{n+1}\right)$.

C. $\frac{5}{3}\left(1 + \frac{2}{3}\left(\frac{2}{3}\right)^{n}\right)$.

D. $\frac{3}{5}\left(1 + \frac{2}{3}\left(\frac{2}{3}\right)^{n}\right)$.

E. $1 - \left(-\frac{2}{3}\right)^{n+1}$.

(b) Find the closed form for the nth partial sum for the geometric sequence $B_n = \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} 2 \cdot (-1)^n$.

A. 2^{n+1}.

B. $1 - (-1)^{n+1}$.

C. $1 + (-1)^n$.

D. $2(1 + (-1)^n)$.

E. $2(1 - (-1)^{n+1})$.

(c) Find the closed form for the nth partial sum for the geometric sequence $C_n = \sum_{i=0}^{n} c_i = \sum_{i=0}^{n} -3 \cdot (1.2)^n$.

Activity 8.3.11 Given the closed forms you found in Activity 8.3.10, which of the following limits are defined? If defined, what is the limit?

A. $\lim_{n\to\infty} A_n$.

B. $\lim_{n\to\infty} B_n$.

C. $\lim_{n\to\infty} C_n$.

Partial Sum Sequence (SQ3)

Definition 8.3.12 Given a sequence a_n, we define the limit of the series

$$\sum_{n=k}^{\infty} a_n := \lim_{n\to\infty} A_n$$

where $A_n = \sum_{i=k}^{n} a_i$. We call $\sum_{n=k}^{\infty} a_n$ an *infinite series.* $\Diamond$

Partial Sum Sequence (SQ3)

Activity 8.3.13 Which of the following series are infinite?

A. $\displaystyle\sum_{n=0}^{\infty} 3(0.8)^n$.

B. $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{5}{4}\right)^n$.

C. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$.

D. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2}(81)^n$.

E. $\displaystyle\sum_{n=0}^{\infty} 10\left(-\frac{1}{5}\right)^n$.

Partial Sum Sequence (SQ3)

Activity 8.3.14 Let $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n} - \frac{1}{n+1}\right\} = 1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \ldots$. Let $A_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$.

Which of the following is the best strategy for evaluating $A_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$?

A. Compute $A_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$, then evaluate the sum.

B. Rewrite $A_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) - \frac{1}{5}$, then simplify.

Activity 8.3.15 Recall from Activity 8.3.14 that $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n} - \frac{1}{n+1}\right\}$ and $A_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$.

Compute the following partial sums:

1. A_3.
2. A_{10}.
3. A_{100}.

Partial Sum Sequence (SQ3)

Activity 8.3.16 Recall from Activity 8.3.14 that $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n} - \frac{1}{n+1}\right\}$ and $A_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$.

Which of the following is equal to A_n?

A. $n - \frac{1}{n+1}$.

B. $1 - \frac{1}{n}$.

C. $1 - \frac{1}{n+1}$.

D. $1 - \frac{1}{i}$.

E. $1 - \frac{1}{i+1}$.

Partial Sum Sequence (SQ3)

Definition 8.3.17 Given a sequence $\{x_n\}_1^\infty$ and a sequence of the form $\{s_n\}_1^\infty := \{x_n - x_{n+1}\}_1^\infty$ we call the series $S_n = \sum_{i=1}^{n} s_i = \sum_{i=1}^{n}(x_i - x_{i+1})$ to be a *telescoping series.* $\Diamond$

Partial Sum Sequence (SQ3)

Activity 8.3.18 Given a telescoping series $S_n = \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} (x_i - x_{i+1})$, find:

1. S_2.
2. S_{10}.
3. Choose S_n from the following options:

A. $x_1 - x_n$

B. $x_1 - x_{n+1}$

C. $x_1 - x_{n-1}$

D. $x_1 - x_n + 1$

E. $x_1 - x_n - 1$

Activity 8.3.19 For each of the following telescoping series, find the closed form for the nth partial sum.

1. $S_n = \displaystyle\sum_{i=1}^{n}(2^{-i} - (2^{-i-1}))$.
2. $S_n = \displaystyle\sum_{i=1}^{n}(i^2 - (i+1)^2)$.
3. $S_n = \displaystyle\sum_{i=1}^{n}\left(\frac{1}{2i+1} - \frac{1}{2i+3}\right)$.

Activity 8.3.20 Given the closed forms you found in Activity 8.3.19, determine which of the following telescoping series converge. If so, to what value does it converge?

A. $\displaystyle\sum_{i=1}^{\infty}(2^{-i}-(2^{-i-1}))$.

B. $\displaystyle\sum_{i=1}^{\infty}(i^2-(i+1)^2)$.

C. $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{2i+1}-\frac{1}{2i+3}\right)$.

Partial Sum Sequence (SQ3)

Activity 8.3.21 Consider the partial sum sequence $A_n = (-2) + \left(\dfrac{2}{3}\right) + \left(-\dfrac{2}{9}\right) + \cdots + \left(-2 \cdot \left(-\dfrac{1}{3}\right)^n\right)$.

(a) Find a closed form for A_n.

(b) Does $\{A_n\}$ converge? If so, to what value?

Partial Sum Sequence (SQ3)

Activity 8.3.22 Consider the partial sum sequence $B_n = \sum_{i=1}^{n} \left(\frac{1}{5\,i+2} - \frac{1}{5\,i+7} \right)$.

(a) Find a closed form for B_n.

(b) Does $\{B_n\}$ converge? If so, to what value?

8.4 Geometric Series (SQ4)

Learning Outcomes

- Determine if a geometric series converges, and if so, the value it converges to.

Activity 8.4.1 Recall from Section 8.3 that for any real numbers a, r and $S_n = \sum_{i=0}^{n} ar^i$ that:

$$\begin{aligned} S_n = \sum_{i=0}^{n} ar^i &= a + ar + ar^2 + \cdots ar^n \\ (1-r)S_n = (1-r)\sum_{i=0}^{n} ar^i &= (1-r)(a + ar + ar^2 + \cdots ar^n) \\ (1-r)S_n = (1-r)\sum_{i=0}^{n} ar^i &= a - ar^{n+1} \\ S_n &= a\frac{1-r^{n+1}}{1-r}. \end{aligned}$$

(a) Using Definition 8.3.12, for which values of r does $\sum_{n=0}^{\infty} ar^n$ converges?

A. $|r| > 1$.

B. $|r| = 1$.

C. $|r| < 1$.

D. The series converges for every value of r.

(b) Where possible, determine what value $\sum_{n=0}^{\infty} ar^n$ converges to.

Fact 8.4.2 ***Geometric series*** *are sums of the form*

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots,$$

where a *and* r *are real numbers. When* $|r| < 1$ *this series converges to the value* $\dfrac{a}{1-r}$*. Otherwise, the geometric series diverges.*

Geometric Series (SQ4)

Activity 8.4.3 Consider the infinite series

$$5 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots .$$

(a) Complete the following rearrangement of terms.

$$\begin{aligned} 5 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots &= ? + \left(3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots\right) \\ &= ? + \sum_{n=0}^{\infty} ? \cdot \left(\frac{1}{?}\right)^n \end{aligned}$$

(b) Since $|\frac{1}{?}| < 1$, this series converges. Use the formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ to find the value of this series.

A. $\dfrac{7}{2}$

B. $\dfrac{13}{2}$

C. 8

D. 10

Geometric Series (SQ4)

Activity 8.4.4 Complete the following calculation, noting $|0.6| < 1$:

$$\sum_{n=2}^{\infty} 2(0.6)^n = \left(\sum_{n=0}^{\infty} 2(0.6)^n\right) - ? - ?$$
$$= \left(\frac{?}{1 - ?}\right) - ? - ?$$

What does this simplify to?

A. 1.1

B. 1.4

C. 1.8

D. 2.1

Observation 8.4.5 Given a series that appears to be mostly geometric such as

$$3 + (1.1)^3 + (1.1)^4 + \cdots (1.1)^n + \cdots$$

we can always rewrite it as the sum of a standard geometric series with some finite modification, in this case:

$$-0.31 + \sum_{n=0}^{\infty} (1.1)^n$$

Thus the original series converges if and only if $\displaystyle\sum_{n=0}^{\infty} (1.1)^n$ converges.

When the series diverges as in this example, then the reason why ($|1.1| \geq 1$) can be seen without any modification of the original series.

Geometric Series (SQ4)

Activity 8.4.6 For each of the following modified geometric series, determine without rewriting if they converge or diverge.

(a) $-7+\left(-\dfrac{3}{7}\right)^2+\left(-\dfrac{3}{7}\right)^3+\cdots.$

(b) $-6+\left(\frac{5}{4}\right)^3+\left(\frac{5}{4}\right)^4+\cdots.$

(c) $4+\displaystyle\sum_{n=4}^{\infty}\left(\frac{2}{3}\right)^n.$

(d) $8-1+1-1+1-1+\cdots.$

Geometric Series (SQ4)

Activity 8.4.7 Find the value of each of the following convergent series.

(a) $-1 + \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^n$.

(b) $-7 + \left(-\dfrac{3}{7}\right)^2 + \left(-\dfrac{3}{7}\right)^3 + \cdots$.

(c) $4 + \displaystyle\sum_{n=4}^{\infty} \left(\frac{2}{3}\right)^n$.

8.5 Basic Convergence Tests (SQ5)

Learning Outcomes

- Use the divergence, alternating series, and integral tests to determine if a series converges or diverges.

Basic Convergence Tests (SQ5)

Activity 8.5.1 Which of the following series seem(s) to diverge? It might be helpful to write out the first several terms.

