

Calculus for Team-Based Inquiry Learning

2022 Edition

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Chapter 1

Limits (LT)

Learning Outcomes

How do we measure “close-by” values?

By the end of this chapter, you should be able to...

1. Find limits from the graph of a function.
2. Infer the value of a limit based on nearby values of the function.
3. Compute limits of functions given algebraically, using proper limit properties.
4. Determine where a function is and is not continuous.
5. Determine limits of functions at infinity.
6. Determine limits of functions approaching vertical asymptotes.

Readiness Assurance. Before beginning this chapter, you should be able to...

a Use function notation and evaluate functions

- Review: [Khan Academy](#)¹
- Practice:
 - [Evaluate functions](#)²
 - [Evaluate functions from their graphs](#)³
 - [Function notation word problems](#)⁴

b Find the domain of a function

¹<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions#x2f8bb11595b61c86:evaluating-functions>

²https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:evaluating-functions/e/functions_1?modal=1

³<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:evaluating-functions/e/evaluate-functions-from-their-graph?modal=1>

⁴<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:interpreting-function-notation/e/function-notation-in-context?modal=1>

- Review: [Khan Academy](#)⁵
 - Practice: [Determine the domain of functions](#)⁶
- c Determine vertical asymptotes, horizontal asymptotes, and holes (removable discontinuities) of rational functions
- Review:
 - [Discontinuities of rational functions](#)⁷
 - [Finding horizontal asymptotes](#)⁸
 - Practice:
 - [Rational functions: zeros, asymptotes, and undefined points](#)⁹
 - [Finding horizontal asymptotes](#)¹⁰
- d Perform basic operations with polynomials
- Review:
 - [Adding and subtracting polynomials](#)¹¹
 - [Multiplying polynomials](#)¹²
 - Practice:
 - [Add polynomials](#)¹³
 - [Subtract polynomials](#)¹⁴
 - [Multiply binomials by polynomials](#)¹⁵
- e Factor quadratic expressions
- Review: [Khan Academy](#)¹⁶

⁵<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions#x2f8bb11595b61c86:determining-the-domain-of-a-function>

⁶<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:determining-the-domain-of-a-function/e/domain-of-algebraic-functions?modal=1>

⁷<https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:rational-functions#x9e81a4f98389efdf:discontinuities-of-rational-functions>

⁸https://www.softschools.com/math/calculus/finding_horizontal_asymptotes_of_rational_functions/

⁹<https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:rational-functions/x9e81a4f98389efdf:discontinuities-of-rational-functions/e/points-of-discontinuity-of-rational-functions?modal=1>

¹⁰https://www.softschools.com/math/calculus/finding_horizontal_asymptotes_of_rational_functions/

¹¹<https://www.khanacademy.org/math/get-ready-for-precalculus/x65c069afc012e9d0:get-ready-for-polynomials#x65c069afc012e9d0:adding-and-subtracting-polynomials>

¹²<https://www.khanacademy.org/math/get-ready-for-precalculus/x65c069afc012e9d0:get-ready-for-polynomials/x65c069afc012e9d0:multiplying-binomials-by-polynomials/a/multiplying-binomials-by-polynomials-review>

¹³https://www.khanacademy.org/math/get-ready-for-precalculus/x65c069afc012e9d0:get-ready-for-polynomials/x65c069afc012e9d0:adding-and-subtracting-polynomials/e/adding_and_subtracting_polynomials?modal=1

¹⁴<https://www.khanacademy.org/math/get-ready-for-precalculus/x65c069afc012e9d0:get-ready-for-polynomials/x65c069afc012e9d0:adding-and-subtracting-polynomials/e/subtract-polynomials?modal=1>

¹⁵https://www.khanacademy.org/math/get-ready-for-precalculus/x65c069afc012e9d0:get-ready-for-polynomials/x65c069afc012e9d0:multiplying-binomials-by-polynomials/e/multiplying_polynomials?modal=1

¹⁶<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:quadratics-multiplying-factoring/x2f8bb11595b61c86:factor-quadratics-intro/a/factoring-simple-quadratics-review?modal=1>

- Practice: [Factoring quadratics intro](#)¹⁷
- f Represent intervals using number lines, inequalities, and interval notation
- Review: [Varsity Tutor](#)¹⁸

1.1 Limits graphically (LT1)

Learning Outcomes

- Find limits from the graph of a function.

Activity 1.1.1 In [Figure 1](#) the graph of a function is given, but something is wrong. The graphic card failed and one portion did not render properly. We can't see what is happening in the neighborhood of $x = 2$.

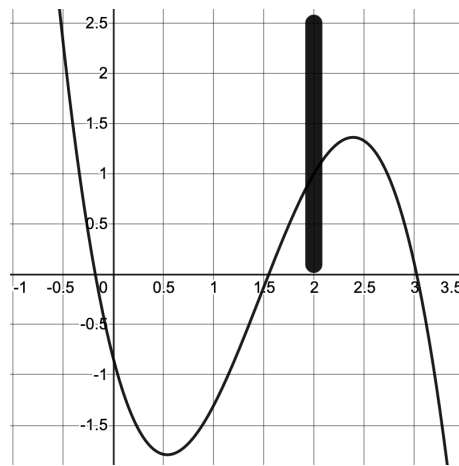


Figure 1 A graph of a function that has not been rendered properly.

- (a) Imagine moving along the graph toward the missing portion from the left, so that you are climbing up and to the right toward the obscured area of the graph. What y -value are you approaching?
- A 0.5
B 1
C 1.5
D 2
E 2.5
- (b) Think of the same process, but this time from the right. You're falling down and to the left this time as you come close to the missing portion. What y -value are you approaching?
- A 0.5
B 1

¹⁷https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:quadratics-multiplying-factoring/x2f8bb11595b61c86:factor-quadratics-intro/e/factoring_polynomials_1?modal=1

¹⁸https://www.varsitytutors.com/hotmath/hotmath_help/topics/interval-notation

- C 1.5
- D 2
- E 2.5

Activity 1.1.2 In [Figure 2](#) the graphic card is working again and we can see more clearly what is happening in the neighborhood of $x = 2$.

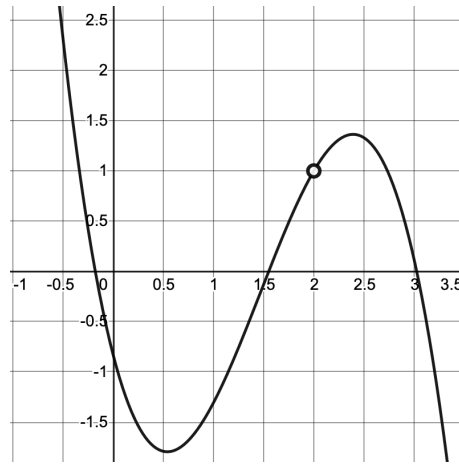


Figure 2 A graph of a function that has rendered properly

- (a) What is the value of $f(2)$?
 - (b) What is the y -value that is approached as we move toward $x = 2$ from the left?
 - (c) What is the y -value that is approached as we move toward $x = 2$ from the right?
- A 0.5
 - B 1
 - C 1.5
 - D 2
 - E 2.5

Remark 1.1.3 When studying functions in algebra, we often focused on the *value* of a function given a specific x -value. For instance, finding $f(2)$ for some function $f(x)$. In calculus, and here in [Activity 1.1.1](#) and [Activity 1.1.2](#), we have instead been exploring what is happening as we *approach* a certain value on a graph. This concept in mathematics is known as finding a limit.

Activity 1.1.4 Based on [Activity 1.1.1](#) and [Activity 1.1.2](#), write your first draft of the definition of a limit. What is important to include? (You can use concepts of limits from your daily life to motivate or define what a limit is.)

Definition 1.1.5 Given a function f , a fixed input $x = a$, and a real number L , we say that f **has limit L as x approaches a** , and write

$$\lim_{x \rightarrow a} f(x) = L$$

provided that we can make $f(x)$ as close to L as we like by taking x sufficiently close (but not equal) to a . If we cannot make $f(x)$ as close to a single value as we would like as x approaches a , then we say that f **does not have a limit as x approaches a** . \diamond

Activity 1.1.6

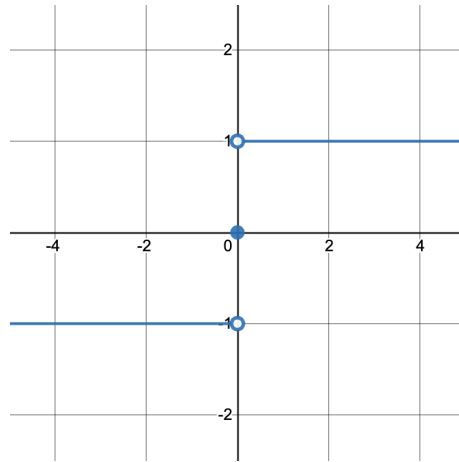


Figure 3 A piecewise-defined function

What is the limit as x approaches 0 in [Figure 3](#)?

- A The limit is 1
- B The limit is -1
- C The limit is 0
- D The limit is not defined

Definition 1.1.7 We say that f *has limit L_1 as x approaches a from the left* and write

$$\lim_{x \rightarrow a^-} f(x) = L_1$$

provided that we can make the value of $f(x)$ as close to L_1 as we like by taking x sufficiently close to a while always having $x < a$. We call L_1 the left-hand limit of f as x approaches a . Similarly, we say L_2 *is the right-hand limit of f as x approaches a* and write

$$\lim_{x \rightarrow a^+} f(x) = L_2$$

provided that we can make the value of $f(x)$ as close to L_2 as we like by taking x sufficiently close to a while always having $x > a$. \diamond

Activity 1.1.8 Refer again to [Figure 3](#) from [Activity 1.1.6](#).

- (a) Which of the following best matches the definition of right and left limits? (Note that DNE is short for "does not exist.")

- A The left limit is -1. The right limit is 1.

- B The left limit is 1. The right limit is -1.
- C The left limit DNE. The right limit is 1.
- D The left limit is -1. The right limit DNE.
- E The left limit DNE. The right limit DNE.

(b) What do you think the overall limit equals?

- A The limit is 1
- B The limit is -1
- C The limit is 0
- D The limit is not defined

Activity 1.1.9 Consider the following graph:

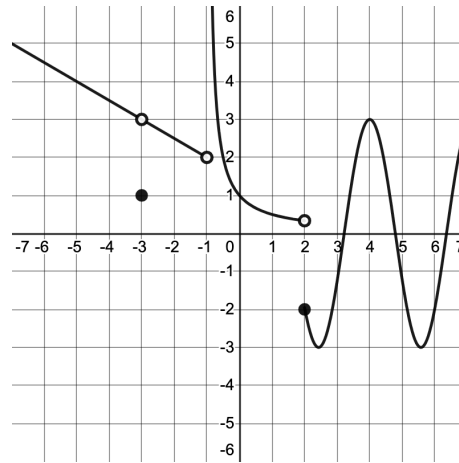


Figure 4 Another piecewise-defined function

- (a) Find $\lim_{x \rightarrow -3^-} f(x)$ and $\lim_{x \rightarrow -3^+} f(x)$.
- (b) Find $\lim_{x \rightarrow -1^-} f(x)$ and $\lim_{x \rightarrow -1^+} f(x)$.
- (c) Find $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$.
- (d) Find $\lim_{x \rightarrow 4^-} f(x)$ and $\lim_{x \rightarrow 4^+} f(x)$.
- (e) For which x -values does the *overall* limit exist? Select all. If the limit exists, find it. If it does not, explain why.
 - A -3
 - B -1
 - C 2
 - D 4

Activity 1.1.10 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

1. $\lim_{x \rightarrow 5^-} f(x)$ is finite, but $\lim_{x \rightarrow 5^+} f(x)$ is infinite.
2. $\lim_{x \rightarrow -3} f(x) = -4$, but $f(-3) = 0$.

3. $\lim_{x \rightarrow -1^-} f(x) = -1$ but $\lim_{x \rightarrow -1^+} f(x) \neq -1$.

Theorem 1.1.11 Suppose that:

- for an interval around $x = a$, we have that $f(x) \leq g(x) \leq h(x)$;
- the limit as x approaches a of $f(x)$ is equal to the same value L as the limit of $h(x)$, so $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$.

Then, the limit of $g(x)$ as $x \rightarrow a$ is also L , so $\lim_{x \rightarrow a} g(x) = L$

Activity 1.1.12 In this activity we will explore a mathematical theorem, the Squeeze Theorem [Theorem 1.1.11](#).

- The part of the theorem that starts with “Suppose...” forms the assumptions of the theorem, whilst the part of the theorem that starts with “Then...” is the conclusion of the theorem. What are the assumptions of the Squeeze Theorem? What is the conclusion?
- The assumptions of the Squeeze Theorem can be restated informally as “the function g is squeezed between the functions f and h around a ”. Explain in your own words how the two assumptions result into a “squeezing effect”.
- Let’s see an example of the application of this theorem. First examine the following picture. Explain why, from the picture, it seems that both assumptions of the theorem hold.



Figure 5 A pictorial example of the Squeeze Theorem.

- Match the functions $f(x)$, $g(x)$, $h(x)$ in the picture to the functions $\cos(x)$, 1 , $\frac{\sin(x)}{x}$.
- Using trigonometry, one can show algebraically that $\cos(x) \leq \frac{\sin(x)}{x} \leq 1$ for x values close to zero. Moreover, $\lim_{x \rightarrow 0} \cos(x) = \cos(0) = 1$ (we say that cosine is a continuous function). Use these facts and the Squeeze Theorem, to find the limit $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$.

1.2 Limits numerically (LT2)

Learning Outcomes

- Infer the value of a limit based on nearby values of the function.

Activity 1.2.1

Table 6

x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	0.1695	0.1699	0.1667	?	0.1667	0.1664	0.1639

Based on the values of Table 6, what is the best approximation for $\lim_{x \rightarrow 7} f(x)$?

- A the limit is approximately 7
- B the limit is approximately 0.17
- C the limit is approximately 0.16
- D the limit is approximately 0.1667
- E the limit is approximately 6.9999

Remark 1.2.2 Notice that the value we obtained in Activity 1.2.1 is only an approximation, based on the trends that we have seen within the table.

Activity 1.2.3

Table 7

x	1.25	1.5	1.75	2	2.25	2.5	2.75
$f(x)$	-0.7606	-0.13	0.4881	?	1.3119	1.33	0.9606

In Activity 1.1.1's Figure 1 we found an approximation to the limit of the function as x tends to 2. Now let us say you are also given a table of numerical values (Table 7) for the function. Given this new information which of the choices below best describes the limit of the function as x tends to 2?

- A There is not enough information because we do not know the value of the function at $x = 2$
- B Can be approximated to be 1 because the data in the table and the graph show that from the left and the right the function approaches 1 as x goes to 2
- C Can be approximated to be 1 because the values appear to approach 1 and the graph appears to approach 1, but we should zoom in on the graph to be sure
- D Cannot be approximated because the function might not exist at $x = 2$.

Activity 1.2.4

Table 8

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	-0.4	-0.49	-0.499	?	0.499	0.49	0.4

Based on Table 8, what information can be inferred about $\lim_{x \rightarrow 1^-} f(x)$, $\lim_{x \rightarrow 1^+} f(x)$, and $\lim_{x \rightarrow 1} f(x)$?

- A $\lim_{x \rightarrow 1^-} f(x) = -0.5$, $\lim_{x \rightarrow 1^+} f(x) = 0.5$, and $\lim_{x \rightarrow 1} f(x) = 0$
- B $\lim_{x \rightarrow 1^-} f(x) = -0.5$, $\lim_{x \rightarrow 1^+} f(x) = 0.5$, and $\lim_{x \rightarrow 1} f(x)$ does not exist
- C $\lim_{x \rightarrow 1^-} f(x) = 0.5$, $\lim_{x \rightarrow 1^+} f(x) = -0.5$, and $\lim_{x \rightarrow 1} f(x)$ does not exist
- D $\lim_{x \rightarrow 1^-} f(x) = 0.5$, $\lim_{x \rightarrow 1^+} f(x) = -0.5$, and $\lim_{x \rightarrow 1} f(x) = 0$

Activity 1.2.5 Consider the following function $f(x) = 3x^3 + 2x^2 - 5x + 20$.

- (a) Of the following options, at which values given would you evaluate $f(x)$ to best determine $\lim_{x \rightarrow 2} f(x)$ numerically?
- A 1.9, 1.99, 2.0, 2.01, 2.1
 - B 1.98, 1.99, 2.0, 2.01, 2.02
 - C 1.8, 1.9, 2.0, 2.1, 2.2
 - D 1.0, 1.5, 2.0, 2.5, 3.0
- (b) Use the values that you chose in part A to calculate an approximation for $\lim_{x \rightarrow 2} f(x)$.
- (c) Which value best describes the limit that you obtained in part (b)?
- A the approximate value is 41.25
 - B the approximate value is 41.5
 - C approximate value is 41.75
 - D approximate value is 42

Activity 1.2.6 In Figure 9 is the graph for $f(x) = \sin\left(\frac{1}{x}\right)$. Several values for $f(x)$ in the neighborhood of $x = 0$ are approximated in Table 10.

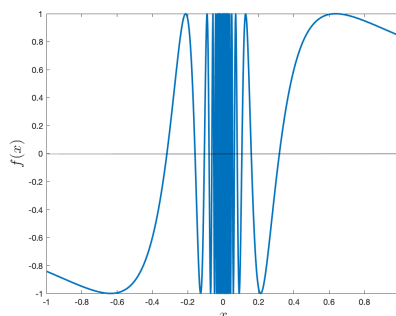


Figure 9 Graph of $f(x) = \sin(1/x)$.

Table 10

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.54402	0.50637	-0.82688	?	0.82688	-0.50637	-0.54402

- (a) Based on the graph and table what is the best explanation for the limit as x tends to zero?
- A the limit does not exist because the left and right limits have opposite values.
 - B the limit does not exist because we do not have enough information to answer the question.
 - C the limit does not exist because the function is oscillating between -1 and 1.
 - D the limit does not exist because you are dividing by zero when $x = 0$ for $f(x)$.
- (b) Would your conclusion that resulted from Activity 1.2.6 change if the function was $f(x) = \cos(1/x)$ or $f(x) = \tan(1/x)$?

Activity 1.2.7 Use technology to complete the following table of values.

$$f(x) = \frac{x^2 - x - 12}{x^2 + 16x + 39}$$

x	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$							

Then explain how to use it to make an educated guess as to the value of the limit

$$\lim_{x \rightarrow -3} \frac{x^2 - x - 12}{x^2 + 16x + 39}$$

Activity 1.2.8 In this activity you will study the velocity of Usain Bolt in his Beijing 100 meters dash. He completed 100 meters in 9.69 seconds for an overall average speed of $100/9.69 = 10.32$ meters per second (about 23 miles per hour). But this is the average velocity on the whole interval. How fast was he at different instances? What was his maximum velocity? Let's explore this. The table [Table 11](#) shows his split times recorded every 10 meters.

Table 11

t (seconds)	1.85	2.87	3.78	4.65	5.5	6.32	7.14	7.96	8.79	9.69
d (meters)	10	20	30	40	50	60	70	80	90	100

- What was the average velocity on the first 50 meters? On the second 50 meters?
- What was the average velocity between 30 and 50 meters? Between 50 and 70 meters?
- What was the average velocity between 40 and 50 meters? Between 50 and 60 meters?
- What is your best estimate for the Usain's velocity at the instant when he passed the 50 meters mark? This is your estimate for the instantaneous velocity.
- Using the table of values, explain why 50 meters is NOT the best guess for when the instantaneous velocity was the largest. What other point would be more reasonable?

1.3 Limits analytically (LT3)

Learning Outcomes

- Compute limits of functions given algebraically, using proper limit properties.

Remark 1.3.1 Recall that in [Activity 1.2.5](#) we used numerical methods and table of values to find the limit of a relatively simple degree three polynomial at a point. This was inefficient, "there's gotta be a better way!"

Activity 1.3.2 Given $f(x) = 3x^2 - \frac{1}{2}x + 4$, evaluate $f(2)$ and approximate $\lim_{x \rightarrow 2} f(x)$ numerically (or graphically). What do you think is more likely?

$$\text{A } \lim_{x \rightarrow 2} f(x) = f(2)$$

B $\lim_{x \rightarrow 2} f(x) \approx f(2)$

C $\lim_{x \rightarrow 2} f(x) \neq f(2)$

Activity 1.3.3 The table below gives values of a few different functions.

Table 12

x	6.99	6.999	7.001	7.01
f(x)	13.99	13.999	14.001	14.01
g(x)	22.97	22.997	23.003	23.03
3f(x)	41.97	41.997	42.003	42.03
f(x)+g(x)	36.96	36.996	37.004	37.04
f(x)g(x)	321.350	321.935	322.065	322.650

Using the table above, which of the following is *least* likely to be true?

A $\lim_{x \rightarrow 7} f(x) = 14$ and $\lim_{x \rightarrow 7} g(x) = 23$

B $\lim_{x \rightarrow 7} 3f(x) = 3 \lim_{x \rightarrow 7} f(x)$

C $\lim_{x \rightarrow 7} (f(x) + g(x)) = \lim_{x \rightarrow 7} f(x) + \lim_{x \rightarrow 7} g(x)$

D $\lim_{x \rightarrow 7} (f(x)g(x)) = f(7) \left(\lim_{x \rightarrow 7} g(x) \right)$

Remark 1.3.4 In [Activity 1.3.3](#) we observed that limits seem to be "well-behaved" when combined with standard operations on functions. The next theorems, known as **Limit Laws**, tell us how limits interact with combinations of functions.

Theorem 1.3.5 Limit Laws, I. Let f and g be functions defined on an open interval I containing the real number c satisfying

$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = K,$$

for L and K some real numbers. Then we have the following limits.

1. *Constant Law:* $\lim_{x \rightarrow c} b = b$, for b any constant real number;
2. *Identity Law:* $\lim_{x \rightarrow c} x = c$;
3. *Sum/Difference Law:* $\lim_{x \rightarrow c} (f(x) \pm g(x)) = L \pm K$;
4. *Scalar Multiple Law:* $\lim_{x \rightarrow c} b \cdot f(x) = bL$, for b any constant real number;
5. *Product Law:* $\lim_{x \rightarrow c} f(x) \cdot g(x) = LK$
6. *Quotient Law:* if $K \neq 0$, then $\lim_{x \rightarrow c} f(x)/g(x) = L/K$

Activity 1.3.6 If $\lim_{x \rightarrow 2} f(x) = 2$ and $\lim_{x \rightarrow 2} g(x) = -3$, which of the following statements are true? Select all that apply!

A $\lim_{x \rightarrow 2} (f(x) \cdot g(x)) = -6$

B $\lim_{x \rightarrow 2} (f(x) + g(x)) = -1$

C $\lim_{x \rightarrow 2} (f(x) - g(x)) = -2$

$$\text{D } \lim_{x \rightarrow 2} (f(x)/g(x)) = -2/3$$

Theorem 1.3.7 Limit Laws, II. Let f and g be functions defined on an open interval I containing c satisfying

$$\lim_{x \rightarrow c} f(x) = L, \quad \lim_{x \rightarrow L} g(x) = K, \text{ and } g(L) = K.$$

Then we have the following limits as well.

1. *Power Law:* $\lim_{x \rightarrow c} f(x)^n = L^n$, for n a positive integer;
2. *Root Law:* $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, for n a positive integer;
3. *Composition Law:* $\lim_{x \rightarrow c} g(f(x)) = K$.

Activity 1.3.8 Below you are given the graphs of two functions. Compute the limits below (if possible).

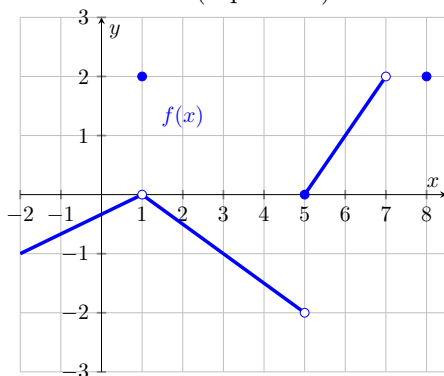


Figure 13 The graph of $f(x)$.

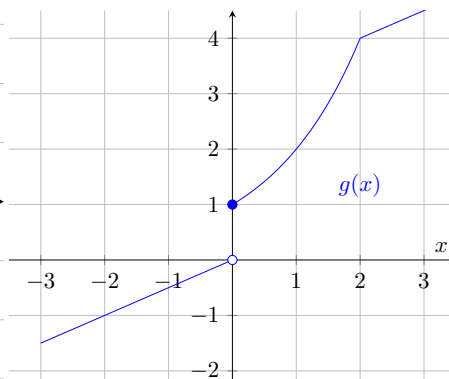


Figure 14 The graph of $g(x)$.

- (a) $\lim_{x \rightarrow 1} f(x) + g(x)$.
- (b) $\lim_{x \rightarrow 5^+} 3f(x)$.
- (c) $\lim_{x \rightarrow 0^+} f(x)g(x)$.
- (d) (Challenge) $\lim_{x \rightarrow 1} g(x)/f(x)$.
- (e) (Challenge) $\lim_{x \rightarrow 0^+} f(g(x))$.

Activity 1.3.9 Given $p(x) = -3x^2 - 5x + 7$, which of the following limit laws would use to determine $\lim_{x \rightarrow 2} p(x)$? Choose all that apply.

- A Sums/Difference Law
- B Scalar Multiple Law
- C Product Law
- D Identity Law
- E Power Law
- F Constant Law

Theorem 1.3.10 Limits of Polynomials. If $p(x)$ is a polynomial and c is a real number, then $\lim_{x \rightarrow c} p(x) = p(c)$. This is also known as the **Direct Substitution Property** for polynomials.

Activity 1.3.11 Given $p(x) = -3x^2 - 5x + 7$ and $q(x) = x^4 - x^2 + 3$, which of the following describes the most efficient way to determine $\lim_{x \rightarrow -1} \frac{p(x)}{q(x)}$?

- A Sums/difference, scalar multiple, and product laws
- B [Theorem 1.3.10](#) and the quotient law
- C Power, sums/difference, scalar multiple, and constant laws
- D Quotient and root law

Theorem 1.3.12 Limits of Rational Functions. If $p(x)$ and $q(x)$ are polynomials, c is a real number, and $q(c) \neq 0$ then $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$.

Activity 1.3.13 Consider taking the limit of a rational function $\frac{p(x)}{q(x)}$ as $x \rightarrow c$. If $q(c) = 0$, is it possible for $\lim_{x \rightarrow c} \frac{p(x)}{q(x)}$ to equal a number?

- A No, because $\frac{p(x)}{q(x)}$ is not defined at $x = c$ since $q(c) = 0$.
- B Yes, because if you graph $f(x) = \frac{x^2-1}{x-1}$, the value $f(1)$ is not defined, but the graph shows that the limit of $f(x)$ does exist as $x \rightarrow 1$.
- C No, because if you graph $g(x) = \frac{x^2+1}{x-1}$, the value $g(1)$ is not defined and the graph shows that the limit of $\lim_{x \rightarrow c} g(x)$ does not exist.
- D Yes, because we can use [Theorem 1.3.12](#).

Activity 1.3.14 Let $f(x) = 2x$ and $g(x) = x$, which of the following statements is true?

- A $\lim_{x \rightarrow 0} (f(x)/g(x)) = 0$
- B $\lim_{x \rightarrow 0} (f(x)/g(x)) = 2$
- C $\lim_{x \rightarrow 0} (f(x)/g(x))$ cannot be determined
- D $\lim_{x \rightarrow 0} (f(x)/g(x))$ does not exist

Remark 1.3.15 When we compute the limit of a ratio where both the numerator and denominator have limit equal to zero, we have to compute the value of a $\frac{0}{0}$ **indeterminate form**. The value of an indeterminate form can be any real number or even infinity or not existent, we just do not know yet! We can usually determine the value of an indeterminate form using some algebraic manipulations of the expression given.

Definition 1.3.16 A function $f(x)$ has a **hole** at $x = c$ if $f(c)$ does not exist but $\lim_{x \rightarrow c} f(x)$ does exist and is equal to a real number. \diamond

Example 1.3.17 The function $f(x) = \frac{x^2-1}{x-1}$ has a hole at $x = 1$ because $f(1)$ is not defined but

$$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \rightarrow 1} (x+1) = 2,$$

so the limit exists and is equal to a real number. Notice that $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ is also an example of a limit giving an indeterminate form $\frac{0}{0}$ which we could then compute using an algebraic manipulation of the function given. \square

Activity 1.3.18 Determine the following limits and explain your reasoning.

$$\lim_{x \rightarrow -6} \frac{x^2 - 6x + 5}{x^2 - 3x - 18}$$

$$\lim_{x \rightarrow -1} \frac{x^2 - 1}{x^2 + 3x + 2}$$

$$\lim_{x \rightarrow 5} \frac{x - 5}{\sqrt{x + 31} - 6}$$

Activity 1.3.19 In activity [Activity 1.2.8](#) you studied the velocity of Usain Bolt in his Beijing 100 meters dash. We will now study this situation analytically. To make our computations simpler, we will approximate that he could run 100 meters in 10 seconds and we will consider the model $d = f(t) = t^2$, where d is the distance in meters and t is the time in seconds.

Note 1.3.20 The average velocity is the ratio distance covered over time elapsed. If we consider the interval that starts at $t = a$ and has width h , written $[a, a + h]$, the average velocity on this interval is $\frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$. The instantaneous velocity at time $t = a$ is given by:

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

- (a) Compute the average velocity on the interval $[5, 6]$. We think of this interval as $[5, 5 + h]$ for the value of $h = 1$.
- (b) Compute the average velocity starting at 5 seconds, but now with $h = 0.5$ seconds.
- (c) We want to study the instantaneous velocity at $a = 5$ seconds. Find an expression for the average velocity on the interval $[5, 5 + h]$, where h is an unspecified value.
- (d) Expand your expression. When $h \neq 0$, you can simplify it!
- (e) Recall that the instantaneous velocity is the limit of your expression as $h \rightarrow 0$. Find the instantaneous velocity given by this model at $t = 5$ seconds.
- (f) The model $d = f(t) = t^2$ does not really capture the real-world situation. Think of at least one reason why this model does not fit the scenario of Usain Bolt's 100 meters dash.

1.4 Continuity (LT4)

Learning Outcomes

- Determine where a function is and is not continuous.

Remark 1.4.1 A continuous function is one whose values change smoothly, with no jumps or gaps in the graph. We'll explore the idea first, and arrive at a mathematical definition soon.

Activity 1.4.2 Which of the following scenarios best describes a continuous function?

- A The age of a person reported in years
- B The price of postage for a parcel depending on its weight
- C The volume of water in a tank that is gradually filled over time
- D The number of likes on my latest TikTok depending on the time since I posted it

Remark 1.4.3 How would you use the language of limits to clarify the definition of continuity?

Activity 1.4.4 A function f defined on $-4 < x < 4$ has the graph pictured below. Use the graph to answer each of the following questions.

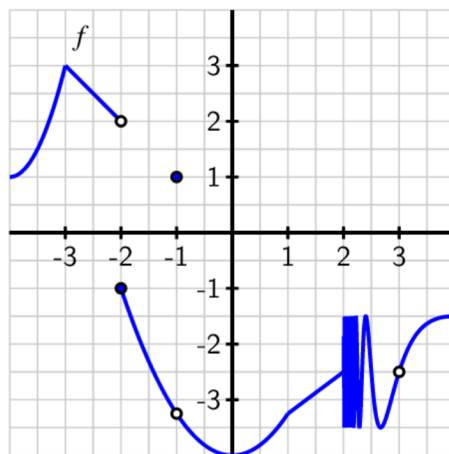


Figure 15

- (a) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether the limit $\lim_{x \rightarrow a} f(x)$ exists. If the limit does not exist, be ready to explain why not.
- (b) For each of the values of a where the limit of f exists, determine the value of $f(a)$ at each such point.
- (c) For each such a value, is $f(a)$ equal to $\lim_{x \rightarrow a} f(x)$?
- (d) Use your understanding of continuity to determine whether f is continuous at each value of a .
- (e) Any revisions would you want to make to your definition of continuity that you arrived at toward the end of [Remark 1.4.3](#)?

Definition 1.4.5 A function f is **continuous** at $x = a$ provided that

- 1 f has a limit as $x \rightarrow a$
- 2 f is defined at $x = a$ (equivalently, a is in the domain of f), and
- 3 $\lim_{x \rightarrow a} f(x) = f(a)$.



Activity 1.4.6 Suppose that some function $h(x)$ is continuous at $x = -3$. Use [Definition 1.4.5](#) to decide which of the following quantities are equal to each other.

A $\lim_{x \rightarrow -3^+} h(x)$

B $\lim_{x \rightarrow -3^-} h(x)$

C $\lim_{x \rightarrow -3} h(x)$

D $h(-3)$

Activity 1.4.7 Consider the function f whose graph is pictured below (it's the same graph from [Activity 1.4.4](#)). In the questions below, consider the values $a = -3, -2, -1, 0, 1, 2, 3$.

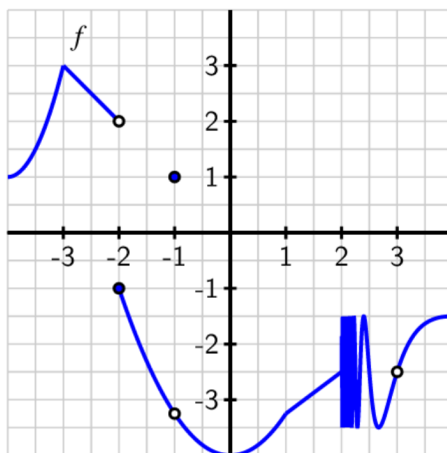


Figure 16

- For which values of a do we have $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$?
- For which values of a is $f(a)$ not defined?
- For which values of a does f have a limit at a , yet $f(a) \neq \lim_{x \rightarrow a} f(x)$?
- For which values of a does f fail to be continuous? Give a complete list of intervals on which f is continuous.

Activity 1.4.8 Which condition is *stronger*, meaning it implies the other?

A f has a limit at $x = a$, or

B f is continuous at $x = a$?

Activity 1.4.9 Previously, you have used graphs, tables, and formulas to answer questions about limits. Which of those are suitable for answering questions about continuity?

A Graphs only

B Formulas only

C Graphs and formulas only

D Tables and formulas only

Activity 1.4.10 Consider the function f whose graph is pictured below.

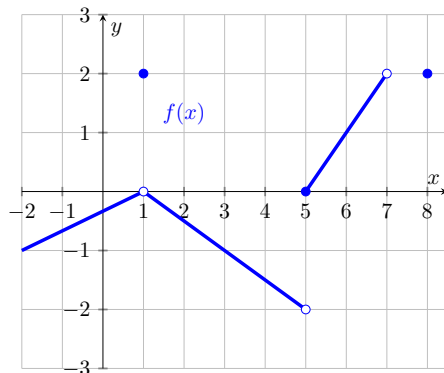


Figure 17 The graph of $f(x)$.

Give a list of x -values where $f(x)$ is not continuous. Be prepared to defend your answer based on [Definition 1.4.5](#).

Remark 1.4.11 When $\lim_{x \rightarrow a} f(x)$ exists but is not equal to $f(a)$, we say that f has a **removable discontinuity** at $x = a$. This is because if $f(a)$ were redefined to be equal to $\lim_{x \rightarrow a} f(x)$, the redefined function would be continuous at $x = a$, thus “removing” the discontinuity.

When the left and right limit exist separately, but are not equal, the discontinuity is not removable and is called a **jump discontinuity**.

Activity 1.4.12

1. Determine the value of b to make $h(x)$ continuous at $x = 5$.

$$h(x) = \begin{cases} b - x, & x < 5 \\ -x^2 + 6x - 6, & x \geq 5 \end{cases}$$

2. Classify the type of discontinuity present at $x = -6$ for the function $f(x)$.

$$f(x) = \begin{cases} -8x - 46, & x < -6 \\ 6, & x = -6 \\ 4x + 30, & x > -6 \end{cases}$$

Answer.

1. To make $h(x)$ continuous at $x = 5$, let $b = 4$.
2. The function $f(x)$ has a jump discontinuity.

Theorem 1.4.13 If f and g are continuous at $x = a$ and c is a real number, then the functions $f + g$, $f - g$, cf , and fg are also continuous at $x = a$. Moreover, f/g is continuous at $x = a$ provided that $g(a) \neq 0$.

Activity 1.4.14 Answer the questions below about piecewise functions. It may be helpful to look at some graphs.

- (a) Which values of c , if any, could make the following function continuous on the real line?

$$g(x) = \begin{cases} x + c & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

- (b) Which values of c , if any, could make the following function continuous on the real line?

$$h(x) = \begin{cases} 4 & x \leq c \\ x^2 & x > c \end{cases}$$

- (c) Which values of c , if any, could make the following function continuous on the real line?

$$k(x) = \begin{cases} x & x \leq c \\ x^2 & x > c \end{cases}$$

Theorem 1.4.15 *Suppose that:*

- *the function f is continuous on the interval $[a, b]$;*
- *you pick a value N such that $f(a) \leq N \leq f(b)$ or $f(b) \leq N \leq f(a)$.*

Then, there is some input c in the interval $[a, b]$ such that $f(c) = N$.

Activity 1.4.16 In this activity we will explore a mathematical theorem, the Intermediate Value Theorem. Here is the statement of the theorem.

- (a) To get an idea for the theorem, draw a continuous function $f(x)$ on the interval $[0, 10]$ such that $f(0) = 8$ and $f(10) = 2$. Find an input c where $f(c) = 5$.
- (b) Now try to draw a graph similar to the previous one, but that does not have any input corresponding to the output 5. Then, find where your graph violates these conditions: $f(x)$ is continuous on $[0, 10]$, $f(0) = 8$, and $f(10) = 2$.
- (c) The part of the theorem that starts with “Suppose...” forms the assumptions of the theorem, whilst the part of the theorem that starts with “Then...” is the conclusion of the theorem. What are the assumptions of the Intermediate Value Theorem? What is the conclusion?
- (d) Apply the Intermediate Value Theorem to show that the function $f(x) = x^3 + x - 3$ has a zero (so crosses the x -axis) at some point between $x = -1$ and $x = 2$. (Hint: What interval of x values is being considered here? What is N ? Why is N between $f(a)$ and $f(b)$?)

1.5 Limits with infinite inputs (LT5)

Learning Outcomes

- Determine limits of functions at infinity.

Activity 1.5.1 Consider the graph of the polynomial function $f(x) = x^3$. We want to think about what the long term behavior of this function might be. Which of the following best describes its behavior?

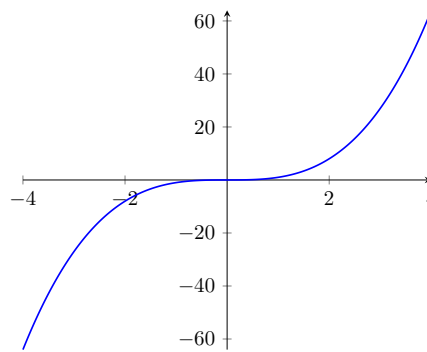


Figure 18 The graph of x^3 .

- A As x gets larger, the function x^3 gets smaller and smaller
- B As x gets more and more negative, the function x^3 gets more and more negative
- C As x gets more and more positive, the function x^3 gets more and more negative
- D As x gets smaller, the function x^3 gets smaller and smaller

Remark 1.5.2 We say that “the limit as x tends to negative infinity of x^3 is negative infinity” and that “the limit as x tends to positive infinity of x^3 is positive infinity”. In symbols, we write

$$\lim_{x \rightarrow +\infty} x^3 = +\infty, \quad \lim_{x \rightarrow -\infty} x^3 = -\infty.$$

Activity 1.5.3 Consider the graph of the rational function $f(x) = 1/x^3$. We want to think about what the long term behavior of this function might be. Which of the following best describes its behavior?

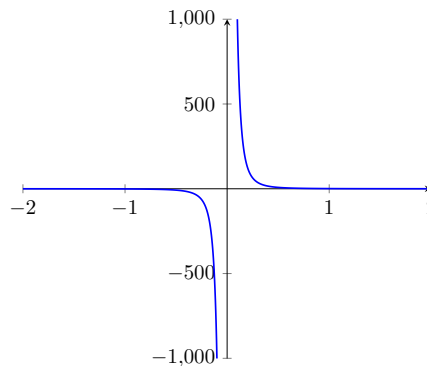


Figure 19 The graph of $1/x^3$.

- A As x tends to positive infinity, the function $1/x^3$ tends to positive infinity
- B As x tends to negative infinity, the function $1/x^3$ tends to 0
- C As x tends to positive infinity, the function $1/x^3$ tends to negative infinity
- D As x tends to 0, the function $1/x^3$ tends to 0

Definition 1.5.4 A function has a **horizontal asymptote** at $y = b$ when

$$\lim_{x \rightarrow +\infty} f(x) = b$$

or

$$\lim_{x \rightarrow -\infty} f(x) = b$$

This means that we can make the output of $f(x)$ as close as we want to b , as long as we take x a large enough positive number ($x \rightarrow \infty$) or a large enough negative number ($x \rightarrow -\infty$). \diamond

Remark 1.5.5 We say that the function $1/x^3$ has horizontal asymptote $y=0$ because the limit as x tends to positive infinity of $1/x^3$ is 0. Alternatively, we could also justify it by saying that the limit as x goes to negative infinity is 0.

Activity 1.5.6 Which of the following functions have horizontal asymptotes? Select all!

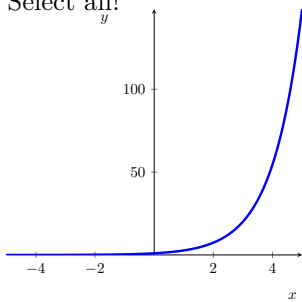
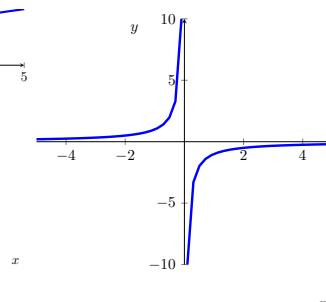
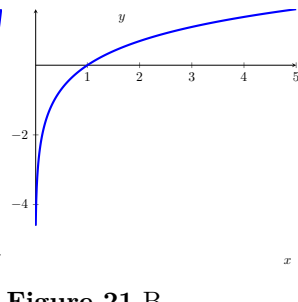


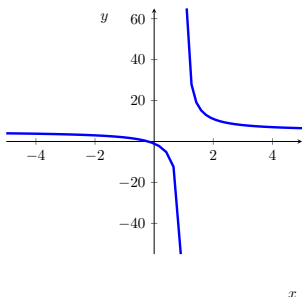
Figure 21 B



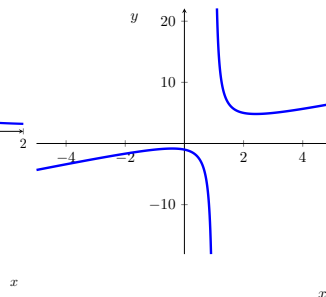
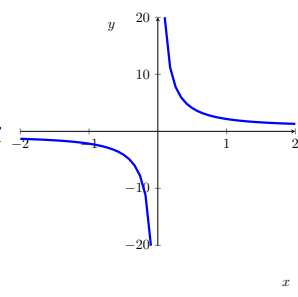
x

Figure 20 A

Figure 22 C



x



x

Figure 23 D

Figure 24 E

Figure 25 F

Activity 1.5.7 Recall that a rational function is a ratio of two polynomials. For any given rational function, what are all the possible behaviors as x tends to $+$ or $-$ infinity?

- A The only possible limit is 0
- B The only possible limits are 0 or $\pm\infty$
- C The only possible limits are 0, 1 or $\pm\infty$
- D The only possible limits are any constant number or $\pm\infty$

Activity 1.5.8 In this activity we will examine functions whose limit as x approaches positive and negative infinity is a nonzero constant.

- (a) Graph the following functions and consider their limits as x approaches positive and negative infinity. Which function(s) have a limit that is nonzero and constant? Find each of these limits.

- A $f(x) = \frac{x^3 - x + 3}{2x^3 - 6x + 1}$
- B $f(x) = \frac{x^2 - 3}{5x^3 - 2x^2 + 5}$
- C $f(x) = \frac{x^4 - 3x - 2}{3x^3 - 5x + 1}$
- D $f(x) = \frac{10x^5 - 3x + 2}{5x^5 - 3x^2 + 1}$
- E $f(x) = \frac{-8x^2 - 5x + 1}{2x^2 - 2x + 3}$

- (b) Conjecture a rule for how to determine that a rational function has a nonzero constant limit as x approaches positive and negative infinity. Test your rule by creating a rational function whose limit as $x \rightarrow \infty$ equals 3 and then check it graphically.

Activity 1.5.9 What about when the limit is not a nonzero constant? How do we recognize those? In this activity you will first conjecture the general behavior of rational functions and then test your conjectures.

- (a) Consider a rational function $r(x) = \frac{p(x)}{q(x)}$. Looking at the numerator $p(x)$ and the denominator $q(x)$, when does the function $r(x)$ have limit equal to 0 as $x \rightarrow \infty$?

- A When the ratio of the leading terms is a constant.
- B When the degree of the numerator is greater than the degree of the denominator.
- C When the degree of the numerator is less than the degree of the denominator.
- D When the degree of the numerator is equal to the degree of the denominator.

- (b) Consider a rational function $r(x) = \frac{p(x)}{q(x)}$. Looking at the numerator $p(x)$ and the denominator $q(x)$, when does the function $r(x)$ have limit approaching infinity as $x \rightarrow \infty$?

- A When the ratio of the leading terms is a constant.
- B When the degree of the numerator is greater than the degree of the denominator.
- C When the degree of the numerator is less than the degree of the denominator.
- D When the degree of the numerator is equal to the degree of the denominator.

- (c) Conjecture a rule for the each of the previous two parts of the activity. Test your rules by creating a rational function whose limit as $x \rightarrow \infty$ equals 0 and another whose limit as $x \rightarrow \infty$ is infinite. Then check them graphically.

Activity 1.5.10 Explain how to find the value of each limit.

1.

$$\lim_{x \rightarrow -\infty} -\frac{6x^4 + 7x^3 - 7}{6x - x^4 + 9} \quad \text{and} \quad \lim_{x \rightarrow +\infty} -\frac{6x^4 + 7x^3 - 7}{6x - x^4 + 9}$$

2.

$$\lim_{x \rightarrow -\infty} -\frac{7x^4 - 5x^3 + 8}{3(2x^5 + 3x^2 - 3)} \text{ and } \lim_{x \rightarrow +\infty} -\frac{7x^4 - 5x^3 + 8}{3(2x^5 + 3x^2 - 3)}$$

3.

$$\lim_{x \rightarrow -\infty} \frac{3x^6 + x^3 - 8}{7x - 6x^5 + 7} \text{ and } \lim_{x \rightarrow +\infty} \frac{3x^6 + x^3 - 8}{7x - 6x^5 + 7}$$

Answer.

1.

$$\lim_{x \rightarrow -\infty} -\frac{6x^4 + 7x^3 - 7}{6x - x^4 + 9} = 6 \text{ and } \lim_{x \rightarrow +\infty} -\frac{6x^4 + 7x^3 - 7}{6x - x^4 + 9} = 6$$

2.

$$\lim_{x \rightarrow -\infty} -\frac{7x^4 - 5x^3 + 8}{3(2x^5 + 3x^2 - 3)} = 0 \text{ and } \lim_{x \rightarrow +\infty} -\frac{7x^4 - 5x^3 + 8}{3(2x^5 + 3x^2 - 3)} = 0$$

3.

$$\lim_{x \rightarrow -\infty} \frac{3x^6 + x^3 - 8}{7x - 6x^5 + 7} = +\infty \text{ and } \lim_{x \rightarrow +\infty} \frac{3x^6 + x^3 - 8}{7x - 6x^5 + 7} = -\infty$$

Activity 1.5.11 What is your best guess for the limit as x goes to $+\infty$ of the function graphed below?

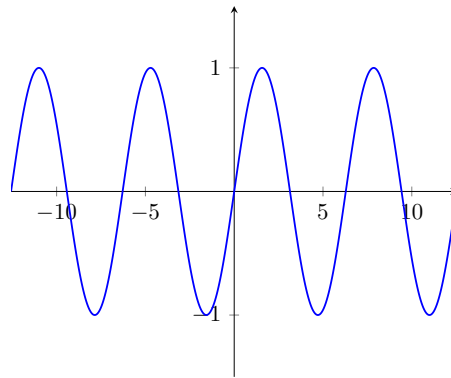


Figure 26 A mysterious periodic function.

- A The limit is 0
- B The limit is 1
- C The limit is -1
- D The limit is $+\infty$
- E The limit DNE

Warning 1.5.12 For a periodic function, a function whose outputs repeat periodically, there is not one distinguished long term behavior, so the limit DNE. Notice that this is different from the limit being $+\infty$ in which case the outputs have a clear behavior: they are getting larger and larger. Unfortunately, you will find in many cases that the notation DNE is used both for a limit equal to infinity and a limit that does not tend to one distinguished value. Beware!

Activity 1.5.13 Compute the following limits.

(a) $\lim_{x \rightarrow -\infty} \frac{x^3 - x + 83}{1}$

(b) $\lim_{x \rightarrow -\infty} \frac{1}{x^3 - x + 83}$

(c) $\lim_{x \rightarrow +\infty} \frac{x + 3}{2 - x}$

(d) $\lim_{x \rightarrow -\infty} \frac{\pi - 3x}{\pi x - 3}$

(e) (Challenge) $\lim_{x \rightarrow +\infty} \frac{3e^x + 2}{2e^x + 3}$

(f) (Challenge) $\lim_{x \rightarrow -\infty} \frac{3e^x + 2}{2e^x + 3}$

Activity 1.5.14 The graph below represents the function $f(x) = \frac{2(x+3)(x+1)}{x^2 - 2x - 3}$.

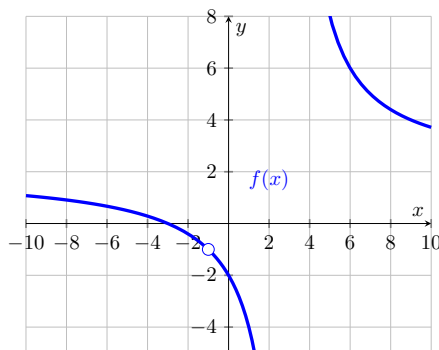


Figure 27 The graph of $f(x)$

- Find the horizontal asymptote of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.
- Use limit notation to describe the behavior of $f(x)$ at its horizontal asymptotes.
- Come up with the formula of a rational function that has horizontal asymptote $y = 3$.
- What do you think is happening at $x = -1$? We will come back to this in the next section!

Note 1.5.15 An exponential function $P(t) = ab^t$ exhibiting exponential decay will have the long term behavior $P(t) \rightarrow 0$ as $t \rightarrow \infty$. If we shift the graph up by c units, we obtain the new function $Q(t) = ab^t + c$, with the long term behavior $\lim_{t \rightarrow \infty} Q(t) = c$. A cooling object can be represented by the exponential decay model $Q(t) = ab^t + c$.

Activity 1.5.16 In this activity you will explore an exponential model for a cooling object.

Consider a cup of coffee initially at 100 degrees Fahrenheit. The said cup of coffee was forgotten this morning on the kitchen counter where the thermostat is set at 72 degrees Fahrenheit. From previous observations, we can assume

that a cup of coffee loses 10 percent of its temperature each minute.

- (a) In the long run, what temperature do you expect the coffee to tend to? Write your observation with limit notation.
- (b) In the model $Q(t) = ab^t + c$, your previous answer gives you the value of one of the parameters in this model. Which one?
- (c) From the information given, we notice that the cup of coffee has decay rate of 10 % or $r = -0.1$. When an exponential model has decay rate r , its exponential base b has value $b = 1 + r$. Use this to find the value of b for the exponential model described in this scenario.
- (d) Assume that the initial temperature corresponds to input $t = 0$. Use the data about the initial temperature to find the value of the parameter a in the model $Q(t) = ab^t + c$
- (e) You should have found that this scenario has exponential model $Q(t) = 28(0.9)^t + 72$. If you go back to drink the cup of coffee 30 minutes after it was left on the counter, what temperature will the coffee have reached?

1.6 Limits with infinite outputs (LT6)

Learning Outcomes

- Determine limits of functions approaching vertical asymptotes.

Activity 1.6.1 Consider the graph in [Figure 28](#).

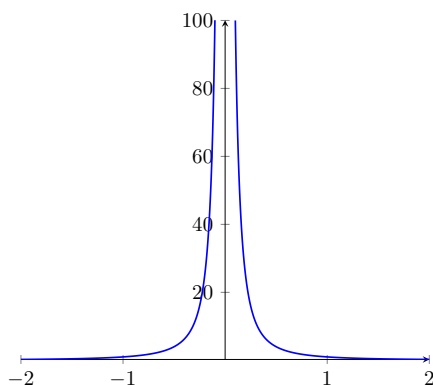


Figure 28 The graph of $1/x^2$.

- (a) Which of the following best describes the limit as x approaches zero in the graph?
 - A The limit is 0
 - B The limit is positive infinity
 - C The limit does not exist
 - D This limit is negative infinity
- (b) Which of the following best describes the relationship between the line $x = 0$ and the graph of the function?
 - A The line $x = 0$ is a horizontal asymptote for the function

- B The function is not continuous at the point $x = 0$
- C The function is moving away from the line $x = 0$
- D The function is getting closer and closer to the line $x = 0$
- E The function has a jump in outputs around $x = 0$

Definition 1.6.2 A function has a **vertical asymptote** at $x = a$ when

$$\lim_{x \rightarrow a} f(x) = +\infty$$

or

$$\lim_{x \rightarrow a} f(x) = -\infty$$

The limit being equal to positive infinity means that we can make the output of $f(x)$ as large a positive number as we want as long as we are sufficiently close to $x = a$. Similarly, the limit being equal to negative infinity means that we can make the output of $f(x)$ as large a negative number as we want as long as we are sufficiently close to $x = a$. \diamond

Activity 1.6.3 Select all of the following graphs which illustrate functions with vertical asymptotes.

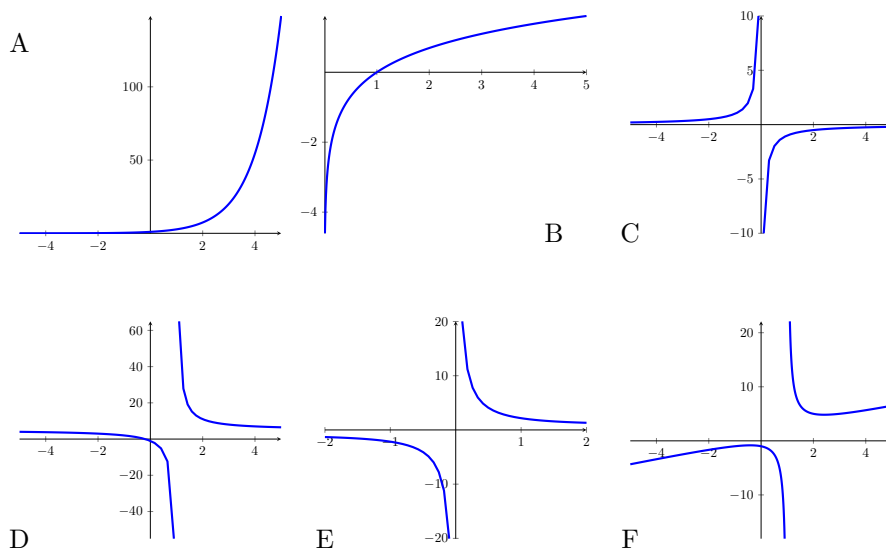


Figure 29 Choices for vertical asymptotes

Remark 1.6.4 If $x = a$ is a vertical asymptote for the function $f(x)$, the function $f(x)$ is not defined at $x = a$. As $f(a)$ does not exist, the function is NOT continuous at $x = a$. Moreover, the function's output tends to plus or minus infinity and so the limit is not equal to a number.

Activity 1.6.5 Notice that as x goes to 0, the value of x^2 goes to 0 but the value of $1/x^2$ goes to infinity. What is the best explanation for this behavior?

- A When dividing by an increasingly small number we get an increasing big number
- B When dividing by an increasingly large number we get an increasing small number
- C A rational function always has a vertical asymptote
- D A rational function always has a horizontal asymptote

Remark 1.6.6 Informally, we say that the limit of " $\frac{1}{0}$ " is infinite. Notice that this could be either positive or negative infinity, depending on how whether the outputs are becoming more and more positive or more and more negative as we approach zero.

Activity 1.6.7 Consider the rational function $f(x) = \frac{2}{x-3}$. Which of the following options best describes the limits as x approaches 3 from the right and from the left?

- A As $x \rightarrow 3^+$, the limit DNE, but as $x \rightarrow 3^-$ the limit is $-\infty$
- B As $x \rightarrow 3^+$, the limit is $+\infty$, but as $x \rightarrow 3^-$ the limit is $-\infty$
- C As $x \rightarrow 3^+$, the limit is $+\infty$, but as $x \rightarrow 3^-$ the limit is $+\infty$
- D As $x \rightarrow 3^+$, the limit is $-\infty$, but as $x \rightarrow 3^-$ the limit is $-\infty$
- E As $x \rightarrow 3^+$, the limit DNE and as $x \rightarrow 3^-$ the limit DNE

Remark 1.6.8 When considering a ratio of functions $f(x)/g(x)$, the inputs a where $g(a) = 0$ are not in the domain of the ratio. If $g(a) = 0$ but $f(a)$ is not equal to 0, then $x = a$ is a vertical asymptote

Activity 1.6.9 Consider the function $f(x) = \frac{x^2-1}{x-1}$. The line $x = 1$ is NOT a vertical asymptote for $f(x)$. Why?

- A When x is not equal to 1, we can simplify the fraction to $x - 1$, so the limit is 1.
- B When x is not equal to 1, we can simplify the fraction to $x + 1$, so the limit is 2.
- C The function is always equal to $x + 1$.
- D The function is always equal to $x - 1$.

Remark 1.6.10 Recall the definition of a hole from [Definition 1.3.16](#). In [Activity 1.6.9](#) we have a hole at $x = 1$.

Activity 1.6.11 Find all the vertical asymptotes of the following rational function

(a) $y = \frac{3x-4}{7x+1}$

(b) $y = \frac{x^2+10x+24}{x^2-2x+1}$

(c) $y = \frac{(x^2-4)(x^2+1)}{x^6}$

(d) $y = \frac{2x+1}{2x^2+8x-10}$

Activity 1.6.12 Explain and demonstrate how to find the value of each limit.

1.

$$\lim_{x \rightarrow -3^-} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

2.

$$\lim_{x \rightarrow -3^+} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

3.

$$\lim_{x \rightarrow -3} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

Answer.

1.

$$\lim_{x \rightarrow -3^-} \frac{(x+4)^2(x-2)}{(x+3)(x-5)} = -\infty$$

2.

$$\lim_{x \rightarrow -3^+} \frac{(x+4)^2(x-2)}{(x+3)(x-5)} = +\infty$$

3.

$$\lim_{x \rightarrow -3} \frac{(x+4)^2(x-2)}{(x+3)(x-5)} \text{ does not exist}$$

Activity 1.6.13 The graph below represents the function $f(x) = \frac{(x+2)(x+4)}{x^2+3x-4}$.

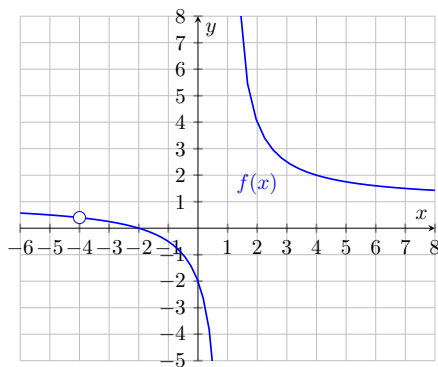


Figure 30 The graph of $f(x)$

- Explain the behavior of $f(x)$ at $x = -4$.
- Find the vertical asymptote(s) of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.
- Find the horizontal asymptote(s) of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.
- Use limit notation to describe the behavior of $f(x)$ at its asymptotes.

Activity 1.6.14 Consider the following rational function.

$$r(x) = \frac{5(x-3)(x-6)^3}{6(x+2)^3(x-3)}$$

- Explain how to find the horizontal asymptote(s) of $r(x)$, if there are any. Then express your findings using limit notation.
- Explain how to find the hole(s) of $r(x)$, if there are any. Then express your findings using limit notation.
- Explain how to find the vertical asymptote(s) of $r(x)$, if there are any. Then express your findings using limit notation.
- Draw a rough sketch of $r(x)$ that showcases all the limits that you have found above.

Answer.

1. The horizontal asymptote is $\frac{5}{6}$
2. The vertical asymptote is -2
3. There is a hole at $(3, -\frac{9}{50})$

Activity 1.6.15 You want to draw a function with all these properties.

- $\lim_{x \rightarrow 3} f(x) = 5$
- $f(3) = 0$
- $\lim_{x \rightarrow 0^-} f(x) = -\infty$
- $\lim_{x \rightarrow 0^+} f(x) = 0$
- $\lim_{x \rightarrow +\infty} f(x) = 2$

Before you start drawing, consider the following guiding questions.

- (a) At which x values will the limit not exist?
- (b) What are the asymptotes of this function?
- (c) At which x values will the function be discontinuous?
- (d) Draw the graph of one function with all the properties above. Make sure that your graph is a function! You only need to draw a graph, writing a formula would be very challenging!

Chapter 2

Derivatives (DF)

Learning Outcomes

Need a question for this chapter

By the end of this chapter, you should be able to...

1. Estimate the value of a derivative using difference quotients, and draw corresponding secant and tangent lines on the graph of a function.
2. Find derivatives using the definition of derivative as a limit.
3. Compute basic derivatives using algebraic rules.
4. Compute derivatives using the Product and Quotient Rules.
5. Compute derivatives using the Chain Rule.
6. Compute derivatives using a combination of algebraic derivative rules.
7. Compute derivatives of implicitly-defined functions.
8. Compute derivatives of inverse functions.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Write equations of lines using slope-intercept and/or point-slope form ([Math is Fun](#)¹)
- b Find the average rate of change of a function over some interval ([Khan Academy](#)²)
- c Evaluate functions at variable expressions ([Purple Math](#)³)
- d Use the laws of exponents to rewrite a given expression ([Khan Academy \(1\)](#)⁴ and [Khan Academy \(2\)](#)⁵)

¹<https://www.mathsisfun.com/algebra/line-equation-point-slope.html>

²<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:average-rate-of-change/v/average-rate-of-change-example-3>

³<https://www.purplemath.com/modules/fcnot2.htm>

⁴<https://www.khanacademy.org/math/get-ready-for-ap-calc/xa350bf684c056c5c:get-ready-for-differentiation-1/xa350bf684c056c5c:rational-exponents/v/basic-fractional-exponents>

⁵<https://www.khanacademy.org/math/get-ready-for-ap-calc/>

- e Compose and decompose functions ([Khan Academy](#)⁶)
- f Recall special trig values on the unit circle ([Khan Academy \(1\)](#)⁷ and [Khan Academy \(2\)](#)⁸)

2.1 Derivatives graphically and numerically (DF1)

Learning Outcomes

- Estimate the value of a derivative using difference quotients, and draw corresponding secant and tangent lines on the graph of a function.

Activity 2.1.1 In this activity you will study the velocity of a ball falling under gravity. The height of the ball (in feet) is given by the formula $f(t) = 64 - 16(t-1)^2$, where t is measured in seconds. We want to study the velocity at the instant $t = 2$, so we will look at smaller and smaller intervals around $t = 2$. For your convenience, below you will find a table of values for $f(t)$. Recall that the average velocity is given by the change in height over the change in time.

Table 31

t (seconds)	1	1.5	1.75	2	2.25	2.5	3
$f(t)$ (feet)	64	60	55	48	39	28	0

- (a) To start we will look at an interval of length one before $t = 2$ and after $t = 2$, so we consider the intervals $[1, 2]$ and $[2, 3]$. What was the average velocity on the interval $[1, 2]$? What about on the interval $[2, 3]$?
- (b) Now let's consider smaller intervals of length 0.5. What was the average velocity on the interval $[1.5, 2]$? What about on the interval $[2, 2.5]$?
- (c) What was the average velocity on the interval $[1.75, 2]$? What about on the interval $[2, 2.25]$?
- (d) If we wanted to approximate the velocity at the instant $t = 2$, what would be your best estimate for this instantaneous velocity?

Observation 2.1.2 If we want to study the velocity at the instant $t = 2$, it is helpful to study the average velocity on small intervals around $t = 2$. If we consider the interval $[2, 2 + h]$, where h is the width of the interval, the average velocity is given by the difference quotient

$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}.$$

xa350bf684c056c5c:get-ready-for-differentiation-1/xa350bf684c056c5c:properties-of-exponents/v/simplifying-exponent-expression-with-division

⁶<https://www.khanacademy.org/math/get-ready-for-ap-calc/xa350bf684c056c5c:get-ready-for-differentiation-2/xa350bf684c056c5c:composing-functions/v/function-composition>

⁷<https://www.khanacademy.org/math/get-ready-for-ap-calc/xa350bf684c056c5c:get-ready-for-limits-and-continuity/xa350bf684c056c5c:special-trig-values-in-q1/v/solving-triangle-unit-circle>

⁸<https://www.khanacademy.org/math/get-ready-for-ap-calc/xa350bf684c056c5c:get-ready-for-limits-and-continuity/xa350bf684c056c5c:unit-circle-introduction/v/unit-circle-definition-of-trig-functions-1>

Observation 2.1.3 We want to be able to consider intervals before and after $t = 2$. A positive value of h will give an interval after $t = 2$ for example, the interval $[2, 3]$ for $h = 1$. A negative value of h will give an interval before $t = 2$ for example, the interval $[1, 2]$ corresponds $h = -1$. In the formula above, it looks like the interval would be $[2, 1]$, but the standard notation in an interval is to write the smallest number first. This does not change the difference quotient because

$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2) - f(2+h)}{2 - (2+h)}.$$

Activity 2.1.4 Consider the height of the ball falling under gravity as in Table 31 .

- What was the average velocity on the interval $[2, 2+h]$ for $h = 1$ and $h = -1$?
- What was the average velocity on the interval $[2, 2+h]$ for $h = 0.5$ and $h = -0.5$?
- What was the average velocity on the interval $[2, 2+h]$ for $h = 0.25$ and $h = -0.25$?
- What is your best estimate for the limiting value of these velocities as $h \rightarrow 0$? Notice that this is your estimate for the instantaneous velocity at $t = 2$!

Definition 2.1.5 The instantaneous velocity at $t = a$ is the limit as $h \rightarrow 0$ of the difference quotient $\frac{f(a+h) - f(a)}{h}$. In the activity above the instantaneous velocity at $t = 2$ is given by the limit

$$v(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

◇

Definition 2.1.6 The slope of the secant line to $f(x)$ through the points $x = a$ and $x = b$ is given by the difference quotient

$$\frac{f(b) - f(a)}{b - a}.$$

◇

Activity 2.1.7 In this activity you will study the slope of a graph at a point. The graph of the function $g(x)$ is given below. For your convenience, below you will find a table of values for $g(x)$.

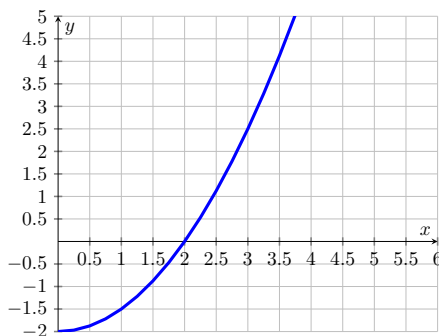


Figure 32 The graph of $g(x)$

Table 33

x	1	1.5	2	2.5	3
$g(x)$	-1.5	-0.875	0	1.125	2.5

- (a) What is the slope of the line through $(1, g(1))$ and $(2, g(2))$? Draw this line on the graph of $g(x)$.
- (b) What is the slope of the line through $(1.5, g(1.5))$ and $(2, g(2))$? Draw this line on the graph of $g(x)$.
- (c) Draw the line tangent to $g(x)$ at $x = 2$. What would be your best estimate for the slope of this tangent line?
- (d) Notice that the slope of the tangent line at $x = 2$ is positive. What feature of the graph of $f(x)$ around $x = 2$ do you think causes the tangent line to have positive slope?
- A The function $f(x)$ is concave up
 B The function $f(x)$ is increasing
 C The function $f(x)$ is concave down
 D The function $f(x)$ is decreasing

Observation 2.1.8 The slope of the secant line to $f(x)$ through the points $x = a$ and $x = b$ is given by the difference quotient $\frac{f(b)-f(a)}{b-a}$. As the point $x = b$ gets closer to $x = a$, the slope of the secant line tends to the slope of the tangent line. Letting $b = a + h$, we say that the slope of the tangent line at $x = a$ is given by the limit

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Definition 2.1.9 The derivative of $f(x)$ at $x = a$, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

◇

Observation 2.1.10 In [Activity 2.1.1](#) and [Activity 2.1.4](#) you studied a ball falling under gravity and estimated the instantaneous velocity as a limiting value of average velocities on smaller and smaller intervals. Drawing the corresponding secant lines, we see how the secant lines approximate better the tangent line, showing graphically what we previously saw numerically. Here is a Desmos animation showing the secant lines approaching the tangent line <https://www.desmos.com/calculator/bzs1bxz7fa>.

Activity 2.1.11 Suppose that the function $f(x)$ gives the position of an object at time x . Which of the following quantities are the same? Select all that apply!

- A The value of the derivative of $f(x)$ at $x = a$
 B The slope of the tangent line to $f(x)$ at $x = a$
 C The instantaneous velocity of the object at $x = a$
 D The difference quotient $\frac{f(a+h)-f(a)}{h}$
 E The limit $\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$

Observation 2.1.12 We can use the difference quotient $\frac{f(a+h)-f(a)}{h}$ for small values of h to estimate $f'(a)$, the value of the derivative at $x = a$.

Activity 2.1.13 Suppose that you know that the function $g(x)$ has values $g(-0.5) = 7$, $g(0) = 4$, and $g(0.5) = 2$. What is your best estimate for $g'(0)$?

- A $g'(0) \approx -3$
- B $g'(0) \approx -2$
- C $g'(0) \approx -6$
- D $g'(0) \approx -4$
- E $g'(0) \approx -5$

Activity 2.1.14 Suppose that you know that the function $f(x)$ has value $f(1) = 3$ and has derivative at $x = 1$ given by $f'(1) = 2$. Which of the following scenarios is most likely?

- A $f(2) = 3$ because the function is constant
- B $f(2) = 2$ because the derivative is constant
- C $f(2) \approx 1$ because the function's output decreases by about 2 units for each increase by 1 unit in the input
- D $f(2) \approx 5$ because the function's output increases by about 2 units for each increase by 1 unit in the input

Observation 2.1.15 We can use the derivative at $x = a$ to estimate the increase/decrease of the function $f(x)$ close to $x = a$. A positive derivative at $x = a$ suggests that the output values are increasing around $x = a$ approximately at a rate given by the value of the derivative. A negative derivative at $x = a$ suggests that the output values are decreasing around $x = a$ approximately at a rate given by the value of the derivative.

Activity 2.1.16 In this activity you will study the absolute value function $f(x) = |x|$. The absolute value function is a piecewise defined function which outputs x when x is positive (or zero) and outputs $-x$ when x is negative. So the absolute value always outputs a number which is positive (or zero). Here is the graph of this function.

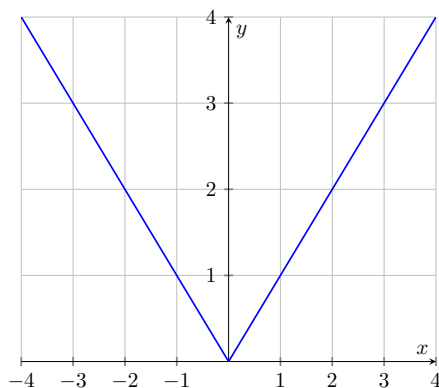


Figure 34 The graph of $|x|$

- (a) What do you think is the slope of the function for any x value smaller than zero?

- A 0
- B 1
- C -1
- D DNE

(b) What do you think is the slope of the function for any x value greater than zero?

- A 0
- B 1
- C -1
- D DNE

(c) What do you think is the slope of the function at zero?

- A 0
- B 1
- C -1
- D DNE

Observation 2.1.17 Because the derivative at a point is defined in terms of a limit, the quantity $f'(a)$ might not exist! In that case we say that $f(x)$ is not differentiable at $x = a$. This might happen when the slope on the left of the point is different from the slope on the right, like in the case of the absolute value function. We call this behavior a corner in the graph.

Activity 2.1.18 Consider the graph of function $h(x)$.

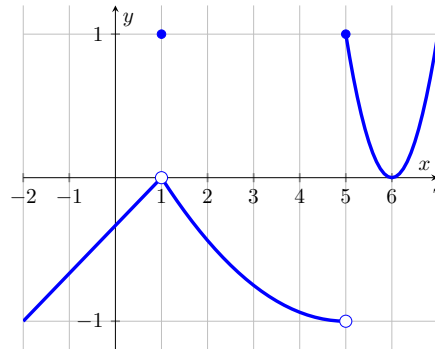


Figure 35 The graph of $h(x)$.

(a) For which of the following points a is $f'(a)$ positive? Select all that apply!

- A -1
- B 1
- C 2
- D 5
- E 6

(b) For which of the following points a is $f'(a)$ negative? Select all that apply!

- A -1

- B 1
- C 2
- D 5
- E 6

(c) For which of the following points a is $f'(a)$ zero? Select all that apply!

- A -1
- B 1
- C 2
- D 5
- E 6

(d) For which of the following points a the quantity $f'(a)$ does NOT exist? Select all that apply!

- A -1
- B 1
- C 2
- D 5
- E 6

Activity 2.1.19 Sketch the graph of a function $f(x)$ that satisfies the following criteria. (You do not need to define the function algebraically.)

- Defined and continuous on the interval $[-5, 5]$.
- $f'(x)$ does not exist at $x = 0$
- $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} < 0$
- The slope tangent to the graph of $f(x)$ at $x = 3$ is zero
- The rate of change of $f(x)$ when $x = -1$ is positive

Activity 2.1.20 You are given the graph of the function $f(x)$.

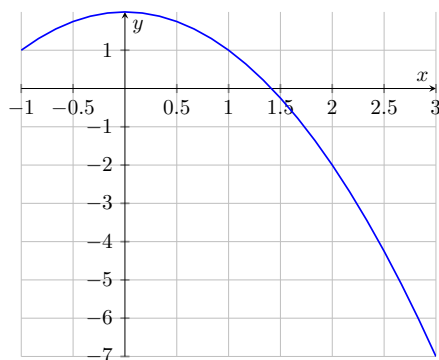


Figure 36 The graph of $f(x)$

- (a) Using the graph, estimate the slope of the tangent line at $x = 2$. Make sure you can carefully describe your process for obtaining this estimate!
- (b) If you call your approximation for the slope m , which one of the following

expression gives you the equation of the tangent line at $x = 2$?

- A $y - 2 = m(x - 2)$
- B $y + 2 = m(x - 2)$
- C $y - 2 = m(x + 2)$
- D $y + 2 = m(x + 2)$

(c) Find the equation of the tangent line at $x = 2$.

2.2 Derivatives analytically (DF2)

Learning Outcomes

- Find derivatives using the definition of derivative as a limit.

