Linear Algebra for Team-Based Inquiry Learning

2023 Edition

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July 22, 2023

Website: Linear Algebra for Team-Based Inquiry Learning [1]

[1] teambasedinquirylearning.github.io/linear-algebra/

[2] CreativeCommons.org/licenses/by-nc-sa/4.0

TBIL Resource Library

This work is made available as part of the TBIL Resource Library[3], a product of NSF DUE Award #2011807[4].

[3]sites.google.com/southalabama.edu/tbil
[4]nsf.gov/awardsearch/showAward?AWD_ID=2011807

For Instructors

If you are adopting this text in your class, please fill out this short form[5] so we can track usage, let you know about updates, etc.

[5] `forms.gle/Ktfbma6iBn2gN1W78`

Video Resources

Videos are available at the end of each section. A complete playlist of videos aligned with this text is available on YouTube[6].

[6] www.youtube.com/watch?v=kpOK7RhFEiQ&list=PLwXCBkIf7xBMo3zMnD7WVt39rANLlSdmj

Slideshows

Slides for each section are available in HTML format.

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   (g) EV7 slides[17]
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[7] teambasedinquirylearning.github.io/linear-algebra/2023/LE1.slides.html
[8] teambasedinquirylearning.github.io/linear-algebra/2023/LE2.slides.html
[9] teambasedinquirylearning.github.io/linear-algebra/2023/LE3.slides.html
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[21] teambasedinquirylearning.github.io/linear-algebra/2023/AT4.slides.html
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6. Applications
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[24] teambasedinquirylearning.github.io/linear-algebra/2023/MX1.slides.html
[25] teambasedinquirylearning.github.io/linear-algebra/2023/MX2.slides.html
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[27] teambasedinquirylearning.github.io/linear-algebra/2023/MX4.slides.html
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Contents

Chapter 1

Systems of Linear Equations (LE)

Learning Outcomes

How can we solve systems of linear equations?

By the end of this chapter, you should be able to...

1. Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.
2. Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.
3. Determine the number of solutions for a system of linear equations or a vector equation.
4. Compute the solution set for a system of linear equations or a vector equation with infinitly many solutions.

Readiness Assurance. Before beginning this chapter, you should be able to...

1. Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
   - Review: Khan Academy[1]
2. Find the unique solution to a two-variable system of linear equations by back-substitution.
   - Review: Khan Academy[2]
3. Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.
   - Review: YouTube[3]

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[1] bit.ly/2l21etm

[2] www.khanacademy.org/math/algebra-basics/alg-basics-systems-of-equations/alg-basics-solving-systems-with-substitution/v/practice-using-substitution-for-systems

[3] youtu.be/xnfUZ-NTsCE

1.1 Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Learning Outcomes

- Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

1.1.1 Class Activities

Definition 1.1.1 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation). ♢

Remark 1.1.2 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition 1.1.3 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{alignedat}{4} a_{11}x_1 + {} & a_{12}x_2 + \ldots + {} & a_{1n}x_n & = b_1 \\ a_{21}x_1 + {} & a_{22}x_2 + \ldots + {} & a_{2n}x_n & = b_2 \\ \vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots \\ a_{m1}x_1 + {} & a_{m2}x_2 + \ldots + {} & a_{mn}x_n & = b_m \end{alignedat}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \,\middle|\, \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

♢

Remark 1.1.4 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:	Verbose standard form:	Concise standard form:
$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$	$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$	$$\begin{alignedat}{4} x_1 & & &+ 3x_3 &&= 3 \\ 3x_1 &- 2x_2 & &+ 4x_3 &&= 0 \\ &- x_2 & &+ x_3 &&= -2 \end{alignedat}$$

Remark 1.1.5 It will often be convenient to think of a system of equations as a vector equation.

By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

is equivalent to the system of equations

$$\begin{alignedat}{4} x_1 & & &+ 3x_3 &&= 3 \\ 3x_1 &- 2x_2 & &+ 4x_3 &&= 0 \\ &- x_2 & &+ x_3 &&= -2 \end{alignedat}$$

Definition 1.1.6 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**. ♢

Fact 1.1.7 *All linear systems are one of the following:*

1. Consistent with one solution: *its solution set contains a single vector, e.g.* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
2. Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g.* $\left\{ \begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$
3. Inconsistent*: its solution set is the empty set, denoted by either* $\{\}$ *or* $\emptyset$.

Activity 1.1.8 All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is the empty set.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 1.1.9 Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

(a) Find three different solutions for this system.

(b) Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$ for the linear system.

Activity 1.1.10 Consider the following linear system.

$$\begin{alignedat}{4} x_1 + 2x_2 & & & - & x_4 &= & 3 \\ & & x_3 & + & 4x_4 &= & -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \left[\begin{array}{c} ? \\ a \\ ? \\ b \end{array}\right] \middle| a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation 1.1.11 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{alignedat}{5} -2x_1 &- 4x_2 &+ x_3 &- 4x_4 &= -8 \\ x_1 &+ 2x_2 &+ 2x_3 &+ 12x_4 &= -1 \\ x_1 &+ 2x_2 &+ x_3 &+ 8x_4 &= 1 \end{alignedat}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Remark 1.1.12 The only important information in a linear system are its coefficients and constants.

Original linear system:	Verbose standard form:	Coefficients/constants:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$1 \quad 0 \quad 3 \mid 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3 \ -2 \quad 4 \mid 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$0 \ -1 \quad 1 \mid -2$

Definition 1.1.13 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{alignedat}{5} a_{11}x_1 &+ a_{12}x_2 &+ \ldots &+ a_{1n}x_n &= b_1 \\ a_{21}x_1 &+ a_{22}x_2 &+ \ldots &+ a_{2n}x_n &= b_2 \\ \vdots & \quad \vdots & & \quad \vdots & \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 &+ \ldots &+ a_{mn}x_n &= b_m \end{alignedat}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$\diamond$

Example 1.1.14 The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.