A. $\displaystyle\sum_{n=0}^{\infty} n^2.$

B. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}.$

C. $\displaystyle\sum_{n=0}^{\infty} (-1)^n.$

D. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}.$

E. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}.$

Basic Convergence Tests (SQ5)

Fact 8.5.2 *If the series* $\sum a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Fact 8.5.3 The Divergence (n^{th} term) Test. *If the* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum a_n$ *diverges.*

Activity 8.5.4 Which of the series from Activity 8.5.1 diverge by Fact 8.5.3?

Basic Convergence Tests (SQ5)

Fact 8.5.5 *If* $a_n > 0$ *for all* n*, then* $\sum a_n$ *is convergent if and only if the sequence of partial sums is bounded from above.*

Activity 8.5.6 Consider the so-called *harmonic series*, $\sum_{n=1}^{\infty} \frac{1}{n}$, and let S_n be its n^{th} partial sum.

(a) Determine which of the following inequalities hold(s).

A. $\frac{1}{3} + \frac{1}{4} < \frac{1}{2}$.

B. $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$.

C. $S_4 \geq S_2 + \frac{1}{2}$.

D. $S_4 \leq S_2 + \frac{1}{2}$.

E. $S_4 = S_2 + \frac{1}{2}$.

(b) Determine which of the following inequalities hold(s).

A. $\frac{1}{2} < \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$.

B. $\frac{1}{2} > \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$.

C. $S_8 = S_4 + \frac{1}{2}$.

D. $S_8 \geq S_4 + \frac{1}{2}$.

E. $S_8 \leq S_4 + \frac{1}{2}$.

Activity 8.5.7 In Activity 8.5.6, we found that $S_4 \geq S_2 + \dfrac{1}{2}$ and $S_8 \geq S_4 + \dfrac{1}{2}$. Based on these inequalities, which statement seems most likely to hold?

A. The harmonic series converges.

B. The harmonic series diverges.

Basic Convergence Tests (SQ5)

Activity 8.5.8 Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(a) We want to compare this series to an improper integral. Which of the following is the best candidate?

A. $\displaystyle\int_1^{\infty} x^2\, dx$.

B. $\displaystyle\int_1^{\infty} \frac{1}{x^3}\, dx$.

C. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$.

D. $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx$.

E. $\displaystyle\int_1^{\infty} x\, dx$.

(b) Select the true statements below.

A. The sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using left Riemann sums where $\Delta x = 1$.

B. The sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using right Riemann sums where $\Delta x = 1$.

C. The sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using left Riemann sums where $\Delta x = 1$.

D. The sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using right Riemann sums where $\Delta x = 1$.

(c) Using the Riemann sum interpretation of the series, identify which of the following inequalities holds.

A. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx$.

B. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \geq \int_1^{\infty} \frac{1}{x^2}\, dx$.

C. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} \geq \int_1^{\infty} \frac{1}{x^2}\, dx$.

D. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx$.

(d) What can we say about the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$?

A. This improper integral converges.

B. This improper integral diverges.

(e) What do you think is true about the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$?

A. The series converges.

B. The series diverges.

Fact 8.5.9 The Integral Test. *Let $\{a_n\}$ be a sequence of positive numbers. If $f(x)$ is continuous, positive, and decreasing, and there is some positive integer N such that $f(n) = a_n$ for all $n \geq N$, then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.*

Activity 8.5.10 Consider the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

(a) Recall that the harmonic series diverges. What value of p corresponds to the harmonic series?

A. $p = -1$.

B. $p = 1$.

C. $p = -2$.

D. $p = 2$.

E. $p = 0$.

(b) From Fact 8.5.9, what can we conclude about the p-series with $p = 2$?

A. There is not enough information to draw a conclusion.

B. This series converges.

C. This series diverges.

Fact 8.5.11 The p-Test. *The series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges for* $p > 1$, *and diverges otherwise.*

Activity 8.5.12 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

(a) If we aim to use the integral test, what is an appropriate choice for $f(x)$?

A. $\frac{1}{x^2}$.

B. x^2+1.

C. $\frac{1}{x^2+1}$.

D. x^2.

E. $\frac{1}{x}$.

(b) Does the series converge or diverge by Fact 8.5.9?

Activity 8.5.13 Prove Fact 8.5.11.

Activity 8.5.14 Which of the following statements seem(s) most likely to be true?

A. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ diverges.

B. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges.

C. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ converges.

D. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ diverges.

Fact 8.5.15 The Alternating Series Test (Leibniz's Theorem). *The series* $\sum(-1)^{n+1}u_n$ *converges if all of the following conditions are satisfied:*

1. *u_n is always positive,*
2. *there is an integer N such that $u_n \geq u_{n+1}$ for all $n \geq N$, and*
3. $\lim_{n\to\infty} u_n = 0$.

Activity 8.5.16 What conclusions can you now make?

A. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ diverges.

B. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges.

C. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ converges.

D. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ diverges.

Activity 8.5.17 For each of the following series, use the *Divergence*, *Alternating Summation* or *Integral* test to determine if the series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2\left(n^2+2\right)}{n^2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3\,(-1)^n}{4\,n}$.

Fact 8.5.18 The Alternating Series Estimation Theorem. *If the alternating series $\sum a_n = \sum (-1)^{n+1} u_n$ converges to L and has n^{th} partial sum S_n, then for $n \geq N$ (as in the alternating series test):*

1. *$|L - S_n|$ is less than $|a_{n+1}|$, and*
2. *$(L - S_n)$ has the same sign as a_{n+1}.*

Activity 8.5.19 Consider the so-called *alternating harmonic series*, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

(a) Use the alternating series test to determine if the series converges.

(b) If so, estimate the series using the first 3 terms.

8.6 Comparison Tests (SQ6)

Learning Outcomes

- Use the direct comparison and limit comparison tests to determine if a series converges or diverges.

Comparison Tests (SQ6)

Activity 8.6.1 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, with infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$. Suppose $\{b_n\}_{n=1}^{\infty}$ is a sequence where each $b_n = 3a_n$, whith infinite series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 3a_n = 3a_1 + 3a_2 + \cdots$.

(a) If $\sum_{n=1}^{\infty} a_n = 5$ what can be said about $\sum_{n=1}^{\infty} b_n$?

A. $\sum_{n=1}^{\infty} b_n$ converges but the value cannot be determined.

B. $\sum_{n=1}^{\infty} b_n$ converges to $3 \cdot 5 = 15$.

C. $\sum_{n=1}^{\infty} b_n$ converges to some value other than 15.

D. $\sum_{n=1}^{\infty} b_n$ diverges.

E. It cannot be determined whether $\sum_{n=1}^{\infty} b_n$ converges or diverges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, what can be said about $\sum_{n=1}^{\infty} b_n$?

A. $\sum_{n=1}^{\infty} b_n$ converges but the value cannot be determined.

B. $\sum_{n=1}^{\infty} b_n$ converges and the value can be determined.

C. $\sum_{n=1}^{\infty} b_n$ diverges.

D. It cannot be determined whether $\sum_{n=1}^{\infty} b_n$ converges or diverges.

Comparison Tests (SQ6)

Activity 8.6.2 Using Fact 8.4.2, we know the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

(a) What can we say about the series

$$3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots?$$

A. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots$ converges to $3 \cdot 2 = 6$.

B. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots$ converges to some value other than 6.

C. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots$ diverges.

(b) What do you think we can say about the series

$$\frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3+(0.1)^n}{2^n} + \cdots?$$

A. $3 + \frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3+(0.1)^n}{2^n} + \cdots$ converges to $3 \cdot 2 = 6$.

B. $3 + \frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3+(0.1)^n}{2^n} + \cdots$ converges to some value other than 6.

C. $3 + \frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3+(0.1)^n}{2^n} + \cdots$ diverges.

Activity 8.6.3 From Fact 8.4.2, we know

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

diverges.

(a) What can we say about the series

$$5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots ?$$

A. $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots$ converges to a known value we can compute.

B. $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots$ converges to some unknown value.

C. $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots$ diverges.

(b) What do you think we can say about the series

$$4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots ?$$

A. $4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots$ converges to a known value we can compute.

B. $4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots$ converges to some unknown value.

C. $4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots$ diverges.

Fact 8.6.4 The Limit Comparison Test. *Let* $\sum a_n$ *and* $\sum b_n$ *be series with positive terms. If*

$$\lim_{n\to\infty} \frac{b_n}{a_n} = c$$

for some positive (finite) constant c*, then* $\sum a_n$ *and* $\sum b_n$ *either both converge or both diverge.*

Comparison Tests (SQ6)

Activity 8.6.5 Recall that

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges.

(a) Let $b_n = \frac{1}{n}$. Compute $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2^n}}$.

A. $-\infty$.

B. 0.

C. $\dfrac{1}{2}$.

D. 1.

E. ∞.

(b) Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

(c) Let $b_n = \dfrac{1}{n^2}$. Compute $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{2^n}}$.

A. ∞.

B. $\ln(2)$.

C. 1.

D. $\dfrac{1}{2}$.

E. $-\infty$.

(d) Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

(e) Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n\to\infty} \frac{b_n}{a_n}$$

diverges, can we conclude that $\sum b_n$ converges or diverges?

Activity 8.6.6 We wish to determine if $\sum_{n=1}^{\infty} \frac{1}{4^n - 1}$ converges or diverges using Fact 8.6.5.

