Linear Algebra for Team-Based Inquiry Learning

2024 Edition PREVIEW

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Steven Clontz
University of South Alabama

Drew Lewis

Contributing Authors

Jessalyn Bolkema
California State University, Dominguez Hills

Jeff Ford
Gustavus Adolphus College

Jordan Kostiuk
Brown University

Sharona Krinsky
California State University, Los Angeles

Jennifer Nordstrom
Linfield University

Kate Owens
College of Charleston

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Website: Linear Algebra for Team-Based Inquiry Learning[1]

[1] linear.tbil.org

[2] github.com/TeamBasedInquiryLearning/linear-algebra/blob/main/LICENSE.md

Contents

Chapter 1: Systems of Linear Equations (LE)

1.1 Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Activity 1.1.1 Consider the pairs of lines described by the equations below. Decide which of these are parallel, identical, or transverse (i.e., intersect in a single point).

(a)

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 5x_2 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 6x_2 &= -2 \end{aligned}$$

(c)

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 6x_2 &= 3 \end{aligned}$$

Activity 1.1.11 All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is the empty set.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 1.1.12 Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

(a) Find three different solutions for this system.

(b) Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \left[\begin{array}{c} ? \\ a \end{array} \right] \middle| a \in \mathbb{R} \right\}$ for the linear system.

Activity 1.1.13 Consider the following linear system.

$$\begin{alignedat}{4} x_1 + 2x_2 & & & {}-{} & x_4 &= 3 \\ & & x_3 & {}+{} & 4x_4 &= -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \left[\begin{array}{c} ? \\ a \\ ? \\ b \end{array} \right] \middle| a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Activity 1.1.18 Consider the following augmented matrices. For each of them, decide how many variables and how many equations the corresponding linear system has.

(a)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

(b)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 3 & -1 & 7 & -1 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

(c)

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 7 & -1 \\ 3 & 0 & 7 & -1 \end{array}\right]$$

(d)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

1.2 Row Reduction of Matrices (LE2)

Activity 1.2.2 Consider whether these matrix manipulations (A) *must keep* or (B) *could change* the solution set for the corresponding linear system.

(a) Swapping two rows, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 4x+5y&=6 \\ x+2y&=3 \end{aligned}$$

(b) Swapping two columns, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 5 & 4 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 2x+y&=3 \\ 5x+4y&=6 \end{aligned}$$

(c) Add a constant to every term of a row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1+6 & 2+6 & 3+6 \\ 4 & 5 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 7x+8y&=9 \\ 4x+5y&=6 \end{aligned}$$

(d) Multiply a row by a nonzero constant, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 6 & 9 \\ 4 & 5 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 3x+6y&=9 \\ 4x+5y&=6 \end{aligned}$$

(e) Add a constant multiple of one row to another row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4+3 & 5+6 & 6+9 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

(f) Replace a column with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 4 & 0 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

(g) Replace a row with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

Row Reduction of Matrices (LE2)

Activity 1.2.4 Each of the following linear systems has the same solution set.

A)

$$\begin{aligned} x+2y+z&=3\\ -x-y+z&=1\\ 2x+5y+3z&=7 \end{aligned}$$

B)

$$\begin{aligned} 2x+5y+3z&=7\\ -x-y+z&=1\\ x+2y+z&=3 \end{aligned}$$

C)

$$\begin{aligned} x-z&=1\\ y+2z&=4\\ y+z&=1 \end{aligned}$$

D)

$$\begin{aligned} x+2y+z&=3\\ y+2z&=4\\ 2x+5y+3z&=7 \end{aligned}$$

E)

$$\begin{aligned} x-z&=1\\ y+2z&=4\\ z&=3 \end{aligned}$$

F)

$$\begin{aligned} x+2y+z&=3\\ y+2z&=4\\ y+z&=1 \end{aligned}$$

Sort these six equivalent linear systems from most complicated to simplest (in your opinion).

