Linear Algebra for Team-Based Inquiry Learning

2024 Edition PREVIEW

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Website: Linear Algebra for Team-Based Inquiry Learning[1]

[1] linear.tbil.org

[2] github.com/TeamBasedInquiryLearning/linear-algebra/blob/main/LICENSE.md

Contents

Chapter 1

Systems of Linear Equations (LE)

Learning Outcomes

How can we solve systems of linear equations?
By the end of this chapter, you should be able to...

1. Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

2. Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.

3. Determine the number of solutions for a system of linear equations or a vector equation.

4. Compute the solution set for a system of linear equations or a vector equation with infinitely many solutions.

1.1 Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Learning Outcomes

- Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

Definition 1.1.1 A **Euclidean vector** is an ordered list of real numbers

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

We will find it useful to almost always typeset Euclidean vectors vertically, but the notation $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ is also valid when vertical typesetting is inconvenient. The set of all Euclidean vectors with n components is denoted as $\mathbb{R}^n$, and vectors are often described using the notation $\vec{v}$.

Each number in the list is called a **component**, and we use the following definitions for the sum of two vectors, and the product of a real number and a vector:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \qquad c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

$\Diamond$

Example 1.1.2 Following are some examples of addition and scalar multiplication in $\mathbb{R}^4$.

$$\begin{bmatrix} 3 \\ -3 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0 \\ -3+2 \\ 0+7 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$$

$$-4\begin{bmatrix} 0 \\ 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4(0) \\ -4(2) \\ -4(-2) \\ -4(3) \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 8 \\ -12 \end{bmatrix}$$

$\square$

Definition 1.1.3 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector whose components can be plugged into the equation). $\Diamond$

Remark 1.1.4 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition 1.1.5 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{alignedat}{5}
a_{11}x_1 &+ a_{12}x_2 &+ \ldots &+ a_{1n}x_n &&= b_1 \\
a_{21}x_1 &+ a_{22}x_2 &+ \ldots &+ a_{2n}x_n &&= b_2 \\
\vdots & \quad \vdots & & \quad \vdots && \quad \vdots \\
a_{m1}x_1 &+ a_{m2}x_2 &+ \ldots &+ a_{mn}x_n &&= b_m
\end{alignedat}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \;\middle|\; \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

$\Diamond$

Remark 1.1.6 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:	Verbose standard form:	Concise standard form:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$x_1 \qquad\quad + 3x_3 = 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$\qquad - \ \ x_2 + \ \ x_3 = -2$

Remark 1.1.7 It will often be convenient to think of a system of equations as a vector equation.

By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

is equivalent to the system of equations

$$\begin{aligned} x_1 \phantom{{}-2x_2} + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ - x_2 + x_3 &= -2 \end{aligned}$$

Definition 1.1.8 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**. $\Diamond$

Fact 1.1.9 *All linear systems are one of the following:*

1. Consistent with one solution: *its solution set contains a single vector, e.g.* $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
2. Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g.* $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\,\middle|\,a\in\mathbb{R}\right\}$
3. Inconsistent*: its solution set is the empty set, denoted by either* $\{\}$ *or* $\emptyset$*.*

Activity 1.1.10 All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is the empty set.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 1.1.11 Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

(a) Find three different solutions for this system.

(b) Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \left[\begin{array}{c} ? \\ a \end{array} \right] \middle| \, a \in \mathbb{R} \right\}$ for the linear system.

Activity 1.1.12 Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & & &- & x_4 &= & 3 \\ & & & & x_3 &+ & 4x_4 &= & -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \left[\begin{array}{c} ? \\ a \\ ? \\ b \end{array}\right] \middle|\, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation 1.1.13 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{alignedat}{5} -2x_1 &-{}& 4x_2 &+{}& x_3 &-{}& 4x_4 &={}& -8 \\ x_1 &+{}& 2x_2 &+{}& 2x_3 &+{}& 12x_4 &={}& -1 \\ x_1 &+{}& 2x_2 &+{}& x_3 &+{}& 8x_4 &={}& 1 \end{alignedat}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Remark 1.1.14 The only important information in a linear system are its coefficients and constants.

Original linear system:	Verbose standard form:	Coefficients/constants:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$1 \quad 0 \quad 3 \mid 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3 \quad -2 \quad 4 \mid 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$0 \quad -1 \quad 1 \mid -2$

Definition 1.1.15 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= \ b_1 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= \ b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \ &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= b_m
\end{aligned}$$

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]$$

$\Diamond$

Example 1.1.16 The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{alignedat}{4} x_1 & & & {}+{} & 3x_3 & {}={} & 3 \\ 3x_1 & {}-{} & 2x_2 & {}+{} & 4x_3 & {}={} & 0 \\ & {}-{} & x_2 & {}+{} & x_3 & {}={} & -2 \end{alignedat}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Vector equation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

□

1.2 Row Reduction of Matrices (LE2)

Learning Outcomes

- Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.

Definition 1.2.1 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent (even though they're not *equal*), which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \neq \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

$\diamond$

Row Reduction of Matrices (LE2)

Activity 1.2.2 Consider whether these matrix manipulations (A) *must keep* or (B) *could change* the solution set for the corresponding linear system.

(a) Swapping two rows, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right] \qquad \begin{array}{r} x+2y=3 \\ 4x+5y=6 \end{array} \qquad \begin{array}{r} 4x+5y=6 \\ x+2y=3 \end{array}$$

(b) Swapping two columns, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 5 & 4 & 6 \end{array}\right] \qquad \begin{array}{r} x+2y=3 \\ 4x+5y=6 \end{array} \qquad \begin{array}{r} 2x+y=3 \\ 5x+4y=6 \end{array}$$

(c) Add a constant to every term of a row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1+6 & 2+6 & 3+6 \\ 4 & 5 & 6 \end{array}\right] \qquad \begin{array}{r} x+2y=3 \\ 4x+5y=6 \end{array} \qquad \begin{array}{r} 7x+8y=9 \\ 4x+5y=6 \end{array}$$

(d) Multiply a row by a nonzero constant, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 6 & 9 \\ 4 & 5 & 6 \end{array}\right] \qquad \begin{array}{r} x+2y=3 \\ 4x+5y=6 \end{array} \qquad \begin{array}{r} 3x+6y=9 \\ 4x+5y=6 \end{array}$$

(e) Add a constant multiple of one row to another row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4+3 & 5+6 & 6+9 \end{array}\right] \qquad \begin{array}{r} x+2y=3 \\ 4x+5y=6 \end{array} \qquad \begin{array}{r} ?\,x+\,?\,y=\,? \\ ?\,x+\,?\,y=\,? \end{array}$$

(f) Replace a column with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 4 & 0 & 6 \end{array}\right] \qquad \begin{array}{r} x+2y=3 \\ 4x+5y=6 \end{array} \qquad \begin{array}{r} ?\,x+\,?\,y=\,? \\ ?\,x+\,?\,y=\,? \end{array}$$

(g) Replace a row with zeros, for example:

Row Reduction of Matrices (LE2)

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x + 2y &= 3 & \quad ?\,x + \,?\,y &= \,? \\ 4x + 5y &= 6 & \quad ?\,x + \,?\,y &= \,? \end{aligned}$$

Definition 1.2.3 The following three **row operations** produce equivalent augmented matrices.

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \end{array}\right]$$

Observe that we will use the following notation: (Combination of old rows) $\to$ (New row). $\Diamond$

Activity 1.2.4 Each of the following linear systems has the same solution set.

A)

$$\begin{alignedat}{4} x &+{}& 2y &+{}& z &= 3 \\ -x &-{}& y &+{}& z &= 1 \\ 2x &+{}& 5y &+{}& 3z &= 7 \end{alignedat}$$

B)

$$\begin{alignedat}{4} 2x &+{}& 5y &+{}& 3z &= 7 \\ -x &-{}& y &+{}& z &= 1 \\ x &+{}& 2y &+{}& z &= 3 \end{alignedat}$$

C)

$$\begin{alignedat}{4} x & & &-{}& z &= 1 \\ & & y &+{}& 2z &= 4 \\ & & y &+{}& z &= 1 \end{alignedat}$$

D)

$$\begin{alignedat}{4} x &+{}& 2y &+{}& z &= 3 \\ & & y &+{}& 2z &= 4 \\ 2x &+{}& 5y &+{}& 3z &= 7 \end{alignedat}$$

E)

$$\begin{alignedat}{4} x & & &-{}& z &= 1 \\ & & y &+{}& 2z &= 4 \\ & & & & z &= 3 \end{alignedat}$$

F)

$$\begin{alignedat}{4} x &+{}& 2y &+{}& z &= 3 \\ & & y &+{}& 2z &= 4 \\ & & y &+{}& z &= 1 \end{alignedat}$$

Sort these six equivalent linear systems from most complicated to simplest (in your opinion).

Activity 1.2.5 Here we've written the sorted linear systems from Activity 1.2.4 as augmented matrices.

$$\left[\begin{array}{ccc|c} 2 & 5 & 3 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 2 & 5 & 3 & 7 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

Assign the following row operations to each step used to manipulate each matrix to the next:

$R_3 - 1R_2 \to R_3$ $\qquad$ $R_2 + 1R_1 \to R_2$ $\qquad$ $R_1 \leftrightarrow R_3$

$R_3 - 2R_1 \to R_3$ $\qquad$ $R_1 - 2R_3 \to R_1$

Definition 1.2.6 A matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write $\mathrm{RREF}(A)$ for the reduced row echelon form of that matrix. $\diamondsuit$

Activity 1.2.7 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Activity 1.2.8 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$D = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad E = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad F = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Remark 1.2.9 In practice, if we simply need to convert a matrix into reduced row echelon form, we use technology to do so.

However, it is also important to understand the **Gauss-Jordan elimination** algorithm that a computer or calculator uses to convert a matrix (augmented or not) into reduced row echelon form. Understanding this algorithm will help us better understand how to interpret the results in many applications we use it for in Chapter 2.

Activity 1.2.10 Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

A. Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

B. Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

C. Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

D. Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Activity 1.2.11 Consider the matrix

$$\left[\begin{array}{cccc} \boxed{1} & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{array}\right].$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

B. Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

C. Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

D. Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

Activity 1.2.12 Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

B. Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

C. Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

D. Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

Observation 1.2.13 The steps for the Gauss-Jordan elimination algorithm may be summarized as follows:

1. Ignoring any rows that already have marked pivots, identify the leftmost column with a nonzero entry.
2. Use row operations to obtain a pivot of value 1 in the topmost row that does not already have a marked pivot.
3. Mark this pivot, then use row operations to change all values above and below the marked pivot to 0.
4. Repeat these steps until the matrix is in RREF.

In particular, *once a pivot is marked, it should remain in the same position.* This will keep you from undoing your progress towards an RREF matrix.

Activity 1.2.14 Complete the following RREF calculation (multiple row operations may be needed for certain steps):

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & ? & ? & ? \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Activity 1.2.15 Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF(A).

Activity 1.2.16 Consider the non-augmented and augmented matrices

$$A = \left[\begin{array}{cccc} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right].$$

Can RREF(A) be used to find RREF(B)?

A. Yes, RREF(A) and RREF(B) are exactly the same.

B. Yes, RREF(A) may be slightly modified to find RREF(B).

C. No, a new calculuation is required.

Activity 1.2.17 Free browser-based technologies for mathematical computation are available online.

- Go to `https://sagecell.sagemath.org/`.
- In the dropdown on the right, you can select a number of different languages. Select "Octave" for the Matlab-compatible syntax used by this text.
- Type `rref([1,3,2;2,5,7])` and then press the Evaluate button to compute the RREF of $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 7 \end{bmatrix}$.

Activity 1.2.18 In the HTML version of this text, code cells are often embedded for your convenience when RREFs need to be computed.

Try this out to compute RREF $\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 0 & 6 \end{array}\right]$.

1.3 Counting Solutions for Linear Systems (LE3)

Learning Outcomes

- Determine the number of solutions for a system of linear equations or a vector equation.

Remark 1.3.1 We will frequently need to know the reduced row echelon form of matrices during the remainder of this course, so unless you're told otherwise, feel free to use technology (see Activity 1.2.17) to compute RREFs efficiently.

Activity 1.3.2 Consider the following system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11. \end{aligned}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

Activity 1.3.3 Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

Activity 1.3.4 What contradictory equations besides $0 = 1$ may be obtained from the RREF of an augmented matrix?

A. $x = 0$ is an obtainable contradiction

B. $x = y$ is an obtainable contradiction

C. $0 = 17$ is an obtainable contradiction

D. $0 = 1$ is the only obtainable contradiction

Activity 1.3.5 Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

(a) Find its corresponding augmented matrix A and find $\text{RREF}(A)$.

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. One C. Infinitely-many

Fact 1.3.6 *We will see in Section 1.4 that the intuition established here generalizes: a* consistent *system with more variables than equations (ignoring* $0 = 0$*) will always have infinitely many solutions.*

Fact 1.3.7 *By finding* $\text{RREF}(A)$ *from a linear system's corresponding augmented matrix* A*, we can immediately tell how many solutions the system has.*

- *If the linear system given by* $\text{RREF}(A)$ *includes the contradiction* $0 = 1$*, that is, the row* $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$*, then the system is* inconsistent*, which means it has* zero *solutions and its solution set is written as* $\emptyset$ *or* $\{\}$*.*

- *If the linear system given by* $\text{RREF}(A)$ *sets each variable of the system to a single value; that is,* $x_1 = s_1$*,* $x_2 = s_2$*, and so on; then the system is* consistent *with exactly* one *solution* $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix}$*, and its solution set is* $\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix} \right\}$.

- *Otherwise, the system must have more variables than non-trivial equations (equations other than* $0 = 0$*). This means it is* consistent *with* infinitely-many *different solutions. We'll learn how to find such solution sets in Section 1.4.*

Counting Solutions for Linear Systems (LE3)

Activity 1.3.8 For each vector equation, write an explanation for whether each solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

(a)

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 4 \end{bmatrix}$$

(b)

$$x_1 \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 13 \end{bmatrix}$$

(c)

$$x_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -5 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ -9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

1.4 Linear Systems with Infinitely-Many Solutions (LE4)

Learning Outcomes

- Compute the solution set for a system of linear equations or a vector equation with infinitely many solutions.

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.1 Consider this simplified linear system found to be equivalent to the system from Activity 1.3.5:

$$\begin{aligned} x_1 + 2x_2 \quad\quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Earlier, we determined this system has infinitely-many solutions.

(a) Let $x_1 = a$ and write the solution set in the form $\left\{ \left[\begin{array}{c} a \\ ? \\ ? \end{array}\right] \middle| \, a \in \mathbb{R} \right\}$.

(b) Let $x_2 = b$ and write the solution set in the form $\left\{ \left[\begin{array}{c} ? \\ b \\ ? \end{array}\right] \middle| \, b \in \mathbb{R} \right\}$.

(c) Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$ caused this?

Definition 1.4.2 Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables. $\diamondsuit$

Activity 1.4.3 Find the solution set for the system

$$\begin{alignedat}{6} 2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3 \\ -x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3 \\ x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2 \end{alignedat}$$

by doing the following.

(a) Row-reduce its augmented matrix.

(b) Assign letters to the free variables (given by the non-pivot columns):

$$? = a$$

$$? = b$$

(c) Solve for the bound variables (given by the pivot columns) to show that

$$? = 1 + 5a + 2b$$

$$? = 1 + 2a + 3b$$

$$? = 3 + 3b$$

(d) Replace x_1 through x_5 with the appropriate expressions of a, b in the following set-builder notation.

$$\left\{ \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right] \middle| \; a, b \in \mathbb{R} \right\}$$

Remark 1.4.4 Don't forget to correctly express the solution set of a linear system. Systems with zero or one solutions may be written by listing their elements, while systems with infinitely-many solutions may be written using set-builder notation.

- *Inconsistent*: $\emptyset$ or $\{\}$
  - (not 0 or $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)
- *Consistent* with *one solution*: e.g. $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$
  - (not just $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$)
- *Consistent* with *infinitely-many solutions*: e.g. $\left\{\begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R}\right\}$
  - (not just $\begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix}$)

Activity 1.4.5 Consider the following system of linear equations.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ -5 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 13 \\ -13 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ -12 \end{bmatrix}.$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Activity 1.4.6 Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x_1 & & & - & 2\,x_3 & = & -3 \\ 5\,x_1 & + & x_2 & - & 7\,x_3 & = & -18 \\ 5\,x_1 & - & x_2 & - & 13\,x_3 & = & -12 \\ x_1 & + & 3\,x_2 & + & 7\,x_3 & = & -12 \end{array}$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Chapter 2

Euclidean Vectors (EV)

Learning Outcomes

What is a space of Euclidean vectors?
By the end of this chapter, you should be able to...

1. Determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.
2. Determine if a set of Euclidean vectors spans $\mathbb{R}^n$ by solving appropriate vector equations.
3. Determine if a subset of $\mathbb{R}^n$ is a subspace or not.
4. Determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.
5. Explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.
6. Compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.
7. Find a basis for the solution set of a homogeneous system of equations.

2.1 Linear Combinations (EV1)

Learning Outcomes

- Determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.

Note 2.1.1 We've been working with **Euclidean vector spaces** of the form

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

There are other kinds of **vector spaces** as well (e.g. polynomials, matrices), which we will investigate in Section 3.5. But understanding the structure of *Euclidean* vectors on their own will be beneficial, even when we turn our attention to other kinds of vectors.

Likewise, when we multiply a vector by a real number, as in $-3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}$, we refer to this real number as a **scalar**.

Definition 2.1.2 A **linear combination** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is given by $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$\Diamond$

Definition 2.1.3 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R}\right\}.$$

$\Diamond$

Activity 2.1.4 Consider $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(a) Sketch the four Euclidean vectors

$$1\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad -2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

in the xy plane by placing a dot at the (x, y) coordinate associated with each vector.

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ 2 \end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$$

in the xy plane by plotting their (x, y) coordinates as dots. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

Remark 2.1.5 It is important to remember that

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} \neq \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}.$$

For example,

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a set containing exactly two vectors, while

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

is a set containing infinitely-many vectors.

Linear Combinations (EV1)

Activity 2.1.6 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

(a) Sketch the following five Euclidean vectors in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}=? \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=? \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=? \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}=?$$

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}-1\\1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

in the xy plane. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

Activity 2.1.7 Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$ in the xy plane. What best describes this sketch?

A. A line

B. A plane

C. A parabola

D. A cube

Activity 2.1.8 Consider the following questions to discover whether a Euclidean vector belongs to a span.

(a) The Euclidean vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to which of these vector equations?

A. $x_1 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

B. $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$

C. $x_1 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 0$

(b) Use technology to find RREF of the corresponding augmented matrix, and then use that matrix to find the solution set of the vector equation.

(c) Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Observation 2.1.9 The following are all equivalent statements:

- The vector $\vec{b}$ belongs to $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$.
- The vector $\vec{b}$ is a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_n$.
- The vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ is consistent.
- The linear system corresponding to $\left[\vec{v}_1 \ \ldots \ \vec{v}_n \,|\, \vec{b}\right]$ is consistent.
- RREF $\left[\vec{v}_1 \ \ldots \ \vec{v}_n \,|\, \vec{b}\right]$ doesn't have a row $[0 \ \cdots \ 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity 2.1.10 Consider this claim about a vector equation:

$\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$.

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is true.

(c) Since your statement was true, use the solution set to describe a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$ that equals $\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$.

Activity 2.1.11 Consider this claim about a vector equation:

$$\begin{bmatrix} -5 \\ -1 \\ -7 \end{bmatrix} \text{ belongs to span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is false, to conclude that the vector does not belong to the span.

2.2 Spanning Sets (EV2)

Learning Outcomes

- Determine if a set of Euclidean vectors spans $\mathbb{R}^n$ by solving appropriate vector equations.

Observation 2.2.1 Any single non-zero vector/number x in $\mathbb{R}^1$ spans $\mathbb{R}^1$, since $\mathbb{R}^1 = \{cx \,|\, c \in \mathbb{R}\}$.

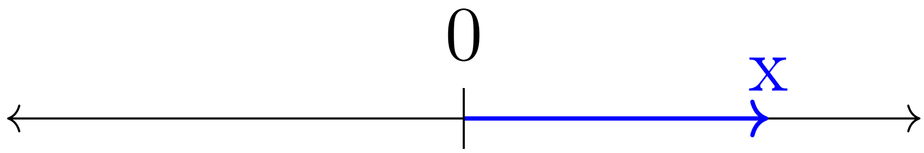


Figure 1 An $\mathbb{R}^1$ vector

Activity 2.2.2 How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

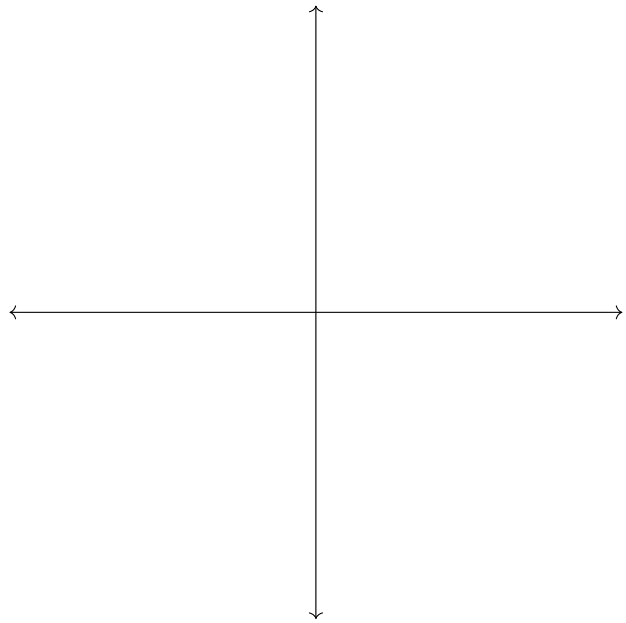


Figure 2 The xy plane $\mathbb{R}^2$

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Activity 2.2.3 How many vectors are required to span $\mathbb{R}^3$?