(a) Compute

$$\lim_{n\to\infty} \frac{\frac{1}{4^n-1}}{\frac{1}{4^n}}.$$

(b) Does the geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converge or diverge by Fact 8.4.2?

(c) Does $\sum_{n=1}^{\infty} \frac{1}{4^n - 1}$ converge or diverge?

Comparison Tests (SQ6)

Activity 8.6.7 We wish to determine if $\displaystyle\sum_{n=2}^{\infty} \frac{2}{\sqrt{n+3}}$ converges or diverges using Fact 8.6.5.

(a) To which of the following should we compare $\{a_n\} = \left\{\dfrac{2}{\sqrt{n+3}}\right\}$?

A. $\left\{\dfrac{1}{n}\right\}$.

B. $\left\{\dfrac{1}{\sqrt{n}}\right\}$.

C. $\left\{\dfrac{1}{n^2}\right\}$.

D. $\left\{\dfrac{1}{2^n}\right\}$.

(b) Compute $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n}$.

(c) Compute $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$.

(d) What is true about $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n}$ and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$?

A. Their values are reciprocals.

B. Their values negative reciprocals.

C. They are both positive finite constants.

D. Only one value is a finite positive constant.

E. One value is 0 and the other value is infinite.

(e) Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge?

(f) Using your chosen sequence and Fact 8.6.5, does $\displaystyle\sum_{n=2}^{\infty} \frac{2}{\sqrt{n+3}}$ converge or diverge?

Comparison Tests (SQ6)

Activity 8.6.8 We wish to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+8n+5}$ converges or diverges using Fact 8.6.5.

(a) To which of the following should we compare $\{x_n\} = \left\{\dfrac{3}{n^2+8n+5}\right\}$?

A. $\left\{\dfrac{1}{n}\right\}$.

B. $\left\{\dfrac{1}{\sqrt{n}}\right\}$.

C. $\left\{\dfrac{1}{n^2}\right\}$.

D. $\left\{\dfrac{1}{2^n}\right\}$.

(b) Using your chosen sequence and Fact 8.6.5, does $\dfrac{3}{n^2+8n+5}$ converge or diverge?

Comparison Tests (SQ6)

Activity 8.6.9 Use Fact 8.6.5 to determine if the series $\displaystyle\sum_{n=5}^{\infty} \frac{2}{4^n}$ converges or diverges.

Activity 8.6.10 Consider sequences $\{a_n\}, \{b_n\}$ where $a_n \geq b_n \geq 0$.

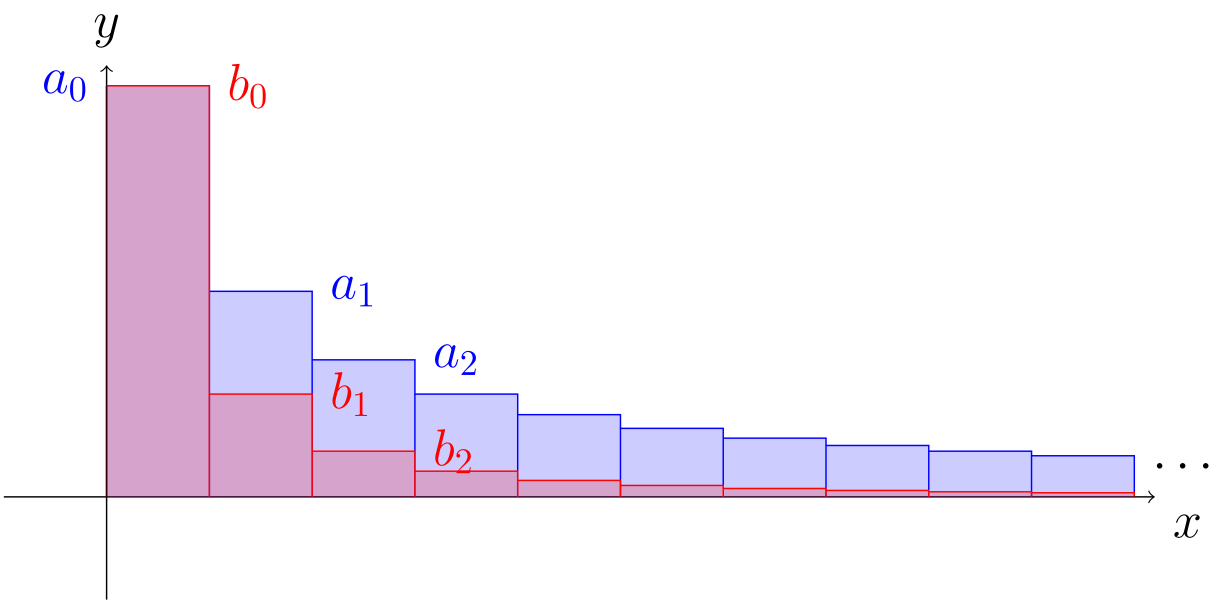


Figure 116 Plots of $\{a_n\}, \{b_n\}$

(a) Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n$ converges. What could be said about $\{b_n\}$?

A. $\displaystyle\sum_{n=0}^{\infty} b_n$ converges.

B. $\displaystyle\sum_{n=0}^{\infty} b_n$ diverges.

C. Whether or not $\displaystyle\sum_{n=0}^{\infty} b_n$ converges or diverges cannot be determined with this information.

(b) Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+1}$ which diverges. Which of the following statements are true?

A. $0 \leq \dfrac{1}{2n^2} \leq \dfrac{1}{n+1}$ for each $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2}$ is a convergent p-series where $p = 2$.

B. $0 \leq \dfrac{1}{2n} \leq \dfrac{1}{n+1}$ for each $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ is a divergent p-series where $p = 1$.

(c) Suppose that $\sum_{n=0}^{\infty} a_n$ was some series that diverges. What could be said about $\{b_n\}$?

A. $\sum_{n=0}^{\infty} b_n$ converges.

B. $\sum_{n=0}^{\infty} b_n$ diverges.

C. Whether or not $\sum_{n=0}^{\infty} b_n$ converges or diverges cannot be determined with this information.

(d) Suppose that $\sum_{n=0}^{\infty} b_n$ diverges. What could be said about $\{a_n\}$?

A. $\sum_{n=0}^{\infty} a_n$ converges.

B. $\sum_{n=0}^{\infty} a_n$ diverges.

C. Whether or not $\sum_{n=0}^{\infty} a_n$ converges or diverges cannot be determined with this information.

(e) Suppose that $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{3^n}$ which converges. Which of the following statements are true?

A. $0 \leq \frac{1}{3^n} \leq \frac{1}{2^n}$ for each n and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent geometric series where $|r| = \frac{1}{2} <$ 1.

B. $0 \leq \frac{1}{3^n} \leq 1$ for each n and $\sum_{n=0}^{\infty} 1$ diverges by the Divergence Test.

(f) Suppose that $\sum_{n=0}^{\infty} b_n$ was some series that converges. What could be said about $\{a_n\}$?

Comparison Tests (SQ6)

A. $\sum_{n=0}^{\infty} a_n$ converges.

B. $\sum_{n=0}^{\infty} a_n$ diverges.

C. Whether or not $\sum_{n=0}^{\infty} a_n$ converges or diverges cannot be determined with this information.

Comparison Tests (SQ6)

Fact 8.6.11 *Supppose we have sequences* $\{a_n\}, \{b_n\}$ *so that for some* k *we have that* $0 \leq b_n \leq a_n$ *for each* $k \geq n$*. Then we have the following results:*

- *If* $\displaystyle\sum_{k=n}^{\infty} a_n$ *converges, then so does* $\displaystyle\sum_{k=n}^{\infty} b_n$.
- *If* $\displaystyle\sum_{k=n}^{\infty} b_n$ *diverges, then so does* $\displaystyle\sum_{k=n}^{\infty} a_n$.

Activity 8.6.12 Suppose that you were handed positive sequences $\{a_n\}, \{b_n\}$. For the first few values $a_n \geq b_n$, but after that what happens is unclear until $n = 100$. Then for any $n \geq 100$ we have that $a_n \leq b_n$.

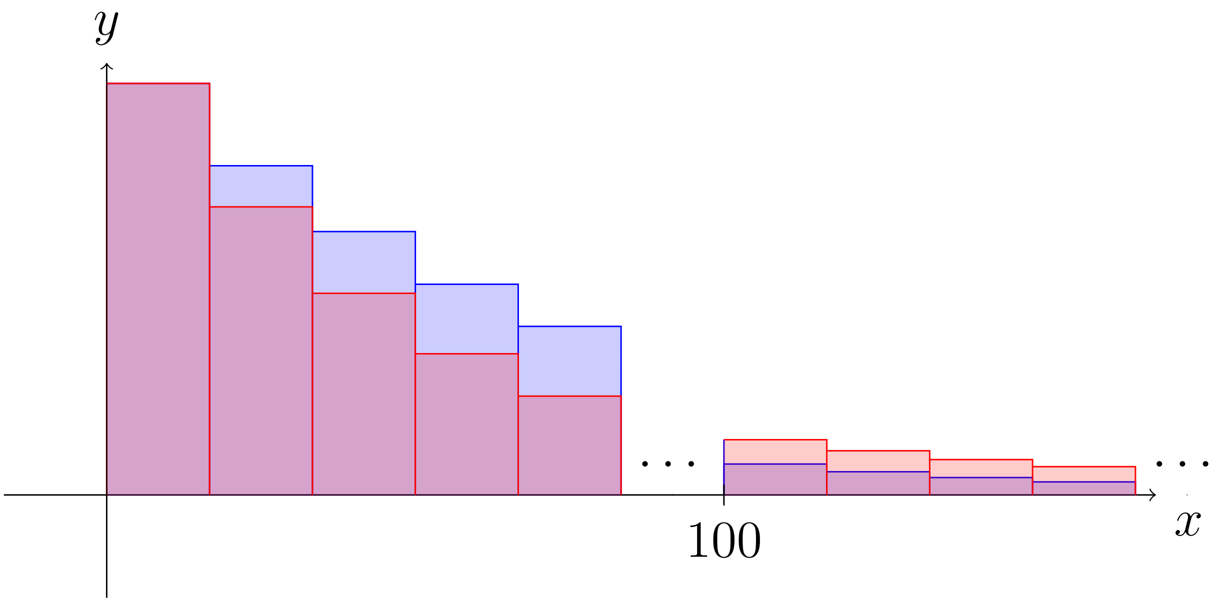


Figure 117 Plots of $\{a_n\}, \{b_n\}$

(a) How might we best utilize Fact 8.6.12 to determine the convergence of $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=0}^{\infty} b_n$?