Observation 2.2.1 Recall that $f'(a)$, the derivative of $f(x)$ at $x = a$, was defined as the limit as $h \rightarrow 0$ of the difference quotient on the interval $[a, a + h]$ as in [Definition 2.1.9](#). If $f'(a)$ exists, then say that $f(x)$ is differentiable at a .

If for some interval I , $f'(x)$ exists for every point x in I , then we say that $f(x)$ is differentiable on I .

Activity 2.2.2 For the function $f(x) = x - x^2$ use the limit definition of the derivative at a point to compute $f'(2)$.

- A $f'(2) = \lim_{h \rightarrow 0} \frac{(2+h) - (2+h)^2 - 2 + 4}{h} = -3$
- B The limit $f'(2) = \lim_{h \rightarrow 0} \frac{(2+h) - (2+h)^2 - 2}{h}$ simplifies algebraically to $\lim_{h \rightarrow 0} \frac{-3h - h^2}{h}$ which does not exist, thus $f'(2)$ is not defined.
- C The limit $f'(2) = \lim_{h \rightarrow 0} \frac{(2+h) - (2+h)^2 - 2}{h}$ simplifies algebraically to $\lim_{h \rightarrow 0} \frac{h - h^2}{h}$ which does not exist, thus $f'(2)$ is not defined.
- D $f'(2) = \lim_{h \rightarrow 0} \frac{(2+h) - (2^2 + h^2) - 2 + 4}{h} = 1$

Activity 2.2.3 Consider the function $f(x) = 3 - 2x$. Which of the following best summarizes the average rates of changes of on f on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$?

- A The average rate of change on the intervals $[1, 4]$ and $[3, 7]$ are equal to the slope of $f(x)$, but the average rate of change of f cannot be determined on $[5, 5 + h]$ without a specific value of h .
- B The average rate of change on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$ are all different values.
- C The average rate of change on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$ are all equal to -2 .

Activity 2.2.4 Can you find $f'(\pi)$ when $f(x) = 3 - 2x$ without computing?

- A No, because we cannot compute the value $f(\pi)$.
- B No, because we cannot compute the average rate of change on the interval

$$[\pi, \pi + h].$$

- C Yes, $f'(\pi) = 3$ because the intercept of the tangent line at any point is equal to the constant intercept of $f(x)$.
- D Yes, $f'(\pi) = -2$ because the slope of the tangent line at any point is equal to the constant slope of $f(x)$.

Definition 2.2.5 Let $f(x)$ be function that is differentiable on an open interval I . The derivative function of $f(x)$, denoted $f'(x)$, is given by the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

At any particular input $x = a$, the derivative function outputs $f'(a)$, the value of derivative at the point $x = a$. \diamond

Activity 2.2.6 In this activity you will consider $f(x) = -x^2 + 4$ and compute its derivative function $f'(x)$ using the limit definition of the derivative function [Definition 2.2.5](#).

- (a) What expression do you get when you simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{-(x+h)^2 + 4 - (-x^2 + 4)}{h}?$$

- A $\frac{x^2 + h^2 + 4 - x^2 - 4}{h} = \frac{h^2}{h}$
- B $\frac{-x^2 - h^2 + 4 + x^2 - 4}{h} = \frac{-h^2}{h}$
- C $\frac{-x^2 - 2xh - h^2 + 4 + x^2 - 4}{h} = \frac{-2xh - h^2}{h}$
- D $\frac{x^2 + 2xh + h^2 + 4 - x^2 - 4}{h} = \frac{2xh + h^2}{h}$

- (b) After taking the limit as $h \rightarrow 0$, which of the following is your result for the derivative function $f'(x)$?

- A $f'(x) = x$
- B $f'(x) = -x$
- C $f'(x) = 2x$
- D $f'(x) = -2x$

Activity 2.2.7 Using the limit definition of the derivative, find $f'(x)$ for $f(x) = -x^2 + 2x - 4$. Which of the following is an accurate expression for $f'(x)$?

- A $f'(x) = 2x + 2$
- B $f'(x) = -2x$
- C $f'(x) = -2x + 2$
- D $f'(x) = -2x - 2$

Activity 2.2.8 Using the limit definition of the derivative, you want to find $f'(x)$ for $f(x) = \frac{1}{x}$. We will do this by first simplifying the difference quotient

and then taking the limit as $h \rightarrow 0$.

(a) What expression do you get when you simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}?$$

A $\frac{\frac{1}{x+h}}{h} = \frac{1}{(x+h)h}$

B $\frac{\frac{h}{x+h}}{h} = \frac{h}{h(x+h)}$

C $\frac{\frac{x-(x+h)}{(x+h)x}}{h} = \frac{-h}{h(x+h)x}$

D $\frac{\frac{x-(x+h)}{(x+h)x}}{h} = \frac{-h^2}{(x+h)x}$

E $\frac{\frac{h}{(x+h)x}}{h} = \frac{h}{h(x+h)x}$

(b) After taking the limit as $h \rightarrow 0$, which of the following is your result for the derivative function $f'(x)$?

A $f'(x) = 0$

B $f'(x) = 1/x$

C $f'(x) = -1/x$

D $f'(x) = 1/x^2$

E $f'(x) = -1/x^2$

Activity 2.2.9 Find $f'(x)$ using the limit definition of the derivative. Then evaluate at $x = 8$.

$$f(x) = x^2 - 5x - 5$$

Answer.

$$f'(x) = 2x - 5$$

$$f'(8) = 11$$

Definition 2.2.10 Once we have computed the first derivative $f'(x)$, the second derivative of $f(x)$ is the first derivative of $f'(x)$ or

$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}.$$

◇

Activity 2.2.11 Consider the function $f(x) = -x^2 + 2x - 4$. Earlier you saw that $f'(x) = -2x + 2$. What is the second derivative of $f(x)$?

A $f''(x) = 2$

B $f'(x) = -2$

C $f'(x) = 2x$

D $f'(x) = -2x$

Remark 2.2.12 The first derivative encodes information about the rate of change of the original function. In particular,

- If $f' > 0$, then f is increasing;
- If $f' < 0$, then f is decreasing;
- If $f' = 0$, then f has a horizontal tangent line (and it might have a max or min or it might just be changing pace);

The second derivative is the derivative of the derivative. It encodes information about the rate of change of the rate of change of the original function. In particular,

- If $f'' > 0$, then f is concave up;
- If $f'' < 0$, then f is concave down;
- If $f'' = 0$, then f might have an inflection point (or it might just be changing pace);

Activity 2.2.13 Consider the function $f(x) = -x^2 + 2x - 4$. Earlier you saw that $f'(x) = -2x + 2$ and $f''(x) = -2$. What does this tell you about the graph of $f(x)$ for $x > 1$?

- A The graph is increasing and concave up
- B The graph is increasing and concave down
- C The graph is decreasing and concave up
- D The graph is decreasing and concave down

Observation 2.2.14 We have two ways to compute analytically the derivative at a point. For example, to compute $f'(1)$, the derivative of $f(x)$ at $x = 1$, we have two methods

1. We can directly compute $f'(1)$ by finding the difference quotient on the interval $[1, 1 + h]$ and then taking the limit as $h \rightarrow 0$.
2. We can first find the derivative function $f'(x)$ by computing the difference quotient on the interval $[x, x + h]$, then taking the limit as $h \rightarrow 0$, and finally evaluating the expression for $f'(x)$ at the input $x = 1$.

The latter approach is more convenient when you want to consider the value of the derivative function at multiple points!

Activity 2.2.15 Consider the function $f(x) = \frac{1}{x^2}$. You will find $f'(1)$ in two ways!

- (a) Using the limit definition of the derivative at a point, compute the difference quotient on the interval $[1, 1 + h]$ and then take the limit as $h \rightarrow 0$. What do you get?
 - A -1
 - B 1
 - C 2
 - D -2
- (b) Now, using the limit definition of the derivative function, find $f'(x)$. Which of the following is your result for the derivative function $f'(x)$?
 - A $f'(x) = -1/x^3$

- B $f'(x) = 1/x^3$
- C $f'(x) = -2/x^3$
- D $f'(x) = 2/x^3$

(c) Make sure that your answers match! So if you plug in $x = 1$ in $f'(x)$, you should get the same number you got when you computed $f'(1)$.

Activity 2.2.16 In this activity you will study (again!) the velocity of a ball falling under gravity. A ball is tossed vertically in the air from a window. The height of the ball (in feet) is given by the formula $f(t) = 64 - 16(t - 1)^2$, where t is the seconds after the ball is launched. Recall that in [Activity 2.1.1](#), you used numerical methods to approximate the instantaneous velocity of $f(t)$ to calculate $v(2)$!

- (a) Using the limit definition of the derivative function, find the velocity function of the ball $v(t) = f'(t)$.
- (b) Using the velocity function $v(t)$, what is $v'(1)$, the instantaneous velocity at $t = 1$?

- A -32 feet per second
- B 32 feet per second
- C 0 feet per second
- D -16 feet per second
- E 16 feet per second

(c) What behavior would explain your finding?

- A After 1 second the ball is falling at a speed of 32 meters per second.
- B After 1 second the ball is moving upwards at a speed of 32 meters per second.
- C After 1 second the ball reaches its highest point and it stops for an instant.
- D After 1 second the ball is falling at a speed of 16 meters per second.
- E After 1 second the ball is moving upwards at a speed of 16 meters per second.

Observation 2.2.17 A function can only be differentiable at $x = a$ if it is also continuous at $x = a$. But not all continuous functions are differentiable: when we have a corner in the graph of a the function, the function is continuous at the corner point, but it is not differentiable at that point!

Activity 2.2.18 In [Observation 2.1.17](#), we said that a function is not differentiable when the limit that defines it does not exist. In this activity we will study differentiability analytically.

- (a) Consider the following continuous function

$$g(x) = \begin{cases} x + 2 & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

Consider the interval $[2, 2 + h]$. When $h < 0$, the interval falls under the first definition of $g(x)$ and the derivative is always equal to 1. What is the derivative function for x values greater than 2? Show that at $x = 2$ the value of this derivative is not equal to 1 and so $g(x)$ is not differentiable at $x = 2$.

- (b) Consider the following discontinuous function

$$g(x) = \begin{cases} x + 2 & x \leq 2 \\ x & x > 2 \end{cases}$$

On both sides of $x = 2$ it seems that the slope is the same, but this function is still not differentiable at $x = 2$. Notice that $g(2) = 4$. When $h > 0$, the interval $[2, 2 + h]$ falls under the second definition of $g(x)$, but $g(2)$ is always fixed at 4. Compute the difference quotient $\frac{g(2+h)-g(2)}{h}$ assuming that $h > 0$ and notice that this does not simplify as expected! Moreover, if you take the limit as $h \rightarrow 0$, you will get infinity and not the expected slope of 1!

- (c) Consider the following function

$$g(x) = \begin{cases} ax + 2 & x \leq 2 \\ bx^2 & x > 2 \end{cases}$$

where a, b are some nonzero parameters you will find. Find an equation in a, b that needs to be true if we want the function to be continuous at $x = 2$. Also, find an equation in a, b that needs to be true if we want the function to be differentiable at $x = 2$. Solve the system of two linear equations... you should find that $a = -2$ and $b = -1/2$ are the only values that make the function differentiable (and continuous!).

2.3 Elementary derivative rules (DF3)

Learning Outcomes

- Compute basic derivatives using algebraic rules.

Observation 2.3.1 If we recall from the previous section we learned about how to find the derivative of functions using the limit definition of a derivative. From the activities we see that this gets cumbersome when the functions are more complicated. In this section we will discuss a more convenient way to calculate derivatives.

Activity 2.3.2 In this activity we will try to deduce a rule for finding the derivative of a power function. Note, a power function is one in which the function is of the form $f(x) = x^n$ where n is any real number.

- (a) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x$?

- A -1
- B 1
- C 0
- D Does not exist

- (b) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^2$?

- A 0
- B $-2x$
- C $2x$

D $2x + 1$

- (c) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^3$?

A $3x^2$

B $-3x^2$

C $3x^2 - 3x$

D $-3x^2 + 3x$

- (d) WITHOUT using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^4$? (See if you can find a pattern from the first three tasks of this activity.)

A $3x^2$

B $3x^3$

C $4x^2$

D $4x^3$

Theorem 2.3.3 The Power Rule. *The derivative of the power function $f(x) = x^n$, for any real number n is*

$$f'(x) = nx^{n-1}.$$

Observation 2.3.4 So far we have been using $f'(x)$ or f prime to denote a derivative of the function $f(x)$. However there are other ways to denote the derivative of a function. Besides $f'(x)$ there is also $\frac{df}{dx}$, pronounced "dee-f dee-x." Both notations can be used interchangeably. If you want to take the derivative of $f'(x)$ or $\frac{df}{dx}$, the notation is $f''(x)$ or $\frac{d^2f}{dx^2}$; this is known as second derivative of $f(x)$.

Activity 2.3.5 Using [Theorem 2.3.3](#), which of the following statement(s) are true? For those statements that are wrong, give the correct derivative.

A The derivative of $y = x^{10}$ is $y' = 10x^{11}$.

B The derivative of $y = x^{-8}$ is $y' = -8x^{-9}$.

C The derivative of $y = x^{100}$ is $y' = 100x^{99}$.

D The derivative of $y = x^{-17}$ is $y' = -17x^{-16}$.

Theorem 2.3.6 The Derivative of a Constant Function. *If $f(x) = c$, for some constant $c \in \mathbb{R}$, then $f'(x) = 0$.*

Activity 2.3.7 Using [Theorem 2.3.6](#), which of the following statement(s) are true? Note: Pay attention to the independent variable (the input) of the function.

A The derivative of $y(x) = 10$ is $y'(x) = 9$.

B The derivative of $y(t) = x$ is $y'(t) = 0$.

C The derivative of $y(a) = x^2$ is $y'(a) = 2x$.

D The derivative of $y(x) = -5$ is $y'(x) = -4$.

Theorem 2.3.8 The Scalar Multiple Rule. *If c is a constant and $f(x)$ is a differentiable function, then using the Scalar Multiple Law for limits, we*

have that

$$\frac{d}{dx} [cf(x)] = c \frac{d}{dx} [f(x)]$$

Activity 2.3.9 What is the derivative of the function $y(x) = 12x^{2/3}$?

A $y'(x) = 8x^{5/3}$.

B $y'(x) = 18x^{-1/3}$.

C $y'(x) = 8x^{-1/3}$.

D $y'(x) = 18x^{5/3}$.

Theorem 2.3.10 The Sum/Difference Rule. If $f(x)$ and $g(x)$ are both differentiable, then using the Sum/Differences Law for limits, we have that

$$\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)]$$

Activity 2.3.11 What are the first and second derivatives for the arbitrary parabola given by $f(x) = ax^2 + bx + c$, where a, b, c are any real numbers and $a \neq 0$?

A $f'(x) = 2ax + bx + c$, $f''(x) = 2a + b$.

B $f'(x) = 2x + 1$, $f''(x) = 2$.

C $f'(x) = 2ax + b$, $f''(x) = 2a$.

D $f'(x) = ax + b$, $f''(x) = a$.

Activity 2.3.12 We have power functions for every real value of p . In particular, we can look at power functions with fractional exponents like $f(x) = x^{\frac{1}{4}} = \sqrt[4]{x}$ or with negative exponents like $g(x) = x^{-4} = \frac{1}{x^4}$. What is the derivative of these two functions?

A $f'(x) = \frac{1}{4\sqrt[4]{x^3}}$, $g'(x) = \frac{-4}{x^3}$.

B $f'(x) = \frac{1}{4}\sqrt[4]{x^3}$, $g'(x) = \frac{-4}{x^5}$.

C $f'(x) = \frac{1}{4}\sqrt[4]{x^3}$, $g'(x) = \frac{-4}{x^3}$.

D $f'(x) = \frac{1}{4\sqrt[4]{x^3}}$, $g'(x) = \frac{-4}{x^5}$.

Theorem 2.3.13 The Derivative of an Exponential Function. The derivative of the exponential function $f(x) = b^x$, for any positive number b is

$$f'(x) = \ln(b) b^x.$$

Observation 2.3.14 A special case of [Theorem 2.3.13](#) is when $b = e$, where e is the base of the natural logarithm function. In this case let $f(x) = e^x$. Then

$$f'(x) = \ln(e) e^x = e^x.$$

So $f(x) = e^x$ is a special function for which $f'(x) = f(x)$.

Activity 2.3.15 The first derivative of the function $g(x) = x^e + e^x$ is given by $g'(x) = ex^{e-1} + e^x$. What is the second derivative of $g(x)$?

A $g''(x) = x^e + e^x$.

B $g''(x) = e(e-1)x^{e-2} + e^x.$

C $g''(x) = ex^{e-1} + e^x.$

D $g''(x) = e^x.$

Theorem 2.3.16 The Derivative of the Sine and Cosine Functions. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Activity 2.3.17 The derivative of $f(x) = 7 \sin(x) + 2e^x + 3x^{1/3} - 2$ is,

A $f'(x) = 7 \cos(x) + 2e^x + x^{-2/3} - 2x.$

B $f'(x) = 7 \cos(x) + 2e^x + -2x^{-2/3} - 2.$

C $f'(x) = -7 \sin(x) + e^x + x^{-2/3}.$

D $f'(x) = -7 \cos(x) + 2e^x \ln(x) + x^{-2/3}.$

E $f'(x) = 7 \cos(x) + 2e^x + x^{-2/3}.$

Theorem 2.3.18 The Derivative of the Natural Log Function. If $f(x) = \ln(x)$, then

$$f'(x) = \frac{1}{x}.$$

Activity 2.3.19 Which of the following statements is NOT true?

A The derivative of $y = 2 \ln(x)$ is $y' = \frac{2}{x}.$

B The derivative of $y = \frac{\ln(x)}{2}$ is $y' = \frac{1}{2x}.$

C The derivative of $y = \frac{2}{3} \ln(x)$ is $y' = \frac{3}{2x}.$

D The derivative of $y = \ln(x^2)$ is $y' = \frac{2}{x}.$

Activity 2.3.20 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (constant multiple, sum and difference, etc.) you are using in your work.

1.

$$g(x) = 2 \cos(x) - 3e^x$$

2.

$$h(w) = \sqrt[5]{w^7} + \frac{6}{w^5}$$

3.

$$f(t) = -4t^5 + 5t^3 + t - 8$$

Answer.

1.

$$g'(x) = -2 \sin(x) - 3e^x$$

2.

$$h'(w) = \frac{7}{5} w^{\frac{2}{5}} - \frac{30}{w^6}$$

3.

$$f'(t) = -20t^4 + 15t^2 + 1$$

Activity 2.3.21 Suppose that the temperature (in degrees Fahrenheit) of a cup of coffee, t minutes after forgetting it on a bench outside, is given by the

function

$$f(t) = 40(0.5)^t + 50$$

Find $f(1)$ and $f'(1)$ and interpret your result in the context of this problem

Activity 2.3.22 In this activity you will use our first derivative rules to study the slope of tangent lines

- (a) The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of these points.
- (b) Find the equations of the two lines tangent to the parabola $y = (x - 2)^2$ which pass through the origin. You will want to think about slope in two ways: as the derivative at $x = a$ and the rise over the run in a linear function through the origin and the point $(a, f(a))$. Use a graph to check your work and sketch the tangent lines on your graph.

Activity 2.3.23 Find the values of the parameters a, b, c for the quadratic polynomial $q(x) = ax^2 + bx + c$ that best approximates the graph of $f(x) = e^x$ at $x = 0$. This means choosing a, b, c such that

- $q(0) = f(0)$
- $q'(0) = f'(0)$
- $q''(0) = f''(0)$

Hint: find the values of $f(0), f'(0), f''(0)$. The values of $q(0), q'(0), q''(0)$ at zero will involve some parameters. You can solve for these parameters using the equations above.

2.4 The product and quotient rules (DF4)

Learning Outcomes

- Compute derivatives using the Product and Quotient Rules.

Activity 2.4.1 Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

- (a) Determine $f'(t)$ and $g'(t)$.
- (b) Let $p(t) = 2t^2(t^3 + 4t)$ and observe that $p(t) = f(t) \cdot g(t)$. Rewrite the formula for p by distributing the $2t^2$ term. Then, compute $p'(t)$ using the sum and constant multiple rules.
- (c) True or false: $p'(t) = f'(t) \cdot g'(t)$.

Theorem 2.4.2 Product Rule. *If f and g are differentiable functions, then their product $P(x) = f(x) \cdot g(x)$ is also a differentiable function, and*

$$P'(x) = f(x)g'(x) + g(x)f'(x).$$

Activity 2.4.3 The product rule is a powerful tool, but sometimes it isn't necessary; a more elementary rule may suffice. Which of the following products could you find the derivative of without using the product rule? Select all that apply.

- A $f(x) = e^x \sin x$
- B $f(x) = \sqrt{x}(x^3 + 3x - 3)$

C $f(x) = (4)(x^5)$

D $f(x) = x \ln x$

Activity 2.4.4 Find the derivative of the following functions by using the product rule.

(a) $f(x) = (x^2 + 3x) \sin x$

(b) $f(x) = e^x \cos x$

(c) $f(x) = x^2 \ln x$

Activity 2.4.5 Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

(a) Determine $f'(t)$ and $g'(t)$. (These were found previously in [Activity 2.4.1](#).)

(b) Let $q(t) = \frac{t^3+4t}{2t^2}$ and observe that $q(t) = \frac{g(t)}{f(t)}$. Rewrite the formula for q by dividing each term in the numerator by the denominator and simplify to write q as a sum of constant multiples of powers of t . Then, compute $q'(t)$ using the sum and constant multiple rules.

(c) True or false: $q'(t) = \frac{g'(t)}{f'(t)}$.

Theorem 2.4.6 Quotient Rule. *If f and g are differentiable functions, then their quotient $Q(x) = \frac{f(x)}{g(x)}$ is also a differentiable function for all x where $g(x) \neq 0$ and*

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Activity 2.4.7 Just like with the product rule, there are times where we can find the derivative of a quotient using elementary rules rather than the quotient rule. Which of the following quotients could you find the derivative of without using the quotient rule? Select all that apply.

A $f(x) = \frac{6}{x^3}$

B $f(x) = \frac{2}{\ln x}$

C $f(x) = \frac{e^x}{\sin x}$

D $f(x) = \frac{x^3 + 3x}{x}$

Activity 2.4.8 Find the derivative of the following functions using the quotient rule (or, if applicable, an elementary rule).

(a) $f(x) = \frac{6}{x^3}$

(b) $f(x) = \frac{2}{\ln x}$

(c) $f(x) = \frac{e^x}{\sin x}$

(d) $f(x) = \frac{x^3+3x}{x}$

Activity 2.4.9 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (product,

quotient, sum and difference, etc.) you are using in your work.

1.

$$f(w) = -\frac{3w^2 + 5w - 2}{\sin(w)}$$

2.

$$g(t) = \frac{t^2 + 6t + 1}{t^2}$$

3.

$$h(t) = -2(t^2 + 3t + 3) \cos(t)$$

Answer.

1.

$$f'(w) = \frac{(3w^2 + 5w - 2) \cos(w)}{\sin(w)^2} - \frac{6w + 5}{\sin(w)}$$

2.

$$g'(t) = -\frac{2(3t + 1)}{t^3}$$

3.

$$h'(t) = -2(2t + 3) \cos(t) + 2(t^2 + 3t + 3) \sin(t)$$

Note 2.4.10 We have found the derivatives of $\sin x$ and $\cos x$, but what about the other trigonometric functions? It turns out that the quotient rule along with some trig identities can help us! (See [Khan Academy](#)¹ for a reminder of the reciprocal and Pythagorean identities.)

Activity 2.4.11 Consider the function $f(x) = \tan x$, and remember that $\tan x = \frac{\sin x}{\cos x}$.

(a) What is the domain of f ?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{(\cos x)(\cos x) + (\sin x)(\sin x)}{(\cos x)^2}.$$

(c) Which Pythagorean identity might be useful here? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec x = \frac{1}{\cos x}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of x is $f'(x)$ defined? How does this set compare to the domain of f ?

Activity 2.4.12 Let $g(x) = \cot x$, and recall that $\cot x = \frac{\cos x}{\sin x}$.

(a) What is the domain of $g(x)$?

(b) Use the quotient rule to develop a formula for $g'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $g'(x)$ only in terms of the cosecant function.

(d) What is the domain of $g'(x)$? How does this compare to the domain of

¹KhanAcademy.org

$$g'(x)?$$

Activity 2.4.13 Let $h(x) = \sec x$, and recall that $\sec x = \frac{1}{\cos x}$.

- What is the domain of $h(x)$?
- Use the quotient rule to develop a formula for $h'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.
- Use other relationships among trigonometric functions to write $h'(x)$ only in terms of the tangent and secant functions.
- What is the domain of $h'(x)$? How does this compare to the domain of $h(x)$?

Activity 2.4.14 Let $p(x) = \csc x$, and recall that $\csc x = \frac{1}{\sin x}$.

- What is the domain of $p(x)$?
- Use the quotient rule to develop a formula for $p'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.
- Use other relationships among trigonometric functions to write $h'(x)$ only in terms of the cotangent and cosecant functions.
- What is the domain of $p'(x)$? How does this compare to the domain of $p(x)$?

Remark 2.4.15 We can now summarize the derivatives for all six trigonometric functions.

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = (\sec x)^2$
- $\frac{d}{dx} \cot x = -(\csc x)^2$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \csc x = -\csc x \cot x$

Activity 2.4.16 Consider the functions

$$f(x) = 3 \cos(x), \quad g(x) = x^2 + 3e^x$$

and the function $h(x)$ for which a table of values is given.

x	-1	0	2
$h(x)$	-4	-1	3
$h'(x)$	0	-1	1

In answering the following questions, be sure to explicitly denote which derivative rules (product, quotient, sum and difference, etc.) you are using in your work.

- Find the derivative of $f(x) \cdot g(x)$.

2. Find the derivative of $\frac{f(x)}{g(x)}$.
3. Find the value of the derivative of $f(x) \cdot h(x)$ at $x = -1$.
4. Find the value of the derivative of $\frac{g(x)}{h(x)}$ at $x = 0$.
5. Consider the function

$$r(x) = 3 \cos(x) \cdot x.$$

Find $r'(x)$, $r''(x)$, $r'''(x)$, and $r^{(4)}(x)$ so the first, second, third, and fourth derivative of $r(x)$. What pattern do you notice? What do you expect the twelfth derivative of $r(x)$ to be?

Answer.

1. $(fg)'(x) = 3(2x + 3e^x) \cos(x) - 3(x^2 + 3e^x) \sin(x)$
2. $(f/g)'(x) = -\frac{3((2x + 3e^x) \cos(x) + (x^2 + 3e^x) \sin(x))}{(x^2 + 3e^x)^2}$
3. $-12 \sin(1)$
4. 0
5. We have

- $r(x) = 3 \cos(x) \cdot x$
- $r'(x) = -3x \sin(x) + 3 \cos(x)$
- $r''(x) = -3x \cos(x) - 6 \sin(x)$
- $r'''(x) = 3x \sin(x) - 9 \cos(x)$
- $r^{(iv)}(x) = 3x \cos(x) + 12 \sin(x)$

The functions involved repeat every four derivatives. So the twelfth derivative should have the summand $3x \cos(x)$. The second summand has a coefficient that increases by 3 every time, so the twelfth derivative should have coefficient 36. Putting this together $r^{(xii)} = 3x \cos(x) + 36 \sin(x)$

Activity 2.4.17

- (a) Differentiate $y = \frac{e^x}{x}$, $y = \frac{e^x}{x^2}$, $y = \frac{e^x}{x^3}$. Simplify your answers as much as possible.
- (b) What do you expect the derivative of $y = \frac{e^x}{x^n}$ to be? Prove your guess!
- (c) What do your answers above tell you about the shape of the graph of $y = \frac{e^x}{x^n}$? Study how the sign of the numerator and the denominator change in the first derivative to determine when the behavior changes!

Activity 2.4.18 The quantity q of skateboards sold depends on the selling price p of a skateboard, so we write $q = f(p)$. You are given that

$$f(140) = 15000, \quad f'(140) = -100$$

- (a) What does the data provided tell you about the sales of skateboards?
- (b) The total revenue, R , earned by the sale of skateboards is given by $R =$

$q \cdot p = f(p) \cdot p$. Explain why.

(c) Find

$$\left. \frac{dR}{dp} \right|_{p=140}.$$

(d) What is the sign of the quantity above? What do you think would happen to the revenue if the price was changed from \$ 140 to \$ 141?

Activity 2.4.19 Let $f(v)$ be the gas consumption in liters per kilometer (l/km) of a car going at velocity v kilometers per hour (km/hr). So if the car is going at velocity v , then $f(v)$ tells you how many liters of gas the car uses to go one kilometer. You are given the following data

$$f(50) = 0.04, \quad f'(50) = 0.0004$$

- (a) Let $g(v)$ be the distance (in kilometers) that the same car covers per liter of gas at velocity v . What are the units of the output of $g(v)$? How can you write $g(v)$ in terms of $f(v)$? Find $g(50)$ and $g'(50)$.
- (b) Let $h(v)$ be the gas consumption over time, so the liters of gas consumed per hour by the same car going at velocity v . What are the units of the output of $h(v)$? How can you write $h(v)$ in terms of $f(v)$? Find $h(50)$ and $h'(50)$.
- (c) How would you explain the practical meaning of your findings to a driver who knows no calculus?

2.5 The chain rule (DF5)

Learning Outcomes

- Compute derivatives using the Chain Rule.

Note 2.5.1 When we consider the composition $f \circ g$ of the function f with the function g , we mean the composite function $f(g(x))$, where the function g is applied first and then f is applied to the output of g . We also call f the outside function whilst g is the inside function.

Activity 2.5.2

(a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Which of the following is a formula for $f(g(x))$?

- A $-4x^2 + 4x + 4$
- B $4x^2 - 4x + 5$
- C $-2x^2 + 9$
- D $-2x^2 + 4$

(b) One of the options above is a formula for $g(f(x))$. Which one?

Activity 2.5.3

(a) Consider the composite function $f(g(x)) = \sqrt{e^x}$. Which function is the outside function $f(x)$ and which one is the inside function $g(x)$?

- A $f(x) = x^2, g(x) = e^x$
- B $f(x) = \sqrt{x}, g(x) = e^x$

C $f(x) = e^x$, $g(x) = \sqrt{x}$

D $f(x) = e^x$, $g(x) = x^2$

(b) Using properties of the exponents, we can rewrite the original function as $e^{\frac{x}{2}}$. Using this new expression, what is your new inside function and your new outside function?

(c) Consider the function $e^{\sqrt{x}}$. In this case, what are the inside and outside functions in this case?

Activity 2.5.4 In this activity we will build the intuition for the chain rule using a real-world scenario and differential notation for derivatives. Consider the following scenario.

My neighborhood is being invaded! The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. How fast is the squirrel population growing per week?

(a) The scenario gives you information regarding the rate of growth of $s(a)$, the squirrel population as a function of acorn availability (measured in bushels). What is the current value of $\frac{ds}{da}$?

A 2

B 100

C 200

D 50

(b) The scenario gives you information regarding the rate of growth of $a(t)$, the acorn availability as a function of time (measured in weeks). What is the current value of $\frac{da}{dt}$?

A 2

B 100

C 200

D 50

(c) Given all the information provided, what is your best guess for the value of $\frac{ds}{dt}$, the rate at which the squirrel population is growing per week?

A 2

B 100

C 200

D 50

(d) Given your answers above, what is the relationship between $\frac{ds}{da}$, $\frac{da}{dt}$, $\frac{ds}{dt}$?

Definition 2.5.5 When looking at the composite function $f(g(x))$, we have that

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Using differential notation, if we consider the composite function $(v \circ u)(x)$, we have that

$$\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}$$

◇

Warning 2.5.6 It's important to consider the input of a function when taking the derivative! In fact, $f'(g(x))$ and $f'(x)$ are different functions. Also computing $\frac{dv}{dx}$ gives a different result than computing $\frac{dv}{du}$.

Activity 2.5.7

- (a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Notice that $f(g(x)) = -4x^2 + 4x + 4$. Which of the following is the derivative function of the composite function $f(g(x))$?

- A $-8x + 4$
- B $-4x$
- C $-2x$
- D 2

- (b) One of the options above is a formula for $f'(x) \cdot g'(x)$. Which one? Notice that this is not the same as the derivative of $f(g(x))$!

Activity 2.5.8 Consider the composite function $h(x) = \sqrt{e^x} = e^{\frac{x}{2}}$. For each of the two expressions, find the derivative using the chain rule. Which of the following expressions are equal to $h'(x)$? Select all!

- A $\frac{1}{2}(e^x)^{-\frac{1}{2}} \cdot e^x$
- B $\frac{1}{2}(e^x)^{\frac{3}{2}} \cdot e^x$
- C $\frac{1}{2}e^{-\frac{x}{2}}$
- D $e^{\frac{x}{2}} \cdot \frac{1}{2}$
- E $\frac{1}{2}\sqrt{e^x}$
- F $\sqrt{e^x} \cdot e^x$

Activity 2.5.9 Below you are given the graphs of two functions: $a(x)$ and $b(x)$. Use the graphs to compute the required values of the functions and of their derivatives, when possible (there are points where the derivative of these functions is not defined!). Notice that to compute the derivative at a point, you first want to find the general formula for the derivative as a function of x and then plug in the input you want to study.

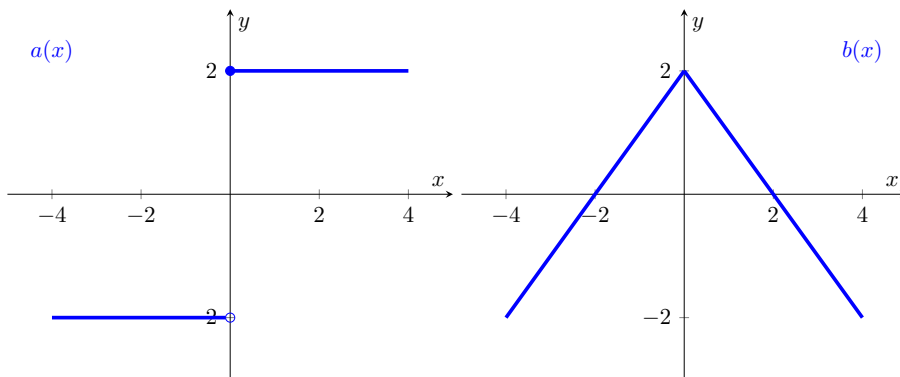


Figure 37 The graphs of $a(x)$ and $b(x)$

- (a) Notice that the derivative of $a \circ b$ is given by $a'(b(x)) \cdot b'(x)$, so the derivative of $a \circ b$ at $x = 4$ is given by the quantity $a'(b(4)) \cdot b'(4) = a'(-2) \cdot b'(4)$ because $b(4) = -2$. Using the graphs to find the slopes, what is the final value of the derivative of $a \circ b$ at $x = 4$?

A 0
 B -1
 C 1
 D -2
 E 2
 F The derivative does not exist at this point.

- (b) Which of the following values is equal to the derivative of $a \circ b$ at $x = 2$?

A 0
 B -1
 C 1
 D -2
 E 2
 F The derivative does not exist at this point.

- (c) Which of the following values is equal to the derivative of $b \circ a$ (different order!) at $x = -2$?

A 0
 B -1
 C 1
 D -2
 E 2
 F The derivative does not exist at this point.

Activity 2.5.10 In this activity you will study the derivative of $\cos^n(x)$ for different powers n .

- (a) Consider the function $\cos^2(x) = (\cos(x))^2$. Combining power and chain rule, what do you get if you differentiate $\cos^2(x)$?

A $-\cos^2(x) \sin(x)$
 B $-\cos^2(x) \sin(x)$
 C $2 \cos(x) \sin(x)$
 D $-2 \cos(x) \sin(x)$

- (b) Consider the function $\cos^3(x)$. Find its derivative.

- (c) Consider the function $\cos^n(x)$, for n any number. Find the general formula for its derivative.

Activity 2.5.11 In this activity you will study the derivative of $b^{\cos(x)}$ for different bases b .

- (a) Consider the function $e^{\cos(x)}$. Combining exponential and chain rule, what do you get if you differentiate $e^{\cos(x)}$?

A $e^{\cos(x)}$

- B $-e^{\cos(x)} \sin(x)$
 C $e^{-\sin(x)}$
 D $e^{\cos(x)} \sin(x)$

(b) Consider the function $2^{\cos(x)}$. Find its derivative.

(c) Consider the function $b^{\cos(x)}$, for b any positive number. Find the general formula for its derivative.

Remark 2.5.12 Remember that exponential and power functions obey very different differentiation rules. This behavior continues when we consider composite function. The composite power function $f(x)^3$ has derivative

$$3[f(x)]^2 \cdot f'(x)$$

but the composite exponential function $3^{f(x)}$ has derivative

$$\ln(3) 3^{f(x)} \cdot f'(x)$$

Activity 2.5.13 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (product, quotient, sum and difference, etc.) you are using in your work.

1.

$$f(x) = -(4x - 3e^x + 4)^3$$

2.

$$k(w) = 9 \cos\left(w^{\frac{7}{5}}\right)$$

3.

$$h(y) = -3 \sin(-5y^2 + 2y - 5)$$

4.

$$g(t) = 9 \cos(t)^{\frac{7}{5}}$$

Answer.

1.

$$f'(x) = 3(4x - 3e^x + 4)^2(3e^x - 4)$$

2.

$$k'(w) = -\frac{63}{5} w^{\frac{2}{5}} \sin\left(w^{\frac{7}{5}}\right)$$

3.

$$h'(y) = 6(5y - 1) \cos(-5y^2 + 2y - 5)$$

4.

$$g'(t) = -\frac{63}{5} \cos(t)^{\frac{2}{5}} \sin(t)$$

Activity 2.5.14 Notice that

$$\left(\frac{f(x)}{g(x)}\right) = (f(x) \cdot g(x)^{-1})$$

Use this observation, the chain rule, the product rule, and the power rule (and some algebra) to find in a new way the quotient rule!

Activity 2.5.15 Remember my neighborhood squirrel invasion? The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. Considering this information as pertaining to the moment $t = 0$, you are given the following possible model for the squirrel:

$$s(a(t)) = 2a(t) + 10 = 2[50 \sin(2t) + 60] + 10.$$

- (a) Check that the model satisfies the data $\frac{ds}{da} = 2$ and $\frac{da}{dt}|_{t=0} = 100$
- (b) Find the derivative function $\frac{ds}{dt}$ and check that $\frac{ds}{dt}|_{t=0} = 200$.
- (c) According to this model, what is the maximum and minimum squirrel population? What is the fastest rate of increase and decrease of the squirrel population? When will these extremal scenarios occur?

Activity 2.5.16 Suppose that a fish population at t months is approximated by

$$P(t) = 100 \cdot 4^{0.05t}$$

- (a) Find $P(10)$ and use units to explain what this value tells us about the population.
- (b) Find $P'(10)$ and use units to explain what this value tells us about the population. You can use the approximation $\ln(4) = 1.4$.

2.6 Differentiation strategy (DF6)

Learning Outcomes

- Compute derivatives using a combination of algebraic derivative rules.

Activity 2.6.1 Consider the functions defined below:

$$f(x) = \sin((x^2 + 3x) \cos(2x))$$

$$g(x) = \sin(x^2 + 3x) \cos(2x)$$

- (a) What do you notice that is similar about these two functions?
- (b) What do you notice that is different about these two functions?
- (c) Imagine that you are sorting functions into different categories based on how you would differentiate them. In what category (or categories) might these functions fall?

Remark 2.6.2 We've learned a lot of rules to help us take derivatives of functions, and these rules are applied based on the algebraic structure of the function. (Note that we often see that algebraic structure come up in the names of the rules. Sum/difference, power, product, and quotient rules for example!) To take a derivative, we need to examine how the function is built and then proceed accordingly. Up until now we have not seen many situations where these rules were mixed together too much. Now we turn our attention to those types of situations.

Below are some questions you might ask yourself as you take the derivative of a function, especially one where multiple rules might need to be used:

1. How is this function built algebraically? What kind of function is this?

What is the big picture?

2. Where do you start?
3. Is there an easier or more convenient way to write the function?
4. Are there products or quotients involved?
5. Is it a composition of two (or more) functions? If so, what are the outside and inside functions?
6. What derivatives rules will be needed along the way?

Activity 2.6.3 Consider the function $f(x) = x^3\sqrt{3-8x^2}$.

- (a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

- A Chain rule
- B Power rule
- C Product rule
- D Quotient rule
- E Sum and difference rule

- (b) What other rules would be needed along the way? Select all that apply.

- A Chain rule
- B Power rule
- C Product rule
- D Quotient rule
- E Sum and difference rule

- (c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.4 Consider the function $f(x) = \left(\frac{\ln x}{(3x-4)^3}\right)^5$.

- (a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

- A Chain rule
- B Power rule
- C Product rule
- D Quotient rule
- E Sum and difference rule

- (b) What other rules would be needed along the way? Select all that apply.

- A Chain rule
- B Power rule
- C Product rule
- D Quotient rule
- E Sum and difference rule

- (c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.5 Consider the function $f(x) = \sin(\cos(\tan(2x^3 - 1)))$.

- (a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A Chain rule
 B Power rule
 C Product rule
 D Quotient rule
 E Sum and difference rule

- (b) What other rules would be needed along the way? Select all that apply.

A Chain rule
 B Power rule
 C Product rule
 D Quotient rule
 E Sum and difference rule

- (c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.6 Consider the function $f(x) = \frac{x^2 e^x}{2x^3 - 5x + \sqrt{x}}$.

- (a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A Chain rule
 B Power rule
 C Product rule
 D Quotient rule
 E Sum and difference rule

- (b) What other rules would be needed along the way? Select all that apply.

A Chain rule
 B Power rule
 C Product rule
 D Quotient rule
 E Sum and difference rule

- (c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.7 Find the derivative of the following functions. For each, include an explanation of the steps involved that references the algebraic structure of the function.

(a) $f(x) = e^{5x}(x^2 + 7^x)^3$

(b) $f(x) = \left(\frac{3x+1}{2x^6-1}\right)^5$

(c) $f(x) = \sqrt{\cos(2x^2 + x)}$

(d) $f(x) = \tan(xe^x)$

2.7 Differentiating implicitly defined functions (DF7)

Learning Outcomes

- Compute derivatives of implicitly-defined functions.

Observation 2.7.1 Many of the equations that has been discussed so far fall under the category of an explicit equation. An explicit equation is one in which the relationship between x and y is given explicitly, such as $y = f(x)$. In this section we will examine when the relationship between x and y is given implicitly. An implicit expression is when the relationship has some equation like $f(x, y) = g(x, y)$ where both sides may depend on both x and y .

Observation 2.7.2 Note that if we are taking the derivative with respect to x , then

$$\frac{d}{dx}(f(x)) = f'(x).$$

However,

$$\frac{d}{dx}(g(y)) = g'(y) \frac{dy}{dx}.$$

Activity 2.7.3 For this activity we want to find the equation of a tangent line for a circle with radius 5 centered at the origin, $x^2 + y^2 = 25$, at the point $(-3, -4)$.

- (a) The derivative with respect to x for the equation of the circle is given by which expression.

A $2x + 2y \frac{dy}{dx} = 25$

B $2x + y \frac{dy}{dx} = 0$

C $2x + 2y \frac{dy}{dx} = 0$

D $2x + 2 \frac{dy}{dx} = 25$

- (b) Solving for $\frac{dy}{dx}$ gives?

A $\frac{dy}{dx} = \frac{25 - 2x}{2y}$

B $\frac{dy}{dx} = -\frac{2x}{y}$

C $\frac{dy}{dx} = -\frac{x}{y}$

D $\frac{dy}{dx} = \frac{25 - 2x}{2}$

- (c) Plug the point $(-3, -4)$ into the expression found above for the derivative to get the slope of the tangent line.

- (d) Use the value for the slope of the tangent line to obtain the equation of the tangent line.

Activity 2.7.4 The curve given in [Figure 38](#) is an example of an astroid. The equation of this astroid is $x^{2/3} + y^{2/3} = 3^{2/3}$. What is the derivative with respect x for this astroid? (Solve for $\frac{dy}{dx}$).

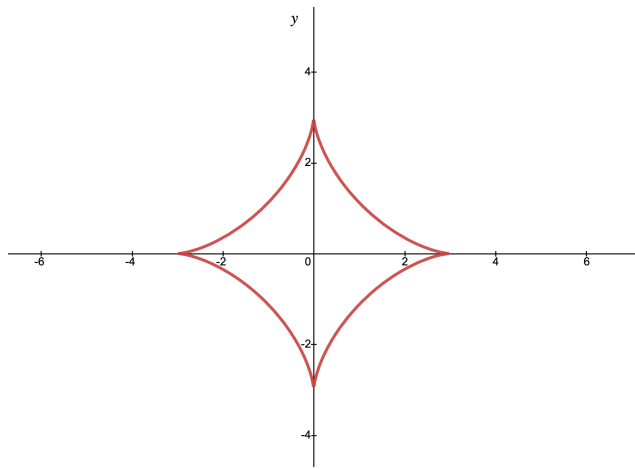


Figure 38 Graph of $x^{2/3} + y^{2/3} = 3^{2/3}$.

- A $\frac{dy}{dx} = \frac{x^{-1/3}}{y^{-1/3}}$
- B $\frac{dy}{dx} = \frac{y^{-1/3}}{x^{-1/3}}$
- C $\frac{dy}{dx} = \frac{3^{-1/3} - x^{-1/3}}{y^{-1/3}}$
- D $\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$

Activity 2.7.5 An example of a lemniscate is given in [Figure 39](#). The equation of this lemniscate is $(x^2 + y^2)^2 = x^2 - y^2$. What is the derivative with respect x for this lemniscate? (Solve for $\frac{dy}{dx}$).

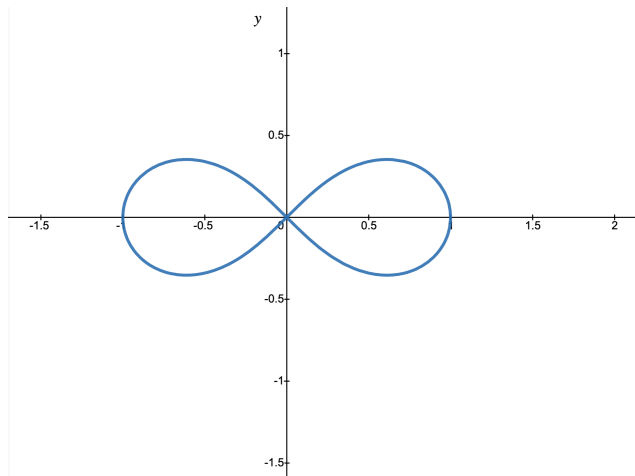


Figure 39 Graph of $(x^2 + y^2)^2 = x^2 - y^2$.

- A $\frac{dy}{dx} = \frac{x(1 - 2(x^2 + y^2))}{y + 2(x^2 + y^2)}$
- B $\frac{dy}{dx} = \frac{x(1 - 2(x^2 + y^2))}{y(1 + 2(x^2 + y^2))}$

$$\text{C } \frac{dy}{dx} = \frac{y(1 + 2(x^2 + y^2))}{x(1 - 2(x^2 + y^2))}$$

$$\text{D } \frac{dy}{dx} = \frac{y + 2(x^2 + y^2)}{x(1 - 2(x^2 + y^2))}$$

Activity 2.7.6 Explain how to use implicit differentiation to find $\frac{dy}{dx}$ for each of the following equations.

1.

$$-5x^5 - 5 \cos(y) = 3y^4 + 2$$

2.

$$-5ye^x + 5 \sin(x) = 0$$

Answer.

1.

$$\frac{dy}{dx} = -\frac{25x^4}{12y^3 - 5 \sin(y)}$$

2.

$$\frac{dy}{dx} = -(ye^x - \cos(x))e^{(-x)}$$

Activity 2.7.7 To take the derivative of some explicit equations you might need to make it an implicit equation. For this activity we will find the derivative of $y = x^x$. Make the equation an implicit equation by taking natural logarithm of both sides, this gives $\ln(y) = x \ln(x)$. Knowing this, what is $\frac{dy}{dx}$? This process to find a derivative is known as logarithmic differentiation.

$$\text{A } \frac{dy}{dx} = x^x(\ln(x) + 1)$$

$$\text{B } \frac{dy}{dx} = \frac{(\ln(x) + 1)}{x^x}$$

$$\text{C } \frac{dy}{dx} = x^x(\ln(x) + x)$$

$$\text{D } \frac{dy}{dx} = \frac{(\ln(x) + x)}{x^x}$$

Activity 2.7.8 Valerie is building a square chicken coop with side length x . Because she needs a separate place for the rooster, she needs to put fence around the square and also along the diagonal line shown. The fence costs \$20 per linear meter, and she has a budget of \$900.

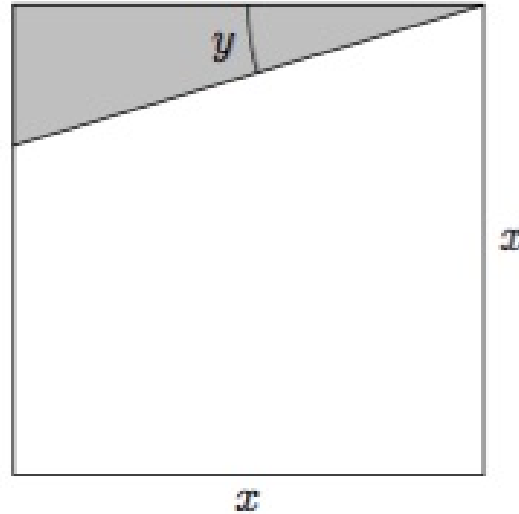


Figure 40 A diagram of the chicken coop.

- (a) Which of the following equations gives the relationship between x and y ?
Make sure you can explain why!

A $20x + \frac{80x}{\cos(y)} = 900$

B $80x + \frac{20x}{\cos(y)} = 900$

C $80x + \frac{20x}{\sin(y)} = 900$

D $20x + \frac{80x}{\sin(y)} = 900$

- (b) If Valerie builds the coop with $y = \pi/3$ (and wants to use her whole budget), find what side length x she needs to use.
- (c) Find the slope of the curve at this point and interpret what this value tells Valerie.

2.8 Differentiating inverse functions (DF8)

Learning Outcomes

- Compute derivatives of inverse functions.

Remark 2.8.1 Let f^{-1} be the inverse function of f . The relationship between a function and its inverse can be expressed with the equation

$$f(f^{-1}(x)) = x.$$

Activity 2.8.2 In this activity you will use implicit differentiation and the

inverse function identity above to find the derivative of $y = \ln(x)$.

- (a) Suppose that $y = \ln(x)$. Then we have that

$$e^y = x.$$

Using implicit implicit differentiation, what do you get?

- A $\frac{dy}{dx} = \frac{x}{y}$
- B $\frac{dy}{dx} = \frac{1}{e^x}$
- C $\frac{dy}{dx} = \frac{x}{e^y}$
- D $\frac{dy}{dx} = \frac{1}{e^y}$

- (b) Notice that we started with the relationship $e^y = x$. Use this to simplify $\frac{dy}{dx}$. You should get that when $y = \ln(x)$ we have that $\frac{dy}{dx} = \frac{1}{x} \dots$ as expected!

Activity 2.8.3 In this activity we will try to find a general formula for the derivative of the inverse function. Let g be the inverse function of f . We have also used the notation f^{-1} before, but for the purpose of this problem, let's use g to avoid too many exponents. We can express the relationship " g is the inverse of f " with the equation

$$f(g(x)) = x.$$

- (a) Looking at the equation $f(g(x)) = x$, what is the derivative with respect to x of the right hand side of the equation?

- A x
- B 1
- C 0
- D x^2

- (b) Looking at the equation $f(g(x)) = x$, what is the derivative with respect to x of the left hand side of the equation?

- A $f'(g(x))$
- B $f'(g'(x))$
- C $f(g(x)) g'(x)$
- D $f'(g(x)) g'(x)$

- (c) Setting the two sides of the equation equal after differentiating, we can solve for $g'(x)$. What do you get?

- A $g'(x) = \frac{x}{f(g(x))}$
- B $g'(x) = \frac{x}{f'(g(x))}$
- C $g'(x) = \frac{1}{f(g(x))}$
- D $g'(x) = \frac{1}{f'(g(x))}$

Observation 2.8.4 In the above activity you should have found that the derivative of $g = f^{-1}$, the inverse function of f , is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Notice that because of the chain rule, the derivative of f has to be evaluated at $f^{-1}(x)$

Activity 2.8.5 In this problem you will apply the general formula for the derivative of the inverse function to find the values of some derivatives graphically.

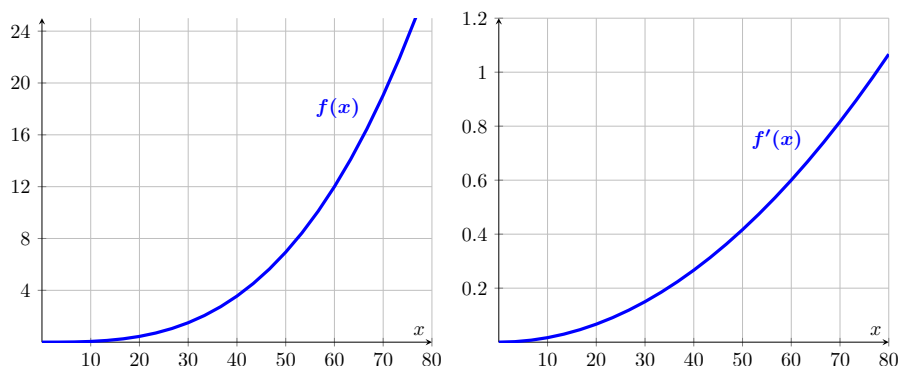


Figure 41 The graphs of $f(x)$ and $f'(x)$.

- (a) The derivative of the inverse function at $x = 12$ given by $(f^{-1})'(12) = \frac{1}{f'(f^{-1}(12))}$. Using the graphs, what is your best approximation for this quantity?

- A $(f^{-1})'(12) \approx \frac{1}{0.2} = 5$
- B $(f^{-1})'(12) \approx \frac{1}{0.6} \approx 1.67$
- C $(f^{-1})'(12) \approx \frac{1}{0.4} = 2.5$
- D $(f^{-1})'(12) \approx \frac{1}{0.1} = 10$

- (b) What is your best estimate for $(f^{-1})'(6)$?

- A $(f^{-1})'(6) \approx \frac{1}{0.2} = 5$
- B $(f^{-1})'(6) \approx \frac{1}{0.6} \approx 1.67$
- C $(f^{-1})'(6) \approx \frac{1}{0.4} = 2.5$
- D $(f^{-1})'(6) \approx \frac{1}{0.1} = 10$

Activity 2.8.6 Use the general formula for the derivative of the inverse function to find...

- (a) The derivative of the inverse function of $f(x) = e^x$... This should match what you found before!

- (b) The derivative of the inverse function of $f(x) = \frac{1}{x}$... This should match something you have seen before! See if you can explain why this holds.

Definition 2.8.7 We can only invert the function $y = \sin(x)$ on the restricted domain $[-\pi/2, \pi/2]$ (Why?). On this domain we define arcsine by the condition

$$x = \sin^{-1}(y) \quad \text{when} \quad y = \sin(x).$$

◇

Activity 2.8.8 In this activity you will study the arcsine function.

- (a) Consider the values of $y = \sin(x)$ given in the table below for an angle x between $-\pi/2$ and $\pi/2$. Fill in the corresponding values for the inverse function arcsine $x = \sin^{-1}(y)$. In other words, you need to provide the angle whose sine is given. You can use the unit circle to help you remember which angles yield the given values of sine. The first entry is provided: a sine value of -1 corresponds to the angle $-\pi/2$.

Table 42

$y = \sin(x)$	-1	$-\sqrt{3}/2$	$-1/2$	0	$1/2$	$\sqrt{3}/2$	1
$x = \sin^{-1}(y)$	$-\pi/2$						

- (b) From the graph of $y = \sin(x)$ and your table above, graph the arcsine function $y = \sin^{-1}(x)$

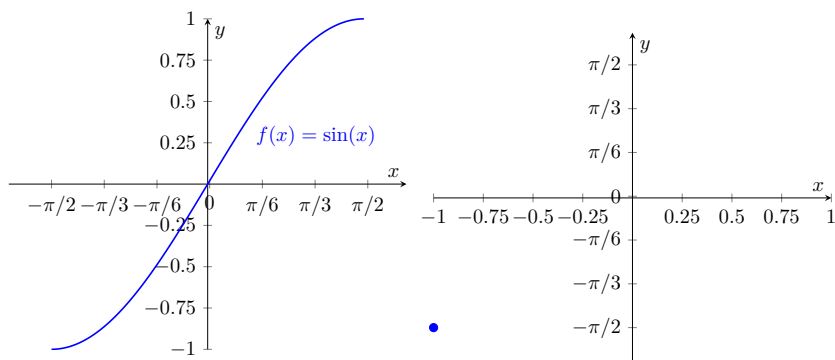


Figure 43 The graphs of $\sin(x)$ and one point on $\sin^{-1}(x)$.

- (c) Let's now work with the function arccosine. Again, we need to restrict the domain of cosine to be able to invert the function (Why?). The convention is to restrict cosine to the domain $[0, \pi]$ in order to define arccosine. Given this restriction, what are the domain and range of arccosine? Create a table of values and graph the function arccosine.
- (d) Let's now work with the function arctangent. Again, we need to restrict the domain of tangent to be able to invert the function (Why?). The convention is to restrict tangent to the domain $(-\pi/2, \pi/2)$ in order to define arctangent. Given this restriction, what are the domain and range of arctangent? Create a table of values and graph the function arctangent.

Activity 2.8.9 In this activity you will find a formula for the derivative of arctangent.

- (a) Differentiate the implicit equation $\tan(y) = x$, what do you get for $\frac{dy}{dx}$?

A $\frac{dy}{dx} = \frac{x}{\tan(y)}$

B $\frac{dy}{dx} = \frac{1}{\tan(y)}$

C $\frac{dy}{dx} = \frac{x}{\sec^2(y)}$

D $\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

(b) For what function $y = g(x)$ have you found the derivative $\frac{dy}{dx}$?

(c) We want to rewrite $\frac{dy}{dx}$ only in terms of x . Notice that

$$\tan^2(y) = \frac{\sin^2(y)}{\cos^2(y)} = \frac{1 - \cos^2(y)}{\cos^2(y)}.$$

Multiplying out by the denominator, isolating, and solving for $\cos^2(y)$, we get that

A $\cos^2(y) = \frac{\tan^2(y)}{\cos^2(y)}$

B $\cos^2(y) = \frac{1}{\tan^2(y) + 1}$

C $\cos^2(y) = \frac{1 - \cos^2(y)}{\tan^2(y)}$

D $\cos^2(y) = \frac{1}{\tan^2(y) - 1}$

(d) Finally, rewrite $\frac{dy}{dx}$ as $\frac{dy}{dx} = \cos^2(y)$ and use the fact that $\tan(y) = x$ to get a nice formula for the derivative of the arctangent function of x .

Remark 2.8.10 Consider the functions $y = \tan^{-1}(x)$. Using your algebra above, you should have found that

$$\frac{d}{dx} \left(\tan^{-1}(x) \right) = \frac{1}{1 + x^2}.$$

In a similar fashion, one can find that

$$\frac{d}{dx} \left(\sin^{-1}(x) \right) = \frac{1}{\sqrt{1 - x^2}}, \quad \frac{d}{dx} \left(\cos^{-1}(x) \right) = -\frac{1}{\sqrt{1 - x^2}}.$$

Activity 2.8.11 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (product, quotient, sum and difference, etc.) you are using in your work.

(a)

$$k(t) = \frac{\arctan(-4t)}{\ln(-4t)}$$

(b)

$$j(u) = -5 \arcsin(u) \log(u^6 + 2)$$

(c)

$$n(x) = \ln(-\arcsin(x) + 4 \arctan(x))$$

Answer.

1.

$$k'(t) = -\frac{\arctan(-4t)}{t \log(-4t)^2} - \frac{4}{(16t^2 + 1) \log(-4t)}$$

2.

$$j'(u) = -\frac{30u^5 \arcsin(u)}{u^6 + 2} - \frac{5 \log(u^6 + 2)}{\sqrt{-u^2 + 1}}$$

3.

$$n'(x) = \frac{\frac{1}{\sqrt{-x^2+1}} - \frac{4}{x^2+1}}{\arcsin(x) - 4 \arctan(x)}$$

Activity 2.8.12

- Find the equation of the tangent line to $y = \tan^{-1}(x)$ at $x = 0$. Draw the function and the tangent on Desmos to check your work!
- Find the equation of the tangent line to $y = \sin^{-1}(x)$ at $x = 0.5$. Draw the function and the tangent on Desmos to check your work!
- Find the equation of the tangent line to $y = \cos^{-1}(x)$ at $x = -0.5$. Draw the function and the tangent on Desmos to check your work!

Activity 2.8.13 Let $y = f(v)$ be the gas consumption (in ml/km) of a car at velocity v (in km/hr). We use the notation: ml for milliliters, km for kilometers, and hr for hours. Also consider the function $g(y)$, where $v = g(y)$ is the function that gives the velocity v (in km/hr) when the gas consumption is y (in ml/km). You are given the graphs of $f(v), f'(v)$ below.

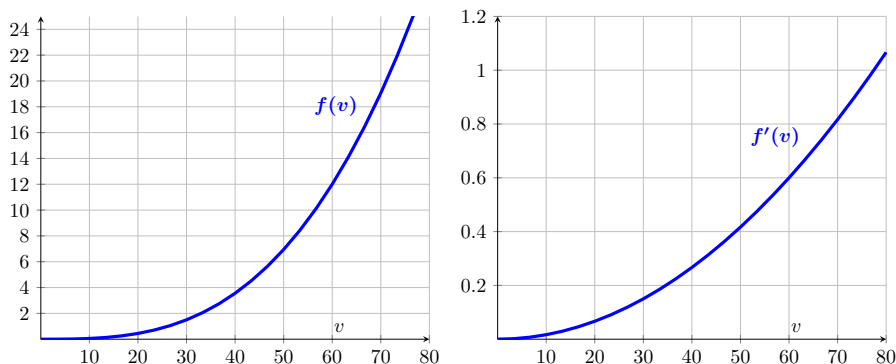


Figure 44 The graphs of $f(v), f'(v)$.

- Estimate $f^{-1}(6)$. What does this value mean in the context of the problem?
- Using your answer from part (a), estimate the derivative of the inverse function of $f(x)$ at $x = 6$ i.e., compute $(f^{-1})'(6)$.
- What is the relationship between the functions f and g ?
- Use the relationship between the functions f and g to estimate $g(12)$ and $g'(12)$. What do these values mean in the context of the problem?

Chapter 3

Applications of Derivatives (AD)

Learning Outcomes

How can we use derivatives to solve application questions?

By the end of this chapter, you should be able to...

1. Use derivatives to answer questions about rates of change and equations of tangents.
2. Use tangent lines to approximate functions.
3. Model and analyze scenarios using related rates.
4. Use the Extreme Value Theorem to find the absolute maximum and minimum values of a continuous function on a closed interval.
5. Determine where a differentiable function is increasing and decreasing and classify the critical points as local extrema.
6. Determine the intervals of concavity of a twice differentiable function and find all of its points of inflection.
7. Sketch the graph of a differentiable function whose derivatives satisfy given criteria.
8. Apply optimization techniques to solve various problems.
9. Compute the values of indeterminate limits using L'Hopital's Rule.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Find the derivative of a function using methods from [Section 2.3](#), [Section 2.4](#), [Section 2.5](#), and [Section 2.6](#).
- b Find the second derivative of a function. ([Khan Academy](#)¹)

¹<https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-2-new/ab-3-6/a/second-derivatives-review>

- c Determine whether values are in the domain of a function ([Khan Academy²](#))
- d Find relative and absolute extrema given the graph of a function ([Khan Academy³](#))
- e Find the intervals where a graph is increasing or decreasing ([Khan Academy⁴](#))
- f Determine the zeros of a polynomial ([Khan Academy⁵](#))
- g Modeling with multiple variables ([Khan Academy⁶](#))

3.1 Tangents, motion, and marginals (AD1)

Learning Outcomes

- Use derivatives to answer questions about rates of change and equations of tangents.

Definition 3.1.1 The tangent line of a function $f(x)$ at $x = a$ is the linear function $L(x)$

$$L(x) = f'(a)(x - a) + f(a).$$

Notice that this is the linear function with slope $f'(a)$ and passing through $(a, f(a))$ in point-slope form. \diamond

Activity 3.1.2 For the following functions, find the required tangent line.

- (a) Find the tangent line to $f(x) = \ln(x)$ at $x = 1$

- A $L(x) = x$
- B $L(x) = x + 1$
- C $L(x) = x - 1$
- D $L(x) = -x + 1$

- (b) Find the tangent line to $f(x) = e^x$ at $x = 0$

- A $L(x) = x$
- B $L(x) = x + 1$
- C $L(x) = x - 1$
- D $L(x) = -x + 1$

²<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:determining-the-domain-of-a-function/v/determine-values-domain>

³<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:maximum-and-minimum-points/v/identifying-relative-and-absolute-maxima-and-minima>

⁴<https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:intervals-where-a-function-is-positive-negative-increasing-or-decreasing/v/increasing-decreasing-positive-and-negative-intervals>

⁵<https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:poly-graphs/x2ec2f6f830c9fb89:poly-zeros/v/polynomial-zeros-introduction>

⁶<https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:modeling-with-multiple-variables/v/modeling-with-multiple-variables-pancakes>

Activity 3.1.3 Let $f(x) = -2x^4 + 4x^2 - x + 5$. Find an equation of the line tangent to the graph at the point $(-2, -9)$.

Definition 3.1.4 If a particle has position function $s = f(t)$, where t is measured in seconds and s is measured in meters, then

- $v(t) = f'(t)$ is the velocity of the particle
- $a(t) = f''(t)$ is the acceleration of the particle

◇

Activity 3.1.5 A particle moves on a vertical line so that its y coordinate at time t is

$$y = t^3 - 9t^2 + 24t + 3$$

for $t \geq 0$. Here t is measured in seconds and y is measured in feet.

- (a) Find the velocity and acceleration functions.
- (b) Sketch graphs of the position, velocity and acceleration functions for $0 \leq t \leq 5$.
- (c) When is the particle moving upward and when is it moving downward?
- (d) When is the particle's velocity increasing?
- (e) Find the total distance that the particle travels in the time interval $0 \leq t \leq 5$. Careful: the total distance is not the same as the displacement (the change in position)! Compute how much the particle moves up and add it to how much the particle moves down.

Activity 3.1.6 Suppose the position of an object in miles is modeled by the following function:

$$s(t) = -t^3 - 3t^2 - 5t + 8.$$

Explain and demonstrate how to find the object's position, velocity, and acceleration at 2 seconds. Use appropriate units for each.

Answer.

1. Position: -22 miles
2. Velocity: -29 miles per second
3. Acceleration: -18 miles per s^2

Observation 3.1.7 In some cases, we want to also consider the speed of a particle, which is the absolute value of the velocity. In symbols $|v(t)| = |f'(t)|$ is the speed of the particle. A particle is speeding up when the speed is increasing.

Activity 3.1.8 Consider the speed of a particle. What is the behavior of the speed in relation to velocity and acceleration?

- A The speed is always positive and it is increasing when the velocity and the acceleration have the same sign.
- B The speed is positive when the velocity is positive and negative when the velocity is negative.
- C The speed is positive when the acceleration is positive and negative when the acceleration is negative.
- D The speed is always positive and it is increasing when the velocity and the acceleration have opposite signs.

Definition 3.1.9 In a parametric motion on a curve C given by $x = f(t)$ and $y = g(t)$ we have that

- $\frac{dx}{dt} = f'(t)$ is the rate of change of $f(t)$, one component of the slope (or velocity)
- $\frac{dy}{dt} = g'(t)$ is the rate of change of $g(t)$, one component of the slope (or velocity)
- $\frac{dy}{dx}$ is the actual slope (or velocity) of the object and by the chain rule $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

◇

Activity 3.1.10 An airplane is cruising at a fixed height and traveling in a pattern described by the parametric equations

$$x = 4t, \quad y = -t^4 + 4t - 1,$$

where x, y have units of miles, and t is in hours.

- Find the slope of the curve.
- What is the slope of the curve at $(0, -1)$.
- Write the equation of the tangent line to the curve at $(0, -1)$.

Definition 3.1.11 If $C(x)$ is the cost of producing x items and $R(x)$ is the revenue from selling x items, then $P(x) = R(x) - C(x)$ is the profit. We can study their derivatives, the marginals

- $C'(x)$ is the marginal cost
- $R'(x)$ is the marginal revenue
- $P'(x) = R'(x) - C'(x)$ is the marginal profit

◇

Activity 3.1.12 The manager of a computer shop has to decide how many computers to store in the back of the shop. If she stores a large number, she has to pay extra in storage costs. If she stores only a small number, she will have to reorder more often, which will involve additional handling costs. She has found that if she stores x computers, the storage and handling costs will be C dollars, where

$$C(x) = 10x^3 - 900x^2 + 16000x + 210000$$

- What is the fixed cost of the computer shop, the cost when no computers are in storage? In practical terms this may account for rent and utilities expenses.
- Find the marginal cost
- Now suppose that x computers give revenue $R(x) = 1000x$. What is the marginal revenue? What is the real world interpretation of your finding?
- Find a formula for the profit function $P(x)$ and find the marginal profit using the marginal revenue and the marginal cost.

Activity 3.1.13 The quantity q of skateboards sold depends on the selling

price p of a skateboard, so we write $q = f(p)$. You are given that

$$f(140) = 15000, \quad f'(140) = -100$$

- (a) What does the data above tell you about the change in demand for skateboards depending on the selling price?
- (b) The total revenue earned by the sale of skateboards $R(p)$ is given by , $R(p) = q(p) \cdot p$. Explain why.
- (c) Find

$$\left. \frac{dR}{dp} \right|_{p=140}$$

- (d) What is the sign of the quantity above? What do you think would happen to the revenue if the price was changed from \$140 to \$141?

Activity 3.1.14 A gizmo is sold for \$63 per item. Suppose that the number of items produced is equal to the number of items sold and that the cost (in dollars) of producing x gizmos is given by the following function:

$$C(x) = 4x^3 + 10x^2 + 7x + 4.$$

Explain and demonstrate how to find the marginal revenue, the marginal cost, and the marginal profit in this situation.

Answer. The marginal cost is $12x^2 + 20x + 7$, the marginal revenue is 63, and the marginal profit is $-12x^2 - 20x + 56$.

Definition 3.1.15 A cooling object has temperature modelled by

$$y = ae^{-kt} + c,$$

where a, c, k are positive constants determined by the local conditions. ◇

Activity 3.1.16 Consider a cup of coffee initially at 100°F. The said cup of coffee was forgotten this morning in my living room where the thermostat is set at 72°F. I also observed that when I initially prepared the coffee, the temperature was decreasing at a rate of 3.8 degrees per minute.

- (a) In the long run, what temperature do you expect the coffee to tend to? Use this information in the model $y = ae^{-kt} + c$ to determine the value of c .
- (b) Using the initial temperature of the coffee and your value of c , find the value of a in the model $y = ae^{-kt} + c$.
- (c) The scenario also gives you information about the value of the rate of change at $t = 0$. Use this additional information to determine the model $y = ae^{-kt} + c$ completely.
- (d) You should find that the temperature model for this coffee cup is $y = 72 + 38e^{-0.1t}$. Explain how the values of each parameter connects to the information given.

3.2 Linear approximation (AD2)

Learning Outcomes

- Use tangent lines to approximate functions.

Definition 3.2.1 The tangent line approximation (or linear approximation, or linearization) of a function $f(x)$ at $x = a$ is the tangent line $L(x)$ at $x = a$. In formulas, $L(x)$ is the linear function

$$L(x) = f'(a)(x - a) + f(a).$$

Notice that this is obtained by writing the tangent line to $f(x)$ at $(a, f(a))$ in point-slope form and calling the resulting linear function $L(x)$. \diamond

Activity 3.2.2 Without using a calculator, we will use calculus to approximate $\ln(1.1)$.

- (a) Find the equation of the tangent line to $\ln(x)$ at $x = 1$. This will be your linear approximation $L(x)$. What do you get for $L(x)$?

- A $L(x) = x$
- B $L(x) = x + 1$
- C $L(x) = x - 1$
- D $L(x) = -x + 1$

- (b) As 1.1 is close to 1, we can use $L(1.1)$ to approximate $\ln(1.1)$. What approximation do you get?

- A $\ln(1.1) \approx 1.1$
- B $\ln(1.1) \approx 2.1$
- C $\ln(1.1) \approx 0.1$
- D $\ln(1.1) \approx -0.1$

- (c) Sketch the tangent line $L(x)$ on the same plane as the graph of $\ln(x)$. What do you notice? What's the relationship between the two graphs?

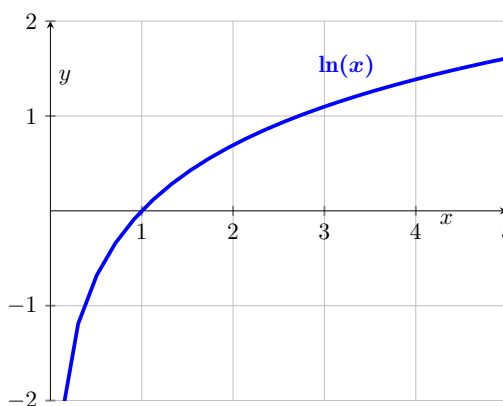


Figure 45 The graph of $\ln(x)$

Activity 3.2.3 Using the equation of the tangent line to the graph of $\ln(x)$ at $x = 1$ and the shape of this graph, you can show that for all values of x , we

have that $\ln(x) \leq x - 1$.

- (a) Compute the second derivative of $\ln(x)$. What do you notice about the sign of the second derivative of $\ln(x)$? What does this tell you about the shape of the graph?
- (b) Conclude that because the graph of $\ln(x)$ has a certain shape, the graph will bend below the tangent line and so the tangent line approximation $L(x) = x - 1$ is always greater than $\ln(x)$.

Activity 3.2.4 In this activity you will approximate power functions near $x = 1$.

- (a) Find the tangent line approximation to x^2 at $x = 1$.

A $L(x) = 2x$

B $L(x) = 2x + 1$

C $L(x) = 2x - 1$

D $L(x) = -2x + 1$

- (b) Show that for any constant k , the tangent line approximation to x^k at $x = 1$ is $L(x) = kx - k + 1$.
- (c) Someone claims that the square root of 1.1 is about 1.05. Use the linear approximation to check this estimate. Do you think this estimate is about right? Why or why not?
- (d) Is the actual value $\sqrt{1.1}$ above or below 1.05? What feature of the graph of \sqrt{x} makes this an over or under estimate?

Remark 3.2.5 If a function $f(x)$ is *concave up* around $x = a$, then the function is turning upwards from its tangent line. So when we use a linear approximation, the value of the approximation will be below the actual value of the function and the approximation is an underestimate. If a function $f(x)$ is *concave down* around $x = a$, then the function is turning downwards from its tangent line. So when we use a linear approximation, the value of the approximation will be above the actual value of the function and the approximation is an overestimate.

Activity 3.2.6 Suppose f has a continuous positive second derivative and Δx is a small increment in x (like h in the limit definition of the derivative). Which one is larger...

$$f(1 + \Delta x) \quad \text{or} \quad f'(1)\Delta x + f(1) \quad ?$$

Activity 3.2.7 A certain function $p(x)$ satisfies $p(7) = 49$ and $p'(7) = 8$.