Activity 1.2.5 Here we've written the sorted linear systems from Activity 1.2.4 as augmented matrices.

$$\left[\begin{array}{ccc|c} 2 & 5 & 3 & 7\\ -1 & -1 & 1 & 1\\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3\\ -1 & -1 & 1 & 1\\ 2 & 5 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3\\ 0 & 1 & 2 & 4\\ 2 & 5 & 3 & 7 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3\\ 0 & \boxed{1} & 2 & 4\\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1\\ 0 & \boxed{1} & 2 & 4\\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1\\ 0 & \boxed{1} & 2 & 4\\ 0 & 0 & -1 & -3 \end{array}\right]$$

Assign the following row operations to each step used to manipulate each matrix to the next:

$R_3 - 1R_2 \to R_3$ $\qquad$ $R_2 + 1R_1 \to R_2$ $\qquad$ $R_1 \leftrightarrow R_3$

$R_3 - 2R_1 \to R_3$ $\qquad$ $R_1 - 2R_3 \to R_1$

Activity 1.2.7 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 2 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 3\\ 0 & 0 & 1 & -1 \end{array}\right]$$

Row Reduction of Matrices (LE2)

Activity 1.2.8 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$D = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad E = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad F = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Activity 1.2.10 Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

A. Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

B. Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

C. Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

D. Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Activity 1.2.11 Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

B. Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

C. Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

D. Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

Row Reduction of Matrices (LE2)

Activity 1.2.12 Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

B. Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

C. Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

D. Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

Activity 1.2.14 Complete the following RREF calculation (multiple row operations may be needed for certain steps):

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & ? & ? & ? \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Activity 1.2.15 Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF(A).

Activity 1.2.16 Consider the non-augmented and augmented matrices

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix} \qquad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right].$$

Can RREF(A) be used to find RREF(B)?

A. Yes, RREF(A) and RREF(B) are exactly the same.

B. Yes, RREF(A) may be slightly modified to find RREF(B).

C. No, a new calculuation is required.

Row Reduction of Matrices (LE2)

Activity 1.2.17 Free browser-based technologies for mathematical computation are available online.

- Go to https://sagecell.sagemath.org/.
- In the dropdown on the right, you can select a number of different languages. Select "Octave" for the Matlab-compatible syntax used by this text.
- Type `rref([1,3,2;2,5,7])` and then press the Evaluate button to compute the RREF of $\left[\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 5 & 7 \end{array}\right]$.

Activity 1.2.18 In the HTML version of this text, code cells are often embedded for your convenience when RREFs need to be computed.

Try this out to compute RREF $\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 0 & 6 \end{array}\right]$.

1.3 Counting Solutions for Linear Systems (LE3)

Activity 1.3.1

(a) Without referring to your Activity Book, which of the four criteria for a matrix to be in Reduced Row Echelon Form (RREF) can you recall?

(b) Which, if any, of the following matrices are in RREF? You may refer to the Activity Book now for criteria that you may have forgotten.

$$P = \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & -3 \\ 0 & 3 & 3 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad Q = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Counting Solutions for Linear Systems (LE3)

Activity 1.3.3 Consider the following system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11. \end{aligned}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

Activity 1.3.4 Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

Activity 1.3.5 What contradictory equations besides $0 = 1$ may be obtained from the RREF of an augmented matrix?

A. $x = 0$ is an obtainable contradiction

B. $x = y$ is an obtainable contradiction

C. $0 = 17$ is an obtainable contradiction

D. $0 = 1$ is the only obtainable contradiction

Activity 1.3.6 Consider the following linear system.

$$x_1 + 2x_2 + 3x_3 = 1$$

Counting Solutions for Linear Systems (LE3)

$$2x_1 + 4x_2 + 8x_3 = 0$$

(a) Find its corresponding augmented matrix A and find RREF(A).

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. One C. Infinitely-many

Activity 1.3.9 For each vector equation, write an explanation for whether each solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

(a)

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 4 \end{bmatrix}$$

(b)

$$x_1 \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 13 \end{bmatrix}$$

(c)

$$x_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -5 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ -9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Counting Solutions for Linear Systems (LE3)

Activity 1.3.10 In Fact 1.1.10, we stated, but did not prove the assertion that all linear systems are one of the following:

1. *Consistent with one solution:* its solution set contains a single vector, e.g. $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$
2. *Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle|\, a \in \mathbb{R}\right\}$
3. *Inconsistent*: its solution set is the empty set, denoted by either $\{\}$ or $\emptyset$.