A. Since a_n is sometimes greater than, and sometimes less than b_n, there is no way to utilize Fact 8.6.12.

B. Since initially, we have $b_n \leq a_n$, we can utilize Fact 8.6.12 by assuming $a_n \geq b_n$.

C. Since we can rewrite $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{99} a_n + \sum_{n=100}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{99} b_n + \sum_{n=100}^{\infty} b_n$ and $\sum_{n=0}^{99} a_n, \sum_{n=0}^{99} b_n$ are necessarily finite, we can compare $\sum_{n=100}^{\infty} a_n, \sum_{n=100}^{\infty} b_n$ with Fact 8.6.12.

Fact 8.6.13 The Direct Comparison Test. *Let* $\sum a_n$ *and* $\sum b_n$ *be series with positive terms. If there is a* k *such that* $b_n \leq a_n$ *for each* $n \geq k$*, then:*

- *If* $\sum a_n$ *converges, then so does* $\sum b_n$*.*
- *If* $\sum b_n$ *diverges, then so does* $\sum a_n$*.*

Comparison Tests (SQ6)

Activity 8.6.14 Suppose we wish to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converged using Fact 8.6.14.

(a) Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n}$ converge or diverge?

(b) For which value k is $\dfrac{1}{3n} \leq \dfrac{1}{2n+3}$ for each $n \geq k$?

A. $\dfrac{1}{3n} \leq \dfrac{1}{2n+3}$ for each $n \geq k = 0$.

B. $\dfrac{1}{3n} \leq \dfrac{1}{2n+3}$ for each $n \geq k = 1$.

C. $\dfrac{1}{3n} \leq \dfrac{1}{2n+3}$ for each $n \geq k = 2$.

D. $\dfrac{1}{3n} \leq \dfrac{1}{2n+3}$ for each $n \geq k = 3$.

E. There is no k for which $\dfrac{1}{3n} \leq \dfrac{1}{2n+3}$ for each $n \geq k$.

(c) Use Fact 8.6.14 and compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$ to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n}$ to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converges or diverges.

Comparison Tests (SQ6)

Activity 8.6.15 Suppose we wish to determine if $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ converged using Fact 8.6.14.

(a) Which series should we compare $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ to best utilize Fact 8.6.14?

A. $\sum_{n=1}^{\infty} \frac{1}{n}$.

B. $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

C. $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

D. $\sum_{n=1}^{\infty} \frac{1}{n+5}$.

E. $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$.

F. $\sum_{n=1}^{\infty} \frac{1}{2^n+5}$.

(b) Using your chosen series and Fact 8.6.14, does $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ converge or diverge?

Comparison Tests (SQ6)

Activity 8.6.16 For each of the following series, determine if it converges or diverges, and explain your choice.

(a) $$\sum_{n=4}^{\infty} \frac{3}{\log(n) + 2}.$$

(b) $$\sum_{n=3}^{\infty} \frac{1}{n^2 + 2\,n + 1}.$$

8.7 Ratio and Root Tests (SQ7)

Learning Outcomes

- Use the ratio and root tests to determine if a series converges or diverges.

Ratio and Root Tests (SQ7)

Activity 8.7.1 Consider the series $\sum_{n=0}^{\infty} \frac{2^n}{3^n - 2}$.

(a) Which of these series most closely resembles $\sum_{n=0}^{\infty} \frac{2^n}{3^n - 2}$?

A. $\sum_{n=0}^{\infty} \frac{2}{3}$.

B. $\sum_{n=0}^{\infty} \frac{2}{3} n$.

C. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$.

(b) Based on your previous choice, do we think this series is more likely to converge or diverge?

(c) Find $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = \lim_{n\to\infty} \frac{2^{n+1}(3^n - 2)}{(3^{n+1} - 2)2^n} = \lim_{n\to\infty} \frac{2 \cdot 2^n(3^n - 2)}{3(3^n - \frac{2}{3})2^n}$.

A. $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 0$.

B. $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = \frac{2}{3}$.

C. $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 1$.

D. $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 2$.

E. $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 3$.

Ratio and Root Tests (SQ7)

Activity 8.7.2 Consider the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{3}{2^n}$.

(a) Does $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{3}{2^n}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

A. 2.

B. $\frac{1}{2}$.

C. $\frac{2^n}{2^n+1}$.

D. $\frac{9}{2^{2n+1}}$.

E. $\frac{9}{2^{n+2}}$.

(c) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

A. $-\infty$.

B. 0.

C. $\frac{1}{2}$.

D. 2.

E. ∞.

Ratio and Root Tests (SQ7)

Activity 8.7.3 Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n+1}$.

(a) Does $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n+1}$ converge?

(b) Find $\dfrac{a_{n+1}}{a_n}$.

A. $\dfrac{n+1}{2}$.

B. $\dfrac{(n^2+1)(n+1)}{(n+2)n^2}$.

C. $\dfrac{(n+1)^2}{n+2}$.

D. $\dfrac{1}{2}$.

E. $\dfrac{(n+1)n^2}{n+2}$.

(c) Find $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

A. $-\infty$.

B. 0.

C. $\dfrac{1}{2}$.

D. 2.

E. ∞.

Ratio and Root Tests (SQ7)

Activity 8.7.4 Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

(a) Does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

(c) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Ratio and Root Tests (SQ7)

Activity 8.7.5 Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(a) Does $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

(c) Find $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Ratio and Root Tests (SQ7)

Fact 8.7.6 The Ratio Test. *Let* $\sum a_n$ *be a series and suppose that* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$. *Then*

1. $\sum a_n$ *converges if* ρ *is less than 1, and*
2. $\sum a_n$ *diverges if* ρ *is greater than 1.*
3. *If* $\rho = 1$, *we cannot determine if* $\sum a_n$ *converges or diverges* with this method.

Ratio and Root Tests (SQ7)

Fact 8.7.7 The Root Test. *Let N be an integer and let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$. Then*

1. *$\sum a_n$ converges if ρ is less than 1, and*

2. *$\sum a_n$ diverges if ρ is greater than 1.*

3. *If $\rho = 1$, we cannot determine if $\sum a_n$ converges or diverges* with this method.

Activity 8.7.8 Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n!}$.

(a) Which of the following is a_n?

A. n^2.

B. $n!$.

C. $\dfrac{n^2}{n!}$.

(b) Which of the following is a_{n+1}?

A. $\dfrac{n^2}{n!}$.

B. $(n+1)^2$.

C. $(n+1)!$.

D. $\dfrac{(n+1)^2}{(n+1)!}$.

E. $\dfrac{n^2+1}{n!+1}$.

(c) Which of the following is $\left|\dfrac{a_{n+1}}{a_n}\right|$?

A. $\dfrac{(n+1)^2 n^2}{(n+1)!n!}$.

B. $\dfrac{(n+1)^2 n!}{(n+1)!n^2}$.

C. $\dfrac{(n+1)!n!}{(n+1)^2 n^2}$.

D. $\dfrac{(n+1)!n^2}{(n+1)^2 n!}$.

(d) Using the fact $(n+1)! = (n+1) \cdot n!$, simplify $\left|\dfrac{a_{n+1}}{a_n}\right|$ as much as possible.

(e) Find $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

(f) Does $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n!}$ converge?

Activity 8.7.9

(a) What is a_n?

(b) Which of the following is $\sqrt[n]{|a_n|}$?

A. $\dfrac{n+1}{9}$.

B. $\dfrac{n}{9}$.

C. n.

D. 9.

E. $\dfrac{1}{9}$.

(c) Find $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$.

(d) Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{9^n}$ converge?

Ratio and Root Tests (SQ7)

Activity 8.7.10 For each series, use the *ratio* or *root* test to determine if the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)(n!)}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{4^n(n!)(n!)}{(2n)!}$

Ratio and Root Tests (SQ7)

Activity 8.7.11 Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$.

(a) Use the root test to check for convergence of this series.

(b) Use the ratio test to check for convergence of this series.

(c) Use the comparison (or limit comparison) test to check for convergence of this series.

(d) Find the sum of this series.

Ratio and Root Tests (SQ7)

Activity 8.7.12 Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$. Recall that $\displaystyle\sqrt[n]{\frac{n}{3^n}} = \left(\frac{n}{3^n}\right)^{1/n} = \frac{n^{1/n}}{(3^n)^{1/n}}$.

(a) Let $\displaystyle\alpha = \lim_{n\to\infty} \ln(n^{1/n}) = \lim_{n\to\infty} \frac{1}{n}\ln(n)$. Find α.