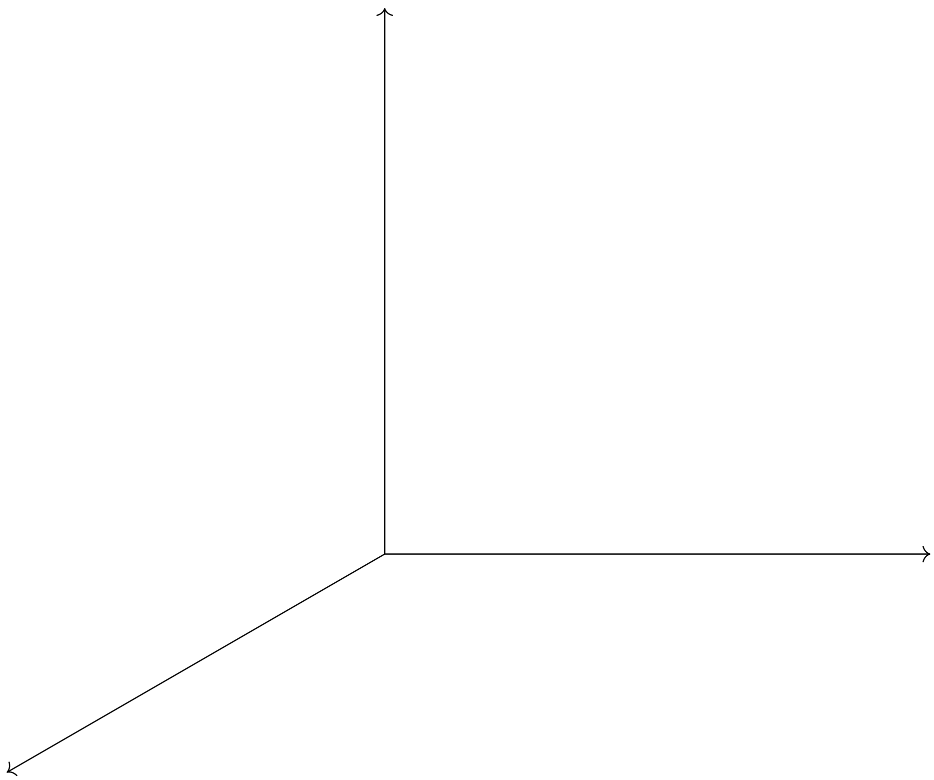


Figure 3 $\mathbb{R}^3$ space

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Fact 2.2.4 *At least n vectors are required to span $\mathbb{R}^n$.*

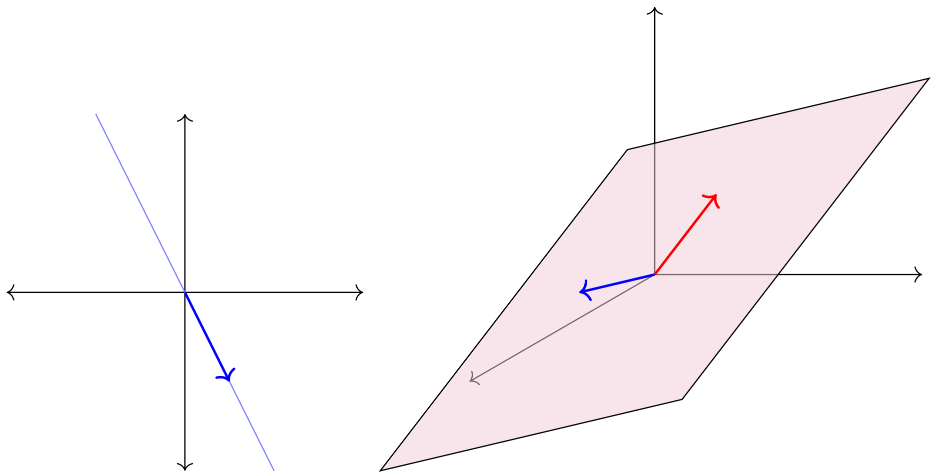


Figure 4 *Failed attempts to span $\mathbb{R}^n$ by $< n$ vectors*

Activity 2.2.5 Consider the question: Does every vector in $\mathbb{R}^3$ belong to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$?

(a) Determine if $\begin{bmatrix}7\\-3\\-2\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(b) Determine if $\begin{bmatrix}2\\5\\7\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(c) An arbitrary vector $\begin{bmatrix}?\\?\\?\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$ provided the equation

$$x_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}-2\\0\\1\end{bmatrix}+x_3\begin{bmatrix}-2\\-2\\2\end{bmatrix}=\begin{bmatrix}?\\?\\?\end{bmatrix}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) We're guaranteed at least one solution if the RREF of the corresponding augmented matrix has no contradictions; likewise, we have no solutions if the RREF corresponds to the contradiction $0 = 1$. Given

$$\left[\begin{array}{ccc|c}1&-2&-2&?\\-1&0&-2&?\\0&1&2&?\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&2&?\\0&1&2&?\\0&0&0&?\end{array}\right]$$

we may conclude that the set does not span all of $\mathbb{R}^3$ because...

Spanning Sets (EV2)

A. the row $[0\,1\,2\,|\,?]$ prevents a contradiction.

B. the row $[0\,1\,2\,|\,?]$ allows a contradiction.

C. the row $[0\,0\,0\,|\,?]$ prevents a contradiction.

D. the row $[0\,0\,0\,|\,?]$ allows a contradiction.

Fact 2.2.6 *The set* $\{\vec{v}_1, \dots, \vec{v}_m\}$ *spans all of* $\mathbb{R}^n$ *exactly when the vector equation*

$$x_1\vec{v}_1 + \cdots x_m\vec{v}_m = \vec{w}$$

is consistent for every *vector* $\vec{w}$.

Likewise, the set $\{\vec{v}_1, \dots, \vec{v}_m\}$ *fails to span all of* $\mathbb{R}^n$ *exactly when the vector equation*

$$x_1\vec{v}_1 + \cdots x_m\vec{v}_m = \vec{w}$$

is inconsistent for some *vector* $\vec{w}$.

Note these two possibilities are decided based on whether or not RREF$[\vec{v}_1 \dots \vec{v}_m]$ *has either all pivot rows, or at least one non-pivot row (a row of zeroes):*

$$\left[\begin{array}{ccc|} 1 & -2 & -2 \\ -1 & 0 & -2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

Activity 2.2.7 Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ and the question "Does $\mathbb{R}^4 = \operatorname{span} S$?"

(a) Rewrite this question in terms of the solutions to a vector equation.

(b) Answer your new question, and use this to answer the original question.

Activity 2.2.8 Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three Euclidean vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

A. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

B. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the same as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

C. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

2.3 Subspaces (EV3)

Learning Outcomes

- Determine if a subset of $\mathbb{R}^n$ is a subspace or not.

Observation 2.3.1 Recall that if $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is subset of vectors in $\mathbb{R}^n$, then $\operatorname{span}(S)$ is the set of all linear combinations of vectors in S. In EV2 (Section 2.2), we learned how to decide whether $\operatorname{span}(S)$ was equal to all of $\mathbb{R}^n$ or something strictly smaller.

Activity 2.3.2 Let S denote a set of vectors in $\mathbb{R}^n$ and suppose that $\vec{u}, \vec{v} \in \text{span}(S)$, $c \in \mathbb{R}$ and that $\vec{w} \in \mathbb{R}^n$. Which of the following vectors might *not* belong to $\text{span}(S)$?

A. $\vec{0}$

B. $\vec{u} + \vec{w}$

C. $\vec{u} + \vec{v}$

D. $c\vec{u}$

Definition 2.3.3 A **homogeneous** system of linear equations is one of the form:

$$\begin{array}{rrrrl} a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n & = 0 \\ a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n & = 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n & = 0 \end{array}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

$\Diamond$

Activity 2.3.4 Consider the homogeneous vector equation $x_1\vec{v}_1+\cdots+x_n\vec{v}_n = \vec{0}$.

(a) Is this equation consistent?

A. no.

B. yes.

C. more information is required.

(b) Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are both solutions, we know that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}.$$

Therefore by adding these equations:

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

(c) Similarly, if $c \in \mathbb{R}$, then since multiplying by c yields:

$$(ca_1)\vec{v}_1 + \cdots + (ca_n)\vec{v}_n = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

D. The empty set.

Observation 2.3.5 If S is any set of vectors in $\mathbb{R}^n$, then the set $\operatorname{span}(S)$ has the following properties:

- the set $\operatorname{span}(S)$ is non-empty.
- the set $\operatorname{span}(S)$ is closed under addition: for any $\vec{u}, \vec{v} \in \operatorname{span}(S)$, the sum $\vec{u} + \vec{v}$ is also in $\operatorname{span}(S)$.
- the set $\operatorname{span}(S)$ is closed under scalar multiplication: for any $\vec{u} \in \operatorname{span}(S)$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in $\operatorname{span}(S)$.

Likewise, if W is the solution set to a homogenous vector equation, it too satisfies:

- the set W is non-empty.
- the set W is closed under addition: for any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v}$ is also in W.
- the set $\operatorname{span}(S)$ is closed under scalar multiplication: for any $\vec{u} \in W$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in W.

Definition 2.3.6 A subset W of a vector space is called a **subspace** provided that it satisfies the following properties:

- the subset is non-empty.
- the subset is **closed under addition**: for any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v}$ is also in W.
- the subset is **closed under scalar multiplication**: for any $\vec{u} \in W$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in W.

$\Diamond$

Observation 2.3.7 Note the similarities between a planar subspace spanned by two non-colinear vectors in $\mathbb{R}^3$, and the Euclidean plane $\mathbb{R}^2$. While they are not the same thing (and shouldn't be referred to interchangably), algebraists call such similar spaces **isomorphic**; we'll learn what this means more carefully in a later chapter.

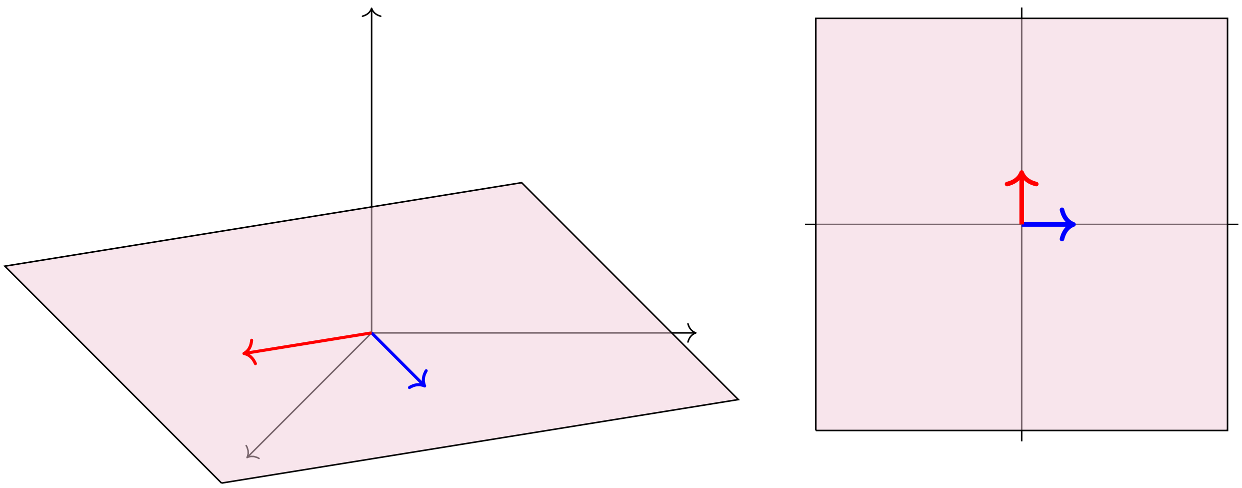


Figure 5 A planar subset of $\mathbb{R}^3$ compared with the plane $\mathbb{R}^2$.

Activity 2.3.8 Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

(a) Is W the empty set?

(b) Let's assume that $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are in W. What are we allowed to assume?

A. $x + 2y + z = 0$.

B. $a + 2b + c = 0$.

C. Both of these.

D. Neither of these.

(c) Which equation must be verified to show that $\vec{v} + \vec{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to W?

A. $(x + a) + 2(y + b) + (z + c) = 0$.

B. $x + a + 2y + b + z + c = 0$.

C. $x + 2y + z = a + 2b + c$.

(d) Use the assumptions from (a) to verify the equation from (b).

(e) Is W is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

(f) Show that $k\vec{v} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$ also belongs to W for any $k \in \mathbb{R}$ by verifying $(kx) + 2(ky) + (kz) = 0$ under these assumptions.

(g) Is W is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

Activity 2.3.9 Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 4 \right\}$.

(a) Is W the empty set?

(b) Which of these statements is valid?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so W *is* a subspace.

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so W *is not* a subspace.

C. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so W *is* a subspace.

D. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so W *is not* a subspace.

(c) Which of these statements is valid?

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so W *is* a subspace.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so W *is not* a subspace.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so W *is* a subspace.

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so W *is not* a subspace.

Remark 2.3.10 In summary, *any one* of the following is enough to prove that a nonempty subset W is *not* a subspace:

- Find specific values for $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v} \notin W$.
- Find specific values for $c \in \mathbb{R}, \vec{v} \in W$ such that $c\vec{v} \notin W$.
- Show that $\vec{0} \notin W$.

If you cannot do any of these, then W can be proven to be a subspace by doing *all* of the following:

1. Show that W is non-empty.
2. For all $\vec{v}, \vec{w} \in W$ (not just specific values), $\vec{u} + \vec{v} \in W$.
3. For all $\vec{v} \in W$ and $c \in \mathbb{R}$ (not just specific values), $c\vec{v} \in W$.

Activity 2.3.11 Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}.$$

(a) Show R isn't a subspace by showing that $\vec{0} \notin R$.

(b) Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

(c) Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Activity 2.3.12 Consider the following two sets of Euclidean vectors:

$$U = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle|\, 7\,x + 4\,y = 0 \right\} \qquad W = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle|\, 3\,xy^2 = 0 \right\}$$

Explain why one of these sets is a subspace of $\mathbb{R}^2$ and one is not.

Activity 2.3.13 Consider the following attempted proof that

$$U = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle| x + y = xy \right\}$$

is closed under scalar multiplication.

> Let $\left[\begin{array}{c} x \\ y \end{array}\right] \in U$, so we know that $x + y = xy$. We want to show $k\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} kx \\ ky \end{array}\right] \in U$, that is, $(kx) + (ky) = (kx)(ky)$. This is verified by the following calculation:
>
> $$\begin{aligned} (kx) + (ky) &= (kx)(ky) \\ k(x + y) &= k^2xy \\ 0[k(x + y)] &= 0[k^2xy] \\ 0 &= 0 \end{aligned}$$

Is this reasoning valid?

A. Yes

B. No

Remark 2.3.14 Proofs of an equality LEFT = RIGHT should generally be of one of these forms:

1. Using a chain of equalities:

$$\begin{aligned} \text{LEFT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{RIGHT} \end{aligned}$$

   Alternatively:

$$\begin{aligned} \text{LEFT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{SAME} \end{aligned} \qquad\qquad \begin{aligned} \text{RIGHT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{SAME} \end{aligned}$$

2. When the assumption THIS = THAT is already known or assumed to be true :

$$\begin{aligned} & \text{THIS} = \text{THAT} \\ \Rightarrow \quad & \cdots = \cdots \\ \Rightarrow \quad & \cdots = \cdots \\ \Rightarrow \quad & \text{LEFT} = \text{RIGHT} \end{aligned}$$

2.4 Linear Independence (EV4)

Learning Outcomes

- Determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.

Activity 2.4.1 Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}.$$

Which of the following is true?

A. $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

B. $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

C. $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Definition 2.4.2 We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

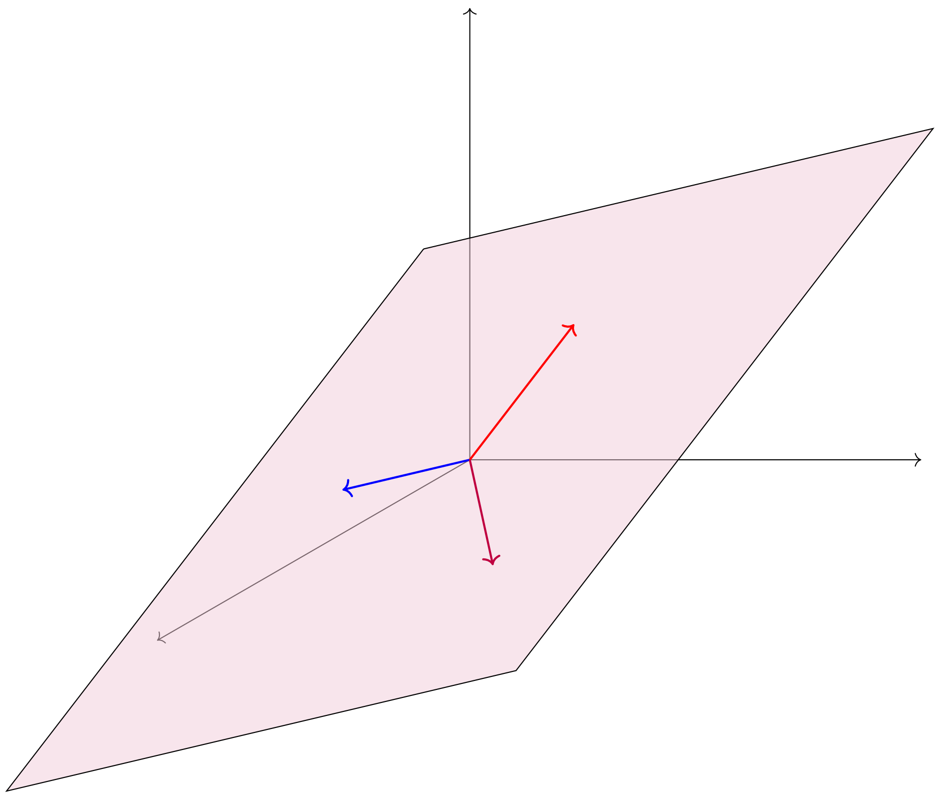


Figure 6 A linearly dependent set of three vectors

You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay in the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

◊

Activity 2.4.3 Consider the following three vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}.$$

(a) Let $\vec{w} = 3\vec{v}_1 - \vec{v}_2 - 5\vec{v}_3 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ is...

A. linearly dependent: at least one vector is a linear combination of others

B. linearly independent: no vector is a linear combination of others

(b) Find

$$\text{RREF}\left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{array}\right] = \text{RREF}\begin{bmatrix} -2 & 1 & -2 & ? \\ 0 & 3 & 5 & ? \\ 0 & 0 & 4 & ? \end{bmatrix} = ?.$$

What does this tell you about solution set for the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{w} = \vec{0}$?

A. It is inconsistent.

B. It is consistent with one solution.

C. It is consistent with infinitely many solutions.

(c) Which of these might explain the connection?

A. A pivot column establishes linear independence and creates a contradiction.

B. A non-pivot column both describes a linear combination and reveals the number of solutions.

C. A pivot row describes the bound variables and prevents a contradiction.

D. A non-pivot row prevents contradictions and makes the vector equation solvable.

Linear Independence (EV4)

Fact 2.4.4 *For any vector space, the set* $\{\vec{v}_1, \dots \vec{v}_n\}$ *is linearly dependent if and only if the vector equation* $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ *is consistent with infinitely many solutions.*

Likewise, the set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ *is linearly independent if and only the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = 0$$

has exactly one solution: $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

Activity 2.4.5 Find

$$\mathrm{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Observation 2.4.6 Compare the following results:

- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly independent if and only if RREF $\left[\begin{array}{ccc} \vec{v}_1 & \dots & \vec{v}_n \end{array}\right]$ has all pivot *columns*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly dependent if and only if RREF $\left[\begin{array}{ccc} \vec{v}_1 & \dots & \vec{v}_n \end{array}\right]$ has at least one non-pivot *column*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ spans $\mathbb{R}^m$ if and only if RREF $\left[\begin{array}{ccc} \vec{v}_1 & \dots & \vec{v}_n \end{array}\right]$ has all pivot *rows*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ fails to span $\mathbb{R}^m$ if and only if RREF $\left[\begin{array}{ccc} \vec{v}_1 & \dots & \vec{v}_n \end{array}\right]$ has at least one non-pivot *row*.

Activity 2.4.7

(a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

(i) "The set of vectors $\left\{\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix},\begin{bmatrix}5\\5\\3\\1\end{bmatrix},\begin{bmatrix}9\\11\\6\\3\end{bmatrix}\right\}$ is linearly *independent.*"

(ii) "The set of vectors $\left\{\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix},\begin{bmatrix}5\\5\\3\\1\end{bmatrix},\begin{bmatrix}9\\11\\6\\3\end{bmatrix}\right\}$ is linearly *dependent.*"

(b) Explain how to determine which of these statements is true.

Activity 2.4.8 What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

A. 3

B. 4

C. 5

D. You can have infinitely many vectors and still be linearly independent.

Activity 2.4.9 Is is possible for the set of Euclidean vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ to be linearly independent?

A. Yes

B. No

2.5 Identifying a Basis (EV5)

Learning Outcomes

- Explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.

Identifying a Basis (EV5)

Activity 2.5.1 Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S, i.e. find scalars such that

$$\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} = ? \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} + ? \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix} + ? \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Find a *different* way to express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S.

(c) Consider another vector $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \end{bmatrix}$. Without computing the RREF of another matrix, how many ways can this vector be written as a linear combination of the vectors in S?

A. Zero.

B. One.

C. Infinitely-many.

D. Computing a new matrix RREF is necessary.

Activity 2.5.2 Let's review some of the terminology we've been dealing with...

(a) If every vector in a vector space can be constructed as one or more linear combination of vectors in a set S, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(b) If the zero vector $\vec{0}$ can be constructed as a *unique* linear combination of vectors in a set S (the combination multiplying every vector by the scalar value 0), we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(c) If every vector of a vector space can either be constructed as a *unique* linear combination of vectors in a set S, or not at all, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

Definition 2.5.3 A **basis** of a vector space V is a set of vectors S contained in V for which

1. *Every* vector in the vector space can be expressed as a linear combination of the vectors in S.

2. For each vector $\vec{v}$ in the vector space, there is only *one* way to write it as a linear combination of the vectors in S.

These two properties may be expressed more succintly as the statement "Every vector in V can be expressed *uniquely* as a linear combination of the vectors in S". $\diamondsuit$

Observation 2.5.4 In terms of a vector equation, a set $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of a vector space if the vector equation

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{w}$$

has a *unique* solution for *every* vector $\vec{w}$ in the vector space.

Put another way, a basis may be thought of as a minimal set of "building blocks" that can be used to construct any other vector of the vector space.

Identifying a Basis (EV5)

Activity 2.5.5 Let S be a basis (Definition 2.5.3) for a vector space. Then...

A. the set S must both span the vector space and be linearly independent.

B. the set S must span the vector space but could be linearly dependent.

C. the set S must be linearly independent but could fail to span the vector space.

D. the set S could fail to span the vector space and could be linearly dependent.

Activity 2.5.6 The vectors

$$\hat{\imath} = (1,0,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ used frequently in multivariable calculus.

Find the unique linear combination of these vectors

$$?\,\hat{\imath} + \,?\,\hat{\jmath} + \,?\,\hat{k}$$

that equals the vector

$$(3,-2,4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

in xyz space.

Definition 2.5.7 The **standard basis** of $\mathbb{R}^n$ is the set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

In particular, the standard basis for $\mathbb{R}^3$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. $\Diamond$

Activity 2.5.8 Take the RREF of an appropriate matrix to determine if each of the following sets is a basis for $\mathbb{R}^4$.

(a)

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(b)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(c)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(d)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(e)

$$\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

Activity 2.5.9 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ has a pivot in every row (because it spans), and has a pivot in every column (because it's linearly independent).

What is $\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Identifying a Basis (EV5)

Fact 2.5.10 *The set* $\{\vec{v}_1, \dots, \vec{v}_m\}$ *is a basis for* $\mathbb{R}^n$ *if and only if* $m = n$ *and*

$$\mathrm{RREF}[\vec{v}_1 \, \dots \, \vec{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for $\mathbb{R}^n$ *must have exactly* n *vectors and its square matrix must row-reduce to the so-called* ***identity matrix*** *containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)*

2.6 Subspace Basis and Dimension (EV6)

Learning Outcomes

- Compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.

Observation 2.6.1 Recall from section Section 2.3 that a **subspace** of a vector space is the result of spanning a set of vectors from that vector space.

Recall also that a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in Figure 14 are needed to span the planar subspace.

Activity 2.6.2 Consider the subspace of $\mathbb{R}^4$ given by $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$.

(a) Mark the column of $\operatorname{RREF}\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

(b) What would be the result of removing the vector that gave us this column?