1. Explain how to find the local linearization $L(x)$ of $p(x)$ at 7.
2. Explain how to estimate the value of $p(6.951)$.
3. Suppose that $p'(7) = 0$ and you know that $p''(x) < 0$ for $x < 7$. Explain how to determine if your estimate of $p(6.951)$ is too large or too small.
4. Suppose that $p''(x) > 0$ for $x > 7$. Use this fact and the additional information above to sketch an accurate graph of $y = p(x)$ near $x = 7$.

Activity 3.2.8 Let's find the quadratic polynomial

$$q(x) = ax^2 + bx + c$$

where a, b, c are parameters to be determined so that $q(x)$ best approximates

the graph of $f(x) = \ln(x)$ at $x = 1$.

- (a) We want to choose a, b, c such that our quadratic polynomial resembles $f(x)$ at $x = 1$. First thing, we want $f(1) = q(1)$. What equation in a, b, c does this condition give you?

A $a + b + c = 1$

B $a + b + c = 0$

C $c = 0$

D $c = 1$

- (b) We also want $f'(1) = q'(1)$. What equation in a, b, c does this condition give you?

- (c) Finally, we want $f''(1) = q''(1)$. What equation in a, b, c does this condition give you?

- (d) Find a solution to this system of linear equations! Your answer will give you values of a, b, c that can be used to draw a quadratic approximating the natural logarithm. You can check your answer on Desmos <https://www.desmos.com/calculator/bad2xrwml>

Observation 3.2.9 A linear approximation $L(x)$ to $f(x)$ at $x = a$ is a linear function with

$$L(a) = f(a), \quad L'(a) = f'(a).$$

A quadratic approximation $Q(x)$ to $f(x)$ at $x = a$ is a quadratic function with

$$Q(a) = f(a), \quad Q'(a) = f'(a), \quad Q''(a) = f''(a).$$

Activity 3.2.10 Find the linear approximation $L(x)$ of $\cos(x)$ at $x = 0$. Then find the quadratic approximation $Q(x)$ of $\cos(x)$ at $x = 0$. Graph both and compare the two approximations!

Activity 3.2.11 Suppose the function $p(x)$ satisfies $p(-2) = 5$, $p'(-2) = 1$, and $p''(x) < 0$ for x values nearby -2 .

1. Explain and demonstrate how to find the linearization $L(x)$ of $p(x)$ at $x = -2$.
2. Explain and demonstrate how to estimate the value of $p(-2.03)$ using this linearization.
3. Explain why your estimate of $p(-2.03)$ is greater than or less than the actual value.
4. Sketch a possible graph of $p(x)$ and its linearization $L(x)$ nearby $x = -2$ to illustrate your findings.

Answer.

1. $L(x) = x + 7$
2. $p(-2.03) \approx 4.97$
3. The estimate is greater than the actual value.

3.3 Related rates (AD3)

Learning Outcomes

- Model and analyze scenarios using related rates.

Remark 3.3.1 In most of our applications of the derivative so far, we have been interested in the instantaneous rate at which one variable, say y , changes with respect to another, say x , leading us to compute and interpret $\frac{dy}{dx}$. We next consider situations where several variable quantities are related, but where each quantity is implicitly a function of time, which will be represented by the variable t . Through knowing how the quantities are related, we will be interested in determining how their respective rates of change with respect to time are related.

Example 3.3.2 In a sense, the chain rule is our first example of related rates: recall that when y is a function of x , which in turn is a function of t , we are considering the composite function $y(x(t))$, and we learned that by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Notice that the chain rule gives a relationship between three rates: $\frac{dy}{dt}, \frac{dy}{dx}, \frac{dx}{dt}$. \square

Activity 3.3.3 Remember the squirrels taking over my neighborhood? The population s grows based on acorn availability a , at a rate of 2 squirrels per bushel. The acorn availability a is currently growing at a rate of 100 bushels per week. What is $\frac{ds}{dt}$ in this situation?

- A 2
- B 100
- C 200
- D Not enough information

Example 3.3.4 In a more serious example, suppose that air is being pumped into a spherical balloon so that its volume increases at a constant rate of 20 cubic inches per second. Since the balloon's volume and radius are related, by knowing how fast the volume is changing, we ought to be able to discover how fast the radius is changing. We are interested in questions such as: can we determine how fast the radius of the balloon is increasing at the moment the balloon's diameter is 12 inches? \square

Activity 3.3.5 A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why? Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change. Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note as well that in the setting of this problem, *both* V and r are changing as time t changes, and thus both V and r may be viewed as *implicit* functions of t , with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$. Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$. At this point in the problem, by differentiating we have “related the rates” of change of V and r . Recall that we are given in the problem that the balloon is being inflated at

a constant *rate* of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why? From part (c), we know the value of $\frac{dV}{dt}$ at every value of t . Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$? How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

Remark 3.3.6 In problems where two or more quantities can be related to one another, and all of the variables involved are implicitly functions of time, t , we are often interested in how their rates are related; we call these *related rates* problems. Once we have an equation establishing the relationship among the variables, we differentiate implicitly with respect to time to find connections among the rates of change.

Remark 3.3.7 A guide to solving related rate problems.

1. *Picture it!* Draw a diagram to represent the situation.
2. *What do we know?* Make a list of all quantities you are given in the problem, choosing clearly defined variable names for them. If a quantity is changing (a rate), then it should be labeled as a derivative.
3. *What do we want to know?* Make a list of all quantities to be determined. Again, choose clearly defined variable names.
4. *How are the variables related to each other?* Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.
5. *How are the rates related?* Differentiate implicitly with respect to time. This will give an equation that relates the rates together.
6. *Time to evaluate!* Evaluate the derivatives and variables at the information relevant to the instant at which a certain rate of change is sought.

Remark 3.3.8 Volume formulas.

- A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$
- A vertical cylinder of radius r and height h has volume $V = \pi r^2 h$
- A cone of radius r and height h has volume $V = \frac{\pi}{3} r^2 h$

Activity 3.3.9 A vertical cylindrical water tank has a radius of 1 meter. If water is pumped out at a rate of 3 cubic meters per minute, at what rate will the water level drop?

- (a) Draw a figure to represent the situation. Introduce variables that measure the radius of the water's surface, the water's depth in the tank, and the volume of the water. Label your diagram.
- (b) What information about rates of changes does the problem give you?
- (c) Recall that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$. What is the related rates equation in the context of the vertical cylindrical tank? What derivative rules did you use to find this equation?

$$\text{A } \frac{dV}{dt} = \pi 2r \frac{dh}{dt}$$

- B $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$
 C $\frac{dV}{dt} = \pi \frac{dr}{dt} h$
 D $\frac{dV}{dt} = \pi 2r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$
 E $\frac{dV}{dt} = \pi 2rh + \pi r^2$

(d) Which variable(s) have a constant value in this situation? Why?

- A The variable measuring the radius of the water's surface
 B The variable measuring the depth of the water
 C The variable measuring the volume of the water

(e) Which variable(s) have a constant rate of change in this situation? Why?

- A The variable measuring the radius of the water's surface
 B The variable measuring the depth of the water
 C The variable measuring the volume of the water

(f) Using your finding above, find at what rate the water level is dropping.

(g) If the full tank contains 12 cubic centimeters of water, how long does it take to empty the tank?

(h) Confirm your finding in the previous part by finding the initial water level for 12 cubic centimeters of water and determine how long it takes for the water level to reach 0.

Activity 3.3.10 A water tank has the shape of an inverted circular cone (the cone points downwards) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

- (a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.
- (b) Say that r is the radius and h the depth of the water at a given time, t . What proportional equation relates the radius and height of the water, and why?
- (c) Determine an equation that relates the volume of water in the tank at time t to the depth h of the water at that time.
- (d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time t .
- (e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.
- (f) When is the water rising most rapidly?
- A $h = 3$
 B $h = 4$

C $h = 5$

D The water level rises at a constant rate

Remark 3.3.11 Recall that in a right triangle with sides a, b and hypotenuse c we have the relationship

$$a^2 + b^2 = c^2,$$

also known in the western world as the Pythagorean theorem (even though this result was well known well before his time by other civilizations).

Activity 3.3.12 Notice that by differentiating each variable in the equation above with respect to t , we get a relationship between $\frac{da}{dt}$, $\frac{db}{dt}$, $\frac{dc}{dt}$. Find this related rates equation!

Activity 3.3.13 A rectangle has one side of 8 cm. How fast is the diagonal of the rectangle changing at the instant when the other side is 6 cm and increasing at a rate of 3 cm per minute?

Activity 3.3.14 A 10 m ladder leans against a vertical wall and the bottom of the ladder slides away at a rate of 0.5 m/sec. When is the top of the ladder sliding the fastest down the wall?

A When the bottom of the ladder is 4 meters from the wall.

B When the bottom of the ladder is 8 meters from the wall.

C The top of the ladder is sliding down at a constant rate.

Activity 3.3.15 Suppose a car was 75 miles east of a town, traveling west at 75 mph. A second car was 120 miles north of the same town, traveling south at 70 mph. At this exact moment, how fast is the distance between the cars changing?

Answer. $\frac{-14025}{15\sqrt{89}} \approx -99.11$ mph.

3.4 Extreme values (AD4)

Learning Outcomes

- Use the Extreme Value Theorem to find the absolute maximum and minimum values of a continuous function on a closed interval.

Remark 3.4.1 In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications—say to identify a point at which maximum profit or minimum cost occurs—or in theory to characterize the behavior of a function or a family of related functions.

Example 3.4.2 Consider the simple and familiar example of a parabolic function such as $s(t) = -16t^2 + 32t + 48$ that represents the height of an object tossed vertically: its maximum value occurs at the vertex of the parabola and represents the greatest height the object reaches. This maximum value is an especially important point on the graph, the point at which the curve changes from increasing to decreasing.

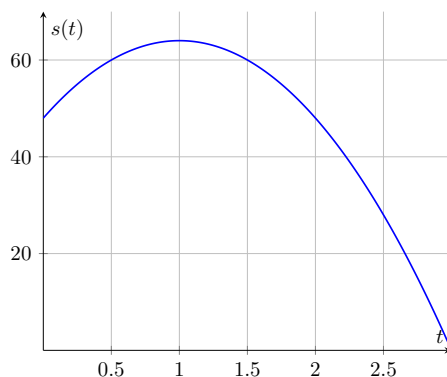


Figure 46 The graph of $s(t) = -16t^2 + 32t + 48$

□

Definition 3.4.3 Given a function f , we say that $f(c)$ is a **global** or **absolute maximum** of f provided that $f(c) \geq f(x)$ for all x in the domain of f , and similarly we call $f(c)$ a **global** or **absolute minimum** of f whenever $f(c) \leq f(x)$ for all x in the domain of f . The absolute maxima and minima are also called the absolute extrema or **extreme values** of the function. ◇

Activity 3.4.4 According to [Definition 3.4.3](#), which of the following statements best describes the absolute extrema of the function in [Figure 46](#)?

- A The absolute maximum is $t = 1$, because this is where the function goes from increasing to decreasing.
- B The absolute maximum is $s(1) = 64$, because $s(t) \leq 64$ for every other input t .
- C The graph has two absolute minima at the endpoints because the endpoints must be absolute extrema.
- D The graph has no absolute minimum.

Observation 3.4.5 From [Activity 3.4.4](#) we notice that there are some issues when determining the absolute minimum and maximum values of a function from its graph alone. The Extreme Value Theorem will guarantee the existence of extrema on a closed interval. Then we will learn about critical points to be able to detect extrema algebraically.

Theorem 3.4.6 If f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on the interval.

Activity 3.4.7 For each of the following figures, decide where the absolute extrema are located.

(a)

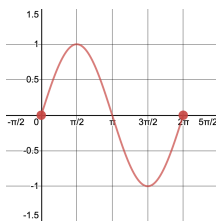


Figure 47

(b)

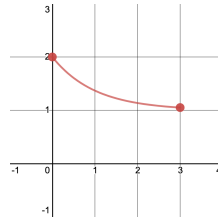


Figure 48

(c)

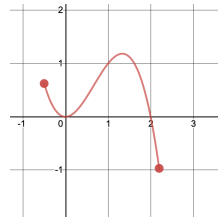


Figure 49

(d)

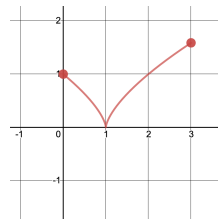


Figure 50

Activity 3.4.8 The Extreme Value Theorem (EVT) guarantees an absolute maximum and absolute minimum for which of the following?

A $f(x) = \frac{x^2}{x^2 - 4x - 5}$ on $[-5, 0]$.

B $f(x) = \frac{x^2}{x^2 - 4x - 5}$ on $[0, 4]$.

C $f(x) = \frac{x^2}{x^2 - 4x - 5}$ on $[4, 6]$.

D $f(x) = \frac{x^2}{x^2 - 4x - 5}$ on $[6, 10]$.

Activity 3.4.9 For the following activity, draw a sketch of a function that has the following properties.

- (a) The function is continuous and has an absolute minimum but no absolute maximum.
- (b) The function is continuous and has an absolute maximum but no absolute minimum.

Definition 3.4.10 We say that x is a **critical point** (or critical number) of $f(x)$ if x is in the domain of $f(x)$ and either $f'(x) = 0$ or $f'(x)$ does not exist.

◇

Activity 3.4.11 Which of the following are critical numbers for $f(x) = \frac{1}{3}x^3 - 2x + 2$?

- A $x = \sqrt{2}$ and $x = -\sqrt{2}$.
- B $x = \sqrt{2}$.
- C $x = 2$ and $x = 0$.
- D $x = 2$.

Remark 3.4.12 The Closed Interval Method. The following is a way of finding the absolute extrema of a continuous f on a closed interval.

- 1 Make a list of all critical points of f in (a, b) . (Do not include any critical points outside of the interval).
- 2 Add the endpoints a and b to the list.
- 3 Evaluate f at all points on your list.
- 4 The smallest value is the absolute minimum. The largest is the absolute maximum.

Activity 3.4.13 What are the absolute extrema for $f(x) = 3x^4 - 4x^3$ on $[-1, 2]$.

- A Absolute maximum is when $x = 0$ and absolute minimum when $x = 1$.
- B Absolute maximum is when $x = 2$ and absolute minimum when $x = -1$.
- C Absolute maximum is when $x = 2$ and absolute minimum when $x = 1$.
- D Absolute maximum is when $x = 0$ and absolute minimum when $x = -1$.

Activity 3.4.14 What is the absolute extrema for $f(x) = x\sqrt{4-x}$ on $[-2, 4]$.

- A Absolute maximum is when $x = -2$ and absolute minimum when $x = \frac{8}{3}$.
- B Absolute maximum is when $x = 4$ and absolute minimum when $x = \frac{8}{3}$.
- C Absolute maximum is when $x = \frac{8}{3}$ and absolute minimum when $x = -2$.
- D Absolute maximum is when $x = 4$ and absolute minimum when $x = -2$.

Activity 3.4.15 Explain how to find the absolute minimum and absolute maximum values of the function $f(x) = -2x^3 + 18x^2 + 42x + 33$ on the interval $[-2, 2]$.

Answer. The absolute extrema may occur at these x -values: $-2, -1, 2$.

- $f(-2) = 37$
- $f(-1) = 11$
- $f(2) = 173$

Thus the absolute minimum is 11 and the absolute maximum is 173.

(The value $f(7) = 523$ is ignored as 7 isn't in the given interval.)

Activity 3.4.16 In this problem you will consider the function $g(x)$.

$$g(x) = \begin{cases} x^3 - 3x & x < 0 \\ x^2 - 4x + 2 & x \geq 0 \end{cases}$$

- (a) What can you say about the point $x = 0$?
- (b) In addition to $x = 0$, find the other two critical points. What are the critical points of $g(x)$?
- A $x = 0, x = 1, x = 2$
 B $x = 0, x = -1, x = 2$
 C $x = 0, x = -1, x = -2$
 D $x = 0, x = 1, x = -2$
- (c) Can you use the Closed Interval Method on $[-4, -1]$? If you can, find the absolute max and min. If you can't, explain why.
- (d) Can you use the Closed Interval Method on $[1, 4]$? If you can, find the absolute max and min. If you can't, explain why.
- (e) Can you use the Closed Interval Method on $[-1, 1]$? If you can, find the absolute max and min. If you can't, explain why.

3.5 Derivative tests (AD5)

Learning Outcomes

- Determine where a differentiable function is increasing and decreasing and classify the critical points as local extrema.

Definition 3.5.1 We say that $f(c)$ is a **relative maximum** (or local maximum) of f provided that $f(c) \geq f(x)$ for all x near c . We say that $f(c)$ is a **relative minimum** (or local minimum) of f provided that $f(c) \leq f(x)$ for all x near c . \diamond

Observation 3.5.2 A strategy for finding the extreme values of a function on open intervals will be to consider all the relative maxima and minima and compare them to find which one(s) give the largest and smallest values. But now we have a new problem: how do you find the relative extrema? We can detect relative extrema by computing the first derivative and finding the critical points (or critical numbers) of the function. By finding the critical points, we will produce a list of candidates for the extrema (local and global) of the function.

Activity 3.5.3 We have encountered several terms recently, so we should make sure that we understand how they are related. Which of the following statements are true?

- A In a closed interval an endpoint is always a local extrema but it might or might not be a global extrema.
- B In a closed interval an endpoint is always a global extrema.
- C A critical point is always a local extrema but it might or might not be a global extrema.
- D A local extrema only occurs where the first derivative is equal to zero.
- E A local extrema always occurs at a critical point.
- F A local extrema might occur at a critical point or at an endpoint of a closed interval.

Activity 3.5.4

- (a) Sketch the graph of a continuous function that is increasing on $(-\infty, -2)$, constant on the interval $(3, 5)$, and decreasing on the interval $(-2, 3)$.
- (b) How would you describe the derivative of the function on each interval?
- A For $x < -2$ we have $f'(x) < 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x) > 0$.
- B For $x < -2$ we have $f'(x) > 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x)$ is undefined.
- C For $x < -2$ we have $f'(x) > 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x) = 0$.
- D For $x < -2$ we have $f'(x) < 0$, then $f'(x) < 0$ on the interval $(-2, 3)$, and on the interval $(3, 5)$ we have $f'(x)$ is constant.

Activity 3.5.5 Look back at the graph you made for [Activity 3.5.4](#).

Which of the following best describes what is occurring when graph changes behavior?

- A There is a critical point.
- B There is a relative maximum or minimum.
- C The derivative is undefined.
- D The derivative is equal to zero.

Observation 3.5.6 Critical points detect changes in the behavior of a function. We will use critical points as "break points" in studying the behavior of a function and its derivatives. We then need to understand what happens at these critical points. We do so using the Derivative Tests.

Theorem 3.5.7 The First Derivative Test. Suppose $f(x)$ that $x = c$ is a critical point of $f(x)$ and $f(x)$ is continuous at $x = c$ and.

1. If $f'(x) > 0$ before $x = c$ and $f'(x) < 0$ after $x = c$, then a local maximum occurs at $x = c$.
2. If $f'(x) < 0$ before $x = c$ and $f'(x) > 0$ after $x = c$, then a local minimum occurs at $x = c$.
3. If $f'(x)$ is the same sign on both sides of $x = c$, then neither a local maximum nor a local minimum occur when $x = c$.

Activity 3.5.8 Let $f(x) = x^4 - 4x^3 + 4x^2$

- (a) Find all critical points of $f(x)$. Draw them on the same number line.
- (b) What intervals have been created by subdividing the number line at the critical points?
- (c) Pick an x -value that lies in each interval. Determine whether $f'(x)$ is positive or negative at each point.
- (d) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?
- (e) List all relative maxima and relative minima.

Theorem 3.5.9 The Second Derivative Test. Suppose that $x = c$ is a

critical point of $f(x)$ and that $f''(x)$ is continuous at $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then a relative maximum occurs when $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then a relative minimum occurs when $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then this test is inconclusive.

In case of an inconclusive Second Derivative Test, we need to go back to the First Derivative Test to classify the critical point.

Activity 3.5.10 Consider the function $f(x) = -x^3 + 3x + 4$. Find the open intervals where $f(x)$ is increasing or decreasing. Find the local extrema of $f(x)$.

Answer. The function decreases on $(-\infty, -1)$, increases on $(-1, 1)$, and decreases on $(1, \infty)$. The point $(x, y) = (-1, 2)$ is a relative minimum, and the point $(x, y) = (1, 6)$ is a relative maximum.

Remark 3.5.11 Dealing with discontinuities. Our previous activity dealt with a function that was continuous for all real numbers. Because of that, we could trust our chart to point out relative extrema. Let's now consider what might happen if a function has any discontinuities.

Activity 3.5.12 Draw a function that is increasing on the left of $x = 1$, discontinuous at $x = 1$, $f(1) = \lim_{x \rightarrow 1^+} f(x)$, and decreasing to the right of $x = 1$. Does the derivative of $f(x)$ exist at $x = 1$? Does your graph have a local maximum or minimum at $x = 1$?

Activity 3.5.13 Let $f(x) = \frac{x}{(x-2)^2}$.

- (a) Note that $f(x)$ is not defined for $x = 2$. But the function may be increasing on one side of $x = 2$ and decreasing on the other! So we include $x = 2$ on your number line.
- (b) Find all critical points of $f(x)$. Plot them and any discontinuities for $f(x)$ on the same number line.
- (c) What intervals have been created by subdividing the number line at the critical points and at the discontinuities?
- (d) Pick an x -value that lies in each interval. Determine whether $f'(x)$ is positive or negative each point.
- (e) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?
- (f) List all relative maxima and relative minima.

Activity 3.5.14 For each of the following functions, find the intervals on which $f(x)$ is increasing or decreasing. Then identify any relative extrema using either the First or Second Derivative Test.

- (a) $f(x) = x^3 + 3x^2 + 3x + 1$
- (b) $f(x) = \frac{1}{2}x + \cos x$ on $(0, 2\pi)$
- (c) $f(x) = (x^2 - 9)^{2/3}$
- (d) $f(x) = \ln(2x - 1)$. (Hint: think about the domain of this one before you get started!)
- (e) $f(x) = \frac{x^2}{x^2 - 4}$

Theorem 3.5.15 The Mean Value Theorem. *If f is continuous and differentiable on the closed interval $[a, b]$, then there is some point c in the interval where $f'(c)$ is equal to the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$. In symbols, for some c in (a, b) we have that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Activity 3.5.16

- (a) Suppose f is continuous and differentiable on $[a, b]$ and also suppose that $f(a) = f(b)$. What is the average rate of change of $f(x)$ on $[a, b]$? What does the MVT (Mean Value Theorem) tell you?
- (b) Use part (a) to show with the MVT that $f(x) = (x - 1)^2 + 3$ has a critical point on $[0, 2]$.

3.6 Concavity and inflection (AD6)

Learning Outcomes

- Determine the intervals of concavity of a twice differentiable function and find all of its points of inflection.

Observation 3.6.1 In addition to asking *whether* a function is increasing or decreasing, it is also natural to inquire *how* a function is increasing or decreasing. [Activity 3.6.2](#) describes three basic behaviors that an increasing function can demonstrate on an interval, as pictured in [Figure 51](#)

Activity 3.6.2 Sketch a sequence of tangent lines at various points to each of the following curves in [Figure 51](#).

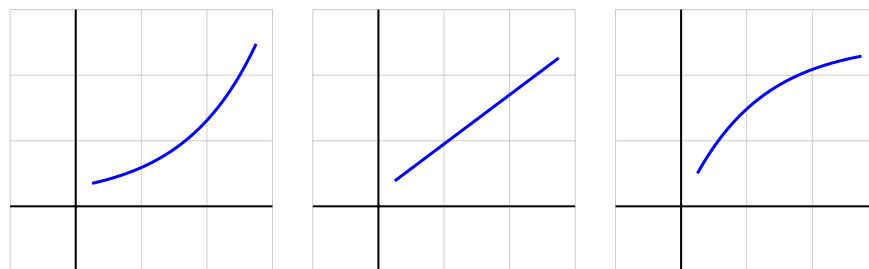


Figure 51 Three increasing functions

- (a) Look at the curve pictured on the left. How would you describe the slopes of the tangent lines as you move from left to right?
 - A The slopes of the tangent lines decrease as you move from left to right.
 - B The slopes of the tangent lines remain constant as you move from left to right.
 - C The slopes of the tangent lines increase as you move from left to right.
- (b) Look at the curve pictured in the middle. How would you describe the slopes of the tangent lines as you move from left to right?

- A The slopes of the tangent lines decrease as you move from left to right.
 - B The slopes of the tangent lines remain constant as you move from left to right.
 - C The slopes of the tangent lines increase as you move from left to right.
- (c) Look at the curve pictured on the right. How would you describe the slopes of the tangent lines as you move from left to right?
- A The slopes of the tangent lines decrease as you move from left to right.
 - B The slopes of the tangent lines remain constant as you move from left to right.
 - C The slopes of the tangent lines increase as you move from left to right.

Remark 3.6.3 On the leftmost curve in [Figure 51](#), as we move from left to right, the slopes of those tangent lines will increase. Therefore, the rate of change of the pictured function is increasing, and this explains why we say this function is *increasing at an increasing rate*.

Observation 3.6.4 We must be extra careful with our language when dealing with negative numbers. For example, it can be tempting to say that “ -100 is bigger than -2 .” But we must remember that “greater than” describes how numbers lie on a number line: -100 is less than -2 because it comes earlier on the number line. It might be helpful to say that “ -100 is ”more negative” than -2 .”

Activity 3.6.5 Sketch a sequence of tangent lines at various points to each of the following curves in [Figure 52](#).



Figure 52 From left to right, three functions that are all decreasing.

- (a) Look at the curve pictured on the left. How would you describe the slopes of the tangent lines as you move from left to right?
- A The slopes of the tangent lines decrease as you move from left to right.
 - B The slopes of the tangent lines remain constant as you move from left to right.
 - C The slopes of the tangent lines increase as you move from left to right.
- (b) Look at the curve pictured in the middle. How would you describe the slopes of the tangent lines as you move from left to right?

- A The slopes of the tangent lines decrease as you move from left to right.
- B The slopes of the tangent lines remain constant as you move from left to right.
- C The slopes of the tangent lines increase as you move from left to right.
- (c) Look at the curve pictured on the right. How would you describe the slopes of the tangent lines as you move from left to right?
- A The slopes of the tangent lines decrease as you move from left to right.
- B The slopes of the tangent lines remain constant as you move from left to right.
- C The slopes of the tangent lines increase as you move from left to right.

Remark 3.6.6 We encountered before the language of concavity: when a curve bends upward, we say its shape is concave up. When a curve bends downwards, we say its shape is concave down.

Activity 3.6.7 Which curve is concave up? Which one is concave down? Why? Try to explain using the graph!

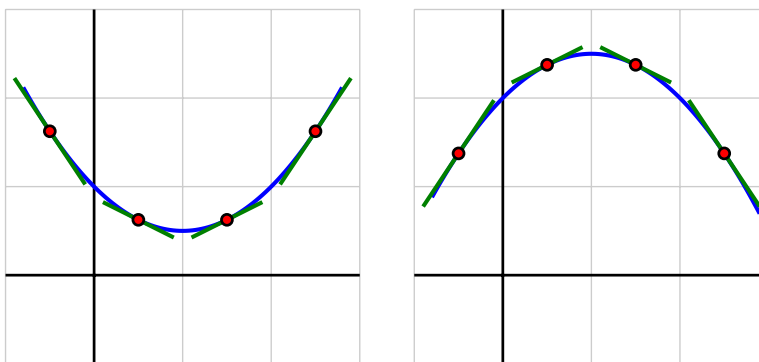


Figure 53 At left, a function that is concave up; at right, one that is concave down.

Definition 3.6.8 Let f be a differentiable function on an interval (a, b) . Then f is **concave up** on (a, b) if and only if f' is increasing on (a, b) ; f is **concave down** on (a, b) if and only if f' is decreasing on (a, b) . \diamond

Activity 3.6.9 Look at how the slopes of the tangent lines change from left to right for each of the two graphs in [Figure 53](#)

- (a) Look at the curve pictured on the left. How would you describe the slopes of the tangent lines as you move from left to right?
- A The slopes of the tangent lines decrease as you move from left to right.
- B The slopes of the tangent lines increase as you move from left to right.
- C The slopes of the tangent lines go from increasing to decreasing as you move from right to left.

- D The slopes of the tangent lines go from decreasing to increasing as you move from right to left.
- (b) Which of the following statements is true about the function on the left in [Figure 53](#)?
- A $f'(x) > 0$ on the entire interval shown.
 - B $f'(x) < 0$ on the entire interval shown.
 - C $f''(x) > 0$ on the entire interval shown.
 - D $f''(x) < 0$ on the entire interval shown.
- (c) Look at the curve pictured on the right. How would you describe the slopes of the tangent lines as you move from left to right?
- A The slopes of the tangent lines decrease as you move from left to right.
 - B The slopes of the tangent lines increase as you move from left to right.
 - C The slopes of the tangent lines go from increasing to decreasing as you move from right to left.
 - D The slopes of the tangent lines go from decreasing to increasing as you move from right to left.
- (d) Which of the following statements is true about the function on the right in [Figure 53](#)?
- A $f'(x) > 0$ on the entire interval shown.
 - B $f'(x) < 0$ on the entire interval shown.
 - C $f''(x) > 0$ on the entire interval shown.
 - D $f''(x) < 0$ on the entire interval shown.

Theorem 3.6.10 Test for Concavity. Suppose that $f(x)$ is twice differentiable on some interval (a, b) . If $f'' > 0$ on (a, b) , then f is concave up on (a, b) . If $f'' < 0$ on (a, b) , then f is concave down on (a, b) .

Observation 3.6.11 In the previous section, we saw in [Activity 3.5.8](#) how to use critical points of the function and the values of the first derivative to identify intervals of increase/decrease of a function. The next activity [Activity 3.6.12](#) uses the critical points of the derivative function and the values of the second derivative as in [Theorem 3.6.10](#) to identify where the original function is concave up/down.

Activity 3.6.12 Let $f(x) = x^4 - 4x^3 + 4x^2$.

- (a) Find all the zeros of $f''(x)$.
- (b) What intervals have been created by subdividing the number line at zeros of $f''(x)$?
- (c) Pick an x -value that lies in each interval. Determine whether $f''(x)$ is positive or negative at each point.
- (d) On which intervals is $f'(x)$ increasing? On which intervals is $f'(x)$ decreasing?
- (e) List all the intervals where $f(x)$ is concave up and all the intervals where $f(x)$ is concave down.

Definition 3.6.13 If $x = c$ is a point where $f''(x)$ changes sign, then the concavity of graph of $f(x)$ changes and we call $x = c$ an **inflection point** of $f(x)$. \diamond

Activity 3.6.14 Use the results from [Activity 3.6.12](#) to identify all of the inflection points of $f(x) = x^4 - 4x^3 + 4x^2$.

Activity 3.6.15 For each of the following functions, describe the open intervals where it is concave up or concave down, and any inflection points.

1. $f(x) = -\frac{1}{4}x^5 - \frac{5}{2}x^4 - \frac{15}{2}x^3$

2. $f(x) = \frac{3}{20}x^5 + x^4 - \frac{5}{2}x^3$

Answer.

1. $f''(x) = -5(x+3)^2x$

- $(-\infty, -3)$ is concave *down*
- -3 is *not* an inflection point
- $(-3, 0)$ is concave *down*
- 0 *is* an inflection point
- $(0, \infty)$ is concave *up*

2. $f''(x) = 3(x+5)(x-1)x$

- $(-\infty, -5)$ is concave *down*
- -5 *is* an inflection point
- $(-5, 0)$ is concave *up*
- 0 *is* an inflection point
- $(0, 1)$ is concave *down*
- 1 *is* an inflection point
- $(1, \infty)$ is concave *up*

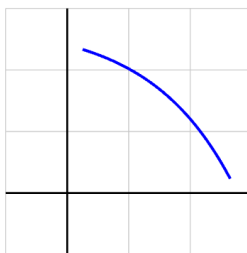
3.7 Graphing with derivatives (AD7)

Learning Outcomes

- Sketch the graph of a differentiable function whose derivatives satisfy given criteria.

Remark 3.7.1 In [Section 3.5](#) and [Section 3.6](#) we learned how the first and second derivatives give us information about the graph of a function. Specifically, we can determine the intervals where a function is increasing, decreasing, concave up, or concave down as well as any relative extrema or inflection points. Now we will put that information together to draw a sketch of the graph.

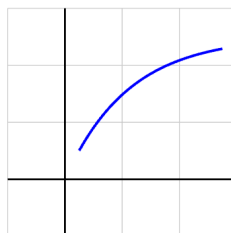
Activity 3.7.2 Which of the following best describes the curve graphed below?

**Figure 54**

- A Increasing and concave up
- B Increasing and concave down
- C Decreasing and concave up
- D Decreasing and concave down

Activity 3.7.3

- (a) Which of the following best describes the curve graphed below?

**Figure 55**

- A $f' > 0$ and $f'' > 0$
- B $f' > 0$ and $f'' < 0$
- C $f' < 0$ and $f'' > 0$
- D $f' < 0$ and $f'' < 0$

- (b) For the other three answer choices above that you did not select, sketch a portion of a curve that matches each description.

Activity 3.7.4 For each prompt that follows, sketch a possible graph of a function on the interval $-3 < x < 3$ that satisfies the stated properties.

- (a) One function $f(x)$ such that f is increasing on $-3 < x < 3$, concave up on $-3 < x < 0$, and concave down on $0 < x < 3$.
- (b) One function $g(x)$ such that g is increasing on $-3 < x < 3$, concave down on $-3 < x < 0$, and concave up on $0 < x < 3$.
- (c) One function $h(x)$ such that h is decreasing on $-3 < x < 3$, concave up on $-3 < x < -1$, neither concave up nor concave down on $-1 < x < 1$, and concave down on $1 < x < 3$.
- (d) One function $p(x)$ such that p is decreasing and concave down on $-3 < x < 0$ and is increasing and concave down on $0 < x < 3$.

Observation 3.7.5 For an accurate sketch, we must keep in mind other characteristics of a function, such as the domain and horizontal and vertical asymptotes (when they exist). The next problem [Activity 3.7.6](#) includes those aspects in addition to increasing, decreasing, and concavity.

Activity 3.7.6 The following chart describes the values of $f(x)$ and its first and second derivatives at or between a few given values of x , where \nexists denotes that $f(x)$ does not exist at that value of x .

x	-8	-6	-3	0	2	5	8	11	13
$f(x)$	3	5	\nexists	-5	\nexists	4	\nexists	-5	-3
$f'(x)$	+	+	-	-	-	-	+	+	+
$f''(x)$	+	-	-	+	-	+	+	-	-

Assume that $f(x)$ has vertical asymptotes at each x -value where $f(x)$ does not exist, that $\lim_{x \rightarrow -\infty} f(x) = 1$, and that $\lim_{x \rightarrow \infty} f(x) = -1$.

- List all the asymptotes of $f(x)$ and mark them on the graph.
- Does $f(x)$ have any relative maxima or relative minima? If so, at what point(s)?
- Does $f(x)$ have any inflection points? If so, at what point(s)?
- Use the information provided to sketch a reasonable graph of $f(x)$. Watch changes in behavior due to changes in the sign of each derivative.

Remark 3.7.7 A guide to curve sketching.

- Identify the domain of the function.
- Identify any vertical or horizontal asymptotes, if they exist.
- Find $f'(x)$. Then use it to determine the intervals where the function is increasing and the intervals where the function is decreasing. State any relative extrema.
- Find $f''(x)$. Then use it to determine the intervals where the function is concave up and the intervals where the function is concave down. State any inflection points.
- Put everything together and draw sketch.

Activity 3.7.8 Sketch the graph of each of the following functions using the guide to curve sketching found in [Remark 3.7.7](#)

- $f(x) = x^4 - 4x^3 + 10$
- $f(x) = \frac{x^2 - 4}{x^2 - 9}$
- $f(x) = x + 2 \cos x$ on the interval $[0, 2\pi]$
- $f(x) = \frac{x^2 + x - 2}{x + 3}$
- $f(x) = \frac{x}{\sqrt{x^2 + 2}}$
- $f(x) = x^6 + \frac{12}{5}x^5 - 12x^4 + 10$

3.8 Applied optimization (AD8)

Learning Outcomes

- Apply optimization techniques to solve various problems.

Activity 3.8.1 The box. Help your company design a box with maximum volume given the following constraints:

- The box must be made from the following material: an 8 by 8 inches piece of cardboard.
 - To create the box, you are asked to cut the same size square from each corner of the 8 by 8 inches piece of cardboard and to fold the remaining cardboard.
- (a) Draw a diagram illustrating how the box is created.
- (b) Explain why the volume of the box is a *function* of the side length x of the cutout squares.
- (c) Express the volume of the box V as a function of the length of the cuts x .
- (d) What is a realistic the domain of the function $V(x)$?
- (e) What cut length x maximizes the volume of the box?

Remark 3.8.2 Approaching optimization problems.

- Draw a diagram and introduce variables. First understand what quantities are allowed to vary in the problem and then to represent those values with variables. Constructing a diagram with the variables labeled is almost always an essential first step. Sometimes drawing several diagrams can be especially helpful to get a sense of the situation.
- Identify the quantity to be optimized as well as any key relationships among the variables. Write down equations that involve the variables that have been introduced: one should represent the quantity whose minimum or maximum is sought, and others could show how multiple variables in the problem are interrelated.
- Determine a function of a single variable that models the quantity to be optimized; this may involve using other relationships among variables to eliminate one or more variables in the function formula. For example, in [Activity 3.8.1](#), we initially found that $V = l \times w \times h$, but then the additional relationships that $l = w = 8 - 2x$ and $h = x$ yield $V(x) = (8 - 2x)^2x$, and thus we have written the volume as a function of the single variable x .
- Decide the domain on which to consider the function being optimized. Often the physical constraints of the problem will limit the possible values that the independent variable can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.
- Use calculus to identify the absolute maximum and/or minimum of the quantity being optimized. This always involves finding the critical numbers of the function first. Then, depending on the domain, we either

construct a first derivative sign chart (for an open or unbounded interval we use the First or Second Derivative Tests) or evaluate the function at the endpoints and critical numbers (for a closed, bounded interval and a continuous function we use the Closed Interval Method).

- Finally, we make certain we have answered the question: what are the optimal points and what optimal values do we obtain at these points?

Activity 3.8.3 According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by “girth” we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

- (a) Let x represent the length of one side of the square end and y the length of the longer side. Label these quantities appropriately on the image shown in [Figure 56](#).

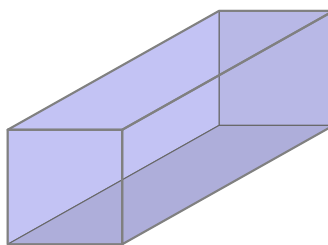


Figure 56 A rectangular parcel with a square end.

- (b) What is the quantity to be optimized in this problem?
- A maximize volume (V)
 - B maximize the girth plus length (L)
 - C minimize volume (V)
 - D minimize the girth plus length (L)
- (c) Which formula below represents the quantity you want to optimize in terms of x and y ?
- A $V = x^2y$
 - B $V = xy^2$
 - C $L = 2x + y$
 - D $L = 4x + y$
- (d) The problem statement tells us that the parcel’s girth plus length (L) may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length (L) to equal 108 inches. What equation do we have involving x and y ?
- A $108 = 4x + y$
 - B $108 = 2x + y$
 - C $108 = x^2 + y$
 - D $108 = xy^2$

- (e) The equation above gives the relationship between x and y . For ease of notation, solve this equation for y as a function on x and then find a formula for the volume of the parcel as a function of a the single variable x . What is the formula for $V(x)$?

A $V(x) = x^2(108 - 4x)$

B $V(x) = x(108 - 4x)^2$

C $V(x) = x^2(108 - 2x)$

D $V(x) = x(108 - 2x)^2$

- (f) Over what domain should we consider this function? To answer this question, notice that the constraints that girth plus length is 108 inches produces intervals of possible values for x and y .

A $0 \leq x \leq 108$

B $0 \leq y \leq 108$

C $0 \leq x \leq 27$

D $0 \leq y \leq 27$

- (g) Find the absolute maximum of the volume of the parcel on the domain you just established. Make sure to justify that you have found the absolute maximum using calculus!

Remark 3.8.4 Notice that a critical point might or might not be an absolute maximum or minimum, so finding the critical points is not enough to answer an optimization problem. Moreover, some of the critical points might be outside of the domain imposed by the context and thus they cannot be feasible optimal points.

Activity 3.8.5 Revenue = Number of tickets \times Price of ticket. Waterford movie theater currently charges \\$8 for a ticket. At this price, the theater sells 200 tickets daily. The general manager wonders if they can generate more revenue by increasing the price of a tickets. A survey shows that they will lose 20 customers for every dollar increase in the ticket price.

- (a) If the price of a movie ticket is increased by d dollars, write a formula for the price P in terms of d .
- (b) If the price of a ticket is increased by one dollar, how many many customers will the theater lose?
- (c) Write a formula for the number of tickets sold T as a function of a ticket price increase of d dollars.
- (d) Consider the new price of a ticket P and the new number of tickets sold T . Write a formula for the revenue R earned by ticket sales as a function of a price increase of d dollars.
- (e) What is a realistic domain for $R(d)$?
- (f) What increase in price d should the general manager choose to maximize the revenue? What would the movie ticket cost and what would be the revenue at that price?
- (g) Now suppose that the cost of running the business if the price is increased by d dollars is given by $C(d) = 10d^3 - 40d^2 + 40d + 600$. If the manager decides that they will definitely increase the price, what price increase d maximizes the profit? (Recall that Profit = Revenue - Cost).

Activity 3.8.6 Modeling given a geometric shape. The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter.

- (a) What choice of dimensions will make the rectangular area in the center as large as possible?
- (b) What should the dimensions so the total area enclosed by the running track is maximized?

Activity 3.8.7 Modeling in algebraic situations.

- (a) Find the coordinates of the point on the curve $y = \sqrt{x}$ closest to the point $(1, 0)$
- (b) The sum of two positive numbers is 48. What is the smallest possible value of the sum of their squares?

Activity 3.8.8 Suppose that if a widget is priced at \$176, then you are able to sell 672 units each day. According to a survey of customers, increasing this price by \$1 will result in losing 4 daily sales, and vice versa. Your manager asks you how to adjust the price of a widget to maximize the revenue (widgets sold times price). Write an explanation of what this change in price should be and why.

Answer. Decrease the cost by \$4.

3.9 Limits and Derivatives (AD9)

Learning Outcomes

- Compute the values of indeterminate limits using L'Hopital's Rule.

Remark 3.9.1 When we compute a limit algebraically, we often encounter the indeterminate form

$$\frac{0}{0}$$

but this means that limit can equal any number, infinity, or might even not exist. When we encounter an indeterminate form, we just do not know (yet) what the value of the limit is.

Activity 3.9.2 We can compute limits that give indeterminate forms via algebraic manipulations. Consider

$$\lim_{x \rightarrow 1} \frac{2x - 2}{x^2 - 1}.$$

- (a) Verify that this limit gives an indeterminate form of the type $\frac{0}{0}$.
- (b) In the limit, you can cancel common factors. After you simplify the fraction, what is this limit equal to?

A 2

B 1

C $\frac{1}{2}$

D The limit does not exist.

Remark 3.9.3 Consider the limits

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Notice that these limits give indeterminate forms of the type $\frac{0}{0}$. However, these limits are equal to the limit definition of $f'(a)$, the derivative of $f(x)$ at $x = a$. If you can compute $f'(a)$, then you can compute the value of the limit.

Activity 3.9.4 Use the limit definition of the derivative to compute the following limits. Each limit is $f'(a)$, the derivative of some function $f(x)$ at $x = a$. You need to determine the function and the point to then find the value of the limit: $f'(a)$.

- (a) Notice that $\lim_{x \rightarrow 0} \frac{e^{2+x} - e^2}{x}$ is the limit definition of the derivative of e^x at $x = 2$ (where x was used for h). Given these observations, what is this limit equal to?

- A 2
- B e
- C e^2
- D The limit does not exist.

- (b) Consider $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$. This limit is also the limit definition of some derivative at a point. What is the value of this limit?

- A 1
- B 0
- C $\ln(2)$
- D The limit does not exist.

Activity 3.9.5 Compute the following limits using the limit definition of the derivative at a point.

(a) $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$

(b) $\lim_{x \rightarrow 0} \frac{\tan(x)}{x}$

(c) $\lim_{x \rightarrow 0} \frac{\cos(\frac{\pi}{3} + x) - \frac{1}{2}}{x}$

Remark 3.9.6 When we compute a limit algebraically, we might encounter the indeterminate form

$$\frac{\infty}{\infty}$$

but this means that limit can equal any number, infinity, or might even not exist. When we encounter an indeterminate form, we just do not know (yet) what the value of the limit is.

Activity 3.9.7 We can compute limits that give indeterminate forms via algebraic manipulations. Consider

$$\lim_{x \rightarrow +\infty} \frac{2x^2 + 1}{x^2 - 1}.$$

- (a) Verify that this limit gives an indeterminate form of the type $\frac{\infty}{\infty}$.
- (b) You can manipulate this fraction algebraically by dividing numerator and denominator by x^2 . Notice that $\pm \frac{1}{x^2} \rightarrow 0$ as $x \rightarrow \infty$. So after you manipulate and simplify the fraction, what is this limit equal to?

A 2

B 1

C $\frac{1}{2}$

D The limit does not exist.

Theorem 3.9.8 L' Hospital's Rule. *If for the limit of $\frac{f(x)}{g(x)}$ as $x \rightarrow a$ we have one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f(x), g(x)$ are both differentiable around $x = a$, then*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit exists!

Activity 3.9.9 Look back at some limits that gave you an indeterminate form. Can you use l'Hospital's Rule to find the limit? If using the L'Hospital's Rule is appropriate, then try to compute the limit this way. It should give you the same result.

Activity 3.9.10 In [Activity 1.1.12](#), when we started to study limits, we encountered the Squeeze Theorem and computed the limit $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$ using this theorem. Let's find new ways to compute this limit.

- (a) Thinking about x as the length of an interval h , this limit is actually equal to the value of some derivative, so $f'(a) = \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$. What function $f(x)$ and what point $x = a$ would lead to this limit? Use these to find $f'(a)$, the value of this limit (in a new way!).
- (b) Verify, one more time, that this limit is indeed an indeterminate form. Then use L'Hospital's Rule to find this limit (again, in another way!).

Activity 3.9.11 For the following limits, check if they give an indeterminate form. If they do, try to use l'Hospital's Rule. Does it help? It may or may not, or you may just need to use the rule repeatedly. Either way, try to compute the value of the following limits.

(a) $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$

(b) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

(c) $\lim_{x \rightarrow \infty} \frac{3x^2 + 3}{x^2 + 2x}$

(d) $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{-x}$

(e) $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x}$

(f) $\lim_{x \rightarrow 0} \frac{\sin^2(3x)}{5x^3 - 3x^2}$

Activity 3.9.12 For each limit, explain if L'Hopital's Rule may be applied. If it can, explain how to use this rule to find the limit.

1.

$$\lim_{x \rightarrow \infty} \frac{-8x + 3e^x}{7x - 3e^x}$$

2.

$$\lim_{x \rightarrow 0} \frac{6 \cos(8x)}{4x - 7}$$

3.

$$\lim_{x \rightarrow 0} \frac{-9 \cos(3x) + 9}{-3x}$$

4.

$$\lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x^2 - 13x + 36}$$

Answer.

1.

$$\lim_{x \rightarrow \infty} \frac{-8x + 3e^x}{7x - 3e^x} = -1$$

2. L'Hôpital's Rule does not apply.

3.

$$\lim_{x \rightarrow 0} \frac{-9 \cos(3x) + 9}{-3x} = 0$$

4.

$$\lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x^2 - 13x + 36} = -\frac{7}{5}$$

Activity 3.9.13 There are situations in which using L'Hospital does not help the situation and you need some algebra skills! Consider the function $r(x) = \frac{x}{\sqrt{x^2+2}}$ and suppose that we want to find the limits as x tends to $\pm\infty$.

- (a) Check that the limit as $x \rightarrow +\infty$ gives an indeterminate form $\frac{\infty}{\infty}$. Then try to use L'Hospital's Rule... what happens? What if you use it again?
- (b) We need to use algebra to handle this limit. Informally, we would like to cancel the highest powers at the numerator and denominator. Look at the denominator, $\sqrt{x^2+2}$. We want to factor out an x^2 under the square root. What do you get?

A $\sqrt{x^2 \left(1 + \frac{2}{x}\right)}$

B $\sqrt{x^2 \left(1 + \frac{2}{x^2}\right)}$

C $\sqrt{x^2(1+x)}$

D $\sqrt{x^2(1+x^2)}$

- (c) Now we need to be careful when computing $\sqrt{x^2}$ as this is equal to $|x|$ (so it is $+x$ when we have a positive input and $-x$ when we have a negative output. So we have the two limits.

$$\lim_{x \rightarrow +\infty} \frac{x}{|x|\sqrt{1 + \frac{2}{x^2}}}$$

$$\lim_{x \rightarrow -\infty} \frac{x}{|x|\sqrt{1 + \frac{2}{x^2}}}$$

Thinking about what happens to the absolute values as you go towards positive or negative infinity, find the values of these two limits... You will find different results!

Chapter 4

Definite and Indefinite Integrals (IN)

Learning Outcomes

By the end of this chapter, you should be able to...

1. Use geometric formulas to compute definite integrals.
2. Approximate definite integrals using Riemann sums.
3. Determine basic antiderivatives.
4. Solve basic initial value problems.
5. Evaluate a definite integral using the Fundamental Theorem of Calculus.
6. Find the derivative of an integral using the Fundamental Theorem of Calculus.
7. Use definite integrals to find area under a curve.
8. Use definite integral(s) to compute the area bounded by several curves.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Find the derivative of a function using elementary derivative rules, as in [Section 2.3](#). ([CheckIt](#)¹)
- b Use sigma notation for sums. ([Khan Academy](#)² and [Active Calculus](#)³)
- c Find the area of a rectangle in the coordinate plane. ([Example at Khan Academy](#)⁴[Example at virtual nerd](#)⁵)

¹<https://stevenclontz.github.io/cal1-tbil-2021-checkit/#/bank/DF3/1/>

²<https://www.khanacademy.org/math/get-ready-for-ap-calc/xa350bf684c056c5c:get-ready-for-integration-and-accumulation-of-change/xa350bf684c056c5c:summation-notation/v/sigma-notation-sum>

³<https://activecalculus.org/single/sec-4-2-Riemann.html#jCJ>

⁴<https://www.khanacademy.org/math/cc-sixth-grade-math/cc-6th-geometry-topic/cc-6th-quadrilaterals-on-plane/v/dimensions-of-rectangle-from-coordinates-example>

⁵<https://virtualnerd.com/middle-math/integers-coordinate-plane/coordinate-plane/calculate-area-rectangle-vertices>

- d Find the intersection of two graphs. ([Khan Academy](#)⁶)
- e Find the area of plane shapes, such as rectangles, triangles, circles, and trapezoids. ([Math is fun](#)⁷)

4.1 Geometry of definite integrals (IN1)

Learning Outcomes

- Use geometric formulas to compute definite integrals.

Definition 4.1.1 The **definite integral** for a positive function $f(x) \geq 0$ between the points $x = a$ and $x = b$ is the area between the function and the x -axis. We denote this quantity as $\int_a^b f(x) dx$ ◇

Remark 4.1.2 For some functions which have known geometric shapes (like pieces of lines or circles) we can already compute these area exactly and we will do so in this section. But for most functions we do not know quite yet how to compute these areas. In the next section, we will see that because we can compute the areas of rectangles quite easily, we can always try to approximate a shape with rectangles, even if this could be a very coarse approximation.

Activity 4.1.3 Consider the linear function $f(x) = 2x$. Sketch a graph of this function. Consider the area between the x -axis and the function on the interval $[0, 1]$. What is $\int_0^1 f(x) dx$?

- A 1
- B 2
- C 3
- D 4

Activity 4.1.4 Consider the linear function $f(x) = 4x$. What is $\int_0^1 f(x) dx$?

- A 1
- B 2
- C 3
- D 4

Activity 4.1.5 Consider the linear function $f(x) = 2x + 2$. Notice that on the interval $[0, 1]$, the shape formed between the graph and the x -axis is a trapezoid. What is $\int_0^1 f(x) dx$?

- A 1
- B 2
- C 3
- D 4

⁶<https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:eq/x2ec2f6f830c9fb89:quad-sys/v/line-and-parabola-system>

⁷<https://www.mathsisfun.com/area.html>

Activity 4.1.6 Consider the function $f(x) = \sqrt{4 - x^2}$. Notice that on the domain $[-2, 2]$, the shape formed between the graph and the x -axis is a semicircle. What is $\int_{-2}^2 f(x) dx$?

- A π
- B 2π
- C 3π
- D 4π

Definition 4.1.7 If a function $f(x) \leq 0$ on $[a, b]$, then we define the integral between a and b to be

$$\int_a^b f(x) dx = (-1) \times \text{area between the graph of } f \text{ and the } x\text{-axis on the interval } [a, b].$$

So the definite integral for a negative function is the "negative" of the area between the graph and the x -axis. \diamond

Activity 4.1.8 Explain how to use geometric formulas for area to compute the following definite integrals. For each part, sketch the function to support your explanation.

1.

$$\int_1^6 (-3x + 6) dx$$

2.

$$\int_2^6 (-3x + 6) dx$$

3.

$$\int_1^5 \left(-\sqrt{-(x-1)^2 + 16} \right) dx$$

Activity 4.1.9 The graph of $g(t)$ and the areas A_1, A_2, A_3 are given below.

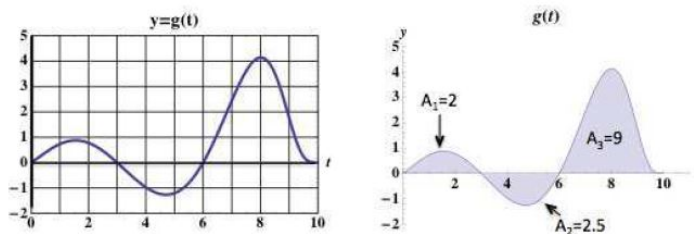


Figure 57

- (a) Find $\int_3^3 g(t) dt$
- (b) Find $\int_3^6 g(t) dt$
- (c) Find $\int_0^{10} g(t) dt$
- (d) Suppose that $g(t)$ gives the velocity in fps at time t (in seconds) of a particle moving in the vertical direction. A positive velocity indicates that the particle is moving up, a negative velocity indicates that the particle is moving down. If the particle started at a height of 3ft, at

what height would it be after 3 seconds? After 6 seconds? After 10 seconds? At what time does the particle reach the highest point in this time interval?

4.2 Approximating definite integrals (IN2)

Learning Outcomes

- Approximate definite integrals using Riemann sums.

Activity 4.2.1 Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

- (a) On the left-hand axes provided in [Figure 58](#), sketch a labeled graph of the velocity function $v(t) = 3$.

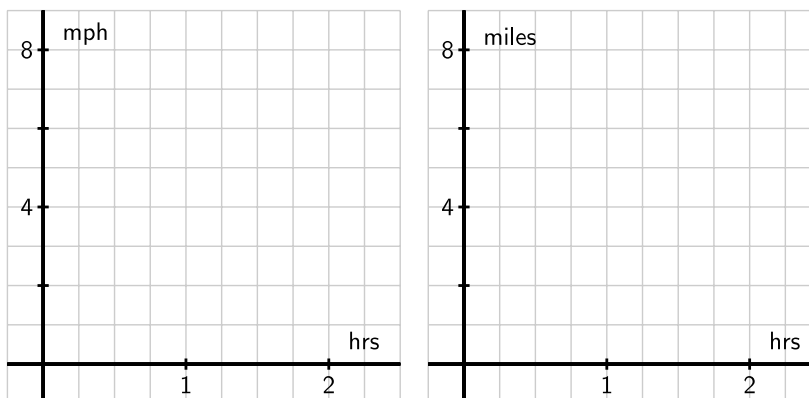


Figure 58 At left, axes for plotting $y = v(t)$; at right, for plotting $y = s(t)$.

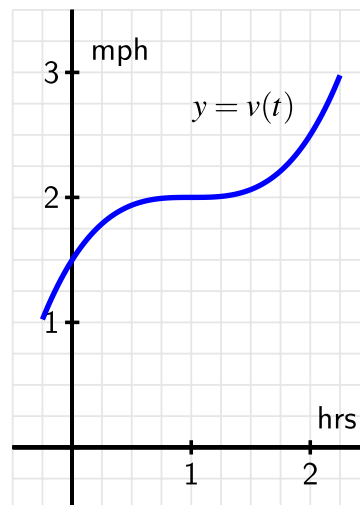
Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question (d).

- (b) How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?
- (c) Find an algebraic formula, $s(t)$, for the position of the person at time t , assuming that $s(0) = 0$. Explain your thinking.
- (d) On the right-hand axes provided in [Figure 58](#), sketch a labeled graph of the position function $y = s(t)$.
- (e) For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v .

Activity 4.2.2 Suppose that a person is walking in such a way that her velocity varies slightly according to the information given in [Table 59](#) and graph given in [Figure 60](#).

Table 59 Velocity data for the person walking.

t	$v(t)$
0.00	1.500
0.25	1.789
0.50	1.938
0.75	1.992
1.00	2.000
1.25	2.008
1.50	2.063
1.75	2.211
2.00	2.500

**Figure 60** The graph of $y = v(t)$.

- Using the grid, graph, and given data appropriately, estimate the distance traveled by the walker during the two hour interval from $t = 0$ to $t = 2$. You should use time intervals of width $\Delta t = 0.5$, choosing a way to use the function consistently to determine the height of each rectangle in order to approximate distance traveled.
- How could you get a better approximation of the distance traveled on $[0, 2]$? Explain, and then find this new estimate.
- Now suppose that you know that v is given by $v(t) = 0.5t^3 - 1.5t^2 + 1.5t + 1.5$. Remember that v is the derivative of the walker's position function, s . Find a formula for s so that $s' = v$.
- Based on your work in (c), what is the value of $s(2) - s(0)$? What is the meaning of this quantity?

Activity 4.2.3 Explain how to approximate the area under the curve $f(x) = -9x^3 + 3x - 9$ on the interval $[4, 10]$ using a right Riemann sum with 3 rectangles of uniform width.

Answer. -31014 square units

4.3 Elementary antiderivatives (IN3)

Learning Outcomes

- Determine basic antiderivatives.

Definition 4.3.1 If g and G are functions such that $G' = g$, we say that G is an **antiderivative** of g . \diamond

Activity 4.3.2 Consider the function $f(x) = \cos x$. Which of the following could be $F(x)$, an antiderivative of $f(x)$?

- $\sin x$
- $\cos x$

- C $\tan x$
- D $\sec x$

Activity 4.3.3 Consider the function $f(x) = x^2$. Which of the following could be $F(x)$, an antiderivative of $f(x)$?

- A $2x$
- B $\frac{1}{3}x^3$
- C x^3
- D $\frac{2}{3}x^3$

Remark 4.3.4 We now note that whenever we know the derivative of a function, we have a *function-derivative pair*, so we also know the antiderivative of a function. For instance, in [Activity 4.3.2](#) we could use our prior knowledge that

$$\frac{d}{dx}[\sin(x)] = \cos(x),$$

to determine that $F(x) = \sin(x)$ is an antiderivative of $f(x) = \cos(x)$. F and f together form a function-derivative pair. Every elementary derivative rule leads us to such a pair, and thus to a known antiderivative.

In the following activity, we work to build a list of basic functions whose antiderivatives we already know.

Activity 4.3.5 Use your knowledge of derivatives of basic functions to complete [Table 61](#) of antiderivatives. For each entry, your task is to find a function F whose derivative is the given function f .

Table 61 Familiar basic functions and their antiderivatives.

given function, $f(x)$	antiderivative, $F(x)$
k , (k is constant)	
x^n , $n \neq -1$	
$\frac{1}{x}$, $x > 0$	
$\sin(x)$	
$\cos(x)$	
$\sec(x) \tan(x)$	
$\csc(x) \cot(x)$	
$\sec^2(x)$	
$\csc^2(x)$	
e^x	
a^x ($a > 1$)	
$\frac{1}{1+x^2}$	
$\frac{1}{\sqrt{1-x^2}}$	

Activity 4.3.6 Using this information, which of the following is an antiderivative for $f(x) = 5 \sin(x) - 4x^2$?

- A $F(x) = -5 \cos(x) + \frac{4}{3}x^3$.
- B $F(x) = 5 \cos(x) + \frac{4}{3}x^3$.
- C $F(x) = -5 \cos(x) - \frac{4}{3}x^3$.
- D $F(x) = 5 \cos(x) - \frac{4}{3}x^3$.

Activity 4.3.7 Find the general antiderivative for each function.

(a)

$$f(x) = -4 \sec^2(x)$$

(b)

$$f(x) = \frac{8}{\sqrt{x}}$$

Activity 4.3.8 Find each indefinite integral.

(a)

$$\int (-9x^4 - 7x^2 + 4) dx$$

(b)

$$\int 3e^x dx$$

4.4 Initial Value Problems (IN4)

Learning Outcomes

- Solve basic initial value problems.

Note 4.4.1 In this section we will discuss the relationship between antiderivatives and solving simple differential equations. A differential equation is an equation that has a derivative. For this section we will focus on differential equations of the form

$$\frac{dy}{dx} = f(x).$$

Our goal is to find a relationship of $y(x)$ that satisfies the differential equation. We can solve for $y(x)$ by finding the antiderivative of $f(x)$.

Activity 4.4.2 Which of the following equations for $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^2 + 2x.$$

A $y(x) = \frac{x^3}{3} + x^2 + 4$

B $y(x) = 2x + 2$

C $y(x) = \frac{x^3}{3} + x^2 + 10$

D $y(x) = \frac{x^3}{3} + x^2$

E $y(x) = 2x$

Remark 4.4.3 In [Activity 4.4.2](#) there are more than one solution that satisfies the differential equation. In fact there is a family of functions that satisfies the differential equation, that is

$$f(x) = \frac{x^3}{3} + x^2 + c_1,$$

where c_1 is an arbitrary constant yet to be defined. To find c_1 we have to

have some initial value for the differential equation, $y(x_0) = y_0$, where the point (x_0, y_0) is the starting point for the differential equation. In general this section we will focus on solving initial value problems (differential equation with an initial condition) of the form,

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

Activity 4.4.4 Which of the following equations for $y(x)$ satisfies the differential equation and initial condition,

$$\frac{dy}{dx} = x^2 + 2x, \quad y(3) = 16.$$

A $y(x) = \frac{x^3}{3} + x^2 - 4$

B $y(x) = \frac{x^3}{3} + x^2 + 2$

C $y(x) = \frac{x^3}{3} + x^2 - 2$

D $y(x) = \frac{x^3}{3} + x^2 + 16$

Activity 4.4.5 Which of the following functions satisfies the initial value problem,

$$\frac{dy}{dx} = \sin(x), \quad y(0) = 1.$$

A $y(x) = \cos(x)$

B $y(x) = \cos(x) + 2$

C $y(x) = \cos(x) + 1$

D $y(x) = -\cos(x)$

E $y(x) = -\cos(x) + 2$

Activity 4.4.6 One of the applications of initial value problems is calculating the distance traveled from a point based on the velocity of the object. Given that the velocity of the of an object in km/hr is approximated by $v(t) = \cos(t) + 1$, what is the approximate distance travelled by the object after 1 hour?

A $s(1) \approx 1$ km

B $s(1) \approx 0.1585$ km

C $s(1) \approx 1.8415$ km

D $s(1) \approx 2.3415$ km

Activity 4.4.7 So far we have only been going from velocity to position of an object. Recall that to find the acceleration of an object, you can take the derivative of the velocity of an object. Let us say we have the acceleration of a falling object in m/s^2 given by $a(t) = -9.8$. What is the velocity of the falling object, if the initial velocity is given by $v(0) = 0$ m/s.

A $v(t) = -9.8t$ m

B $v(t) = -9.8t$ m/s

C $v(t) = 9.8t$ m/s

D $v(t) = 9.8t + 1$ m

What is the position of the object, if the initial position is given by $s(0) = 10$ m.

A $s(t) = 4.9t + 10$ m

B $s(t) = -4.9t^2 + 14.9$ m

C $s(t) = -4.9t^2 + 10$ m

D $s(t) = 4.9t + 5.1$ m

Activity 4.4.8 Let $f'(x) = -12x - 6$. Find $f(x)$ such that $f(5) = -179$.

Answer.

$$-6x^2 - 6x + 1$$

4.5 FTC for definite integrals (IN5)

Learning Outcomes

- Evaluate a definite integral using the Fundamental Theorem of Calculus.

Activity 4.5.1 Find the area between $f(x) = \frac{1}{2}x + 2$ and the x -axis from $x = 2$ to $x = 6$.

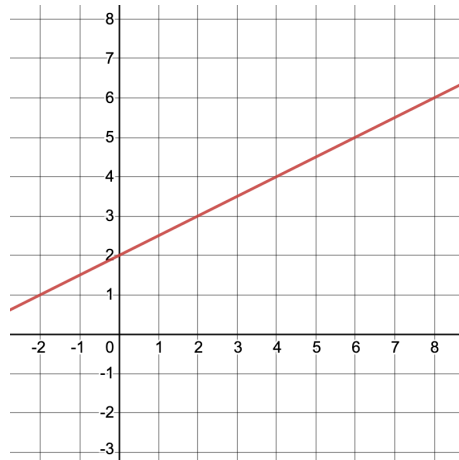
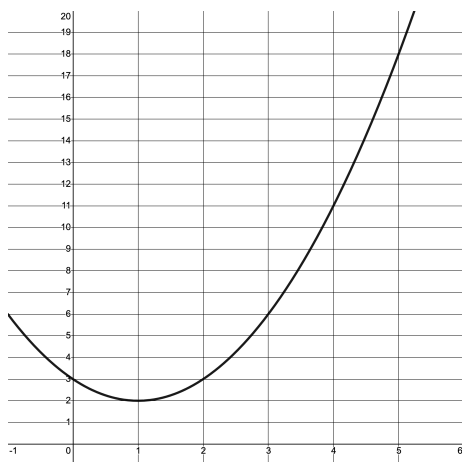


Figure 62

Activity 4.5.2 Approximate the area under the curve $f(x) = (x - 1)^2 + 2$ on the interval $[1, 5]$ using a left Riemann sum with four uniform subdivisions. Draw your rectangles on the graph.

**Figure 63**

Definition 4.5.3 Let $f(x)$ be a continuous function on the interval $[a, b]$. Divide the interval into n subdivisions of equal width, Δx , and choose a point x_i in each interval. Then, the definite integral of $f(x)$ from a to b is

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \int_a^b f(x) dx$$

◇

Activity 4.5.4 How does $\int_2^6 \left(\frac{1}{2}x + 2\right) dx$ relate to [Activity 4.5.1](#)? Could you use [Activity 4.5.1](#) to find $\int_0^4 \left(\frac{1}{2}x + 2\right) dx$? What about $\int_1^7 \left(\frac{1}{2}x + 2\right) dx$?

Remark 4.5.5 Properties of Definite Integrals.

1. If f is defined at $x = a$, then $\int_a^a f(x) dx = 0$.
2. If f is integrable on $[a, b]$, then $\int_a^b f(x) dx = -\int_b^a f(x) dx$.
3. If f is integrable on $[a, b]$ and c is in $[a, b]$, then $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.
4. If f is integrable on $[a, b]$ and k is a constant, then kf is integrable on $[a, b]$ and $\int_a^b kf(x) dx = k \int_a^b f(x) dx$.
5. If f and g are integrable on $[a, b]$, then $f \pm g$ are integrable on $[a, b]$ and $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$.

Activity 4.5.6 Suppose that $\int_1^5 f(x) dx = 10$ and $\int_5^7 f(x) dx = 4$. Find each

of the following.

(a) $\int_1^7 f(x) dx$

(b) $\int_5^1 f(x) dx$

(c) $\int_7^7 f(x) dx$

(d) $3 \int_5^7 f(x) dx$

Observation 4.5.7 We've been looking at two big things in this chapter: antiderivatives and the area under a curve. In the early days of the development of calculus, they were not known to be connected to one another. The integral sign wasn't originally used in both instances. (Gottfried Leibniz introduced it as an elongated S to represent the sum when finding the area.) Connecting these two seemingly separate problems is done by the Fundamental Theorem of Calculus

Theorem 4.5.8 The Fundamental Theorem of Calculus. *If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then*

$$\int_a^b f(x) dx = F(b) - F(a)$$

Activity 4.5.9 Evaluate the following definite integrals. Include a sketch of the graph with the area you've found shaded in. Approximate the area to check to see if your definite integral answer makes sense. (Note: Just a guess, you don't have to use Riemann sums. Use the grid to help.)

(a) $\int_0^2 (x^2 + 3) dx$

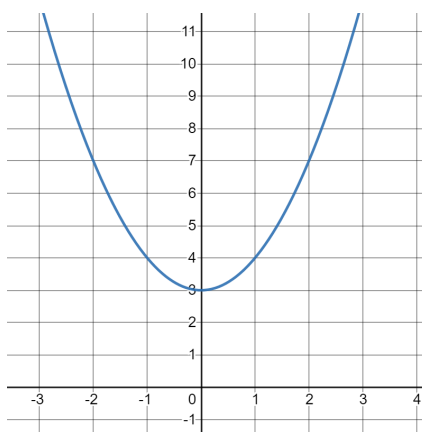
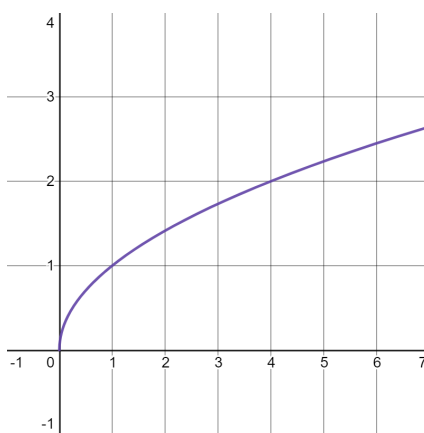
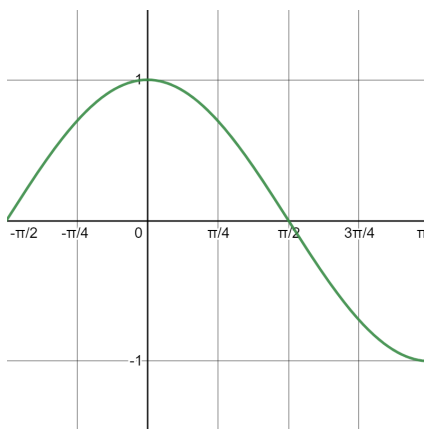


Figure 64

(b) $\int_1^9 (\sqrt{x}) dx$

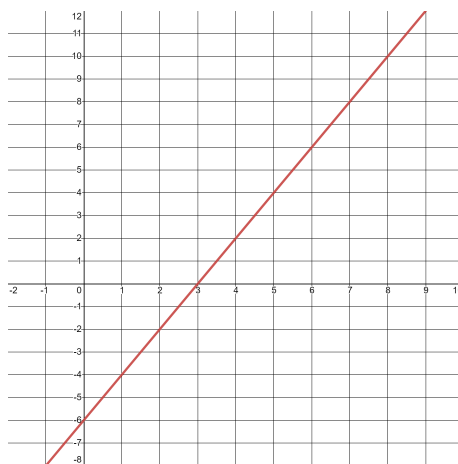
**Figure 65**

(c) $\int_{-\pi/4}^{\pi/2} (\cos x) \, dx$

**Figure 66**

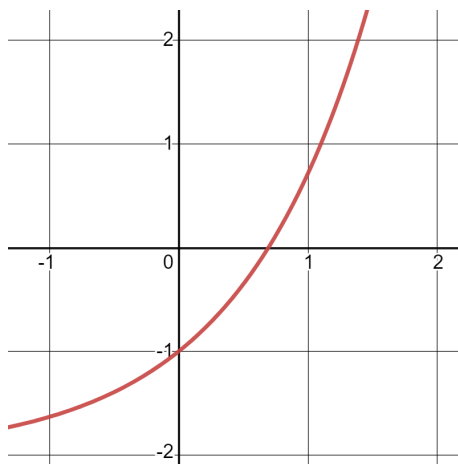
Activity 4.5.10 Find the area between $f(x) = 2x - 6$ on the interval $[0, 8]$ using

1. geometry
2. the definite integral

**Figure 67**

What do you notice?

Activity 4.5.11 Find the area bounded by the curves $f(x) = e^x - 2$, the x -axis, $x = 0$, and $x = 1$.

**Figure 68**

Activity 4.5.12 Set up a definite integral that represents the shaded area. Then find the area of the given region using the definite integral.

(a) $y = \frac{1}{x^2}$

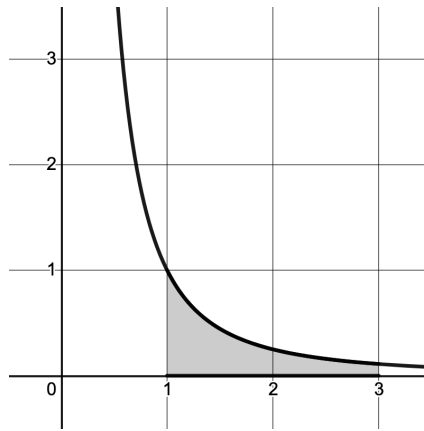


Figure 69

(b) $y = 3x^2 - x^3$

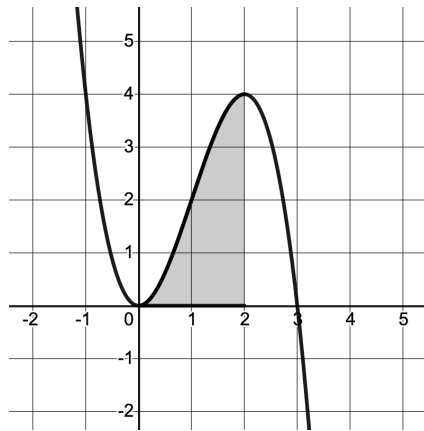


Figure 70

Activity 4.5.13 Explain how to compute the exact value of each of the following definite integrals using the Fundamental Theorem of Calculus. Leave all answers in exact form, with no decimal approximations.

1.

$$\int_{-3}^{-2} (-9x^3 - 9x^2 + 1) dx$$

2.

$$\int_{\frac{7}{6}\pi}^{\frac{5}{4}\pi} (-3 \sin(x)) dx$$

3.

$$\int_2^6 (3e^x) dx$$

Answer.

1.

$$\frac{361}{4}$$

2.

$$\frac{3}{2}\sqrt{3} - \frac{3}{2}\sqrt{2}$$

3.

$$3e^6 - 3e^2$$

4.6 FTC for derivatives of integrals (IN6)

Learning Outcomes

- Find the derivative of an integral using the Fundamental Theorem of Calculus.

Note 4.6.1 In this section we extend the Fundamental Theorem of Calculus discussed in [Section 4.5](#) to include taking the derivatives of integrals. We will call this addition to the Fundamental Theorem of Calculus (FTC) part II. First we will introduce part II and then discuss the implications of this addition.

Theorem 4.6.2 The Fundamental Theorem of Calculus (Part II). *If a function f is continuous on the closed interval $[a, b]$, then the area function*

$$A(x) = \int_a^x f(t) dt \quad \text{for } a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b) . The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Activity 4.6.3 For the following activity we will explore the Fundamental Theorem of Calculus Part II.