(b) Recall that $\displaystyle\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n^{1/n})} = e^{\alpha}$. Find $\displaystyle\lim_{n\to\infty} n^{1/n}$.

(c) Find $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{n}{3^n}} = \lim_{n\to\infty} \left(\frac{n}{3^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{(3^n)^{1/n}}$.

(d) Does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$ converge?

Activity 8.7.13 Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$.

(a) Use the root test to check for convergence of this series.

(b) Use the ratio test to check for convergence of this series.

(c) Use the comparison (or limit comparison) test to check for convergence of this series.

8.8 Absolute Convergence (SQ8)

Learning Outcomes

- Determine if a series converges absolutely or conditionally.

Absolute Convergence (SQ8)

Activity 8.8.1 Recall the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ from Activity 8.7.5.

(a) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge or diverge?

(b) Does the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ converge or diverge?

Absolute Convergence (SQ8)

Activity 8.8.2 Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

(a) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge or diverge?

(b) Does the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right|$ converge or diverge?

Absolute Convergence (SQ8)

Definition 8.8.3 Given a series

$$\sum a_n$$

we say that $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges. $\diamondsuit$

Absolute Convergence (SQ8)

Activity 8.8.4 Consider the series: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$.

(a) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$ converge or diverge? (Recall Fact 8.7.6.)

(b) Compute $|a_n|$.

(c) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$ converge absolutely?

Fact 8.8.5 *Notice that Fact 8.7.6 and Fact 8.7.7 both involve taking absolute values to determine convergence. As such, series that are convergent by either the Ratio Test or the Root Test are also absolutely convergent (by applying the same test after taking the absolute value).*

Absolute Convergence (SQ8)

Activity 8.8.6 Consider the series: $\sum_{n=1}^{\infty} -n$.

(a) Does the series $\sum_{n=1}^{\infty} -n$ converge or diverge?

(b) Compute $|a_n|$.

(c) Does the series $\sum_{n=1}^{\infty} -n$ converge absolutely?

Absolute Convergence (SQ8)

Activity 8.8.7 For each of the following series, determine if the series is *convergent*, and if the series is *absolutely convergent*.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(-1)^n}{n^3+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$

Activity 8.8.8 If you know a series $\sum a_n$ is absolutely convergent, what can you conclude about whether or not $\sum a_n$ is convergent?

A. We cannot determine if $\sum a_n$ is convergent.

B. $\sum a_n$ is convergent since it "grows slower" than $\sum |a_n|$ (and falls slower than $\sum -|a_n|$).

Absolute Convergence (SQ8)

Fact 8.8.9 *If* $\sum a_n$ *is absolutely convergent, then it must be convergent.*

Activity 8.8.10 Find 3 series that are convergent but not absolutely convergent (recall Fact 8.5.15, Section 8.6).

8.9 Series Convergence Strategy (SQ9)

Learning Outcomes

- Identify appropriate convergence tests for various series.

Series Convergence Strategy (SQ9)

Activity 8.9.1 Which test for convergence is the best first test to apply to any series $\displaystyle\sum_{k=1}^{\infty} a_k$?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

Activity 8.9.2 In which of the following scenarios can we successfully apply the Direct Comparison Test to determine the convergence of the series $\sum a_k$?

A. When we find a convergent series $\sum b_k$ where $0 \leq a_k \leq b_k$

B. When we find a divergent series $\sum b_k$ where $0 \leq a_k \leq b_k$

C. When we find a convergent series $\sum b_k$ where $0 \leq b_k \leq a_k$

D. When we find a divergent series $\sum b_k$ where $0 \leq b_k \leq a_k$

Activity 8.9.3 Which test(s) for convergence would we use for a series $\sum a_k$ where a_k involves k^{th} powers?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

Activity 8.9.4 Which test(s) for convergence would we use for a series of the form $\sum ar^k$?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

Activity 8.9.5 Which test(s) for convergence would we use for a series $\sum a_k$ where a_k involves factorials and powers?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

Activity 8.9.6 Which test(s) for convergence would we use for a series $\sum a_k$ where a_k is a rational function?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

Activity 8.9.7 Which test(s) for convergence would we use for a series of the form $\sum(-1)^k a_k$?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

Series Convergence Strategy (SQ9)

Fact 8.9.8 *Here is a strategy checklist when dealing with series:*

1. *The divergence test: unless $a_n \to 0$, $\sum a_n$ diverges*

2. *Geometric Series: $\sum ar^k$ converges if $-1 < r < 1$ and diverges otherwise*

3. *p-series: $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise*

4. *Series with no negative terms: try the ratio test, root test, integral test, or try to compare to a known series with the comparison test or limit comparison test*

5. *Series with some negative terms: check for absolute convergence*

6. *Alternating series: use the alternating series test (Leibniz's Theorem)*

7. *Anything else: consider the sequence of partial sums, possibly rewriting the series in a different form, hope for the best*

Activity 8.9.9 Consider the series $\displaystyle\sum_{k=3}^{\infty} \frac{2}{\sqrt{k-2}}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test
B. Geometric Series
C. Integral Test
D. Direct Comparison Test
E. Limit Comparison Test
F. Ratio Test
G. Root Test
H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.
B. This series is divergent.

Series Convergence Strategy (SQ9)

Activity 8.9.10 Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+2k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Series Convergence Strategy (SQ9)

Activity 8.9.11 Consider the series $\sum_{k=0}^{\infty} \frac{2k^2+1}{k^3+k+1}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test
B. Geometric Series
C. Integral Test
D. Direct Comparison Test
E. Limit Comparison Test
F. Ratio Test
G. Root Test
H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.
B. This series is divergent.

Activity 8.9.12 Consider the series $\displaystyle\sum_{k=0}^{\infty} \frac{100^k}{k!}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Activity 8.9.13 Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{5^k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Activity 8.9.14 Consider the series $\sum_{k=1}^{\infty} \frac{k^3 - 1}{k^5 + 1}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Series Convergence Strategy (SQ9)

Activity 8.9.15 Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{3^{k-1}}{7^k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Activity 8.9.16 Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Activity 8.9.17 Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Activity 8.9.18 Consider the series $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A. Divergence Test

B. Geometric Series

C. Integral Test

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

G. Root Test

H. Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A. This series is convergent.

B. This series is divergent.

Series Convergence Strategy (SQ9)

Activity 8.9.19 Determine which of the following series is *convergent* and which is *divergent*. Justify both choices with an appropriate test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4\,(-1)^{n+1}\,n^2}{2\,n^3 + 4\,n^2 + 5}.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3 \cdot 3^n n^4}.$

Chapter 9

Power Series (PS)

Learning Outcomes

How do we use series to understand functions?
By the end of this chapter, you should be able to...

1. Approximate functions defined as power series.
2. Determine the interval of convergence for a given power series.
3. Compute power series by manipulating known exponential/ trigonometric/binomial power series.
4. Determine a Taylor or Maclaurin series for a function.

9.1 Power Series (PS1)

Learning Outcomes

- Approximate functions defined as power series.

Power Series (PS1)

Activity 9.1.1 Suppose we could define a function as an "infinite-length polynomial":

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

(a) Would $f(1)$ be well-defined as a finite real number?

A. No, the sum would diverge towards ∞.

B. No, the sum would oscillate between 0 and 1.

C. Yes, the sum would be 0.

D. Yes, the sum would be 1.

(b) Would $f(-1)$ be well-defined as a finite real number?

A. No, the sum would diverge towards ∞.

B. No, the sum would oscillate between 0 and 1.

C. Yes, the sum would be 0.

D. Yes, the sum would be 1.

(c) Would $f(1/2)$ be well-defined as a finite real number?

A. No, the sum would diverge towards ∞.

B. Yes, the sum would be approximately 1.

C. Yes, the sum would be approximately 2.

D. Yes, the sum would be exactly 2.

(d) When is $f(x)$ well-defined as a finite real number?

A. Its value is $\frac{x}{1-x}$ when $|x| < 1$.

B. Its value is $\frac{x}{1-x}$ when $x < 1$.

C. Its value is $\frac{1}{1-x}$ when $|x| < 1$.

D. Its value is $\frac{1}{1-x}$ when $x < 1$.

Definition 9.1.2 Given a sequence of numbers a_n and a number c, we may define a function $f(x)$ as a **power series**:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots.$$

The above power series is said to be **centered at** c. Often power series are centered at 0; in this case, they may be written as:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

The domain of this function (often referred to as the **domain of convergence** or **interval of convergence**) is exactly the set of x-values for which the series converges. $\Diamond$

Activity 9.1.3 In Section 9.2 we will learn how to prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for each real value x. Thus the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

has the domain of all real numbers.

(a) To estimate $f(2)$, use technology to compute the first few terms as follows:

$$\begin{aligned} f(2) = \sum_{n=0}^{\infty} \frac{2^n}{n!} &= 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \frac{2^4}{24} + \frac{2^5}{120} + \cdots \\ &= ? + \cdots \\ &\approx ? \end{aligned}$$

Which of these choices is the closest to this value?

A. $\sqrt{2} \approx 1.414$.

B. $e^2 \approx 7.389$.

C. $\sin(2) \approx 0.909$.

D. $\cos(2) \approx -0.416$.

(b) Estimate $f(-1)$ in a similar fashion:

$$\begin{aligned} f(-1) = \sum_{n=0}^{\infty} \frac{?}{n!} &= ? + ? + ? + ? + ? + ? + \cdots \\ &= ? + \cdots \\ &\approx ? \end{aligned}$$

Which of these choices is the closest to this value?