A. The set still spans W, and remains linearly dependent.

B. The set still spans W, but is now also linearly independent.

C. The set no longer spans W, and remains linearly dependent.

D. The set no longer spans W, but is now linearly independent.

Definition 2.6.3 Let W be a subspace of a vector space. A **basis** for W is a linearly independent set of vectors that spans W (but not necessarily the entire vector space). $\Diamond$

Observation 2.6.4 So given a set $S = \{\vec{v}_1, \dots, \vec{v}_m\}$, to compute a basis for the subspace $\operatorname{span} S$, simply remove the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\vec{v}_1 \dots \vec{v}_m]$. For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}\right\}$ as a basis.

Activity 2.6.5

(a) Find a basis for $\operatorname{span} S$ where

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

(b) Find a basis for $\operatorname{span} T$ where

$$T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Observation 2.6.6 Even though we found different bases for them, span S and span T are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} = T.$$

Thus the basis for a subspace is not unique in general.

Fact 2.6.7 *Any non-trivial real vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.*

For example,

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \;\; \textit{and} \;\; \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \;\; \textit{and} \;\; \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for $\mathbb{R}^3$*, and they all contain three vectors.*

Definition 2.6.8 The **dimension** of a vector space or subspace is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors. $\Diamond$

Activity 2.6.9 Consider the following subspace W of $\mathbb{R}^4$:

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\-5\\5\end{bmatrix}, \begin{bmatrix}12\\-3\\15\\-18\end{bmatrix}\right\}.$$

(a) Explain and demonstrate how to find a basis of W.

(b) Explain and demonstrate how to find the dimension of W.

Activity 2.6.10 The dimension of a subspace may be found by doing what with an appropriate RREF matrix?

A. Count the rows.

B. Count the non-pivot columns.

C. Count the pivots.

D. Add the number of pivot rows and pivot columns.

2.7 Homogeneous Linear Systems (EV7)

Learning Outcomes

- Find a basis for the solution set of a homogeneous system of equations.

Remark 2.7.1 Recall that a **homogeneous** system of linear equations is one of the form:

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= 0 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \ \ &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= 0
\end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & 0
\end{array}\right]$$

Activity 2.7.2 Consider the homogeneous vector equation $x_1\vec{v}_1+\cdots+x_n\vec{v}_n = \vec{0}$.

(a) Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are both solutions, we know that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}.$$

Therefore by adding these equations,

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0}$$

shows that $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ is also a solution. Thus the solution set of a homogeneous system is...

A. Closed under addition.

B. Not closed under addition.

C. Linearly dependent.

D. Linearly independent.

(b) Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is also closed under scalar multiplication, and therefore...

A. A basis for $\mathbb{R}^n$.

B. A subspace of $\mathbb{R}^n$.

C. All of $\mathbb{R}^n$.

D. The empty set.

Activity 2.7.3 Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \quad\quad + \; x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \; x_4 &= 0 \end{aligned}$$

(a) Find its solution set (a subspace of $\mathbb{R}^4$).

(b) Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

(c) Rewrite this solution space in the form

$$\operatorname{span} \left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}.$$

(d) Which of these choices best describes the set of two vectors $\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}$ used in this span?

A. The set is linearly dependent.

B. The set is linearly independent.

C. The set spans all of $\mathbb{R}^4$.

D. The set fails to span the solution space.

Fact 2.7.4 *The coefficients of the free variables in the solution space of a linear system always yield linearly independent vectors that span the solution space.*

Thus if

$$\left\{ a\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+b\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix} \middle| \, a,b\in\mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\right\}$$

is the solution space for a homogeneous system, then

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\right\}$$

is a basis for the solution space.

Activity 2.7.5 Consider the homogeneous system of equations

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 - 4x_4 &= 0 \\ -2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity 2.7.6 Consider the homogeneous vector equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis for its solution space.

Activity 2.7.7 Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 + 6x_2 + 4x_3 &= 0 \\ x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

(a) Find its solution space.

(b) Which of these is the best choice of basis for this solution space?

A $\{\}$

B $\{\vec{0}\}$

C The basis does not exist

Activity 2.7.8 To create a computer-animated film, an animator first models a scene as a subset of $\mathbb{R}^3$. Then to transform this three-dimensional visual data for display on a two-dimensional movie screen or television set, the computer could apply a linear tranformation that maps visual information at the point $(x, y, z) \in \mathbb{R}^3$ onto the pixel located at $(x + y, y - z) \in \mathbb{R}^2$.

(a) What homoegeneous linear system describes the positions (x, y, z) within the original scene that would be aligned with the pixel $(0, 0)$ on the screen?

(b) Solve this system to describe these locations.

Chapter 3

Algebraic Properties of Linear Maps (AT)

Learning Outcomes

How can we understand linear maps algebraically?
By the end of this chapter, you should be able to...

1. Determine if a map between Euclidean vector spaces is linear or not.
2. Translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
3. Compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.
4. Determine if a given linear map is injective and/or surjective.
5. Explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property, but nonetheless isn't a vector space.
6. Answer questions about vector spaces of polynomials or matrices.

3.1 Linear Transformations (AT1)

Learning Outcomes

- Determine if a map between Euclidean vector spaces is linear or not.

Definition 3.1.1 A **linear transformation** (also called a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$, and
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, and $\vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result. $\Diamond$

Definition 3.1.2 Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

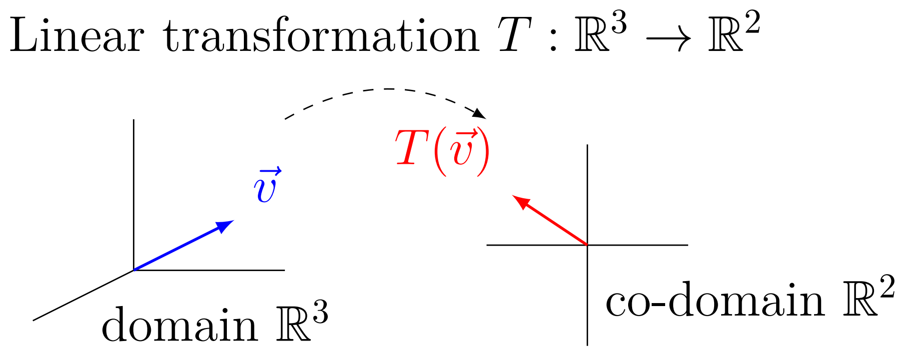


Figure 7 A linear transformation with a domain of $\mathbb{R}^3$ and a co-domain of $\mathbb{R}^2$

$\Diamond$

Observation 3.1.3 One example of a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ is the projection of three-dimesional data onto a two-dimensional screen, as is necessary for computer animiation in film or video games.

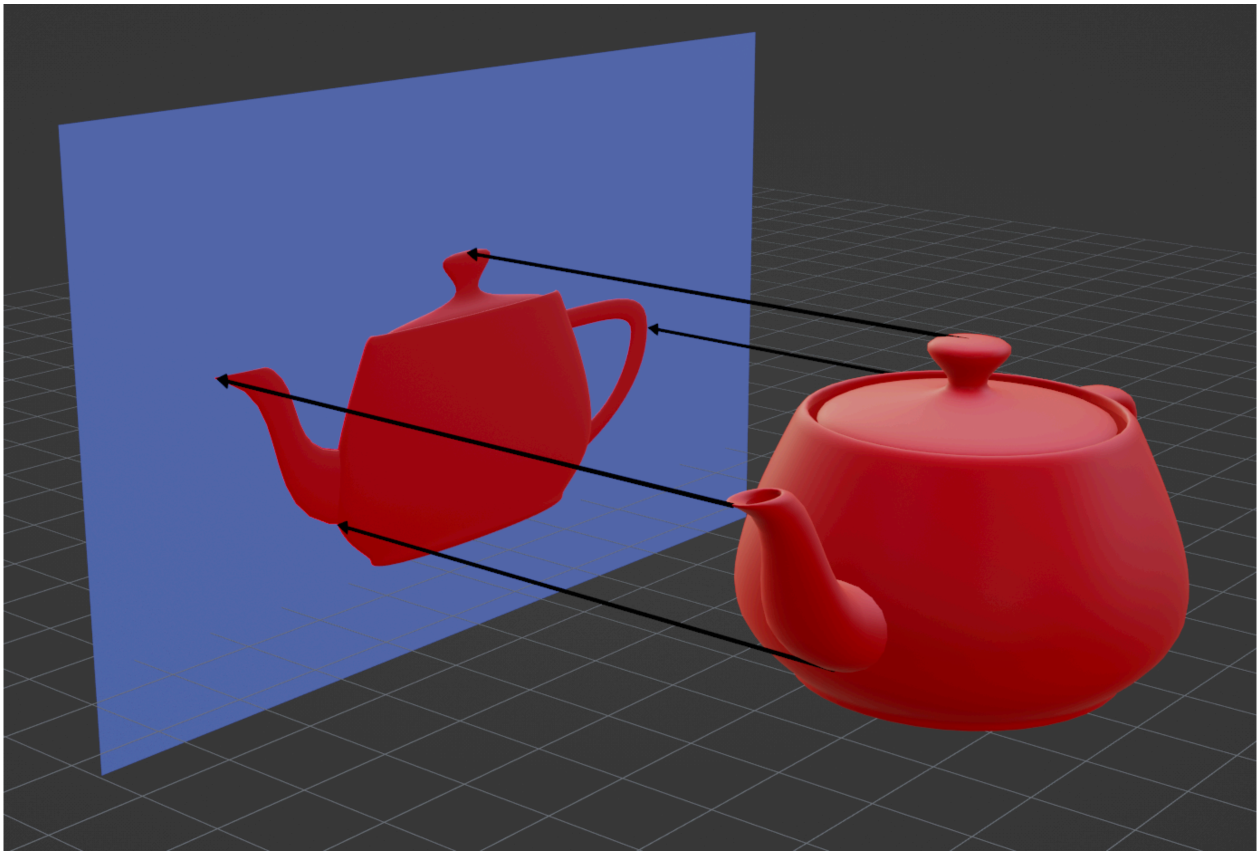


Figure 8 A projection of a $3D$ teapot onto a $2D$ screen

Activity 3.1.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}.$$

(a) Compute the result of adding vectors before a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right)$$

A. $\begin{bmatrix} x-u+z-w \\ 3y-3v \end{bmatrix}$

B. $\begin{bmatrix} x+u-z-w \\ 3y+3v \end{bmatrix}$

C. $\begin{bmatrix} x+u \\ 3y+3v \\ z+w \end{bmatrix}$

D. $\begin{bmatrix} x-u \\ 3y-3v \\ z-w \end{bmatrix}$

(b) Compute the result of adding vectors after a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix}$$

A. $\begin{bmatrix} x-u+z-w \\ 3y-3v \end{bmatrix}$

B. $\begin{bmatrix} x+u-z-w \\ 3y+3v \end{bmatrix}$

C. $\begin{bmatrix} x+u \\ 3y+3v \\ z+w \end{bmatrix}$

D. $\begin{bmatrix} x-u \\ 3y-3v \\ z-w \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

(d) Compute the result of scalar multiplcation before a T transformation:

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right)$$

A. $\begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$

B. $\begin{bmatrix} cx + cz \\ -3cy \end{bmatrix}$

C. $\begin{bmatrix} x + c \\ 3y + c \\ z + c \end{bmatrix}$

D. $\begin{bmatrix} x - c \\ 3y - c \\ z - c \end{bmatrix}$

(e) Compute the result of scalar multiplcation after a T transformation:

$$cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

A. $\begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$

B. $\begin{bmatrix} cx + cz \\ -3cy \end{bmatrix}$

C. $\begin{bmatrix} x + c \\ 3y + c \\ z + c \end{bmatrix}$

D. $\begin{bmatrix} x - c \\ 3y - c \\ z - c \end{bmatrix}$

(f) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Activity 3.1.5 Let $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

(a) Compute

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = S\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(b) Compute

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + S\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0^2 \\ 1+3 \\ 1-2^0 \end{bmatrix} + \begin{bmatrix} 2+3 \\ 2^2 \\ 3+3 \\ 3-2^2 \end{bmatrix}$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Activity 3.1.6 Fill in the $?$s, assuming $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear:

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(? \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = ?\, T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Remark 3.1.7 In summary, *any one* of the following is enough to prove that $T : V \to W$ is *not* a linear transformation:

- Find specific values for $\vec{v}, \vec{w} \in V$ such that $T(\vec{v} + \vec{w}) \neq T(\vec{v}) + T(\vec{w})$.
- Find specific values for $\vec{v} \in V$ and $c \in \mathbb{R}$ such that $T(c\vec{v}) \neq cT(\vec{v})$.
- Show $T(\vec{0}) \neq \vec{0}$.

If you cannot do any of these, then T can be proven to be a linear transformation by doing *both* of the following:

1. For all $\vec{v}, \vec{w} \in V$ (not just specific values), $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.
2. For all $\vec{v} \in V$ and $c \in \mathbb{R}$ (not just specific values), $T(c\vec{v}) = cT(\vec{v})$.

(Note the similarities between this process and showing that a subset of a vector space is or is not a subspace: Remark 2.3.10.)

Activity 3.1.8

(a) Consider the following maps of Euclidean vectors $P : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2\,x - 3\,y - 3\,z \\ 3\,x + 4\,y + 4\,z \\ 3\,x + 4\,y + 5\,z \end{bmatrix} \quad \text{and} \quad Q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 4\,y + 9\,z \\ y - 2\,z \\ 8\,y^2 - 3\,xz \end{bmatrix}.$$

Which do you *suspect*?

A. P is linear, but Q is not.

B. Q is linear, but P is not.

C. Both maps are linear.

D. Neither map is linear.

(b) Consider the following map of Euclidean vectors $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2\,y \\ 9\,xy \end{bmatrix}.$$

Prove that S *is not* a linear transformation.

(c) Consider the following map of Euclidean vectors $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 8\,x - 6\,y \\ 6\,x - 4\,y \end{bmatrix}.$$

Prove that T *is* a linear transformation.

3.2 Standard Matrices (AT2)

Learning Outcomes

- Translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

Remark 3.2.1 Recall that a linear map $T : V \to W$ satisfies

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Activity 3.2.2 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

D. $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Standard Matrices (AT2)

Activity 3.2.3 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Standard Matrices (AT2)

Activity 3.2.4 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity 3.2.5 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What piece of information would help you compute $T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right)$?

A. The value of $T\left(\begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}\right)$.

B. The value of $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

C. The value of $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

D. Any of the above.

Fact 3.2.6 *Consider any basis* $\{\vec{b}_1, \ldots, \vec{b}_n\}$ *for* V*. Since every vector* $\vec{v}$ *can be written as a linear combination of basis vectors,* $\vec{v} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$*, we may compute* $T(\vec{v})$ *as follows:*

$$T(\vec{v}) = T(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n) = x_1T(\vec{b}_1) + \cdots + x_nT(\vec{b}_n).$$

Therefore any linear transformation $T : V \to W$ *can be defined by just describing the values of* $T(\vec{b}_i)$*.*

Put another way, the images of the basis vectors completely ***determine*** *the transformation* T*.*

Definition 3.2.7 Since a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by its action on the standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$, it is convenient to store this information in an $m \times n$ matrix, called the **standard matrix** of T, given by $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\vec{e}_1\right) = T\left(\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right] \quad T\left(\vec{e}_2\right) = T\left(\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -1 \\ 4 \end{smallmatrix}\right] \quad T\left(\vec{e}_3\right) = T\left(\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}\right]$$

Then the standard matrix corresponding to T is

$$\left[\begin{array}{ccc} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{array}\right] = \left[\begin{array}{ccc} 3 & -1 & 5 \\ 2 & 4 & 0 \end{array}\right].$$

$\Diamond$

Activity 3.2.8 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\vec{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\vec{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\vec{e}_1) \ \cdots \ T(\vec{e}_n)]$ for T.

Activity 3.2.9 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

(a) Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

(b) Find the standard matrix for T.

Fact 3.2.10 *Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\vec{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an array of the coefficients of the x_i:*

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Since the formula for a linear transformation T and its standard matrix A may both be used to compute the transformation of a vector $\vec{x}$, we will often write $T(\vec{x})$ and $A\vec{x}$ interchangably:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + by_2 + cx_3 + dx_4 \\ ex_1 + fy_2 + gx_3 + hx_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Activity 3.2.11 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

(b) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity 3.2.12 Compute the following linear transformations of vectors given their standard matrices.

(a)

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1=\begin{bmatrix}4&3\\0&-1\\1&1\\3&0\end{bmatrix}$$

(b)

$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2=\begin{bmatrix}4&3&0&-1\\1&1&3&0\end{bmatrix}$$

(c)

$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3=\begin{bmatrix}4&3&0\\0&-1&3\\5&1&1\\3&0&0\end{bmatrix}$$

3.3 Image and Kernel (AT3)

Learning Outcomes

- Compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.

Image and Kernel (AT3)

Activity 3.3.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes the set of all vectors that transform into $\vec{0}$?

A. $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

Definition 3.3.2 Let $T : V \to W$ be a linear transformation, and let $\vec{z}$ be the additive identity (the "zero vector") of W. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = \vec{z} \right\}$$

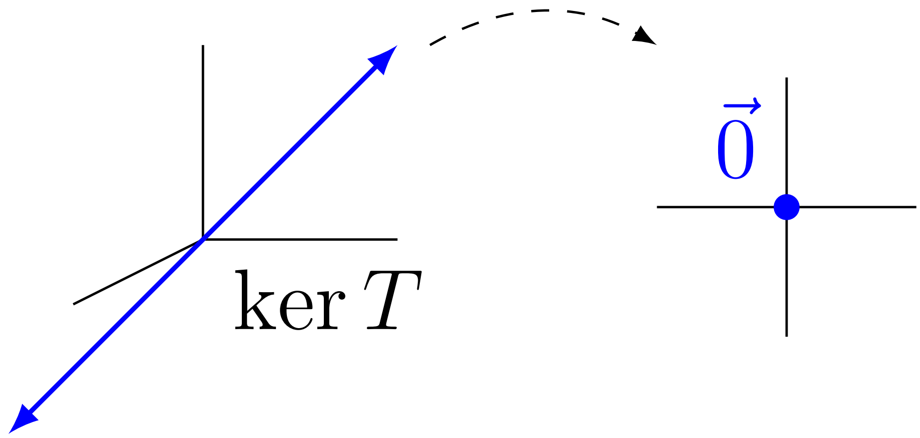


Figure 9 The kernel of a linear transformation

$\Diamond$

Activity 3.3.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

Activity 3.3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

(a) Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

(b) Use $\text{RREF}(A)$ to solve this homogeneous system of equations and find a basis for the kernel of T.

Activity 3.3.5 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

Find a basis for the kernel of T.

Image and Kernel (AT3)

Activity 3.3.6 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

Definition 3.3.7 Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \vec{w} \in W \;\middle|\; \text{there is some } \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w} \right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.

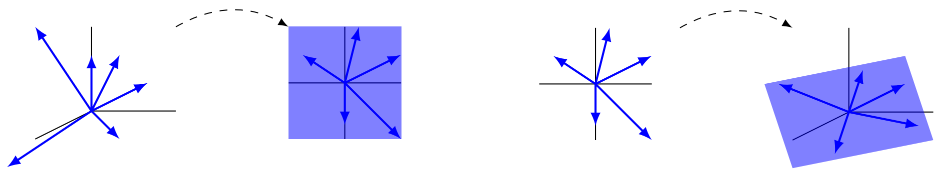


Figure 10 The image of a linear transformation

$\Diamond$

Activity 3.3.8 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

A. $\left\{\begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

B. $\left\{\begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

C. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

D. $\left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$

Activity 3.3.9 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix}.$$

Consider the question: Which vectors $\vec{w}$ in $\mathbb{R}^3$ belong to $\operatorname{Im} T$?

(a) Determine if $\begin{bmatrix} 12 \\ 3 \\ 3 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(b) Determine if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(c) An arbitrary vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ belongs to $\operatorname{Im} T$ provided the equation

$$x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) + x_4 T(\vec{e}_4) = \vec{w}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) Based on this, how do $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ relate to each other?

A. The set $\operatorname{Im} T$ contains $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ but is not equal to it.

B. The set $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ contains $\operatorname{Im} T$ but is not equal to it.

C. The set $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ are equal to each other.

D. There is no relation between these two sets.

Observation 3.3.10 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \left[\begin{array}{cccc} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{array}\right].$$

Since the set $\left\{\left[\begin{array}{c} 3 \\ -1 \\ 2 \end{array}\right], \left[\begin{array}{c} 4 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{c} 7 \\ 0 \\ 3 \end{array}\right], \left[\begin{array}{c} 1 \\ 2 \\ -1 \end{array}\right]\right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A = \left[\begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ and only using the vectors corresponding to pivot columns:

$$\left\{\left[\begin{array}{c} 3 \\ -1 \\ 2 \end{array}\right], \left[\begin{array}{c} 4 \\ 1 \\ 1 \end{array}\right]\right\}$$

Image and Kernel (AT3)

Fact 3.3.11 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.*

- *The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel.*

- *The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in* $\mathrm{RREF}\, A$ *to get a basis for the image.*

Activity 3.3.12 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Image and Kernel (AT3)

Activity 3.3.13 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the kernel of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Image and Kernel (AT3)

Activity 3.3.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the image of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Observation 3.3.15 Combining these with the observation that the number of columns is the dimension of the domain of T, we have the **rank-nullity theorem**:

$$\text{The dimension of the domain of } T \text{ equals } \dim(\ker T) + \dim(\operatorname{Im} T).$$

The dimension of the image is called the **rank** of T (or A) and the dimension of the kernel is called the **nullity**.

Image and Kernel (AT3)

Activity 3.3.16 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x - y + 5\,z + 3\,w \\ -x - 4\,z - 2\,w \\ y - 2\,z - w \end{bmatrix}.$$

(a) Explain and demonstrate how to find the image of T and a basis for that image.

(b) Explain and demonstrate how to find the kernel of T and a basis for that kernel.

(c) Explain and demonstrate how to find the rank and nullity of T, and why the rank-nullity theorem holds for T.

3.4 Injective and Surjective Linear Maps (AT4)

Learning Outcomes

- Determine if a given linear map is injective and/or surjective.

Definition 3.4.1 Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

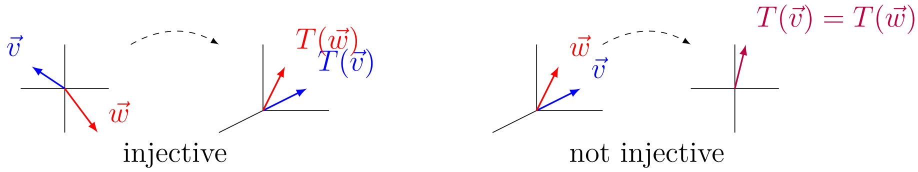


Figure 11 An injective transformation and a non-injective transformation

$\diamondsuit$

Activity 3.4.2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

Activity 3.4.3 Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\y\\0\end{bmatrix}\qquad\text{with standard matrix }\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v})=T(\vec{w})$ whenever $\vec{v}=\vec{w}$.

B. Yes, because $T(\vec{v})\neq T(\vec{w})$ whenever $\vec{v}\neq\vec{w}$.

C. No, because $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\neq T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)$.

Definition 3.4.4 Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $\vec{v} \in V$ with $T(\vec{v}) = \vec{w}$.