- (a) Given that $A(x) = \int_a^x t^3 dt$, then by the Fundamental Theorem of Calculus Part I,

A $A(x) = x^3 - a^3$

B $A(x) = a^4 - x^4$

C $A(x) = \frac{1}{4}(x^4 - a^4)$

D $A(x) = 3x^2$

- (b) Using what you found for $A(x)$, what is $A'(x)$

A $A'(x) = 3x^2$

B $A'(x) = 4a^3 - 4x^3$

C $A'(x) = x^3$

D $A'(x) = 6x$

- (c) Use the Fundamental Theorem of Calculus Part II to find $A'(x)$. What do you notice between what you got above and using FTC Part II? Which method do you prefer?

A $A'(x) = 3x^2$

B $A(x) = 4a^3 - 4x^3$

C $A(x) = x^3$

D $A(x) = 6x$

Activity 4.6.4 Given $A(x) = \int_x^b e^t dt$, what is $A'(x)$?

- A $A'(x) = -e^x$
- B $A'(x) = e^x$
- C $A'(x) = e^b - e^x$
- D $A'(x) = e^x - e^b$

Observation 4.6.5 For the first two activities we have only explored when the function of the limits of the integrand are x . Now we want to see what happens when the limits are more complicated. To do this we will follow a similar procedure as that done in activity 1.

Activity 4.6.6 Recall that by the Fundamental Theorem of Calculus Part I, $\int_a^b f(t) dt = F(b) - F(a)$.

- (a) Let $A(x) = \int_x^{x^2} f(t) dt$ and re-write using FTC Part I.
- (b) Using what you got find $A'(x)$. Explain what derivative rule(s) you used.
- (c) Using what you found what is the derivative of $A(x) = \int_x^{x^2} (t+2) dt$?

- A $A'(x) = 2x(x+2) - (x+2)$
- B $A'(x) = (x+2) - 2x(x^2+2)$
- C $A'(x) = (x^2+2) - (x+2)$
- D $A'(x) = 2x(x^2+2) - (x+2)$

Remark 4.6.7 Now we have some thoughts of how to generalize the FTC Part II when the limits are more complicated.

Theorem 4.6.8 The Fundamental Theorem of Calculus (Part II): Modified. If

$$A(x) = \int_{g(x)}^{h(x)} f(t) dt$$

then

$$A'(x) = \frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x),$$

where $g(x)$ and $h(x)$ are continuous differentiable functions.

Activity 4.6.9 Given $A(x) = \int_{x^3}^{x^5} (\sin(t) - 2) dt$, what is $A'(x)$?

- A $A'(x) = -e^x$
- B $A'(x) = e^x$
- C $A'(x) = e^b - e^x$
- D $A'(x) = e^x - e^b$

4.7 Area under curves (IN7)

Learning Outcomes

- Use definite integrals to find area under a curve.

Remark 4.7.1 A geometrical interpretation of

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \int_a^b f(x) dx$$

(Definition 4.5.3) defines $\int_a^b f(x) dx$ as the **net area** between the graph of $y = f(x)$ and the x -axis. By net area, we mean the area above the x -axis (when $f(x)$ is positive) minus the area below the x -axis (when $f(x)$ is negative).

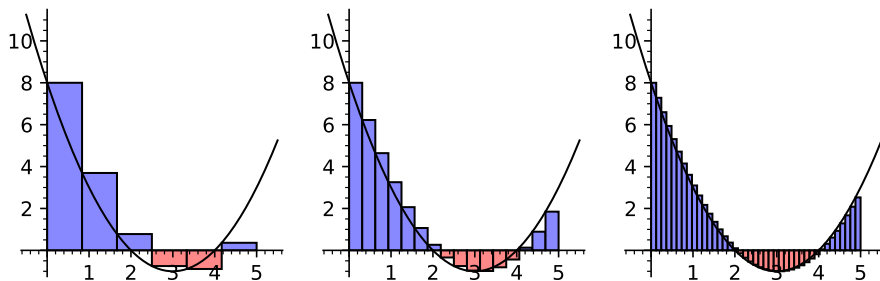


Figure 71 Improving approximations of $\int_0^5 (x-2)(x-4) dx$

Activity 4.7.2

- Write the net area between $f(x) = 6x^2 - 18x$ and the x -axis from $x = 2$ to $x = 7$ as a definite integral.
- Evaluate this definite integral to verify the net area is equal to 265 square units.

Observation 4.7.3 In order to find the total area between a curve and the x -axis, one must break up the definite integral at points where $f(x) = 0$, that is, wherever $f(x)$ may change from positive to negative, or vice versa.

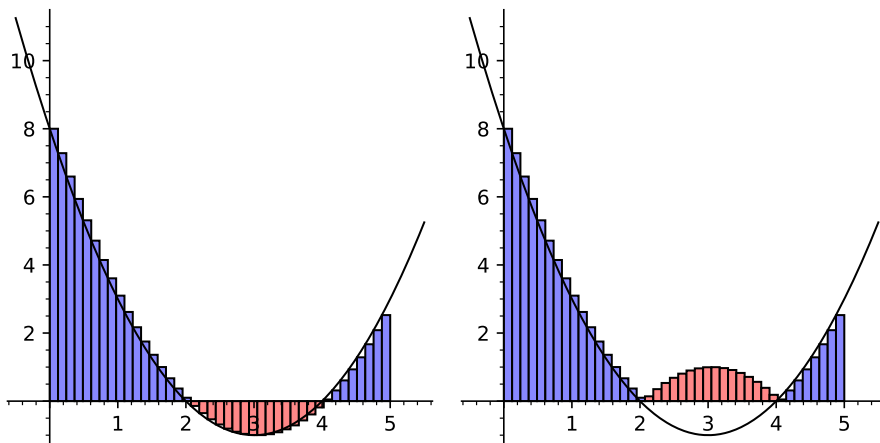


Figure 72 Partitioning $\int_0^5 (x-2)(x-4) dx$ at $x = 2$ and $x = 4$.

Since $f(x) = (x-2)(x-4)$ is zero when $x = 2$ and $x = 4$, we may compute the total area between $y = (x-2)(x-4)$ and the x -axis using absolute values as follows:

$$\text{Area} = \left| \int_0^2 (x-2)(x-4) dx \right| + \left| \int_2^4 (x-2)(x-4) dx \right| + \left| \int_4^5 (x-2)(x-4) dx \right|$$

Activity 4.7.4 Follow these steps to find the total area between $f(x) = 6x^2 - 18x$ and the x -axis from $x = 2$ to $x = 7$.

- Find all values for x where $f(x) = 6x^2 - 18x$ is equal to 0.
- Only one such value is between $x = 2$ and $x = 7$. Use this value to fill in the ? below, then verify that its value is 279 square units.

$$\text{Area} = \left| \int_2^? (6x^2 - 18x) dx \right| + \left| \int_?^7 (6x^2 - 18x) dx \right|$$

Activity 4.7.5 Answer the following questions concerning $f(x) = 6x^2 - 96$.

- What is the total area between $f(x) = 6x^2 - 96$ and the x -axis from $x = -1$ to $x = 9$?
- What is the net area between $f(x) = 6x^2 - 96$ and the x -axis from $x = -1$ to $x = 9$?

Answer.

$$1. \quad \left| \int_{-1}^4 (6x^2 - 96) dx \right| + \left| \int_4^9 (6x^2 - 96) dx \right| = 1200$$

$$2. \quad \int_{-1}^9 (6x^2 - 96) dx = 500$$

4.8 Area between curves (IN8)

Learning Outcomes

- Use definite integral(s) to compute the area bounded by several curves.

Remark 4.8.1 In [Section 4.7](#), we learned how to find the area between a curve and the x -axis ($f(x) = 0$) using a definite integral. What if we want the area between any two functions? What if the x -axis is not one of the boundaries?

In this section, we'll investigate how a definite integral may be used to represent the area between two curves.

Activity 4.8.2 Consider the functions given by $f(x) = 5 - (x - 1)^2$ and $g(x) = 4 - x$.

- Use algebra to find the points where the graphs of f and g intersect.
- Sketch an accurate graph of f and g on the xy plane, labeling the curves by name and the intersection points with ordered pairs.
- Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the x -axis on the interval between the intersection points of f and g . Shade this area in your sketch.
- Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the x -axis on the interval between the intersection points of f and g . Shade this area in your sketch.
- Let's denote the area between $y = f(x)$ and the x -axis as A_f and the area between $y = g(x)$ and the x -axis as A_g . How could we use A_f and

A_g to find exact area between f and g between their intersection points?

- A We could find $A_f + A_g$ to find the area between the curves.
- B We could find $A_f - A_g$ to find the area between the curves.
- C We could find $A_g - A_f$ to find the area between the curves.

Note 4.8.3 We've seen from [Activity 4.8.2](#) that a natural way to think about the area between two curve is as the area beneath the upper curve minus the area beneath the lower curve.

Activity 4.8.4 We now look for a general way of writing definite integrals for the area between two given curves, $f(x)$ and $g(x)$. Consider this area, illustrated in [Figure 73](#).

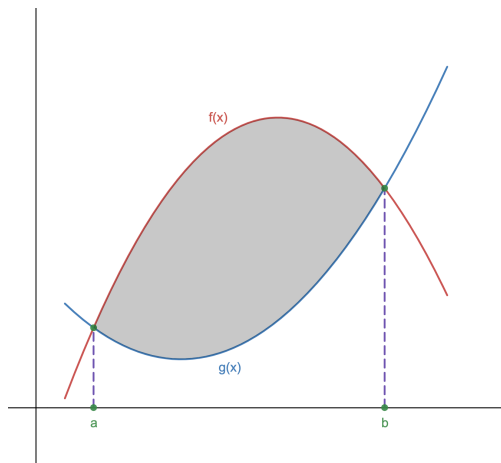


Figure 73 Area between $f(x)$ and $g(x)$.

(a) How could we represent the shaded area in [Figure 73](#)?

- A $\int_b^a f(x) dx - \int_b^a g(x) dx$
- B $\int_a^b f(x) dx - \int_a^b g(x) dx$
- C $\int_b^a g(x) dx - \int_b^a f(x) dx$
- D $\int_a^b g(x) dx - \int_a^b f(x) dx$

(b) The two definite integrals above can be rewritten as one definite integral using the sum and difference property of definite integrals:

If f and g are continuous functions, then

$$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

Use the property above to represent the shaded area in [Figure 73](#) using one definite integral.

- A $\int_b^a (f(x) - g(x)) dx$

$$\text{B } \int_a^b (f(x) - g(x)) dx$$

$$\text{C } \int_b^a (g(x) - f(x)) dx$$

$$\text{D } \int_a^b (g(x) - f(x)) dx$$

Fact 4.8.5 If two curves $y = f(x)$ and $y = g(x)$ intersect at $(a, g(a))$ and $(b, g(b))$, and for all x such that $a \leq x \leq b$, $f(x) \geq g(x)$, then the area between the curves is $A = \int_a^b (f(x) - g(x)) dx$.

Activity 4.8.6 In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

- (a) The finite region bounded by $y = \sqrt{x}$ and $y = \frac{1}{4}x$.
- (b) The finite region bounded by $y = 12 - 2x^2$ and $y = x^2 - 8$.
- (c) The area bounded by the y -axis, $f(x) = \cos(x)$, and $g(x) = \sin(x)$, where we consider the region formed by the first positive value of x for which f and g intersect.
- (d) The finite regions between the curves $y = x^3 - x$ and $y = x^2$.

Activity 4.8.7 Let \mathbf{R} be the finite region bounded by the graphs of $y = (x + 5)^2 - 1$ and $y = 7x + 34$.

Sketch an illustration of \mathbf{R} , and then explain how to express the area of \mathbf{R} in the following two ways:

1. As a definite integral with respect to x .
2. As a definite integral with respect to y .

(Do not evaluate either definite integral.)

Answer.

1.

$$\int_{-5}^2 \left(-(x + 5)^2 + 7x + 35 \right) dx$$

2.

$$\int_{-1}^{48} \left(\frac{1}{7}y + \sqrt{y + 1} + \frac{1}{7} \right) dy$$

Chapter 5

Techniques of Integration (TI)

Learning Outcomes

How do we use various techniques to integrate less simple functions?

By the end of this chapter, you should be able to...

1. Evaluate various integrals via the substitution method.
2. Compute integrals using integration by parts.
3. Compute integrals involving products of trigonometric functions.
4. Use trigonometric substitution to compute indefinite integrals.
5. I can integrate functions using a table of integrals.
6. I can integrate functions using the method of partial fractions.
7. I can select appropriate strategies for integration.
8. I can compute improper integrals.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Recognize when to apply the chain rule, and use it successfully.
 - Review: [Khan Academy](#)¹
- b Recognize when to apply the product rule, and use it successfully.
 - Review: [Khan Academy](#)²
- c Perform basic definite integrals.
 - Review: [Khan Academy](#)³
- d Compute basic antiderivatives.

¹<https://www.khanacademy.org/math/differential-calculus/dc-chain/dc-chain-rule/v/chain-rule-introduction?modal=1>

²<https://www.khanacademy.org/math/differential-calculus/dc-diff-intro/dc-product-rule/v/applying-the-product-rule-for-derivatives?modal=1>

³<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-7/v/connecting-the-first-and-second-fundamental-theorems-of-calculus?modal=1>

- Review: [Khan Academy](#)⁴
- e Differentiate trigonometric functions.
 - Review: [Khan Academy](#)⁵
- f Compute limits.
 - Review: [Khan Academy](#)⁶
 - Review: [Khan Academy](#)⁷
- g Apply the pythagorean identity.
 - Review: [Khan Academy](#)⁸

5.1 Substitution method (TI1)

Learning Outcomes

- Evaluate various integrals via the substitution method.

Activity 5.1.1 Answer the following.

- (a) Using the chain rule, which of these is derivative of e^{x^3} with respect to x ?

A e^{3x^2}

C $3x^2e^{x^3}$

B $x^3e^{x^3-1}$

D $\frac{1}{4}e^{x^4}$

- (b) Based on this result, which of these would you suspect to equal $\int x^2e^{x^3} dx$?

A $e^{x^3+1} + C$

C $3e^{x^3} + C$

B $\frac{1}{3x}e^{x^3+1} + C$

D $\frac{1}{3}e^{x^3} + C$

Activity 5.1.2 Recall that if u is a function of x , then $\frac{d}{dx}[u^7] = 7u^6u'$ by the Chain Rule.

For each question, choose from the following.

A $\frac{1}{7}u^7 + C$

B $u^7 + C$

C $7u^7 + C$

D $\frac{6}{7}u^7 + C$

- (a) What is $\int 7u^6u' dx$?

- (b) What is $\int u^6u' dx$?

- (c) What is $\int 6u^6u' dx$?

⁴<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-7/v/antiderivatives-and-indefinite-integrals?modal=1>

⁵<https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-1-new/ab-2-10/v/derivatives-of-tanx-and-cotx?modal=1>

⁶<https://www.khanacademy.org/math/ap-calculus-bc/bc-limits-new/bc-1-2/v/introduction-to-limits-hd?modal=1>

⁷<https://www.khanacademy.org/math/ap-calculus-bc/bc-diff-contextual-applications-new/bc-4-7/v/introduction-to-l-hopital-s-rule?modal=1>

⁸<https://www.khanacademy.org/math/trigonometry/unit-circle-trig-func/pythagorean-identity/v/using-the-pythagorean-trig-identity?modal=1>

Activity 5.1.3 Based on these activities, which of these choices seems to be a viable strategy for integration?

- A Memorize an integration formula for every possible function.
- B Attempt to rewrite the integral in the form $\int g'(u)u' dx = g(u) + C$.
- C Keep differentiating functions until you come across the function you want to integrate.

Fact 5.1.4 By the chain rule,

$$\frac{d}{dx}[g(u) + C] = g'(u)u'.$$

There is an dual integration technique reversing this process, known as the **substitution method**.

This technique involves choosing an appropriate function u in terms of x to rewrite the integral as follows:

$$\int f(x) dx = \cdots = \int g'(u)u' dx = g(u) + C.$$

Observation 5.1.5 Recall that $u' = \frac{du}{dx}$. This allows for the following:

$$\int g'(u)u' dx = \int g'(u)\frac{du}{dx} dx = \int g'(u)du = g(u) + C.$$

Therefore rather than dealing with equations like $u' = \frac{du}{dx} = x^2$, we will prefer to write $du = x^2 dx$.

Activity 5.1.6 Consider $\int x^2 e^{x^3} dx$, which we conjectured earlier to be $\frac{1}{3}e^{x^3} + C$.

Suppose we decided to let $u = x^3$.

- (a) Compute $\frac{du}{dx} = ?$, and rewrite it as $du = ? dx$.
- (b) Multiply both sides of the equation $du = ? dx$ so that its right-hand side appears in $\int x^2 e^{x^3} dx$.
- (c) Show why $\int x^2 e^{x^3} dx$ may now be rewritten as $\int \frac{1}{3}e^u du$.
- (d) Solve $\int \frac{1}{3}e^u du$ in terms of u , then replace u with x^3 to confirm our original conjecture.

Example 5.1.7 Here is how one might write out the explanation of how to find $\int x^2 e^{x^3} dx$ from start to finish:

$$\int x^2 e^{x^3} dx$$

$$\text{Let } u = x^3$$

$$du = 3x^2 dx$$

$$\frac{1}{3}du = x^2 dx$$

$$\begin{aligned} \int x^2 e^{x^3} dx &= \int e^{(x^3)}(x^2 dx) \\ &= \int e^u \frac{1}{3} du \\ &= \frac{1}{3}e^u + C \end{aligned}$$

$$= \frac{1}{3}e^{x^3} + C$$

□

Activity 5.1.8 Which step of the previous example do you think was the most important?

- A Choosing $u = x^3$.
- B Finding $du = 3x^2 dx$ and $\frac{1}{3}du = x^2 dx$.
- C Substituting $\int x^2 e^{x^3} dx$ with $\int \frac{1}{3}e^u du$.
- D Integrating $\int \frac{1}{3}e^u du = \frac{1}{3}e^u + C$.
- E Unsubstituting $\frac{1}{3}e^u + C$ to get $\frac{1}{3}e^{x^3} + C$.

Activity 5.1.9 Below are two correct solutions to the same integral, using two different choices for u . Which method would you prefer to use yourself?

$\int x\sqrt{4x+4} dx \quad \text{Let } u = x+1$ $4u = 4x+4$ $x = u-1$ $du = dx$ $\int x\sqrt{4x+4} dx = \int (u-1)\sqrt{4u} du$ $= \int (2u^{3/2} - 2u^{1/2}) du$ $= \frac{4}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$ $= \frac{4}{5}(x+1)^{5/2}$ $- \frac{4}{3}(x+1)^{3/2} + C$	$\int x\sqrt{4x+4} dx \quad \text{Let } u = \sqrt{4x+4}$ $u^2 = 4x+4$ $x = \frac{1}{4}u^2 - 1$ $dx = \frac{1}{2}u du$ $\int x\sqrt{4x+4} dx = \int \left(\frac{1}{4}u^2 - 1\right)(u) \left(\frac{1}{2}u du\right)$ $= \int \left(\frac{1}{8}u^4 - \frac{1}{2}u^2\right) du$ $= \frac{1}{40}u^5 - \frac{1}{6}u^3 + C$ $= \frac{1}{40}(4x+4)^{5/2}$ $- \frac{1}{6}(4x+4)^{3/2} + C$
--	--

Activity 5.1.10 Suppose we wanted to try substitution method to find $\int e^x \cos(e^x + 3) dx$. Which of these choices for u appears to be most useful?

- A $u = x$, so $du = dx$
- B $u = e^x$, so $du = e^x dx$
- C $u = e^x + 3$, so $du = e^x dx$
- D $u = \cos(x)$, so $du = -\sin(x) dx$
- E $u = \cos(e^x + 3)$, so $du = e^x \sin(e^x + 3) dx$

Activity 5.1.11 Complete the following solution using your choice from the previous activity to find $\int e^x \cos(e^x + 3) dx$.

$\int e^x \cos(e^x + 3) dx$ $\int e^x \cos(e^x + 3) dx = \int ? du$	<p>Let $u = ?$</p> <p>$du = ? dx$</p>
---	---

$$= \dots$$

$$= \sin(e^x + 3) + C$$

Activity 5.1.12 Complete the following integration by substitution to find $\int \frac{x^3}{x^4+4} dx$.

$$\int \frac{x^3}{x^4+4} dx$$

Let $u = ?$

$$du = ? dx$$

$$? du = ? dx$$

$$\int \frac{x^3}{x^4+4} dx = \int \frac{?}{?} du$$

$$= \dots$$

$$= \frac{1}{4} \ln|x^4+4| + C$$

Activity 5.1.13 Given that $\int \frac{x^3}{x^4+4} dx = \frac{1}{4} \ln|x^4+4| + C$, what is the value of $\int_0^2 \frac{x^3}{x^4+4} dx$?

A $\frac{8}{20}$

C $\frac{1}{4} \ln(20) - \frac{1}{4} \ln(4)$

B $-\frac{8}{20}$

D $\frac{1}{4} \ln(4) - \frac{1}{4} \ln(20)$

Activity 5.1.14 What's wrong with the following computation?

$$\int_0^2 \frac{x^3}{x^4+4} dx$$

Let $u = x^4 + 4$

$$du = 4x^3 dx$$

$$\frac{1}{4} du = x^3 dx$$

$$\begin{aligned} \int_0^2 \frac{x^3}{x^4+4} dx &= \int_0^2 \frac{1/4}{u} du \\ &= \left[\frac{1}{4} \ln|u| \right]_0^2 \\ &= \frac{1}{4} \ln 2 - \frac{1}{4} \ln 0 \end{aligned}$$

A The wrong u substitution was made.

B The antiderivative of $\frac{1/4}{u}$ was wrong.

C The x values 0, 2 were plugged in for the variable u .

Activity 5.1.15 When $x = 2$, we know that $u = 2^4 + 4 = 20$. Likewise when $x = 0$, we know that $u = 0^4 + 4 = 4$. Use these facts to complete the below computation.

$$\int_0^2 \frac{x^3}{x^4+4} dx$$

Let $u = x^4 + 4$

$$du = 4x^3 dx$$

$$\frac{1}{4} du = x^3 dx$$

$$\begin{aligned}
\int_{x=0}^{x=2} \frac{x^3}{x^4 + 4} dx &= \int_{u=?}^{u=?} \frac{1/4}{u} du \\
&= \dots \\
&= \frac{1}{4} \ln(20) - \frac{1}{4} \ln(4)
\end{aligned}$$

Example 5.1.16 Here is how one might write out the explanation of how to find $\int_1^3 x^2 e^{x^3} dx$ from start to finish by leaving bounds in terms of x :

$$\begin{aligned}
\int_1^3 x^2 e^{x^3} dx & \qquad \text{Let } u = x^3 \\
& \qquad \qquad \qquad du = 3x^2 dx \\
& \qquad \qquad \qquad \frac{1}{3} du = x^2 dx \\
\int_1^3 x^2 e^{x^3} dx &= \int_{x=1}^{x=3} e^{(x^3)} (x^2 dx) \\
&= \int_{x=1}^{x=3} e^u \frac{1}{3} du \\
&= \left[\frac{1}{3} e^u \right]_{x=1}^{x=3} \\
&= \left[\frac{1}{3} e^{x^3} \right]_{x=1}^{x=3} \\
&= \frac{1}{3} e^{3^3} - \frac{1}{3} e^{1^3} \\
&= \frac{1}{3} e^{27} - \frac{1}{3} e
\end{aligned}$$

□

Example 5.1.17 Here is how one might write out the explanation of how to find $\int_1^3 x^2 e^{x^3} dx$ from start to finish by transforming bounds into terms of u :

$$\begin{aligned}
\int_1^3 x^2 e^{x^3} dx & \qquad \text{Let } u = x^3 \\
& \qquad \qquad \qquad du = 3x^2 dx \\
& \qquad \qquad \qquad \frac{1}{3} du = x^2 dx \\
\int_1^3 x^2 e^{x^3} dx &= \int_{x=1}^{x=3} e^{(x^3)} (x^2 dx) \\
&= \int_{u=1^3}^{u=3^3} e^u \frac{1}{3} du \\
&= \left[\frac{1}{3} e^u \right]_1^{27} \\
&= \frac{1}{3} e^{27} - \frac{1}{3} e
\end{aligned}$$

□

Activity 5.1.18 Use substitution to show that

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^2 - 2e.$$

Activity 5.1.19 Use substitution to show that

$$\int_0^{\pi/4} \sin(2\theta) d\theta = \frac{1}{2}.$$

Activity 5.1.20 Use substitution to show that

$$\int u^5(u^3 + 1)^{1/3} du = \frac{1}{7}(u^3 + 1)^{7/3} - \frac{1}{4}(u^3 + 1)^{4/3} + C$$

Activity 5.1.21 Consider $\int (3x - 5)^2 dx$.

- Solve this integral using substitution.
- Replace $(3x - 5)^2$ with $(9x^2 - 30x + 25)$ in the original integral, then solve using the reverse power rule.
- Which method did you prefer?

Activity 5.1.22 Consider $\int \tan(x) dx$.

- Replace $\tan(x)$ in the integral with a fraction involving sine and cosine.
- Use substitution to solve the integral.

5.1.1 Videos



YouTube: https://www.youtube.com/watch?v=-9amHammH_I

Figure 74 Video: Evaluate various integrals via the substitution method

5.2 Integration by Parts (TI2)

Learning Outcomes

- Compute integrals using integration by parts.

Activity 5.2.1 Answer the following.

- Using the product rule, which of these is derivative of $x^3 e^x$ with respect to x ?

A $3x^2 e^x$

C $3x^2 e^{x-1}$

B $3x^2 e^x + x^3 e^x$

D $\frac{1}{4} x^4 e^x$

- Based on this result, which of these would you suspect to equal $\int 3x^2 e^x + x^3 e^x dx$?

A $x^3 e^x + C$

C $6x e^x + 3x^2 e^x + C$

B $x^3 e^x + \frac{1}{4} x^4 e^x + C$

D $6x e^x + 3x^2 e^x + 3x^2 e^x + x^3 e^x + C$

Activity 5.2.2 Answer the following.

(a) Which differentiation rule is easier to implement?

A Product Rule

B Chain Rule

(b) Which differentiation strategy do expect to be easier to reverse?

A Product Rule

B Chain Rule

Activity 5.2.3 Answer the following.

(a) Which of the following equations is equivalent to the formula $\frac{d}{dx}[uv] = u'v + uv'$?

A $uv' = -\frac{d}{dx}(uv) - vu'$

C $uv' = \frac{d}{dx}(uv) + vu'$

B $uv' = -\frac{d}{dx}(uv) + vu'$

D $uv' = \frac{d}{dx}(uv) - vu'$

(b) Which of these is the most concise result of integrating both sides with respect to x ?

A $\int (uv') dx = uv - \int (vu') dx$

B $\int (u) dv = uv - \int (v) du$

C $\int (uv') dx = uv - \int (vu') dx + C$

D $\int (u) dv = uv - \int (v) du + C$

Fact 5.2.4 By the product rule, $\frac{d}{dx}[uv] = u'v + uv'$ and, subsequently, $uv' = \frac{d}{dx}[uv] - u'v$. There is a dual integration technique reversing this process, known as **integration by parts**.

This technique involves using algebra to rewrite an integral of a product of functions in the form $\int (u) dv$ and then using the equality

$$\int (u) dv = uv - \int (v) du.$$

Activity 5.2.5 Consider $\int xe^x dx$. Suppose we decided to let $u = x$.

(a) Compute $\frac{du}{dx} = ?$, and rewrite it as $du = ? dx$.

(b) What is the best candidate for dv ?

A $dv = x dx$

C $dv = x$

B $dv = e^x$

D $dv = e^x dx$

(c) Given that $dv = e^x dx$, find $v = ?$.

(d) Show why $\int xe^x dx$ may now be rewritten as $xe^x - \int e^x dx$.

(e) Solve $\int e^x dx$, and then give the most general antiderivative of $\int xe^x dx$.

Example 5.2.6 Here is how one might write out the explanation of how to find $\int x e^x dx$ from start to finish:

$$\begin{aligned} \int x e^x dx & \\ u = x & \qquad \qquad \qquad dv = e^x dx \\ du = 1 \cdot dx & \qquad \qquad \qquad v = e^x \\ \int x e^x dx = x e^x - \int e^x dx & \\ = x e^x - e^x + C & \end{aligned}$$

□

Activity 5.2.7 Which step of the previous example do you think was the most important?

- A Choosing $u = x$ and $dv = e^x dx$.
- B Finding $du = 1 dx$ and $v = e^x dx$.
- C Applying integration by parts to rewrite $\int x e^x dx$ as $x e^x - \int e^x dx$.
- D Integrating $\int e^x dx$ to get $x e^x - e^x + C$.

Activity 5.2.8 Consider the integral $\int x^9 \ln(x) dx$. Suppose we proceed using integration by parts. We choose $u = \ln(x)$ and $dv = x^9 dx$. What is du ? What is v ? What do you get when plugging these pieces into integration by parts? Does the new integral $\int v du$ seem easier or harder to compute than the original integral $\int x^9 \ln(x) dx$?

- A The original integral is easier to compute.
- B The new integral is easier to compute.
- C Neither integral seems harder than the other one.

Activity 5.2.9 Consider the integral $\int x^9 \ln(x) dx$ once more. Suppose we still proceed using integration by parts. However, this time we choose $u = x^9$ and $dv = \ln(x) dx$. Do you prefer this choice or the choice we made in [Activity 5.2.8](#)?

- A We prefer the substitution choice of $u = \ln(x)$ and $dv = x^9 dx$.
- B We prefer the substitution choice of $u = x^9$ and $dv = \ln(x) dx$.
- C We do not have a strong preference, since these choices are of the same difficulty.

Activity 5.2.10 Consider the integral $\int x \cos(x) dx$. Suppose we proceed using integration by parts. Which of the following candidates for u and dv would best allow you to evaluate this integral?

- A $u = \cos(x)$, $dv = x dx$
- B $u = \cos(x) dx$, $dv = x$
- C $u = x dx$, $dv = \cos(x)$
- D $u = x$, $dv = \cos(x) dx$

Activity 5.2.11 Evaluate the integral $\int x \cos(x) dx$ using integration by parts.

Activity 5.2.12 Now use integration by parts to evaluate the integral $\int_{\frac{\pi}{6}}^{\pi} x \cos(x) dx$.

Activity 5.2.13 Consider the integral $\int x \arctan(x) dx$. Suppose we proceed using integration by parts. Which of the following candidates for u and dv would best allow you to evaluate this integral?

- A $u = x dx, dv = \arctan(x)$
- B $u = \arctan(x), dv = x dx$
- C $u = x \arctan(x), dv = dx$
- D $u = x, dv = \arctan(x) dx$

Activity 5.2.14 Consider the integral $\int e^x \cos(x) dx$. Suppose we proceed using integration by parts. Which of the following candidates for u and dv would best allow you to evaluate this integral?

- A $u = e^x, dv = \cos(x) dx$
- B $u = \cos(x), dv = e^x dx$
- C $u = e^x dx, dv = \cos(x)$
- D $u = \cos(x) dx, dv = e^x$

Activity 5.2.15 Suppose we started using integration by parts to solve the integral $\int e^x \cos(x) dx$ as follows:

$$\begin{aligned} \int e^x \cos(x) dx \\ \begin{array}{ll} u = \cos(x) & dv = e^x dx \\ du = -\sin(x) dx & v = e^x \end{array} \\ \int e^x \cos(x) dx = \cos(x)e^x - \int e^x(-\sin(x) dx) \\ = \cos(x)e^x + \int e^x \sin(x) dx \end{aligned}$$

We will have to use integration by parts a second time to evaluate the integral $\int e^x \sin(x) dx$. Which of the following candidates for u and dv would best allow you to continue evaluating the original integral $\int e^x \cos(x) dx$?

- A $u = e^x, dv = \sin(x) dx$
- B $u = \sin(x), dv = e^x dx$
- C $u = e^x dx, dv = \sin(x)$
- D $u = \sin(x) dx, dv = e^x$

Activity 5.2.16 Use integration by parts to show that $\int_0^{\frac{\pi}{4}} x \sin(2x) dx = \frac{1}{4}$.

Activity 5.2.17 Consider the integral $\int t^5 \sin(t^3) dt$.

- (a) Use the substitution $x = t^3$ to rewrite the integral in terms of x .
- (b) Use integration by parts to evaluate the integral in terms of x .
- (c) Replace x with t^3 to finish evaluating the original integral.

Activity 5.2.18 Use integration by parts to show that $\int \ln(z) dz = z \ln(z) - z + C$.

Activity 5.2.19 Given that $\int \ln(z) dz = z \ln(z) - z + C$, evaluate $\int (\ln(z))^2 dz$.

Activity 5.2.20 Consider the antiderivative $\int (\sin(x))^2 dx$.

(a) Noting that $\int (\sin(x))^2 dx = \int (\sin(x))(\sin(x)) dx$ and letting $u = \sin(x)$, $dv = \sin(x) dx$, what equality does integration by parts yield?

A $\int (\sin(x))^2 dx = \sin(x) \cos(x) + \int (\cos(x))^2 dx.$

B $\int (\sin(x))^2 dx = -\sin(x) \cos(x) + \int (\cos(x))^2 dx.$

C $\int (\sin(x))^2 dx = \sin(x) \cos(x) - \int (\cos(x))^2 dx.$

D $\int (\sin(x))^2 dx = -\sin(x) \cos(x) - \int (\cos(x))^2 dx.$

(b) Using the fact that $(\cos(x))^2 = 1 - (\sin(x))^2$ to rewrite the above equality.

(c) Solve algebraically for $\int (\sin(x))^2 dx$.

Activity 5.2.21 Modifying the approach from [Activity 5.2.20](#), use parts to find $\int (\cos(x))^2 dx$.

5.2.1 Videos



YouTube: <https://www.youtube.com/watch?v=m4mJd4kpwr4>

Figure 75 Video: Compute integrals using integration by parts

5.3 Integration of trigonometry (TI3)

Learning Outcomes

- Compute integrals involving products of trigonometric functions.

Activity 5.3.1 Consider $\int \sin(x) \cos(x) dx$. Which strategy would you choose

to evaluate this integral?

- A The method of substitution
- B Integration by parts
- C Other (specify your strategy)

Activity 5.3.2 Consider $\int \sin^2(x) \cos(x) dx$. Which strategy would you choose to evaluate this integral?

- A The method of substitution
- B Integration by parts
- C Other (specify your strategy)

Activity 5.3.3 Consider $\int \sin^2(x) \cos^2(x) dx$. Which strategy would you choose to evaluate this integral?

- A The method of substitution
- B Integration by parts
- C Other (specify your strategy)

Activity 5.3.4 Let's consider $\int \sin^2(x) \cos^2(x) dx$ again.

- A Use the fact that $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$ and $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ to rewrite the integrand using the above identities to an integral involving $\cos^2(2x)$.
- B Use the above identities to rewrite this new integrand as one involving $\cos(4x)$.
- C Evaluate the integral.

Activity 5.3.5 Consider $\int \sin^m(x) \cos^n(x) dx$, where m is even and n is odd. Which strategy would you choose to evaluate this integral?

- A Rewrite sine in terms of cosine by using a double-angle/half-angle identity.
- B Rewrite cosine in terms of sine by using a double-angle/half-angle identity.
- C Rewrite cosine in terms of sine by using a pythagorean identity.

Activity 5.3.6 Consider $\int \sin^m(x) \cos^n(x) dx$, where m and n are both odd and $m \geq n$. Which strategy would you choose to evaluate this integral?

- A Rewrite sine in terms of cosine by using a pythagorean identity.
- B Rewrite cosine in terms of sine by using a double-angle/half-angle identity.
- C Rewrite cosine in terms of sine by using a pythagorean identity.

Activity 5.3.7 Consider $\int \sin^m(x) \cos^n(x) dx$, where m and n are both even

and $m \geq n$. Which strategy would you choose to evaluate this integral?

- A Rewrite sine in terms of cosine by using a pythagorean identity.
- B Rewrite cosine in terms of sine by using a pythagorean identity.
- C Rewrite everything in terms of cosine by using a double-angle/half-angle identity.

Activity 5.3.8 Consider $\int \sin^4(x) \cos^4(x) dx$. Which would be the most useful way to rewrite the integral?

- A $\int (1 - \cos^2(x))^2 \cos^4(x) dx$
- B $\int \sin^4(x)(1 - \sin^2(x))^2 dx$
- C $\int \left(\frac{1 - \cos(2x)}{2}\right)^2 \left(\frac{1 + \cos(2x)}{2}\right)^2 dx$

Activity 5.3.9 Consider $\int \sin^3(x) \cos^5(x) dx$. Which would be the most useful way to rewrite the integral?

- A $\int (1 - \cos^2(x)) \sin(x) \cos^5(x) dx$
- B $\int \sin^3(x) \left(\frac{1 + \cos(2x)}{2}\right)^2 \cos(x) dx$
- C $\int \sin^3(x)(1 - \sin^2(x))^2 \cos(x) dx$

Remark 5.3.10 We might also use some other trigonometric identities to manipulate our integrands, listed in [Appendix B](#).

Activity 5.3.11 Consider $\int \sin(\theta) \sin(3\theta) d\theta$.

- A Find an identity from the table above which could be used to transform our integrand.
- B Rewrite the integrand using the selected identity.
- C Evaluate the integral.

5.3.1 Videos



YouTube: <https://www.youtube.com/watch?v=1iPsINVuHuQ>

Figure 76 Video: Compute integrals involving products of trigonometric functions

5.4 Trigonometric Substitution (TI4)

Learning Outcomes

- Use trigonometric substitution to compute indefinite integrals.

Activity 5.4.1 Consider the integral $\int \frac{\sqrt{1-x^2}}{x} dx$. If one were to use the method of substitution to evaluate this integral, what would be the best choice of u ?

A $u = 1 - x^2$.

B $u = \sqrt{1 - x^2}$.

C $u = \frac{1}{x}$.

D There is no u for which the method of substitution will work.

Activity 5.4.2 It would be good if $\sqrt{1-x^2}$ were a term that can be written more simply. Which of the following mathematical facts could help us simplify this integrand?

A For any real number, x^2 must be positive (note: this isn't even true).

B $u = \sqrt{1 - x^2}$.

C The Pythagorean theorem allows us to write $(\cos(\theta))^2$ as $1 - (\sin(\theta))^2$.

D The ratio of an even function and an odd function is odd.

Activity 5.4.3 Suppose we let $x = \sin(\theta)$, then $\frac{dx}{d\theta} = \cos(\theta)$ and $dx = \cos(\theta)d\theta$.

Which of the following best represents the integral $\int \sqrt{1-x^2} dx$?

A $\int \sqrt{1 - \sin^2(\theta)} dx$.

B $\int \sqrt{1 - \sin^2(\theta)} \cos(\theta) d\theta$.

C $\int \sqrt{1 - \sin^2(\theta)} \cos(\theta) d\theta$.

Activity 5.4.4 Use the fact from [Activity 5.4.2](#) to rewrite your choice of integral from [Activity 5.4.3](#) to an integral without any $\sin(\theta)$ terms.

A $\int \cos(\theta) d\theta$.

B $\int \cos^2(\theta) d\theta$.

C $\int \sqrt{\cos(\theta)} d\theta$.

Activity 5.4.5 Use the fact that $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$ to rewrite the integral from [Activity 5.4.4](#).

A $\int \sqrt{\frac{1+\cos(2\theta)}{2}} d\theta$.

B $\int \frac{1 + \cos(2\theta)}{2} d\theta.$

C $\int \left(\frac{1 + \cos(2\theta)}{2} \right)^2 d\theta.$

Activity 5.4.6 Evaluate the integral in terms of θ that you found in [Activity 5.4.5](#).

Activity 5.4.7 Consider the fact that $x = \sin(\theta)$ and thus $\theta = \arcsin(x)$. Use this to rewrite the solution you found in [Activity 5.4.6](#) in terms of x .

Activity 5.4.8 Consider the integral $\int \sqrt{9 - x^2} dx$; how can we best modify our approach in [Activity 5.4.3](#) and adapt it here?

A Let $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$, $\int \sqrt{9 - x^2} dx = \int \sqrt{9 - \sin^2(\theta)} \cos(\theta)d\theta.$

B Let $x = 3 \sin(\theta)$, $dx = 3 \cos(\theta)d\theta$, $\int \sqrt{9 - x^2} dx = \int \sqrt{9 - (3 \sin(\theta))^2} \cos(\theta)d\theta.$

C Let $x = \sin(3\theta)$, $dx = 3 \cos(3\theta)d\theta$, $\int \sqrt{9 - x^2} dx = \int \sqrt{9 - \sin^2(3\theta)} 3 \cos(3\theta)d\theta.$

Activity 5.4.9 Using your choice of substitution in [Activity 5.4.8](#) and the fact that $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$ to rewrite the integral from [Activity 5.4.8](#)

A $\int \sqrt{\frac{9 + 9 \cos(2\theta)}{2}} \cos(\theta)d\theta.$

B $\int \frac{9 + 9 \cos(2\theta)}{2} \cos(\theta)d\theta.$

C $\int \left(\frac{9 + 9 \cos(2\theta)}{2} \right)^2 \cos(\theta)d\theta.$

Activity 5.4.10 Evaluate the integral that you found in [Activity 5.4.9](#). (Recall that since $x = 3 \sin(\theta)$, $\theta = \arcsin\left(\frac{x}{3}\right)$.)

Activity 5.4.11 Consider the integral $\int \frac{1}{1 + x^2} dx$. Which of the following facts would best simplify this expression?

A $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}.$

B $(\cos(\theta))^2 = 1 - (\sin(\theta))^2$

C Since $\sin^2(\theta) + \cos^2(\theta) = 1$, $\tan^2(\theta) + 1 = \sec^2(\theta).$

Activity 5.4.12 Suppose we let $x = \tan(\theta)$, $dx = \sec^2(\theta)$. Which of the following best represents the integral $\int \frac{1}{1 + x^2} dx$?

A $\int \frac{1}{1 + \tan^2(\theta)} dx.$

B $\int \frac{1}{1 + \tan^2(\theta) \sec^2(\theta)} d\theta.$

C $\int \frac{1}{1 + \tan^2(\theta)} \sec^2(\theta) d\theta.$

Activity 5.4.13 Using the fact from [Activity 5.4.11](#), simplify and evaluate your choice of integral from [Activity 5.4.12](#). (Recall that since $x = \tan(\theta)$, $\theta = \arctan(x)$.)

Activity 5.4.14 Consider the integral $\int \frac{1}{\frac{1}{4} + x^2} dx$; how can we best modify our approach in [Activity 5.4.12](#) and adapt it here?

A Let $x = \frac{1}{2} \tan(\theta)$, $dx = \frac{1}{2} \sec^2(\theta) d\theta$, $\int \frac{1}{\frac{1}{4} + x^2} dx = \int \frac{1}{\frac{1}{4} + (\frac{1}{2} \tan(\theta))^2} \frac{1}{2} \sec^2(\theta) d\theta$.

B Let $x = \frac{1}{4} \tan(\theta)$, $dx = \frac{1}{4} \sec^2(\theta) d\theta$, $\int \frac{1}{\frac{1}{4} + x^2} dx = \int \frac{1}{\frac{1}{4} + (\frac{1}{4} \tan(\theta))^2} \frac{1}{4} \sec^2(\theta) d\theta$.

C Let $x = \tan(\frac{1}{2}\theta)$, $dx = \sec^2(\frac{1}{2}\theta) d\theta$, $\int \frac{1}{\frac{1}{4} + x^2} dx = \int \frac{1}{\frac{1}{4} + (\tan(\frac{1}{2}\theta))^2} \frac{1}{2} \sec^2(\frac{1}{2}\theta) d\theta$.

Activity 5.4.15 Evaluate the choice of integral you found in [Activity 5.4.8](#), using the fact that if $x = \frac{1}{2} \tan(\theta)$, $\theta = \arctan(2x)$.

Activity 5.4.16 Consider the integral $\int \sqrt{x^2 - 1} dx$. Which of the following facts *best* simplifies this integral?

A $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$.

B $\tan^2(\theta) + 1 = \sec^2(\theta)$.

C Since $\sin^2(\theta) = 1 - \cos^2(\theta)$, $\tan^2(\theta) = \sec^2(\theta) - 1$.

Activity 5.4.17 Using your choice in [Activity 5.4.16](#), pick an appropriate trigonometric substitution for x and rewrite $\int \sqrt{x^2 - 1} dx$ as an integral in terms of θ . (Remember to compute dx as well!)

Activity 5.4.18 Evaluate the integral you found in [Activity 5.4.17](#), using the fact that if $x = \sec(\theta)$, $\theta = \operatorname{arcsec}(x)$.

Hint. To better understand the relationship between x and functions of θ , consider this right triangle where $\sec(\theta) = x$.

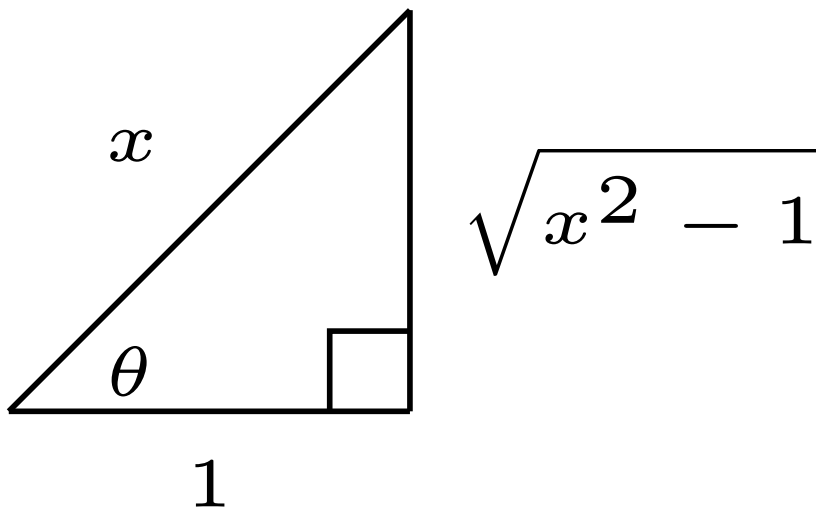


Figure 77 A right triangle where $\sec(\theta) = x$.

Activity 5.4.19 If we wish to simplify an expression that contained a $k^2 - x^2$ expression, which would be the best substitution?

A $x = k \sin(\theta)$

B $x = k \tan(\theta)$

C $x = k \sec(\theta)$

Activity 5.4.20 If we wish to simplify an expression that contained a $k^2 + x^2$ expression, which would be the best substitution?

A $x = k \sin(\theta)$

B $x = k \tan(\theta)$

C $x = k \sec(\theta)$

Activity 5.4.21 If we wish to simplify an expression that contained a $x^2 - k^2$ expression, which would be the best substitution?

A $x = k \sin(\theta)$

B $x = k \tan(\theta)$

C $x = k \sec(\theta)$

Activity 5.4.22 Consider the unit circle $x^2 + y^2 = 1$. Find a function $f(x)$ so that $y = f(x)$ is the graph of the upper-half semicircle of the unit circle.

Activity 5.4.23

- Find the area under the curve $y = f(x)$ from [Activity 5.4.22](#).
- How does this value compare to what we know about areas of circles?

5.4.1 Videos



YouTube: <https://www.youtube.com/watch?v=FTFtiiDARf0>

Figure 78 Video: Use trigonometric substitution to compute indefinite integrals

5.5 Tables of Integrals (TI5)

Learning Outcomes

- I can integrate functions using a table of integrals.

Activity 5.5.1 Consider the integral $\int \sqrt{16 - 9x^2} dx$. Which of the following substitutions appears most promising to find an antiderivative for this integral?

A $u = 16 - 9x^2$

B $u = 9x^2$

C $u = 3x$

D $u = x$

Activity 5.5.2 The form of which entry from [Appendix A](#) best matches the form of the integral $\int \sqrt{16 - 9x^2} \, dx$?

A b.

B c.

C g.

D h.

Activity 5.5.3 For each of the following integrals, identify which entry from [Appendix A](#) best matches the form of that integral.

(a) The integral $\int \frac{25x^2}{\sqrt{25x^2 - 9}} \, dx$

(b) The integral $\int \frac{81x^2}{\sqrt{16 - x^2}} \, dx$

(c) The integral $\int \frac{1}{10x\sqrt{100 - x^2}} \, dx$

(d) The integral $\int \frac{7}{\sqrt{25x^2 - 9}} \, dx$

(e) The integral $\int \frac{1}{\sqrt{25x^2 + 16}} \, dx$

Example 5.5.4 Here is how one might write out the explanation of how to find $\int \frac{3}{x\sqrt{49x^2 - 4}} \, dx$ from start to finish:

$$\int \frac{3}{x\sqrt{49x^2 - 4}} \, dx$$

Let $u^2 = 49x^2$

Let $a^2 = 4$

$u = 7x$

$du = 7 \, dx$

$\frac{1}{7} du = dx$

$a = 2$

$$\begin{aligned} \int \frac{3}{x\sqrt{49x^2 - 4}} \, dx &= 3 \int \frac{1}{x\sqrt{49x^2 - 4}} (dx) \\ &= 3 \int \frac{1}{\frac{u}{7}\sqrt{u^2 - a^2}} \left(\frac{1}{7} du\right) \\ &= 3 \int \frac{1}{u\sqrt{u^2 - a^2}} du \quad \text{which best matches f.} \\ &= 3 \left(\frac{1}{a} \operatorname{arcsec} \left(\frac{u}{a} \right) \right) + C \\ &= \frac{3}{2} \operatorname{arcsec} \left(\frac{7x}{2} \right) + C \end{aligned}$$

□

Activity 5.5.5 Which step of the previous example do you think was the most important?

A Choosing $u^2 = 49x^2$ and $a^2 = 4$.

B Finding $u = 7x$, $du = 7 dx$, $\frac{1}{7} du = dx$, and $a = 2$.

C Substituting $\frac{3}{x\sqrt{49x^2-4}} dx$ with $3 \int \frac{1}{u\sqrt{u^2-a^2}} du$ and finding the best match of f from [Appendix A](#).

D Integrating $3 \int \frac{1}{u\sqrt{u^2-a^2}} du = 3(\frac{1}{a} \operatorname{arcsec}(\frac{u}{a})) + C$.

E Unsubstituting $3(\frac{1}{a} \operatorname{arcsec}(\frac{u}{a})) + C$ to get $\frac{3}{2} \operatorname{arcsec}(\frac{7x}{2}) + C$.

Activity 5.5.6 Consider the integral $\int \frac{1}{\sqrt{64-9x^2}} dx$. Suppose we proceed using [Appendix A](#). We choose $u^2 = 9x^2$ and $a^2 = 64$.

(a) What is u ?

(b) What is du ?

(c) What is a ?

(d) What do you get when plugging these pieces into the integral $\int \frac{1}{\sqrt{64-9x^2}} dx$?

(e) Is this a good substitution choice or a bad substitution choice?

Activity 5.5.7 Consider the integral $\int \frac{1}{\sqrt{64-9x^2}} dx$ once more. Suppose we still proceed using [Appendix A](#). However, this time we choose $u^2 = x^2$ and $a^2 = 64$. Do you prefer this choice of substitution or the choice we made in [Activity 5.5.6](#)?

A We prefer the substitution choice of $u^2 = x^2$ and $a^2 = 64$.

B We prefer the substitution choice of $u^2 = 9x^2$ and $a^2 = 64$.

C We do not have a strong preference, since these substitution choices are of the same difficulty.

Activity 5.5.8 Use the appropriate substitution and entry from [Appendix A](#) to show that $\int \frac{7}{x\sqrt{4+49x^2}} dx = -\frac{7}{2} \ln \left| \frac{2+\sqrt{49x^2+4}}{7x} \right| + C$.

Activity 5.5.9 Use the appropriate substitution and entry from [Appendix A](#) to show that $\int \frac{3}{5x^2\sqrt{36-49x^2}} dx = -\frac{\sqrt{36-49x^2}}{60x} + C$.

Activity 5.5.10 Evaluate the integral $\int 8\sqrt{4x^2-81} dx$. Be sure to specify which entry is used from [Appendix A](#) at the corresponding step.

5.5.1 Videos



YouTube: <https://www.youtube.com/watch?v=r8KqUVrHsdQ>

Figure 79 Video: I can integrate functions using a table of integrals

5.6 Partial fractions (TI6)

Learning Outcomes

- I can integrate functions using the method of partial fractions.

Activity 5.6.1 Using the method of substitution, which of these is the integral

of $\int \frac{1}{(ax+b)} dx$?

- A $\frac{1}{a} \ln(ax+b) + C$.
- B $\frac{1}{a} \ln|ax+b| + C$.
- C $\frac{1}{x} (\ln(ax) + \ln(b)) + C$.
- D $\frac{1}{a} (\ln|ax| + \ln|b|) + C$.

Activity 5.6.2 Which of the following is equivalent to the rational expression

$$\frac{3x+3}{2x^2+3x}?$$

- A $\frac{3}{2x} + \frac{1}{x}$.
- B $\frac{3}{x} + \frac{3}{2x+3}$.
- C $\frac{1}{x} + \frac{1}{2x+3}$.
- D $\frac{3}{2x^2} + 1$.

Activity 5.6.3 Based on your choice in [Activity 5.6.2](#), which of these is the

integral for $\int \frac{3x+3}{2x^2+3x} dx$?

- A $\frac{3}{2} \ln|x| + \ln|2x+3| + C$.
- B $3 \ln|x| + \frac{3}{2} \ln|2x+3| + C$.
- C $\ln|x| + \frac{1}{2} \ln|2x+3| + C$.
- D $-\frac{3}{2}x^{-1} + x + C$.

Activity 5.6.4 Which of the following is an antiderivative for $\frac{1}{(x+k)^2+h^2}$?

- A $\arctan(x)$.
- B $\arctan(x+k)$.
- C $\frac{1}{h} \arctan\left(\frac{x+h}{h}\right)$.
- D $\frac{1}{h^2} \arctan\left(\frac{x+k}{h^2}\right)$.

E $\frac{1}{h} \arctan\left(\frac{x+k}{h}\right).$

Activity 5.6.5 Consider the irreducible quadratic $x^2 - 4x + 80$. Which of the following is the completed square form of this quadratic expression?

A $(x - 4)^2 + 80$.

B $(x - 2)^2 + 80$.

C $(x - 4)^2 + 64$.

D $(x - 2)^2 + 76$.

E $(x - 4)^2 + 76$.

Activity 5.6.6 Using the fact that $\int \frac{1}{x^2 - 8x + 80} dx = \int \frac{1}{(x - 4)^2 + 64} dx$, evaluate the integral $\int \frac{1}{x^2 - 8x + 80} dx$.

Activity 5.6.7 Suppose that one could write $\frac{4x^2 - 5x - 2}{x^2(x - 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2}$. Recall that:

$$\begin{aligned}\frac{4x^2 - 5x - 2}{x^2(x - 2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2} = \frac{Ax(x - 2) + B(x - 2) + Cx^2}{x^2(x - 2)} \\ \frac{4x^2 - 5x - 2}{x^2(x - 2)} &= \frac{Ax(x - 2) + B(x - 2) + Cx^2}{x^2(x - 2)} \\ 4x^2 - 5x - 2 &= Ax(x - 2) + B(x - 2) + Cx^2.\end{aligned}$$

What are the values of A, B and C ?

A $A = 4, B = -5, C = -2$.

B $A = 3, B = 1, C = 1$.

C $A = 1, B = 1, C = 1$.

D $A = 1, B = 3, C = -1$.

E $A = 2, B = 1, C = 1$.

Fact 5.6.8 Partial Fraction Decomposition. Let $\frac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q .

A Linear Terms: Let $(x - a)$ divide $q(x)$, where $(x - a)^n$ is the highest power of $(x - a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x - a)} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}.$$

B Quadratic Terms: Let $x^2 + bx + c$ be an irreducible quadratic that divides $q(x)$, where $(x^2 + bx + c)^n$ is the highest power of $x^2 + bx + c$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + bx + c)^n}.$$

Activity 5.6.9 Which of the following is the form of the partial fraction decomposition of $\frac{x^3 - 7x^2 - 7x + 15}{x^3(x+5)}$?

- A $\frac{A}{x} + \frac{B}{x+5}$.
- B $\frac{A}{x^3} + \frac{B}{x+5}$.
- C $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+5}$.
- D $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x+5}$.

Activity 5.6.10 Which of the following is the form of the partial fraction decomposition of $\frac{x^2+1}{(x-3)^2(x^2+4)^2}$?

- A $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x^2+4} + \frac{D}{(x^2+4)^2}$.
- B $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{(x^2+4)^2}$.
- C $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$.
- D $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}$.

Activity 5.6.11 Consider that the partial decomposition of $\frac{x^2+5x+3}{(x+1)^2x}$ is

$$\frac{x^2+5x+3}{(x+1)^2x} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x}.$$

What equality do we obtain if we multiply both sides of the above equation by $(x+1)^2x$?

- A $x^2 + 5x + 3 = Ax(x+1) + Bx + C(x+1)^2$.
- B $x^2 + 5x + 3 = A(x+1) + B(x+1)^2 + Cx$.
- C $x^2 + 5x + 3 = Ax(x+1) + Bx + C(x+1)$.
- D $x^2 + 5x + 3 = A(x+1) + Bx + Cx^2$.
- E $x^2 + 5x + 3 = Ax(x+1) + Bx^2 + C(x+1)^2$.

Activity 5.6.12 Notice that $x^2+5x+3 = Ax(x+1) + Bx + C(x+1)^2$. Which of the following values can we determine by setting $x = 0$? Select all that apply.

- A A
- B B
- C C

Which of the following values can we determine by setting $x = -1$? Select all that apply.

A A B B C C

Activity 5.6.13 Rewrite $Ax(x+1) + x + 3(x+1)^2$ so that $Ax(x+1) + x + 3(x+1)^2 = ?x^2 + ?x + ?$. Knowing that

$$1x^2 + 5x + 3 = Ax(x+1) + x + 3(x+1)^2 = ?x^2 + ?x + ?,$$

can we solve for A ?

Activity 5.6.14 By using the form of the decomposition $\frac{x^2 + 5x + 3}{(x+1)^2x} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x}$ and the coefficients found in [Activity 5.6.12](#) and [Activity 5.6.13](#), evaluate $\int \frac{x^2 + 5x + 3}{(x+1)^2x} dx$.

Activity 5.6.15 Given that $\frac{x^3 - 7x^2 - 7x + 15}{x^3(x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+5}$, what are A, B, C , and D ?

A $A = 1, B = -7, C = -7, D = 15$.B $A = -1, B = -2, C = 3, D = 2$.C $A = 1, B = -2, C = -3, D = 2$.D $A = -1, B = 2, C = 3, D = -2$.

Activity 5.6.16 Given your choice in [Activity 5.6.15](#) Find $\int \frac{x^3 - 7x^2 - 7x + 15}{x^3(x+5)} dx$.

Activity 5.6.17 Consider the rational expression $\frac{2x^3 + 2x + 4}{x^4 + 2x^3 + 4x^2}$. Which of the following is the partial fraction decomposition of this rational expression?

A $\frac{1}{x} + \frac{1}{x^2} + \frac{2x-1}{x^2+2x+4}$.B $\frac{2}{x} + \frac{0}{x^2} + \frac{-1}{x^2+2x+4}$.C $\frac{0}{x} + \frac{1}{x^2} + \frac{-1}{x^2+2x+4}$.D $\frac{0}{x} + \frac{1}{x^2} + \frac{2x-1}{x^2+2x+4}$.

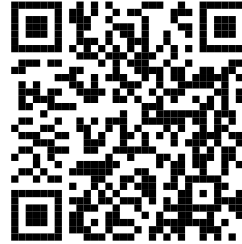
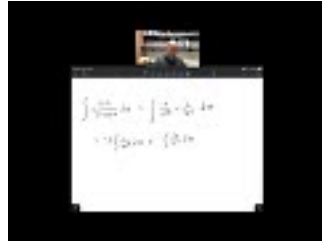
Activity 5.6.18 Given your choice in [Activity 5.6.17](#) Find $\int \frac{2x^3 + 2x + 4}{x^4 + 2x^3 + 4x^2} dx$.

Activity 5.6.19 Given that $\frac{2x+5}{x^2+3x+2} = \frac{-1}{x+2} + \frac{3}{x+1}$, find $\int_0^3 \frac{2x+5}{x^2+3x+2} dx$.

Remark 5.6.20 This is all well and good, but given a rational function, how might one actually find coefficients for the partial fractions if we weren't given them?

Activity 5.6.21 Evaluate $\int \frac{4x^2 - 3x + 1}{(2x+1)(x+2)(x-3)} dx$.

5.6.1 Videos



YouTube: <https://www.youtube.com/watch?v=3Bb2l-LszM>

Figure 80 Video: I can integrate functions using the method of partial fractions

5.7 Integration strategy (TI7)

Learning Outcomes

- I can select appropriate strategies for integration.

Activity 5.7.1 Consider the integral $\int e^t \tan(e^t) \sec^2(e^t) dt$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

- A The method of substitution
- B The method of integration by parts
- C Trigonometric substitution
- D Using a table of integrals
- E The method of partial fractions

Activity 5.7.2 Consider the integral $\int \frac{2x+3}{1+x^2} dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

- A The method of substitution
- B The method of integration by parts
- C Trigonometric substitution
- D Using a table of integrals
- E The method of partial fractions

Activity 5.7.3 Consider the integral $\int \frac{x}{\sqrt[3]{1-x^2}} dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

- A The method of substitution
- B The method of integration by parts
- C Trigonometric substitution
- D Using a table of integrals
- E The method of partial fractions

Activity 5.7.4 Consider the integral $\int \frac{1}{2x\sqrt{1-36x^2}} dx$. Which strategy is a

reasonable first step to make progress towards evaluating this integral?

- A The method of substitution
- B The method of integration by parts
- C Trigonometric substitution
- D Using a table of integrals
- E The method of partial fractions

Activity 5.7.5 Consider the integral $\int t^5 \cos(t^3) dt$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

- A The method of substitution
- B The method of integration by parts
- C Trigonometric substitution
- D Using a table of integrals
- E The method of partial fractions

Activity 5.7.6 Consider the integral $\int \frac{1}{1+e^x} dx$. Which strategy is a reasonable first step to make progress towards evaluating this integral?

- A The method of substitution
- B The method of integration by parts
- C Trigonometric substitution
- D Using a table of integrals
- E The method of partial fractions

5.7.1 Videos



YouTube: <https://www.youtube.com/watch?v=DIAMb55ZGQ8>

Figure 81 Video: I can select appropriate strategies for integration

5.8 Improper integrals (TI8)

Learning Outcomes

- I can compute improper integrals.

Activity 5.8.1 Consider $f(x) = \frac{1}{x^2}$. Compute the following definite integrals.

(a) $\int_{0.1}^1 \frac{1}{x^2} dx.$

(b) $\int_{0.01}^1 \frac{1}{x^2} dx.$

(c) $\int_{0.001}^1 \frac{1}{x^2} dx.$

Activity 5.8.2

(a) What do you notice about $\int_a^1 \frac{1}{x^2} dx$ as a approached 0 in [Activity 5.8.1](#)?

A $\int_a^1 \frac{1}{x^2} dx$ approaches a finite, non-zero constant.

C $\int_a^1 \frac{1}{x^2} dx$ approaches ∞ .

D $\int_a^1 \frac{1}{x^2} dx$ approaches $-\infty$.

B $\int_a^1 \frac{1}{x^2} dx$ approaches 0.

E There is not enough information.

(b) Why can't $a = 0$?

Activity 5.8.3 We still are considering $f(x) = \frac{1}{x^2}$. Compute the following definite integrals.

(a) $\int_1^{10} \frac{1}{x^2} dx.$

(b) $\int_1^{100} \frac{1}{x^2} dx.$

(c) $\int_1^{1000} \frac{1}{x^2} dx.$

Activity 5.8.4 What do you notice about $\int_1^a \frac{1}{x^2} dx$ as a approached ∞ in [Activity 5.8.3](#)?

A $\int_1^a \frac{1}{x^2} dx$ approaches a finite, non-zero constant.

B $\int_1^a \frac{1}{x^2} dx$ approaches 0.

C $\int_1^a \frac{1}{x^2} dx$ approaches ∞ .

D $\int_1^a \frac{1}{x^2} dx$ approaches $-\infty$.

E There is not enough information.

Activity 5.8.5 Compute the following definite integrals.

(a) $\int_{0.1}^1 \frac{1}{\sqrt{x}} dx.$

(b) $\int_{0.01}^1 \frac{1}{\sqrt{x}} dx.$

(c) $\int_{0.001}^1 \frac{1}{\sqrt{x}} dx.$

Activity 5.8.6

- (a) What do you notice about the integral $\int_a^1 \frac{1}{\sqrt{x}} dx$ as a approached 0 in [Activity 5.8.5](#)?

A $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches a finite, non-zero constant.

B $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches 0.

C $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches ∞ .

D $\int_a^1 \frac{1}{\sqrt{x}} dx$ approaches $-\infty$.

E There is not enough information.

- (b) How does this compare to what you found in [Activity 5.8.1](#)?

Activity 5.8.7 Compute the following definite integrals.

(a) $\int_1^{10} \frac{1}{\sqrt{x}} dx.$

(b) $\int_1^{100} \frac{1}{\sqrt{x}} dx.$

(c) $\int_1^{1000} \frac{1}{\sqrt{x}} dx.$

Activity 5.8.8

- (a) What do you notice the integral $\int_1^a \frac{1}{\sqrt{x}} dx$ as a approached ∞ in [Activity 5.8.7](#)?

A $\int_1^a \frac{1}{\sqrt{x}} dx$ approaches a finite, non-zero constant.

B $\int_1^a \frac{1}{\sqrt{x}} dx$ approaches 0.

C $\int_1^a \frac{1}{\sqrt{x}} dx$ approaches ∞ .

D $\int_1^a \frac{1}{\sqrt{x}} dx$ approaches $-\infty$.

E There is not enough information.

- (b) How does this compare to what you found in [Activity 5.8.3](#)?

Definition 5.8.9 For a function $f(x)$ and a constant c , we let $\int_c^\infty f(x) dx$

denote

$$\int_c^\infty f(x)dx = \lim_{a \rightarrow \infty} \left(\int_c^a f(x)dx \right).$$

If this limit is a defined real number, then we say $\int_c^\infty f(x)dx$ is *convergent*, otherwise, it is *divergent*.

Similarly,

$$\int_{-\infty}^c f(x)dx = \lim_{a \rightarrow -\infty} \left(\int_a^c f(x)dx \right).$$

◇

Activity 5.8.10 Which of the following is $\int_1^a \frac{1}{x^2} dx$?

A $-\frac{1}{x} + C$

B $-\frac{1}{a} + C$

C $-\frac{1}{x} + 1$

D $-\frac{1}{a}$

E $-\frac{1}{a} + 1$

Activity 5.8.11 Using your choice in [Activity 5.8.10](#), what is

$$\lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^2} dx?$$

A 0

B 1

C 2

D ∞

E $-\infty$

Activity 5.8.12

(a) Find $\int_{\pi/2}^a \cos(x)dx$.

(b) Which of the following is true about $\int_{\pi/2}^\infty \cos(x)dx$?

A $\int_{\pi/2}^\infty \cos(x)dx$ is convergent.

B $\int_{\pi/2}^\infty \cos(x)dx$ is divergent

C More information is needed.

Definition 5.8.13 For a function $f(x)$ with a vertical asymptote at $x = b$ and

a constant c , we let $\int_c^b f(x)dx$ denote

$$\int_c^b f(x)dx = \lim_{a \rightarrow b^-} \left(\int_c^a f(x)dx \right).$$

Similarly,

$$\int_b^c f(x)dx = \lim_{a \rightarrow b^+} \left(\int_a^c f(x)dx \right).$$

◇

Activity 5.8.14

(a) Find $\int_a^1 \frac{1}{\sqrt{x}} dx$.

(b) Using the above result and an appropriate limit, find $\int_0^1 \frac{1}{\sqrt{x}} dx$.

Activity 5.8.15

(a) Find $\int_a^1 \frac{1}{x^2} dx$.

(b) Using the above result and an appropriate limit, find $\int_0^1 \frac{1}{x^2} dx$.

Fact 5.8.16 Suppose that $0 < p$ and $p \neq 1$. Applying the integration power rule gives us the indefinite integral $\int \frac{1}{x^p} dx = \frac{1}{(1-p)} x^{1-p} + C$.

Activity 5.8.17

(a) If $0 < p < 1$, which of the following statements must be true? Select all that apply.

- A $1 - p < 0$.
- B $1 - p > 0$.
- C $1 - p < 1$.
- D $\int_1^\infty \frac{1}{x^p} dx$ converges.
- E $\int_1^\infty \frac{1}{x^p} dx$ diverges.

(b) If $p > 1$, which of the following statements must be true? Select all that apply.

- A $1 - p < 0$.
- B $1 - p > 0$.
- C $1 - p < 1$.
- D $\int_1^\infty \frac{1}{x^p} dx$ converges.
- E $\int_1^\infty \frac{1}{x^p} dx$ diverges.

Activity 5.8.18

(a) If $0 < p < 1$, which of the following statements must be true?

A $\int_0^1 \frac{1}{x^p} dx$ converges.

B $\int_0^1 \frac{1}{x^p} dx$ diverges.

(b) If $p > 1$, which of the following statements must be true?

A $\int_0^1 \frac{1}{x^p} dx$ converges.

B $\int_0^1 \frac{1}{x^p} dx$ diverges.

Activity 5.8.19 Consider when $p = 1$. Then $\frac{1}{x^p} = \frac{1}{x}$ and $\int \frac{1}{x^p} dx = \int \frac{1}{x} dx = \ln|x| + C$.

(a) What can we conclude about $\int_1^\infty \frac{1}{x} dx$?

A $\int_1^\infty \frac{1}{x} dx$ converges.

B $\int_1^\infty \frac{1}{x} dx$ diverges.

C There is not enough information to determine whether this integral converges or diverges.

(b) What can we conclude about $\int_0^1 \frac{1}{x} dx$?

A $\int_0^1 \frac{1}{x} dx$ converges.

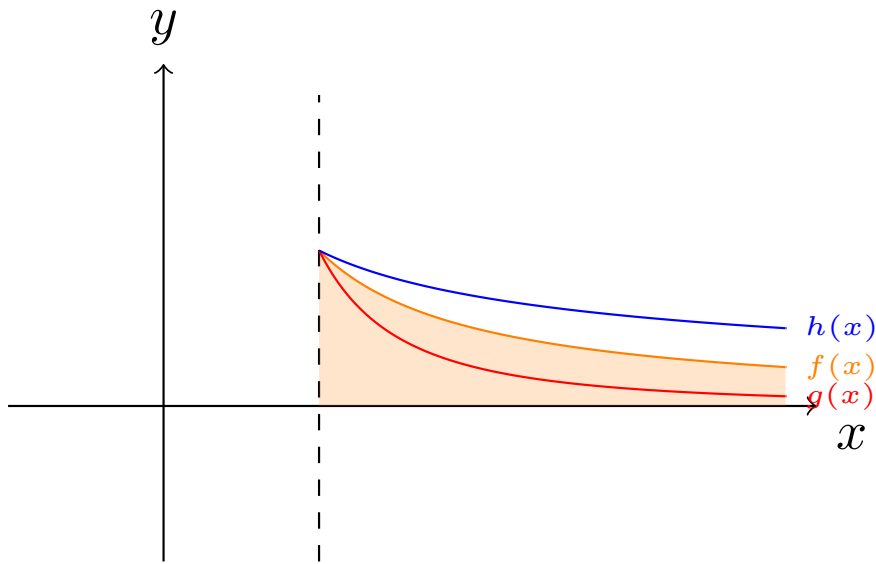
B $\int_0^1 \frac{1}{x} dx$ diverges.

C There is not enough information to determine whether this integral converges or diverges.

Fact 5.8.20 Let $c, p > 0$.

- $\int_0^c \frac{1}{x^p} dx$ is defined if and only if $p < 1$.
- $\int_c^\infty \frac{1}{x^p} dx$ is defined if and only if $p > 1$.

Activity 5.8.21 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

**Figure 82** Plots of $f(x), g(x), h(x)$

If $\int_1^\infty f(x)dx$ is convergent, what can we say about $g(x), h(x)$?

A $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are both convergent.

B $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are both divergent.

C Whether or not $\int_1^\infty g(x)dx$ and $\int_1^\infty h(x)dx$ are convergent or divergent cannot be determined.

D $\int_1^\infty g(x)dx$ is convergent and $\int_1^\infty h(x)dx$ is divergent.

E $\int_1^\infty g(x)dx$ is convergent and $\int_1^\infty h(x)dx$ could be either convergent or divergent.

Activity 5.8.22 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

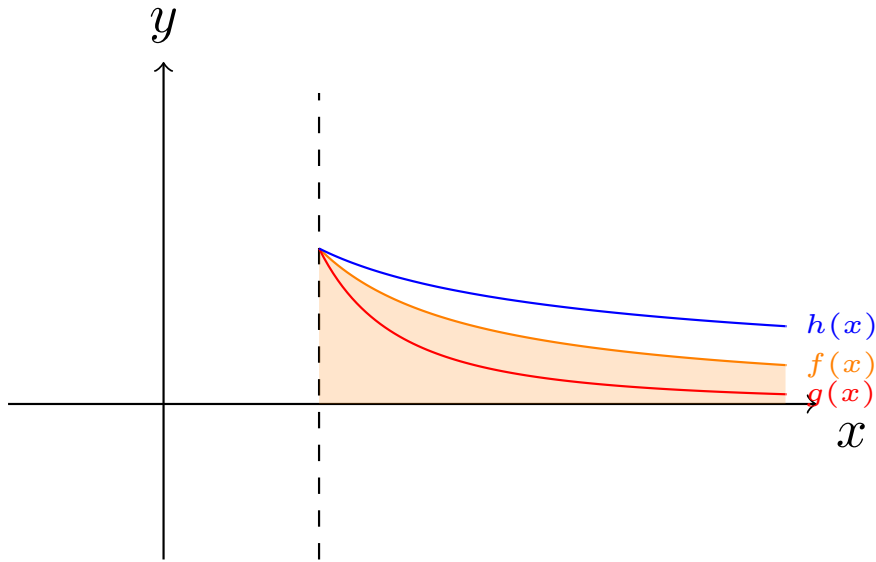


Figure 83 Plots of $f(x), g(x), h(x)$

If $\int_1^{\infty} f(x)dx$ is divergent, what can we say about $g(x), h(x)$?

A $\int_1^{\infty} g(x)dx$ and $\int_1^{\infty} h(x)dx$ are both convergent.

B $\int_1^{\infty} g(x)dx$ and $\int_1^{\infty} h(x)dx$ are both divergent.

C Whether or not $\int_1^{\infty} g(x)dx$ and $\int_1^{\infty} h(x)dx$ are convergent or divergent cannot be determined.

D $\int_1^{\infty} g(x)dx$ could be either convergent or divergent and $\int_1^{\infty} h(x)dx$ is divergent.

E $\int_1^{\infty} g(x)dx$ is convergent and $\int_1^{\infty} h(x)dx$ is divergent.