A. $\frac{1}{\sqrt{1}} \approx 1.000$.

B. $\frac{1}{e^1} \approx 0.369$.

C. $\frac{1}{\sin(1)} \approx 1.188$.

D. $\frac{1}{\cos(1)} \approx 1.851$.

Activity 9.1.4 The function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n$$

is centered at 0. Likewise, graphing the polynomial that uses the first six terms

$$f_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

alongside the graph of e^x reveals the illustration given in the following figure.

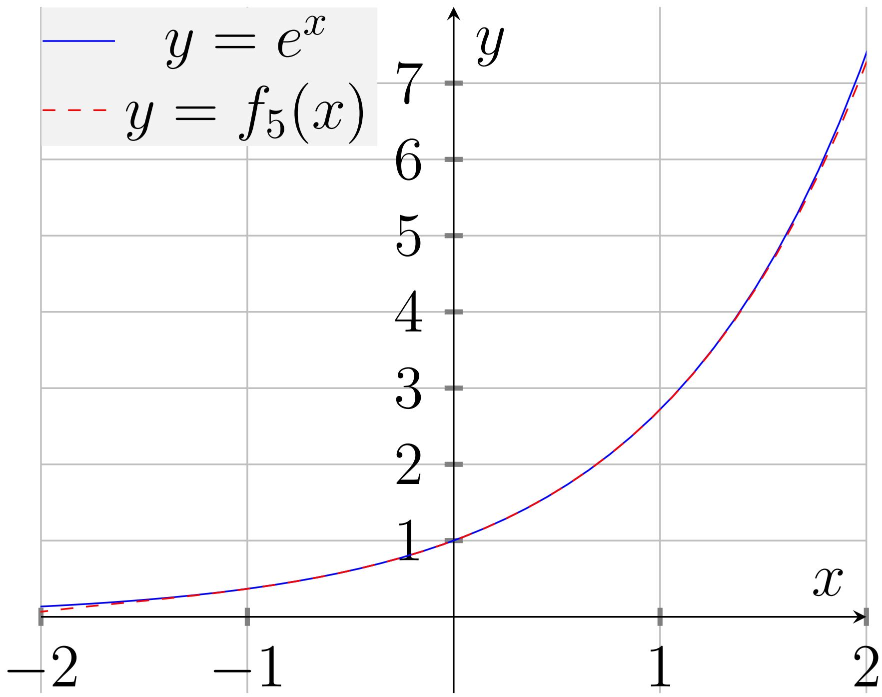


Figure 118 Plots of $y = f_5(x), y = e^x$.

What might we conclude?

A. $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$ near $x = 0$.

B. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$ near $x = 0$.

C. $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$ for all x.

D. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$ for all x.

Definition 9.1.5 Given a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots,$$

let

$$f_N(x) = \sum_{n=0}^{N} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_N(x-c)^N$$

be its **degree** N **polynomial approximation** for x nearby c.

For example,

$$\begin{aligned} g_3(x) = \sum_{n=0}^{3} n^2(x-1)^n &= 0 + (x-1) + 4(x-1)^2 + 9(x-1)^3 \\ &= -6 + 20x - 23x^3 + 9x^3 \end{aligned}$$

is a degree 3 approximation of $g(x) = \sum_{n=0}^{\infty} n^2(x-1)^n$ valid for x values nearby 1. $\Diamond$

Activity 9.1.6 Consider a function $p(x)$ defined by $p(x) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^n$.

(a) Find $p_3(x)$, the degree 3 polynomial approximation for $p(x)$.

(b) Use $p_3(x)$ to estimate $p(-1)$.

9.2 Convergence of Power Series (PS2)

Learning Outcomes

- Determine the interval of convergence for a given power series.

Activity 9.2.1 Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ where x is a real number.

(a) If $x = 2$, then $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}$. What can be said about this series?

A. The techniques we have learned so far allow us to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges.

B. The techniques we have learned so far allow us to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}$ diverges.

C. None of the techniques we have learned so far allow us to conclude whether $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges or diverges.

(b) If $x = -100$, then $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!}$. What can be said about this series?

A. The techniques we have learned so far allow us to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!}$ converges.

B. The techniques we have learned so far allow us to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!}$ diverges.

C. None of the techniques we have learned so far allow us to conclude whether $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!}$ converges or diverges.

(c) Suppose that x were some arbitrary real number. What can be said about this series?

A. The techniques we have learned so far allow us to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges.

B. The techniques we have learned so far allow us to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ diverges.

C. None of the techniques we have learned so far allow us to conclude whether $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges or diverges.

Remark 9.2.2 Consider a power series $\sum c_n(x-a)^n$. Recall from Fact 8.7.6 that if

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| < 1$$

then $\sum c_n(x-a)^n$ converges.

Then recall:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| &= \lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)}{c_n}\right| \\ &= \lim_{n\to\infty}|x-a|\left|\frac{c_{n+1}}{c_n}\right| \\ &= |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|.\end{aligned}$$

Convergence of Power Series (PS2)

Activity 9.2.3 Consider $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n$.

(a) Letting $c_n = \frac{1}{n^2+1}$, find $\displaystyle\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$.

(b) For what values of x is $|x| \displaystyle\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$?

A. $x < 1$.

B. $0 \le x < 1$.

C. $-1 < x < 1$.

(c) If $x = 1$, does $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n$ converge?

(d) If $x = -1$, does $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n$ converge?

(e) Which of the following describe the values of x for which $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n$ converges?

A. $(-1, 1)$.

B. $[-1, 1)$.

C. $(-1, 1]$.

D. $[-1, 1]$.

Activity 9.2.4 Consider $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{5^n}(x-2)^n$.

(a) Letting $c_n = \frac{2^n}{5^n}$, find $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$.

(b) For what values of x is $\displaystyle|x-2|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1$?

A. $-\frac{2}{5} < x < \frac{2}{5}$.

B. $\frac{8}{5} < x < \frac{12}{5}$.

C. $-\frac{5}{2} < x < \frac{5}{2}$.

D. $-\frac{1}{2} < x < \frac{9}{2}$.

(c) If $x = \frac{9}{2}$, does $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{5^n}(x-2)^n$ converge?

(d) If $x = -\frac{1}{2}$, does $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{5^n}(x-2)^n$ converge?

(e) Which of the following describe the values of x for which $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{5^n}(x-2)^n$ converges?

A. $(-\frac{1}{2}, \frac{9}{2})$.

B. $[-\frac{1}{2}, \frac{9}{2})$.

C. $(-\frac{1}{2}, \frac{9}{2}]$.

D. $[-\frac{1}{2}, \frac{9}{2}]$.

Convergence of Power Series (PS2)

Activity 9.2.5 Consider $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n!}\left(x+\frac{1}{2}\right)^n$.

(a) Letting $c_n = \frac{n^2}{n!}$, find $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$.

(b) For what values of x is $\displaystyle\left|x+\frac{1}{2}\right| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1$?

A. $0 \leq x < \infty$.

B. All real numbers.

(c) What describes the values of x for which $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n!}\left(x+\frac{1}{2}\right)^n$ converges?

Fact 9.2.6 *Given the power series* $\sum c_n(x-a)^n$, *the* ***center of convergence*** *is* $x = a$. *The* ***radius of convergence*** *is*

$$r = \frac{1}{\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|}.$$

If $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = 0$, *we say that* $r = \infty$.

The ***interval of convergence*** *represents all possible values of* x *for which* $\sum c_n(x-a)^n$ *converges, which is of the form:*

- $(a-r, a+r)$
- $[a-r, a+r)$
- $(a-r, a+r]$
- $[a-r, a+r]$

Depending on if $\sum c_n(x-a)^n$ *converges when* $x = a-r$ *or* $x = a+r$.

If $r = \infty$ *the interval of convergence is all real numbers.*

Convergence of Power Series (PS2)

Activity 9.2.7 Find the center of convergence, radius of convergence, and interval of convergence for the series:

$$\sum_{n=0}^{\infty} \frac{3^n \left(-1\right)^n \left(x-1\right)^n}{n!}.$$

Convergence of Power Series (PS2)

Activity 9.2.8 Find the center of convergence, radius of convergence, and interval of convergence for the series:

$$\sum_{n=0}^{\infty} \frac{3^n (x+2)^n}{n}.$$

Convergence of Power Series (PS2)

Activity 9.2.9 Consider the power series $\sum_{n=0}^{\infty} \frac{2^n + 1}{n3^n} (x+1)^n$.

(a) What is the center of convergence for this power series?

(b) What is the radius of convergence for this power series?

(c) What is the interval of convergence for this power series?

(d) If $x = -0.5$, does this series converge? (Use the interval of convergence.)

(e) If $x = 1$, does this series converge? (Use the interval of convergence.)

9.3 Manipulation of Power Series (PS3)

Learning Outcomes

- Compute power series by manipulating known exponential/ trigonometric/binomial power series.

Manipulation of Power Series (PS3)

Activity 9.3.1 How might we use the known geometric power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

to find the value of

$$? = \sum_{n=0}^{\infty} nx^{n-1} = 0 + 1 + 2x + 3x^2 + 4x^3 + \dots ?$$

(a) Which operation describes the relationship between these two series?

A. Bifurcation

B. Composition

C. Differentiation

D. Multiplication

(b) What is the result of applying this operation to $\frac{1}{1-x}$?