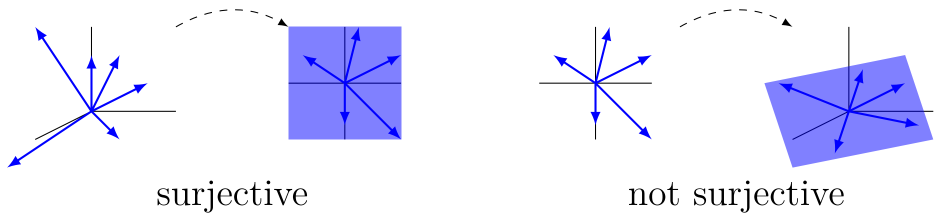


Figure 12 A surjective transformation and a non-surjective transformation

$\Diamond$

Activity 3.4.5 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

B. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity 3.4.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

B. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Activity 3.4.7 Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Fact 3.4.8 *A linear transformation T is injective* if and only if $\ker T = \{\vec{0}\}$. *Put another way, an injective linear transformation may be recognized by its* ***trivial*** *kernel.*

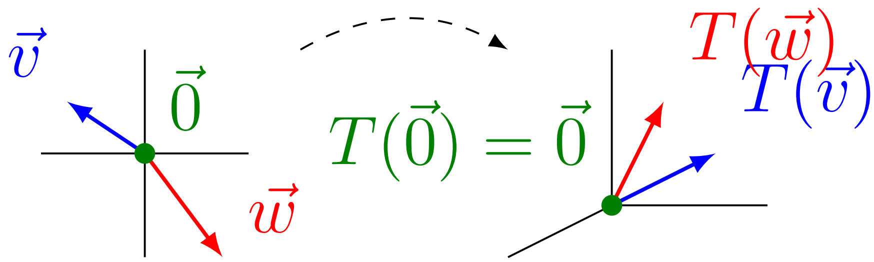


Figure 13 *A linear transformation with trivial kernel, which is therefore injective*

Activity 3.4.9 Let $T : V \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T$ may be spanned by only two vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Fact 3.4.10 *A linear transformation $T : V \to W$ is surjective* if and only if $\operatorname{Im} T = W$. *Put another way, a surjective linear transformation may be recognized by its identical codomain and image.*

surjective, $\operatorname{Im} T = \mathbb{R}^2$ *not surjective,* $\operatorname{Im} T \neq \mathbb{R}^3$

Figure 14 *A linear transformation with identical codomain and image, which is therefore surjective; and a linear transformation with an image smaller than the codomain* $\mathbb{R}^3$*, which is therefore not surjective.*

Definition 3.4.11 A transformation that is both injective and surjective is said to be **bijective**. ◊

Activity 3.4.12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The kernel of T is trivial, i.e. $\ker T = \{\vec{0}\}$.
2. The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.
3. For every $\vec{w} \in \mathbb{R}^m$, the set $\{\vec{v} \in \mathbb{R}^n | T(\vec{v}) = \vec{w}\}$ contains exactly one vector.

Activity 3.4.13 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The columns of A span $\mathbb{R}^m$.
2. The columns of A form a basis for $\mathbb{R}^m$.
3. The columns of A are linearly independent.

Activity 3.4.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. $\operatorname{RREF}(A)$ is the identity matrix.
2. Every column of $\operatorname{RREF}(A)$ has a pivot.
3. Every row of $\operatorname{RREF}(A)$ has a pivot.

Activity 3.4.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$.

2. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for all $\vec{b} \in \mathbb{R}^m$.

3. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has exactly one solution.

Observation 3.4.16 The easiest way to determine if the linear map with standard matrix A is injective is to see if $\mathrm{RREF}(A)$ has a pivot in each column.

The easiest way to determine if the linear map with standard matrix A is surjective is to see if $\mathrm{RREF}(A)$ has a pivot in each row.

Activity 3.4.17 What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Activity 3.4.18 What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

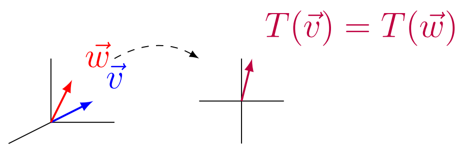
C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

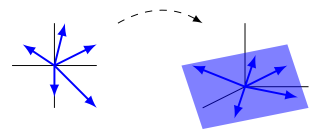
Fact 3.4.19 *The following are true for any linear map $T : V \to W$:*

- *If* $\dim(V) > \dim(W)$*, then T is not injective.*
- *If* $\dim(V) < \dim(W)$*, then T is not surjective.*

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase dimension from its domain to its image.



not injective, $3 > 2$



not surjective, $2 < 3$

Figure 15 *A linear transformation whose domain has a larger dimension than its codomain, and is therefore not injective; and a linear transformation whose domain has a smaller dimension than its codomain, and is therefore not surjective.*

But dimension arguments cannot *be used to prove a map* is *injective or surjective.*

Activity 3.4.20 Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is bijective.

(a) How many pivot rows must RREF A have?

(b) How many pivot columns must RREF A have?

(c) What is RREF A?

Activity 3.4.21 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

A. RREF(A) is the identity matrix.

B. The columns of A form a basis for $\mathbb{R}^n$

C. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.

Observation 3.4.22 The easiest way to show that the linear map with standard matrix A is bijective is to show that $\mathrm{RREF}(A)$ is the identity matrix.

Activity 3.4.23 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.24 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.25 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.26 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

3.5 Vector Spaces (AT5)

Learning Outcomes

- Explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property, but nonetheless isn't a vector space.

Observation 3.5.1 Consider the following applications of properties of the real numbers $\mathbb{R}$:

1. $1 + (2 + 3) = (1 + 2) + 3$.
2. $7 + 4 = 4 + 7$.
3. There exists some $?$ where $5 + ? = 5$.
4. There exists some $?$ where $9 + ? = 0$.
5. $\frac{1}{2}(1 + 7)$ is the only number that is equally distant from 1 and 7.

Activity 3.5.2 Which of the following properites of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

B. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

C. There exists some $\begin{bmatrix} ? \\ ? \end{bmatrix}$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

D. There exists some $\begin{bmatrix} ? \\ ? \end{bmatrix}$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

E. $\frac{1}{2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$ is the only vector whose endpoint is equally distant from the endpoints of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Observation 3.5.3 Consider the following applications of properites of the real numbers $\mathbb{R}$:

1. $3(2(7)) = (3 \cdot 2)(7)$.
2. $1(19) = 19$.
3. There exists some $?$ such that $? \cdot 4 = 9$.
4. $3 \cdot (2+8) = 3 \cdot 2 + 3 \cdot 8$.
5. $(2+7) \cdot 4 = 2 \cdot 4 + 7 \cdot 4$.

Activity 3.5.4 Which of the following properites of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $a\left(b\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ab\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

B. $1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

C. There exists some $?$ such that $? \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

D. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.

E. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

Fact 3.5.5 *Every Euclidean vector space $\mathbb{R}^n$ satisfies the following properties, where $\vec{u}, \vec{v}, \vec{w}$ are Euclidean vectors and a, b are scalars.*

1. *Vector addition is associative:* $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
2. *Vector addition is commutative:* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. *An additive identity exists: There exists some* $\vec{z}$ *where* $\vec{v} + \vec{z} = \vec{v}$.
4. *Additive inverses exist: There exists some* $-\vec{v}$ *where* $\vec{v} + (-\vec{v}) = \vec{z}$.
5. *Scalar multiplication is associative:* $a(b\vec{v}) = (ab)\vec{v}$.
6. *1 is a multiplicative identity:* $1\vec{v} = \vec{v}$.
7. *Scalar multiplication distributes over vector addition:* $a(\vec{u} + \vec{v}) = (a\vec{u}) + (a\vec{v})$.
8. *Scalar multiplication distributes over scalar addition:* $(a + b)\vec{v} = (a\vec{v}) + (b\vec{v})$.

Definition 3.5.6 A **vector space** V is any set of mathematical objects, called **vectors**, and a set of numbers, called **scalars**, with associated addition $\oplus$ and scalar multiplication $\odot$ operations that satisfy the following properties. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors belonging to V, and let a, b be scalars.

We always assume the codomain of our operations is V, i.e. that addition is a map $V \times V \to V$ and that scalar multiplication is a map $\mathbb{R} \times V \to V$.

Likewise, we only consider "real" vector spaces, i.e. those whose scalars come from $\mathbb{R}$. However, one can similarly define vector spaces with scalars from other fields like the complex or rational numbers.

1. Vector addition is associative: $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.

2. Vector addition is commutative: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.

3. An additive identity exists: There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.

4. Additive inverses exist: There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.

5. Scalar multiplication is associative: $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.

6. 1 is a multiplicative identity: $1 \odot \vec{v} = \vec{v}$.

7. Scalar multiplication distributes over vector addition: $a \odot (\vec{u} \oplus \vec{v}) = (a \odot \vec{u}) \oplus (a \odot \vec{v})$.

8. Scalar multiplication distributes over scalar addition: $(a + b) \odot \vec{v} = (a \odot \vec{v}) \oplus (b \odot \vec{v})$.

$\Diamond$

Remark 3.5.7 Consider the set $\mathbb{C}$ of complex numbers with the usual definition for addition: $(a + b\mathbf{i}) \oplus (c + d\mathbf{i}) = (a + c) + (b + d)\mathbf{i}$.

Let $\vec{u} = a + b\mathbf{i}$, $\vec{v} = c + d\mathbf{i}$, and $\vec{w} = e + f\mathbf{i}$. Then

$$\begin{aligned}
\vec{u} \oplus (\vec{v} \oplus \vec{w}) &= (a + b\mathbf{i}) \oplus ((c + d\mathbf{i}) \oplus (e + f\mathbf{i})) \\
&= (a + b\mathbf{i}) \oplus ((c + e) + (d + f)\mathbf{i}) \\
&= (a + c + e) + (b + d + f)\mathbf{i}
\end{aligned}$$

$$\begin{aligned}
(\vec{u} \oplus \vec{v}) \oplus \vec{w} &= ((a + b\mathbf{i}) \oplus (c + d\mathbf{i})) \oplus (e + f\mathbf{i}) \\
&= ((a + c) + (b + d)\mathbf{i}) \oplus (e + f\mathbf{i}) \\
&= (a + c + e) + (b + d + f)\mathbf{i}
\end{aligned}$$

This proves that complex addition is associative: $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$. The seven other vector space properties may also be verified, so $\mathbb{C}$ is an example of a vector space.

Remark 3.5.8 The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{C}$: Complex numbers.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathcal{P}_n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 3.5.9 Consider the set $V = \{(x, y) \mid y = 2^x\}$.

Which of the following vectors is not in V?

A. $(0, 0)$

B. $(1, 2)$

C. $(2, 4)$

D. $(3, 8)$

Activity 3.5.10 Consider the set $V = \{(x, y) \mid y = 2^x\}$ with the operation $\oplus$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2).$$

Let $\vec{u}, \vec{v}$ be in V with $\vec{u} = (1, 2)$ and $\vec{v} = (2, 4)$. Using the operations defined for V, which of the following is $\vec{u} \oplus \vec{v}$?

A. $(2, 6)$

B. $(2, 8)$

C. $(3, 6)$

D. $(3, 8)$

Activity 3.5.11 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c).$$

Let $a = 2, b = -3$ be scalars and $\vec{u} = (1, 2) \in V$.

(a) Verify that

$$(a + b) \odot \vec{u} = \left(-1, \frac{1}{2}\right).$$

(b) Compute the value of

$$(a \odot \vec{u}) \oplus (b \odot \vec{u}).$$

Activity 3.5.12 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c).$$

Let a, b be unspecified scalars in $\mathbb{R}$ and $\vec{u} = (x, y)$ be an unspecified vector in V.

(a) Show that both sides of the equation

$$(a + b) \odot (x, y) = (a \odot (x, y)) \oplus (b \odot (x, y))$$

simplify to the expression $(ax + bx, y^a y^b)$.

(b) Show that V contains an additive identity element $\vec{z} = (\,?\,,\,?\,)$ satisfying

$$(x, y) \oplus (\,?\,,\,?\,) = (x, y)$$

for all $(x, y) \in V$.

That is, pick appropriate values for $\vec{z} = (\,?\,,\,?\,)$ and then simplify $(x, y) \oplus (\,?\,,\,?\,)$ into just (x, y).

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Remark 3.5.13 It turns out $V = \{(x, y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c)$$

satisifes all eight properties from Definition 3.5.6.

Thus, V is a vector space.

Activity 3.5.14 Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2)$$

$$c \odot (x, y) = (x^c, y + c - 1).$$

(a) Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

(b) Show that V does not have an additive identity element $\vec{z} = (z, w)$ by showing that $(0, -1) \oplus (z, w) \neq (0, -1)$ no matter what the values of z, w are.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Activity 3.5.15 Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x, y) = (cx, cy).$$

(a) Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for *all* $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

(b) Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for *some* vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

3.6 Polynomial and Matrix Spaces (AT6)

Learning Outcomes

- Answer questions about vector spaces of polynomials or matrices.

Observation 3.6.1 Nearly every term we've defined for Euclidean vector spaces $\mathbb{R}^n$ was actually defined for all kinds of vector spaces:

- Definition 2.1.2
- Definition 2.1.3
- Definition 2.3.6
- Definition 2.4.2
- Definition 2.5.3
- Definition 3.1.1
- Definition 3.1.2
- Definition 3.3.2
- Definition 3.3.7
- Definition 3.4.1
- Definition 3.4.4
- Definition 3.4.11

Activity 3.6.2 Let V be a vector space with the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Which of these completes the following definition for a bijective linear map $T : V \to \mathbb{R}^3$?

$$T(\vec{v}) = T(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} a+b+c \\ 0 \\ 0 \end{bmatrix}$

C. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Fact 3.6.3 *Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ has a linear bijection T with Euclidean space $\mathbb{R}^n$ that simply swaps its basis with the standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$:*

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

This transformation (in fact, any linear bijection between vector spaces) is called an ***isomorphism****, and V is said to be* ***isomorphic*** *to $\mathbb{R}^n$.*

Note, in particular, that every *vector space of dimension n is isomorphic to $\mathbb{R}^n$.*

Activity 3.6.4 The matrix space $M_{2,2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ has the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) What is the dimension of $M_{2,2}$?

A. 2

B. 3

C. 4

D. 5

(b) Which Euclidean space is $M_{2,2}$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

(c) Describe an isomorphism $T : M_{2,2} \to \mathbb{R}^?$:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} ? \\ \\ \vdots \\ \\ ? \end{bmatrix}$$

Activity 3.6.5 The polynomial space $\mathcal{P}^4 = \left\{a + bx + cx^2 + dx^3 + ex^4 \middle| a, b, c, d, e \in \mathbb{R}\right\}$ has the basis

$$\left\{1, x, x^2, x^3, x^4\right\}.$$

(a) What is the dimension of $\mathcal{P}^4$?

A. 2

B. 3

C. 4

D. 5

(b) Which Euclidean space is $\mathcal{P}^4$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

(c) Describe an isomorphism $T : \mathcal{P}^4 \to \mathbb{R}^{\,?}$:

$$T\left(a + bx + cx^2 + dx^3 + ex^4\right) = \begin{bmatrix} ? \\ \\ \vdots \\ \\ ? \end{bmatrix}$$

Remark 3.6.6 Since any finite-dimensional vector space is isomorphic to a Euclidean space $\mathbb{R}^n$, one approach to answering questions about such spaces is to answer the corresponding question about $\mathbb{R}^n$.

Activity 3.6.7 Consider how to construct the polynomial $x^3 + x^2 + 5x + 1$ as a linear combination of polynomials from the set

$$\left\{x^3 - 2\,x^2 + x + 2, 2\,x^2 - 1, -x^3 + 3\,x^2 + 3\,x - 2, x^3 - 6\,x^2 + 9\,x + 5\right\}.$$

(a) Describe the vector space involved in this problem, and an isomorphic Euclidean space and relevant Eucldean vectors that can be used to solve this problem.

(b) Show how to construct an appropriate Euclidean vector from an approriate set of Euclidean vectors.

(c) Use this result to answer the original question.

Observation 3.6.8 The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional vector spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such vector spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Chapter 4

Matrices (MX)

Learning Outcomes

What algebraic structure do matrices have?
By the end of this chapter, you should be able to...

1. Multiply matrices.

2. Determine if a matrix is invertible, and if so, compute its inverse.

3. Invert an appropriate matrix to solve a system of linear equations.

4. Express row operations through matrix multiplication.

4.1 Matrices and Multiplication (MX1)

Learning Outcomes

- Multiply matrices.

Observation 4.1.1 If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ are linear maps, then the composition map $S \circ T$ computed as $(S \circ T)(\vec{v}) = S(T(\vec{v}))$ is a linear map from $\mathbb{R}^n \to \mathbb{R}^k$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^k \end{tikzcd}$$

Figure 16 The composition of two linear maps.

Activity 4.1.2 Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be defined by the 2×3 standard matrix B and $S:\mathbb{R}^2\to\mathbb{R}^4$ be defined by the 4×2 standard matrix A:

$$B=\begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix} \qquad A=\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

(a) What are the domain and codomain of the composition map $S\circ T$?

A. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

B. The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$

C. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$

D. The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$

(b) What size will the standard matrix of $S\circ T$ be?

A. 4 (rows) $\times$ 3 (columns)

B. 3 (rows) $\times$ 4 (columns)

C. 3 (rows) $\times$ 2 (columns)

D. 2 (rows) $\times$ 4 (columns)

(c) Compute

$$(S\circ T)(\vec{e}_1)=S(T(\vec{e}_1))=S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)=\begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

(d) Compute $(S\circ T)(\vec{e}_2)$.

(e) Compute $(S\circ T)(\vec{e}_3)$.

(f) Use $(S\circ T)(\vec{e}_1),(S\circ T)(\vec{e}_2),(S\circ T)(\vec{e}_3)$ to write the standard matrix for $S\circ T$.

Definition 4.1.3 We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, T was a map $\mathbb{R}^3 \to \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \to \mathbb{R}^4$, so $S \circ T$ gave a map $\mathbb{R}^3 \to \mathbb{R}^4$ with a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

$\Diamond$

Activity 4.1.4 Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

(a) Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

(b) Find the standard matrix AB of $S \circ T$.

(c) Find the standard matrix BA of $T \circ S$.

Activity 4.1.5 Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

(a) Find the domain and codomain of each of the three linear maps corresponding to A, B, and C.

(b) Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Activity 4.1.6 Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) Compute the product BA by hand.

(b) Check your work using technology. Using Octave:

```
B = [3 -4 0 ; 2 0 -1 ; 0 -3 3]
A = [2 7 -1 ; 0 3 2  ; 1 1 -1]
B*A
```

Activity 4.1.7 Of the following three matrices, only two may be multiplied.

$$A = \begin{bmatrix} -1 & 3 & -2 & -3 \\ 1 & -4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ -2 & 4 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

Explain which two can be multiplied and why. Then show how to find their product.

Activity 4.1.8 Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ In Fact 3.2.10 we adopted the notation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Verify that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ in terms of matrix multiplication.

4.2 The Inverse of a Matrix (MX2)

Learning Outcomes

- Determine if a matrix is invertible, and if so, compute its inverse.

Activity 4.2.1 Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Definition 4.2.2 The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position. $\lozenge$

Fact 4.2.3 *For any square matrix A, $IA = AI = A$:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity 4.2.4 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is *injective*, a group that means T is *surjective*, and a group that means T is *bijective*.

A. $T(\vec{x}) = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

B. $T(\vec{x}) = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

C. $T(\vec{x}) = \vec{0}$ has a unique solution.

D. The columns of A span $\mathbb{R}^m$

E. The columns of A are linearly independent

F. The columns of A are a basis of $\mathbb{R}^m$

G. Every column of $\text{RREF}(A)$ has a pivot

H. Every row of $\text{RREF}(A)$ has a pivot

I. $m = n$ and $\text{RREF}(A) = I$

Definition 4.2.5 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection with standard matrix A.

By item (B) from Activity 4.2.4 we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ that defines $T^{-1}(\vec{b})$ as the unique solution $\vec{x}$ satisfying $T(\vec{x}) = \vec{b}$, that is, $T(T^{-1}(\vec{b})) = \vec{b}$.

Furthermore, let

$$A^{-1} = [T^{-1}(\vec{e}_1) \quad \cdots \quad T^{-1}(\vec{e}_n)]$$

be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, and we also say that A is an **invertible** matrix. $\Diamond$

Activity 4.2.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear bijection given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

(a) To find $\vec{x} = T^{-1}(\vec{e}_1)$, we need to find the unique solution for $T(\vec{x}) = \vec{e}_1$. Which of these linear systems can be used to find this solution?

A. $\begin{aligned} 2x_1 \; -1x_2 \; -6x_3 &= x_1 \\ 2x_1 \; +1x_2 \; +3x_3 &= 0 \\ 1x_1 \; +1x_2 \; +4x_3 &= 0 \end{aligned}$

B. $\begin{aligned} 2x_1 \; -1x_2 \; -6x_3 &= x_1 \\ 2x_1 \; +1x_2 \; +3x_3 &= x_2 \\ 1x_1 \; +1x_2 \; +4x_3 &= x_3 \end{aligned}$

C. $\begin{aligned} 2x_1 \; -1x_2 \; -6x_3 &= 1 \\ 2x_1 \; +1x_2 \; +3x_3 &= 0 \\ 1x_1 \; +1x_2 \; +4x_3 &= 0 \end{aligned}$

D. $\begin{aligned} 2x_1 \; -1x_2 \; -6x_3 &= 1 \\ 2x_1 \; +1x_2 \; +3x_3 &= 1 \\ 1x_1 \; +1x_2 \; +4x_3 &= 1 \end{aligned}$

(b) Use that system to find the solution $\vec{x} = T^{-1}(\vec{e}_1)$ for $T(\vec{x}) = \vec{e}_1$.

(c) Similarly, solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$, and solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

(d) Use these to write

$$A^{-1} = [T^{-1}(\vec{e}_1) \quad T^{-1}(\vec{e}_2) \quad T^{-1}(\vec{e}_3)],$$

the standard matrix for T^{-1}.

Activity 4.2.7 Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & 0 & -4 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

by computing how it transforms each of the standard basis vectors for $\mathbb{R}^4$: $T^{-1}(\vec{e}_1)$, $T^{-1}(\vec{e}_2)$, $T^{-1}(\vec{e}_3)$, and $T^{-1}(\vec{e}_4)$.

Activity 4.2.8 Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible?

A. Yes, because its transformation is a bijection.

B. Yes, because its transformation is not a bijection.

C. No, because its transformation is a bijection.

D. No, because its transformation is not a bijection.

Observation 4.2.9 An $n \times n$ matrix A is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Activity 4.2.10 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

(a) Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

(b) If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Observation 4.2.11 $T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1}$ equals the identity matrix I for any invertible matrix A.

4.3 Solving Systems with Matrix Inverses (MX3)

Learning Outcomes

- Invert an appropriate matrix to solve a system of linear equations.

Activity 4.3.1 Consider the following linear system with a unique solution:

$$\begin{array}{rcrcrcrcr} 3x_1 & - & 2x_2 & - & 2x_3 & - & 4x_4 & = & -7 \\ 2x_1 & - & x_2 & - & x_3 & - & x_4 & = & -1 \\ -x_1 & & & + & x_3 & & & = & -1 \\ & - & x_2 & & & - & 2x_4 & = & -5 \end{array}$$

(a) Suppose we let

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 & - & 2x_2 & - & 2x_3 & - & 4x_4 \\ 2x_1 & - & x_2 & - & x_3 & - & x_4 \\ -x_1 & & & + & x_3 & & \\ & - & x_2 & & & - & 2x_4 \end{bmatrix}.$$

Which of these choices would help us solve the given system?