Activity 5.8.23 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

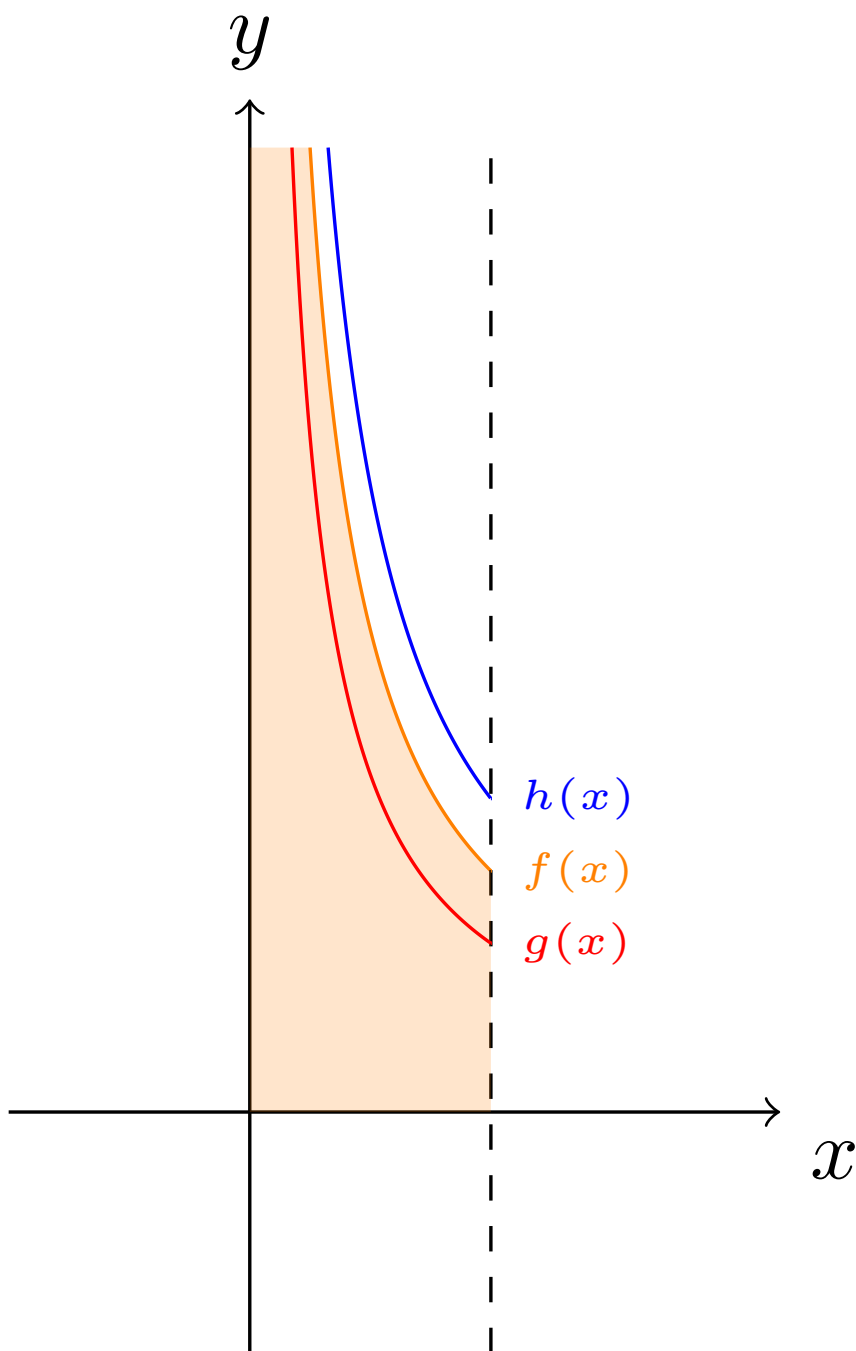


Figure 84 Plots of $f(x)$, $g(x)$, $h(x)$

If $\int_0^1 f(x)dx$ is convergent, what can we say about $g(x)$, $h(x)$?

A $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are both convergent.

B $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are both divergent.

- C Whether or not $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are convergent or divergent cannot be determined.
- D $\int_0^1 g(x)dx$ is convergent and $\int_0^1 h(x)dx$ is divergent.
- E $\int_0^1 g(x)dx$ is convergent and $\int_0^1 h(x)dx$ can either be convergent or divergent.

Activity 5.8.24 Consider the plots of $f(x), g(x), h(x)$ where $0 < g(x) < f(x) < h(x)$.

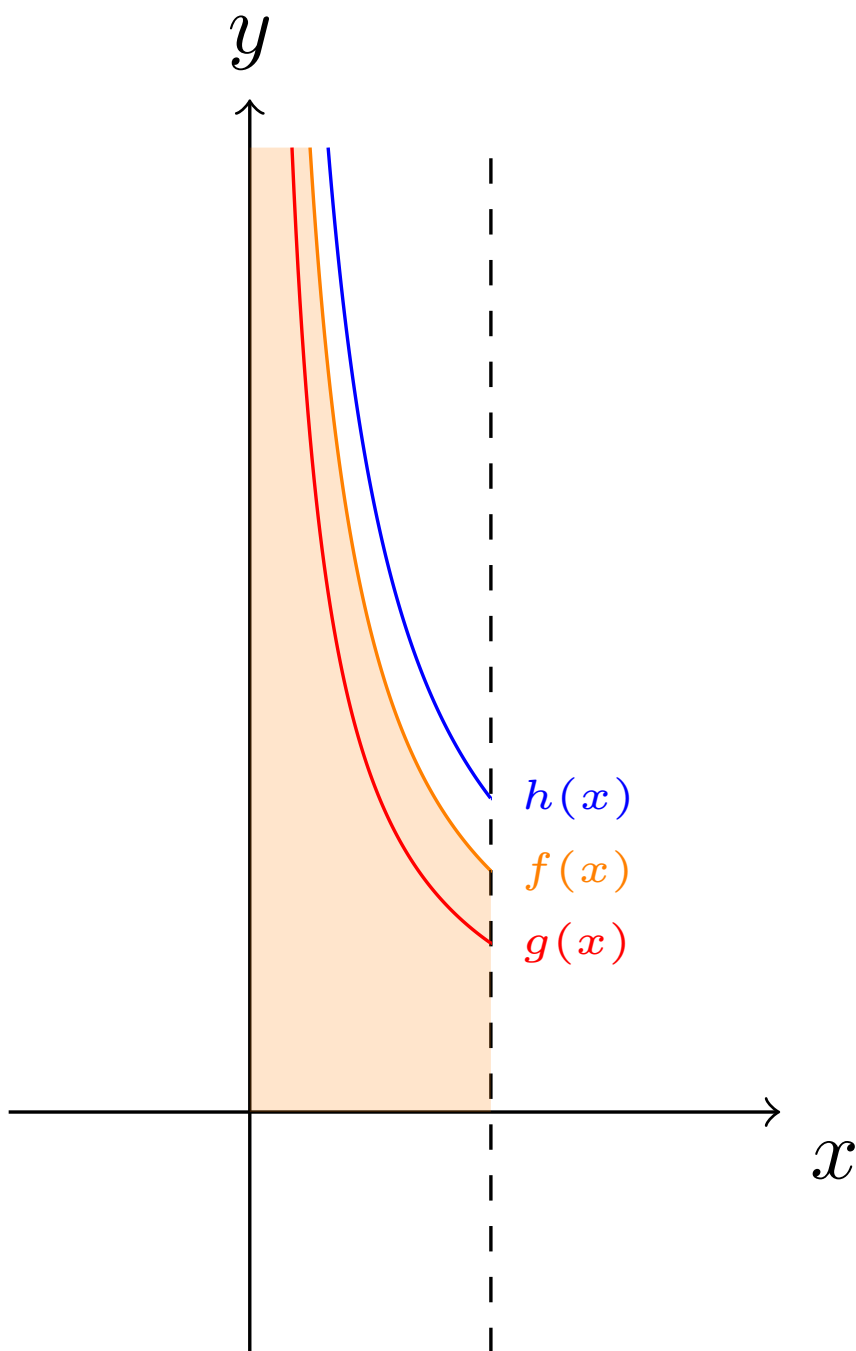


Figure 85 Plots of $f(x)$, $g(x)$, $h(x)$

If $\int_0^1 f(x)dx$ is divergent, what can we say about $g(x)$, $h(x)$?

A $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are both convergent.

B $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are both divergent.

C Whether or not $\int_0^1 g(x)dx$ and $\int_0^1 h(x)dx$ are convergent or divergent cannot be determined.

D $\int_0^1 g(x)dx$ can be either convergent or divergent and $\int_0^1 h(x)dx$ is divergent.

E $\int_0^1 g(x)dx$ is convergent and $\int_0^1 h(x)dx$ is divergent.

Fact 5.8.25 Let $f(x), g(x), h(x)$ be functions such that over a region I , $g(x) < f(x) < h(x)$. Then for the integrals $\int_I f(x)dx$, $\int_I g(x)dx$, $\int_I h(x)dx$:

- If $f(x)$ converges, so does $g(x)$. We need more information to conclude anything about $h(x)$.
- If $f(x)$ diverges, so does $h(x)$. We need more information to conclude anything about $g(x)$.

Activity 5.8.26 Compare $\frac{1}{x^3+1}$ to one of the following functions where $x > 2$ and use this to determine if $\int_2^\infty \frac{1}{x^3+1}dx$ is convergent or divergent.

A $\frac{1}{x}$.

B $\frac{1}{\sqrt{x}}$.

C $\frac{1}{x^2}$.

D $\frac{1}{x^3}$.

5.8.1 Videos



YouTube: <https://www.youtube.com/watch?v=BPKawAzqoas>

Figure 86 Video: I can compute improper integrals, $p > 1$



YouTube: <https://www.youtube.com/watch?v=kYIv3E9Jio4>

Figure 87 Video: I can compute improper integrals, $p < 1$

Chapter 6

Applications of Integration (AI)

Learning Outcomes

How do we use various techniques to integrate less simple functions?

By the end of this chapter, you should be able to...

1. Compute the average value of a function on an interval.
2. Estimate the arclength of a curve with Riemann sums and find an integral which computes the arclength.
3. Compute volumes of solids of revolution.
4. Compute surface areas of surfaces of revolution.
5. Set up integrals to solve problems involving density, mass, and center of mass.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Estimate a definite integral with a Riemann sum.
 - Review: [Khan Academy](#)¹
- b Evaluate the limit of a Riemann sum to find an integral.
 - Review: [Khan Academy](#)²
- c Solve accumulation problems with definite integrals.
 - Review: [Khan Academy](#)³
- d Find the distance between points on a curve.

¹<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/generalizing-a-left-riemann-sum-with-equally-spaced-rectangles?modal=1>

²<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/riemann-sums-and-integrals?modal=1>

³<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-4/v/functions-defined-by-definite-integrals?modal=1>

- Review: [Khan Academy](#)⁴
- e Express the sum of indexed values using summation notation.
- Review: [Khan Academy](#)⁵

6.1 Average Value (AI1)

Learning Outcomes

- Compute the average value of a function on an interval.

Activity 6.1.1 Suppose a car drives due east at 70 mph for 2 hours, and then slows down to 40 mph for an additional hour.

- (a) How far did the car travel in these 3 hours?
- (b) What was their average velocity over these 3 hours?

Activity 6.1.2 Suppose a car drives due east at a velocity of $v(t) = 40 + \sqrt{300t}$ mph, where t is the number of hours after they start driving.

- (a) Which of the following expressions measures how far the car travel travels in these 3 hours?

A $\int_0^2 70x \, dx + \int_2^3 40x \, dx.$

B $\int_0^2 70 \, dx + \int_2^3 40 \, dx.$

C $\int_0^2 70x \, dx + \int_0^1 40x \, dx.$

D $\int_0^2 70 \, dx + \int_0^1 40 \, dx.$

E $70(2) + 40(1).$

- (b) What was their average velocity over these 3 hours?

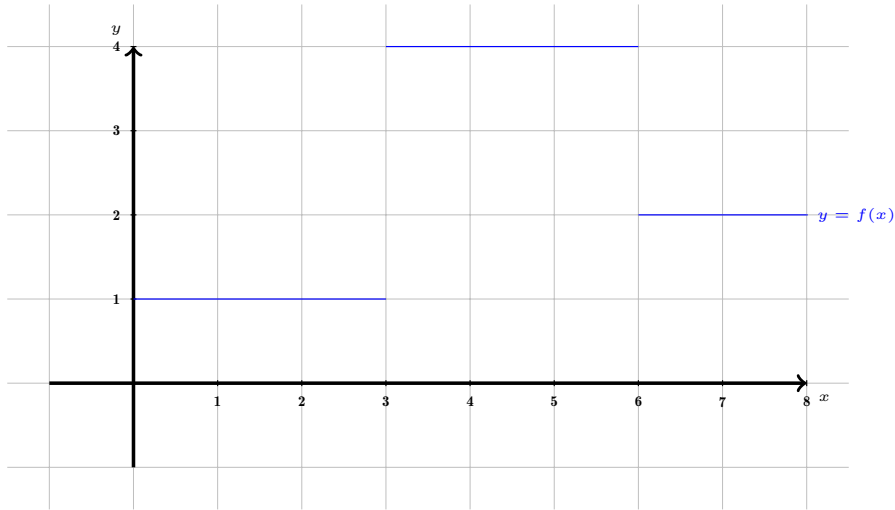
- A 180 mi.
- B 180 mph.
- C 60 mi.
- D 60 mph.

Activity 6.1.3 Which of the following is the average value of $f(x)$ over the

interval $[0, 8]$? Note $f(x) = \begin{cases} 1, & 0 \leq x \leq 3 \\ 4, & 3 < x \leq 6 \\ 2, & 6 < x \leq 8 \end{cases}$.

⁴<https://www.khanacademy.org/math/geometry/hs-geo-analytic-geometry/hs-geo-distance-and-midpoints/v/distance-formula>

⁵<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/sigma-notation-sum>

**Figure 88** Plot of $f(x)$.

- A 4.
- B 2.
- C $\frac{7}{3}$.
- D 2.375.

Activity 6.1.4 Consider a function $f(x)$ which has average value v on the interval $[a, b]$. Which of the following is most likely always true about v ?

- A v is the average of a and b .
- B v is the average of $f(a)$ and $f(b)$.
- C v is the average of $\min\{f(x)\}$ and $\max\{f(x)\}$ on the interval $[a, b]$.

D $\int_a^b f(x)dx = \int_a^b vdx$.

Definition 6.1.5 Given a function $f(x)$ defined on $[a, b]$, its average value v is:

$$v = \frac{1}{b-a} \int_a^b f(x)dx.$$

◇

Activity 6.1.6 Which of the following expressions represent the average value of $f(x) = x \cos(x^2) + x$ on the interval $[\pi, 4\pi]$?

- A $\int_0^{4\pi} (x \cos(x^2) + x)dx$.
- B $\frac{1}{3\pi} \int_0^{4\pi} (x \cos(x^2) + x)dx$.
- C $\frac{1}{4\pi} \int_0^{4\pi} (x \cos(x^2) + x)dx$.
- D $\int_{\pi}^{4\pi} (x \cos(x^2) + x)dx$.

$$\text{E } \frac{1}{3\pi} \int_{\pi}^{4\pi} (x \cos(x^2) + x) dx.$$

$$\text{F } \frac{1}{4\pi} \int_{\pi}^{4\pi} (x \cos(x^2) + x) dx.$$

Activity 6.1.7 Find the average value of $f(x) = x \cos(x^2) + x$ on the interval $[\pi, 4\pi]$ using the chosen expression from [Activity 6.1.6](#).

Activity 6.1.8 Find the average value of $g(t) = \frac{t}{t^2+1}$ on the interval $[0, 4]$.

Activity 6.1.9 A shot of a drug is administered to a patient and the quantity of the drug in the bloodstream over time is $q(t) = 3te^{-0.25t}$ where t is measured in hours and q is measured in mg. What is the average quantity of this drug in the patient's bloodstream over the first 6 hours after injection?

6.1.1 Videos



YouTube: <https://www.youtube.com/watch?v=rwAl4trDIac>

Figure 89 Video: Compute the average value of a function on an interval

6.2 Arclength (AI2)

Learning Outcomes

- Estimate the arclength of a curve with Riemann sums and find an integral which computes the arclength.

Activity 6.2.1 Suppose we wanted to find the arclength of the parabola $y = -x^2 + 6x$ over the interval $[0, 4]$.

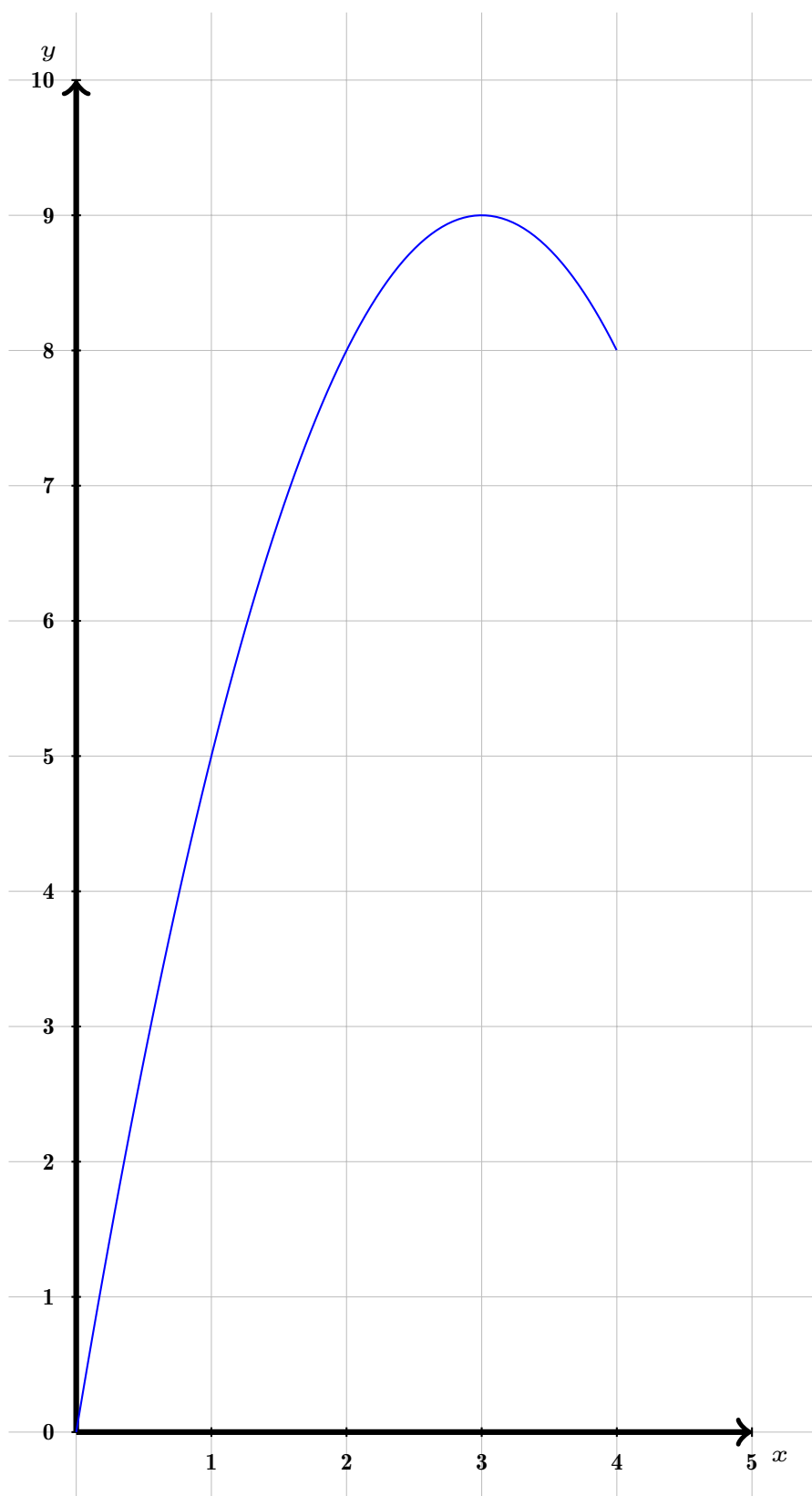


Figure 90 Plot of $y = -x^2 + 6x$ over $[0, 4]$.

- (a) Suppose we wished to estimate this length with two line segments, where $\Delta x = 2$.

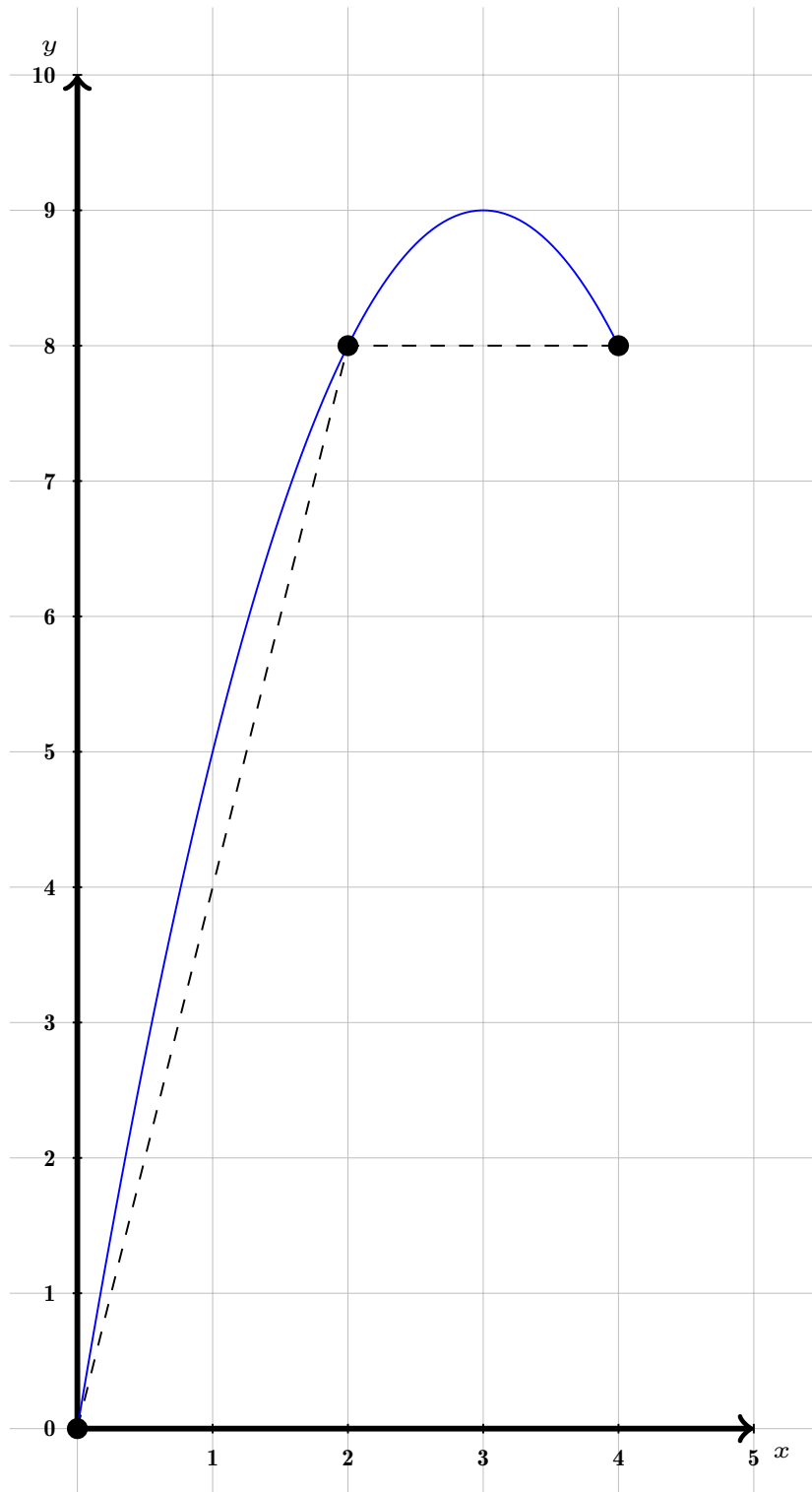


Figure 91 Plot of $y = -x^2 + 6x$ over $[0, 4]$.

Which of the following expressions represents the sum of the lengths of the line segments with endpoints $(0, 0)$, $(2, 8)$ and $(4, 8)$.

A $\sqrt{4+8}$.

B $\sqrt{4^2+8^2}$.

C $\sqrt{2^2+8^2} + \sqrt{4^2+8^2}$.

D $\sqrt{2^2+8^2} + \sqrt{(4-2)^2 + (8-8)^2}$.

- (b) Suppose we wished to estimate this length with four line segments, where $\Delta x = 1$.

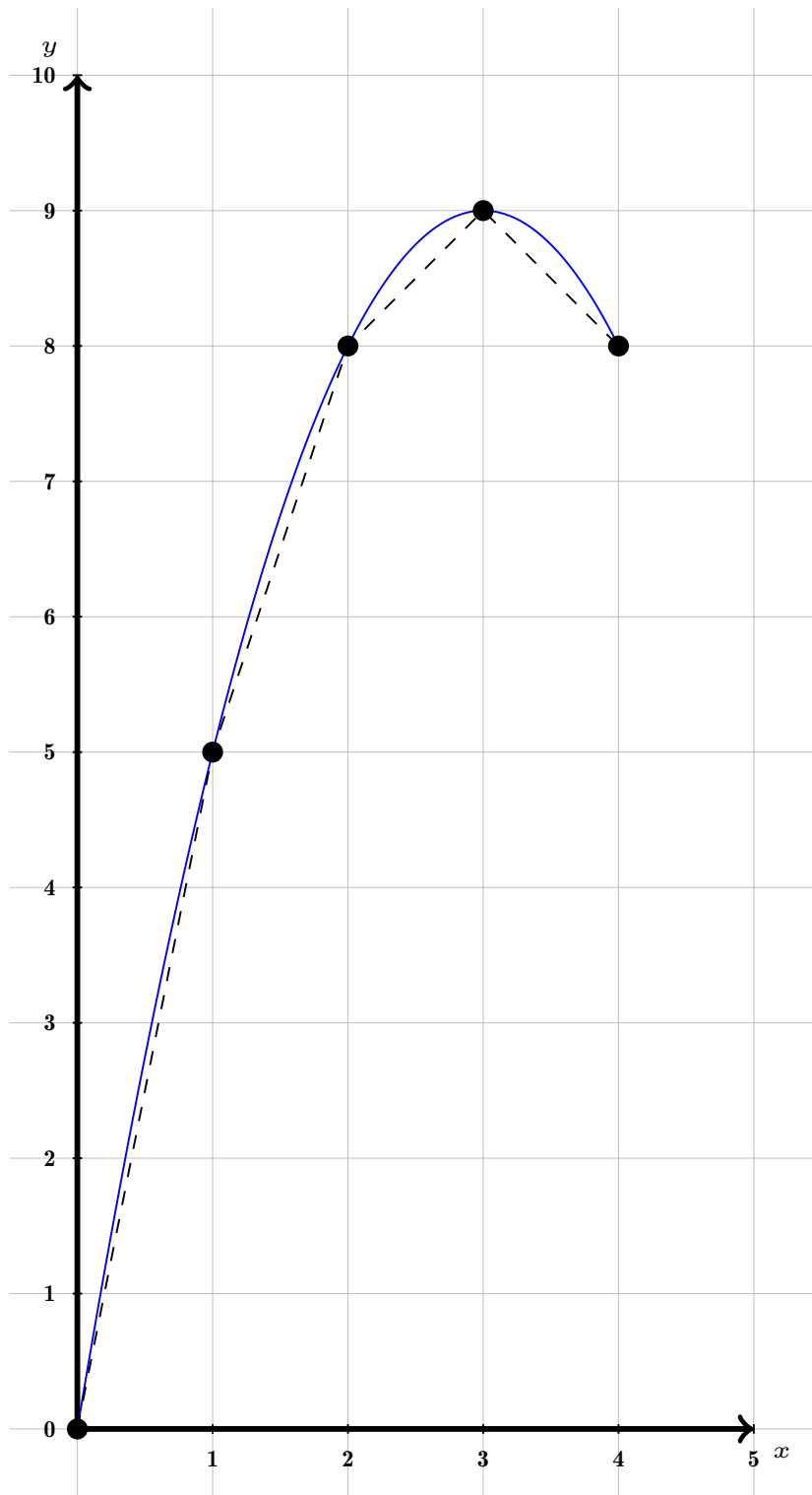


Figure 92 Plot of $y = -x^2 + 6x$ over $[0, 4]$.

Which of the following expressions represents the sum of the lengths of the line segments with endpoints $(0, 0)$, $(1, 5)$, $(2, 8)$, $(3, 9)$ and $(4, 8)$.

A $\sqrt{4^2 + 8^2}$.

$$\text{B } \sqrt{1^2 + (5-0)^2} + \sqrt{1^2 + (8-5)^2} + \sqrt{1^2 + (9-8)^2} + \sqrt{1^2 + (8-9)^2}.$$

$$\text{C } \sqrt{1^2 + 5^2} + \sqrt{2^2 + 8^2} + \sqrt{3^2 + 9^2} + \sqrt{4^2 + 8^2}.$$

- (c) Suppose we wished to estimate this length with n line segments $\Delta x = 4/n$. Let $f(x) = -x^2 + 6x$.

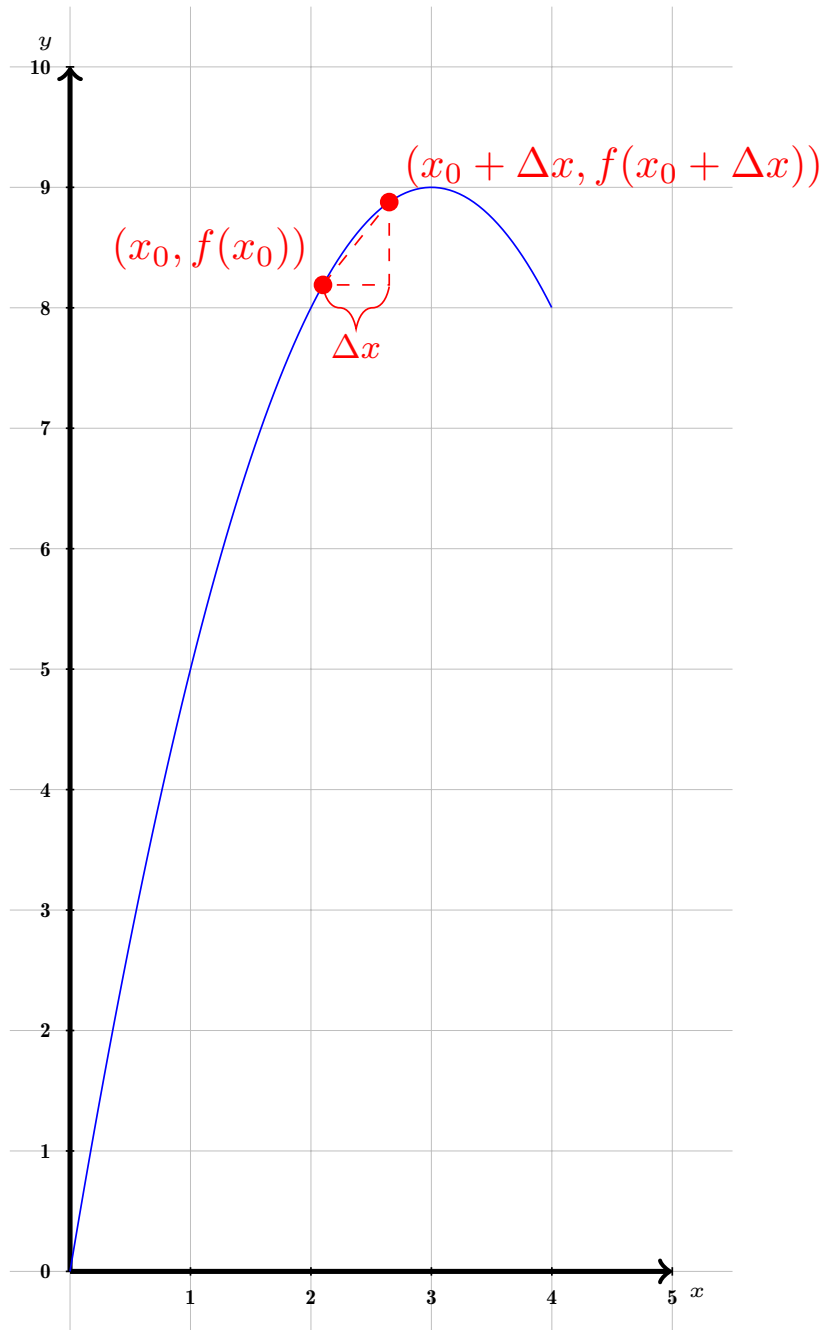


Figure 93 Plot of $y = -x^2 + 6x$ over $[0, 4]$.

Which of the following expressions represents the length of the line segment from $(x_0, f(x_0))$ to $(x_0 + \Delta x, f(x_0 + \Delta x))$?

$$\text{A } \sqrt{x_0^2 + f(x_0)^2}.$$

- B $\sqrt{(x_0 + \Delta x)^2 + f(x_0 + \Delta x)^2}$.
 C $\sqrt{(\Delta x)^2 + (f(x_0 + \Delta x) - f(x_0))^2}$.
 D $\sqrt{(\Delta x)^2 + f(\Delta x)^2}$.

(d) Which of the following Riemann sums best estimates the arclength of the parabola $y = -x^2 + 6x$ over the interval $[0, 4]$? Let $f(x) = -x^2 + 6x$.

- A $\sum \sqrt{x_i^2 + f(x_i)^2}$.
 B $\sum \sqrt{(x_i + \Delta x)^2 + f(x_i + \Delta x)^2}$.
 C $\sum \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}$.
 D $\sum \sqrt{(\Delta x)^2 + f(\Delta x)^2}$.

(e) Note that

$$\begin{aligned}\sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2} &= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2\right)} \\ &= \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2} \Delta x\end{aligned}$$

Which of the following best describes $\lim_{\Delta x \rightarrow 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$?

- A 0.
 B 1.
 C $f'(x_i)$.
 D This limit is undefined.

Fact 6.2.2 Given a differentiable function $f(x)$, the arclength of $y = f(x)$ defined on $[a, b]$ is computed by the integral:

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum \sqrt{(\Delta x)^2 + (f(x_i + \Delta) - f(x_i))^2} &= \lim_{n \rightarrow \infty} \sum \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2} \Delta x \\ &= \int_a^b \sqrt{1 + (f'(x))^2} dx.\end{aligned}$$

Activity 6.2.3 Use [Fact 6.2.2](#) to find an integral which measures the arclength of the parabola $y = -x^2 + 6x$ over the interval $[0, 4]$.

Activity 6.2.4 Consider the curve $y = 2^x - 1$ defined on $[1, 5]$.

(a) Estimate the arclength of this curve with two line segments with $\Delta x = 2$.

x_i	$(x_i, f(x_i))$	$(x_i + \Delta x, f(x_i + \Delta x))$	Length of segment
1			
3			

- (b) Estimate the arclength of this curve with four line segments with $\Delta x = 1$.

x_i	$(x_i, f(x_i))$	$(x_i + \Delta x, f(x_i + \Delta x))$	Length of segment
1			
2			
3			

- (c) Find an integral which computes the arclength of the curve $y = 2^x - 1$ defined on $[1, 5]$.

Activity 6.2.5 Consider the curve $y = 5e^{-x^2}$ over the interval $[-1, 4]$

- (a) Estimate this arclength with 5 line segments with $\Delta x = 1$.
 (b) Find an integral which computes this arclength.

6.2.1 Videos



YouTube: <https://www.youtube.com/watch?v=SFflh9Y0hK8>

Figure 94 Video: Estimate the arclength of a curve with Riemann sums and find an integral which computes the arclength

6.3 Volumes of Revolution (AI3)

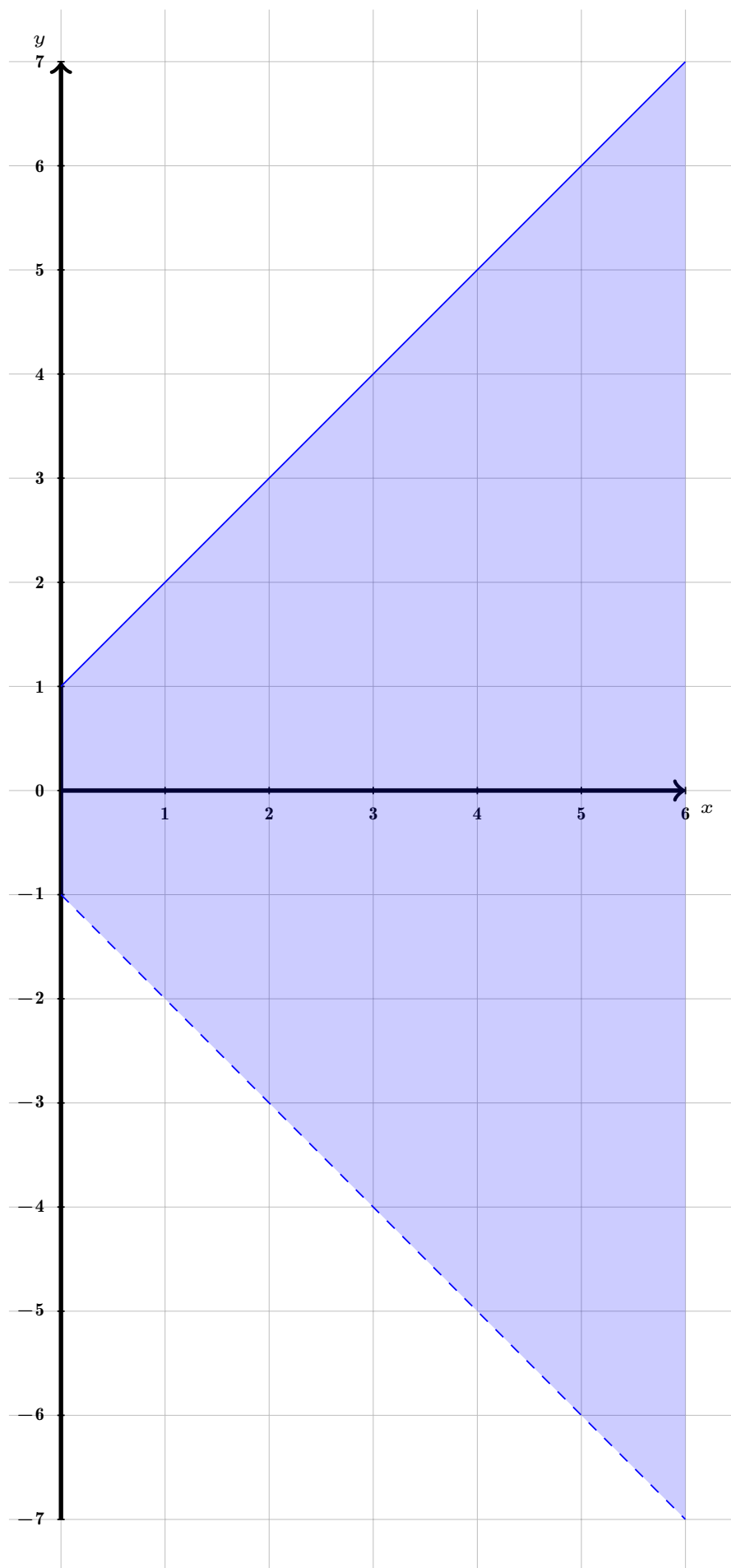
Learning Outcomes

- Compute volumes of solids of revolution.

Activity 6.3.1 Consider the function $f(x) = x + 1$ and the solid of revolution generated by rotating the region bound by:

$$\begin{aligned} 0 &\leq y \leq f(x) \\ 0 &\leq x \leq 6 \end{aligned}$$

about the x -axis.

**Figure 95** Plot of $f(x) = x + 1$ over $[0, 6]$.

We wish to find the volume of this solid.

- (a) Suppose we estimate the volume of this solid by approximating it with two cylinders of “height” $\Delta x = 3$ and radii $f(0) = 1, f(3) = 4$:

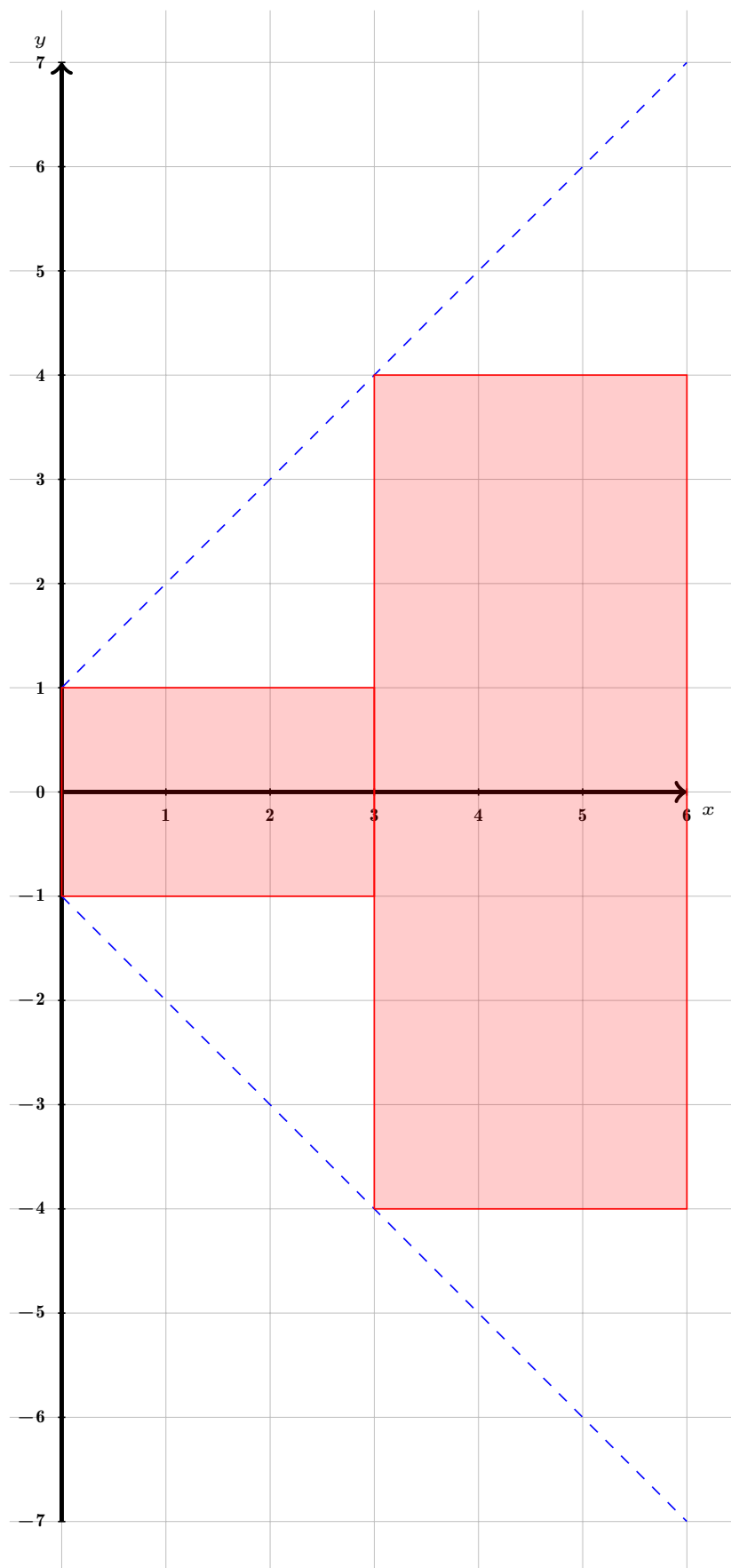


Figure 96 Plot of $f(x) = x + 1$ over $[0, 6]$ and cross section of two cylinders.

What is the volume of the cylinder with radius $f(0) = 1$?

A $\pi \cdot 0^2 \cdot 3.$

B $\pi \cdot 1^2 \cdot 3.$

C $\pi \cdot 3^2 \cdot 3.$

D $\pi \cdot 4^2 \cdot 3.$

E $\pi \cdot 6^2 \cdot 3.$

F $\pi \cdot 7^2 \cdot 3.$

(b)

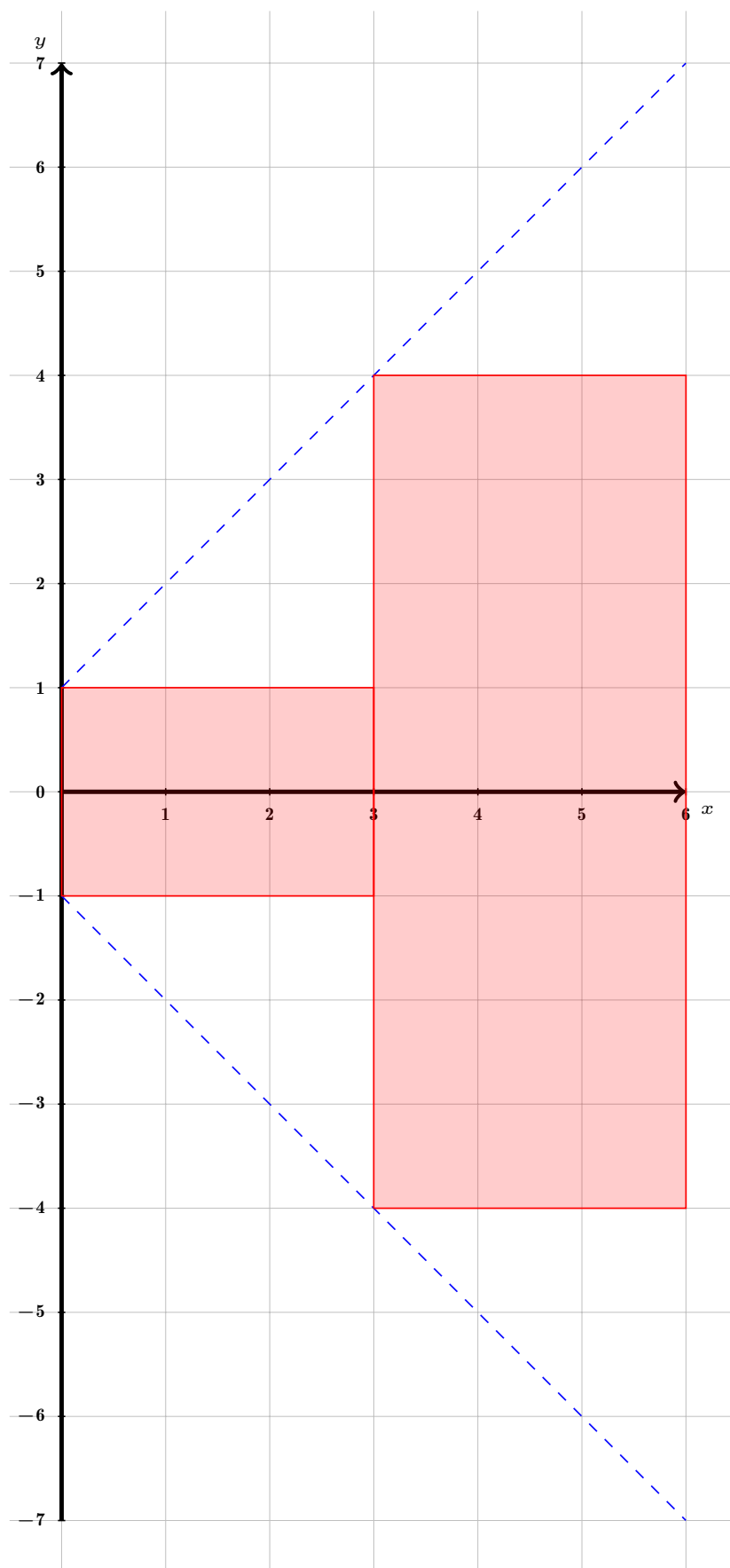


Figure 97 Plot of $f(x) = x + 1$ over $[0, 6]$ and cross section of two cylinders.

What is the volume of the cylinder with radius $f(3) = 4$?

A $\pi \cdot 0^2 \cdot 3$.

B $\pi \cdot 1^2 \cdot 3$.

C $\pi \cdot 3^2 \cdot 3$.

D $\pi \cdot 4^2 \cdot 3$.

E $\pi \cdot 6^2 \cdot 3$.

F $\pi \cdot 7^2 \cdot 3$.

(c) Suppose we now use a “height” of $\Delta x = 2$:

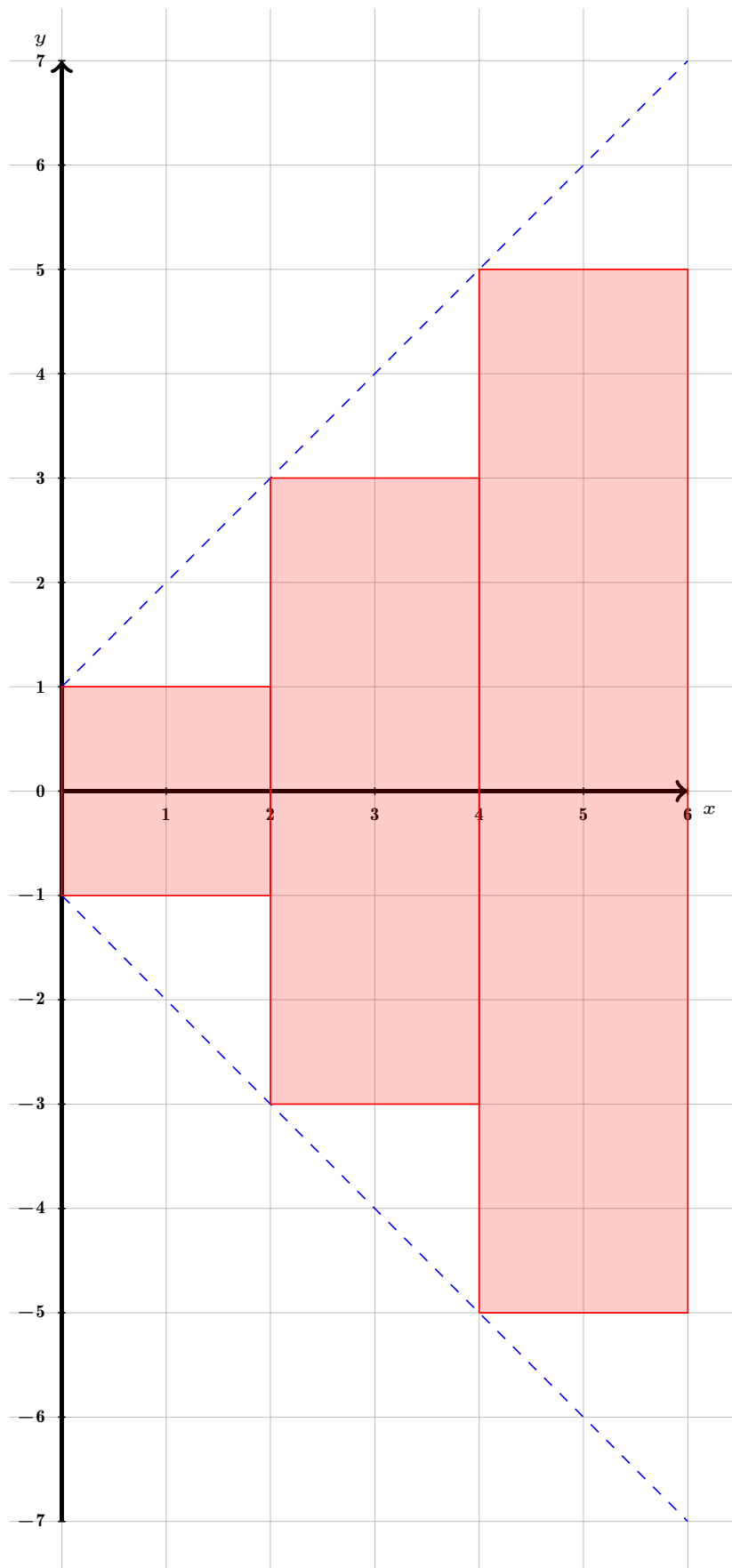


Figure 98 Plot of $f(x) = x + 1$ over $[0, 6]$ and cross section of 3 cylinders.

Fill out the following table:

x_i	Δx	Radius of Cross Section = $f(x_i)$	Estimated Volume
$x_1 = 0$	2		
$x_2 = 2$	2		
$x_3 = 4$	2		

(d) Which of the following Reimann sums best approximates the volume of our solid?

- A $\sum \pi x_i^2 \Delta x$
 B $\sum \pi(x_i + 1)^2 \Delta x$
 C $\sum 2\pi x_i \Delta x$
 D $\sum 2\pi(x_i + 1) \Delta x$

(e) Which of the following integrals computes the volume of our solid?

- A $\int_0^6 \pi x^2 dx$
 B $\int_0^6 \pi(x + 1)^2 dx$
 C $\int_0^6 2\pi x dx$
 D $\int_0^6 2\pi(x + 1) dx$

(f) What is the volume of our solid?

Activity 6.3.2 Suppose we wanted to find the volume of the the solid of revolution generated by rotating the region in the first quadrant bound by the curves:

$$x = 0, y = 2, y = \sqrt{x}$$

about the y -axis.

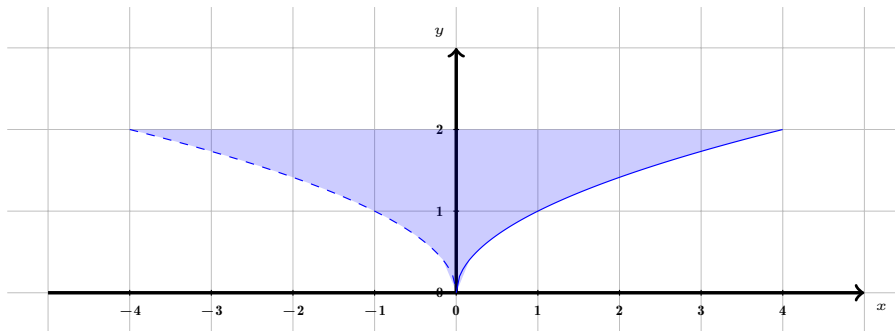
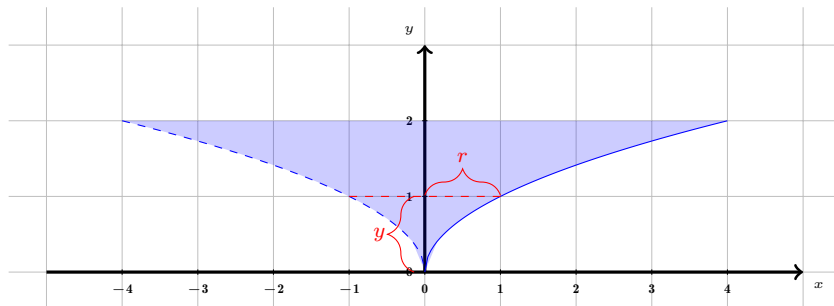


Figure 99 Plot of Bounded region rotated about y -axis.

(a)

**Figure 100** Plot of Bounded region rotated about y -axis.

Consider a cross section at height y , what is the radius of this cross section?

- A $r = 4$.
- B $r = 2$.
- C $r = \sqrt{x}$.
- D $r = x^2$.
- E $r = \sqrt{y}$.
- F $r = y^2$.

(b) Suppose we wanted to estimate this volume with 4 cylinders with $\Delta y = 0.5$. Fill out the following table:

y_i	Δy	Radius of Cross Section	Estimated Volume
$y_1 = 0$	0.5		
$y_2 = 0.5$	0.5		
$y_3 = 1$	0.5		
$y_4 = 1.5$	0.5		

(c) Which of the following Reimann sums best approximates the volume of our solid?

- A $\sum 2\pi y_i^2 \Delta y$
- B $\sum \pi(y_i^2)^2 \Delta y$
- C $\sum 2\pi \sqrt{y_i} \Delta y$
- D $\sum \pi(\sqrt{y_i})^2 \Delta y$

(d) Which of the following integrals computes the volume of our solid?

- A $\int_0^2 \pi y^4 dy$
- B $\int_0^4 \pi y^4 dy$
- C $\int_{-4}^4 \pi y^4 dy$

(e) What is the volume of our solid?

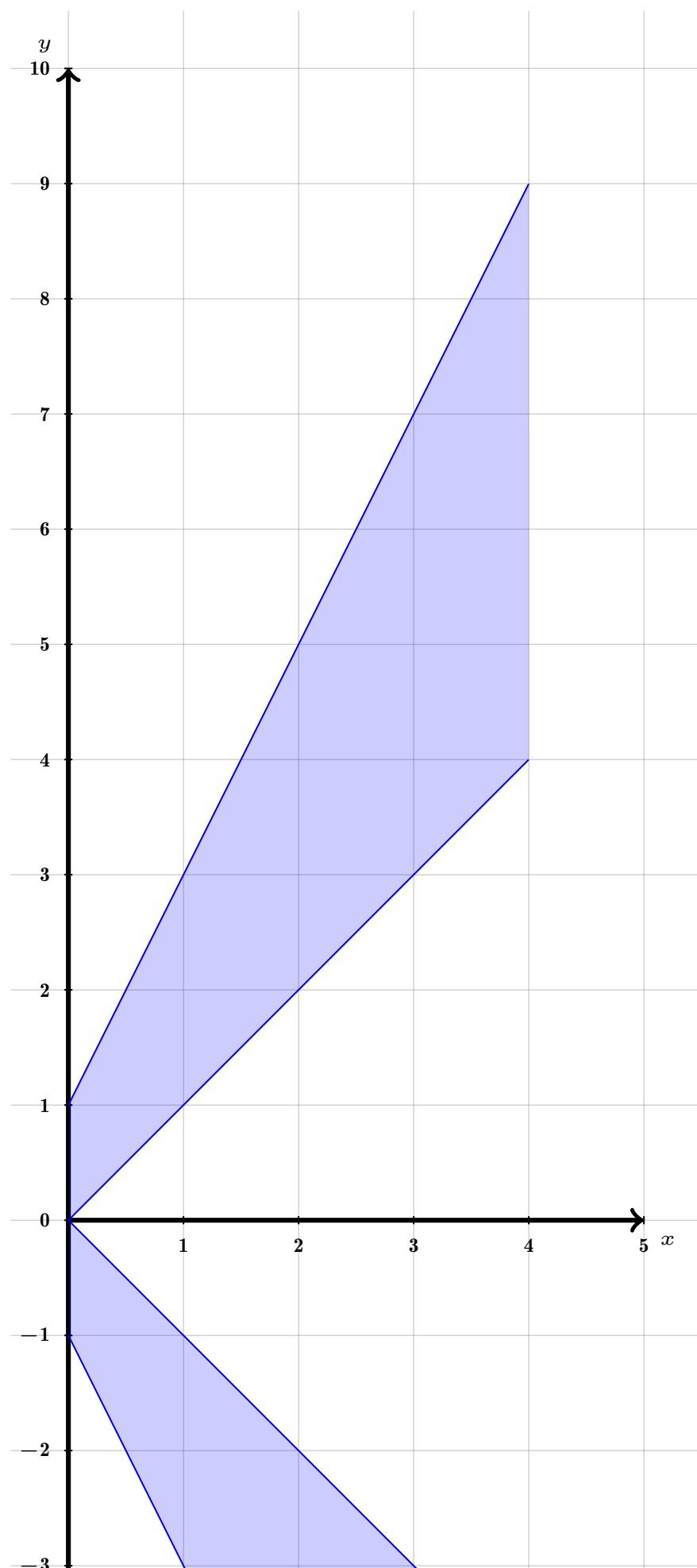
Fact 6.3.3 Volumes of Revolution: Disk Method.. *If $y = r(x)$ is a nonnegative continuous function on $[a, b]$, then the volume of the solid of revolution generated by revolving the curve about the x -axis over this interval is given by*

$$V = \int_a^b \pi r(x)^2 dx.$$

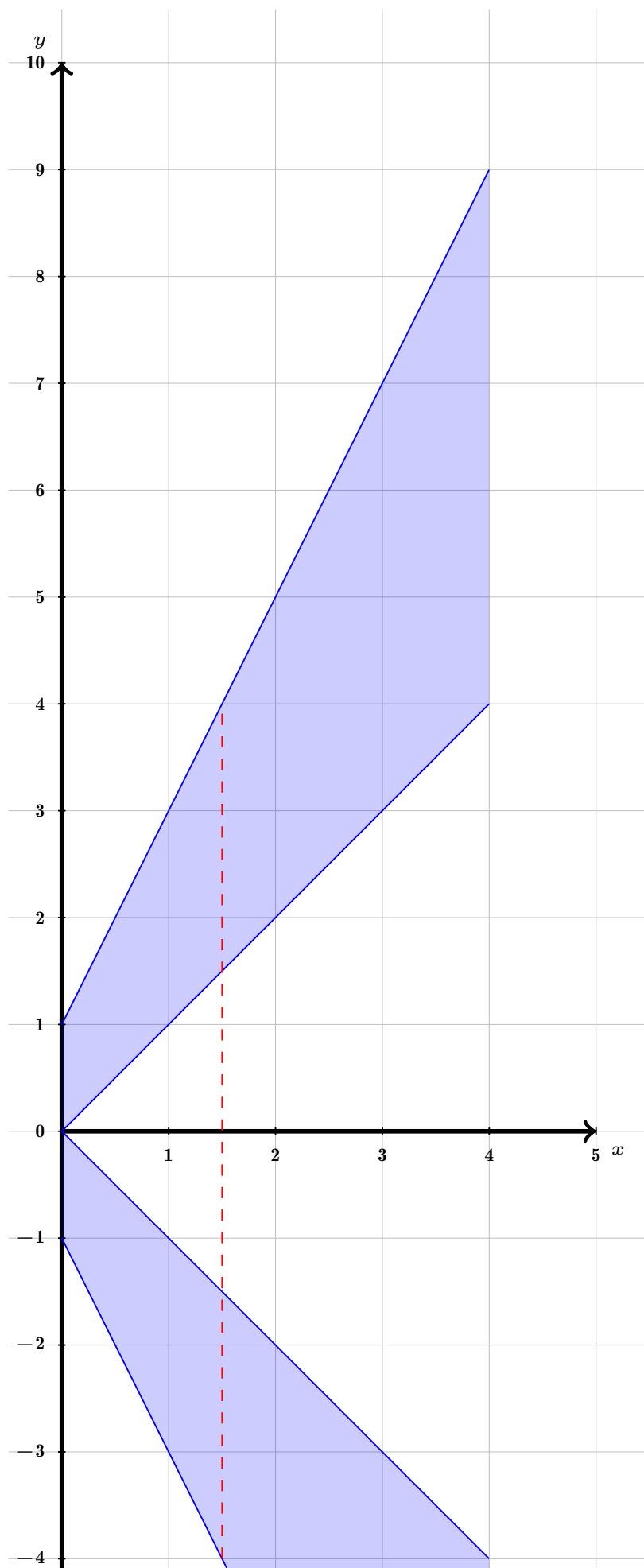
Activity 6.3.4 Consider the solid of revolution generated by rotating the region bound by

$$x = 0, x = 4, y = 0, y = x, y = 2x + 1$$

about the x -axis.



- (a) What is the shape of the cross section made at $x = x_i$?



- A A circle with radius $r = x_i$.
 B A circle with radius $r = 2x_i + 1$.
 C A circle with radius $r = 2x_i + 1 - x_i = x_i + 1$.
 D An annulus with outside radius $R = 2x_i + 1$ and inner radius $r = x_i$.
- (b) Suppose we wanted to estimate this volume with 4 cylinders or annular cylinders with $\Delta x = 1$. What is the volume of the (annular) cylinder formed at the cross section $x_i = 1$?
- (a) $\pi \cdot 1^2 \cdot 1$.
 (b) $\pi \cdot 3^2 \cdot 1$.
 (c) $\pi \cdot 2^2 \cdot 1$.
 (d) $(\pi \cdot 3^2 - \pi \cdot 1^2) \cdot 1$.

- (c) We continue to estimate this volume with 4 cylinders or annular cylinders with $\Delta x = 1$. Fill out the following table:

x_i	Δx	Area of Cross Section	Estimated Volume
$x_1 = 0$	1		
$x_2 = 1$	1	8π	$8\pi \cdot 1$
$x_3 = 2$	1		
$x_4 = 3$	1		

- (d) Which of the following Reimann sums best approximates the volume of our solid?

- A $\sum \pi x_i^2 \Delta x$
 B $\sum \pi (2x_i + 1)^2 \Delta x$
 C $\sum \pi (2x_i + 1 - x_i)^2 \Delta x$
 D $\sum \pi ((2x_i + 1)^2 - x_i^2) \Delta x$

- (e) Which of the following integrals computes the volume of our solid?

- A $\int_0^4 \pi x^2 dx$
 B $\int_0^4 \pi (2x + 1)^2 dx$
 C $\int_0^4 \pi (x + 1)^2 dx$
 D $\int_0^4 \pi ((2x + 1)^2 - x^2) dx$

- (f) What is the volume of our solid?

- (g) How else could we have computed the volume?

- A Found the volume of the region bounded by $f(x) = 2x + 1$ on $[0, 4]$ rotated about the x -axis, subtract the volume of the region bounded by $g(x) = x$ on $[0, 4]$ rotated about the x -axis.
 B Found the volume of the region bounded by $f(x) = 2x + 1 - x = x + 1$ on $[0, 4]$ rotated about the x -axis.

Fact 6.3.5 The Washer Method. If $y = R(x)$ and $y = r(x)$ are nonnegative continuous functions on $[a, b]$ that satisfy $R(x) \geq r(x)$ for all x in $[a, b]$, then the volume of the solid of revolution generated by revolving the region between them about the x -axis over this interval is given by

$$V = \int_a^b \pi[R(x)^2 - r(x)^2] dx.$$

Activity 6.3.6 Suppose we wanted to find the volume of the the solid of revolution generated by rotating the region in the first quadrant bound by the curves:

$$x = 0, x = 3, y = 0, y = \frac{4}{2^x}$$

about the y -axis.

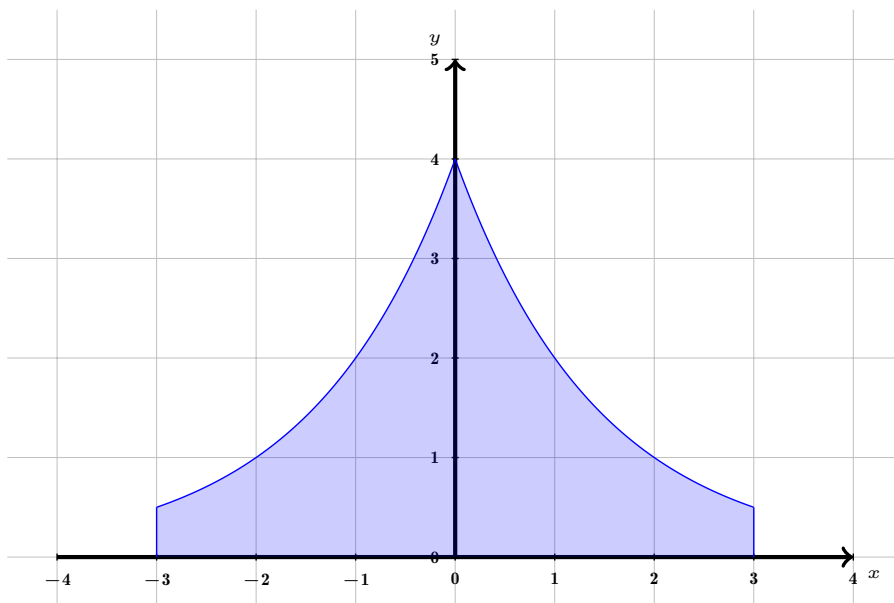


Figure 103 Plot of Bounded region rotated about y -axis.

- (a) Suppose we wanted to estimate this volume with 3 concentric annular cylinders with heights $h_i = f(r_i) = \frac{4}{2^{r_i}}$, where r_i are the inner radii and R_i are the outer radii. Over which values do the r_i, R_i range?
- A $[-3, 3]$.
 - B $[0, 3]$.
 - C $[-4, 4]$.
 - D $[0, 4]$.
- (b) Recall that the volume of an annular cylinder of height h , inner radius r and outer radius R is $V = \pi(R^2 - r^2)h$.

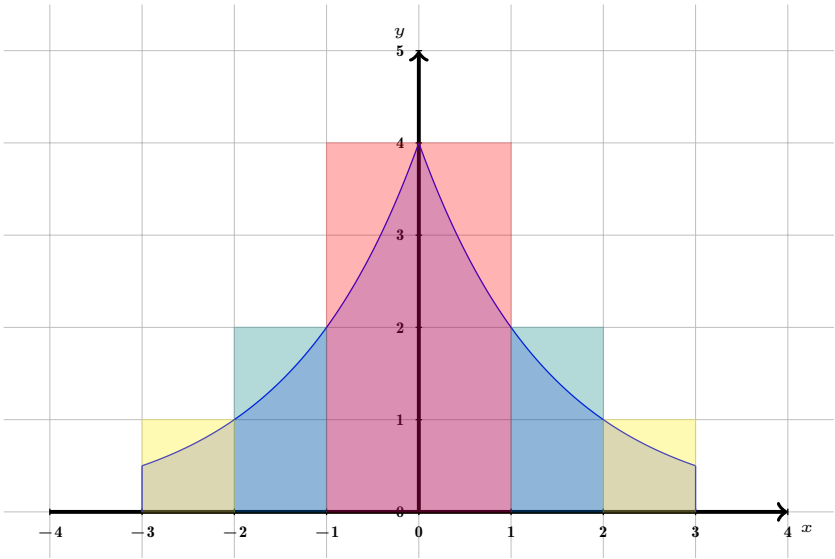


Figure 104 Plot of Bounded region rotated about y -axis.

Fill out the following table:

r_i	Δr_i	$; R_i = r_i + \Delta r$	$h_i = f(r_i) = \frac{4}{2^{r_i}}$	Estimated Volume
$r_1 = 0$	1	$R_1 = 0 + 1 = 1$		
$r_2 = 1$	1	$R_2 = 1 + 1 = 2$		
$r_3 = 2$	1	$R_3 = 2 + 1 = 3$		

- (c) Consider an arbitrary annular cylinder with inner radius r_i , outer radius $r_i + \Delta r$, and height $h_i = \frac{4}{2^{r_i}}$.

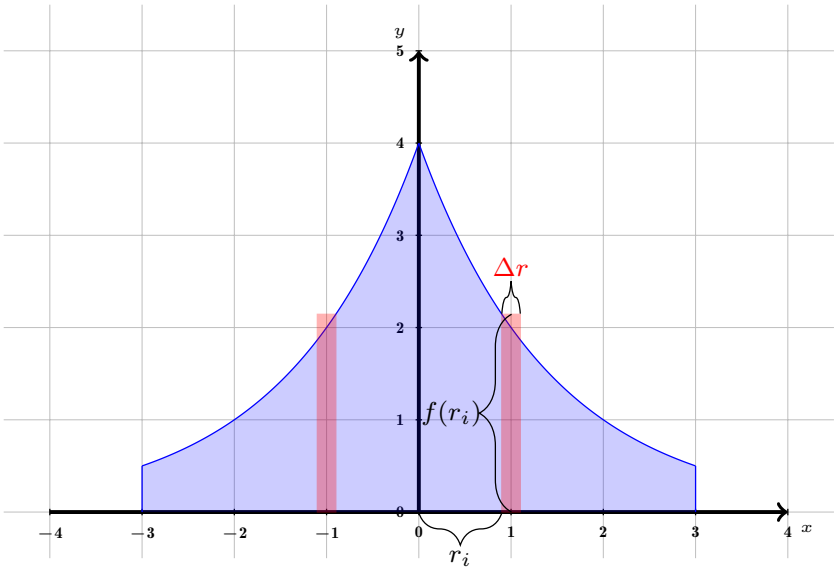


Figure 105 Plot of Bounded region rotated about y -axis.

Which of the following represents the volume of this annular cylinder?

- A $\pi r_i^2 \cdot \frac{4}{2^{r_i}}$.
- B $\pi (r_i + \Delta r)^2 \cdot \frac{4}{2^{r_i + \Delta r}}$.

$$\text{C } \pi((r_i + \Delta r)^2 - r_i^2) \cdot \frac{4}{2^{r_i}}.$$

$$\text{D } \pi((r_i + \Delta r) - r_i)^2 \cdot \frac{4}{2^{r_i}}.$$

- (d) Recall that $\pi((r_i + \Delta r)^2 - r_i^2) = \pi(2(\Delta r)r_i + (\Delta r)^2) = \pi(2r_i + (\Delta r))\Delta r$. Which of the following Riemann sums best estimates the volume of our solid?

$$\text{A } \sum \pi(2r_i + (\Delta r)) \frac{4}{2^{r_i}} \Delta r$$

$$\text{B } \sum \pi r_i^2 \frac{4}{2^{r_i}} \Delta r$$

$$\text{C } \sum \pi(r_i + \Delta r)^2 \frac{4}{2^{r_i}} \Delta r$$

- (e) Which of the following integrals computes the volume of our solid?

$$\text{A } \int_{-3}^3 2\pi r \frac{4}{2^r} dr$$

$$\text{B } \int_0^3 2\pi r \frac{4}{2^r} dr$$

$$\text{C } \int_0^3 2\pi r \frac{4}{2^r} dr$$

- (f) What is the volume of our solid?

Fact 6.3.7 The Shell Method. Let a solid be formed by revolving a region R , bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x) dx.$$

Activity 6.3.8 Consider the region bounded by

$$x = 0, y = 2, x = 4, y = \sqrt{x}.$$

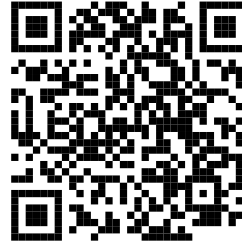
- (a) Find the volume of the solid generated by rotating this region about the x -axis.
- (b) Find the volume of the solid generated by rotating this region about the y -axis.

Activity 6.3.9 Consider the region bounded by

$$x = 0, y = 0, y = x + 3, x = 4.$$

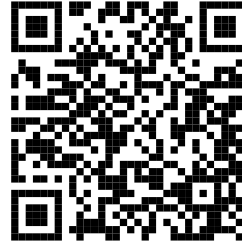
Find the volume of the solid generated by rotating this region about the line $x = 10$. (Hint: Sketch this figure, what is $R(x)$ the outer radius? What is $r(x)$ the inner radius?)

6.3.1 Videos



YouTube: <https://www.youtube.com/watch?v=CBL66Ro9Vcc>

Figure 106 Video: Compute volumes of solids of revolution, washer x -axis



YouTube: <https://www.youtube.com/watch?v=KOPFv12uC68>

Figure 107 Video: Compute volumes of solids of revolution, shell x -axis



YouTube: <https://www.youtube.com/watch?v=Y16cc0Z43Uc>

Figure 108 Video: Compute volumes of solids of revolution, washer y -axis



YouTube: <https://www.youtube.com/watch?v=j8A0rflmA2I>

Figure 109 Video: Compute volumes of solids of revolution, shell y -axis

6.4 Surface Areas of Revolution (AI4)

Learning Outcomes

- Compute surface areas of surfaces of revolution.

Fact 6.4.1 A frustum is the portion of a cone that lies between one or two parallel planes cutting it.