Explain why this fact is a consequence of Fact 1.3.7 above.

1.4 Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.1 Write down any three linear systems and determine if they are consistent, have a single solution, or have infinitely many solutions.

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.2 Consider this simplified linear system found to be equivalent to the system from Activity 1.3.6:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Earlier, we determined this system has infinitely-many solutions.

(a) Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

(b) Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

(c) Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$ caused this?

Activity 1.4.4 Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by doing the following.

(a) Row-reduce its augmented matrix.

(b) Assign letters to the free variables (given by the non-pivot columns):

$$\begin{aligned} ? &= a \\ ? &= b \end{aligned}$$

(c) Solve for the bound variables (given by the pivot columns) to show that

$$\begin{aligned} ? &= 1 + 5a + 2b \\ ? &= 1 + 2a + 3b \\ ? &= 3 + 3b \end{aligned}$$

(d) Replace x_1 through x_5 with the appropriate expressions of a, b in the following set-builder notation.

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.6 Consider the following system of linear equations.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ -5 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 13 \\ -13 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ -12 \end{bmatrix}.$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Activity 1.4.7 Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x_1 & & & - & 2\,x_3 & = & -3 \\ 5\,x_1 & + & x_2 & - & 7\,x_3 & = & -18 \\ 5\,x_1 & - & x_2 & - & 13\,x_3 & = & -12 \\ x_1 & + & 3\,x_2 & + & 7\,x_3 & = & -12 \end{array}$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.8 Consider the following linear system, its augmented matrix A, and $\operatorname{RREF}(A)$:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 4 \\ & & x_2 & - & 2\,x_3 & = & -1 \\ & & x_2 & - & 2\,x_3 & = & -3 \\ x_1 & + & 2\,x_2 & - & 5\,x_3 & = & 0 \end{array}$$

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 1 & 2 & -5 & 0 \end{array}\right], \ \operatorname{RREF}(A) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

All of the following statements are not accurate or otherwise incorrect; identify what is problematic about the statements and correct them.

(a) The matrix A is inconsistent.

(b) The linear system has two bound variables and one free variable.

(c) The solution set to the given linear system is $\{\emptyset\}$.

Activity 1.4.9 Consider the following linear system, its augmented matrix B, and $\operatorname{RREF}(B)$:

$$\begin{array}{rcrcrcrcrcr} 2\,x_1 & - & 2\,x_2 & - & 8\,x_3 & + & 3\,x_4 & - & 9\,x_5 & = & -17 \\ -x_1 & & & + & x_3 & - & x_4 & + & 2\,x_5 & = & 6 \\ 2\,x_1 & - & x_2 & - & 5\,x_3 & + & x_4 & - & 5\,x_5 & = & -10 \\ -x_1 & + & 3\,x_2 & + & 10\,x_3 & & & + & 7\,x_5 & = & 6 \end{array}$$

$$B = \left[\begin{array}{ccccc|c} 2 & -2 & -8 & 3 & -9 & -17 \\ -1 & 0 & 1 & -1 & 2 & 6 \\ 2 & -1 & -5 & 1 & -5 & -10 \\ -1 & 3 & 10 & 0 & 7 & 6 \end{array}\right]$$

$$\operatorname{RREF}(B) = \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -3 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

All of the following statements are not accurate or otherwise incorrect; identify what is problematic about the statements and correct them.

(a) The matrix B is consistent with infinitely many solutions.

(b) The solution set is given by $\left[\begin{array}{c} a+b-3 \\ -3\,a-2\,b+1 \\ a \\ b-3 \\ b \end{array}\right]$.

(c) The variables x_3, x_5 are free. Setting them equal to a, b respectively and solving for the bound variables, the solution set to the linear system is given by $\left\{ \left[\begin{array}{c} a+b-3 \\ -3\,a-2\,b+1 \\ b-3 \end{array} \right] \middle|\, a, b \in \mathbb{R} \right\}$.

Chapter 2: Euclidean Vectors (EV)

2.1 Linear Combinations (EV1)

Activity 2.1.1 Discuss which of the vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$ is a solution to the given vector equation:

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

Linear Combinations (EV1)

Activity 2.1.5 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

(a) Sketch the four Euclidean vectors

$$1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\quad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix},\quad 0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad -2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

in the xy plane by placing a dot at the (x, y) coordinate associated with each vector.