A. Compute $T\left(\begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}\right)$

B. Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

(b) How can we express this in terms of matrix multiplication?

A. $\begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

B. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

C. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

D. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$

(c) How could a matrix equation of the form $A\vec{x} = \vec{b}$ be solved for $\vec{x}$?

A. Multiply: $(\text{RREF } A)(A\vec{x}) = (\text{RREF } A)\vec{b}$

B. Add: $(\text{RREF } A) + A\vec{x} = (\text{RREF } A) + \vec{b}$

C. Multiply: $(A^{-1})(A\vec{x}) = (A^{-1})\vec{b}$

D. Add: $(A^{-1}) + A\vec{x} = (A^{-1}) + \vec{b}$

(d) Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ using the method you chose in (c).

Remark 4.3.2 The linear system described by the augmented matrix $[A \mid \vec{b}]$ has exactly the same solution set as the matrix equation $A\vec{x} = \vec{b}$.

When A is invertible, then we have both $[A \mid \vec{b}] \sim [I \mid \vec{x}]$ and $\vec{x} = A^{-1}\vec{b}$, which can be seen as

$$\begin{aligned} && A\vec{x} &= \vec{b} \\ \Rightarrow && A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ \Rightarrow && \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Activity 4.3.3 Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$$

with a unique solution.

(a) Explain and demonstrate how this problem can be restated using matrix multiplication.

(b) Use the properties of matrix multiplication to find the unique solution.

4.4 Row Operations as Matrix Multiplication (MX4)

Learning Outcomes

- Express row operations through matrix multiplication.

Activity 4.4.1 Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

(a) Which of these tweaks of the identity matrix yields a matrix that doubles the third row of A when left-multiplying? ($2R_3 \to R_3$)

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

A. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) Which of these tweaks of the identity matrix yields a matrix that swaps the first and third rows of A when left-multiplying? ($R_1 \leftrightarrow R_3$)

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Which of these tweaks of the identity matrix yields a matrix that adds 5 times the third row of A to the first row when left-multiplying? ($R_1 + 5R_3 \to R_1$)

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Row Operations as Matrix Multiplication (MX4)

A. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 5 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Fact 4.4.2 *If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.*

- *Scaling a row:* $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- *Swapping rows:* $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- *Adding a row multiple to another row:* $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\mathrm{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices $R_1, R_2, \dots, R_k$.

Activity 4.4.3 What would happen if you *right*-multiplied by the tweaked identity matrix rather than left-multiplied?

A. The manipulated rows would be reversed.

B. Columns would be manipulated instead of rows.

C. The entries of the resulting matrix would be rotated 180 degrees.

Activity 4.4.4 Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$$
$$\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Activity 4.4.5 Let A be *any* 4×4 matrix.

(a) Give a 4×4 matrix M that may be used to perform the row operation $-5R_2 \to R_2$.

(b) Give a 4×4 matrix Y that may be used to perform the row operation $R_2 \leftrightarrow R_3$.

(c) Use matrix multiplication to describe the matrix obtained by applying $-5R_2 \to R_2$ and then $R_2 \leftrightarrow R_3$ to A (note the order).

Chapter 5

Geometric Properties of Linear Maps (GT)

Learning Outcomes

How do we understand linear maps geometrically?
By the end of this chapter, you should be able to...

1. Describe how a row operation affects the determinant of a matrix.
2. Compute the determinant of a 4×4 matrix.
3. Find the eigenvalues of a 2×2 matrix.
4. Find a basis for the eigenspace of a 4×4 matrix associated with a given eigenvalue.

5.1 Row Operations and Determinants (GT1)

Learning Outcomes

- Describe how a row operation affects the determinant of a matrix.

Activity 5.1.1 The image in Figure 46 illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

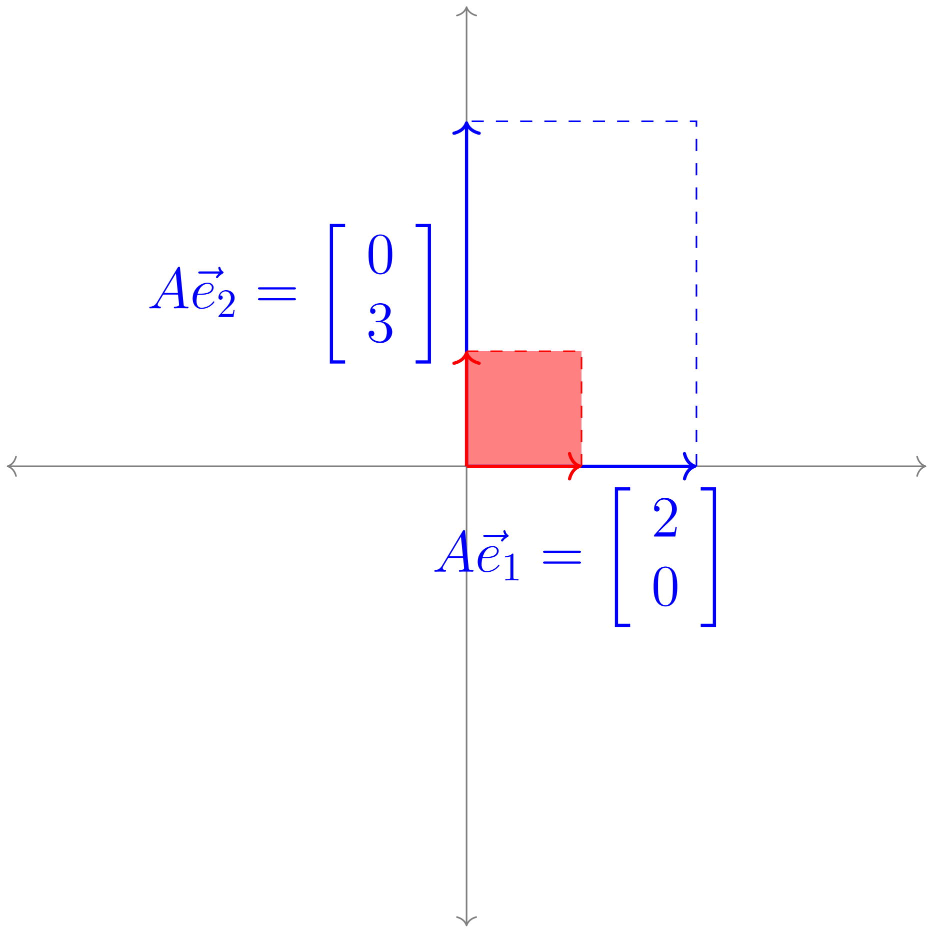


Figure 17 Transformation of the unit square by the matrix A.

(a) What are the lengths of $A\vec{e}_1$ and $A\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity 5.1.2 The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ transforms the unit square.

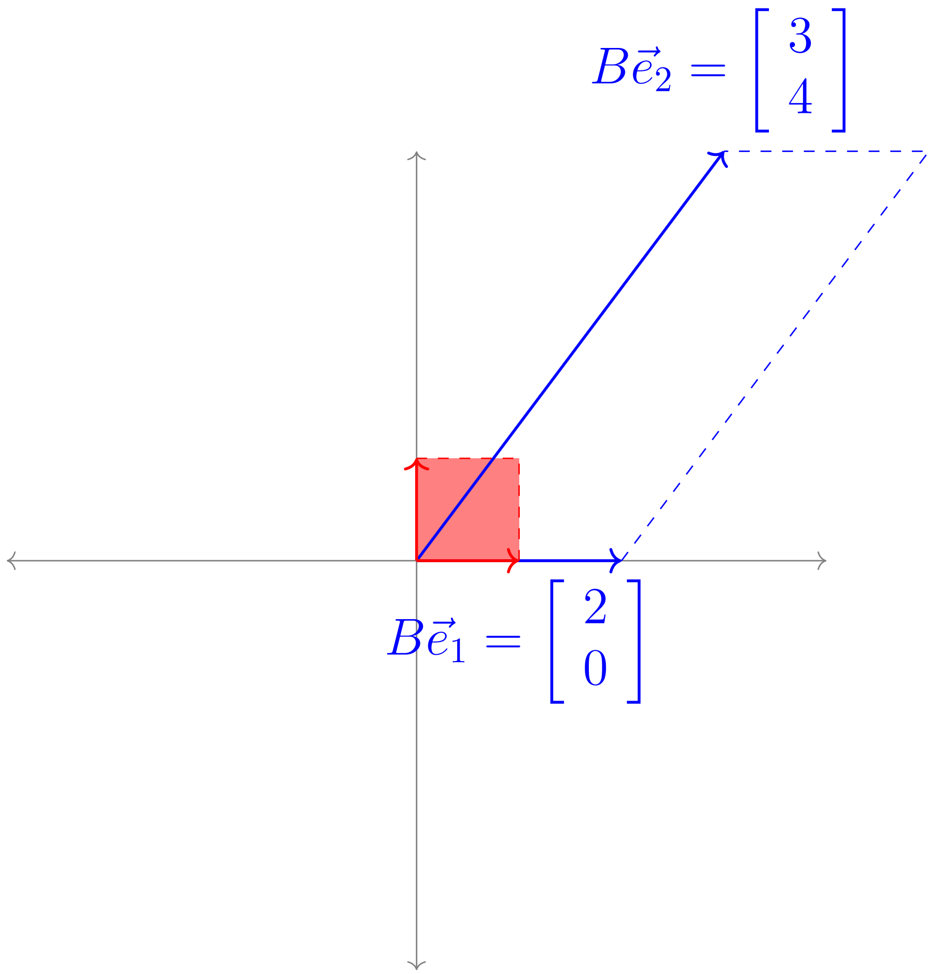


Figure 18 Transformation of the unit square by the matrix B

(a) What are the lengths of $B\vec{e}_1$ and $B\vec{e}_2$?

(b) What is the area of the transformed unit square?

Observation 5.1.3 It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by B.

$$B\vec{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

$$B \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

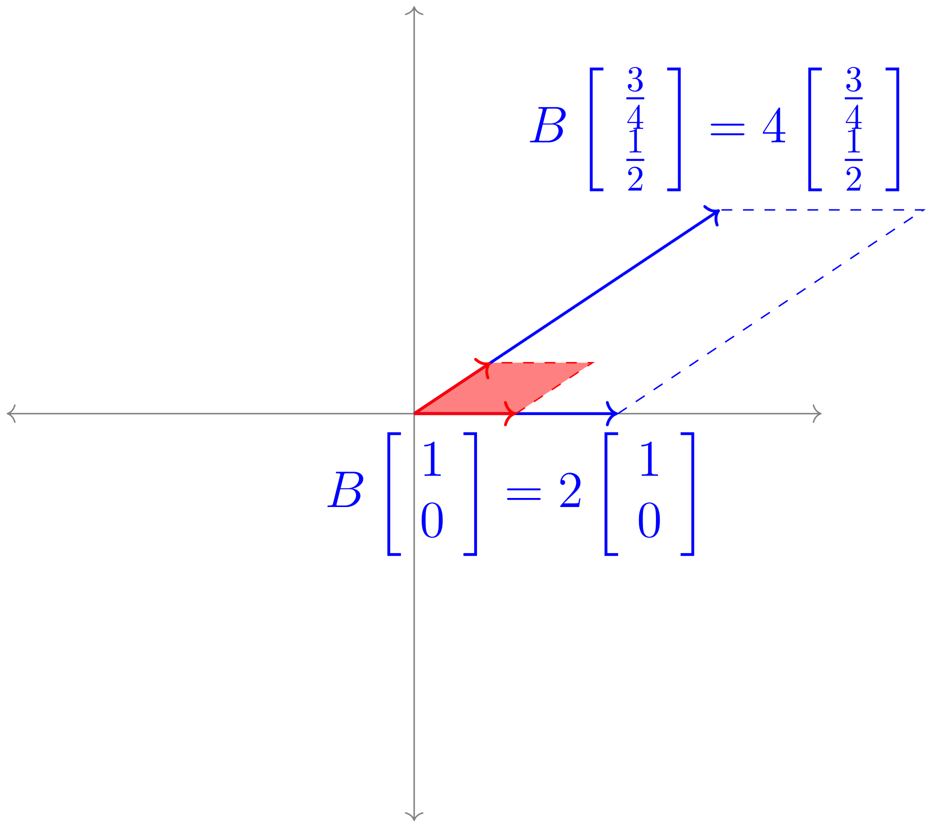


Figure 19 Certain vectors are stretched out without being rotated.

The process for finding such vectors will be covered later in this chapter.

Observation 5.1.4 Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, this factor is 8.

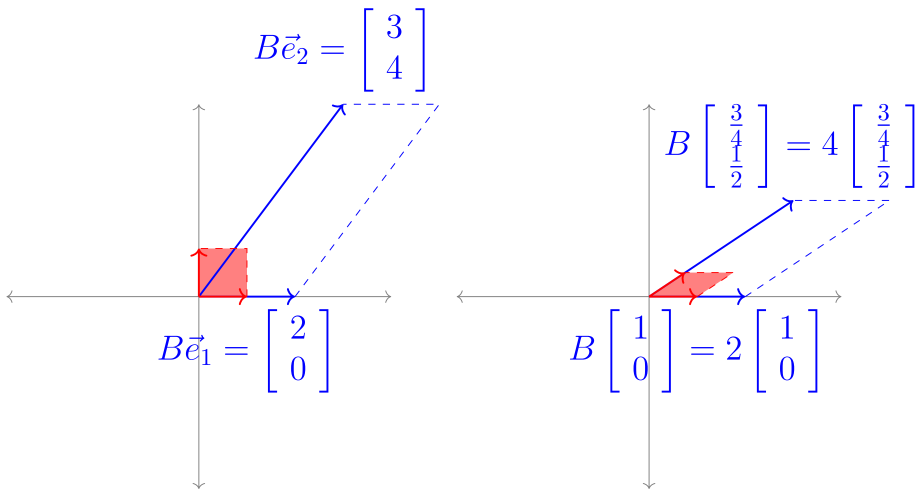


Figure 20 A linear map transforming parallelograms into parallelograms.

Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

Remark 5.1.5 We will define the **determinant** of a square matrix B, or $\det(B)$ for short, to be the factor by which B scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.

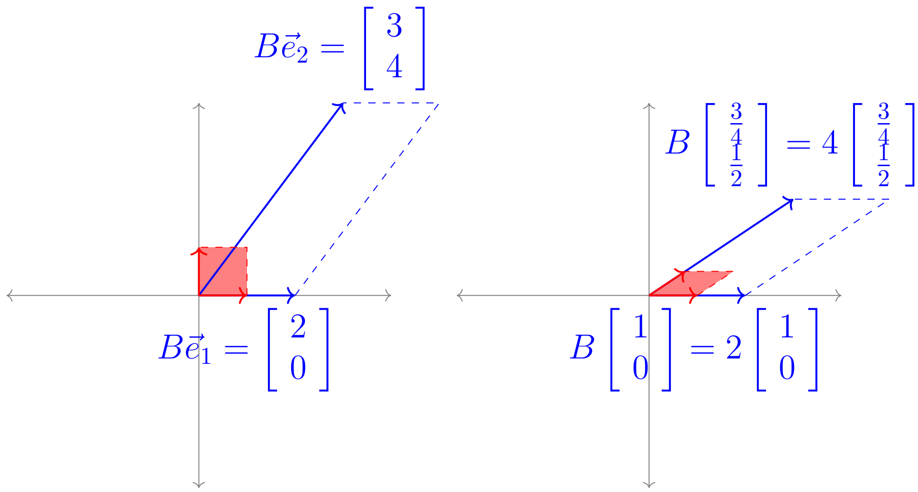


Figure 21 The linear transformation B scaling areas by a constant factor, which we call the **determinant**

Activity 5.1.6 The transformation of the unit square by the standard matrix $[\vec{e}_1 \ \ \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. If $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$ is the area of resulting parallelogram, what is the value of $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$?

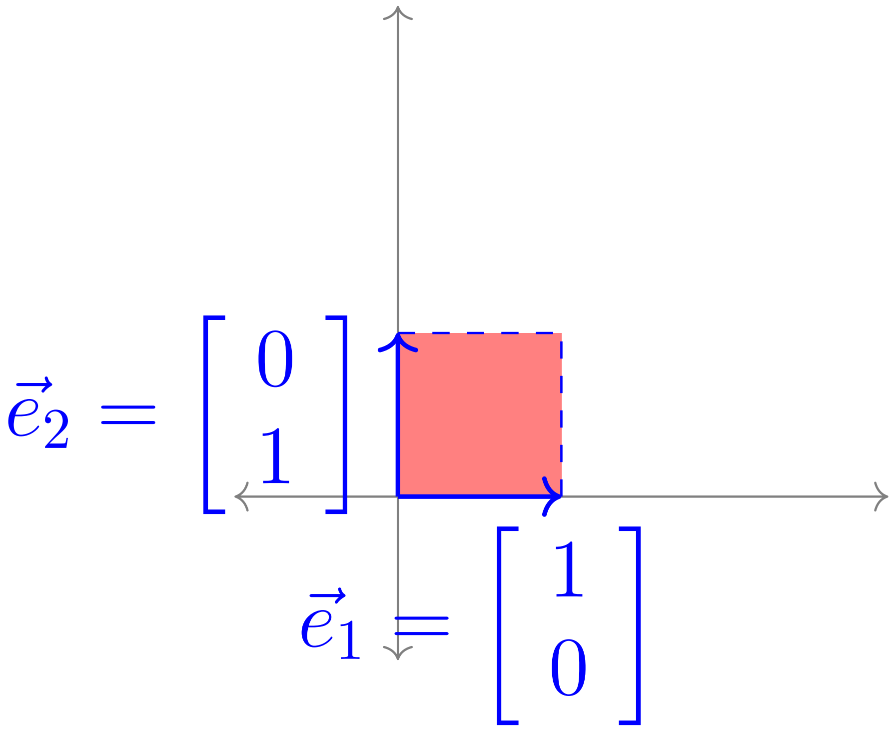


Figure 22 The transformation of the unit square by the identity matrix.

The value for $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$ is:

A. 0

B. 1

C. 2

D. 4

Activity 5.1.7 The transformation of the unit square by the standard matrix $[\vec{v}\ \ \vec{v}]$ is illustrated below: both $T(\vec{e}_1) = T(\vec{e}_2) = \vec{v}$. If $\det([\vec{v}\ \ \vec{v}])$ is the area of the generated parallelogram, what is the value of $\det([\vec{v}\ \ \vec{v}])$?

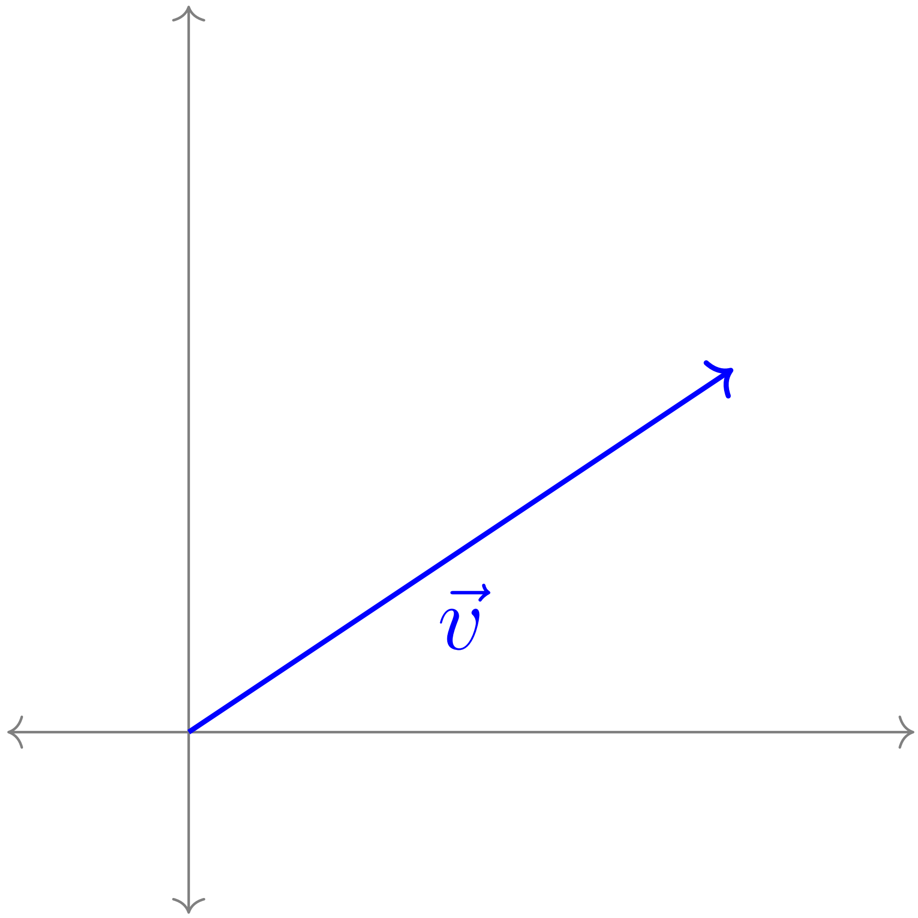


Figure 23 Transformation of the unit square by a matrix with identical columns.

The value of $\det([\vec{v}\ \ \vec{v}])$ is:

A. 0

B. 1

C. 2

D. 4

Activity 5.1.8 The transformations of the unit square by the standard matrices $[\vec{v}\ \ \vec{w}]$ and $[c\vec{v}\ \ \vec{w}]$ are illustrated below. Describe the value of $\det([c\vec{v}\ \ \vec{w}])$.

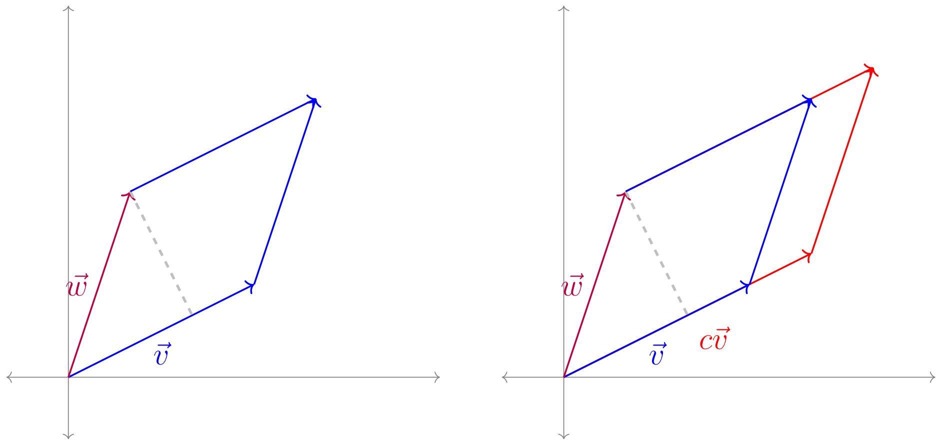


Figure 24 The parallelograms generated by $\vec{v}$ and $\vec{w}/c\vec{w}$

Describe the value of $\det([c\vec{v}\ \ \vec{w}])$:

A. $\det([\vec{v}\ \ \vec{w}])$

B. $c\det([\vec{v}\ \ \vec{w}])$

C. $c^2\det([\vec{v}\ \ \vec{w}])$

D. Cannot be determined from this information.

Remark 5.1.9 Consider the vectors $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{w}$ displayed below. Each pair of vectors generates a parallelogram, and the area of each parallelogram can be described in terms of determinants.