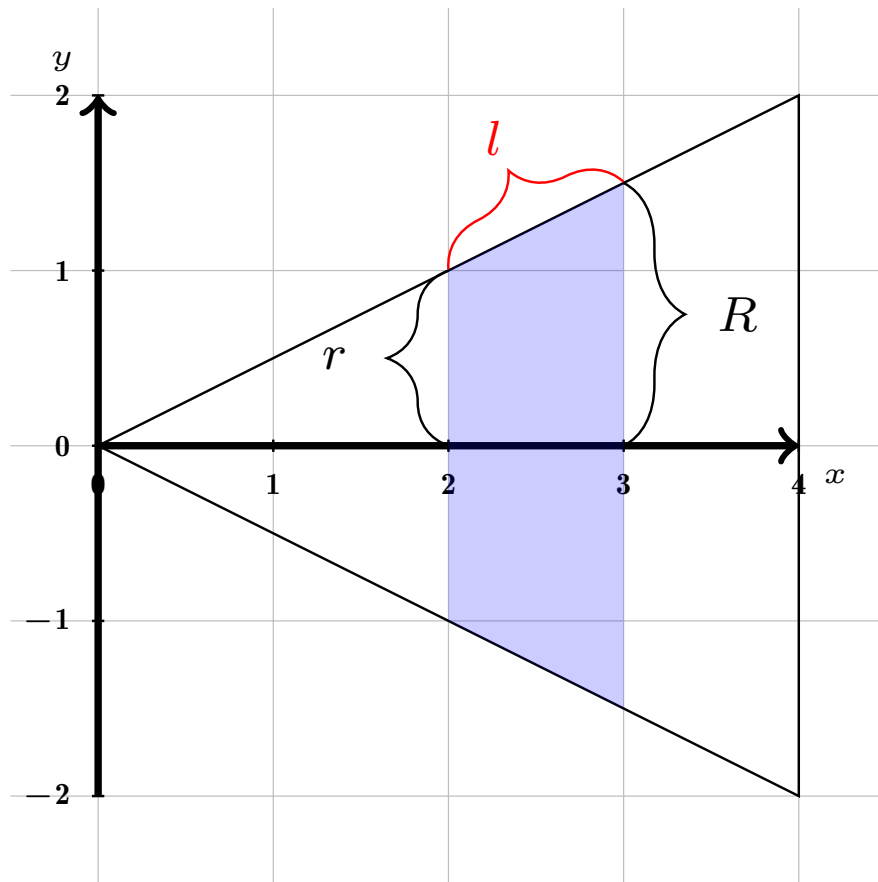


Figure 110 Plot of a frustum.

The surface area of the “side” of the frustum is:

$$2\pi \frac{r + R}{2} \cdot l$$

where r, R are the radii of the bases, and l is the length of the side.

Note that if $r = R$, this reduces to the surface area of a “side” of a cylinder.

Activity 6.4.2 Suppose we wanted to find the surface area of the the solid of revolution generated by rotating

$$y = \sqrt{x}, 0 \leq x \leq 4$$

about the y -axis.

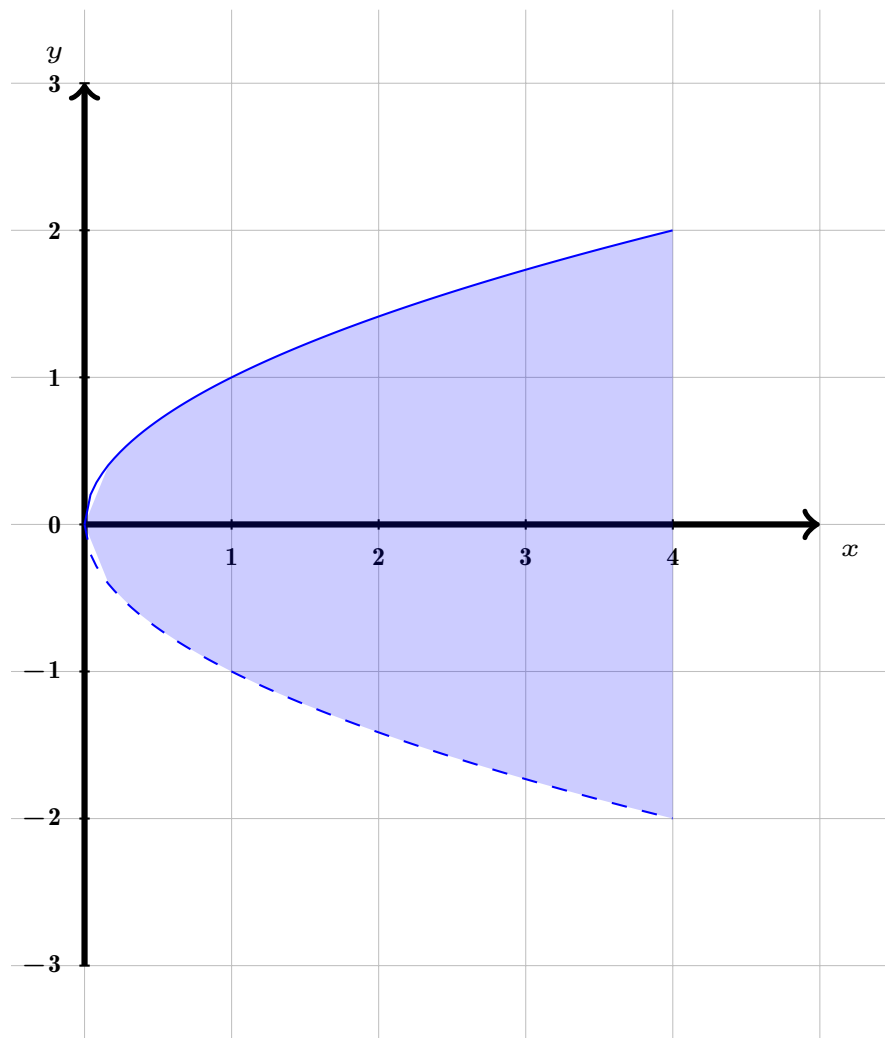


Figure 111 Plot of Bounded region rotated about x -axis.

- (a) Suppose we wanted to estimate the surface area with two fulstrums with $\Delta x = 2$.

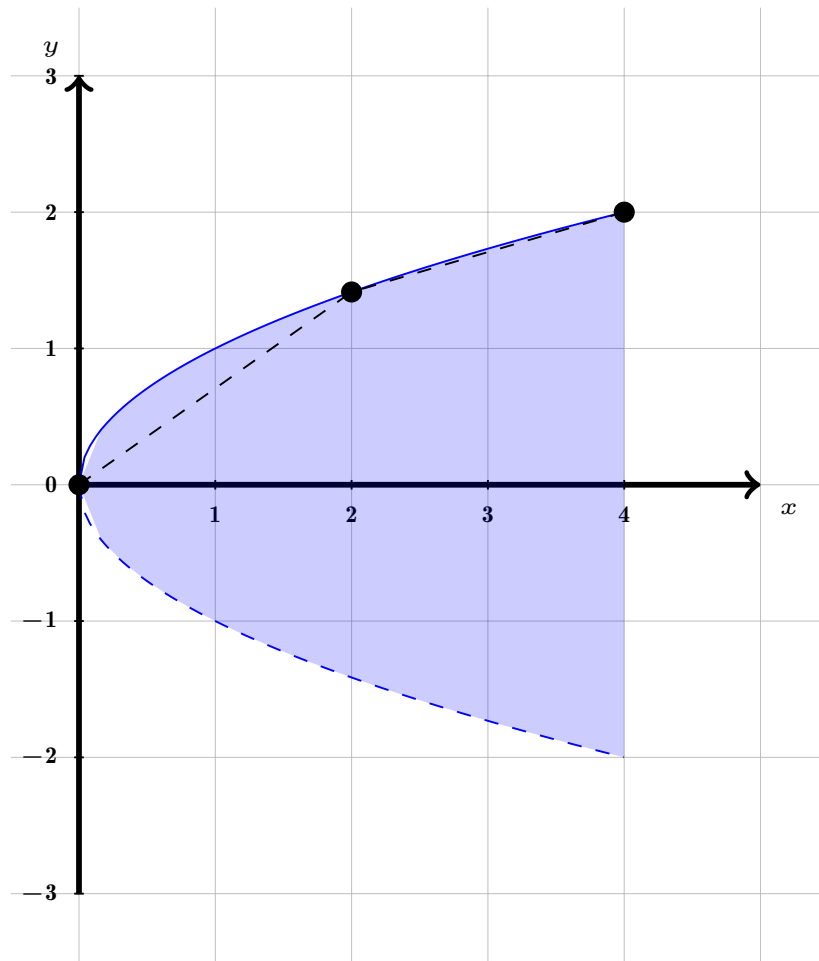
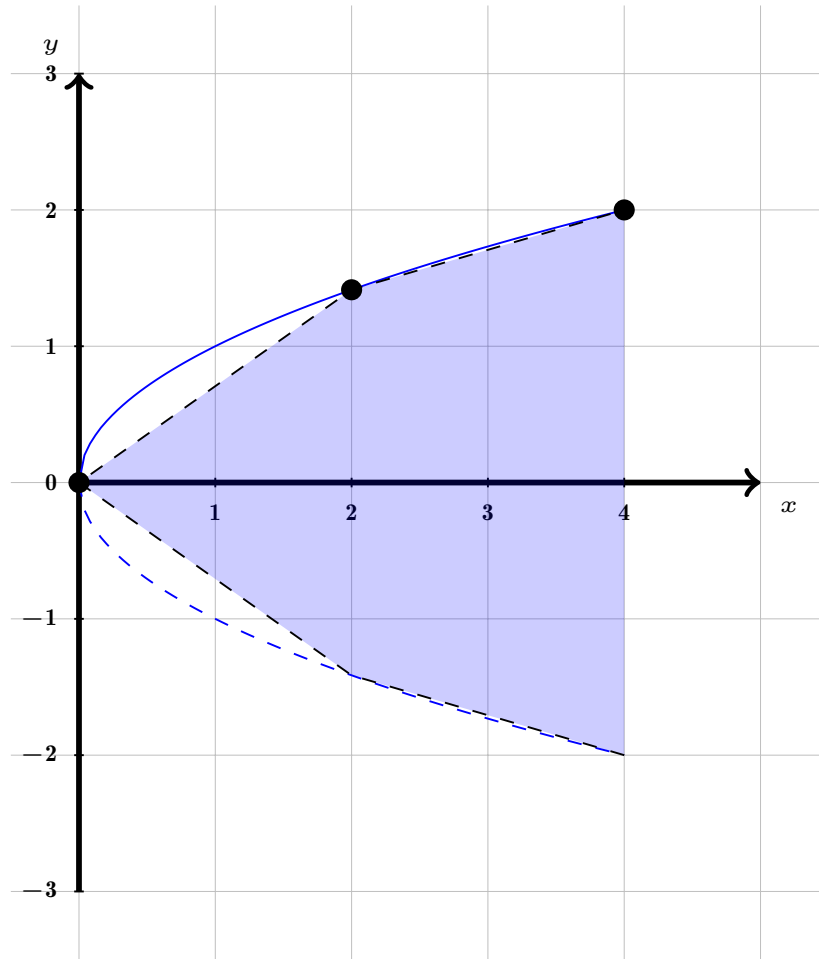


Figure 112 Plot of Bounded region rotated about x -axis.

What is the surface area of the fulstrum formed by rotating the line segment from $(0, 0)$ to $(2, \sqrt{2})$ about the x -axis?

- A $2\pi \frac{0+\sqrt{2}}{2} \cdot 2.$
- B $2\pi \frac{0+\sqrt{2}}{2} \cdot \sqrt{2^2 + \sqrt{2}^2}.$
- C $\pi \sqrt{2}^2 \cdot 2.$
- D $\pi \sqrt{2}^2 \cdot \sqrt{2^2 + \sqrt{2}^2}.$

(b)

**Figure 113** Plot of Bounded region rotated about x -axis.

What is the surface area of the fulstrum formed by rotating the line segment from $(2, \sqrt{2})$ to $(4, 2)$ about the x -axis?

- A $2\pi \frac{4+\sqrt{2}}{2} \cdot \sqrt{2}.$
 B $2\pi \frac{4+\sqrt{2}}{2} \cdot \sqrt{6}.$
 C $2\pi \frac{4+\sqrt{2}}{2} \cdot \sqrt{6 - 2\sqrt{2}}.$

(c) Suppose we wanted to estimate the surface area with four fulstrums with $\Delta x = 1$.

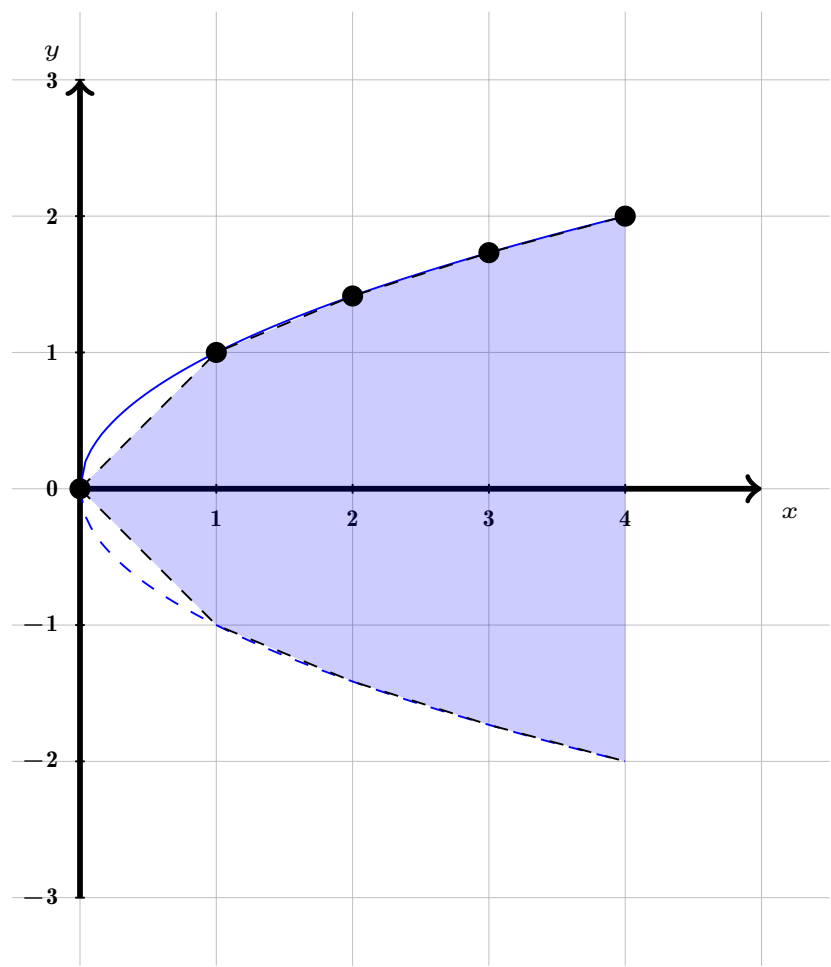


Figure 114 Plot of Bounded region rotated about x -axis.

x_i	Δx	r_i	R_i	l	Estimated Surface Area
$x_1 = 0$	1	0	1	$\sqrt{1^2 + 1^2}$	
$x_2 = 1$	1	1	$\sqrt{2}$	$\sqrt{1^2 + (\sqrt{2} - 1)^2}$	
$x_3 = 2$	1	$\sqrt{2}$	$\sqrt{3}$		
$x_4 = 3$	1	3	2		

(d) Suppose we wanted to estimate the surface area with n fulstrums.

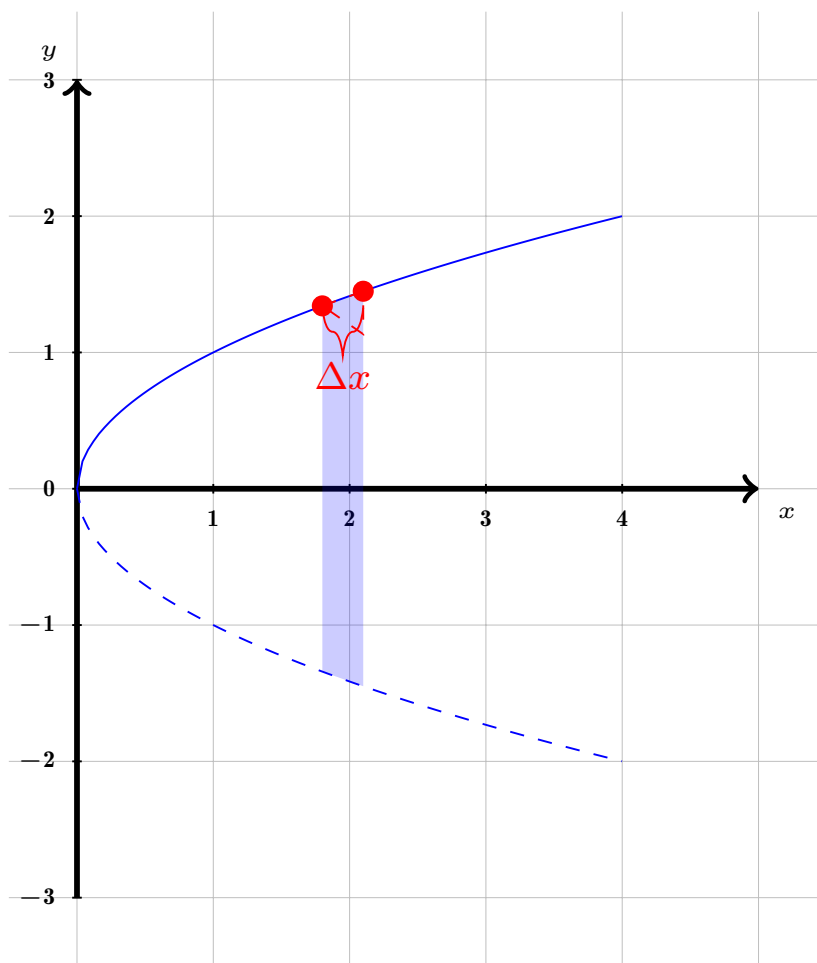


Figure 115 Plot of Bounded region rotated about x -axis.

Let $f(x) = \sqrt{x}$. Which of the following expressions represents the surface area generated by rotating the line segment from $(x_0, f(x_0))$ to $(x_0 + \Delta x, f(x_0 + \Delta x))$ about the x -axis?

- A $\pi \left(\frac{f(x_0) + f(x_0 + \Delta x)}{2} \right)^2 \sqrt{(\Delta x)^2 + (f(x_0 + \Delta x) - f(x_0))^2}.$
- B $2\pi \frac{f(x_0) + f(x_0 + \Delta x)}{2} \sqrt{(\Delta x)^2 + (f(x_0 + \Delta x) - f(x_0))^2}.$
- C $2\pi \frac{f(x_0) + f(x_0 + \Delta x)}{2} \Delta x.$

- (e) Which of the following Riemann sums best estimates the surface area of the solid generated by rotating $y = \sqrt{x}$ over $[0, 4]$ about the x -axis? Let $f(x) = \sqrt{x}$.

- A $\sum \pi \left(\frac{f(x_i) + f(x_i + \Delta x)}{2} \right)^2 \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}.$
- B $\sum 2\pi \frac{f(x_i) + f(x_i + \Delta x)}{2} \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}.$
- C $\sum 2\pi \frac{f(x_i) + f(x_i + \Delta x)}{2} \Delta x.$

Fact 6.4.3 Recall from [Section 6.2](#) that

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2} &= \lim_{\Delta x \rightarrow 0} \sqrt{(\Delta x)^2 \left(1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right)^2 \right)} \\ &= \lim_{\Delta x \rightarrow 0} \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right)^2} \Delta x \\ &= \sqrt{1 + (f'(x))^2} dx,\end{aligned}$$

and that

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_i) + f(x_i + \Delta x)}{2} = f(x_i).$$

Thus given a function $f(x) \geq 0$ over $[a, b]$, the surface area of the solid generated by rotating this function about the x -axis is

$$SA = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

Activity 6.4.4 Consider again the solid generated by rotating $y = \sqrt{x}$ over $[0, 4]$ about the x -axis.

- Find an integral which computes the surface area of this solid.
- If we instead rotate $y = \sqrt{x}$ over $[0, 4]$ about the y -axis, what is an integral which computes the surface area for this solid?

Activity 6.4.5 Consider again the function $f(x) = \ln(x) + 1$ over $[1, 5]$.

- Find an integral which computes the surface area of the solid generated by rotating the above curve about the x -axis.
- Find an integral which computes the surface area of the solid generated by rotating the above curve about the y -axis.

6.4.1 Videos



YouTube: <https://www.youtube.com/watch?v=nUFCAnuFTGc>

Figure 116 Video: Compute surface areas of surfaces of revolution

6.5 Density, Mass, and Center of Mass (AI5)

Learning Outcomes

- Set up integrals to solve problems involving density, mass, and center of mass.

Activity 6.5.1 Consider a rectangular prism with a $10 \text{ m} \times 10 \text{ m}$ square base and height 20 m . Suppose the density of the material in the prism increases with height, following the function $\delta(h) = 10 + h \text{ kg/m}^3$, where h is the height in meters.

- (a) If one were to cut this prism, parallel to the base, into 4 pieces with height 5 m , what would the volume of each piece be?
- (b) Consider the piece sitting on top the slice made at height $h = 5$. Using a density of $\delta(5) = 15 \text{ kg/m}^3$, and the volume you found, estimate the mass of this piece.

A $500 \cdot 5 = 2500 \text{ kg}$.

B $500 \cdot 15 = 7500 \text{ kg}$.

C $500 \cdot 15 \cdot 5 = 37500 \text{ kg}$.

- (c) Is this estimate the actual mass of this piece?

Activity 6.5.2 Consider all 4 slices from [Activity 6.5.1](#).

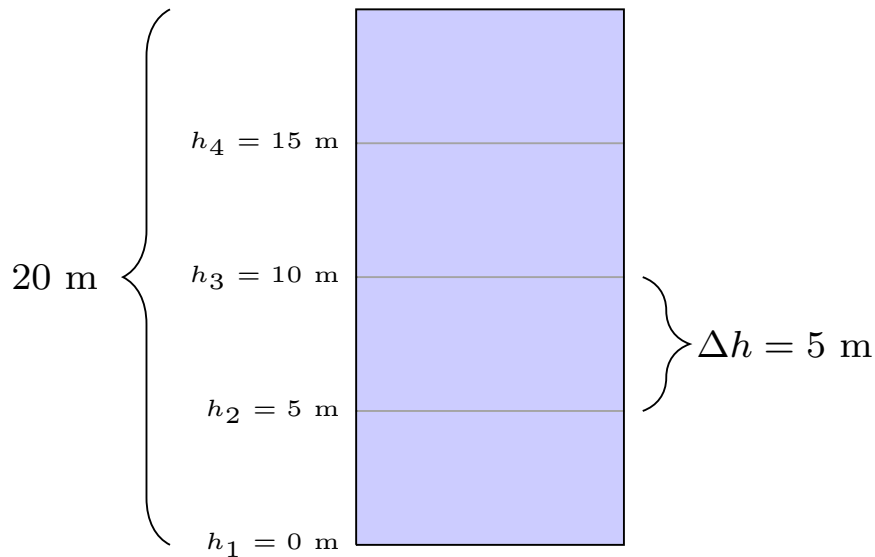


Figure 117 $10 \times 10 \times 20$ prism sliced into 4 pieces.

- (a) Fill out the following table:

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_4 = 15 \text{ m}$	$\delta(15) = 25 \text{ kg/m}^3$	500 m^3	
$h_3 = 10 \text{ m}$	$\delta(10) = 20 \text{ kg/m}^3$	500 m^3	
$h_2 = 5 \text{ m}$	$\delta(5) = 15 \text{ kg/m}^3$	500 m^3	7500 kg
$h_1 = 0 \text{ m}$	$\delta(0) = 10 \text{ kg/m}^3$	500 m^3	

- (b) What is the estimated mass of the rectangular prism?

Activity 6.5.3 Suppose instead that we sliced the prism from [Activity 6.5.1](#) into 5 pices of height 4 m .

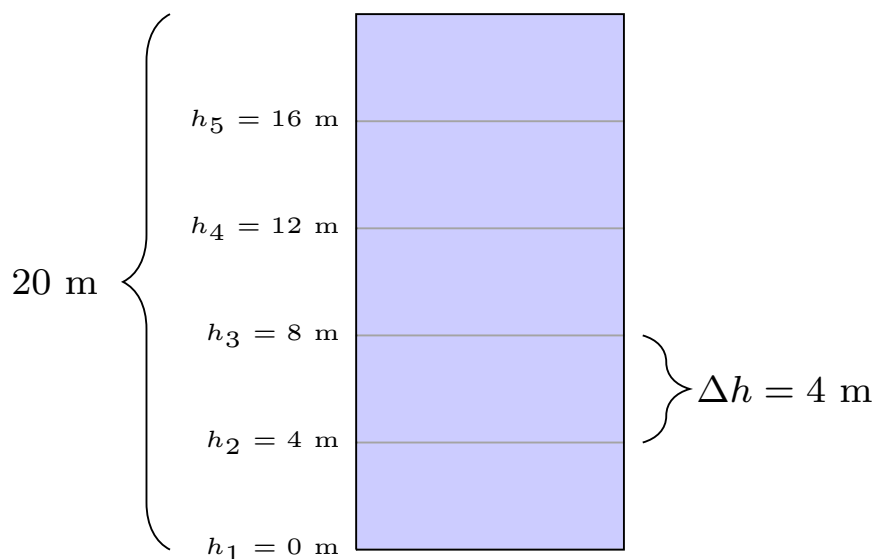


Figure 118 $10 \times 10 \times 20$ prism sliced into 5 pieces.

(a) Fill out the following table:

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_5 = 16$ m	$\delta(16) = 26$ kg/m ³	400 m ³	
$h_4 = 12$ m	$\delta(12) = 22$ kg/m ³	400 m ³	
$h_3 = 8$ m	$\delta(8) = 18$ kg/m ³	400 m ³	
$h_2 = 4$ m	$\delta(4) = 14$ kg/m ³	400 m ³	
$h_1 = 0$ m	$\delta(0) = 10$ kg/m ³	400 m ³	

(b) What is the estimated mass of the rectangular prism?

Activity 6.5.4 Which of the estimates computed in [Activity 6.5.2](#) and [Activity 6.5.3](#) is a better estimate of the mass of the prism?

A [Activity 6.5.2](#), 4 pieces is a better estimate.

B [Activity 6.5.3](#), 5 pieces is a better estimate.

Activity 6.5.5 Suppose now that we slice the prism from [Activity 6.5.1](#) into slices of height Δh m.

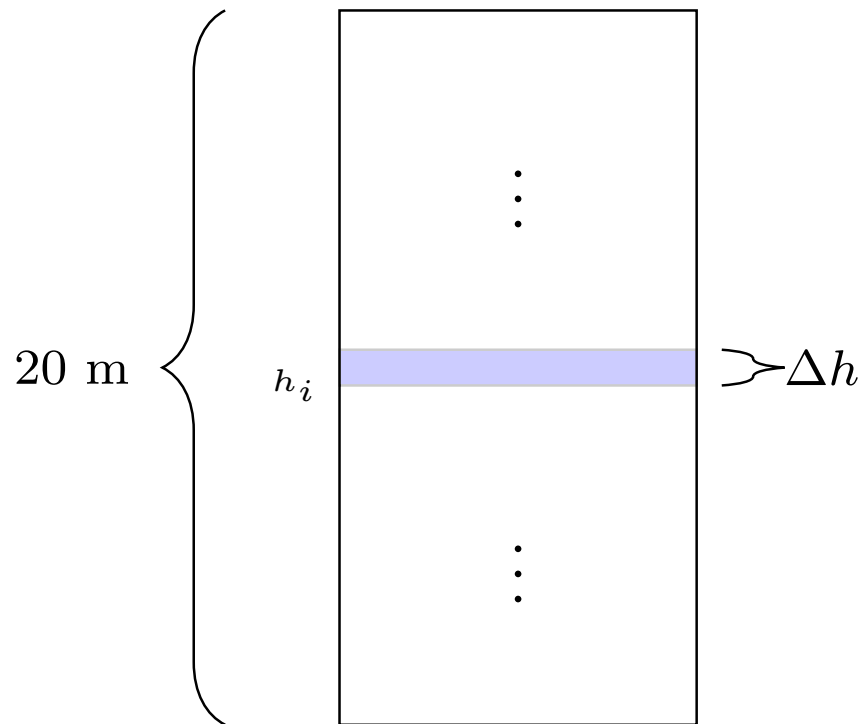


Figure 119 $10 \times 10 \times 20$ prism sliced into many pieces.

- (a) Consider the piece sitting atop the slice made at height h_i . Using $\delta(h_i) = 10 + h_i$ as the estimate for the density of this piece, what is the mass of this piece?
- A $(10 + h)100 \cdot h_i$.
 - B $(10 + \Delta h)100 \cdot h_i$.
 - C $(10 + h_i)100 \cdot \Delta h$.
 - D $(10 + h_i)100 \cdot h$.

Activity 6.5.6 Consider a cylindrical cone with a base radius of 15 inches and a height of 60 inches. Suppose the density of the cone is $\delta(h) = 15 + \sqrt{h}$ oz/in³.

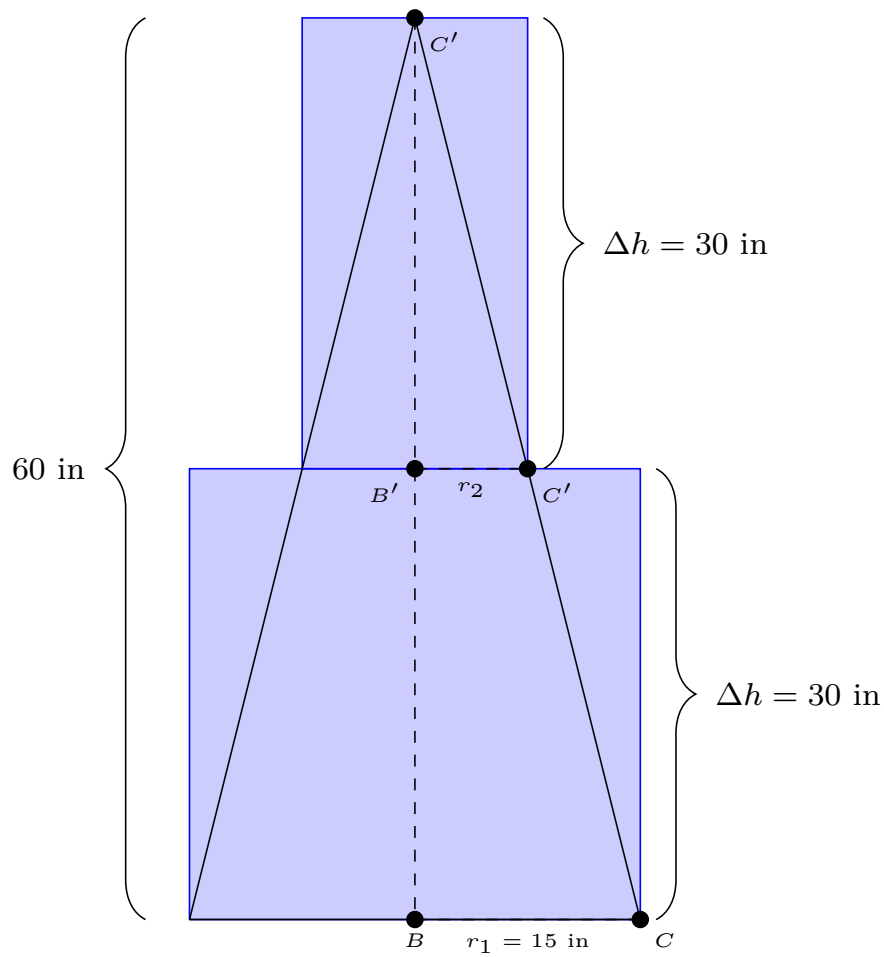


Figure 120 15× 60 cylindrical cone sliced into two pieces.

- (a) Let r_2 be the radius of the circular cross section of the cone, made at height 30 inches. Recall that $\triangle ABC, \triangle AB'C'$ are similar triangles, what is r_2 ?
- A 15 inches.
 - B 7.5 inches.
 - C 30 inches.
 - D 60 inches.
- (b) What is the volume of a cylinder with radius $r_1 = 15$ in and height 30 in?
- (c) What is the volume of a cylinder with radius r_2 and height 30 in?

Activity 6.5.7 Suppose that we estimate the mass of the cone from [Activity 6.5.6](#) with 2 cylinders of height 30 in.

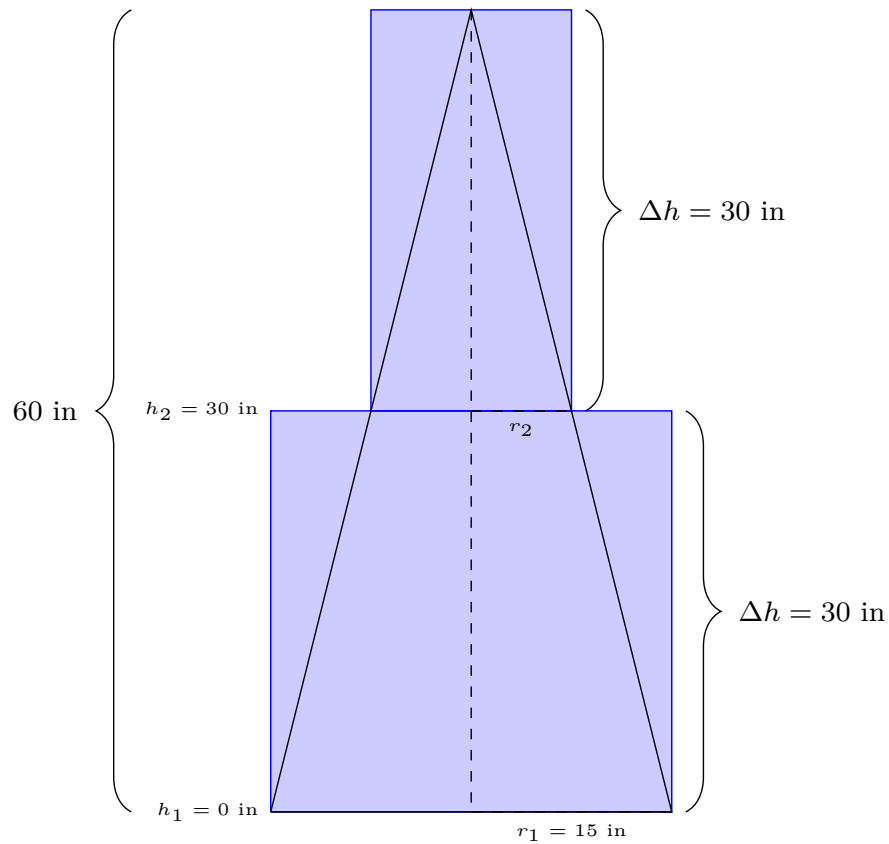


Figure 121 15× 60 cylindrical cone sliced into two pieces.

(a) Fill out the following table:

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_2 = 30$ m	$\delta(30) = 15 + \sqrt{30}$ oz/in ³	$\pi(7.5)^2 \cdot 30$ in ³	
$h_1 = 0$ m	$\delta(0) = 15$ oz/in ³	$\pi(15)^2 \cdot 30$ in ³	

(b) What is the estimated mass of the cone?

Activity 6.5.8 Suppose that we estimate the mass of the cone from [Activity 6.5.6](#) with 3 cylinders of height 20 in.

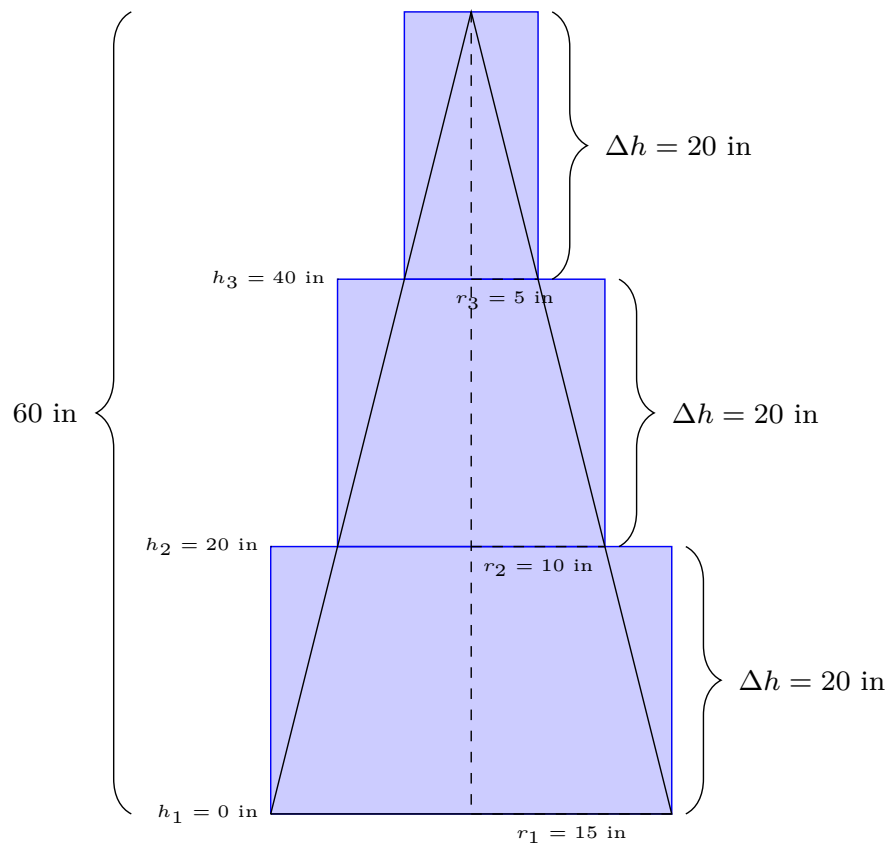


Figure 122 15× 60 cylindrical cone sliced into three pieces.

(a) Fill out the following table:

h_i	$\delta(h_i)$	Volume	Estimated Mass
$h_2 = 40$ m	$\delta(40) = 15 + \sqrt{40}$ oz/in ³	$\pi(5)^2 \cdot 20$ in ³	
$h_2 = 20$ m	$\delta(20) = 15 + \sqrt{20}$ oz/in ³	$\pi(10)^2 \cdot 20$ in ³	
$h_1 = 0$ m	$\delta(0) = 15$ oz/in ³	$\pi(15)^2 \cdot 20$ in ³	

(b) What is the estimated mass of the cone?

Activity 6.5.9 Suppose that we estimate the mass of the cone from [Activity 6.5.6](#) with cylinders of height Δh .

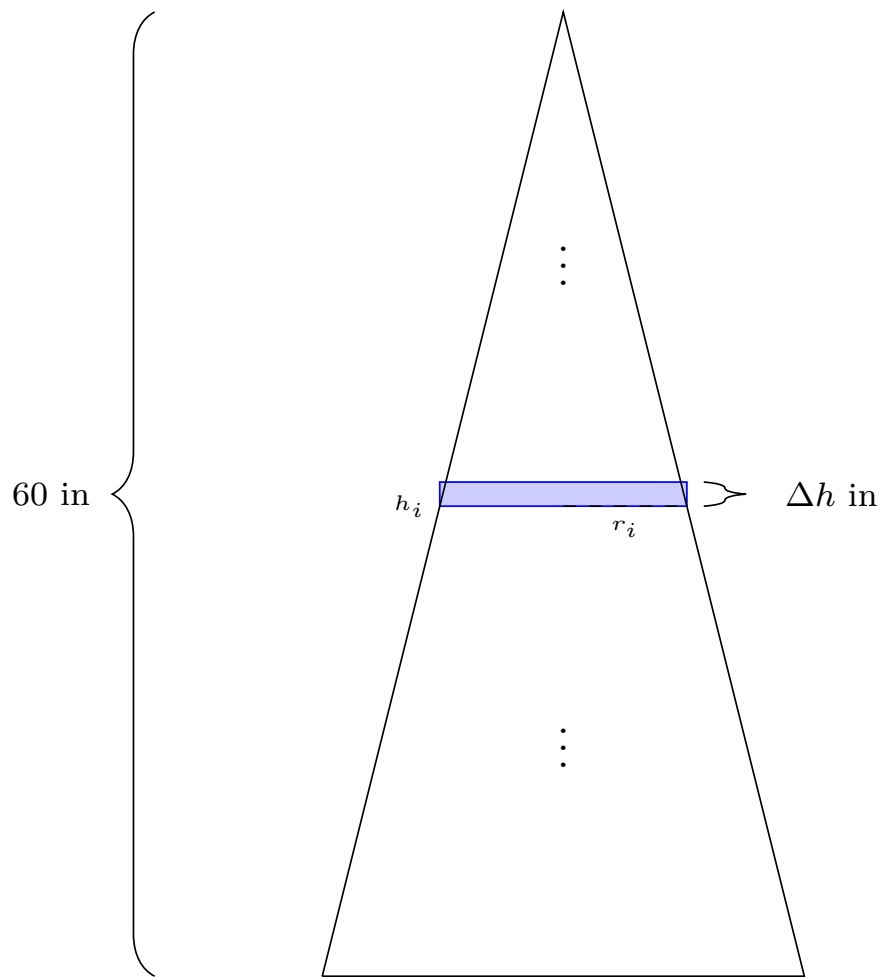


Figure 123 15×60 cylindrical cone sliced into many pieces.

- (a) Consider the piece sitting atop the slice made at height h_i . Using $\delta(h_i) = 15 + \sqrt{h_i}$ as the estimate for the density of this cylinder, what is the mass of this cylinder?

- A $(15 + \sqrt{h})\pi r_i^2 \cdot \Delta h$.
- B $(15 + \sqrt{h_i})\pi r_i^2 \cdot \Delta h$.
- C $(15 + \Delta h)\pi r_i^2 \cdot \Delta h_i$.
- D $(15 + \sqrt{h_i})\pi r^2 \cdot \Delta h$.

Activity 6.5.10 Consider a solid where the cross section of the solid at $x = x_i$ has area $A(x_i)$, and the density when $x = x_i$ is $\delta(x_i)$.

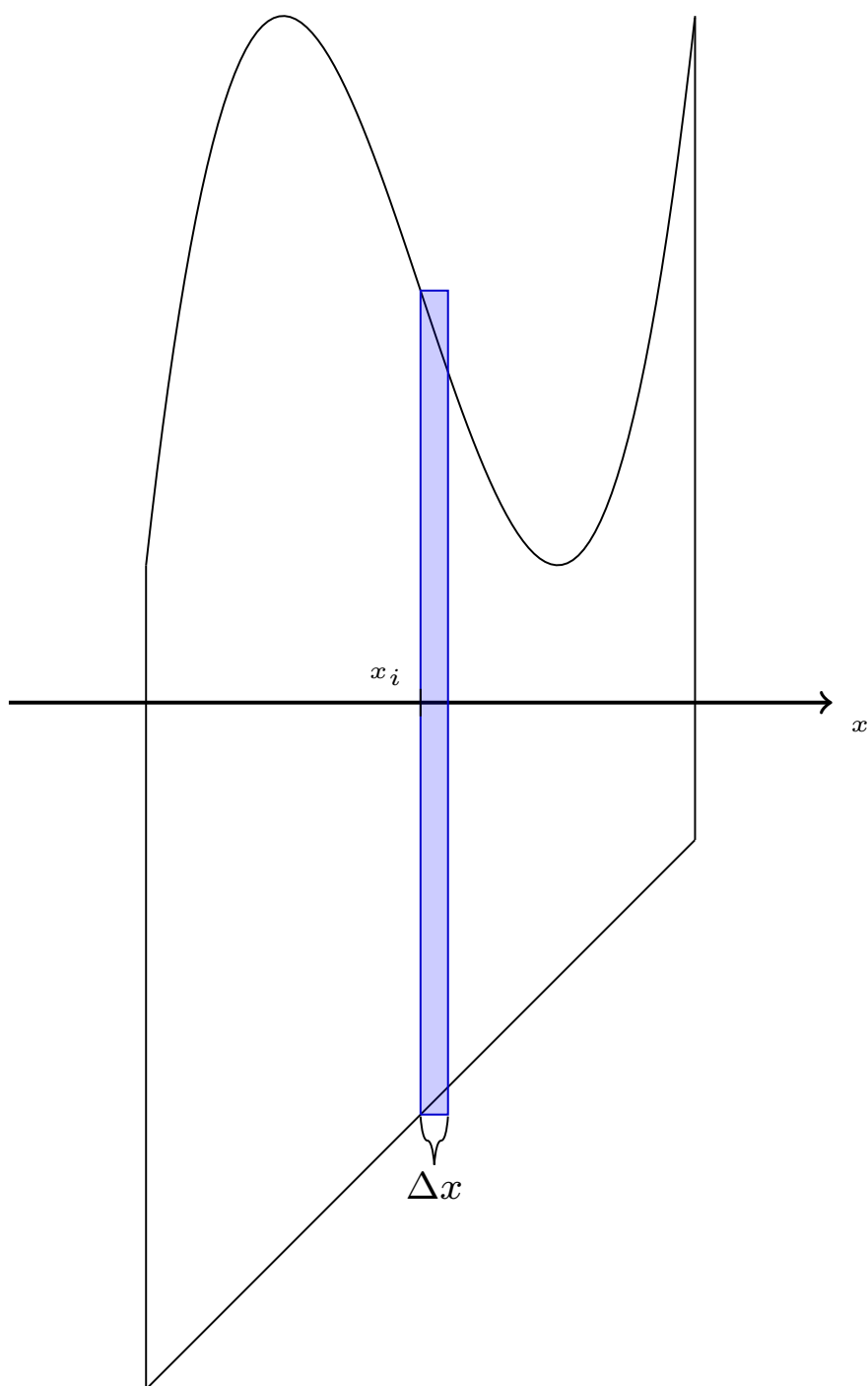


Figure 124 Solid approximated with prisms of width Δx .

- (a) If we used prisms of width Δx to approximate this solid, what is the mass of the slice associated with x_i ?
- A $A(x)\delta(x)\Delta x$.
 - B $\pi A(x)^2\delta(x_i)\Delta x$.
 - C $A(x_i)\delta(x_i)\Delta x$.
 - D $A(x_i)\delta(x_i)\Delta x_i$.

Fact 6.5.11 Consider a solid where the cross section of the solid at $x = x_i$ has area $A(x_i)$, and the density when $x = x_i$ is $\delta(x_i)$. Suppose the interval $[a, b]$ represented the x values of this solid. If one sliced the solid into n pieces of width $\Delta x = \frac{b-a}{n}$, then one can approximate the mass of the solid by

$$\sum_{i=1}^n \delta(x_i) A(x_i) \Delta x.$$

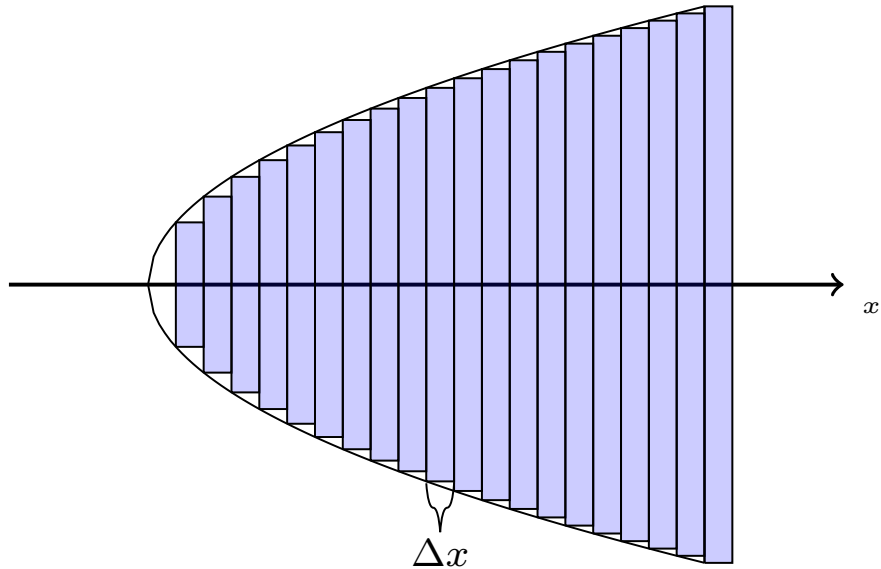


Figure 125 Solid approximated with prisms of width Δx .

We can then find actual mass by taking the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \delta(x_i) A(x_i) \Delta x \right) = \int_a^b \delta(x) A(x) dx.$$

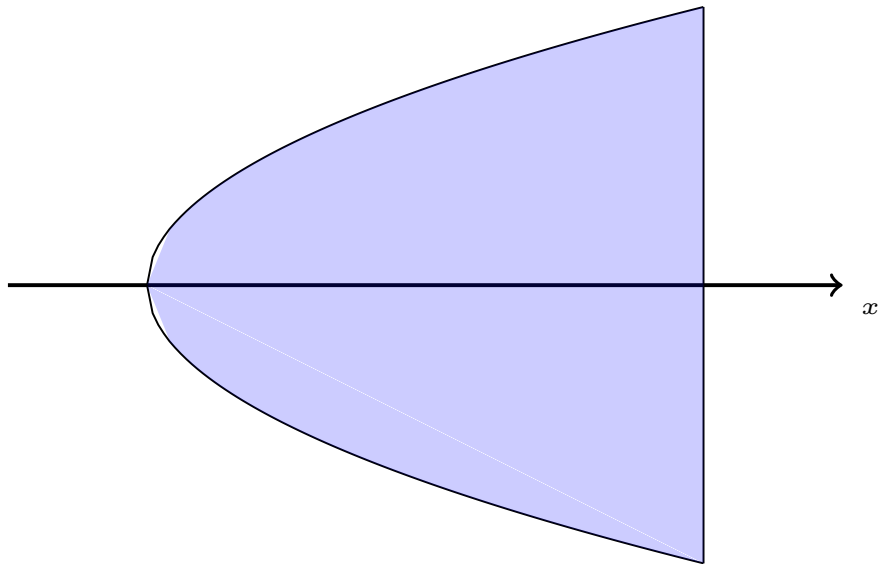


Figure 126 Solid mass.

Activity 6.5.12 Consider that for the prism from [Activity 6.5.1](#), a cross section of height h is $A(h) = 10^2 = 100 \text{ m}^2$. Also recall that the density of the prism is $\delta(h) = 10 + h \text{ kg/m}^3$, where h is the height in meters.

Use [Fact 6.5.11](#) to find the mass of the prism.

Activity 6.5.13 Consider that for the cone from [Activity 6.5.6](#), a cross section of height h is $A(h) = \pi r^2 \text{ in}^2$, where r is the radius of the circular cross-section at height h inches. Also recall that the density of the cone is $\delta(h) = 15 + \sqrt{h} \text{ oz/in}^3$, where h is the height in inches.

(a) When the height is h in, what is r ?

Use similar triangles:

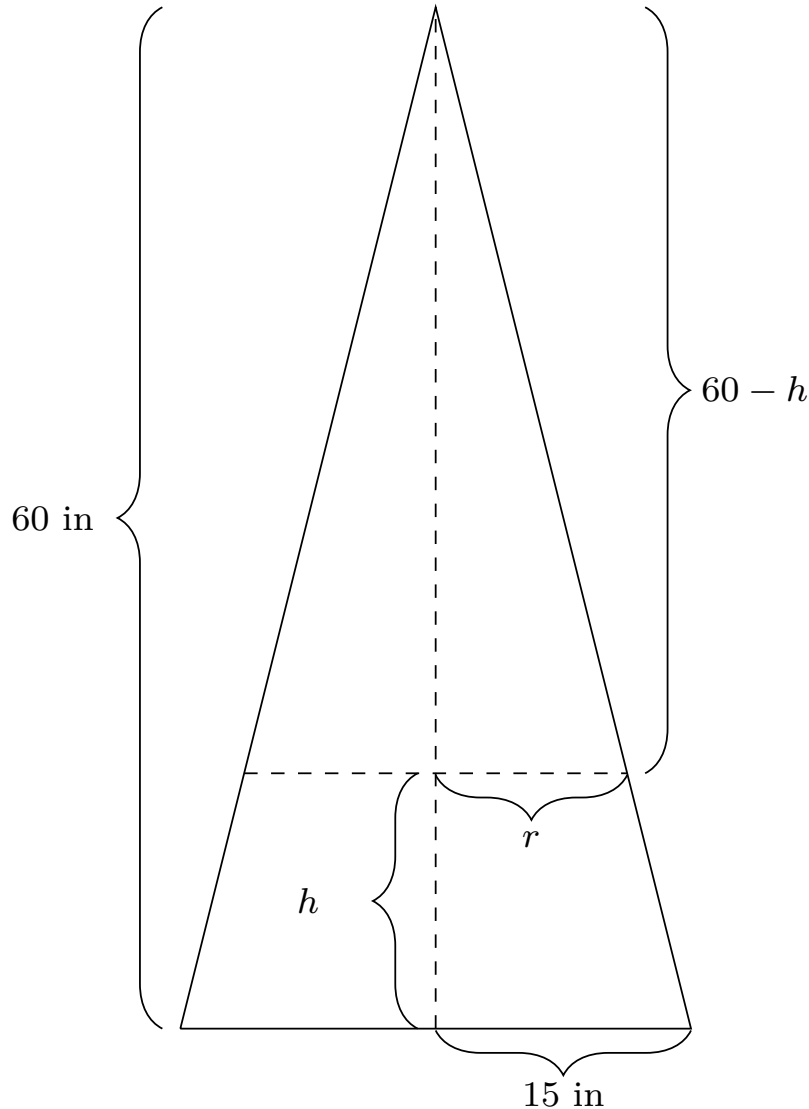


Figure 127 The right triangles in this figure are similar.

(b) Find $A(h)$ as a function of h using this information.

(c) Use [Fact 6.5.11](#) to find the mass of the cone.

Activity 6.5.14 Consider a pyramid with a 8×8 ft square base and a height of 16 ft. Suppose the density of the pyramid is $\delta(h) = 10 + \cos(\pi h)$ lb/ft³ where h is the height in feet.

- (a) When the height is h ft, what is the area of the square cross section at that height, $A(h)$?

Use similar triangles:

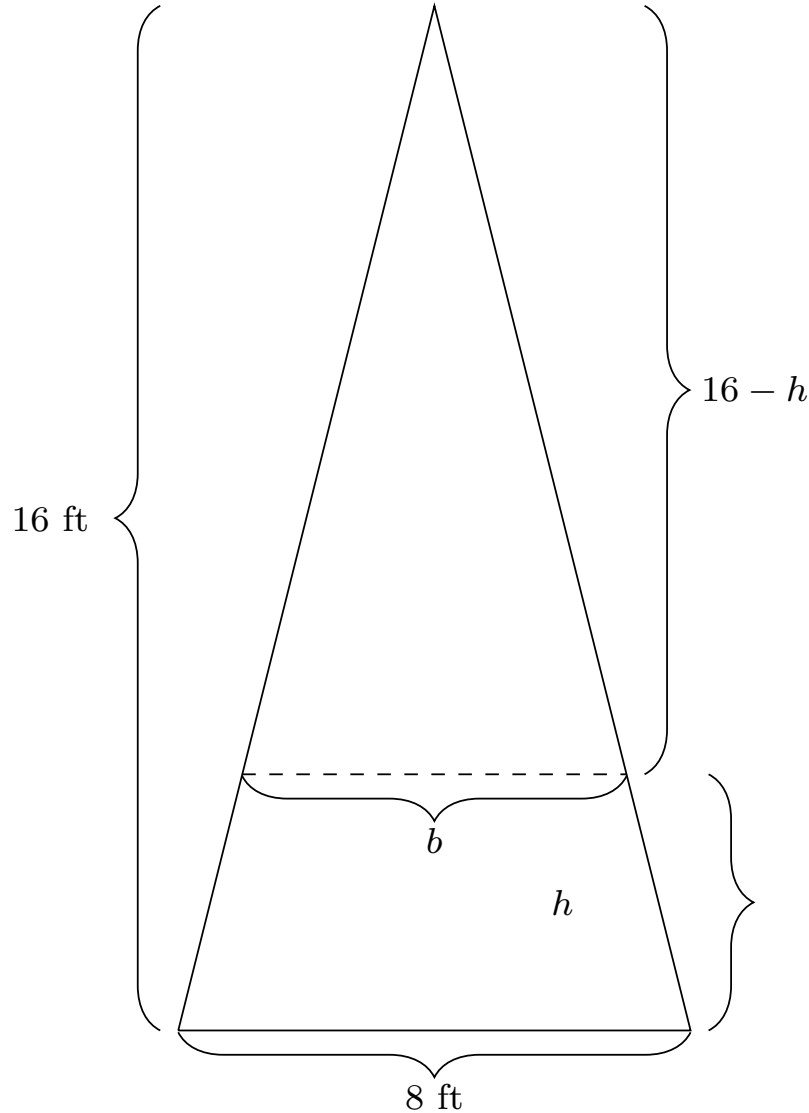


Figure 128 The triangles in this figure are similar.

- (b) Use [Fact 6.5.11](#) to find the mass of the pyramid.

Activity 6.5.15 Consider a board sitting atop the x -axis with six 1×1 blocks each weighing 1 kg are placed upon it. Two blocks atop the 1, three blocks atop the 2, and one block atop the 6.

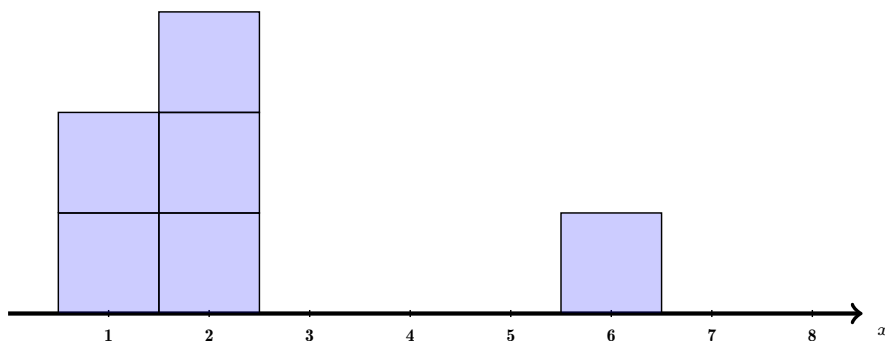


Figure 129 Six 1 kg blocks atop the x -axis.

Which of the following describes the x -value where the center of gravity of these blocks lie?

A $\frac{1+6}{2} = 3.5$.

B $\frac{1+2+6}{3} = 3$.

C $\frac{2 \cdot 1 + 3 \cdot 2 + 1 \cdot 6}{6} \approx 2.3333$.

Activity 6.5.16 Consider a board sitting atop the x -axis with six 1×1 blocks each weighing 1 kg are placed upon it. Two blocks atop the 1, three blocks atop the 2, and one block atop the 8.

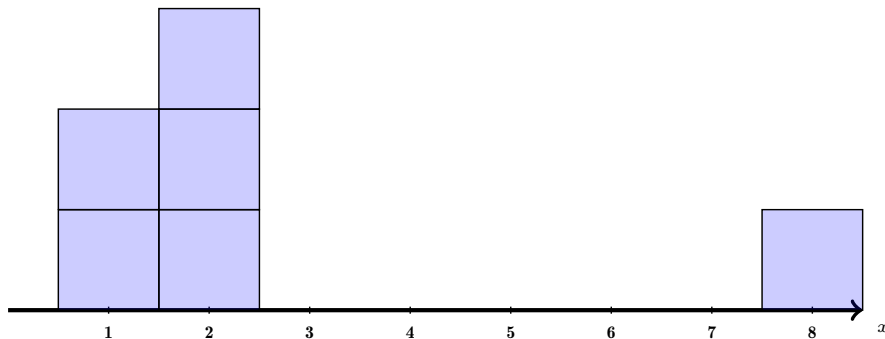


Figure 130 Six 1 kg blocks atop the x -axis.

Find the x -value where the center of gravity of these blocks lie.

Fact 6.5.17 Consider a solid where the cross section of the solid at $x = x_i$ has area $A(x_i)$, and the density when $x = x_i$ is $\delta(x_i)$. Suppose the interval $[a, b]$ represented the x values of this solid. Since each slice has approximated mass $\delta(x_i)A(x_i)\Delta x$, we can approximate the center of mass by taking the weighted “average” of the x_i -values weighted by the associated mass:

$$\frac{\sum_{i=1}^n x_i \delta(x_i) A(x_i) \Delta x}{\sum_{i=1}^n \delta(x_i) A(x_i) \Delta x}.$$

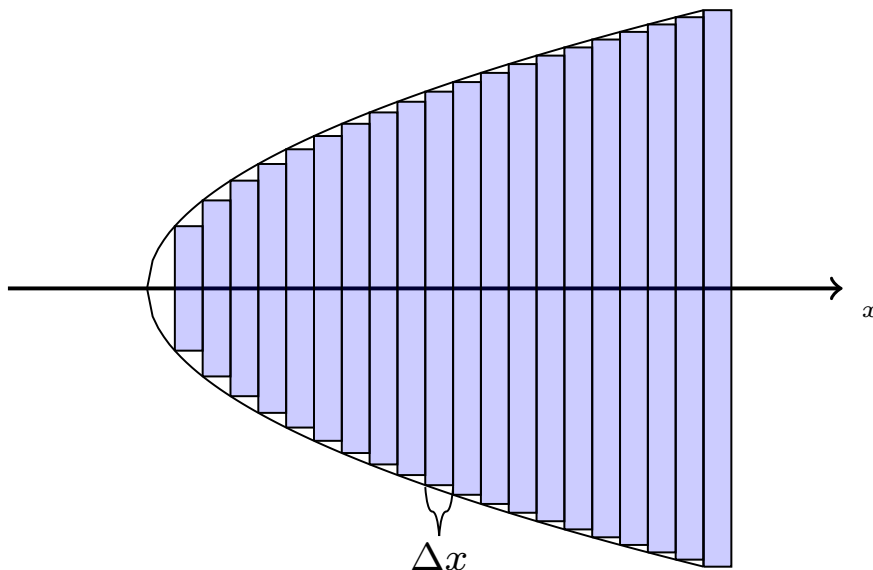


Figure 131 Solid approximated with prisms of width Δx .

We can then find actual center of mass by taking the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \left(\frac{\sum_{i=1}^n x_i \delta(x_i) A(x_i) \Delta x}{\sum_{i=1}^n \delta(x_i) A(x_i) \Delta x} \right) = \frac{\int_a^b x \delta(x) A(x) dx}{\int_a^b \delta(x) A(x) dx} = \frac{\int_a^b x \delta(x) A(x) dx}{\text{The Total Mass}}.$$

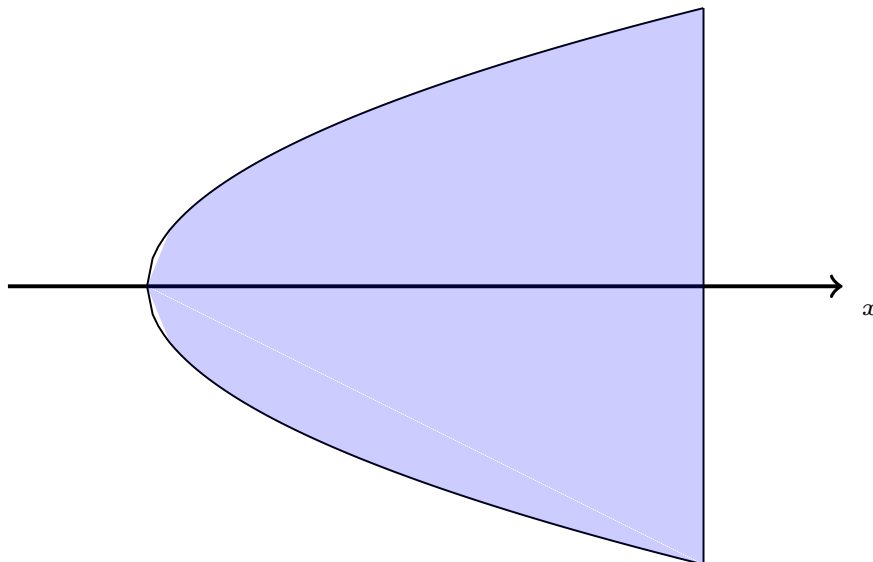


Figure 132 Solid mass.

Activity 6.5.18 Consider that for the prism from [Activity 6.5.12](#), a cross section of height h is $A(h) = 10^2 = 100 \text{ m}^2$. Also recall that the density of the prism is $\delta(h) = 10 + h \text{ kg/m}^3$, where h is the height in meters, and that we found the total mass to be 40000 kg.

Use [Fact 6.5.17](#) to find the height where the center of mass occurs.

Activity 6.5.19 Consider that for the prism from [Activity 6.5.13](#), a cross section of height h is $A(h) = \pi \cdot \left(\frac{60-h}{4}\right)^2 \text{ in}^2$. Also recall that the density of the cone is $\delta(h) = 15 + \sqrt{h} \text{ oz/in}^3$, where h is the height in inches, and that

we found the total mass to be about 142492.6 oz.

Use [Fact 6.5.17](#) to find the height where the center of mass occurs.

Activity 6.5.20 Consider that for the pyramid from [Activity 6.5.14](#), a cross section of height h is $A(h) = \pi \cdot \left(\frac{16-h}{2}\right)^2 \text{ ft}^2$. Also recall that the density of the pyramid is $\delta(h) = 10 + \cos \pi h \text{ lb/ft}^3$, where h is the height in feet, and that we found the total mass to be about 3414.146 lb's.

Use [Fact 6.5.17](#) to find the height where the center of mass occurs.

6.5.1 Videos



YouTube: <https://www.youtube.com/watch?v=6r1X-ISoePU>

Figure 133 Video: Set up integrals to solve problems involving density, mass, and center of mass

6.6 Physics Applications (AI6)

Learning Outcomes

- Set up integrals to solve problems involving work, force, and/or pressure.

Fact 6.6.1 Given a physical object m , the work done on that object is

$$W = Fd = mad,$$

where F is the force applied to the object over a distance of d . Recall that force $F = ma$, where m is the mass of the object, and a is the acceleration applied to it.

Activity 6.6.2 Consider a bucket with 10 kg of water, being pulled against the acceleration of gravity, $g = 9.8 \text{ m/s}^2$, at a constant speed for 20 meters. Using [Fact 6.6.1](#), what is the work needed to pull this bucket up 20 meters in kgm^2/s^2 (or Nm)?

- A $10 \text{ kgm}^2/\text{s}^2$.
- B $20 \text{ kgm}^2/\text{s}^2$.
- C $98 \text{ kgm}^2/\text{s}^2$.
- D $200 \text{ kgm}^2/\text{s}^2$.
- E $1960 \text{ kgm}^2/\text{s}^2$.

Activity 6.6.3 Consider the bucket from [Activity 6.6.2](#) with 10 kg of water, being pulled against the acceleration of gravity, $g = 9.8 \text{ m/s}^2$, at a constant speed for 20 meters. Suppose that halfway up at a height of 10m, 5kg of water spilled out, leaving 5kg left. How much work total does it take to get this

bucket to a height of 20m?

A $980 \text{ kgm}^2/s^2$ or (Nm).

B $1470 \text{ kgm}^2/s^2$ or (Nm).

C $1960 \text{ kgm}^2/s^2$ or (Nm).

Activity 6.6.4 Suppose a 10 kg bucket of water is constantly losing water as it's pulled up, so at a height of h meters, the mass of the bucket is $m(h) = 2 + 8e^{-0.2h}$ kg.

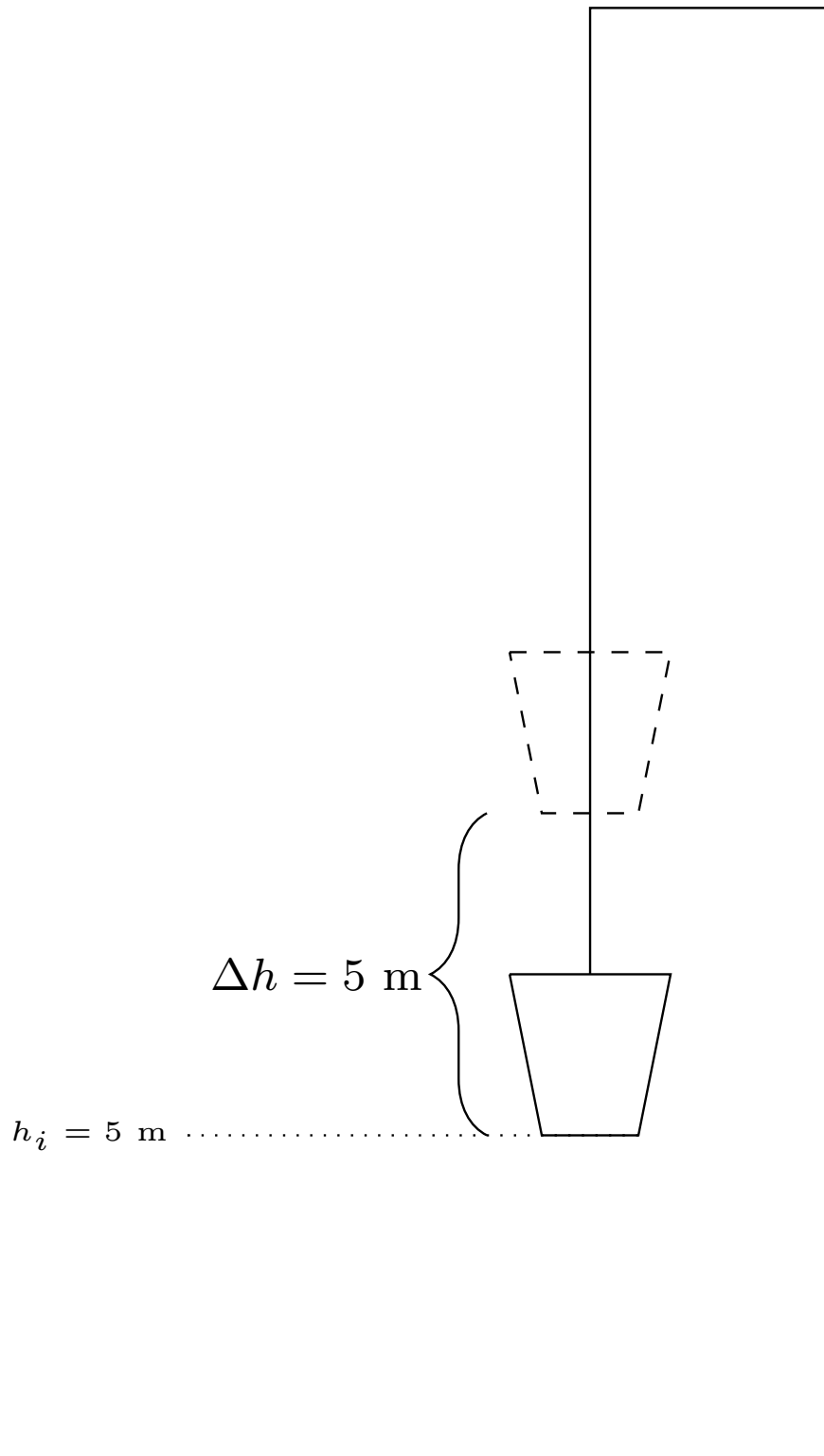


Figure 134 Bucket 5 m in the air, to be hoisted by another 5 meters.

- (a) What is the mass of the bucket at height $h_i = 5 \text{ m}$?
- (b) Assume that the bucket does not lose water to estimate the amount of work needed to lift this bucket up $\Delta h = 5 \text{ meters}$.

Activity 6.6.5 Consider the bucket from [Activity 6.6.4](#). Consider the buckets mass at heights $h_i = 0, 5, 10, 15$ meters.

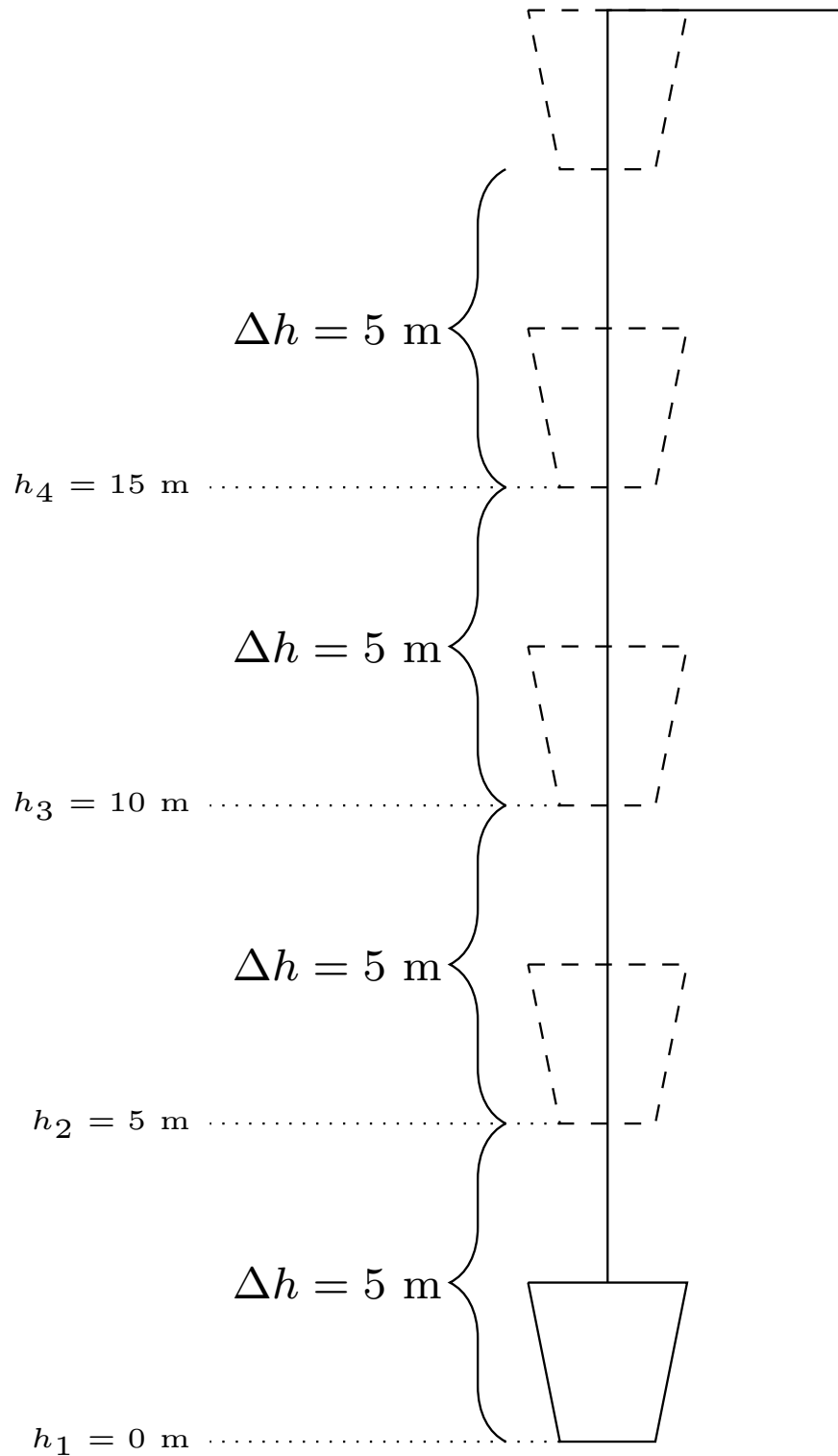


Figure 135 Bucket lifted 5 m at a time.

(a) Fill out the following table, estimating the work it would take to lift the

bucket 20 meters:

h_i	Mass $m(h_i)$	Distance	Estimated Work
$h_4 = 15$ m	$m(15) = 2 + 8e^{-0.2 \cdot 15} \approx 2.398$ kg	5 m	
$h_3 = 10$ m	$m(10) = 2 + 8e^{-0.2 \cdot 10} \approx 3.083$ kg	5 m	
$h_2 = 5$ m	$m(5) = 2 + 8e^{-0.2 \cdot 5} \approx 4.943$ kg	5 m	242.207 Nm
$h_1 = 0$ m	$m(5) = 2 + 8e^{-0.2 \cdot 0} = 10$ kg	5 m	

(b) What is the total estimated work to lift this bucket 20 meters?