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$$

in the xy plane by plotting their (x, y) coordinates as dots. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

Activity 2.1.7 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

(a) Sketch the following five Euclidean vectors in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}=?$$

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}-1\\1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

in the xy plane. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

Activity 2.1.8 Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane. What best describes this sketch?

A. A line

B. A plane

C. A parabola

D. A cube

Linear Combinations (EV1)

Activity 2.1.9 Consider the following questions to discover whether a Euclidean vector belongs to a span.

(a) The Euclidean vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to which of these vector equations?

A. $x_1 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

B. $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$

C. $x_1 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 0$

(b) Use technology to find RREF of the corresponding augmented matrix, and then use that matrix to find the solution set of the vector equation.

(c) Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Activity 2.1.11 Consider this claim about a vector equation:
$\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$.

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is true.

(c) Since your statement was true, use the solution set to describe a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$ that equals $\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$.

Activity 2.1.12 Consider this claim about a vector equation:
$\begin{bmatrix} -5 \\ -1 \\ -7 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} \right\}$.

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

Linear Combinations (EV1)

(b) Explain why the statement you wrote is false, to conclude that the vector does not belong to the span.

Linear Combinations (EV1)

Activity 2.1.13 Before next class, find some time to do the following:

(a) Without referring to your activity book, write down the definition of a linear combination of vectors.

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$. Write down an example $\vec{w_1} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ of a linear combination of $\vec{u}, \vec{v}$. Then write down an example $\vec{w_2} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ that is *not* a linear combination of $\vec{u}, \vec{v}$.

(c) Draw a rough sketch of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, $\vec{w_1} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$, and $\vec{w_2} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$.

2.2 Spanning Sets (EV2)

Activity 2.2.1 Given a set of ingredients and a meal, a recipe is a list of amounts of each ingredient required to prepare the given meal.

(a) Use the words *vector* and *linear combination* to create a new statement that is analogous to one above.

(b) Building on your analogy, what role might the word *span* play?

Spanning Sets (EV2)

Activity 2.2.3 How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

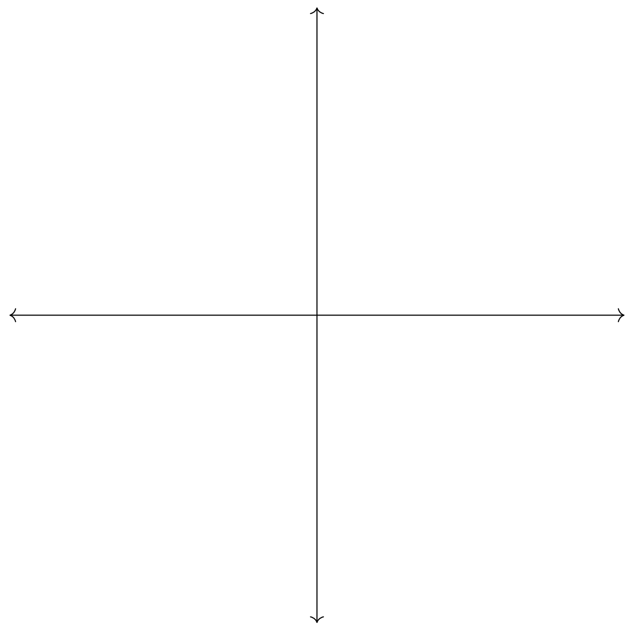


Figure 7 The xy plane $\mathbb{R}^2$

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Activity 2.2.4 How many vectors are required to span $\mathbb{R}^3$?