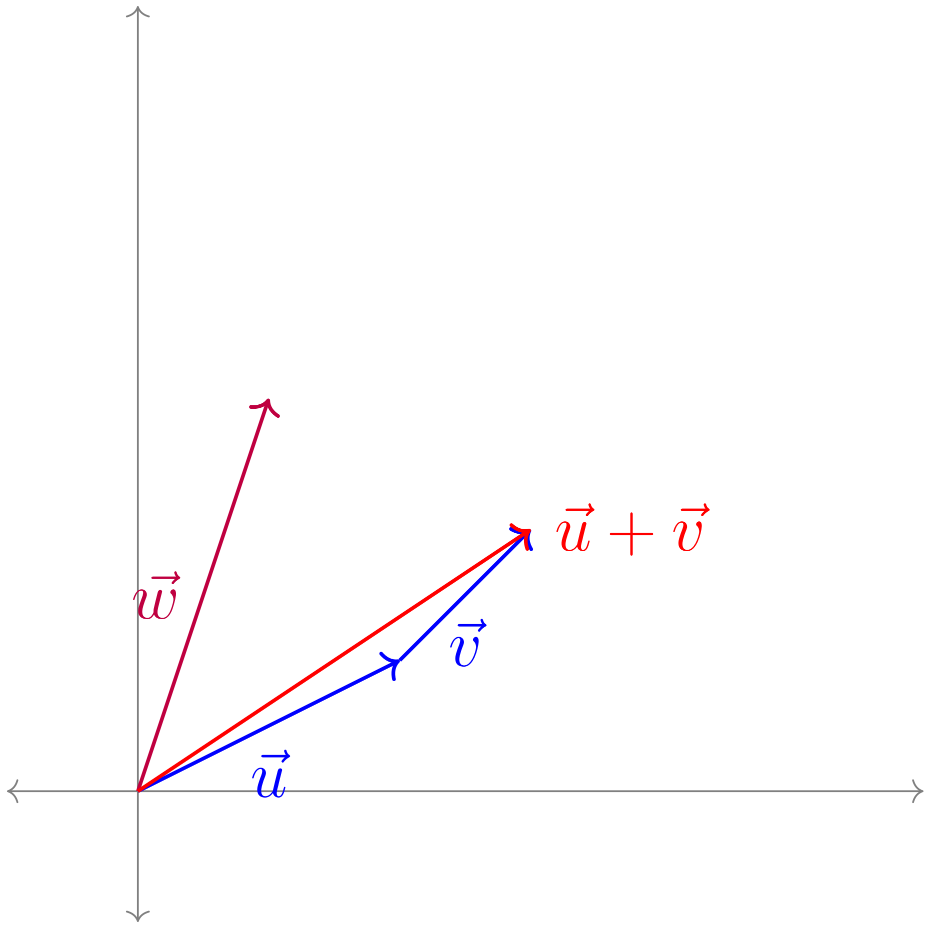


Figure 25 The vectors $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$ and $\vec{w}$

Remark 5.1.10 For example, $\det([\vec{u}\ \ \vec{w}])$ represents the shaded area shown below.

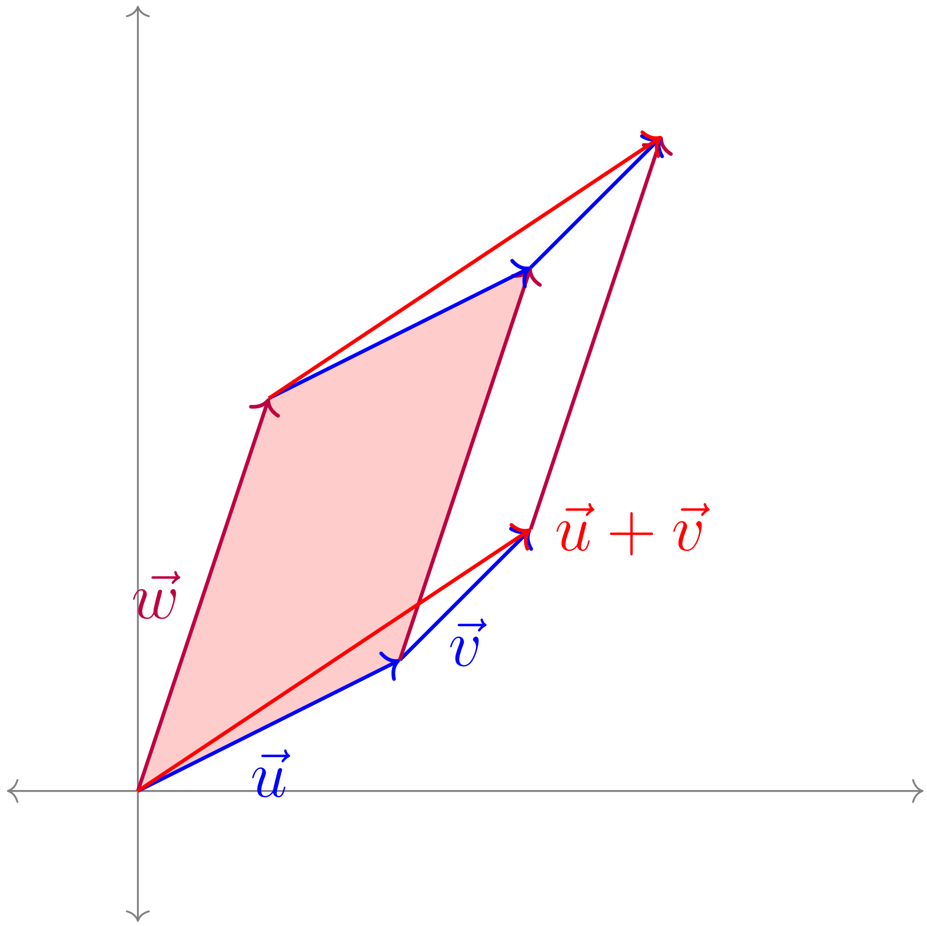


Figure 26 Parallelogram generated by $\vec{u}$ and $\vec{w}$

Remark 5.1.11 Similarly, $\det([\vec{v}\ \ \vec{w}])$ represents the shaded area shown below.

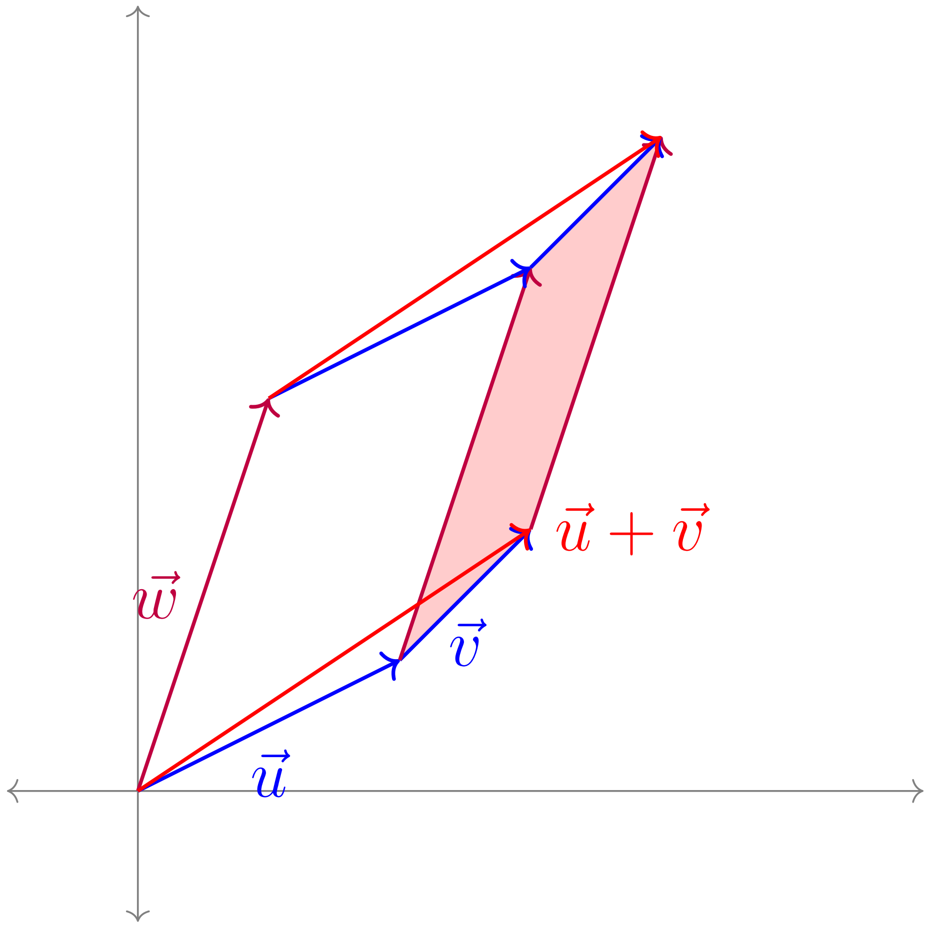


Figure 27 Parallelogram generated by $\vec{v}$ and $\vec{w}$

Activity 5.1.12 The parallelograms generated by the standard matrices $[\vec{u}\ \ \vec{w}]$, $[\vec{v}\ \ \vec{w}]$ and $[\vec{u}+\vec{v}\ \ \vec{w}]$ are illustrated below.

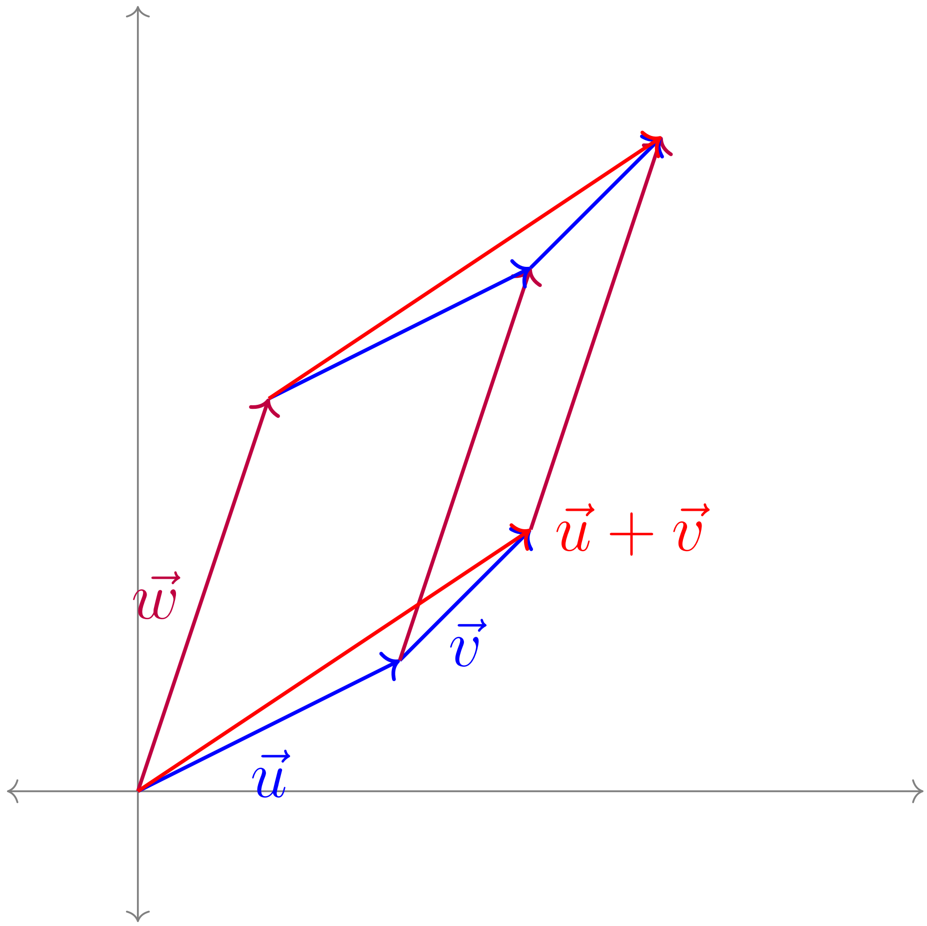


Figure 28 Parallelogram generated by $\vec{u}+\vec{v}$ and $\vec{w}$

Describe the value of $\det([\vec{u}+\vec{v}\ \ \vec{w}])$.

A. $\det([\vec{u}\ \ \vec{w}]) = \det([\vec{v}\ \ \vec{w}])$

B. $\det([\vec{u}\ \ \vec{w}]) + \det([\vec{v}\ \ \vec{w}])$

C. $\det([\vec{u}\ \ \vec{w}])\det([\vec{v}\ \ \vec{w}])$

D. Cannot be determined from this information.

Definition 5.1.13 The **determinant** is the unique function $\det : M_{n,n} \to \mathbb{R}$ satisfying these properties:

1. $\det(I) = 1$
2. $\det(A) = 0$ whenever two columns of the matrix are identical.
3. $\det[\cdots \ \ c\vec{v} \ \ \cdots] = c\det[\cdots \ \ \vec{v} \ \ \cdots]$, assuming no other columns change.
4. $\det[\cdots \ \ \vec{v}+\vec{w} \ \ \cdots] = \det[\cdots \ \ \vec{v} \ \ \cdots] + \det[\cdots \ \ \vec{w} \ \ \cdots]$, assuming no other columns change.

Note that these last two properties together can be phrased as "The determinant is linear in each column." $\Diamond$

Observation 5.1.14 The determinant must also satisfy other properties. Consider $\det([\vec{v} \quad \vec{w}+c\vec{v}])$ and $\det([\vec{v} \quad \vec{w}])$.

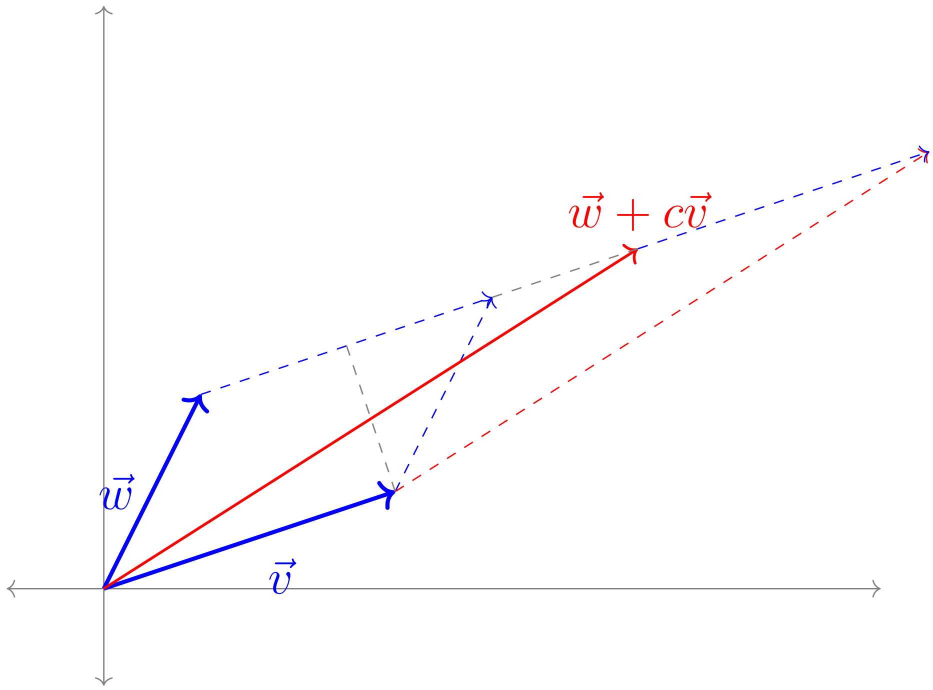


Figure 29 Parallelogram built by $\vec{w}+c\vec{v}$ and $\vec{w}$

The base of both parallelograms is $\vec{v}$, while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
\det([\vec{v}+c\vec{w} \quad \vec{w}]) &= \det([\vec{v} \quad \vec{w}]) + \det([c\vec{w} \quad \vec{w}])\\
&= \det([\vec{v} \quad \vec{w}]) + c\det([\vec{w} \quad \vec{w}])\\
&= \det([\vec{v} \quad \vec{w}]) + c \cdot 0\\
&= \det([\vec{v} \quad \vec{w}])
\end{aligned}$$

Remark 5.1.15 Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad \det A = 8 \qquad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} \quad \det B = -8$$

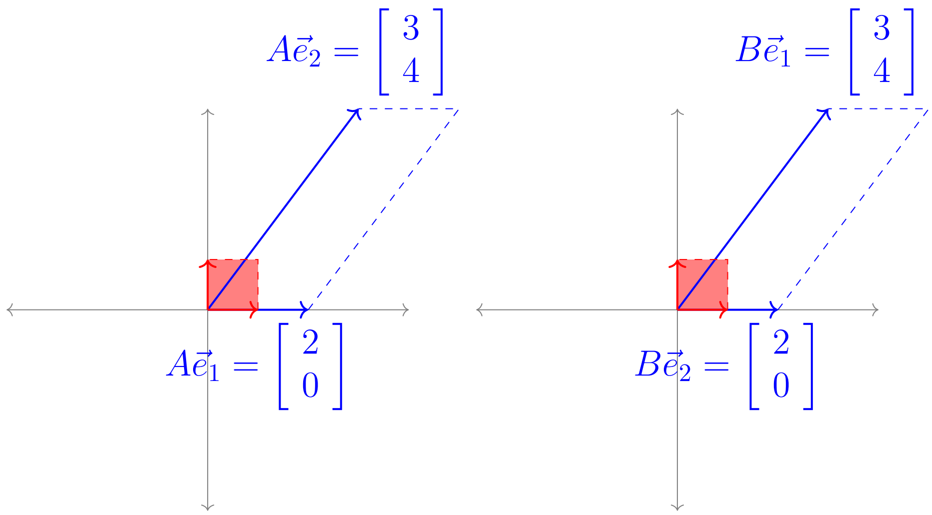


Figure 30 Reflection of a parallelogram as a result of swapping columns.

Observation 5.1.16 The fact that swapping columns multiplies determinants by a negative may be verified by adding and subtracting columns.

$$\begin{aligned}
\det([\vec{v} \quad \vec{w}]) &= \det([\vec{v}+\vec{w} \quad \vec{w}]) \\
&= \det([\vec{v}+\vec{w} \quad \vec{w}-(\vec{v}+\vec{w})]) \\
&= \det([\vec{v}+\vec{w} \quad -\vec{v}]) \\
&= \det([\vec{v}+\vec{w}-\vec{v} \quad -\vec{v}]) \\
&= \det([\vec{w} \quad -\vec{v}]) \\
&= -\det([\vec{w} \quad \vec{v}])
\end{aligned}$$

Fact 5.1.17 *To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:*

1. *Multiplying a column by a scalar multiplies the determinant by that scalar:*
$$c\det([\cdots \quad \vec{v} \quad \cdots]) = \det([\cdots \quad c\vec{v} \quad \cdots])$$

2. *Swapping two columns changes the sign of the determinant:*
$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = -\det([\cdots \quad \vec{w} \quad \cdots \quad \vec{v} \quad \cdots])$$

3. *Adding a multiple of a column to another column does not change the determinant:*
$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = \det([\cdots \quad \vec{v}+c\vec{w} \quad \cdots \quad \vec{w} \quad \cdots])$$

Activity 5.1.18 The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?

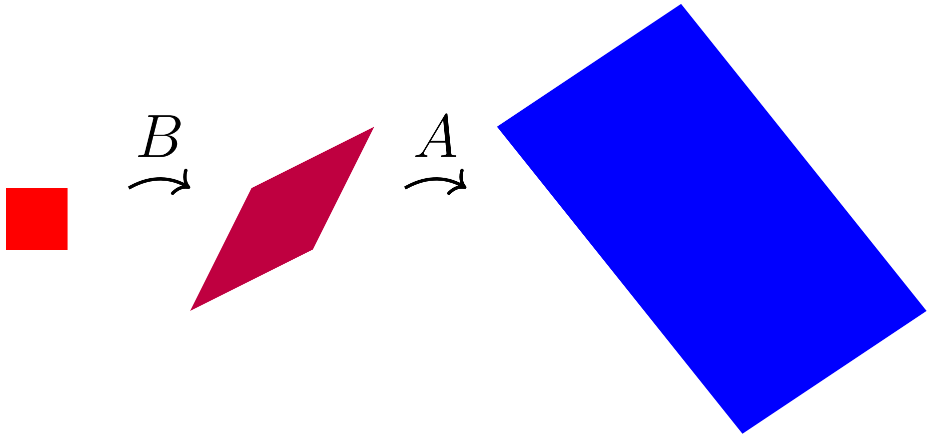


Figure 31 Area changing under the composition of two linear maps

A. 1

B. 7

C. 12

D. Cannot be determined

Fact 5.1.19 *Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that*

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA).$$

Remark 5.1.20 Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of A by c: $\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of A: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$
- Add c times the third row to the first row of A: $\begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

Fact 5.1.21 *The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:*

- *Scaling a row:* $\det\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c$

- *Swapping rows:* $\det\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1$

- *Adding a row multiple to another row:* $\det\begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$ $\det\begin{bmatrix} 1 & 0 & c-1c & 0 \\ 0 & 1 & 0-0c & 0 \\ 0 & 0 & 1-0c & 0 \\ 0 & 0 & 0-0c & 1 \end{bmatrix} = \det(I) = 1$

Activity 5.1.22 Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1+4(9) & 2+4(10) & 3+4(11) & 4+4(12) \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{4,4}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity 5.1.23 Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{4,4}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Activity 5.1.24 Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3(5) & 3(6) & 3(7) & 3(8) \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{4,4}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Row Operations and Determinants (GT1)

Activity 5.1.25 Let A be *any* 4×4 matrix with determinant 2.

(a) Let B be the matrix obtained from A by applying the row operation $R_1 - 5R_3 \to R_1$. What is $\det B$?

A -4 B -2 C 2 D 10

(b) Let M be the matrix obtained from A by applying the row operation $R_3 \leftrightarrow R_1$. What is $\det M$?

A -4 B -2 C 2 D 10

(c) Let P be the matrix obtained from A by applying the row operation $2R_4 \to R_4$. What is $\det P$?

A -4 B -2 C 2 D 10

Remark 5.1.26 Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

1. Multiplying columns by scalars:

$$\det([\cdots \quad c\vec{v} \quad \cdots]) = c\det([\cdots \quad \vec{v} \quad \cdots])$$

2. Swapping two columns:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = -\det([\cdots \quad \vec{w} \quad \cdots \quad \vec{v} \quad \cdots])$$

3. Adding a multiple of a column to another column:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = \det([\cdots \quad \vec{v}+c\vec{w} \quad \cdots \quad \vec{w} \quad \cdots])$$

Remark 5.1.27 The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Swapping rows: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Adding a row multiple to another row: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Fact 5.1.28 *Thus we can also use both row operations to simplify determinants:*

- *Multiplying rows by scalars:*

$$\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$$

- *Swapping two rows:*

$$\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$$

- *Adding multiples of rows/columns to other rows:*

$$\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$$

Activity 5.1.29 Complete the following derivation for a formula calculating 2×2 determinants:

$$\begin{aligned}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ? \det \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & b/a \\ c - c & d - bc/a \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & b/a \\ 0 & d - bc/a \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= ? \det I \\
&= ?
\end{aligned}$$

Observation 5.1.30 So we may compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by using determinant properties to manipulate its rows/columns to reduce the matrix to I:

$$\begin{aligned}
\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \\
&= 2 \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\
&= -2 \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\
&= -2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= -2
\end{aligned}$$

Or we may use a formula:

$$\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = (2)(3) - (4)(2) = -2$$

5.2 Computing Determinants (GT2)

Learning Outcomes

- Compute the determinant of a 4×4 matrix.