Activity 6.6.6 If we estimated the mass and work of the bucket from [Activity 6.6.5](#) at height h_i with intervals of length Δh meters, which of the following best represents the Riemann sum of the work it would take to lift this bucket 20 meters?

- A $\sum h_i \cdot 9.8\Delta h$. Nm
- B $\sum (2 + 8e^{-0.02h}) \cdot 9.8\Delta m$. Nm
- C $\sum (2 + 8e^{-0.02h_i}) \cdot 9.8\Delta h$. Nm
- D $\sum (2 + 8e^{-0.02h_i}) \cdot 9.8\Delta m$. Nm

Activity 6.6.7 Based on the Riemann sum chosen in [Activity 6.6.6](#), which of the following integrals computes the work it would take to lift this bucket 20 meters?

- A $\int_0^{20} h_i \cdot 9.8dh$. Nm
- B $\int_0^{20} (2 + 8e^{-0.02h}) \cdot 9.8dm$. Nm
- C $\int_0^{20} (2 + 8e^{-0.02h}) \cdot 9.8dh$. Nm
- D $\int_0^{20} (2 + 8e^{-0.02h_i}) \cdot 9.8dh$. Nm

Activity 6.6.8 Based on the integral chosen in [Activity 6.6.7](#), compute the work it would take to lift this bucket 20 meters.

Observation 6.6.9 A “how to” for applying integrals to physics.

- A Estimate the value over a piece of the problem with x value x_i over interval of length Δx .
- B Find a Riemann sum using (1) which estimates the value in question.
- C Convert the Riemann sum to an integral and solve.

Activity 6.6.10 Consider a cylindrical tank filled with water, where the base of the cylinder has a 3 meter radius, and a height of 10 meters. Consider a 2 m thick slice of water sitting 6 meters high in the tank. Using the fact that the mass of this water is $1000 \cdot \pi(3)^2 \cdot 2 = 18000\pi$ kg, estimate how much work is needed to lift this slice 4 more meters to the top of the tank.

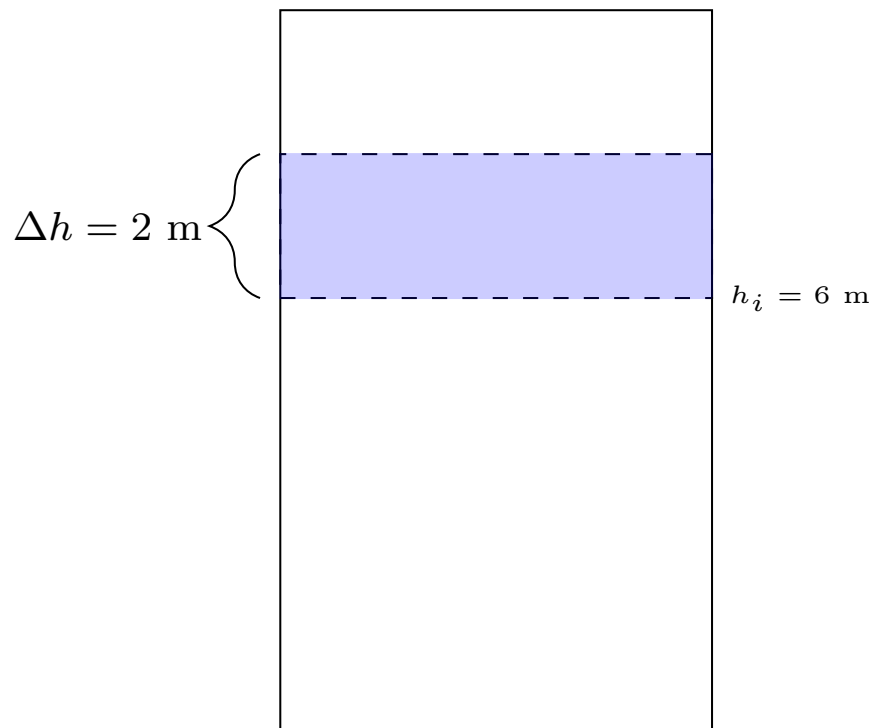


Figure 136 2m thick slice of water lifted 4m.

- A $18000\pi \cdot 4 \text{ Nm}$.
- B $18000\pi \cdot 9.8 \text{ Nm}$.
- C $18000\pi \cdot 4 \cdot 9.8 \text{ Nm}$.
- D $18000\pi \cdot 6 \text{ Nm}$.
- E $18000\pi \cdot 6 \cdot 9.8 \text{ Nm}$.

Activity 6.6.11 Consider the cylindrical tank filled with water, from [Activity 6.6.10](#). Suppose we wished to estimate the amount of work required to pump all the water out of the tank. Suppose we sliced the water into 5 pieces and estimated the work it would take to lift each piece out



Figure 137 2m thick slices of water.

- (a) Fill out the following table, estimating the work it would take to pump all the water out:

h_i	Mass	Distance	Estimated Work
$h_5 = 8 \text{ m}$	$18000\pi \text{ kg}$		
$h_4 = 6 \text{ m}$	$18000\pi \text{ kg}$	4 m	$705600\pi \text{ Nm}$
$h_3 = 4 \text{ m}$	$18000\pi \text{ kg}$		
$h_2 = 2 \text{ m}$	$18000\pi \text{ kg}$		
$h_1 = 0 \text{ m}$	$18000\pi \text{ kg}$	10 m	

- (b) What is the total estimated work to pump out all the water?

Activity 6.6.12 Recall [Activity 6.6.11](#). If we estimated the work needed to lift slices of thickness Δh m at heights h_i m, which of the following Riemann sums best estimates the total work needed to pump all the water from the tank?

- A $\sum 1000 \cdot \pi 3^2 \cdot 9.8(10 - h)\Delta h. \text{ Nm}$
- B $\sum 1000 \cdot \pi 3^2 \cdot 9.8(10 - h_i)\Delta h. \text{ Nm}$
- C $\sum 1000 \cdot \pi (h_i)^2 \cdot 9.8(10 - h)\Delta h. \text{ Nm}$
- D $\sum 1000 \cdot \pi (h_i)^2 \cdot 9.8(10 - h_i)\Delta h. \text{ Nm}$

Activity 6.6.13 Based on the Riemann sum chosen in [Activity 6.6.12](#), which of the following integrals computes the work it would take to pump all the water from the tank?

A $\int_0^{10} 9000\pi \cdot 9.8(10 - h)dh$. Nm

B $\int_0^{10} 1000\pi \cdot 9.8h^2(10 - h)dh$. Nm

Activity 6.6.14 Based on the integral chosen in [Activity 6.6.13](#), compute the work it would take to pump all the water out of the tank.

Activity 6.6.15 Consider a cylindrical truncated-cone tank where the radius on the bottom of the tank is 10 m, the radius at the top of the tank is 100 m, and the height of the tank is 100m.

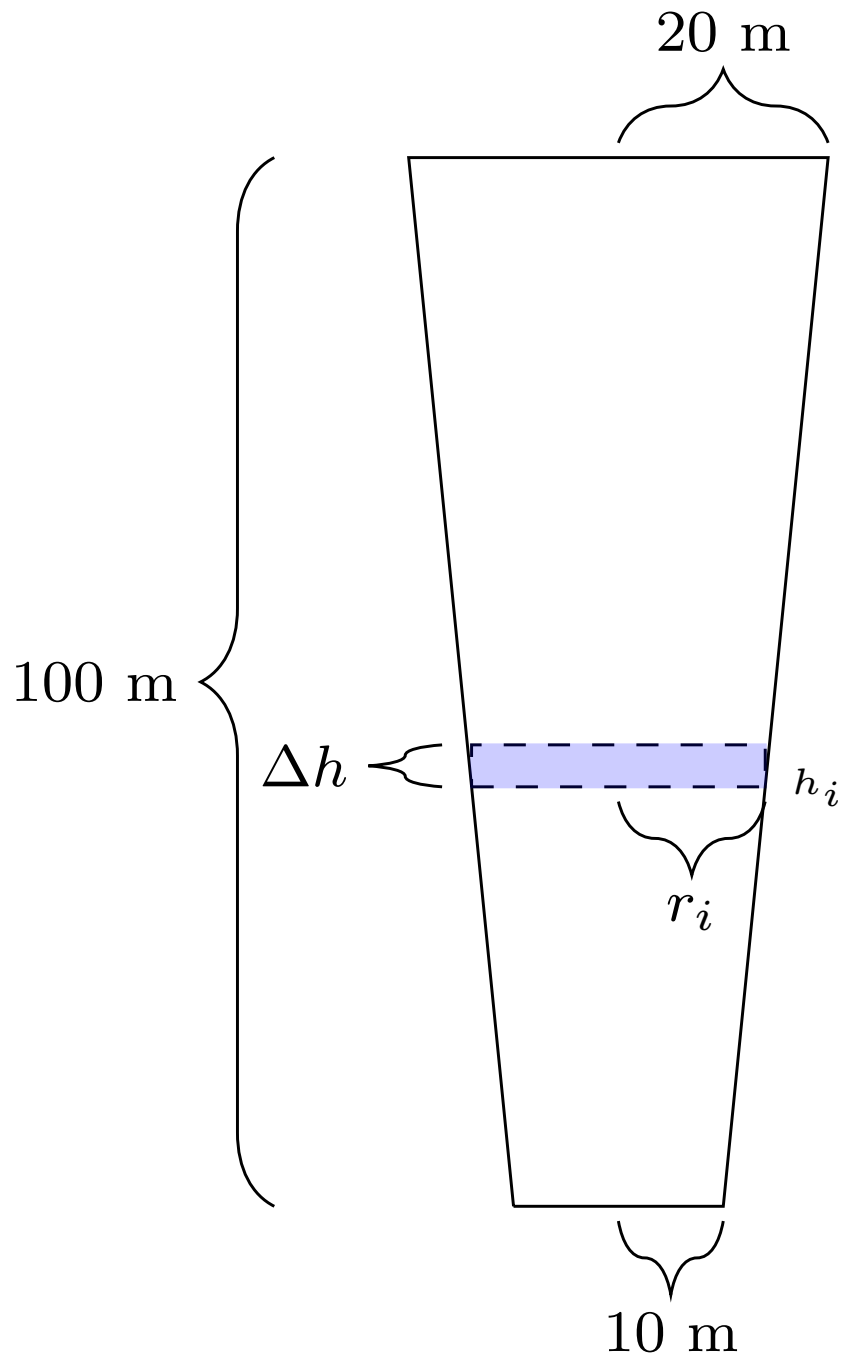


Figure 138 A slice at height h_i of width Δh .

- (a) What is the radius r_i in meters of the cross section made at height h_i meters?
- (b) What is the volume of a cylinder with radius r_i meters with width Δh meters?
- (c) Using the fact that water has density 1000 kg/m^3 , how much mass is the volume of water found in (b)?

- (d) How far must this cylinder of water be lifted to be out of this tank?

Activity 6.6.16 Recall the computations done in [Activity 6.6.15](#).

- (a) Find a Riemann sum which estimates the total work needed to pump all the water out of this tank, using slices at heights h_i m, of width Δh m.
- (b) Use (a) to find an integral expression which computes the amount of work needed to pump all the water out of this tank.
- (c) Evaluate the integral found in (b).

Fact 6.6.17 Recall that pressure is measured as force over area.

$$P = F/A.$$

Rewriting this, we have that $F = PA$.

Activity 6.6.18 Consider a trapezoid-shaped dam that is 60 feet wide at its base and 90 feet wide at its top, and assume the dam is 20 feet tall with water that rises to its top. Water weighs 62.4 pounds per cubic foot, and exerts $P = 62.4d$ lbs/ft² of pressure at depth d ft. Consider a rectangular slice of this dam at height h_i feet and width b_i .

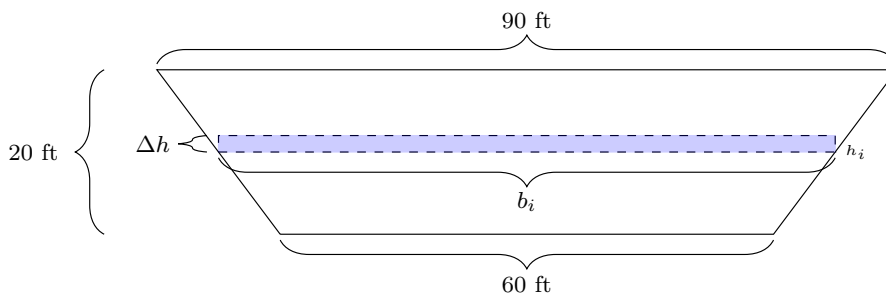


Figure 139 A slice at height h_i of width Δh .

- (a) At a height of h_i feet, what is the base of the rectangle b_i ?
- (b) What is the area of a rectangle with base b_i feet and height Δh feet?
- (c) Using a depth of $20 - h_i$ feet, how much pressure is exerted on this rectangle?
- (d) Using the pressure found in (c), the area in (b) and [Fact 6.6.17](#), how much force is exerted on this rectangle?

Activity 6.6.19 Recall the computations done in [Activity 6.6.18](#).

- (a) Find a Riemann sum which estimates the total force exerted on the dam, using slices at heights h_i m, of width Δh m.
- (b) Use (a) to find an integral expression which computes the amount of force exerted on this dam.
- (c) Evaluate the integral found in (b).

6.6.1 Videos



YouTube: <https://www.youtube.com/watch?v=mky7ZYL642Y>

Figure 140 Video: Set up integrals to solve problems involving work, force, and/or pressure

Chapter 7

Coordinates and Vectors (CO)

Learning Outcomes

How do we use alternative coordinates and vectors to describe points in the plane?

By the end of this chapter, you should be able to...

1. Sketch the graph of a two-dimensional parametric/vector equation, and convert such equations into equations of only x and y .
2. Compute derivatives and tangents related to two-dimensional parametric/vector equations.
3. Compute arclengths related to two-dimensional parametric/vector equations.
4. Convert points and equations between polar and Cartesian coordinates and equations.
5. Compute arclengths of curves given in polar coordinates.
6. Compute areas bounded by curves given in polar coordinates.

7.1 Parametric/vector equations (CO1)

Learning Outcomes

- Sketch the graph of a two-dimensional parametric/vector equation, and convert such equations into equations of only x and y .

Activity 7.1.1 Consider how we might graph the equation $y = 2 - x^2$ in the xy -plane.

- (a) Complete the following chart of xy values by plugging each x value into the equation to produce its y value.

Table 141 Chart of x and y values to graph

x	y
-2	
-1	1
0	
1	
2	

(b) Plot each point (x, y) in your chart in the xy plane.

(c) Connect the dots to obtain a reasonable sketch of the equation's graph.

Activity 7.1.2 Suppose that we are told that at after t seconds, an object is located at the x -coordinate given by $x = t - 2$ and the y -coordinate given by $y = -t^2 + 4t - 2$.

(a) Complete the following chart of txy values by plugging each t value into the equations to produce its x and y values.

Table 142 Chart of x and y values for each t

t	x	y
0		
1	-1	1
2		
3		
4		

(b) Plot each point (x, y) in your chart in the xy plane, labeling it with its t value.

(c) Connect the dots to obtain a reasonable sketch of the equation's graph.

Definition 7.1.3 Graphs in the xy plane can be described by **parametric equations** $x = f(t)$ and $y = g(t)$, where plugging in different values of t into the functions f and g produces different points of the graph.

The t -values may be thought of representing the moment of *time* when an object is located at a particular position, and the graph may be thought of as the path the object travels throughout time. \diamond

Activity 7.1.4 Earlier we obtained the same graphs for the xy equation $y = 2 - x^2$ and the parametric equations $x = t - 2$ and $y = -t^2 + 4t - 2$. Do the following steps to find out why.

(a) Which of the following equations describes t in terms of x ?

- A $t = x - 2$
- B $t = x + 2$
- C $t = 2x$
- D $t = -2x$

(b) Which of these is the result of plugging this choice in for t in the parametric equation for y ?

- A $y = -x + 2^2 + 4x + 2 - 2$
- B $y = -(x + 2)^2 + 4(x + 2) - 2$
- C $y = -x^2 + 2^2 + 4x + 4 \cdot 2 - 2$

- (c) Show how to simplify this choice to obtain the equation $y = 2 - x^2$.

Fact 7.1.5 One method of graphing parametric equations $x = f(t)$ and $y = g(t)$ is to combine them into a single equation only involving x and y , and using your usual graphing techniques.

Activity 7.1.6 Parametric equations have the advantage of describing paths that cannot be described by a function $y = h(x)$. One such example is the graph of $x = 3 \sin(\pi t)$ and $y = -3 \cos(\pi t)$. (Use technology or the approximation $\sqrt{2} \approx 0.707$ to approximate coordinates as needed.)

- (a) Complete the following table.

Table 143 Chart of approximate x and y values

t	x	y
0		
1/4		
1/2		
3/4	2.12	2.12
1		
5/4		
3/2		
7/4		
2		

- (b) Plot these (x, y) points in the xy plane and connect the dots to draw a sketch of the graph.
- (c) What do you obtain by plugging the parametric equations into the expression $x^2 + y^2$?
- A $x^2 + y^2 = 6 \sin(\pi x) \cos(\pi x)$
 B $x^2 + y^2 = -6 \sin(\pi x) \cos(\pi x)$
 C $x^2 + y^2 = 9$
 D $x^2 + y^2 = 0$
- (d) Which of these describes the xy equation and graph given by these parametric equations?
- A a parabola
 B a line
 C a circle
 D a square
- (e) The graph of these parametric equations cannot be described by a function. Why?
- A The graph fails the vertical line test.
 B The graph fails the horizontal line test.
 C The graph doesn't extend vertically to $+\infty$.
 D The graph doesn't extend horizontally to $-\infty$.

Definition 7.1.7 The parametric equations $x = f(t)$ and $y = g(t)$ are sometimes written in the form of the **vector equation** $\vec{r} = \langle f(t), g(t) \rangle$.

For example, the parametric equations $x = 3\sin(\pi t)$ and $y = -3\cos(\pi t)$ may be combined into the single vector equation $\vec{r} = \langle 3\sin(\pi t), -3\cos(\pi t) \rangle$. \diamond

Activity 7.1.8 Consider the vector equation $\vec{r} = \langle 2t - 3, -6t + 13 \rangle$.

(a) What are the corresponding parametric equations?

A $x = 2t - 3$ and $y = -6t + 13$

B $y = 2t - 3$ and $x = -6t + 13$

C $xy = 2t - 3 - 6t + 13$

D Vector equations cannot be converted into parametric equations.

(b) Draw a table of txy values with $t = 0, 1, 2, 3, 4$.

(c) Plot these (x, y) points in the plane and connect the dots to sketch the graph of this vector equation.

(d) Solve for t in terms of x and plug into the y parametric equation to show that this is the vector equation for the line $y = -3x + 4$.

7.2 Parametric/vector derivatives (CO2)

Learning Outcomes

- Compute derivatives and tangents related to two-dimensional parametric/vector equations.

Activity 7.2.1 Consider the parametric equations $x = 2t - 1$ and $y = (2t - 1)(2t - 5)$. The coordinate on this graph at $t = 2$ is $(3, -3)$.

(a) Which of the following equations of x, y describes the graph of these parametric equations?

A $y = 2x(x + 2) = 2x^2 + 2x$

B $y = 2x(x - 2) = 2x^2 - 2x$

C $y = x(x + 4) = x^2 + 4x$

D $y = x(x - 4) = x^2 - 4x$

(b) Which of the following describes the slope tangent to the graph at the point $(3, -3)$?

A $\frac{dy}{dx} = 2x + 4$, which is 10 when $x = 3$.

B $\frac{dy}{dx} = 2x + 4$, which is 8 when $t = 2$.

C $\frac{dy}{dx} = 2x - 4$, which is 2 when $x = 3$.

D $\frac{dy}{dx} = 2x - 4$, which is 0 when $t = 2$.

(c) Note that the parametric equation for y simplifies to $y = 4t^2 - 12t + 5$. What do we get for the derivatives $\frac{dx}{dt}$ of $x = 2t - 1$ and $\frac{dy}{dt}$ for $y = 4t^2 - 12t + 5$?

A $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 8t - 12$.

B $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 8t - 12$.

C $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 6t + 5$.

D $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 6t + 5$.

- (d) It follows that when $t = 2$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 4$. Which of the following conjectures seems most likely?

A The slope $\frac{dy}{dx} = 2$ could also be found by computing $\frac{dx}{dt} + \frac{dy}{dt}$.

B The slope $\frac{dy}{dx} = 2$ could also be found by computing $\frac{dy/dt}{dx/dt}$.

C The slope $\frac{dy}{dx} = 2$ is always equal to $\frac{dx}{dt}$.

D The slope $\frac{dy}{dx} = 2$ is always equal to $\frac{dy}{dt}$.

Fact 7.2.2 Suppose x is a function of t , and y may be thought of as a function of either x or t . Then the Chain Rule requires that

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

This provides the slope formula for parametric equations:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Activity 7.2.3 Let's draw the picture of the line tangent to the parametric equations $x = 2t - 1$ and $y = (2t - 1)(2t - 5)$ when $t = 2$.

- (a) Use a t, x, y chart to sketch the parabola given by these parametric equations for $-3 \leq t \leq 3$, including the point $(3, -3)$ when $t = 2$.
- (b) Earlier we determined that the tangent slope was 2. Draw a line with slope 2 passing through $(3, -3)$ and confirm that it appears to be tangent.
- (c) Use the point-slope formula $y - y_0 = m(x - x_0)$ along with the slope 2 and point $(3, -3)$ to find the exact equation for this tangent line.

A $y = 2x - 10$

B $y = 2x - 9$

C $y = 2x - 8$

D $y = 2x - 7$

Activity 7.2.4 Consider the vector equation $\vec{r}(t) = \langle 3t^2 - 9, t^3 - 3t \rangle$.

- (a) What are the corresponding parametric equations and their derivatives?

A $y = 3t^2 - 9$ and $x = t^3 - 3t$; $\frac{dy}{dt} = 9t$ and $\frac{dx}{dt} = 3t - 6$

B $x = 3t^2 - 9$ and $y = t^3 - 3t$; $\frac{dx}{dt} = 9t$ and $\frac{dy}{dt} = 3t - 6$

C $y = 3t^2 - 9$ and $x = t^3 - 3t$; $\frac{dy}{dt} = 6t$ and $\frac{dx}{dt} = 3t^2 - 3$

D $x = 3t^2 - 9$ and $y = t^3 - 3t$; $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 3t^2 - 3$

- (b) The formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ allows us to compute slopes as which of the following functions of t ?

A $\frac{6t}{t^2 + 3}$

B $\frac{6t}{t^2 + 1}$

C $\frac{t^2 - 1}{2t}$

D $\frac{2t}{3t^2 - 1}$

(c) Find the point, tangent slope, and tangent line equation (recall $y - y_0 = m(x - x_0)$) corresponding to the parameter $t = -2$.

A Point $(-3, -4)$, slope $\frac{3}{4}$, EQ $y = \frac{3}{4}x + 7$

B Point $(3, -14)$, slope $-\frac{4}{3}$, EQ $y = -\frac{4}{3}x - 10$

C Point $(1, -11)$, slope $\frac{2}{3}$, EQ $y = \frac{2}{3}x - 11$

D Point $(-1, -9)$, slope $-\frac{1}{6}$, EQ $y = -\frac{1}{6}x + 2$

7.3 Parametric/vector arclength (CO3)

Learning Outcomes

- Compute arclengths related to two-dimensional parametric/vector equations.

Example 7.3.1 In Figure 144, the blue curve is the graph of the parametric equations $x = t^2$ and $y = t^3$ for $1 \leq t \leq 2$. This curve connects the point $(1, 1)$ to the point $(4, 8)$. The red dashed line is the straight line segment connecting these points.

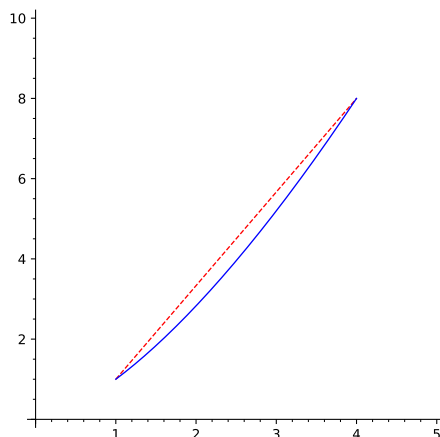


Figure 144 A parametric curve and segment from $(1, 1)$ to $(4, 8)$

□

Activity 7.3.2 Let's first investigate the length of the dashed red line segment in Figure 144.

- (a) Draw a right triangle with the red dashed line segment as its hypotenuse, one leg parallel to the x -axis, and the other parallel to the y -axis.

How long are these legs?

A 3 and 7.

B 4 and 8.

C 3 and 8.

D 4 and 7.

- (b) The Pythagorean theorem states that for a right triangle with leg lengths a, b and hypotenuse length c , we have...

- A $a = b = c$.
- B $a + b = c$.
- C $a^2 = b^2 = c^2$.
- D $a^2 + b^2 = c^2$.

(c) Using the leg lengths and Pythagorean theorem, how long must the red dashed hypotenuse be?

- A $\sqrt{20} \approx 4.47$.
- B $\sqrt{56} \approx 7.48$.
- C $\sqrt{67} \approx 8.19$.
- D $\sqrt{100} = 10$.

(d) Compared with the blue parametric curve connecting the same two points, is the red dashed line segment length an overestimate or underestimate?

- A Overestimate: the blue curve is shorter than the red line.
- B Underestimate: the blue curve is longer than the red line.
- C Exact: the blue curve is exactly as long as the red line.

Fact 7.3.3 Recall that the linear distance between two points (x_1, y_1) and (x_2, y_2) may be computed by the distance formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note that $\Delta x = |x_2 - x_1|$ and $\Delta y = |y_2 - y_1|$ measure leg lengths of a right triangle whose hypotenuse is the distance we want to measure, so we may rewrite this formula as

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This formula will need to be modified to measure a curved path between two points.

Observation 7.3.4 By approximating the curve by several, say N -many, segments connecting points along the curve, we obtain a better approximation than a single line segment, as shown in Figure 145.

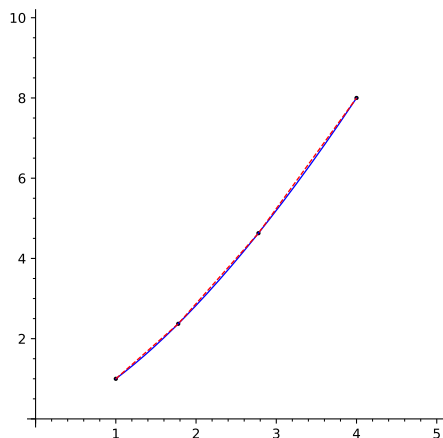


Figure 145 Subdividing a parametric curve where $N = 3$

Activity 7.3.5 How should we modify the distance formula $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ to measure arclength as illustrated in Figure 145?

- (a) Let $\Delta L_1, \Delta L_2, \Delta L_3$ describe the lengths of each of the three segments. Which expression describes the total length of these segments?

A $\Delta L_1 \times \Delta L_2 \times \Delta L_3$

B $\Delta L_1 + 2\Delta L_2 + 3\Delta L_3$

C $\sum_{i=1}^3 \Delta L_i$

- (b) We can let each $\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. But we will find it useful to involve the parameter t as well, or more accurately, the change Δt_i of t between each point of the subdivision.

Which of these is algebraically the same as the above formula for ΔL_i ?

A $\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}$

B $\sqrt{\left[\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2\right]} \Delta t_i$

C $\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} \Delta t_i$

- (c) Finally, we'll want to increase N from 3 so that it limits to ∞ . What can we conclude when that happens?

A Each segment is infinitely small

B $\Delta x_i \rightarrow 0$

C $\Delta t_i \rightarrow 0$

D $\frac{\Delta x_i}{\Delta t_i} \rightarrow \frac{dx}{dt}$

E All of the above.

Observation 7.3.6 Put together, and limiting the subdivisions of the curve $N \rightarrow \infty$, we obtain the Riemann sum

$$\sum_{i=1}^{\infty} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} \Delta t_i.$$

Thus arclength along a parametric curve from $a \leq t \leq b$ may be calculated by using the corresponding definite integral

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Activity 7.3.7 Let's gain confidence in the arclength formula

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

by checking to make sure it matches the distance formula for line segments.

The parametric equations $x = 3t - 1$ and $y = 2 - 4t$ for $1 \leq t \leq 3$ represent the segment of the line $y = -\frac{4}{3}x - \frac{2}{3}$ connecting $(2, -2)$ to $(8, -10)$.

- Find dx/dt and dy/dt , and substitute them into the formula above along with $a = 1$ and $b = 3$.
- Show that the value of this formula is 10.
- Show that the length of the line segment connecting $(2, -2)$ to $(8, -10)$ is 10 by applying the distance formula directly instead.

Activity 7.3.8 For each of these parametric equations, use

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

to write a definite integral that computes the given length. (Do not evaluate the integral.)

- The portion of $x = \sin 3t, y = \cos 3t$ where $0 \leq t \leq \pi/6$.
- The portion of $x = e^t, y = \ln t$ where $1 \leq t \leq e$.
- The portion of $x = t + 1, y = t^2$ between the points $(3, 4)$ and $(5, 16)$.

Activity 7.3.9 Let's see how to modify $\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ to produce the arclength of the graph of a function $y = f(x)$.

- Let $x = t$. How can $\frac{dx}{dt}$ be simplified?

- dx
- dt
- 1
- 0

- Given $x = t$, how should $\frac{dy}{dt}$ and dt be rewritten?

- $\frac{dy}{dt} = \frac{dy}{dx}$ and $dt = dx$
- $\frac{dy}{dt} = \frac{dx}{dt}$ and $dt = dx$
- $\frac{dy}{dt} = \frac{dy}{dx}$ and $dt = 1$
- $\frac{dy}{dt} = \frac{dy}{dt}$ and $dt = 1$

- Write a modified, simplified formula for $\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ with t replaced with x .

7.4 Polar coordinates (CO4)

Learning Outcomes

- Convert points and equations between polar and Cartesian coordinates and equations.

Fact 7.4.1 ‘‘As the crow flies’’ is an idiom used to describe the most direct path between two points. The polar coordinate system is a useful parametrization of

the plane that, rather than describing horizontal and vertical position relative to the origin in the usual way, describes a point in terms of distance from the origin and direction. The origin is also known as the pole (hence polar coordinates). Let \overline{OP} be a line segment from the origin to a given point P in the plane. The length of \overline{OP} is the distance (or radius) r from the origin to P . The polar axis is a ray starting at the origin. To define the "direction" of P , we form an angle θ by letting the polar axis serve as the initial ray and \overline{OP} as the terminal ray. We will set the positive x -axis as the polar axis and assume the movement in the positive direction is counter-clockwise (as in trigonometry). Notice that, unlike in the rectangular (or Cartesian) coordinate system, the polar coordinates (r, θ) for a point are not unique, as we could turn either way to face a given point (or even spin around a number of times before facing that direction). Furthermore, by allowing r to be negative, we can also "walk backwards" to get to a point by facing in the opposite direction. Rather than the grid lines defined by specific values for x and y in the rectangular coordinate system, specific values of r correspond to circles of radius r centered about the origin, and specific values of θ correspond to lines going through the pole (called radial lines).

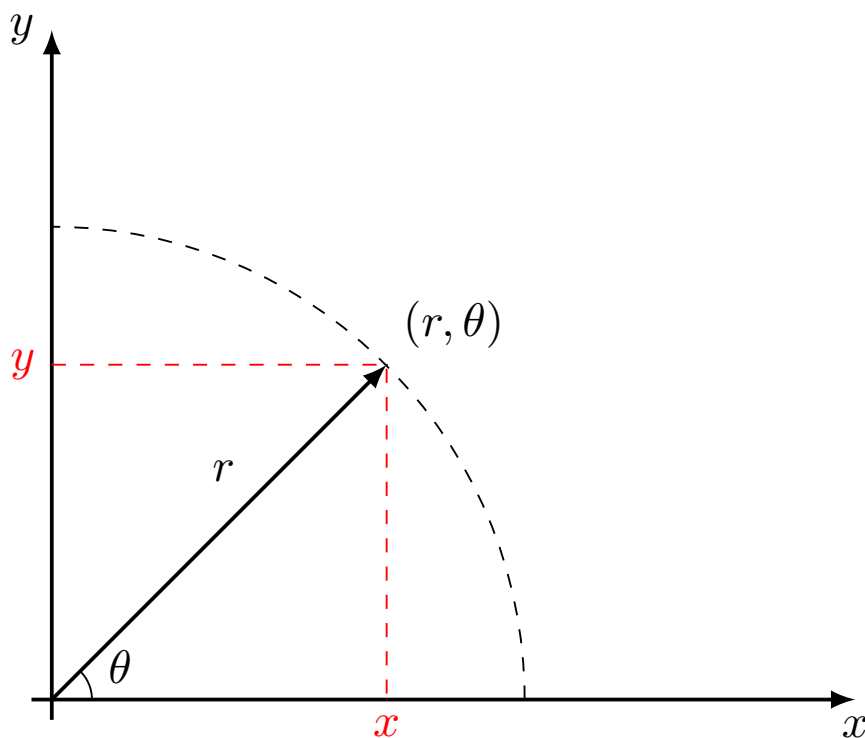


Figure 146 A point in the polar coordinate system.

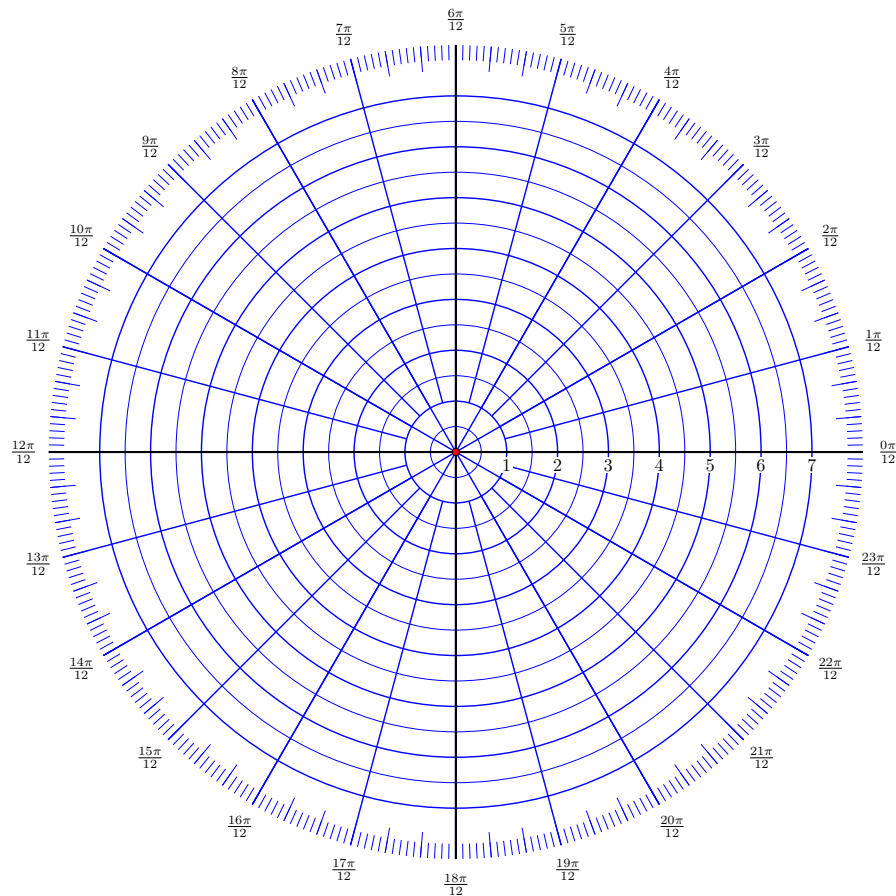


Figure 147 The polar grid.

Activity 7.4.2

- Plot the Cartesian point $P = (x, y) = (\sqrt{3}, -1)$ and draw line segments connecting the origin to P , the origin to $(x, y) = (\sqrt{3}, 0)$, and P to $(x, y) = (\sqrt{3}, 0)$.
- Solve the triangle formed by the line segments you just drew (i.e. find the lengths of all sides and the measures of each angle).
- Find all polar coordinates for the Cartesian point $(x, y) = (\sqrt{3}, -1)$.
- Find Cartesian coordinates for the polar point $(r, \theta) = \left(-\sqrt{2}, \frac{3\pi}{4}\right)$.

Activity 7.4.3 Graph each of the following.

- $r = 1$
- $r = -1$
- $\theta = \frac{\pi}{6}$
- $\theta = \frac{7\pi}{6}$
- $\theta = \frac{-5\pi}{6}$

(f) $1 \leq r < -1, 0 \leq \theta \leq \frac{\pi}{2}$

(g) $-3 \leq r \leq 2, \theta = \frac{\pi}{4}$

(h) $r \leq 0, \theta = \frac{-\pi}{2}$

(i) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$

(j) $r = 3 \sec(\theta)$

Fact 7.4.4 If a polar graph is symmetric about the x -axis, then if the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ also lies on the graph.

Fact 7.4.5 If a polar graph is symmetric about the y -axis, then if the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ also lies on the graph.

Fact 7.4.6 If a polar graph is rotationally symmetric about the origin, then if the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \pi + \theta)$ also lies on the graph.

Activity 7.4.7

(a) Find a polar form of the the Cartesian equation $x^2 + (y - 3) = 9$.

Activity 7.4.8 Find a Cartesian form of each of the given polar equations.

(a) $r^2 = 4r \cos(\theta)$

(b) $r = \frac{4}{2 \cos(\theta) - \sin(\theta)}$

7.5 Polar Arclength (CO5)

Learning Outcomes

- Compute arclengths of curves given in polar coordinates.

Activity 7.5.1 Recall that the length of a parametric curve is given by

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

(a) Let $x(t) = r \cos(\theta)$ and $y(t) = r \sin(\theta)$ and show that the length of a polar curve $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by

$$\int_{\theta=\alpha}^{\theta=\beta} \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

(b) Find an integral computing the arclength of the polar curve defined by $r = 3 \cos(\theta) - 2$ on $\pi/3 \leq \theta \leq \pi$.

(c) Find the length of the cardioid $r = 1 - \cos(\theta)$.

7.6 Polar area (CO6)

Learning Outcomes

- Compute areas bounded by curves given in polar coordinates.

Fact 7.6.1 The area of the “fan-shaped” region between the pole and $r = f(\theta)$ as the angle θ ranges from α to β is given by

$$\int_{\theta=\alpha}^{\theta=\beta} \frac{r^2}{2} d\theta.$$

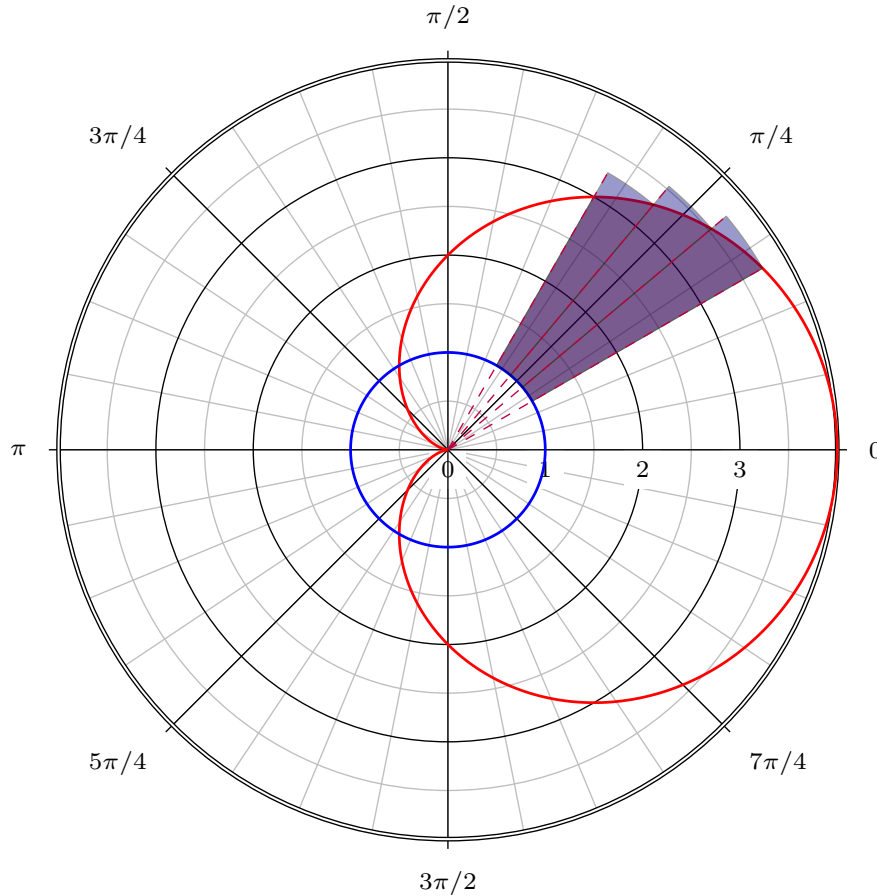


Figure 148 Finding the polar area differential

Activity 7.6.2

- Find an integral computing the area of the region defined by $0 \leq r \leq -\cos(\theta) + 5$ and $\pi/2 \leq \theta \leq 3\pi/4$.
- Find the area enclosed by the cardioid $r = 2(1 + \cos(\theta))$.
- Find the area enclosed by one loop of the 4-petaled rose $r = \cos(2\theta)$.

Chapter 8

Sequences and Series (SQ)

Learning Outcomes

How do we use various techniques to integrate less simple functions?

By the end of this chapter, you should be able to...

1. Define and use explicit and recursive formulas for sequences.
2. Determine if a sequence is convergent, divergent, monotonic, or bounded, and compute limits of convergent sequences.
3. Compute the first few terms of a telescoping or geometric partial sum sequence, and find a closed form for this sequence, and compute its limit.
4. Determine if a geometric series converges, and if so, the value it converges to.
5. Use the divergence, alternating series, and integral tests to determine if a series converges or diverges.
6. Use the direct comparison and limit comparison tests to determine if a series converges or diverges.
7. Use the ratio and root tests to determine if a series converges or diverges.
8. Determine if a series converges absolutely or conditionally.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Compute limits.
 - Review: [Khan Academy](#)¹
 - Review: [Khan Academy](#)²
- b Express the sum of indexed values using summation notation.
 - Review: [Khan Academy](#)³

¹<https://www.khanacademy.org/math/ap-calculus-bc/bc-limits-new/bc-1-2/v/introduction-to-limits-hd?modal=1>

²<https://www.khanacademy.org/math/ap-calculus-bc/bc-diff-contextual-applications-new/bc-4-7/v/introduction-to-l-hopital-s-rule?modal=1>

³<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/sigma-notation-sum>

c Evaluate indefinite integrals.

- Review: [Khan Academy](https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-7/v/antiderivatives-and-indefinite-integrals)⁴

8.1 Sequence Formulas (SQ1)

Learning Outcomes

- Define and use explicit and recursive formulas for sequences.

Activity 8.1.1 Which of the following are sequences?

- A monthly gas bill
- B days in the year
- C how long you wash dishes
- D 1, 1, 2, 3, 5, 8, ...
- E how much you spend on groceries

Activity 8.1.2 Consider the sequence 1, 2, 4, ...

- (a) Which of the choices below reasonably continues this sequence of numbers?

- A 7, 12, 24, ...
- B 7, 11, 16, ...
- C 8, 16, 32, ...
- D 1, 2, 4, ...
- E 7, 12, 20, ...

- (b) Where possible, find a formula that allows us to move from one term to the next one.

Remark 8.1.3 As seen in the previous activity, having too few terms may prevent us from finding a unique way to continue creating a sequence of numbers. In fact, we need sufficiently many terms to uniquely continue a sequence of numbers (and how many terms is sufficient depends on which sequence of numbers you are trying to generate). Sometimes, we do not want to write out all of the terms needed to allow for this. Therefore, we will want to find short-hand notation that allows us to do so.

Definition 8.1.4 Let a_n denote the n th term in a sequence. Then $\{a_n\}_{n=1}^{\infty} = a_1, a_2, \dots, a_n, \dots$ \diamond

Activity 8.1.5 Consider the sequence $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$. Which of the following choices gives a closed formula for this sequence? Select all that apply.

- A $\left\{ \left(\frac{1}{3} \right)^{n-1} \right\}_{n=1}^{\infty}$
- B $\left\{ \left(\frac{1}{3} \right)^n \right\}_{n=1}^{\infty}$

⁴<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-7/v/antiderivatives-and-indefinite-integrals>

C $\left\{ \left(\frac{1}{3} \right)^{n-1} \right\}_{n=2}^{\infty}$

D $\left\{ \left(\frac{1}{3} \right)^{n+1} \right\}_{n=0}^{\infty}$

E $\left\{ \left(\frac{1}{3} \right)^n \right\}_{n=0}^{\infty}$

Activity 8.1.6 Let a_n be the n th term in the sequence $\left\{ \frac{n+1}{n} \right\}_{n=1}^{\infty}$. Which of the following terms corresponds to the 27^{th} term of this sequence?

A $\frac{27}{26}$

B $\frac{26}{27}$

C $\frac{27}{28}$

D $\frac{28}{27}$

E $\frac{29}{28}$

Activity 8.1.7 Let a_n be the n th term in the sequence $\left\{ \frac{n+1}{n} \right\}_{n=2}^{\infty}$. Which of the following terms corresponds to the 27^{th} term of this sequence?

A $\frac{27}{26}$

B $\frac{26}{27}$

C $\frac{27}{28}$

D $\frac{28}{27}$

E $\frac{29}{28}$

Activity 8.1.8 Let a_n be the n th term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. Identify the 81st term of this sequence.

A $\frac{1}{79}$

B $\frac{1}{80}$

C $\frac{1}{81}$

D $\frac{1}{82}$

E $\frac{1}{83}$

Activity 8.1.9 Find a closed form for the sequence $0, 3, 8, 15, 24, \dots$

Activity 8.1.10 Find a closed form for the sequence $\frac{12}{1}, \frac{16}{2}, \frac{20}{3}, \frac{24}{4}, \frac{28}{5}, \dots$

Activity 8.1.11 Let a_n be the n th term in the sequence $1, 1, 2, 3, 5, 8, \dots$. Find a formula for a_n .

Definition 8.1.12 A sequence is recursive if the terms are defined as a function of previous terms (with the necessary initial terms provided). \diamond

Activity 8.1.13 Consider the sequence defined by $a_1 = 6$ and $a_{k+1} = 4a_k - 7$ for $k \geq 1$. What are the first four terms?

Activity 8.1.14 Consider the sequence $2, 7, 22, 67, 202, \dots$. Which of the following offers the best recursive formula for this sequence?

A $3a_n + 1$

B $a_1 = 2, a_k = 3a_{k-1} + 1$ for $k > 1$

C $a_1 = 2, a_2 = 7, a_k = 3a_{k-1} + 1$ for $k > 2$

Activity 8.1.15 Once more, consider the sequence $1, 1, 2, 3, 5, 8, \dots$ from [Activity 8.1.11](#). Suppose $a_1 = 1$ and $a_2 = 1$. Give a recursive formula for a_n for all $n \geq 3$.

Activity 8.1.16 Give a recursive formula that generates the sequence $1, 2, 4, 8, 16, 32, \dots$

Activity 8.1.17

1. Find the first 5 terms of the following sequence:

- $a_n = 3 \cdot 2^n$,

2. Find a closed form for the following sequence:

- $4, 5, 8, 13, 20, \dots$,

3. Find a recursive form for the following sequence:

- $-3, 2, 7, 12, 17, \dots$,

Answer.

1. • $6, 12, 24, 48, 96$ if you start with $n = 1$, or $3, 6, 12, 24, 48$ if you start with $n = 0$,

2. • Answers can vary depending on your choice of starting index. For example, $n^2 + 4$ if $n = 0, 1, \dots$,

3. • Answers can vary depending on your choice of starting index. For example, if $a_0 = -3$ then $a_n = 1a_{n-1} + 5$,

Activity 8.1.18

1. Find the first 5 terms of the following sequence:

- $a_n = 5n + 4$,

2. Find a closed form for the following sequence:

- $0, 1, 4, 9, 16, \dots$,

3. Find a recursive form for the following sequence:

- $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \dots$,

Answer.

1. • 9, 14, 19, 24, 29 if you start with $n = 1$, or 4, 9, 14, 19, 24 if you start with $n = 0$.
2. • Answers can vary depending on your choice of starting index. For example, n^2 if $n = 0, 1, \dots$,
3. • Answers can vary depending on your choice of starting index. For example, if $a_0 = 2$ then $a_n = -\frac{1}{2}a_{n-1} + 0$.

8.1.1 Videos



YouTube: <https://www.youtube.com/watch?v=JmCWeEXQAJw>

Figure 149 Video: Define and use explicit and recursive formulas for sequences

8.2 Sequence Properties and Limits (SQ2)

Learning Outcomes

- Determine if a sequence is convergent, divergent, monotonic, or bounded, and compute limits of convergent sequences.

Activity 8.2.1 We will consider the function $f(x) = \frac{4x+8}{x}$.

- (a) Compute the limit $\lim_{x \rightarrow \infty} \frac{4x+8}{x}$.

- A 0
- B 8
- C 1
- D 4

- (b) Determine on which intervals $f(x)$ is increasing and/or decreasing. (Hint: compute $f'(x)$ first.)

- (c) Which statement best describes $f(x)$ for $x > 0$?

- A $f(x)$ is bounded above by 4
- B $f(x)$ is bounded below by 4
- C $f(x)$ is bounded above and below by 4
- D $f(x)$ is not bounded above
- E $f(x)$ is not bounded below

Definition 8.2.2 Given a sequence $\{x_n\}$:

- $\{x_n\}$ is *monotonically increasing* if $x_{n+1} > x_n$ for every choice of n .
- $\{x_n\}$ is *monotonically non-decreasing* if $x_{n+1} \geq x_n$ for every choice of n .
- $\{x_n\}$ is *monotonically decreasing* if $x_{n+1} < x_n$ for every choice of n .
- $\{x_n\}$ is *monotonically non-increasing* if $x_{n+1} \leq x_n$ for every choice of n .

All of these sequences would be *monotonic*. ◇

Activity 8.2.3 Consider the sequence $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$.

- (a) Compute $x_{n+1} - x_n$.
- (b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.
- A $x_{n+1} - x_n > 0$ for every choice of n .
 - B $x_{n+1} - x_n \geq 0$ for every choice of n .
 - C $x_{n+1} - x_n < 0$ for every choice of n .
 - D $x_{n+1} - x_n \leq 0$ for every choice of n .
- (c) Which of the following (if any) describe $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$?
- A Monotonically increasing.
 - B Monotonically non-decreasing.
 - C Monotonically decreasing.
 - D Monotonically non-increasing.

Activity 8.2.4 Consider the sequence $\left\{ \frac{n^2 + 1}{n} \right\}_{n=1}^{\infty}$.

- (a) Compute $x_{n+1} - x_n$.
- (b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.
- A $x_{n+1} - x_n > 0$ for every choice of n .
 - B $x_{n+1} - x_n \geq 0$ for every choice of n .
 - C $x_{n+1} - x_n < 0$ for every choice of n .
 - D $x_{n+1} - x_n \leq 0$ for every choice of n .
- (c) Which of the following (if any) describe $\left\{ \frac{n^2 + 1}{n} \right\}_{n=1}^{\infty}$?
- A Monotonically increasing.
 - B Monotonically non-decreasing.
 - C Monotonically decreasing.
 - D Monotonically non-increasing.

Activity 8.2.5 Consider the sequence $\left\{ \frac{n+1}{n} \right\}_{n=1}^{\infty}$.

- (a) Compute $x_{n+1} - x_n$.

- (b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.

- A $x_{n+1} - x_n > 0$ for every choice of n .
- B $x_{n+1} - x_n \geq 0$ for every choice of n .
- C $x_{n+1} - x_n < 0$ for every choice of n .
- D $x_{n+1} - x_n \leq 0$ for every choice of n .

- (c) Which of the following (if any) describe $\left\{ \frac{n+1}{n} \right\}_{n=1}^{\infty}$?

- A Monotonically increasing.
- B Monotonically non-decreasing.
- C Monotonically decreasing.
- D Monotonically non-increasing.

Activity 8.2.6 Consider the sequence $\left\{ \frac{2}{3^n} \right\}_{n=0}^{\infty}$.

- (a) Compute $x_{n+1} - x_n$.
- (b) Which of the following is true about $x_{n+1} - x_n$? There can be more or less than one answer.

- A $x_{n+1} - x_n > 0$ for every choice of n .
- B $x_{n+1} - x_n \geq 0$ for every choice of n .
- C $x_{n+1} - x_n < 0$ for every choice of n .
- D $x_{n+1} - x_n \leq 0$ for every choice of n .

- (c) Which of the following (if any) describe $\left\{ \frac{2}{3^n} \right\}_{n=0}^{\infty}$?

- A Monotonically increasing.
- B Monotonically non-decreasing.
- C Monotonically decreasing.
- D Monotonically non-increasing.

Definition 8.2.7 A sequence $\{x_n\}$ is *bounded* if there are real numbers b_u, b_ℓ such that

$$b_\ell \leq x_n \leq b_u$$

for every n ?

◇

Activity 8.2.8 Consider the sequence $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$ from [Activity 8.2.3](#).

- (a) Is there a b_u such that $x_n \leq b_u$ for every n ? If so, what would be one such b_u ?
- (b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n ? If so, what would be one such b_ℓ ?
- (c) Is $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$ bounded?

Activity 8.2.9 Consider the sequence $\left\{\frac{n^2+1}{n}\right\}_{n=1}^{\infty}$ from [Activity 8.2.4](#).

- (a) Is there a b_u such that $x_n \leq b_u$ for every n ? If so, what would be one such b_u ?
- (b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n ? If so, what would be one such b_ℓ ?
- (c) Is $\left\{\frac{n^2+1}{n}\right\}_{n=1}^{\infty}$ bounded?

Activity 8.2.10 Consider the sequence $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$ from [Activity 8.2.5](#).

- (a) Is there a b_u such that $x_n \leq b_u$ for every n ? If so, what would be one such b_u ?
- (b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n ? If so, what would be one such b_ℓ ?
- (c) Is $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$ bounded?

Activity 8.2.11 Consider the sequence $\left\{\frac{2}{3^n}\right\}_{n=1}^{\infty}$ from [Activity 8.2.6](#).

- (a) Is there a b_u such that $x_n \leq b_u$ for every n ? If so, what would be one such b_u ?
- (b) Is there a b_ℓ such that $b_\ell \leq x_n$ for every n ? If so, what would be one such b_ℓ ?
- (c) Is $\left\{\frac{2}{3^n}\right\}_{n=1}^{\infty}$ bounded?

Definition 8.2.12 Given a sequence $\{x_n\}$, we say x_n has *limit* L , denoted

$$\lim_{n \rightarrow \infty} x_n = L$$

if we can make x_n as close to L as we like by making n sufficiently large. If such an L exists, we say $\{x_n\}$ *converges to* L . If no such L exists, we say $\{x_n\}$ *does not converge*. \diamond

Activity 8.2.13

- (a) For each of the following, determine if the sequence converges.

A $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

B $\left\{\frac{n^2+1}{n}\right\}_{n=1}^{\infty}$.

C $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$.

D $\left\{\frac{2}{3^n}\right\}_{n=0}^{\infty}$.

- (b) Where possible, find the limit of the sequence.

Activity 8.2.14

- (a) Determine to what value $\left\{ \frac{4n}{n+1} \right\}_{n=0}^{\infty}$ converges.
- (b) Which of the following is most likely true about $\left\{ \frac{4n(-1)^n}{n+1} \right\}_{n=0}^{\infty}$?

- A $\left\{ \frac{4n(-1)^n}{n+1} \right\}_{n=0}^{\infty}$ converges to 4.
- B $\left\{ \frac{4n(-1)^n}{n+1} \right\}_{n=0}^{\infty}$ converges to 0.
- C $\left\{ \frac{4n(-1)^n}{n+1} \right\}_{n=0}^{\infty}$ converges to -4.
- D $\left\{ \frac{4n(-1)^n}{n+1} \right\}_{n=0}^{\infty}$ does not converge.

Activity 8.2.15 For each of the following sequences, determine which of the properties: *monotonic*, *bounded* and *convergent*, the sequence satisfies. If a

sequence is convergent, determine to what it converges. $\{3n\}_{n=0}^{\infty} \cdot \left\{ \frac{n^3}{3^n} \right\}_{n=0}^{\infty}$.

$\left\{ \frac{n}{n+3} \right\}_{n=1}^{\infty} \cdot \left\{ \frac{(-1)^n}{n+3} \right\}_{n=1}^{\infty}$.

Fact 8.2.16 If a sequence is monotonic and bounded, then it is convergent.

8.2.1 Videos

YouTube: <https://www.youtube.com/watch?v=Bt2sx5amiVo>

Figure 150 Video: Determine if a sequence is convergent, divergent, monotonic, or bounded, and compute limits of convergent sequences

8.3 Partial Sum Sequence (SQ3)**Learning Outcomes**

- Compute the first few terms of a telescoping or geometric partial sum sequence, and find a closed form for this sequence, and compute its limit.

Activity 8.3.1 Consider the sequence $\{a_n\}_{n=0}^{\infty} = \left\{ \frac{1}{2^n} \right\}_{n=0}^{\infty}$.

- (a) Find the first 5 terms of this sequence.
- (b) Compute the following:

- (a) a_0 .

- (b) $a_0 + a_1$.
- (c) $a_0 + a_1 + a_2$.
- (d) $a_0 + a_1 + a_2 + a_3$.
- (e) $a_0 + a_1 + a_2 + a_3 + a_4$.

Activity 8.3.2 Consider the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

(a) Find the first 5 terms of this sequence.

(b) Compute the following:

- (a) a_1 .
- (b) $a_1 + a_2$.
- (c) $a_1 + a_2 + a_3$.
- (d) $a_1 + a_2 + a_3 + a_4$.
- (e) $a_1 + a_2 + a_3 + a_4 + a_5$.

Definition 8.3.3 Given a sequence $\{a_n\}_{n=0}^{\infty}$ define the k^{th} *partial sum* of $\{a_n\}$ to be

$$A_k = \sum_{i=0}^k a_i = a_1 + a_2 + \cdots + a_k.$$

Note that $\{A_n\} = A_0, A_1, A_2, \dots$ is itself a sequence called the *partial sum sequence*. \diamond

Activity 8.3.4 Let $a_n = \frac{2}{3^n}$. Find the following partial sums:

1. A_0 .
2. A_1 .
3. A_2 .
4. A_3 .
5. A_{100} .

Activity 8.3.5 Consider the sequence $a_n = \frac{2}{3^n}$. What is the best way to find the 100th partial sum A_{100} ?

A Sum the first 101 terms of the sequence $\{a_n\}$.

B Find a closed form for the partial sum sequence $\{A_n\}$.

Activity 8.3.6 Expand the following polynomial products, and then reduce to as few summands as possible.

1. $(1-x)(1+x+x^2)$.
2. $(1-x)(1+x+x^2+x^3)$.
3. $(1-x)(1+x+x^2+x^3+x^4)$.
4. $(1-x)(1+x+x^2+\cdots+x^n)$, where n is any nonnegative integer.

Activity 8.3.7 Suppose $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$. Without actually computing this sum, which of the following is equal to $(1 - \frac{1}{2}) S_5$?

- A $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$.

B $1 - \frac{1}{64}$.

C $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32}$.

Activity 8.3.8 Recall from [Activity 8.3.4](#) that $A_{100} = 2 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \cdots + \frac{2}{3^{100}} = 2 \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{100}} \right)$.

(a) Which of the following is equal to $\left(1 - \frac{1}{3}\right) A_{100}$?

A $1 - \frac{1}{3^{101}}$.

B $1 - \frac{1}{3^{100}}$.

C $2 \left(1 - \frac{1}{3^{101}} \right)$.

D $2 \left(1 - \frac{1}{3^{100}} \right)$.

(b) Based on your previous choice, write out an expression for A_{100} .

Activity 8.3.9 Suppose that $\{b_n\}_{n=0}^{\infty} = \{(-2)^n\}_{n=0}^{\infty} = \{1, -2, 4, -8, \dots\}$. Let $B_n = \sum_{i=0}^n b_i$ be the n th partial sum of $\{b_n\}$.

(a) Find simple expressions for the following:

(a) $(1 - (-2))B_{10}$.

(b) $(1 - (-2))B_{30}$.

(c) $(1 - (-2))B_n$. Choose from the following:

A $1 + (-2)^n$.

B $1 - (-2)^n$.

C $1 + (-2)^{n+1}$.

D $1 - (-2)^{n+1}$.

E $1 - 2^n$.

(b) Based on your previous answers, solve for the following:

(a) B_{10} .

(b) B_{30} .

(c) B_n . Choose from the following:

A $\frac{1 - (-2)^{n+1}}{1 - (-2)}$

B $\frac{1 - (-2)^{n+1}}{1 - 2}$

C $\frac{1 - (-2)^{n+1}}{1 + (-2)}$

D $\frac{1 - (-2)^n}{1 - 2}$

E $\frac{1 - (-2)^n}{1 - (-2)}$

Activity 8.3.10 Consider the following sequences:

1. $\{a_n\}_{n=0}^{\infty} = \left\{ \left(-\frac{2}{3} \right)^n \right\}_{n=0}^{\infty}$.
2. $\{b_n\}_{n=0}^{\infty} = \{2 \cdot (-1)^n\}_{n=0}^{\infty}$.
3. $\{c_n\}_{n=0}^{\infty} = \{-3 \cdot (1.2)^n\}_{n=0}^{\infty}$.

(a) Find the closed form for the n th partial sum for the geometric sequence

$$A_n = \sum_{i=0}^n a_i = \sum_{i=0}^n \left(-\frac{2}{3} \right)^i.$$

A $\frac{3}{5} \left(1 - \left(-\frac{2}{3} \right)^{n+1} \right).$

B $\frac{5}{3} \left(1 - \left(-\frac{2}{3} \right)^{n+1} \right).$

C $\frac{5}{3} \left(1 + \frac{2}{3} \left(\frac{2}{3} \right)^n \right).$

D $\frac{3}{5} \left(1 + \frac{2}{3} \left(\frac{2}{3} \right)^n \right).$

E $1 - \left(-\frac{2}{3} \right)^{n+1}.$

(b) Find the closed form for the n th partial sum for the geometric sequence

$$B_n = \sum_{i=0}^n b_i = \sum_{i=0}^n 2 \cdot (-1)^i.$$

A $2^{n+1}.$

B $1 - (-1)^{n+1}.$

C $1 + (-1)^n.$

D $2(1 + (-1)^n).$

E $2(1 - (-1)^{n+1}).$

(c) Find the closed form for the n th partial sum for the geometric sequence

$$C_n = \sum_{i=0}^n c_i = \sum_{i=0}^n -3 \cdot (1.2)^i.$$

Activity 8.3.11 Given the closed forms you found in [Activity 8.3.10](#), which of the following limits are defined? If defined, what is the limit?

A $\lim_{n \rightarrow \infty} A_n.$

B $\lim_{n \rightarrow \infty} B_n.$

C $\lim_{n \rightarrow \infty} C_n.$

Definition 8.3.12 Given a sequence a_n , we define the limit of the series

$$\sum_{n=k}^{\infty} a_n := \lim_{n \rightarrow \infty} A_n$$

where $A_n = \sum_{i=k}^n a_i$. We call $\sum_{n=k}^{\infty} a_n$ an *infinite series*. ◇

Activity 8.3.13 Which of the following series are infinite?

A $\sum_{n=0}^{\infty} 3(0.8)^n$.

B $\sum_{n=0}^{\infty} 2\left(\frac{5}{4}\right)^n$.

C $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$.

D $\sum_{n=0}^{\infty} \frac{1}{2}(81)^n$.

E $\sum_{n=0}^{\infty} 10\left(-\frac{1}{5}\right)^n$.

Activity 8.3.14 Let $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} = 1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \dots$. Let

$$A_n = \sum_{i=1}^n a_i = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1} \right).$$

Which of the following is the best strategy for evaluating $A_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$?

A Compute $A_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$, then evaluate the sum.

B Rewrite $A_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) - \frac{1}{5}$, then simplify.

Activity 8.3.15 Recall from [Activity 8.3.14](#) that $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n} - \frac{1}{n+1} \right\}$

and $A_n = \sum_{i=1}^n a_i = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1} \right)$.

Compute the following partial sums:

1. A_3 .

2. A_{10} .

3. A_{100} .

Activity 8.3.16 Recall from [Activity 8.3.14](#) that $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n} - \frac{1}{n+1} \right\}$

and $A_n = \sum_{i=1}^n a_i = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1} \right)$.

Which of the following is equal to A_n ?

A $n - \frac{1}{n+1}$.

B $1 - \frac{1}{n}$.

C $1 - \frac{1}{n+1}$.

D $1 - \frac{1}{i}$.

E $1 - \frac{1}{i+1}$.

Definition 8.3.17 Given a sequence $\{x_n\}_1^\infty$ and a sequence of the form $\{s_n\}_1^\infty := \{x_n - x_{n+1}\}_1^\infty$ we call the series $S_n = \sum_{i=1}^n s_i = \sum_{i=1}^n (x_i - x_{i+1})$ to be a *telescoping series*. ◇

Activity 8.3.18 Given a telescoping series $S_n = \sum_{i=1}^n s_i = \sum_{i=1}^n (x_i - x_{i+1})$, find:

1. S_2 .
2. S_{10} .
3. S_n . Choose from the following:

A $x_1 - x_n$

B $x_1 - x_{n+1}$

C $x_1 - x_{n-1}$

D $x_1 - x_n + 1$

E $x_1 - x_n - 1$

Activity 8.3.19 For each of the following telescoping series, find the closed form for the n th partial sum.

1. $S_n = \sum_{i=1}^n (2^{-i} - (2^{-i-1}))$.

2. $S_n = \sum_{i=1}^n (i^2 - (i+1)^2)$.

3. $S_n = \sum_{i=1}^n \left(\frac{1}{2i+1} - \frac{1}{2i+3} \right)$.

Activity 8.3.20 Given the closed forms you found in [Activity 8.3.19](#), determine which of the following telescoping series converge. If so, to what value does it converge?

A $\sum_{i=1}^{\infty} (2^{-i} - (2^{-i-1}))$.

B $\sum_{i=1}^{\infty} (i^2 - (i+1)^2)$.

$$C \sum_{i=1}^{\infty} \left(\frac{1}{2i+1} - \frac{1}{2i+3} \right).$$

Activity 8.3.21

- Consider the partial sum sequence $A_n = (-2) + \left(\frac{2}{3}\right) + \left(-\frac{2}{9}\right) + \cdots + \left(-2 \cdot \left(-\frac{1}{3}\right)^n\right)$.
 - Find a closed form for A_n .
 - Does $\{A_n\}$ converge? If so, to what value?
- Consider the partial sum sequence $B_n = \sum_{i=1}^n \left(\frac{1}{5i+2} - \frac{1}{5i+7} \right)$.
 - Find a closed form for B_n .
 - Does $\{B_n\}$ converge? If so, to what value?

Answer.

- $A_n = -2 \cdot \frac{1 - \left(-\frac{1}{3}\right)^{n+1}}{1 - \left(-\frac{1}{3}\right)}$.
 - $\lim_{n \rightarrow \infty} -2 \cdot \frac{1 - \left(-\frac{1}{3}\right)^{n+1}}{1 - \left(-\frac{1}{3}\right)}$ converges to $-\frac{3}{2}$.
- $B_n = \frac{1}{7} - \frac{1}{5n+7}$.
 - $\lim_{n \rightarrow \infty} \frac{1}{7} - \frac{1}{5n+7}$ converges to $\frac{1}{7}$.

8.3.1 Videos

YouTube: <https://www.youtube.com/watch?v=egshCNxf5FE>

Figure 151 Video: Compute the first few terms of a telescoping or geometric partial sum sequence, and find a closed form for this sequence, and compute its limit.

8.4 Geometric Series (SQ4)**Learning Outcomes**

- Determine if a geometric series converges, and if so, the value it converges to.

Activity 8.4.1 Recall from [Section 8.3](#) that for any real numbers a, r and $S_n = \sum_{i=0}^n ar^i$ that:

$$\begin{aligned} S_n &= \sum_{i=0}^n ar^i = a + ar + ar^2 + \cdots + ar^n \\ (1-r)S_n &= (1-r) \sum_{i=0}^n ar^i = (1-r)(a + ar + ar^2 + \cdots + ar^n) \\ (1-r)S_n &= (1-r) \sum_{i=0}^n ar^i = a - ar^{n+1} \\ S_n &= a \frac{1-r^{n+1}}{1-r}. \end{aligned}$$

(a) Using [Definition 8.3.12](#), given each restriction on r , determine if $\sum_{n=0}^{\infty} ar^n$ converges.

- A $r > 1$.
- B $r = 1$.
- C $-1 < r < 1$.
- D $r = -1$.
- E $r < -1$.

(b) Where possible, determine what value $\sum_{n=0}^{\infty} ar^n$ converges to.

Fact 8.4.2 Geometric series are of the form $\sum_{n=0}^{\infty} ar^n$, where a and r are real numbers. A geometric series converges only when $|r| < 1$. In this case, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. Otherwise, the geometric series diverges.

Activity 8.4.3 Consider the infinite series

$$5 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

(a) What of the following approaches will best help us determine the convergence of this series?

- A This is a geometric series of the form $\sum_{n=0}^{\infty} 5 \cdot \left(\frac{1}{2}\right)^n$.
- B This is a geometric series of the form $\sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^n$.
- C We can rewrite

$$5 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 4 + \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right).$$

(b) Using your chosen approach, determine if $5 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges, and if so, to what it converges.

Activity 8.4.4 Consider the infinite series

$$\sum_{n=3}^{\infty} 2(0.7)^n.$$

(a) What of the following approaches will best help us determine the convergence of this series?

A Noticing that $3 \cdot \sum_{n=3}^{\infty} 2(0.7)^n = \sum_{n=0}^{\infty} 2(0.7)^n.$

B Noticing that $2 + \sum_{n=3}^{\infty} 2(0.7)^n = \sum_{n=0}^{\infty} 2(0.7)^n.$

C Noticing that $2n + \sum_{n=3}^{\infty} 2(0.7)^n = \sum_{n=0}^{\infty} 2(0.7)^n.$

D Noticing that $2(0.7)^0 + 2(0.7)^1 + 2(0.7)^2 + \sum_{n=3}^{\infty} 2(0.7)^n = \sum_{n=0}^{\infty} 2(0.7)^n.$

E Noticing that $0 + 1 + 2 + \sum_{n=3}^{\infty} 2(0.7)^n = \sum_{n=0}^{\infty} 2(0.7)^n.$

(b) Using your chosen approach, determine if $\sum_{n=3}^{\infty} 2(0.7)^n$ converges, and if so, to what it converges.

Example 8.4.5 Given a series that appears to be mostly geometric:

$$3 + (1.1)^3 + (1.1)^4 + \cdots (1.1)^n + \cdots$$

we can rewrite it as the sum of a standard geometric series with some modification: $??? + \sum_{n=0}^{\infty} a \cdot r^n$:

$$\begin{aligned} & 3 + (1.1)^3 + (1.1)^4 + \cdots (1.1)^n + \cdots \\ &= 3 - ((1.1)^0 + (1.1)^1 + (1.1)^2) + ((1.1)^0 + (1.1)^1 + (1.1)^2) + (1.1)^3 + \cdots \\ &= 3 - ((1.1)^0 + (1.1)^1 + (1.1)^2) + (1.1)^0 + (1.1)^1 + (1.1)^2 + (1.1)^3 + \cdots \\ &= 3 - ((1.1)^0 + (1.1)^1 + (1.1)^2) + \sum_{n=0}^{\infty} (1.1)^n \\ &= -0.31 + \sum_{n=0}^{\infty} (1.1)^n \end{aligned}$$

which converges if and only if $\sum_{n=0}^{\infty} (1.1)^n$ converges. In this case, it does not. □

Activity 8.4.6 For each of the following modified geometric series, rewrite them in the form $??? + \sum_{n=0}^{\infty} a \cdot r^n$.

1. $-7 + \left(-\frac{3}{7}\right)^2 + \left(-\frac{3}{7}\right)^3 + \cdots$

2. $-6 + \left(\frac{5}{4}\right)^3 + \left(\frac{5}{4}\right)^4 + \dots$

3. $4 + \sum_{n=4}^{\infty} \left(\frac{2}{3}\right)^n$

4. $8 - 1 + 1 - 1 + 1 - 1 + \dots$

Activity 8.4.7 Use your rewritten forms from [Activity 8.4.6](#) to determine which of the modified geometric series converge. If the series converges, find to what value it converges.

A $-7 + \left(-\frac{3}{7}\right)^2 + \left(-\frac{3}{7}\right)^3 + \dots$

B $-6 + \left(\frac{5}{4}\right)^3 + \left(\frac{5}{4}\right)^4 + \dots$

C $4 + \sum_{n=4}^{\infty} \left(\frac{2}{3}\right)^n$

D $8 - 1 + 1 - 1 + 1 - 1 + \dots$

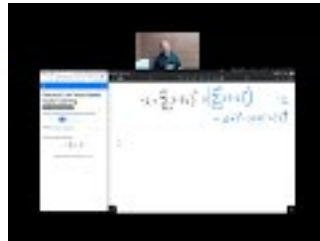
Activity 8.4.8 Find the limit of the following series.

$$-1 + \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^n$$

Answer.

$$-1 + \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^n = 1.$$

8.4.1 Videos



YouTube: https://www.youtube.com/watch?v=e_ol6EmA0_I

Figure 152 Video: Determine if a geometric series converges, and if so, the value it converges to.

8.5 Basic Convergence Tests (SQ5)

Learning Outcomes

- Use the divergence, alternating series, and integral tests to determine if a series converges or diverges.

Activity 8.5.1 Which of the following series seem(s) to diverge? It might be

helpful to write out the first several terms.

A $\sum_{n=0}^{\infty} n^2.$

B $\sum_{n=1}^{\infty} \frac{n+1}{n}.$

C $\sum_{n=0}^{\infty} (-1)^n.$

D $\sum_{n=1}^{\infty} \frac{1}{n}.$

E $\sum_{n=1}^{\infty} \frac{1}{n^2}.$

Fact 8.5.2 If the series $\sum a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.

Fact 8.5.3 The Divergence (n^{th} term) Test. If the $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum a_n$ diverges.

Activity 8.5.4 Which of the series from [Activity 8.5.1](#) diverge by [Fact 8.5.3](#)?

Fact 8.5.5 If $a_n > 0$ for all n , then $\sum a_n$ is convergent if and only if the sequence of partial sums is bounded from above.

Activity 8.5.6 Consider the so-called *harmonic series*, $\sum_{n=1}^{\infty} \frac{1}{n}$, and let S_n be its n^{th} partial sum.

(a) Determine which of the following inequalities hold(s).

A $\frac{1}{3} + \frac{1}{4} < \frac{1}{2}.$

B $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}.$

C $S_4 \geq S_2 + \frac{1}{2}.$

D $S_4 \leq S_2 + \frac{1}{2}.$

E $S_4 = S_2 + \frac{1}{2}.$

(b) Determine which of the following inequalities hold(s).