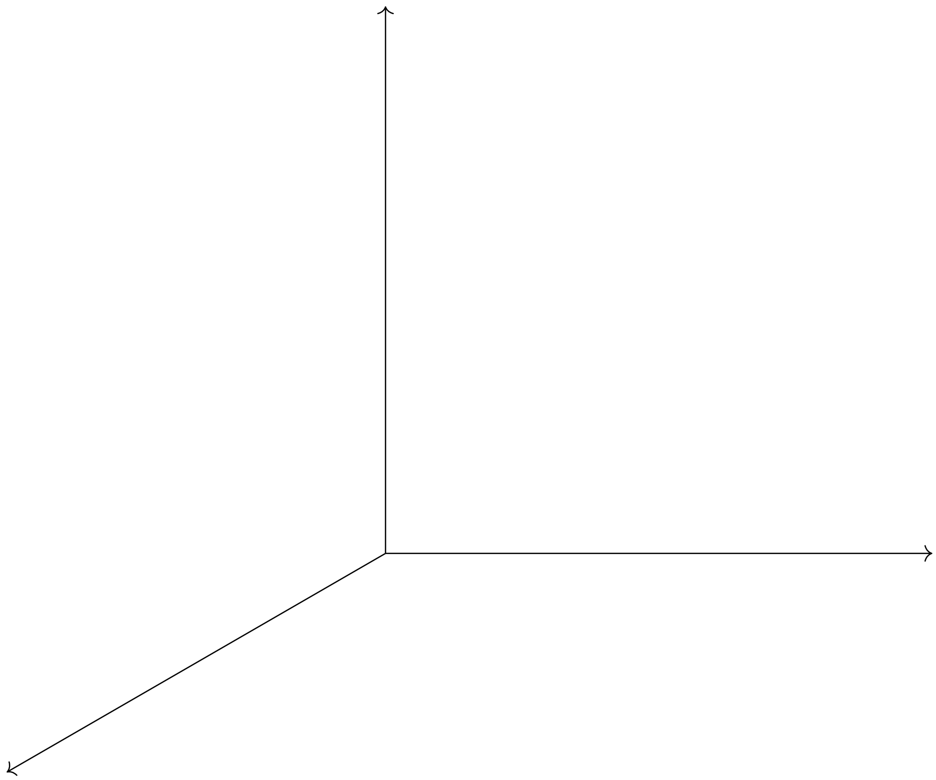


Figure 8 $\mathbb{R}^3$ space

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Activity 2.2.6 Consider the question: Does every vector in $\mathbb{R}^3$ belong to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$?

(a) Determine if $\begin{bmatrix}7\\-3\\-2\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(b) Determine if $\begin{bmatrix}2\\5\\7\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(c) An arbitrary vector $\begin{bmatrix}?\\?\\?\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$ provided the equation

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) We're guaranteed at least one solution if the RREF of the corresponding augmented matrix has no contradictions; likewise, we have no solutions if the RREF corresponds to the contradiction $0 = 1$. Given

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & ? \\ -1 & 0 & -2 & ? \\ 0 & 1 & 2 & ? \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & ? \\ 0 & 1 & 2 & ? \\ 0 & 0 & 0 & ? \end{array}\right]$$

we may conclude that the set does not span all of $\mathbb{R}^3$ because...

A. the row $[0\,1\,2\,|\,?]$ prevents a contradiction.

B. the row $[0\,1\,2\,|\,?]$ allows a contradiction.

C. the row $[0\,0\,0\,|\,?]$ prevents a contradiction.

D. the row $[0\,0\,0\,|\,?]$ allows a contradiction.

Activity 2.2.8 Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ and the question "Does $\mathbb{R}^4 = \operatorname{span} S$?"

(a) Rewrite this question in terms of the solutions to a vector equation.

(b) Answer your new question, and use this to answer the original question.

Activity 2.2.9 Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three Euclidean vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

A. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

B. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the same as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

C. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Spanning Sets (EV2)

Activity 2.2.10 One of our important results in this lesson is Fact 2.2.5, which states that a set of n vectors is required to span $\mathbb{R}^n$. While we developed some geometric intuition for why this true, we did not prove it in class. Before coming to class next time, follow the steps outlined below to convince yourself of this fact using the concepts we learned in this lesson.

(a) Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be a set of vectors living in $\mathbb{R}^n$ and assume that $m < n$. How many rows and how many columns will the matrix $[\vec{v}_1 \cdots \vec{v}_m]$ have?

(b) Given no additional information about the vectors $\vec{v}_1, \ldots, \vec{v}_m$, what is the maximum possible number of pivots in $\text{RREF}[\vec{v}_1\ \ldots\ \vec{v}_m]$?

(c) Conclude that our given set of vector cannot span all of $\mathbb{R}^n$.