Remark 5.2.1 We've seen that row reducing all the way into RREF gives us a method of computing determinants.

However, we learned in Chapter 1 that this can be tedious for large matrices. Thus, we will try to figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

Activity 5.2.2 The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

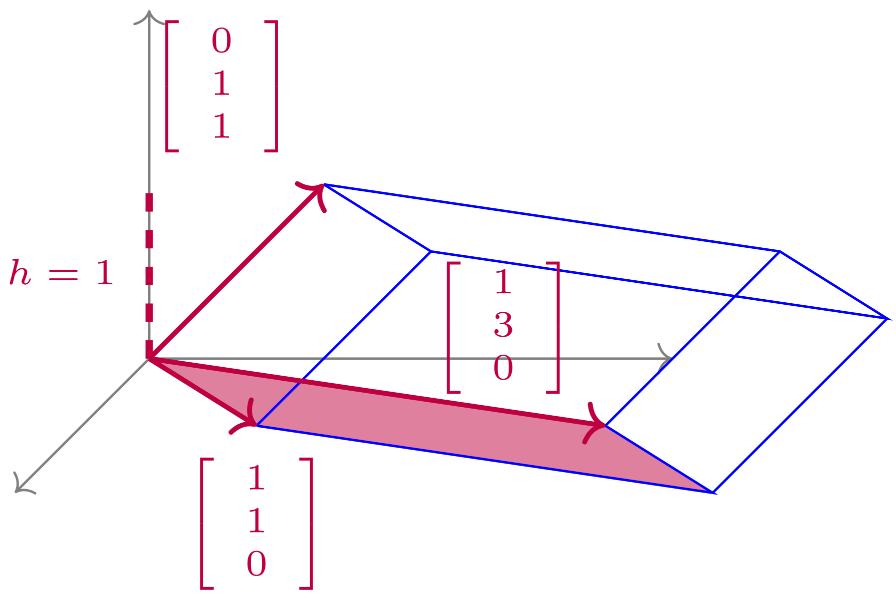


Figure 32 Transformation of the unit cube by the linear transformation.

Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

A. $\det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

B. $\det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

C. $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

D. $\det \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Fact 5.2.3 *If row i contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row i may be removed without changing the value of the determinant.*

$$\det\begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det\begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Put another way, if you have either a column or row from the identity matrix, you can cancel both the column and row containing the 1.

Activity 5.2.4 Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity 5.2.5 Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Factor out a 2 from a column.

(b) Swap rows or columns to put a 1 on the main diagonal.

Activity 5.2.6 Simplify $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Use row/column operations to create two zeroes in the same row or column.

(b) Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Observation 5.2.7 Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\det\begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix}$$

$$= \cdots = -2\det\begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2\det\begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix}$$

$$= \cdots = -2\det\begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2\det[-167]$$

$$= -2(-167)\det(I) = 334$$

Activity 5.2.8 Rewrite

$$\det \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 0 & 1 & 4 \\ -2 & 2 & 3 & 0 \\ -2 & 0 & -3 & -3 \end{bmatrix}$$

as a multiple of a determinant of a 3×3 matrix.

Activity 5.2.9 Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Observation 5.2.10 Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.

For example, since $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$,

$$\begin{aligned}
\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} &= 1\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 0 & 0 \end{bmatrix} + 2\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 1 & 0 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
&= -1\det\begin{bmatrix} 5 & 3 & 2 \\ 5 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 5 & 3 \\ -1 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
&= -\det\begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 5 \\ -1 & 5 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}
\end{aligned}$$

Observation 5.2.11 Recall the formula for a 2×2 determinant found in Observation 5.1.30:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

There are formulas and algorithms for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a 4×4 determinant would require 24 different terms!

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

Activity 5.2.12 Based on the previous activities, which technique is easier for computing determinants?

A. Memorizing formulas.

B. Using row/column operations.

C. Laplace expansion.

D. Some other technique.

Activity 5.2.13 Use your preferred technique to compute $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

5.3 Eigenvalues and Characteristic Polynomials (GT3)

Learning Outcomes

- Find the eigenvalues of a 2×2 matrix.

Activity 5.3.1 An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to $\det(M)\det(M^{-1})$?

A. -1

B. 0

C. 1

D. 4

Fact 5.3.2 *For every invertible matrix M,*

$$\det(M)\det(M^{-1}) = \det(I) = 1$$

so $\det(M^{-1}) = \frac{1}{\det(M)}$.

Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation 5.3.3 Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

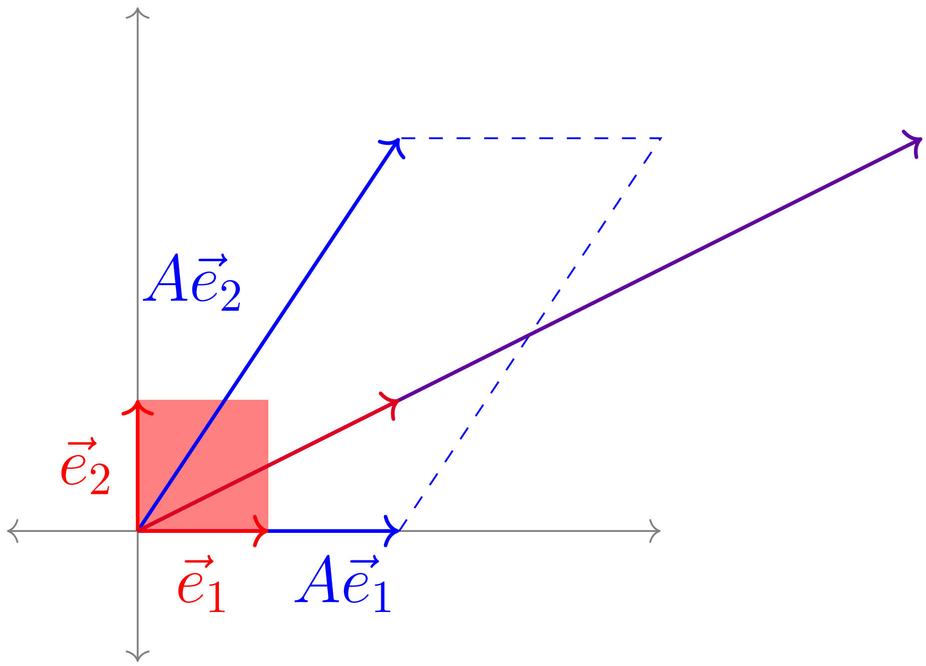


Figure 33 Transformation of the unit square by the linear transformation A

It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It is less obvious (but easily checked once you find it) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Definition 5.3.4 Let $A \in M_{n,n}$. An **eigenvector** for A is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$.

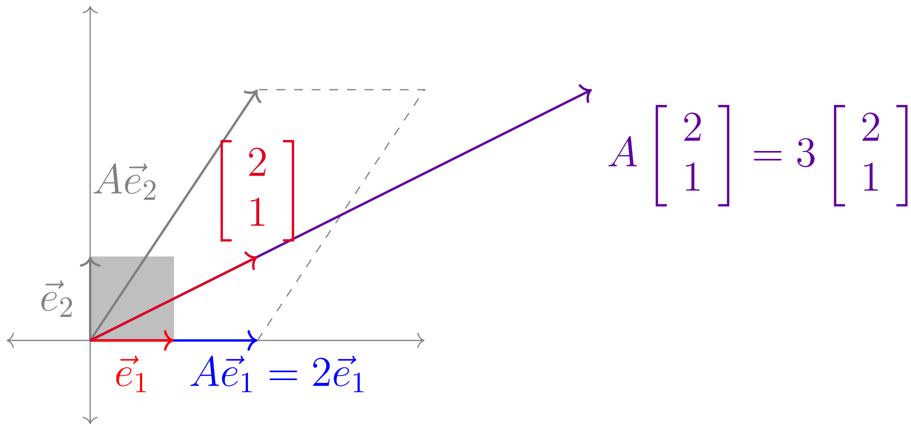


Figure 34 The map A stretches out the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by a factor of 3 (the corresponding eigenvalue).

In other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ. If $\vec{x} \neq \vec{0}$, then we say $\vec{x}$ is a **nontrivial eigenvector** and we call this λ an **eigenvalue** of A. $\diamondsuit$

Activity 5.3.5 Finding the eigenvalues λ that satisfy

$$A\vec{x} = \lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$$

for some nontrivial eigenvector $\vec{x}$ is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

(a) If λ is an eigenvalue, and T is the transformation with standard matrix $A - \lambda I$, which of these must contain a non-zero vector?

A. The kernel of T

B. The image of T

C. The domain of T

D. The codomain of T

(b) Therefore, what can we conclude?

A. A is invertible

B. A is not invertible

C. $A - \lambda I$ is invertible

D. $A - \lambda I$ is not invertible

(c) And what else?

A. $\det A = 0$

B. $\det A = 1$

C. $\det(A - \lambda I) = 0$

D. $\det(A - \lambda I) = 1$

Eigenvalues and Characteristic Polynomials (GT3)

Fact 5.3.6 *The eigenvalues λ for a matrix A are exactly the values that make $A - \lambda I$ non-invertible.*

Thus the eigenvalues λ for a matrix A are the solutions to the equation

$$\det(A - \lambda I) = 0.$$

Definition 5.3.7 The expression $\det(A - \lambda I)$ is called **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix}.$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(5) = \lambda^2 - 5\lambda - 6$$

and its eigenvalues are the solutions $-1, 6$ to $\lambda^2 - 5\lambda - 6 = 0$. $\diamondsuit$

Activity 5.3.8 Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

(a) Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

(b) Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of A.

Activity 5.3.9 Find all the eigenvalues for the matrix $A = \left[\begin{array}{cc} 3 & -3 \\ 2 & -4 \end{array}\right]$.

Activity 5.3.10 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$.

Activity 5.3.11 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

5.4 Eigenvectors and Eigenspaces (GT4)

Learning Outcomes

- Find a basis for the eigenspace of a 4×4 matrix associated with a given eigenvalue.

Activity 5.4.1 It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A-(-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{x}$ such that $A\vec{x} = -2\vec{x}$.

Definition 5.4.2 Since the kernel of a linear map is a subspace of $\mathbb{R}^n$, and the kernel obtained from $A - \lambda I$ contains all the eigenvectors associated with λ, we call this kernel the **eigenspace** of A associated with λ. $\Diamond$

Activity 5.4.3 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ associated with the eigenvalue 3.

Activity 5.4.4 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 5 & -2 & 0 & 4 \\ 6 & -2 & 1 & 5 \\ -2 & 1 & 2 & -3 \\ 4 & 5 & -3 & 6 \end{bmatrix}$ associated with the eigenvalue 1.

Activity 5.4.5 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ associated with the eigenvalue 2.

Appendix A

Applications

A.1 Civil Engineering: Trusses and Struts

Definition A.1.1 In engineering, a **truss** is a structure designed from several beams of material called **struts**, assembled to behave as a single object.

Figure 35 A simple truss

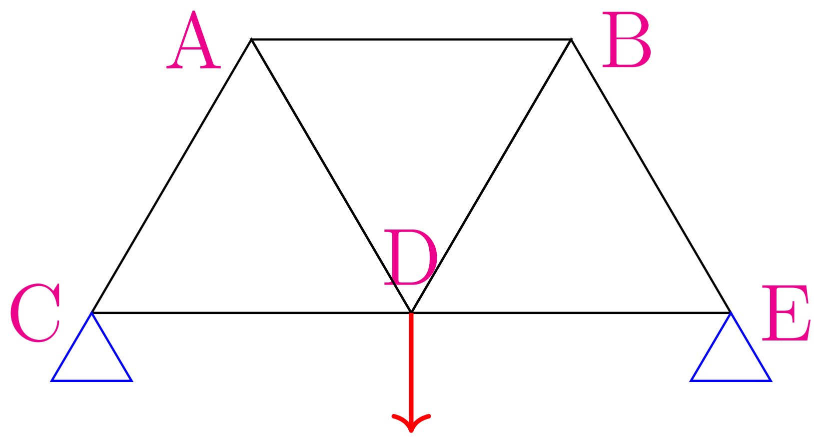


Figure 36 A simple truss

◊

Activity A.1.2 Consider the representation of a simple truss pictured below. All of the seven struts are of equal length, affixed to two anchor points applying a normal force to nodes C and E, and with a $10000N$ load applied to the node given by D.

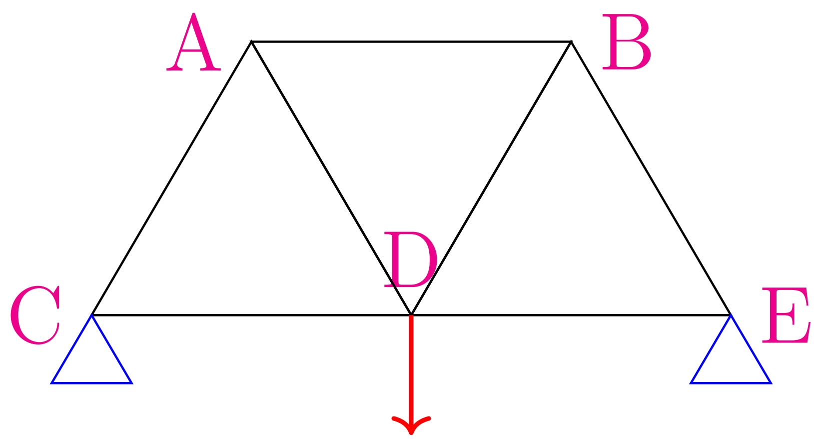


Figure 37 A simple truss

Which of the following must hold for the truss to be stable?

1. All of the struts will experience compression.
2. All of the struts will experience tension.
3. Some of the struts will be compressed, but others will be tensioned.

Observation A.1.3 Since the forces must balance at each node for the truss to be stable, some of the struts will be compressed, while others will be tensioned.

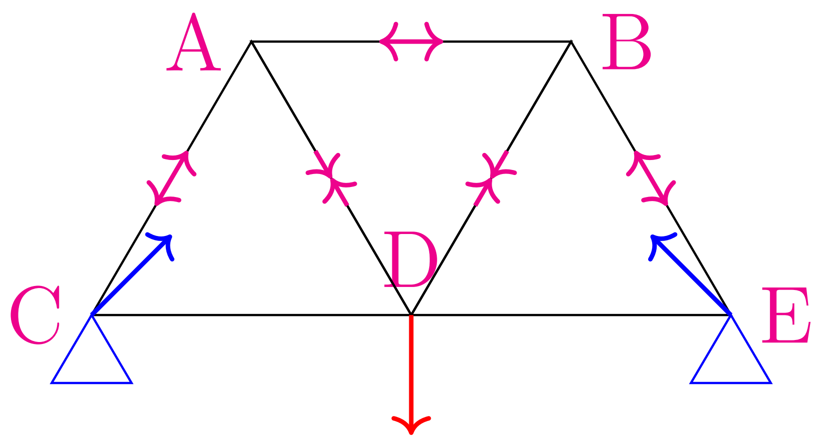


Figure 38 Completed truss

By finding vector equations that must hold at each node, we may determine many of the forces at play.

Remark A.1.4 For example, at the bottom left node there are 3 forces acting.

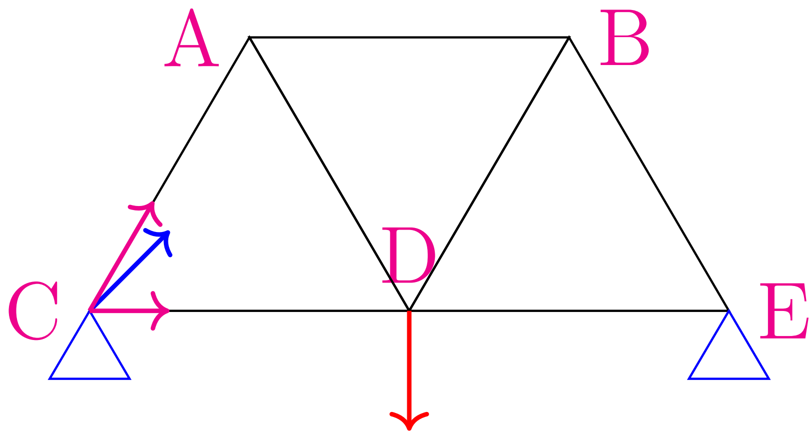


Figure 39 Truss with forces

Let $\vec{F}_{CA}$ be the force on C given by the compression/tension of the strut CA, let $\vec{F}_{CD}$ be defined similarly, and let $\vec{N}_C$ be the normal force of the anchor point on C.

For the truss to be stable, we must have:

$$\vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

Activity A.1.5 Using the conventions of the previous remark, and where $\vec{L}$ represents the load vector on node D, find four more vector equations that must be satisfied for each of the other four nodes of the truss.

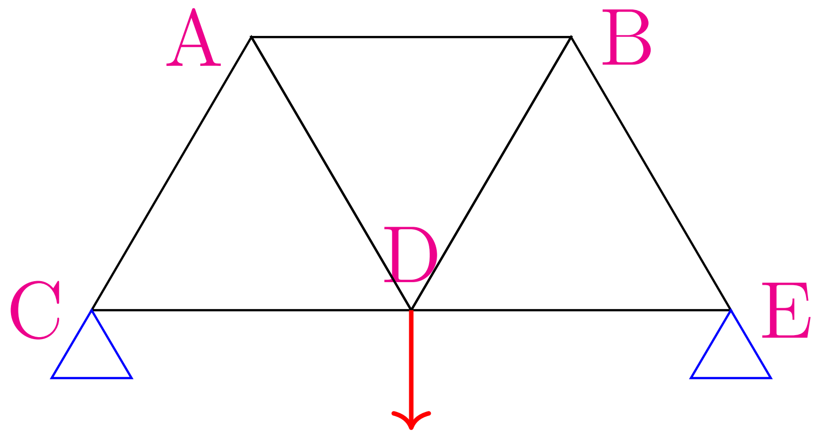


Figure 40 A simple truss

$$A : ?$$

$$B : ?$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$D : ?$$

$$E : ?$$

Remark A.1.6 The five vector equations may be written as follows.

$$A : \vec{F}_{AC} + \vec{F}_{AD} + \vec{F}_{AB} = \vec{0}$$

$$B : \vec{F}_{BA} + \vec{F}_{BD} + \vec{F}_{BE} = \vec{0}$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$D : \vec{F}_{DC} + \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DE} + \vec{L} = \vec{0}$$

$$E : \vec{F}_{EB} + \vec{F}_{ED} + \vec{N}_E = \vec{0}$$

Observation A.1.7 Each vector has a vertical and horizontal component, so it may be treated as a vector in $\mathbb{R}^2$. Note that $\vec{F}_{CA}$ must have the same magnitude (but opposite direction) as $\vec{F}_{AC}$.

$$\vec{F}_{CA} = x \begin{bmatrix} \cos(60°) \\ \sin(60°) \end{bmatrix} = x \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$\vec{F}_{AC} = x \begin{bmatrix} \cos(-120°) \\ \sin(-120°) \end{bmatrix} = x \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Activity A.1.8 To write a linear system that models the truss under consideration with constant load 10000 newtons, how many scalar variables will be required?

- 7: 5 from the nodes, 2 from the anchors
- 9: 7 from the struts, 2 from the anchors
- 11: 7 from the struts, 4 from the anchors
- 12: 7 from the struts, 4 from the anchors, 1 from the load
- 13: 5 from the nodes, 7 from the struts, 1 from the load

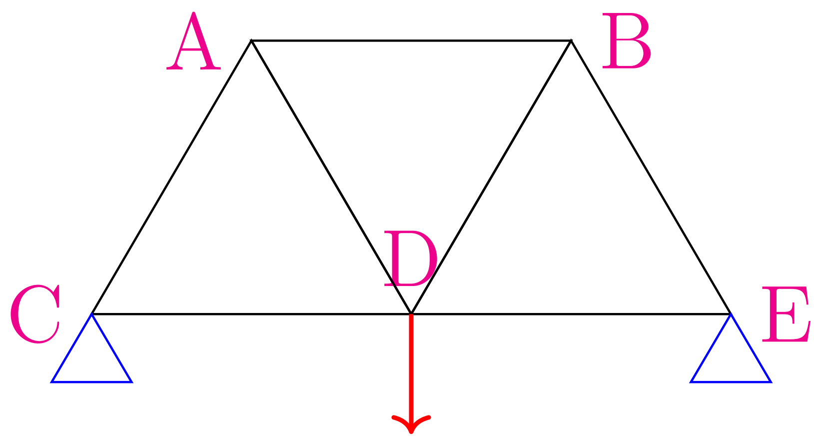


Figure 41 A simple truss

Observation A.1.9 Since the angles for each strut are known, one variable may be used to represent each.

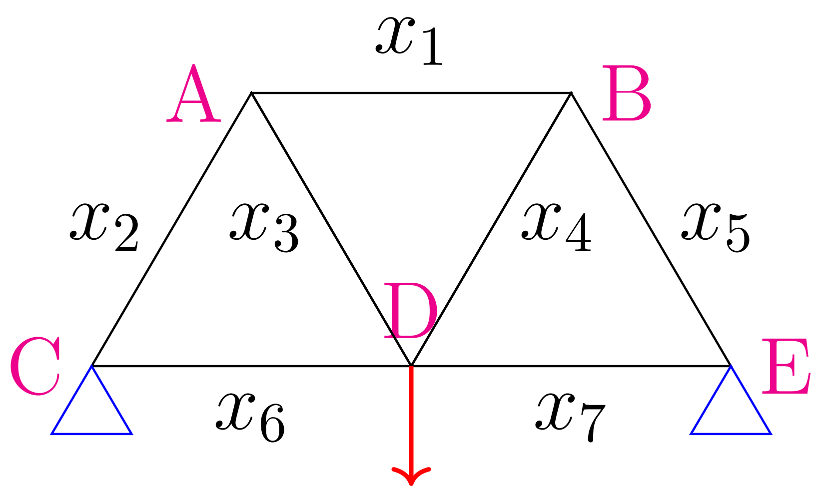


Figure 42 Variables for the truss

For example:

$$\vec{F}_{AB} = -\vec{F}_{BA} = x_1 \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{F}_{BE} = -\vec{F}_{EB} = x_5 \begin{bmatrix} \cos(-60°) \\ \sin(-60°) \end{bmatrix} = x_5 \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Observation A.1.10 Since the angle of the normal forces for each anchor point are unknown, two variables may be used to represent each.

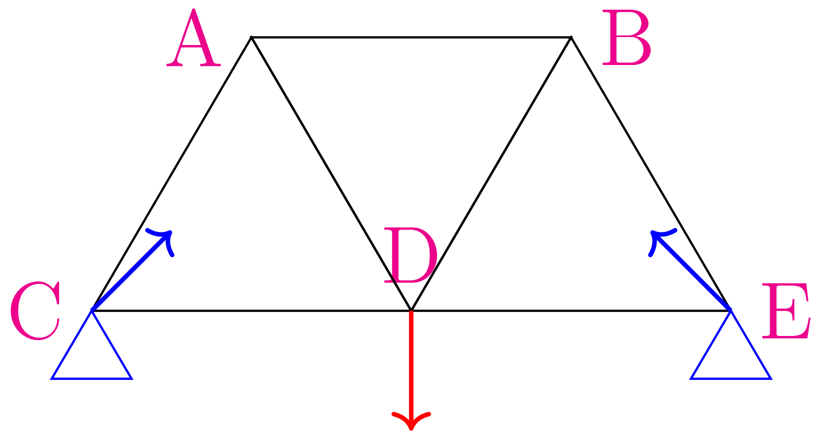


Figure 43 Truss with normal forces

$$\vec{N}_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \vec{N}_D = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

The load vector is constant.

$$\vec{L} = \begin{bmatrix} 0 \\ -10000 \end{bmatrix}$$

Remark A.1.11 Each of the five vector equations found previously represent two linear equations: one for the horizontal component and one for the vertical.