A $\frac{1}{2} < \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}.$

B $\frac{1}{2} > \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}.$

C $S_8 = S_4 + \frac{1}{2}.$

D $S_8 \geq S_4 + \frac{1}{2}.$

E $S_8 \leq S_4 + \frac{1}{2}.$

Activity 8.5.7 In [Activity 8.5.6](#), we found that $S_4 \geq S_2 + \frac{1}{2}$ and $S_8 \geq S_4 + \frac{1}{2}$. Based on these inequalities, which statement seems most likely to hold?

- A The harmonic series converges.
- B The harmonic series diverges.

Activity 8.5.8 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- (a) We want to compare this series to an improper integral. Which of the following is the best candidate?

A $\int_1^{\infty} x^2 dx$.

B $\int_1^{\infty} \frac{1}{x^3} dx$.

C $\int_1^{\infty} \frac{1}{x^2} dx$.

D $\int_1^{\infty} \frac{1}{x} dx$.

E $\int_1^{\infty} x dx$.

- (b) Select the true statements below.

A The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using left Riemann sums where $\Delta x = 1$.

B The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using right Riemann sums where $\Delta x = 1$.

C The sum $\sum_{n=2}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using left Riemann sums where $\Delta x = 1$.

D The sum $\sum_{n=2}^{\infty} \frac{1}{n^2}$ corresponds to approximating the integral chosen above using right Riemann sums where $\Delta x = 1$.

- (c) Using the Riemann sum interpretation of the series, identify which of the following inequalities holds.

A $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2} dx$.

B $\sum_{n=1}^{\infty} \frac{1}{n^2} \geq \int_1^{\infty} \frac{1}{x^2} dx$.

C $\sum_{n=2}^{\infty} \frac{1}{n^2} \geq \int_1^{\infty} \frac{1}{x^2} dx$.

D $\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2} dx$.

(d) What can we say about the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$?

- A This improper integral converges.
- B This improper integral diverges.

(e) What do you think is true about the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

- A The series converges.
- B The series diverges.

Fact 8.5.9 The Integral Test. *Let $\{a_n\}$ be a sequence of positive numbers. If $f(x)$ is continuous, positive, and decreasing, and there is some positive integer N such that $f(n) = a_n$ for all $n \geq N$, then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x) dx$ both converge or both diverge.*

Activity 8.5.10 Consider the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

(a) Recall that the harmonic series diverges. What value of p corresponds to the harmonic series?

- A $p = -1$.
- B $p = 1$.
- C $p = -2$.
- D $p = 2$.
- E $p = 0$.

(b) From [Fact 8.5.9](#), what can we conclude about the p -series with $p = 2$?

- A There is not enough information to draw a conclusion.
- B This series converges.
- C This series diverges.

Fact 8.5.11 The p -Test. *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, and diverges otherwise.*

Activity 8.5.12 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

(a) If we aim to use the integral test, what is an appropriate choice for $f(x)$?

- A $\frac{1}{x^2}$.
- B $x^2 + 1$.
- C $\frac{1}{x^2 + 1}$.
- D x^2 .
- E $\frac{1}{x}$.

(b) Does the series converge or diverge by [Fact 8.5.9](#)?

Activity 8.5.13 Prove [Fact 8.5.11](#).

Activity 8.5.14 Which of the following statements seem(s) most likely to be true?

A $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ diverges.

B $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

C $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges.

D $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ diverges.

Fact 8.5.15 The Alternating Series Test (Leibniz's Theorem). *The series $\sum (-1)^{n+1} u_n$ converges if all of the following conditions are satisfied:*

1. u_n is always positive,
2. there is an integer N such that $u_n \geq u_{n+1}$ for all $n \geq N$, and
3. $\lim_{n \rightarrow \infty} u_n = 0$.

Activity 8.5.16 What conclusions can you now make?

A $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ diverges.

B $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

C $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges.

D $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ diverges.

Activity 8.5.17 For each of the following series, use the *Divergence*, *Alternating Summation* or *Integral* test to determine if the series converges.

1. $\sum_{n=1}^{\infty} \frac{2(n^2 + 2)}{n^2}$.

2. $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

3. $\sum_{n=1}^{\infty} \frac{3(-1)^n}{4n}$.

Answer.

1. $\sum_{n=1}^{\infty} \frac{2(n^2 + 2)}{n^2}$ diverges by the *Divergence* test.

2. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the *Integral* test.
3. $\sum_{n=1}^{\infty} \frac{3(-1)^n}{4n}$ converges by the *Alternating Summation* test.

Fact 8.5.18 The Alternating Series Estimation Theorem. *If the alternating series $\sum a_n = \sum (-1)^{n+1} u_n$ converges to L and has n^{th} partial sum S_n , then for $n \geq N$ (as in the alternating series test):*

1. $|L - S_n|$ is less than $|a_{n+1}|$, and
2. $(L - S_n)$ has the same sign as a_{n+1} .

Activity 8.5.19 Consider the so-called *alternating harmonic series*, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

- (a) Use the alternating series test to determine if the series converges.
- (b) If so, estimate the series using the first 3 terms.

8.5.1 Videos



YouTube: <https://www.youtube.com/watch?v=XPGsG1Usjxg>

Figure 153 Video: Use the divergence, alternating series, and integral tests to determine if a series converges or diverges

8.6 Comparison Tests (SQ6)

Learning Outcomes

- Use the direct comparison and limit comparison tests to determine if a series converges or diverges.

Activity 8.6.1 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, with infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$. Suppose $\{b_n\}_{n=1}^{\infty}$ is a sequence where each $b_n = 3a_n$, with infinite series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 3a_n = 3a_1 + 3a_2 + \cdots$.

- (a) If $\sum_{n=1}^{\infty} a_n = 5$ what can be said about $\sum_{n=1}^{\infty} b_n$?

A $\sum_{n=1}^{\infty} b_n$ converges but the value cannot be determined.

B $\sum_{n=1}^{\infty} b_n$ converges to $3 \cdot 5 = 15$.

C $\sum_{n=1}^{\infty} b_n$ converges to some value other than 15.

D $\sum_{n=1}^{\infty} b_n$ diverges.

E It cannot be determined whether $\sum_{n=1}^{\infty} b_n$ converges or diverges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, what can be said about $\sum_{n=1}^{\infty} b_n$?

A $\sum_{n=1}^{\infty} b_n$ converges but the value cannot be determined.

B $\sum_{n=1}^{\infty} b_n$ converges and the value can be determined.

C $\sum_{n=1}^{\infty} b_n$ diverges.

D It cannot be determined whether $\sum_{n=1}^{\infty} b_n$ converges or diverges.

Activity 8.6.2 Using [Fact 8.4.2](#), we know the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

(a) What can we say about the series

$$3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots?$$

A $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots$ converges to $3 \cdot 2 = 6$.

B $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots$ converges to some value other than 6.

C $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^n} + \cdots$ diverges.

(b) What do you think we can say about the series

$$\frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3 + (0.1)^n}{2^n} + \cdots?$$

A $3 + \frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3 + (0.1)^n}{2^n} + \cdots$ converges to $3 \cdot 2 = 6$.

B $3 + \frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3 + (0.1)^n}{2^n} + \cdots$ converges to some value other than 6.

C $3 + \frac{3.1}{2} + \frac{3.01}{4} + \frac{3.001}{8} + \cdots + \frac{3 + (0.1)^n}{2^n} + \cdots$ diverges.

Activity 8.6.3 From [Fact 8.4.2](#), we know

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

diverges.

(a) What can we say about the series

$$5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots?$$

A $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots$ converges to a known value we can compute.

B $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots$ converges to some unknown value.

C $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots + \frac{5}{n} + \cdots$ diverges.

(b) What do you think we can say about the series

$$4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots?$$

A $4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots$ converges to a known value we can compute.

B $4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots$ converges to some unknown value.

C $4.9 + \frac{4.99}{2} + \frac{4.999}{3} + \frac{4.9999}{4} + \cdots + \frac{5 - (0.1)^n}{n} + \cdots$ diverges.

Fact 8.6.4 The Limit Comparison Test. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = c$$

for some positive (finite) constant c , then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Activity 8.6.5 Recall that

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges.

(a) Let $b_n = \frac{1}{n}$. Compute $\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2^n}}$.

A $-\infty$.

B 0.

C $\frac{1}{2}$.

D 1.

E ∞ .

(b) Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

(c) Let $b_n = \frac{1}{n^2}$. Compute $\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{2^n}}$.

A ∞ .

B $\ln(2)$.

C 1.

D $\frac{1}{2}$.

E $-\infty$.

(d) Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

(e) Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n}$$

diverges, can we conclude that $\sum b_n$ converges or diverges?

Activity 8.6.6 We wish to determine if $\sum_{n=1}^{\infty} \frac{1}{4^n - 1}$ converges or diverges using [Fact 8.6.4](#).

(a) Compute

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{4^n - 1}}{\frac{1}{4^n}}.$$

(b) Does the geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converge or diverge by [Fact 8.4.2](#)?

(c) Does $\sum_{n=1}^{\infty} \frac{1}{4^n - 1}$ converge or diverge?

Activity 8.6.7 We wish to determine if $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n+3}}$ converges or diverges using [Fact 8.6.4](#).

(a) To which of the following should we compare $\{a_n\} = \left\{ \frac{2}{\sqrt{n+3}} \right\}$?

A $\left\{ \frac{1}{n} \right\}$.

B $\left\{ \frac{1}{\sqrt{n}} \right\}$.

C $\left\{ \frac{1}{n^2} \right\}$.

D $\left\{ \frac{1}{2^n} \right\}$.

(b) Compute $\lim_{n \rightarrow \infty} \frac{b_n}{a_n}$.

(c) Compute $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.

(d) What is true about $\lim_{n \rightarrow \infty} \frac{b_n}{a_n}$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$?

- A Their values are reciprocals.
- B Their values negative reciprocals.
- C They are both positive finite constants.
- D Only one value is a finite positive constant.
- E One value is 0 and the other value is infinite.

(e) Does the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge?

(f) Using your chosen sequence and [Fact 8.6.4](#), does $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n+3}}$ converge or diverge?

Activity 8.6.8 We wish to determine if $\sum_{n=1}^{\infty} \frac{3}{n^2 + 8n + 5}$ converges or diverges using [Fact 8.6.4](#).

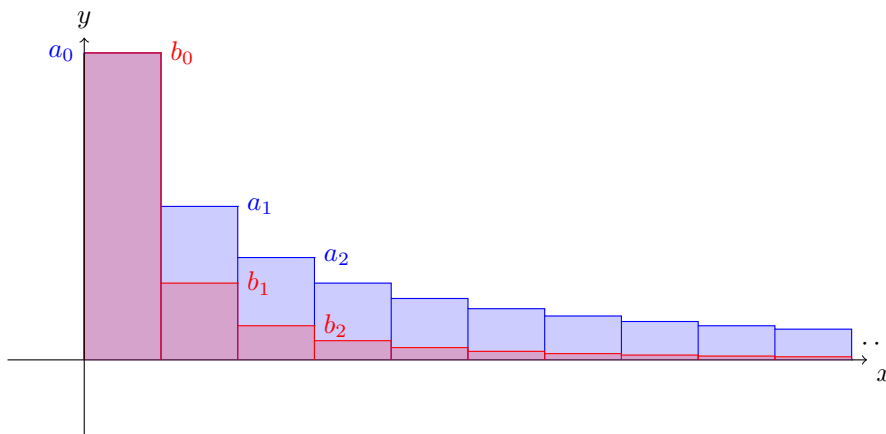
(a) To which of the following should we compare $\{x_n\} = \left\{ \frac{3}{n^2 + 8n + 5} \right\}$?

- A $\left\{ \frac{1}{n} \right\}$.
- B $\left\{ \frac{1}{\sqrt{n}} \right\}$.
- C $\left\{ \frac{1}{n^2} \right\}$.
- D $\left\{ \frac{1}{2^n} \right\}$.

(b) Using your chosen sequence and [Fact 8.6.4](#), does $\frac{3}{n^2 + 8n + 5}$ converge or diverge?

Activity 8.6.9 Use [Fact 8.6.4](#) to determine if the series $\sum_{n=5}^{\infty} \frac{2}{4^n}$ converges or diverges.

Activity 8.6.10 Consider sequences $\{a_n\}, \{b_n\}$ where $a_n \geq b_n \geq 0$.

**Figure 154** Plots of $\{a_n\}, \{b_n\}$

(a) Suppose that $\sum_{n=0}^{\infty} a_n$ converges. What could be said about $\{b_n\}$?

A $\sum_{n=0}^{\infty} b_n$ converges.

B $\sum_{n=0}^{\infty} b_n$ diverges.

C Whether or not $\sum_{n=0}^{\infty} b_n$ converges or diverges cannot be determined with this information.

(b) Suppose that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+1}$ which diverges. Which of the following statements are true?

A $0 \leq \frac{1}{2n^2} \leq \frac{1}{n+1}$ for each $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ is a convergent p -series where $p = 2$.

B $0 \leq \frac{1}{2n} \leq \frac{1}{n+1}$ for each $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ is a divergent p -series where $p = 1$.

(c) Suppose that $\sum_{n=0}^{\infty} a_n$ was some series that diverges. What could be said about $\{b_n\}$?

A $\sum_{n=0}^{\infty} b_n$ converges.

B $\sum_{n=0}^{\infty} b_n$ diverges.

C Whether or not $\sum_{n=0}^{\infty} b_n$ converges or diverges cannot be determined with this information.

(d) Suppose that $\sum_{n=0}^{\infty} b_n$ diverges. What could be said about $\{a_n\}$?

A $\sum_{n=0}^{\infty} a_n$ converges.

B $\sum_{n=0}^{\infty} a_n$ diverges.

C Whether or not $\sum_{n=0}^{\infty} a_n$ converges or diverges cannot be determined with this information.

(e) Suppose that $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{3^n}$ which converges. Which of the following statements are true?

A $0 \leq \frac{1}{3^n} \leq \frac{1}{2^n}$ for each n and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent geometric series where $|r| = \frac{1}{2} < 1$.

B $0 \leq \frac{1}{3^n} \leq 1$ for each n and $\sum_{n=0}^{\infty} 1$ diverges by the Divergence Test.

(f) Suppose that $\sum_{n=0}^{\infty} b_n$ was some series that converges. What could be said about $\{a_n\}$?

A $\sum_{n=0}^{\infty} a_n$ converges.

B $\sum_{n=0}^{\infty} a_n$ diverges.

C Whether or not $\sum_{n=0}^{\infty} a_n$ converges or diverges cannot be determined with this information.

Fact 8.6.11 Suppose we have sequences $\{a_n\}, \{b_n\}$ so that for some k we have that $0 \leq b_n \leq a_n$ for each $k \geq n$. Then we have the following results:

- If $\sum_{k=n}^{\infty} a_n$ converges, then so does $\sum_{k=n}^{\infty} b_n$.
- If $\sum_{k=n}^{\infty} b_n$ diverges, then so does $\sum_{k=n}^{\infty} a_n$.

Activity 8.6.12 Suppose that you were handed positive sequences $\{a_n\}, \{b_n\}$. For the first few values $a_n \geq b_n$, but after that what happens is unclear until $n = 100$. Then for any $n \geq 100$ we have that $a_n \leq b_n$.

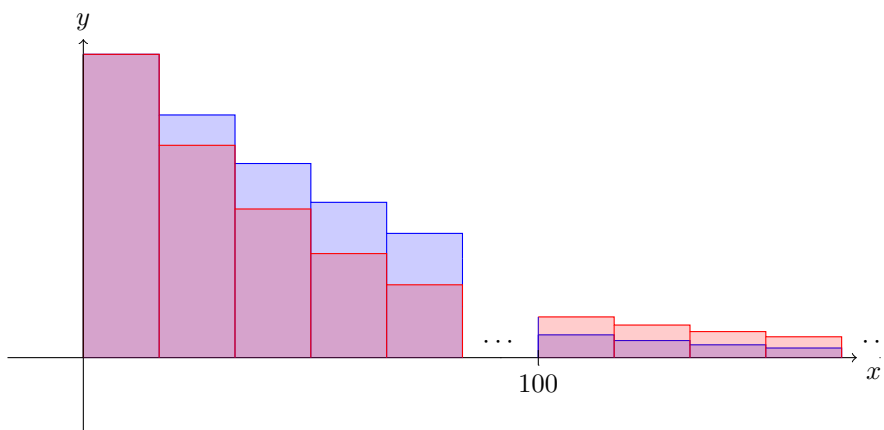


Figure 155 Plots of $\{a_n\}, \{b_n\}$

- (a) How might we best utilize [Fact 8.6.11](#) to determine the convergence of

$$\sum_{n=0}^{\infty} a_n \text{ or } \sum_{n=0}^{\infty} b_n?$$

A Since a_n is sometimes greater than, and sometimes less than b_n , there is no way to utilize [Fact 8.6.11](#).

B Since initially, we have $b_n \leq a_n$, we can utilize [Fact 8.6.11](#) by assuming $a_n \geq b_n$.

C Since we can rewrite $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{99} a_n + \sum_{n=100}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{99} b_n + \sum_{n=100}^{\infty} b_n$ and $\sum_{n=100}^{\infty} b_n$ and $\sum_{n=0}^{99} a_n, \sum_{n=0}^{99} b_n$ are necessarily finite, we can compare $\sum_{n=100}^{\infty} a_n, \sum_{n=100}^{\infty} b_n$ with [Fact 8.6.11](#).

Fact 8.6.13 The Direct Comparison Test. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If there is a k such that $b_n \leq a_n$ for each $n \geq k$, then:

- If $\sum a_n$ converges, then so does $\sum b_n$.
- If $\sum b_n$ diverges, then so does $\sum a_n$.

Activity 8.6.14 Suppose we wish to determine if $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converged using [Fact 8.6.13](#).

- (a) Does $\sum_{n=1}^{\infty} \frac{1}{3n}$ converge or diverge?

- (b) For which value k is $\frac{1}{3n} \leq \frac{1}{2n+3}$ for each $n \geq k$?

A $\frac{1}{3n} \leq \frac{1}{2n+3}$ for each $n \geq k = 0$.

B $\frac{1}{3n} \leq \frac{1}{2n+3}$ for each $n \geq k = 1$.

C $\frac{1}{3n} \leq \frac{1}{2n+3}$ for each $n \geq k = 2$.

D $\frac{1}{3n} \leq \frac{1}{2n+3}$ for each $n \geq k = 3$.

E There is no k for which $\frac{1}{3n} \leq \frac{1}{2n+3}$ for each $n \geq k$.

(c) Use [Fact 8.6.13](#) and compare $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ to $\sum_{n=1}^{\infty} \frac{1}{3n}$ to determine if $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converges or diverges.

Activity 8.6.15 Suppose we wish to determine if $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ converged using [Fact 8.6.13](#).

(a) Which series should we compare $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ to best utilize [Fact 8.6.13](#)?

A $\sum_{n=1}^{\infty} \frac{1}{n}$.

B $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

C $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

D $\sum_{n=1}^{\infty} \frac{1}{n+5}$.

E $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$.

F $\sum_{n=1}^{\infty} \frac{1}{2^n+5}$.

(b) Using your chosen series and [Fact 8.6.13](#), does $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ converge or diverge?

Activity 8.6.16 For each of the following series, determine if it converges or diverges, and explain your choice.

1. $\sum_{n=4}^{\infty} \frac{3}{\log(n)+2}$.

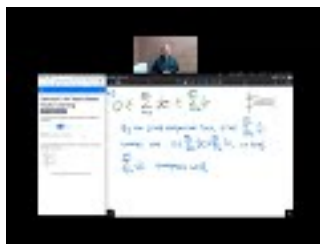
2. $\sum_{n=3}^{\infty} \frac{1}{n^2+2n+1}$.

Answer.

1. $\sum_{n=4}^{\infty} \frac{3}{\log(n)+2}$ diverges.

2. $\sum_{n=3}^{\infty} \frac{1}{n^2+2n+1}$ converges.

8.6.1 Videos



YouTube: <https://www.youtube.com/watch?v=Sb3UUb6-7A>

Figure 156 Video: Use the direct comparison and limit comparison tests to determine if a series converges or diverges

8.7 Ratio and Root Tests (SQ7)

Learning Outcomes

- Use the ratio and root tests to determine if a series converges or diverges.

Activity 8.7.1 Consider the series $\sum_{n=0}^{\infty} \frac{2^n}{3^n - 2}$.

(a) Which of these series most closely resembles $\sum_{n=0}^{\infty} \frac{2^n}{3^n - 2}$?

A $\sum_{n=0}^{\infty} \frac{2}{3}$.

B $\sum_{n=0}^{\infty} \frac{2}{3}n$.

C $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$.

(b) Based on your previous choice, do we think this series is more likely to converge or diverge?

(c) Find $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = \lim_{n \rightarrow \infty} \frac{2^{n+1}(3^n-2)}{(3^{n+1}-2)2^n} = \lim_{n \rightarrow \infty} \frac{2 \cdot 2^n(3^n-2)}{3(3^n-\frac{2}{3})2^n}$.

A $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 0$.

B $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = \frac{2}{3}$.

C $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 1$.

D $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 2$.

E $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1}-2}}{\frac{2^n}{3^n-2}} = 3$.

Activity 8.7.2 Consider the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{3}{2^n}$.

(a) Does $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{3}{2^n}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

A 2.

B $\frac{1}{2}$.

C $\frac{2^n}{2^n + 1}$.

D $\frac{9}{2^{2n+1}}$.

E $\frac{9}{2^{n+2}}$.

(c) Find $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

A $-\infty$.

B 0.

C $\frac{1}{2}$.

D 2.

E ∞ .

Activity 8.7.3 Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n+1}$.

(a) Does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n+1}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

A $\frac{n+1}{2}$.

B $\frac{(n^2+1)(n+1)}{(n+2)n^2}$.

C $\frac{(n+1)^2}{n+2}$.

D $\frac{1}{2}$.

E $\frac{(n+1)n^2}{n+2}$.

(c) Find $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

A $-\infty$.

B 0.

C $\frac{1}{2}$.

D 2.

E ∞ .

Activity 8.7.4 Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

(a) Does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

(c) Find $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

Activity 8.7.5 Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(a) Does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

(b) Find $\frac{a_{n+1}}{a_n}$.

(c) Find $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

Fact 8.7.6 The Ratio Test. Let $\sum a_n$ be a series and suppose that

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$. Then

1. $\sum a_n$ converges if ρ is less than 1, and
2. $\sum a_n$ diverges if ρ is greater than 1.
3. If $\rho = 1$, we cannot determine if $\sum a_n$ converges or diverges with this method.

Fact 8.7.7 The Root Test. Let N be an integer and let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$. Then

1. $\sum a_n$ converges if ρ is less than 1, and
2. $\sum a_n$ diverges if ρ is greater than 1.
3. If $\rho = 1$, we cannot determine if $\sum a_n$ converges or diverges with this method.

Activity 8.7.8 Consider the series $\sum_{n=0}^{\infty} \frac{n^2}{n!}$.

(a) Which of the following is a_n ?

A n^2 .B $n!$.C $\frac{n^2}{n!}$.

(b) Which of the following is a_{n+1} ?

A $\frac{n^2}{n!}$.

B $(n+1)^2$.

C $(n+1)!$.

D $\frac{(n+1)^2}{(n+1)!}$.

E $\frac{n^2+1}{n!+1}$.

(c) Which of the following is $\left| \frac{a_{n+1}}{a_n} \right|$?

A $\frac{(n+1)^2 n^2}{(n+1)! n!}$.

B $\frac{(n+1)^2 n!}{(n+1)! n^2}$.

C $\frac{(n+1)! n!}{(n+1)^2 n^2}$.

D $\frac{(n+1)! n^2}{(n+1)^2 n!}$.

(d) Using the fact $(n+1)! = (n+1) \cdot n!$, simplify $\left| \frac{a_{n+1}}{a_n} \right|$ as much as possible.

(e) Find $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

(f) Does $\sum_{n=0}^{\infty} \frac{n^2}{n!}$ converge?

Activity 8.7.9

(a) What is a_n ?

(b) Which of the following is $\sqrt[n]{|a_n|}$?

A $\frac{n+1}{9}$.

B $\frac{n}{9}$.

C n .

D 9.

E $\frac{1}{9}$.

(c) Find $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$.

(d) Does $\sum_{n=1}^{\infty} \frac{n^n}{9^n}$ converge?

Activity 8.7.10 For each series, use the *ratio* or *root* test to determine if the

series converges or diverges.

(a) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n} \right)^n$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

(c) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)(n!)}$

(d) $\sum_{n=1}^{\infty} \frac{4^n (n!)(n!)}{(2n)!}$

Activity 8.7.11 Consider the series $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$.

- (a) Use the root test to check for convergence of this series.
- (b) Use the ratio test to check for convergence of this series.
- (c) Use the comparison (or limit comparison) test to check for convergence of this series.
- (d) Find the sum of this series.

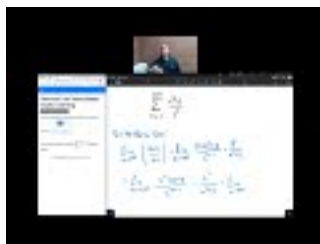
Activity 8.7.12 Consider $\sum_{n=1}^{\infty} \frac{n}{3^n}$. Recall that $\sqrt[n]{\frac{n}{3^n}} = \left(\frac{n}{3^n} \right)^{1/n} = \frac{n^{1/n}}{(3^n)^{1/n}}$.

- (a) Let $\alpha = \lim_{n \rightarrow \infty} \ln(n^{1/n}) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln(n)$. Find α .
- (b) Recall that $\lim_{n \rightarrow \infty} n^{1/n} = \lim_{n \rightarrow \infty} e^{\ln(n^{1/n})} = e^{\alpha}$. Find $\lim_{n \rightarrow \infty} n^{1/n}$.
- (c) Find $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3^n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{3^n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{(3^n)^{1/n}}$.
- (d) Does $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converge?

Activity 8.7.13 Consider the series $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$.

- (a) Use the root test to check for convergence of this series.
- (b) Use the ratio test to check for convergence of this series.
- (c) Use the comparison (or limit comparison) test to check for convergence of this series.

8.7.1 Videos



YouTube: <https://www.youtube.com/watch?v=YmGXeuGeC3I>

Figure 157 Video: Use the ratio and root tests to determine if a series converges or diverges

8.8 Absolute Convergence (SQ8)

Learning Outcomes

- Determine if a series converges absolutely or conditionally.

Activity 8.8.1 Recall the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ from [Activity 8.7.5](#).

- (a) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge or diverge?
- (b) Does the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$ converge or diverge?

Activity 8.8.2 Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

- (a) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge or diverge?
- (b) Does the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right|$ converge or diverge?

Definition 8.8.3 Given a series

$$\sum a_n$$

we say that $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges. \diamond

Activity 8.8.4 Consider the series: $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$.

- (a) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$ converge or diverge? (Recall [Fact 8.7.6](#).)
- (b) Compute $|a_n|$.

(c) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$ converge absolutely?

Fact 8.8.5 Notice that [Fact 8.7.6](#) and [Fact 8.7.7](#) both involve taking absolute values to determine convergence. As such, series that are convergent by either the Ratio Test or the Root Test are also absolutely convergent (by applying the same test after taking the absolute value).

Activity 8.8.6 Consider the series: $\sum_{n=1}^{\infty} -n$.

(a) Does the series $\sum_{n=1}^{\infty} -n$ converge or diverge?

(b) Compute $|a_n|$.

(c) Does the series $\sum_{n=1}^{\infty} -n$ converge absolutely?

Activity 8.8.7 For each of the following series, determine if the series is *convergent*, and if the series is *absolutely convergent*.

(a) $\sum_{n=1}^{\infty} \frac{n^2(-1)^n}{n^3 + 1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$

Activity 8.8.8 If you know a series $\sum a_n$ is absolutely convergent, what can you conclude about whether or not $\sum a_n$ is convergent?

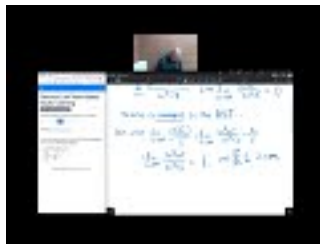
A We cannot determine if $\sum a_n$ is convergent.

B $\sum a_n$ is convergent since it “grows slower” than $\sum |a_n|$ (and falls slower than $\sum -|a_n|$).

Fact 8.8.9 If $\sum a_n$ is absolutely convergent, then it must be convergent.

Activity 8.8.10 Find 3 series that are convergent but not absolutely convergent (recall [Fact 8.5.15](#), [Section 8.6](#)).

8.8.1 Videos



YouTube: <https://www.youtube.com/watch?v=snRKVwCkymQ>

Figure 158 Video: Determine if a series converges absolutely or conditionally

8.9 Series Convergence Strategy (SQ9)

Learning Outcomes

- Identify appropriate convergence tests for various series.

Activity 8.9.1 Which test for convergence is the best first test to apply to any series $\sum_{k=1}^{\infty} a_k$?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

Activity 8.9.2 In which of the following scenarios can we successfully apply the Direct Comparison Test to determine the convergence of the series $\sum a_k$?

- A When we find a convergent series $\sum b_k$ where $0 \leq a_k \leq b_k$
- B When we find a divergent series $\sum b_k$ where $0 \leq a_k \leq b_k$
- C When we find a convergent series $\sum b_k$ where $0 \leq b_k \leq a_k$
- D When we find a divergent series $\sum b_k$ where $0 \leq b_k \leq a_k$

Activity 8.9.3 Which test(s) for convergence would we use for a series $\sum a_k$ where a_k involves k^{th} powers?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

Activity 8.9.4 Which test(s) for convergence would we use for a series of the

form $\sum ar^k$?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

Activity 8.9.5 Which test(s) for convergence would we use for a series $\sum a_k$ where a_k involves factorials and powers?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

Activity 8.9.6 Which test(s) for convergence would we use for a series $\sum a_k$ where a_k is a rational function?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

Activity 8.9.7 Which test(s) for convergence would we use for a series of the form $\sum (-1)^k a_k$?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

Fact 8.9.8 Here is a strategy checklist when dealing with series:

1. The divergence test: unless $a_n \rightarrow 0$, $\sum a_n$ diverges
2. Geometric Series: $\sum ar^k$ converges if $-1 < r < 1$ and diverges otherwise
3. p -series: $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise
4. Series with no negative terms: try the ratio test, root test, integral test, or try to compare to a known series with the comparison test or limit comparison test
5. Series with some negative terms: check for absolute convergence
6. Alternating series: use the alternating series test (Leibniz's Theorem)
7. Anything else: consider the sequence of partial sums, possibly rewriting the series in a different form, hope for the best

Activity 8.9.9 Consider the series $\sum_{k=3}^{\infty} \frac{2}{\sqrt{k}-2}$.

- (a) Which test(s) seem like the most appropriate one(s) to test for conver-

gence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

- | | |
|------------------------------|-----------------------------|
| A This series is convergent. | B This series is divergent. |
|------------------------------|-----------------------------|

Activity 8.9.10 Consider the series $\sum_{k=1}^{\infty} \frac{k}{1+2k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

- | | |
|------------------------------|-----------------------------|
| A This series is convergent. | B This series is divergent. |
|------------------------------|-----------------------------|

Activity 8.9.11 Consider the series $\sum_{k=0}^{\infty} \frac{2k^2+1}{k^3+k+1}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

- | | |
|------------------------------|-----------------------------|
| A This series is convergent. | B This series is divergent. |
|------------------------------|-----------------------------|

Activity 8.9.12 Consider the series $\sum_{k=0}^{\infty} \frac{100^k}{k!}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

A This series is convergent.

B This series is divergent.

Activity 8.9.13 Consider the series $\sum_{k=1}^{\infty} \frac{2^k}{5^k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A Divergence Test

E Limit Comparison Test

B Geometric Series

F Ratio Test

C Integral Test

G Root Test

D Direct Comparison Test

H Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A This series is convergent.

B This series is divergent.

Activity 8.9.14 Consider the series $\sum_{k=1}^{\infty} \frac{k^3 - 1}{k^5 + 1}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A Divergence Test

E Limit Comparison Test

B Geometric Series

F Ratio Test

C Integral Test

G Root Test

D Direct Comparison Test

H Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A This series is convergent.

B This series is divergent.

Activity 8.9.15 Consider the series $\sum_{k=2}^{\infty} \frac{3^{k-1}}{7^k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

A Divergence Test

E Limit Comparison Test

B Geometric Series

F Ratio Test

C Integral Test

G Root Test

D Direct Comparison Test

H Alternating Series Test

(b) Apply an appropriate test to determine the convergence of this series.

A This series is convergent.

B This series is divergent.

Activity 8.9.16 Consider the series $\sum_{k=2}^{\infty} \frac{1}{k^k}$.

(a) Which test(s) seem like the most appropriate one(s) to test for conver-

gence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

- | | |
|------------------------------|-----------------------------|
| A This series is convergent. | B This series is divergent. |
|------------------------------|-----------------------------|

Activity 8.9.17 Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

- | | |
|------------------------------|-----------------------------|
| A This series is convergent. | B This series is divergent. |
|------------------------------|-----------------------------|

Activity 8.9.18 Consider the series $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$.

(a) Which test(s) seem like the most appropriate one(s) to test for convergence or divergence?

- | | |
|--------------------------|---------------------------|
| A Divergence Test | E Limit Comparison Test |
| B Geometric Series | F Ratio Test |
| C Integral Test | G Root Test |
| D Direct Comparison Test | H Alternating Series Test |

(b) Apply an appropriate test to determine the convergence of this series.

- | | |
|------------------------------|-----------------------------|
| A This series is convergent. | B This series is divergent. |
|------------------------------|-----------------------------|

Activity 8.9.19 Determine which of the following series is *convergent* and which is *divergent*. Justify both choices with an appropriate test.

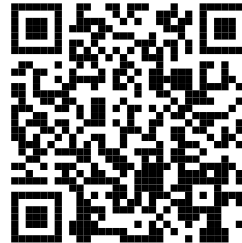
- $\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}n^2}{2n^3 + 4n^2 + 5}$.
- $\sum_{n=1}^{\infty} \frac{n!}{3 \cdot 3^n n^4}$.

Answer.

- $\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}n^2}{2n^3 + 4n^2 + 5}$ is convergent.

2. $\sum_{n=1}^{\infty} \frac{n!}{3 \cdot 3^n n^4}$ is divergent.

8.9.1 Videos



YouTube: <https://www.youtube.com/watch?v=YGzI0qR5GVE>

Figure 159 Video: Identify appropriate convergence tests for various series

Chapter 9

Power Series (PS)

Learning Outcomes

How do we use various techniques to integrate less simple functions?

By the end of this chapter, you should be able to...

1. Approximate functions defined as power series.
2. Determine the interval of convergence for a given power series.
3. Compute power series by manipulating known exponential/trigonometric/binomial power series.
4. Determine a Taylor or Maclaurin series for a function.

Readiness Assurance. Before beginning this chapter, you should be able to...

- a Compute limits.
 - Review: [Khan Academy](#)¹
 - Review: [Khan Academy](#)²
- b Express the sum of indexed values using summation notation.
 - Review: [Khan Academy](#)³
- c Employ the ratio test for convergence.
 - Review: [Khan Academy](#)⁴
- d Power rule for differentiating and integrating.
 - Review: [Khan Academy](#)⁵

¹<https://www.khanacademy.org/math/ap-calculus-bc/bc-limits-new/bc-1-2/v/introduction-to-limits-hd?modal=1>

²<https://www.khanacademy.org/math/ap-calculus-bc/bc-diff-contextual-applications-new/bc-4-7/v/introduction-to-l-hopital-s-rule?modal=1>

³<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/sigma-notation-sum>

⁴<https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-8/v/ratio-test-convergence>

⁵<https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-1-new/ab-2-5/v/power-rule>

- Review: [Khan Academy](https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-8a/v/indefinite-integrals-of-x-raised-to-a-power)⁶

9.1 Power Series (PS1)

Learning Outcomes

- Approximate functions defined as power series.

Activity 9.1.1 Consider the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ where x is a real number.

(a) If $x = 2$, then $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}$. What can be said about this series?

A The techniques we have learned so far allow us to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} \text{ converges.}$$

B The techniques we have learned so far allow us to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} \text{ diverges.}$$

C None of the techniques we have learned so far allow us to conclude

$$\text{whether } \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} \text{ converges or diverges.}$$

(b) If $x = -100$, then $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!}$. What can be said about this series?

A The techniques we have learned so far allow us to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!} \text{ converges.}$$

B The techniques we have learned so far allow us to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!} \text{ diverges.}$$

C None of the techniques we have learned so far allow us to conclude

$$\text{whether } \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!} \text{ converges or diverges.}$$

(c) Suppose that x were some arbitrary real number. What can be said about this series?

A The techniques we have learned so far allow us to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n \text{ converges.}$$

B The techniques we have learned so far allow us to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n \text{ diverges.}$$

⁶<https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-8a/v/indefinite-integrals-of-x-raised-to-a-power>

C None of the techniques we have learned so far allow us to conclude

whether $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges or diverges.

Activity 9.1.2 Since $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for each real value x , we can define

a function $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ which assigns to each input x the sum $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

(a) To estimate $f(2)$, we consider $f_5(x) = \sum_{n=0}^5 \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$. What is $f_5(2)$?

A $f_5(2) = 4 \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \right) = 4 + 4x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{30}x^5$.

B $f_5(2) = 1 + 2 + \frac{1}{2}2 + \frac{1}{6}2 + \frac{1}{24}2 + \frac{1}{120}2 = \frac{133}{30} \approx 4.4333$.

C $f_5(2) = 1 + 2 + \frac{1}{2}2^2 + \frac{1}{6}2^3 + \frac{1}{24}2^4 + \frac{1}{120}2^5 = \frac{109}{15} \approx 7.2667$.

(b) To which of these values is $f_5(2)$ the closest?

A $(2)^2$.

B $\sqrt{2}$.

C e^2 .

D $\ln(2)$.

E $\sin(2)$.

F $\cos(2)$.

(c) Consider the plot of $y = f_5(x)$:

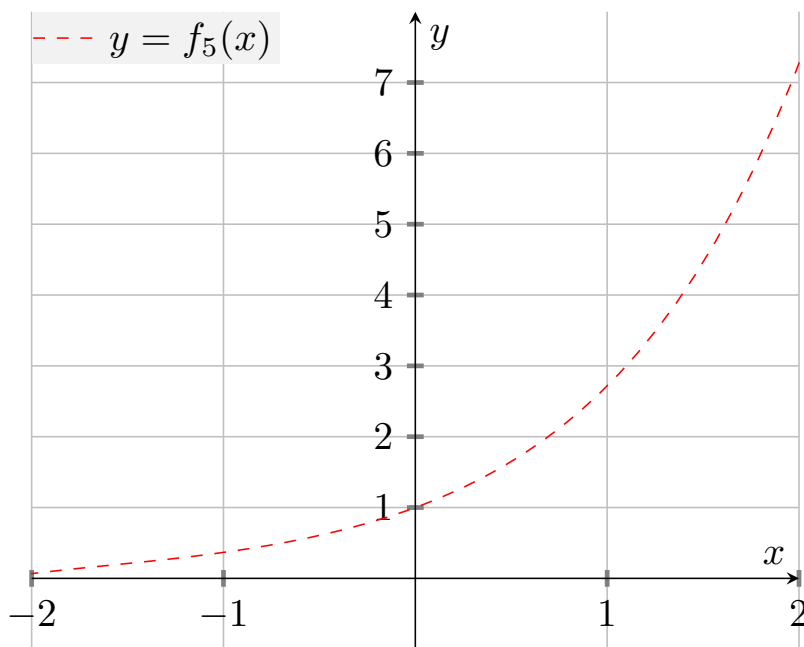


Figure 160 Plot of $f_5(x)$.

Which of the following plots does the plot $y = f_5(x)$ most resemble?

A $y = x^2$.

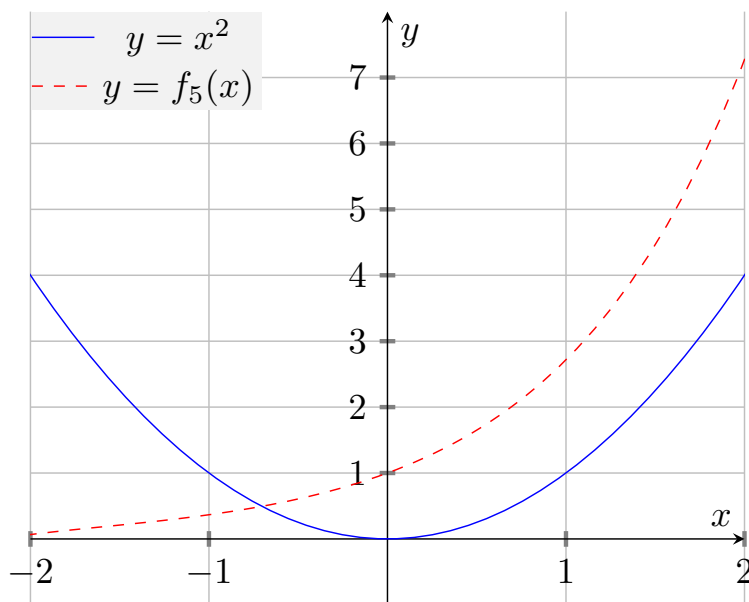


Figure 161 Plots of $y = f_5(x)$, $y = x^2$.

B $y = \sqrt{x}$.

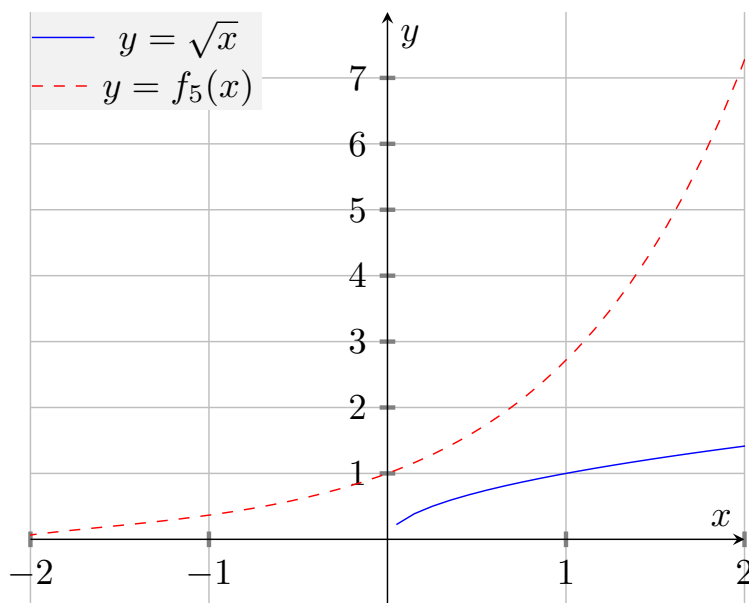


Figure 162 Plots of $y = f_5(x)$, $y = \sqrt{x}$.

C $y = e^x$.

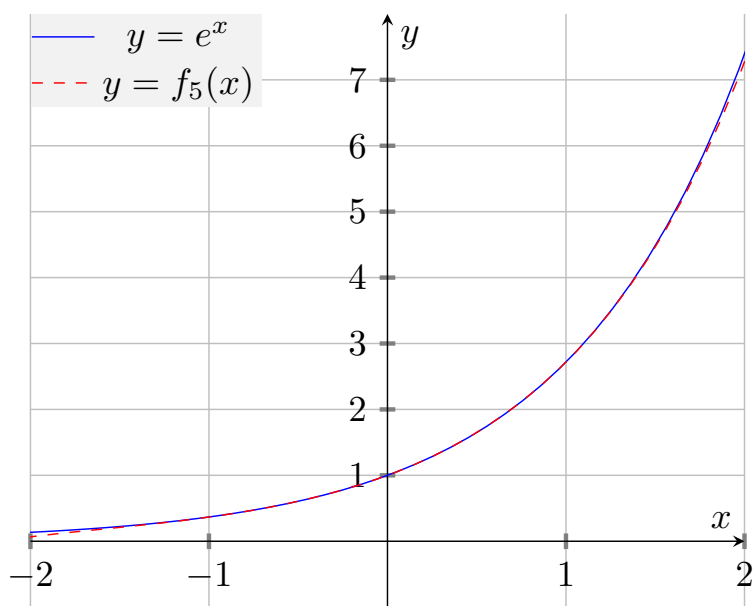


Figure 163 Plots of $y = f_5(x)$, $y = e^x$.

D $y = \ln x$.

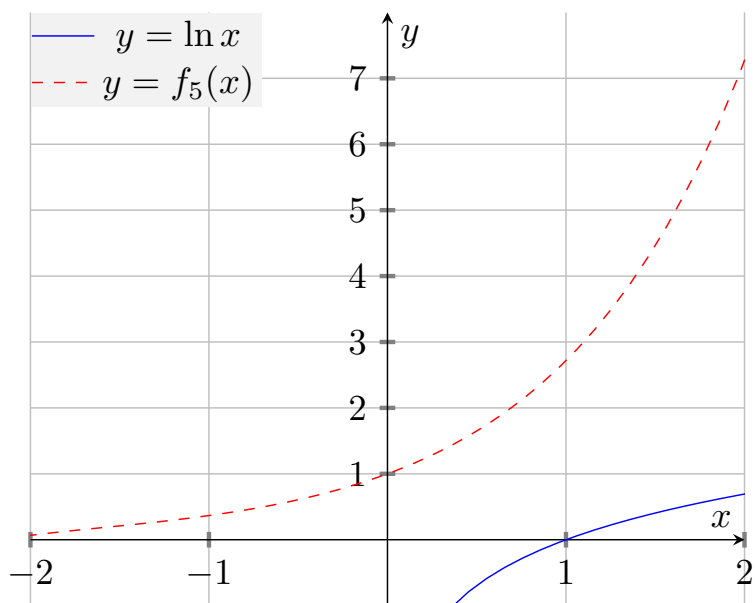


Figure 164 Plots of $y = f_5(x)$, $y = \ln x$.

E $y = \sin x$.

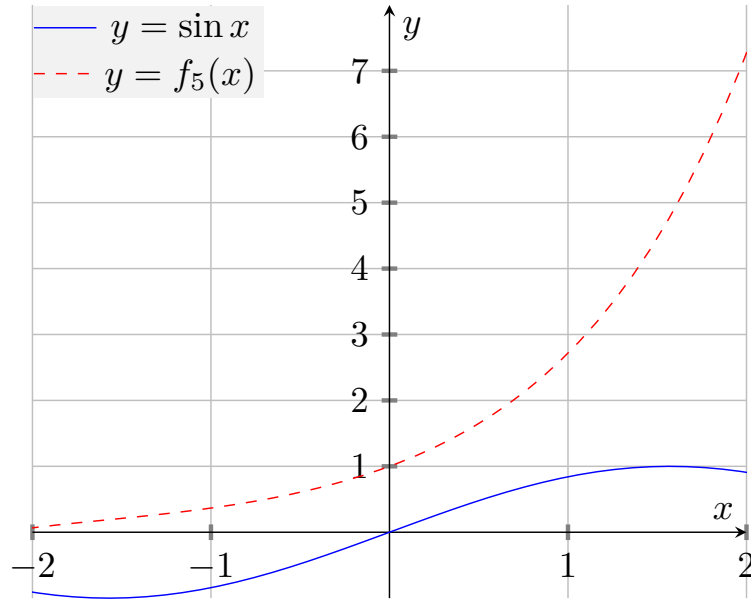


Figure 165 Plots of $y = f_5(x)$, $y = \sin x$.

F $y = \sin x$.

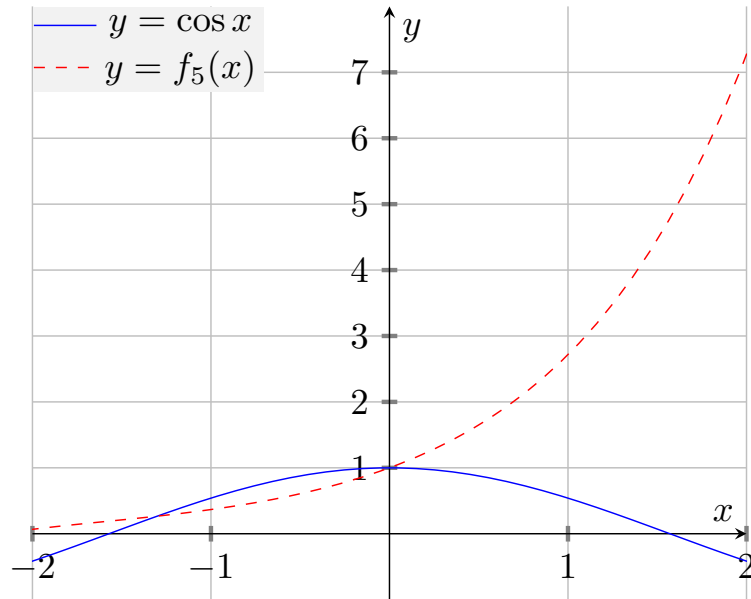


Figure 166 Plots of $y = f_5(x)$, $y = \cos x$.

Definition 9.1.3 For a real number c a *power series* centered at c is a series $\sum_{n=0}^{\infty} a_n(x - c)^n$ where each coefficient a_n is a real number.

For the values of x where $\sum_{n=0}^{\infty} a_n(x - c)^n$ converges, we can define a function

$p(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$. We can define $p_k(x) = \sum_{n=0}^k a_n(x - c)^n$ to be the degree k polynomial approximation of $p(x)$. \diamond

Example 9.1.4 Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. We could compute $\frac{d}{dx} \left[\sum_{n=0}^{\infty} a_n x^n \right]$ by:

$$\frac{d}{dx} \left[\sum_{n=0}^{\infty} a_n x^n \right] = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

(Then since when $n = 0$, $a_0(0)x^{-1} = 0$)

$$= \sum_{n=1}^{\infty} a_n n x^{n-1}$$

(Then let $m = n - 1$, $n = m + 1$)

$$= \sum_{m=0}^{\infty} a_{m+1} (m+1) x^m$$

(Then renaming indices.)

$$= \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n$$

We could also compute $\frac{d}{dx} \left[\sum_{n=0}^{\infty} a_n x^n \right]$ by:

$$\begin{aligned} \frac{d}{dx} \left[\sum_{n=0}^{\infty} a_n x^n \right] &= \frac{d}{dx} [a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots] \\ &= a_1 + a_2 2x + a_3 3x^2 + \cdots a_n n x^{n-1} + a_{n+1} (n+1) x^n + \cdots \\ &= \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n \end{aligned}$$

□

Activity 9.1.5

(a) Given $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, which of the following best represents $f'(x) =$

$$\frac{d}{dx} \left[\sum_{n=0}^{\infty} \frac{1}{n!} x^n \right] ?$$

A $\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$

B $\sum_{n=0}^{\infty} \frac{1}{n!} x^{n-1}.$

C $n \sum_{n=0}^{\infty} \frac{1}{n!} x^{n-1}.$

D $n \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^{n-1}.$

E $n \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1}.$

(b) For which of the following is $y = \frac{dy}{dx}$?

- A $y = x^2$.
- B $y = \sqrt{x}$.
- C $y = e^x$.
- D $y = \ln(x)$.
- E $y = \sin(x)$.
- F $y = \cos(x)$.

Fact 9.1.6 We have that

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x.$$

What we mean by equality is that for any real number x , the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ will converge to e^x .

Activity 9.1.7 Estimate e^{-3} using $e^x \approx f_6(x) = \sum_{n=0}^6 \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6$ by evaluating $f_6(-3)$.

Example 9.1.8 Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. We could compute $\int \sum_{n=0}^{\infty} a_n x^n dx$ by:

$$\begin{aligned} \int \sum_{n=0}^{\infty} a_n x^n dx &= C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \\ (\text{Letting } m = n+1, n = m-1) \\ &= C + \sum_{m=1}^{\infty} \frac{a_{m-1}}{m} x^m \\ (\text{Then, renaming}) \\ &= C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n \end{aligned}$$

We could also compute $\int \sum_{n=0}^{\infty} a_n x^n dx$ by:

$$\begin{aligned} \int \sum_{n=0}^{\infty} a_n x^n dx &= \int (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots) dx \\ &= C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_{n-1}}{n} x^n + \frac{a_n}{n+1} x^{n+1} + \cdots \\ &= C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n \end{aligned}$$

□

Activity 9.1.9 Consider a function $p(x)$ defined by $p(x) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^n$.

- (a) Find $p_4(x)$, the degree 4 polynomial approximation for $p(x)$.

(b) Use $p_4(x)$ to estimate $p(-1)$ by computing $p_4(-1)$.

(c) Use $p_4(x)$ to estimate $\int_3^6 p(x) dx$ by computing $\int_3^6 p_4(x) dx$.

9.1.1 Videos



YouTube: <https://www.youtube.com/watch?v=MpJKiKSVIc0>

Figure 167 Video: Approximate functions defined as power series

9.2 Convergence of Power Series (PS2)

Learning Outcomes

- Determine the interval of convergence for a given power series.

Remark 9.2.1 Consider a power series $\sum c_n(x-a)^n$. Recall from [Fact 8.7.6](#) that if

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| < 1$$

then $\sum c_n(x-a)^n$ converges.

Then recall:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-a)}{c_n} \right| \\ &= \lim_{n \rightarrow \infty} |x-a| \left| \frac{c_{n+1}}{c_n} \right| \\ &= |x-a| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|. \end{aligned}$$

Activity 9.2.2 Consider $\sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n$.

(a) Letting $c_n = \frac{1}{n^2+1}$ find $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$.

(b) For what values of x is $|x| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$?

A $x < 1$.

B $0 \leq x < 1$.

C $-1 < x < 1$.

(c) If $x = 1$, does $\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} x^n$ converge?

(d) If $x = -1$, does $\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} x^n$ converge?

(e) Which of the following describe the values of x for which $\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} x^n$ converges?

A $(-1, 1)$.

B $[-1, 1)$.

C $(-1, 1]$.

D $[-1, 1]$.

Activity 9.2.3 Consider $\sum_{n=0}^{\infty} \frac{2^n}{5^n} (x - 2)^n$.

(a) Letting $c_n = \frac{2^n}{5^n}$ find $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$.

(b) For what values of x is $|x - 2| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$?

A $-\frac{2}{5} < x < \frac{2}{5}$.

B $\frac{8}{5} < x < \frac{12}{5}$.

C $-\frac{5}{2} < x < \frac{5}{2}$.

D $-\frac{1}{2} < x < \frac{9}{2}$.

(c) If $x = \frac{9}{2}$, does $\sum_{n=0}^{\infty} \frac{2^n}{5^n} (x - 2)^n$ converge?

(d) If $x = -\frac{1}{2}$, does $\sum_{n=0}^{\infty} \frac{2^n}{5^n} (x - 2)^n$ converge?

(e) Which of the following describe the values of x for which $\sum_{n=0}^{\infty} \frac{2^n}{5^n} (x - 2)^n$ converges?

A $(-\frac{1}{2}, \frac{9}{2})$.

B $[-\frac{1}{2}, \frac{9}{2})$.

C $(-\frac{1}{2}, \frac{9}{2}]$.

D $[-\frac{1}{2}, \frac{9}{2}]$.

Activity 9.2.4 Consider $\sum_{n=0}^{\infty} \frac{n^2}{n!} \left(x + \frac{1}{2}\right)^n$.

(a) Letting $c_n = \frac{n^2}{n!}$ find $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$.

(b) For what values of x is $\left|x + \frac{1}{2}\right| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$?

A $0 \leq x < \infty$.

B All real numbers.

(c) What describes the values of x for which $\sum_{n=0}^{\infty} \frac{n^2}{n!} \left(x + \frac{1}{2}\right)^n$ converges?

Fact 9.2.5 Given the power series $\sum c_n(x-a)^n$, the center of convergence is $x = a$. The radius of convergence is

$$r = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|}.$$

If $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = 0$, we say that $r = \infty$.

The interval of convergence represents all possible values of x for which $\sum c_n(x-a)^n$ converges, which is of the form:

- $(a-r, a+r)$
- $[a-r, a+r)$
- $(a-r, a+r]$
- $[a-r, a+r]$

Depending on if $\sum c_n(x-a)^n$ converges when $x = a-r$ or $x = a+r$. If $r = \infty$ the interval of convergence is all real numbers.

Activity 9.2.6 Find the center of convergence, radius of convergence, and interval of convergence for the series:

$$\sum_{n=0}^{\infty} \frac{3^n (-1)^n (x-1)^n}{n!}.$$

Answer.

- The center of convergence is 1.
- The radius of convergence is $+\infty$.
- The interval of convergence is $(-\infty, +\infty)$.

Activity 9.2.7 Find the center of convergence, radius of convergence, and interval of convergence for the series:

$$\sum_{n=0}^{\infty} \frac{3^n (x+2)^n}{n}.$$

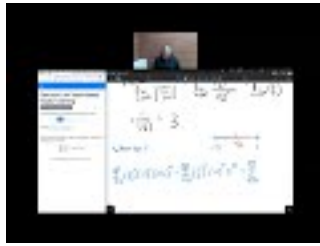
Answer.

- The center of convergence is -2 .
- The radius of convergence is $\frac{1}{3}$.
- The interval of convergence is $[-\frac{7}{3}, -\frac{5}{3})$.

Activity 9.2.8 Consider the power series $\sum_{n=0}^{\infty} \frac{2^n + 1}{n3^n} (x + 1)^n$.

- What is the center of convergence for this power series?
- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?
- If $x = -0.5$, does this series converge? (Use the interval of convergence.)
- If $x = 1$, does this series converge? (Use the interval of convergence.)

9.2.1 Videos



YouTube: <https://www.youtube.com/watch?v=npXONByK-GQ>

Figure 168 Video: Determine the interval of convergence for a given power series

9.3 Manipulation of Power Series (PS3)

Learning Outcomes

- Compute power series by manipulating known exponential/trigonometric/binomial power series.

Activity 9.3.1 Suppose we wished to find the power series for the function $f(x) = e^{2x}$. Recall that $\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- Which of the following represents the power series for $\exp(a) = e^a$?

A $\exp(a) = \sum_{n=0}^{\infty} a \frac{x^n}{n!}$.

B $\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$.

C $\exp(a) = \sum_{n=0}^{\infty} \frac{x^n}{a!}$.

D $\exp(a) = \sum_{a=0}^{\infty} \frac{x^n}{n!}$.

- What is the interval of convergence for a in the series you found in part (a)?

A $(0, 1)$.

- B $(-1, 1)$.
- C $[-1, 1]$.
- D $[0, 1]$.
- E $(-\infty, \infty)$.

(c) Evaluating $a = 2x$, what is the power series for $\exp(a) = \exp(2x) = f(x)$?

- A $\exp(2x) = \sum_{n=0}^{\infty} \frac{2x^n}{n!}$.
- B $\exp(2x) = \sum_{2x=0}^{\infty} \frac{x^n}{n!}$.
- C $\exp(2x) = \sum_{n=0}^{\infty} 2x \frac{x^n}{n!}$.
- D $\exp(2x) = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$.
- E $\exp(2x) = \sum_{n=0}^{\infty} \frac{x^n}{(2x)!}$.

(d) What is the interval of convergence for x in the series you found in part (c)?

- A $(-\infty, \infty)$.
- B $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
- C $\left(0, \frac{1}{2}\right)$.
- D $\left[0, \frac{1}{2}\right]$.
- E $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

Activity 9.3.2 Suppose we wished to find the power series for the function

$$f(x) = \frac{1}{x}. \text{ Recall that } g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

(a) Which of the following represents the power series for $\frac{1}{1-a}$?

- A $g(a) = \sum_{n=0}^{\infty} ax^n$.
- B $g(a) = \sum_{n=0}^{\infty} (ax)^n$.
- C $g(a) = \sum_{n=0}^{\infty} a^n$.
- D $g(a) = \sum_{a=0}^{\infty} x^a$.

(b) What is the radius of convergence for a in the series you found in part (a)?

- A $(0, 1)$.
- B $(-1, 1)$.
- C $[-1, 1]$.
- D $[0, 1]$.
- E $(-\infty, \infty)$.

(c) For what value of a is $\frac{1}{1-a} = \frac{1}{x}$?

- A $a = x - 1$.
- B $a = 1 - x$.
- C $a = x + 1$.
- D $a = -x$.
- E $a = x$.

(d) Evaluating a at the previously found value, which of the following is the power series of $f(x) = \frac{1}{x}$?

- A $f(x) = \sum_{n=0}^{\infty} x^n$.
- B $f(x) = \sum_{n=0}^{\infty} (-x)^n$.
- C $f(x) = \sum_{n=0}^{\infty} (1-x)^n$.
- D $f(x) = \sum_{n=0}^{\infty} (x-1)^n$.
- E $f(x) = \sum_{n=0}^{\infty} (1+x)^n$.

(e) Given that the interval of convergence for a is $-1 < a < 1$, what is the interval of convergence for x ?

- A $-1 < x < 1$.
- B $-2 < x < 0$.
- C $-2 < x < 2$.
- D $0 < x < 2$.
- E $2 < x < 0$.

Activity 9.3.3 Suppose we wished to find the power series for the function

$$f(x) = \frac{1}{3-2x}. \text{ Recall that } g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

(a) For what value of a is $\frac{1}{1-a} = \frac{1}{3-2x}$?

- A $a = 2x - 2$.
- B $a = 2 - 2x$.
- C $a = 2x - 3$.
- D $a = x$.
- E $a = 3 - 2x$.

- (b) Evaluating a at the previously found value, which of the following is the power series of $f(x) = \frac{1}{3-2x}$?

A $f(x) = \sum_{n=0}^{\infty} (3-2x)^n.$

B $f(x) = \sum_{n=0}^{\infty} (2x-3)^n.$

C $f(x) = \sum_{n=0}^{\infty} (2-2x)^n.$

D $f(x) = \sum_{n=0}^{\infty} (2x-2)^n.$

E $f(x) = \sum_{n=0}^{\infty} x^n.$

- (c) Given that the interval of convergence for a is $-1 < a < 1$, what is the interval of convergence for x ?

A $-1 < x < 1.$

B $-1 < x < \frac{3}{2}.$

C $-\frac{1}{2} < x < 1.$

D $\frac{1}{2} < x < \frac{3}{2}.$

E $-\frac{1}{2} < x < \frac{3}{2}.$

Activity 9.3.4 Suppose we wished to find the power series for the function

$$f(x) = \frac{1}{1+x^2}. \text{ Recall that } g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

- (a) For what value of a is $\frac{1}{1-a} = \frac{1}{1+x^2}$?

A $a = x^2.$

B $a = -x^2.$

C $a = 1 + x^2.$

D $a = 1 - x^2.$

E $a = x^2 - 1.$

- (b) Evaluating a at the previously found value, which of the following is the power series of $f(x) = \frac{1}{1+x^2}$?

A $f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$

B $f(x) = \sum_{n=0}^{\infty} (1+x^2)^n.$

C $f(x) = \sum_{n=0}^{\infty} (1-x^2)^n.$

D $f(x) = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$

$$\text{E } f(x) = \sum_{n=0}^{\infty} (x^2 - 1)^n.$$

(c) Given that the interval of convergence for a is $-1 < a < 1$, what is the interval of convergence for x ?

A $-1 < x < 1$.

B $-1 < x < 0$.

C $0 < x < 1$.

D $0 < x < 2$.

E $-2 < x < 0$.

F $-2 < x < 2$.

Activity 9.3.5 What function $f(x)$ has power series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} =$

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots?$$

A $f(x) = (-1)^n e^x$.

B $f(x) = e^{-1}$.

C $f(x) = -e^x$.

D $f(x) = e^{-x}$.

E $f(x) = -e^{-x}$.

Activity 9.3.6 What function $f(x)$ has power series $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!} =$

$$x^3 + x^4 + \frac{x^5}{2} + \frac{x^6}{6} + \cdots?$$

A $f(x) = e^{x+3}$.

B $f(x) = e^{x^3}$.

C $f(x) = e^{3x}$.

D $f(x) = x^3 e^x$.

E $f(x) = 3e^x$.

Activity 9.3.7 What function $f(x)$ has power series $f(x) = \sum_{n=3}^{\infty} x^n = x^3 +$

$$x^4 + \cdots?$$

A $f(x) = \frac{1}{1-3x}$.

B $f(x) = \frac{3}{1-x}$.

C $f(x) = \frac{1}{1-x} - x^2 - x - 1$.

D $f(x) = \frac{x^3}{1-x}$.

Activity 9.3.8 The function $n(x) = e^{-x^2}$ is one whose integrals are very impor-

tant for statistics. However, it does not admit an elementary anti-derivative.

(a) Which of the following best represents the power series for $n(x) = e^{-x^2}$?

A $n(x) = -x^2 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} -\frac{1}{n!} x^{n+2}.$

B $n(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n x^{2n}.$

C $n(x) = x^{-2} \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{n+2} x^{n+2}.$

(b) Which of the following best represents a degree 10 polynomial that approximates $n(x)$?

A $n_{10}(x) = -x^2 - x^3 - \frac{1}{2}x^4 - \frac{1}{6}x^5 - \frac{1}{24}x^6 - \frac{1}{120}x^7 - \frac{1}{720}x^8 - \frac{1}{5040}x^9 - \frac{1}{40320}x^{10}.$

B $n_{10}(x) = x^2 - x^3 + \frac{1}{2}x^4 - \frac{1}{6}x^5 + \frac{1}{24}x^6 - \frac{1}{120}x^7 + \frac{1}{720}x^8 - \frac{1}{5040}x^9 + \frac{1}{40320}x^{10}.$

C $n_{10}(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}.$

(c) Use your choice of $n_{10}(x)$ to estimate $\int_0^1 n(x)dx$ by computing $\int_0^1 n_{10}(x)dx$.

Activity 9.3.9 Recall that

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $-1 < x < 1$.

(a) Which of the following represents an antiderivative of $g(x) = \frac{1}{1-x}$?

A $G(x) = C + \sum_{n=0}^{\infty} x^{n+1}.$

B $G(x) = C + \sum_{n=1}^{\infty} \frac{1}{n} x^{n+1}.$

C $G(x) = C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}.$

D $G(x) = C + \sum_{n=1}^{\infty} \frac{1}{n+1} x^n.$

(b) Find the interval of convergence for this series.

(c) Recall that $\tilde{G}(x) = \ln|1-x|$ is an antiderivative of $g(x) = \frac{1}{1-x}$. For which C is your chosen $G(x) = \ln|1-x|$?

(d) Use $G_4(x)$ to estimate $\int_2^4 \ln|1-x|dx$.

Activity 9.3.10 Recall that the power series for $f(x) = \sin(x)$ is:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

1. Find a power series for $\sin(-5x^2)$.
2. Find a power series for $x^4 \sin(x)$.
3. Find a power series for $F(x)$, an antiderivative of $f(x)$ such that $F(0) = 4$.

Answer.

$$1. \sin(-5x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (-5x^2)^{2n+1}}{(2n+1)!}.$$

$$2. x^4 \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{(2n+1)!}.$$

$$3. F(x) = 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}.$$

Activity 9.3.11 Recall that the power series for $f(x) = -\frac{1}{x-1}$ is:

$$-\frac{1}{x-1} = \sum_{n=0}^{\infty} x^n.$$

1. Find a power series for $\frac{1}{x^4+1}$.
2. Find a power series for $-\frac{x^5}{x-1}$.
3. Find a power series for $f'(x)$.

Answer.

$$1. \frac{1}{x^4+1} = \sum_{n=0}^{\infty} (-x^4)^n.$$

$$2. -\frac{x^5}{x-1} = \sum_{n=0}^{\infty} x^{n+5}.$$

$$3. f'(x) = \sum_{n=0}^{\infty} (n+1)x^n.$$

Activity 9.3.12 Recall that

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $-1 < x < 1$ and $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2} = g(-x^2)$. We computed the power series for $g(-x^2)$ in [Activity 9.3.4](#).

- (a) Anti differentiate this power series and find C to find a power series for $H(x) = \arctan(x)$. Recall that $\arctan(0) = 0$.
- (b) Find the interval of convergence for this series.

Activity 9.3.13

1. Find the power series for $\alpha(x) = \ln|x|$.
2. Find the interval of convergence for this series.

Activity 9.3.14

1. Find the power series for $\beta(x) = \arctan(-3x^2)$.
2. Find the interval of convergence for this series.

9.3.1 Videos

YouTube: <https://www.youtube.com/watch?v=39g1YKi1cnI>

Figure 169 Video: Compute power series by manipulating known exponential/trigonometric/binomial power series

9.4 Taylor Series (PS4)**Learning Outcomes**

- Determine a Taylor or Maclaurin series for a function.

Activity 9.4.1

- (a) Find the 2nd derivative of $f(x) = x^2$.
- (b) Find the 4th derivative of $f(x) = x^4$.
- (c) Find the 9th derivative of $f(x) = x^9$.
- (d) Which of the following represent the n th derivative of $f(x) = x^n$? (Make sure it is consistent with $n = 2, 4, 9$ that you found above.)
 - A n .
 - B $n!$.
 - C x .
 - D nx .
- (e) What then is the $n+1$ st derivative of $f(x) = x^n$? (We are differentiating with respect to x .)

Activity 9.4.2 Suppose $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $p(0) = 2, p'(0) = 1, p''(0) = -2, p'''(0) = 4$.

- (a) What must a_0 be for $p(0) = 2$?
- (b) Using your choice of a_0 , what must a_1 be for $p'(0) = 1$?
- (c) Using your choice of a_0, a_1 , what must a_2 be for $p''(0) = -2$?

(d) Using your choice of a_0, a_1, a_2 , what must a_3 be for $p'''(0) = 4$?

(e) What is $p(x)$?

Activity 9.4.3 Suppose f is a function. Then if $f(x)$ can be represented as a power series, there is a real number c and an interval, I , centered at $x = c$ such that $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$ on I .

(a) Use term-by-term differentiation to find a power series for $f'(x)$.

(b) Use this power series to find $f'(c)$.

(c) Use term-by-term differentiation to find a power series for $f''(x)$.

(d) Use this power series to find $f''(c)$.

(e) Use term-by-term differentiation to find a power series for $f'''(x)$.

(f) Use this power series to find $f'''(c)$.

(g) Deduce that $f^{(n)}(c) = n!a_n$ and hence $a_n = \frac{f^{(n)}(c)}{n!}$.

Fact 9.4.4 If $f(x)$ can be written as a power series, then there is a real number c such that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \dots$ on some interval centered at $x = c$.

Fact 9.4.5 A function can be represented as a power series if and only if it is infinitely differentiable on some open interval.

Definition 9.4.6 The *Taylor series* generated by $f(x)$ and centered at $x = c$ is given by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \dots$, along with its interval of convergence (or *interval of validity*).

When $c = 0$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x^n) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x^2) + \frac{f^{(3)}(0)}{3!}(x^3) + \dots$ is called the *Maclaurin series* generated by f . \diamond

Definition 9.4.7 $f(x) \approx T_k(x) = \sum_{n=0}^k \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \dots + \frac{f^{(k)}(c)}{k!}(x-c)^k$, where $T_k(x)$ is called the k^{th} degree *Taylor polynomial* generated by f and centered at $x = c$.

The k^{th} degree Taylor polynomial can be seen as the “best” polynomial of degree k or less for approximating $f(x)$ for values close to $x = c$. Note that polynomials of degree less than k are their own Taylor polynomials, and the 1^{st} degree Taylor polynomial is also known as the *linearization* of f . \diamond

Activity 9.4.8 Let $f(x)$ be a function such that:

$f(4)$	$f'(4)$	$f''(4)$	$f'''(4)$	$f^{(4)}(4)$	$f^{(5)}(4)$	$f^{(6)}(4)$
0	1	2	3	4	5	6

1. Find a Taylor polynomial for $f(x)$ centered at $x = 4$ of degree 3.
2. Using the table above, find a general closed form for $f^{(n)}(4)$.
3. Use (b) to find a Taylor series for $f(x)$ centered at $x = 4$.

Answer.

1. $f_3(x) = \frac{1}{2}(x-4)^3 + (x-4)^2 + x - 4$.
2. $f^{(n)}(4) = n$.
3. $f(x) = \sum_{n=0}^{\infty} \frac{n}{n!}(x-4)^n$.

Activity 9.4.9 Let $f(x)$ be a function such that:

$f(-2)$	$f'(-2)$	$f''(-2)$	$f'''(-2)$	$f^{(4)}(-2)$	$f^{(5)}(-2)$	$f^{(6)}(-2)$
0	2	-16	54	-128	250	-432

1. Find a Taylor polynomial for $f(x)$ centered at $x = -2$ of degree 3.
2. Using the table above, find a general closed form for $f^{(n)}(-2)$.
3. Use (b) to find a Taylor series for $f(x)$ centered at $x = -2$.

Answer.

1. $f_3(x) = 9(x+2)^3 - 8(x+2)^2 + 2x + 4$.
2. $f^{(n)}(-2) = -2(-1)^n n^3$.
3. $f(x) = \sum_{n=0}^{\infty} \frac{-2(-1)^n n^3}{n!}(x+2)^n$.

Activity 9.4.10

- (a) Find the Maclaurin series for $f(x) = \sin(x)$ about $x = 0$, along with the interval of validity.
- (b) Find the Taylor series generated by $f(x) = \cos(x)$ about $x = 0$.
- (c) Find the Maclaurin series for $f(x) = e^x$ about $x = 0$, along with the interval of validity.

Remark 9.4.11 You might have seen $\sqrt{-1}$ written as i , and know that z is a complex number if $z = a + bi$ for some real numbers a and b . Note that $i^2 = -1$, $i^3 = (i^2)i = -i$, $i^4 = (i^2)^2 = 1$, $i^5 = (i^4)i = i$, and so on. This gives rise to the following notion.

Definition 9.4.12 Euler's Identity. For any real number θ ,

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!} + \dots$$

$$\begin{aligned}
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}$$

◇

Activity 9.4.13 Use Euler's identity to evaluate $e^{i\pi}$.

9.4.1 Videos



YouTube: <https://www.youtube.com/watch?v=bjktfFCafGs>

Figure 170 Video: Determine a Taylor or Maclaurin series for a function

Appendix A

A Short Table of Integrals

- a. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$
- b. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}| + C$
- c. $\int \sqrt{u^2 \pm a^2} du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln|u + \sqrt{u^2 \pm a^2}| + C$
- d. $\int \frac{u^2 du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2} \sqrt{u^2 \pm a^2} \mp \frac{a^2}{2} \ln|u + \sqrt{u^2 \pm a^2}| + C$
- e. $\int \frac{du}{u\sqrt{u^2 + a^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C$
- f. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{u}{a}\right) + C$
- g. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
- h. $\int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin\left(\frac{u}{a}\right) + C$
- i. $\int \frac{u^2}{\sqrt{a^2 - u^2}} du = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin\left(\frac{u}{a}\right) + C$
- j. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$
- k. $\int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C$

Appendix B

List of Trigonometric Identities

a. $[\sin(\alpha)]^2 + [\cos(\alpha)]^2 = 1$

b. $\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$

c. $\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$

d. $\cos(2\alpha) = [\cos(\alpha)]^2 - [\sin(\alpha)]^2 = 2[\cos(\alpha)]^2 - 1 = 1 - 2[\sin(\alpha)]^2$

e. $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$

f. $[\cos(\alpha)]^2 = \frac{1 + \cos(2\alpha)}{2}$

g. $[\sin(\alpha)]^2 = \frac{1 - \cos(2\alpha)}{2}$

h. $\sin(\alpha) \cos(\beta) = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$

i. $\sin(\alpha) \sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$

j. $\cos(\alpha) \sin(\beta) = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}$

k. $\cos(\alpha) \cos(\beta) = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$

l. $\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$

m. $\sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$

n. $\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$

o. $\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$