A. 0

B. $\dfrac{1}{(1-x)^2}$

C. $1 - \dfrac{1}{x}$

D. $\dfrac{x}{1-x^2}$

Manipulation of Power Series (PS3)

Fact 9.3.2 *Whenever a function is defined as a power series:*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

then its derivative and general antiderivative are also defined as power series with the same domain of convergence as $f(x)$, found by differentiating or integrating term-by-term:

$$\begin{aligned}
\frac{d}{dx}[f(x)] &= \sum_{n=0}^{\infty} \frac{d}{dx}\left[a_n (x-c)^n\right] \\
&= \sum_{n=0}^{\infty} n a_n (x-c)^{n-1} \\
\int f(x)\,dx &= C + \sum_{n=0}^{\infty} \left[\int a_n (x-c)^n \,dx\right] \\
&= C + \sum_{n=0}^{\infty} \frac{(x-c)^{n+1}}{n+1}
\end{aligned}$$

Manipulation of Power Series (PS3)

Activity 9.3.3 Let's investigate the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots.$$

(a) What is the value of $\exp(0)$?

A. 0.

B. 1.

C. 2.

D. ∞.

(b) What is the value of $\exp'(x)$?

A. $0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$.

B. $1 + x + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \dots$.

C. $0 + 1 + x + \frac{x^2}{3} + \frac{x^3}{12} + \frac{x^4}{60} + \dots$.

D. $1 + x + \frac{x^2}{3} + \frac{x^3}{12} + \frac{x^4}{60} + \dots$.

(c) What can we conclude from our calculation of $f'(x)$?

A. $\exp'(x) = [\exp(x)]^2$.

B. $\exp'(x) = \exp(x^2)$.

C. $\exp'(x) = 2\exp(x)$.

D. $\exp'(x) = \exp(x)$.

(d) What function do we know of that shares each of these properites?

A. $\exp(x) = \dfrac{1}{1+x}$

B. $\exp(x) = \cos(x)$

C. $\exp(x) = e^x$

D. $\exp(x) = 0$

Fact 9.3.4 *We have that*

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

That is, for any real number x, the series $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ *will converge to* e^x*.*

Fact 9.3.5 *We may similarly determine that*

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

for all real numbers x. However, we will delay until Fact 9.4.6 to prove this fact another way.

Manipulation of Power Series (PS3)

Activity 9.3.6 Suppose we wish to find the power series for the function $f(x) = e^{2x}$ by modifying the power series $\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

(a) Substituting $z = 2x$, what is the power series for $\exp(2x)$?

A. $\exp(2x) = \sum_{n=0}^{\infty} \frac{2x^n}{n!} = 2 + 2x + x^2 + \frac{1}{3}x^3 + \ldots$.

B. $\exp(2x) = \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!} = 2x + 2x^2 + x^3 + \frac{1}{3}x^4 + \ldots$.

C. $\exp(2x) = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$.

D. $\exp(2x) = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!} = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{720} + \ldots$.

(b) What is the interval of convergence for x for this series?

A. $(-\infty, \infty)$.

B. $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

C. $\left(0, \frac{1}{2}\right)$.

D. $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

Manipulation of Power Series (PS3)

Fact 9.3.7 *If a power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

is known, then for any polynomial $g(x)$ the composition $f \circ g$ has a power series given by

$$(f \circ g)(x) = f(g(x)) = \sum_{n=0}^{\infty} a_n (g(x) - c)^n$$

where the domain of convergence is transformed based upon the transformation given by $g(x)$.

For example, if $f(x)$ has the domain of convergence $-2 \leq x < 2$, then $f(2x+4)$ has the domain of convergence:

$$-2 \leq 2x + 4 < 2$$

$$-6 \leq 2x < -2$$

$$-3 \leq x < -1$$

Manipulation of Power Series (PS3)

Activity 9.3.8 Suppose we wish to find the power series for the function $f(x) = \frac{1}{x}$.

(a) Which of the following represents the power series for $g(r) = \frac{1}{1-r}$?

A. $g(r) = \sum_{n=0}^{\infty} rx^n$.

B. $g(r) = \sum_{n=0}^{\infty} (rx)^n$.

C. $g(r) = \sum_{n=0}^{\infty} r^n$.

D. $g(r) = \sum_{r=0}^{\infty} x^r$.

(b) For what value of r is $\frac{1}{1-r} = \frac{1}{x}$?

A. $r = x - 1$.

B. $r = 1 - x$.

C. $r = x + 1$.

D. $r = -x$.

(c) Substituting r with this value, which of the following is a power series for $f(x) = \frac{1}{x}$?

A. $f(x) = \sum_{n=0}^{\infty} (-x)^n$.

B. $f(x) = \sum_{n=0}^{\infty} (1-x)^n$.

C. $f(x) = \sum_{n=0}^{\infty} (x-1)^n$.

D. $f(x) = \sum_{n=0}^{\infty} (1+x)^n$.

(d) Given that the domain of convergence for r in $f(r)$ is $-1 < r < 1$, what should be the domain of convergence for x in $f(x)$?

A. $-1 < x < 1$.

B. $-2 < x < 0$.

C. $-2 < x < 2$.

D. $0 < x < 2$.

Manipulation of Power Series (PS3)

Activity 9.3.9 Suppose we wish to find the power series for the function $f(x) = \frac{1}{3-2x}$. Recall that $g(x) = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.

(a) For what value of r is $\frac{1}{1-r} = \frac{1}{3-2x}$?

A. $r = 2x - 2$.

B. $r = 2 - 2x$.

C. $r = 2x - 3$.

D. $r = 3 - 2x$.

(b) Evaluating r at the previously found value, which of the following is the power series of $f(x) = \frac{1}{3-2x}$?

A. $f(x) = \sum_{n=0}^{\infty} (3 - 2x)^n$.

B. $f(x) = \sum_{n=0}^{\infty} (2x - 3)^n$.

C. $f(x) = \sum_{n=0}^{\infty} (2 - 2x)^n$.

D. $f(x) = \sum_{n=0}^{\infty} (2x - 2)^n$.

(c) Given that the interval of convergence for r is $-1 < r < 1$, what is the interval of convergence for x?

A. $-1 < x < \frac{3}{2}$.

B. $-\frac{1}{2} < x < 1$.

C. $\frac{1}{2} < x < \frac{3}{2}$.

D. $-\frac{1}{2} < x < \frac{3}{2}$.

Manipulation of Power Series (PS3)

Activity 9.3.10 Suppose we wish to find the power series for the function $f(x) = \frac{1}{1+x^2}$. Recall that $g(x) = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.

(a) For what value of r is $\frac{1}{1-r} = \frac{1}{1+x^2}$?

A. $r = x^2$.

B. $r = -x^2$.

C. $r = 1 - x^2$.

D. $r = x^2 - 1$.

(b) Evaluating r at the previously found value, which of the following is the power series of $f(x) = \frac{1}{1+x^2}$?

A. $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

B. $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (1 - x^2)^n$.

C. $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} x^{2n}$.

D. $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (x^2 - 1)^n$.

(c) Given that the interval of convergence for r is $-1 < r < 1$, what is the interval of convergence for x?

A. $-1 < x < 1$.

B. $-1 < x < 0$.

C. $0 < x < 1$.

D. $0 < x < 4$.

(d) How can the power series for $\frac{1}{1+x^2}$ be manipulated to obtain a power series for $\arctan(x)$?

A. Differentiate each term.

B. Integrate each term.

C. Replace x with x^2 in each term.

D. Replace x with $1/x$ in each term.

(e) Which of these power series is the result of this manipulation?

Manipulation of Power Series (PS3)

A. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

B. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{2n-1}$.

C. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n (2n) x^{2n-1}$.

D. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{2n}$.

Activity 9.3.11 What function $f(x)$ has power series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$?

A. $f(x) = (-1)^n e^x$.

B. $f(x) = -e^x$.

C. $f(x) = e^{-x}$.

D. $f(x) = -e^{-x}$.

Manipulation of Power Series (PS3)

Activity 9.3.12 What function $f(x)$ has power series $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!} = x^3 + x^4 + \frac{x^5}{2} + \frac{x^6}{6} + \cdots$?

A. $f(x) = e^{x+3}$.

B. $f(x) = e^{x^3}$.

C. $f(x) = e^{3x}$.

D. $f(x) = x^3 e^x$.

Manipulation of Power Series (PS3)

Fact 9.3.13 *If a power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

is known, then for any polynomial $g(x)$ *the product* fg *has a power series given by*

$$(fg)(x) = f(x)g(x) = \sum_{n=0}^{\infty} a_n g(x)(x - c)^n$$

where the domain of convergence is the same as $f(x)$*.*

Manipulation of Power Series (PS3)

Activity 9.3.14 What function $f(x)$ has power series $f(x) = \sum_{n=3}^{\infty} x^n = x^3 + x^4 + \cdots$?

A. $f(x) = \frac{1}{1-3x}$.

B. $f(x) = \frac{3}{1-x}$.

C. $f(x) = \frac{1}{1-x} - x^2 - x - 1$.

D. $f(x) = \frac{x^3}{1-x}$.

Activity 9.3.15 The function $n(x) = e^{-x^2}$ is one whose integrals are very important for statistics. However, it does not admit an elementary antiderivative.

(a) Which of the following best represents the power series for $n(x) = e^{-x^2}$?

A. $n(x) = -x^2 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} -\frac{1}{n!} x^{n+2}$.

B. $n(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n x^{2n}$.

C. $n(x) = x^{-2} \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{n+2} x^{n+2}$.

(b) Which of the following best represents a degree 10 polynomial that approximates $n(x)$?

A. $n_{10}(x) = -x^2 - x^3 - \frac{1}{2}x^4 - \frac{1}{6}x^5 - \frac{1}{24}x^6 - \frac{1}{120}x^7 - \frac{1}{720}x^8 - \frac{1}{5040}x^9 - \frac{1}{40320}x^{10}$.