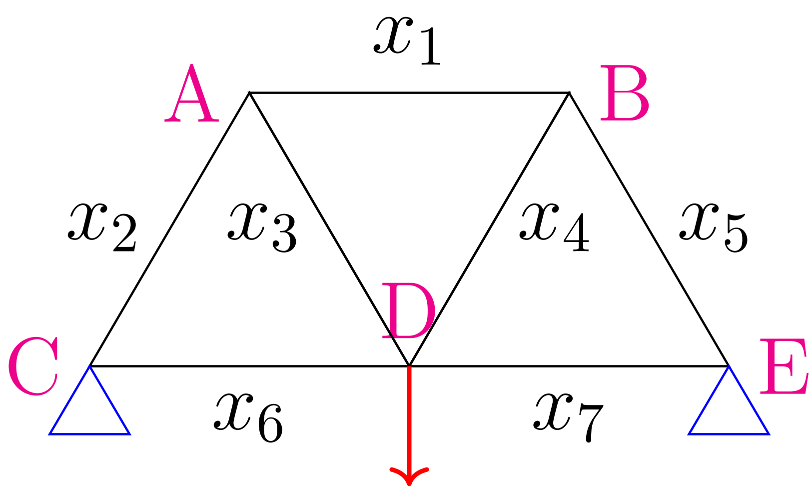


Figure 44 Variables for the truss

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$
$$\Leftrightarrow x_2 \begin{bmatrix} \cos(60°) \\ \sin(60°) \end{bmatrix} + x_6 \begin{bmatrix} \cos(0°) \\ \sin(0°) \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the approximation $\sqrt{3}/2 \approx 0.866$, we have

$$\Leftrightarrow x_2 \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Activity A.1.12 Expand the vector equation given below using sine and cosine of appropriate angles, then compute each component (approximating $\sqrt{3}/2 \approx 0.866$).

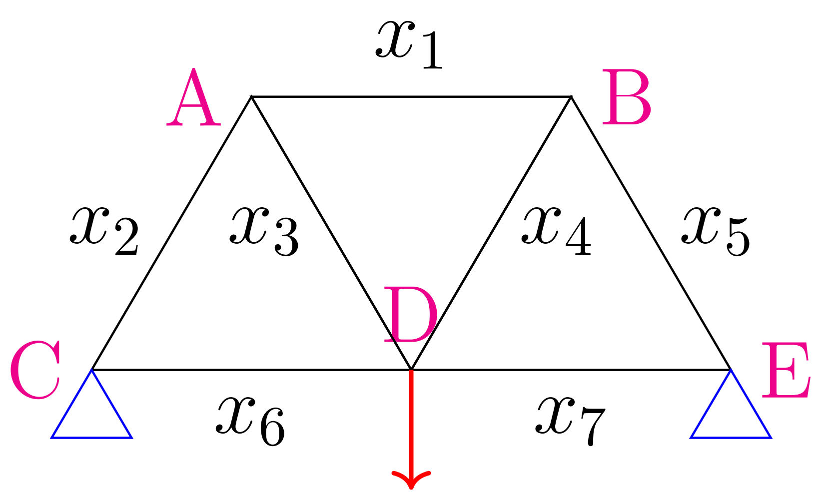


Figure 45 Variables for the truss

$$D : \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DC} + \vec{F}_{DE} = -\vec{L}$$

$$\Leftrightarrow x_3 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} + x_4 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} + x_6 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} + x_7 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\Leftrightarrow x_3 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_4 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_6 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_7 \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Observation A.1.13 The full augmented matrix given by the ten equations in this linear system is given below, where the elevent columns correspond to $x_1, \ldots, x_7, y_1, y_2, z_1, z_2$, and the ten rows correspond to the horizontal and vertical components of the forces acting at $A, \ldots, E$.

$$\left[\begin{array}{ccccccccccc|c}
1 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.866 & -0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.866 & -0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.866 & 0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 \\
0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.866 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

Observation A.1.14 This matrix row-reduces to the following.

$$\sim \left[\begin{array}{ccccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5773.7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5773.7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 2886.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 2886.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5000 \end{array}\right]$$

Observation A.1.15 Thus we know the truss must satisfy the following conditions.

$$x_1 = x_2 = x_5 = -5882.4$$
$$x_3 = x_4 = 5882.4$$
$$x_6 = x_7 = 2886.8 + z_1$$
$$y_1 = -z_1$$
$$y_2 = z_2 = 5000$$

In particular, the negative x_1, x_2, x_5 represent tension (forces pointing into the nodes), and the postive x_3, x_4 represent compression (forces pointing out of the nodes). The vertical normal forces $y_2 + z_2$ counteract the 10000 load.

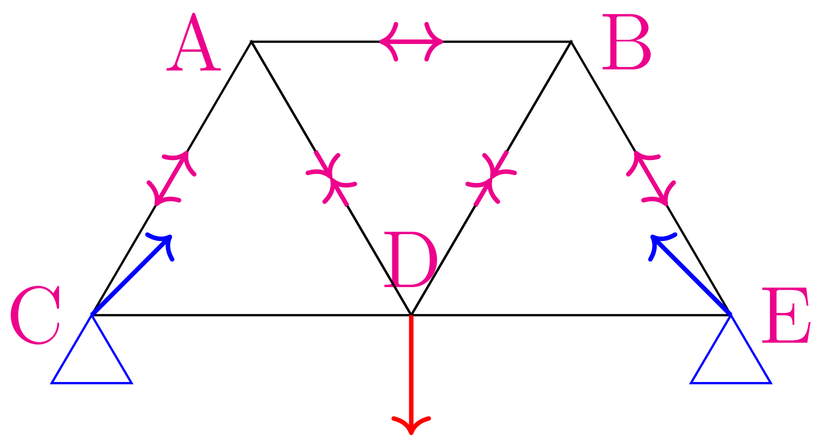


Figure 46 Completed truss

A.2 Computer Science: PageRank

Activity A.2.1 The $978,000,000,000 Problem.
In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.

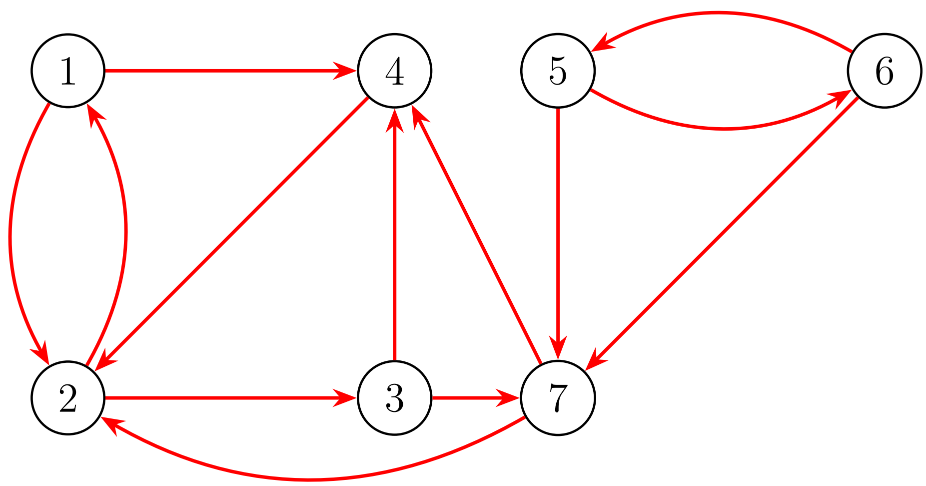


Figure 47 A seven-webpage network

Based on how these pages link to each other, write a list of the 7 webpages in order from most important to least important.

Observation A.2.2 The $978,000,000,000 Idea. Links are endorsements. That is:

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example A.2.3 Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

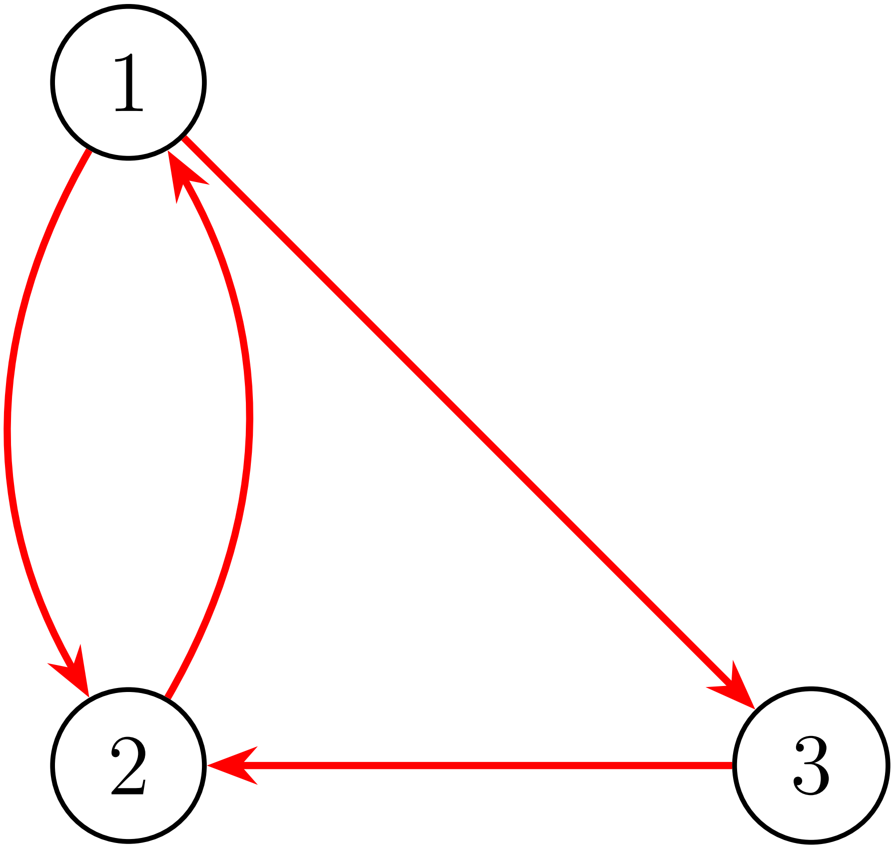


Figure 48 A three-webpage network

1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**:

$$\begin{aligned} x_2 \quad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \quad &= x_3 \end{aligned}$$

$\square$

Observation A.2.4

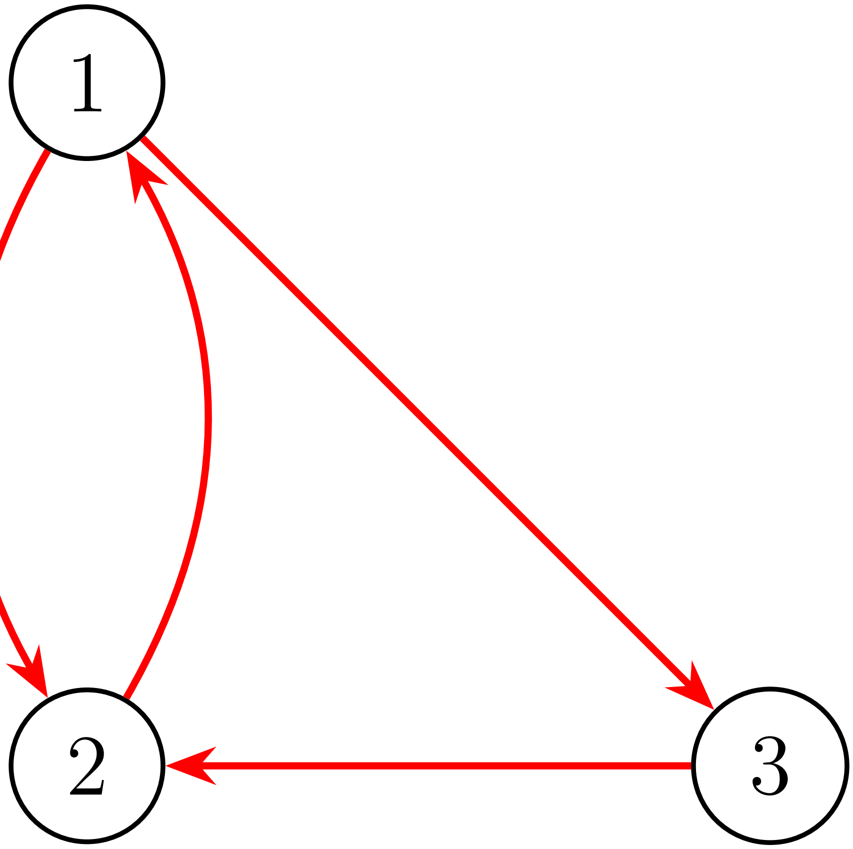


Figure 49 A three-webpage network

$$\begin{aligned} x_2 \quad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \quad &= x_3 \end{aligned} \qquad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

By writing this linear system in terms of matrix multiplication, we obtain the **page rank matrix** $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$ and page rank vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Thus, computing the importance of pages on a network is equivalent to solving the matrix equation $A\vec{x} = 1\vec{x}$.

Activity A.2.5 Thus, our \$978,000,000,000 problem is what kind of problem?

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A. An antiderivative problem

B. A bijection problem

C. A cofactoring problem

D. A determinant problem

E. An eigenvector problem

Activity A.2.6 Find a page rank vector $\vec{x}$ satisfying $A\vec{x} = 1\vec{x}$ for the following network's page rank matrix A.

That is, find the eigenspace associated with $\lambda = 1$ for the matrix A, and choose a vector from that eigenspace.

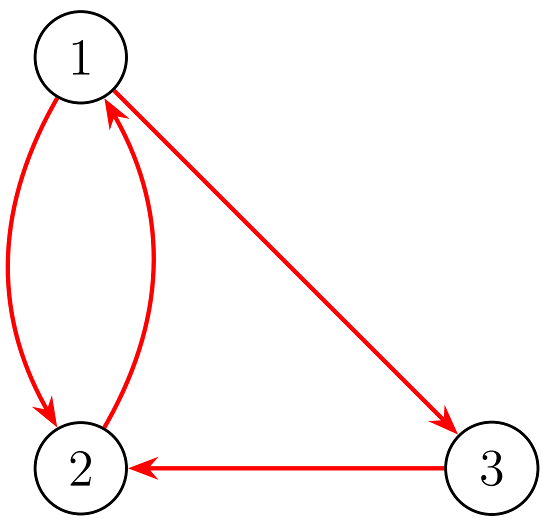


Figure 50 A three-webpage network

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Observation A.2.7 Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.

Activity A.2.8 Compute the 7×7 page rank matrix for the following network.

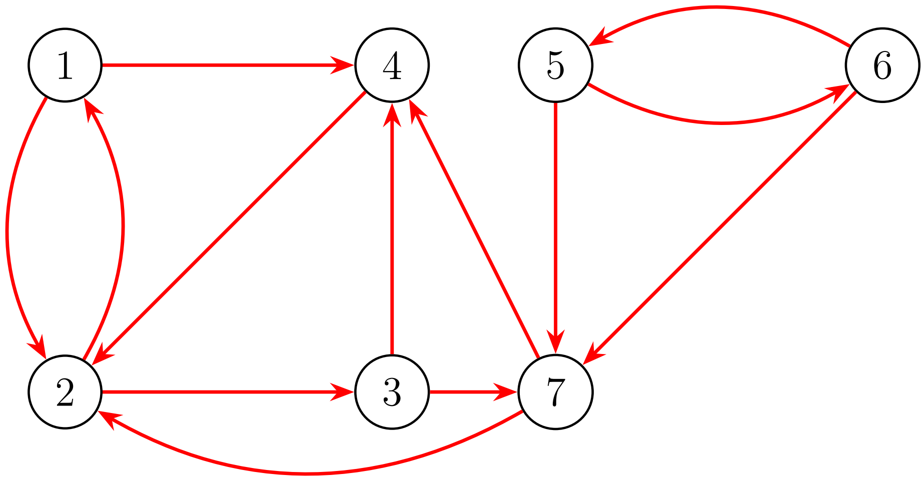


Figure 51 A seven-webpage network

For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.2.9 Find a page rank vector for the given page rank matrix.

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

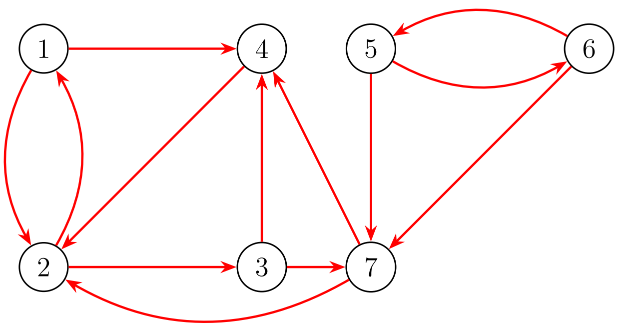


Figure 52 A seven-webpage network

Which webpage is most important?

Observation A.2.10 Since a page rank vector for the network is given by $\vec{x}$, it's reasonable to consider page 2 as the most important page.

$$\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Based upon this page rank vector, here is a complete ranking of all seven pages from most important to least important:

$$2, 4, 1, 3, 7, 5, 6$$

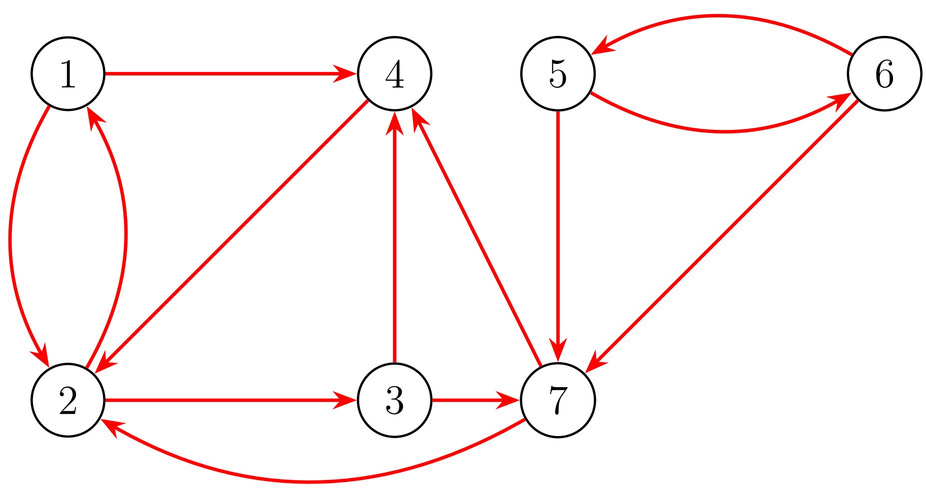


Figure 53 A seven-webpage network

Activity A.2.11 Given the following diagram, use a page rank vector to rank the pages 1 through 7 in order from most important to least important.

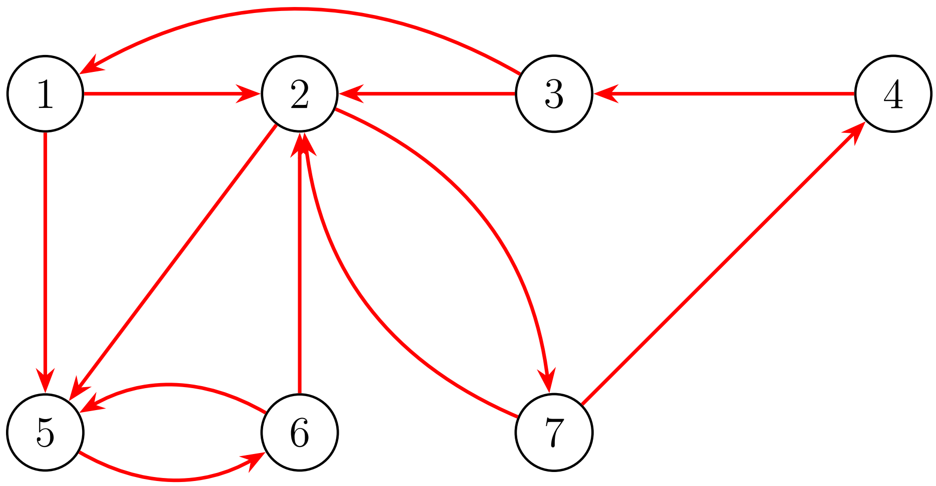


Figure 54 Another seven-webpage network

A.3 Geology: Phases and Components

Definition A.3.1 In geology, a **phase** is any physically separable material in the system, such as various minerals or liquids.

A **component** is a chemical compound necessary to make up the phases; these are usually oxides such as Calcium Oxide (CaO) or Silicon Dioxide (SiO_2).

In a typical application, a geologist knows how to build each phase from the components, and is interested in determining reactions among the different phases. $\diamond$

Observation A.3.2 Consider the 3 components

$$\vec{c}_1 = \mathrm{CaO} \quad \vec{c}_2 = \mathrm{MgO} \quad \text{and } \vec{c}_3 = \mathrm{SiO_2}$$

and the 5 phases:

$$\begin{array}{lll} \vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} & \vec{p}_2 = \mathrm{CaMgSiO_4} & \vec{p}_3 = \mathrm{CaSiO_3} \\ \vec{p}_4 = \mathrm{CaMgSi_2O_6} & \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} & \end{array}$$

Geologists already know (or can easily deduce) that

$$\begin{array}{lll} \vec{p}_1 = 3\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 & \vec{p}_2 = \vec{c}_1 + \vec{c}_2 + \vec{c}_3 & \vec{p}_3 = \vec{c}_1 + 0\vec{c}_2 + \vec{c}_3 \\ \vec{p}_4 = \vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 & \vec{p}_5 = 2\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 & \end{array}$$

since, for example:

$$\vec{c}_1 + \vec{c}_3 = \mathrm{CaO} + \mathrm{SiO_2} = \mathrm{CaSiO_3} = \vec{p}_3$$

Activity A.3.3 To study this vector space, each of the three components $\vec{c}_1, \vec{c}_2, \vec{c}_3$ may be considered as the three components of a Euclidean vector.

$$\vec{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Determine if the set of phases is linearly dependent or linearly independent.

Activity A.3.4 Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = Ca_3MgSi_2O_8 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_2 = CaMgSiO_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{p}_3 = CaSiO_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_4 = CaMgSi_2O_6 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_5 = Ca_2MgSi_2O_7 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

(a) Set up a system of equations equivalent to this vector equation.

(b) Find a basis for its solution space.

(c) Interpret each basis vector as a vector equation and a chemical equation.

Activity A.3.5 We found two basis vectors $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, corresponding to the vector and chemical equations

$$\begin{array}{rc} 2\vec{p}_2 + 2\vec{p}_3 = \vec{p}_1 + \vec{p}_4 & 2\mathrm{CaMgSiO_4} + 2\mathrm{CaSiO_3} = \mathrm{Ca_3MgSi_2O_8} + \mathrm{CaMgSi_2O_6} \\ \vec{p}_2 + \vec{p}_3 = \vec{p}_5 & \mathrm{CaMgSiO_4} + \mathrm{CaSiO_3} = \mathrm{Ca_2MgSi_2O_7} \end{array}$$

Combine the basis vectors to produce a chemical equation among the five phases that does not involve $\vec{p}_2 = \mathrm{CaMgSiO_4}$.