B. $n_{10}(x) = x^2 - x^3 + \frac{1}{2}x^4 - \frac{1}{6}x^5 + \frac{1}{24}x^6 - \frac{1}{120}x^7 + \frac{1}{720}x^8 - \frac{1}{5040}x^9 + \frac{1}{40320}x^{10}$.

C. $n_{10}(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}$.

(c) Use your choice of $n_{10}(x)$ to estimate $\int_0^1 n(x)dx$ by computing $\int_0^1 n_{10}(x)dx$.

Manipulation of Power Series (PS3)

Activity 9.3.16 Recall that

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $-1 < x < 1$.

(a) Which of the following represents an antiderivative of $g(x) = \dfrac{1}{1-x}$?

A. $G(x) = C + \sum_{n=0}^{\infty} x^{n+1}$.

B. $G(x) = C + \sum_{n=1}^{\infty} \frac{1}{n} x^{n+1}$.

C. $G(x) = C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$.

D. $G(x) = C + \sum_{n=1}^{\infty} \frac{1}{n+1} x^{n}$.

(b) Find the interval of convergence for this series.

(c) Recall that $\tilde{G}(x) = \ln|1-x|$ is an antiderivative of $g(x) = \dfrac{1}{1-x}$. For which C is your chosen $G(x) = \ln|1-x|$?

(d) Use $G_4(x)$ to estimate $\displaystyle\int_2^4 \ln|1-x|dx$.

Manipulation of Power Series (PS3)

Activity 9.3.17 Recall that the power series for $f(x) = \sin(x)$ is:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

(a) Find a power series for $\sin\left(-5x^2\right)$.

(b) Find a power series for $x^4 \sin(x)$.

(c) Find a power series for $F(x)$, an antiderivative of $f(x)$ such that $F(0) = 4$.

Manipulation of Power Series (PS3)

Activity 9.3.18 Recall that the power series for $f(x) = -\frac{1}{x-1}$ is:

$$-\frac{1}{x-1} = \sum_{n=0}^{\infty} x^n.$$

(a) Find a power series for $\frac{1}{x^4+1}$.

(b) Find a power series for $-\frac{x^5}{x-1}$.

(c) Find a power series for $f'(x)$.

Activity 9.3.19 Recall that

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $-1 < x < 1$ and $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2} = g(-x^2)$. We computed the power series for $g(-x^2)$ in Activity 9.3.10.

(a) Integrate this power series and find C to find a power series for $H(x) = \arctan(x)$. Recall that $\arctan(0) = 0$.

(b) Find the interval of convergence for this series.

Activity 9.3.20

(a) Find the power series for $\alpha(x) = \ln|x|$.

(b) Find the interval of convergence for this series.

Activity 9.3.21

(a) Find the power series for $\beta(x) = \arctan(-3x^2)$.

(b) Find the interval of convergence for this series.

9.4 Taylor Series (PS4)

Learning Outcomes

- Determine a Taylor or Maclaurin series for a function.

Taylor Series (PS4)

Activity 9.4.1 The following tasks will help us find a mechanism to produce a power series given information about its derivatives.

(a) Find the 2nd derivative of x^2.

A $2x$

B 2

C $4x$

D 4

(b) Find the 3rd derivative of x^3.

A 2

B $3x$

C 6

D $12x$

(c) Find the 4th derivative of x^4.

A 18

B 24

C 32

D 64

(d) Based on these results, which of the following should always equal the nth derivative of x^n with respect to x?

A n

B n^2

C $n!$

D n^n

Taylor Series (PS4)

Activity 9.4.2 Let's use derivatives to rediscover the sequence a_n which gives a power series representation for e^x.

(a) Let's say that

$$e^x = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots.$$

What must a_0 be to satisfy $e^0 = 1$?

(b) Then,

$$\frac{d}{dx}[e^x] = e^x = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \ldots.$$

What must a_1 be to also satisfy $e^0 = 1$?

(c) Then,

$$\frac{d^2}{dx^2}[e^x] = e^x = 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots.$$

What must a_2 be to also satisfy $e^0 = 1$?

(d) Then,

$$\frac{d^3}{dx^3}[e^x] = e^x = 6a_3 + 24a_4 x + \ldots.$$

What must a_3 be to also satisfy $e^0 = 1$?

(e) So this $6a_3$ term was obtained from the fact that the 3rd derivative of x^3 is $3! = 6$.

So finally, we may skip ahead to the nth derivative:

$$\frac{d^n}{dx^n}[e^x] = e^x = n! \cdot a_n + (n+1)! \cdot a_{n+1} \cdot x + \ldots.$$

What must a_n be to also satisfy $e^0 = 1$?

(f) This reveals the power series we previously found for e^x:

$$e^x = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Taylor Series (PS4)

So in general, if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$, then

$$\frac{d^n}{dx^n}[f(x)] = f^{(n)}(x) = n! \cdot a_n + (n+1)! \cdot a_{n+1} \cdot x + \ldots.$$

What must a_n be to produce the correct value for $f^{(n)}(0)$?

Taylor Series (PS4)

Fact 9.4.3 *If $f(x)$ can be written as a power series, then there is a real number c such that*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \dots \end{aligned}$$

on some interval centered at $x = c$.

In fact, the functions that can be represented as power series are exactly those functions which are infinitely differentiable on some open interval.

Definition 9.4.4 The **Taylor series** generated by $f(x)$ and *centered* at $x = c$ is given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \dots \end{aligned}$$

with an interval of convergence determinable by series convergence rules.

When $c = 0$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots \end{aligned}$$

is called the **Maclaurin series** generated by f. $\Diamond$

Activity 9.4.5 Observe that $f(x) = \sin(x)$ is a function such that:

$f(0)$	$f'(0)$	$f''(0)$	$f^{(3)}(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$	$f^{(6)}(0)$	$f^{(7)}(0)$
$\sin(0)$	$\cos(0)$	$-\sin(0)$	$-\cos(0)$	$\sin(0)$	$\cos(0)$	$-\sin(0)$	$-\cos(0)$
0	1	0	-1	0	1	0	-1

(a) Given the zeros appearing for every even derivative above, which of these is a valid simplification of the Maclarin series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for $\sin(x)$?

A $\displaystyle\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

B $\displaystyle\sum_{2n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

C $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n}$

D $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$

(b) Now consider the following consolidated chart:

$f^{(1)}(0)$	$f^{(3)}(0)$	$f^{(5)}(0)$	$f^{(7)}(0)$
$\cos(0)$	$-\cos(0)$	$\cos(0)$	$-\cos(0)$
1	-1	1	-1

Which formula yields these alternating 1s and -1s appearing for $f^{(2n+1)}(0)$?

A $f^{(2n+1)}(0) = (-1)^n$

B $f^{(2n+1)}(0) = (-1)^{n+1}$

C $f^{(2n+1)}(0) = (-1)^{2n}$

D $f^{(2n+1)}(0) = (-1)^{2n+1}$

Taylor Series (PS4)

Fact 9.4.6 *The power series we've introduced for each of the following functions are in fact their Maclaurin series (Taylor series centered at* 0*).*

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = 1 + x + x^2 + x^3 + \dots \\
e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots
\end{aligned}$$

Definition 9.4.7 For a function $f(x)$ with a Taylor series centered at $x = c$,

$$\begin{aligned} f(x) &\approx T_k(x) \\ &= \sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(k)}(c)}{k!}(x-c)^k \end{aligned}$$

where $T_k(x)$ is called the k^{th} degree **Taylor polynomial** generated by f and centered at $x = c$.

The k^{th} degree Taylor polynomial can be seen as the "best" polynomial of degree k or less for approximating $f(x)$ for values close to $x = c$. Note that the 1^{st} degree Taylor polynomial is also known as the **linearization** of f. $\Diamond$

Taylor Series (PS4)

Activity 9.4.8 Let $f(x)$ be a function such that:

$f(4)$	$f'(4)$	$f''(4)$	$f'''(4)$	$f^{(4)}(4)$	$f^{(5)}(4)$	$f^{(6)}(4)$
0	1	2	3	4	5	6

(a) Find a Taylor polynomial for $f(x)$ centered at $x = 4$ of degree 3.

(b) Using the table above, find a general closed form for $f^{(n)}(4)$.

(c) Use (b) to find a Taylor series for $f(x)$ centered at $x = 4$.

Activity 9.4.9 Let $f(x)$ be a function such that:

$f(-2)$	$f'(-2)$	$f''(-2)$	$f'''(-2)$	$f^{(4)}(-2)$	$f^{(5)}(-2)$	$f^{(6)}(-2)$
0	2	-16	54	-128	250	-432

(a) Find a Taylor polynomial for $f(x)$ centered at $x = -2$ of degree 3.

(b) Using the table above, find a general closed form for $f^{(n)}(-2)$.

(c) Use (b) to find a Taylor series for $f(x)$ centered at $x = -2$.

Remark 9.4.10 You might have seen $\sqrt{-1}$ written as i, and know that z is a complex number if $z = a + bi$ for some real numbers a and b. Note that $i^2 = -1$, $i^3 = (i^2)i = -i$, $i^4 = (i^2)^2 = 1$, $i^5 = (i^4)i = i$, and so on. This gives rise to the following notion.

Definition 9.4.11 Euler's Identity. For any real number θ,

$$\begin{aligned}
e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!} + \dots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}$$

$\Diamond$

Taylor Series (PS4)

Activity 9.4.12 Use Euler's identity to evaluate $e^{i\pi}$.