2.3 Subspaces (EV3)

Activity 2.3.1 Consider the linear equation

$$x + 2y + z = 0.$$

(a) Verify that both $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are solutions.

(b) Is the vector $2\vec{v} - 3\vec{w}$ also a solution?

Activity 2.3.3 Let S denote a set of vectors in $\mathbb{R}^n$ and suppose that $\vec{u}, \vec{v} \in \operatorname{span}(S)$, $c \in \mathbb{R}$ and that $\vec{w} \in \mathbb{R}^n$. Which of the following vectors might *not* belong to span(S)?

A. $\vec{0}$

B. $\vec{u} + \vec{w}$

C. $\vec{u} + \vec{v}$

D. $c\vec{u}$

Activity 2.3.5 Consider the homogeneous vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$.

(a) Is this equation consistent?

A. no.

B. yes.

C. more information is required.

(b) Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are both solutions, we know that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}.$$

Therefore by adding these equations:

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

(c) Similarly, if $c \in \mathbb{R}$, then since multiplying by c yields:

$$(ca_1)\vec{v}_1 + \cdots + (ca_n)\vec{v}_n = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

D. The empty set.

Activity 2.3.9 Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

(a) Is W the empty set?

(b) Let's assume that $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are in W. What are we allowed to assume?

A. $x + 2y + z = 0$.

B. $a + 2b + c = 0$.

C. Both of these.

D. Neither of these.

(c) Which equation must be verified to show that $\vec{v} + \vec{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to W?

A. $(x + a) + 2(y + b) + (z + c) = 0$.

B. $x + a + 2y + b + z + c = 0$.

C. $x + 2y + z = a + 2b + c$.

(d) Use the assumptions from (a) to verify the equation from (b).

(e) Is W is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

(f) Show that $k\vec{v} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$ also belongs to W for any $k \in \mathbb{R}$ by verifying $(kx) + 2(ky) + (kz) = 0$ under these assumptions.

(g) Is W is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

Activity 2.3.10 Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 4 \right\}$.

(a) Is W the empty set?

(b) Which of these statements is valid?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so W *is* a subspace.

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so W *is not* a subspace.

C. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so W *is* a subspace.

D. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so W *is not* a subspace.

(c) Which of these statements is valid?

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so W *is* a subspace.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so W *is not* a subspace.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so W *is* a subspace.

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so W *is not* a subspace.

Activity 2.3.12 Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}.$$

(a) Show R isn't a subspace by showing that $\vec{0} \notin R$.

(b) Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

(c) Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Activity 2.3.13 Consider the following two sets of Euclidean vectors:

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, 7x + 4y = 0 \right\} \qquad W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, 3xy^2 = 0 \right\}$$

Explain why one of these sets is a subspace of $\mathbb{R}^2$ and one is not.

Subspaces (EV3)

Activity 2.3.14 Consider the following attempted proof that

$$U = \left\{ \left[\begin{array}{c} x \\ y \end{array} \right] \middle| x + y = xy \right\}$$

is closed under scalar multiplication.

Let $\left[\begin{array}{c} x \\ y \end{array} \right] \in U$, so we know that $x + y = xy$. We want to show $k \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} kx \\ ky \end{array} \right] \in U$, that is, $(kx) + (ky) = (kx)(ky)$. This is verified by the following calculation:

$$\begin{aligned} (kx) + (ky) &= (kx)(ky) \\ k(x + y) &= k^2 xy \\ 0[k(x + y)] &= 0[k^2 xy] \\ 0 &= 0 \end{aligned}$$

Is this reasoning valid?

A. Yes

B. No

Subspaces (EV3)

Activity 2.3.18

(a) Given the set of ingredients $S = \{\text{flour}, \text{yeast}, \text{salt}, \text{water}, \text{sugar}, \text{milk}\}$, how should we think of the subspace $\text{span}(S)$?

(b) What is one meal that lives in the subspace $\text{span}(S)$?

(c) What is one meal that does not live in the subspace $\text{span}(S)$?

Activity 2.3.19 Let

$$W = \left\{ \left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] \middle| x + y = 3z + 2w \right\}.$$

The set W is a subspace. Below are two attempted proofs of the fact that W is closed under vector addition. Both of them are invalid; explain why.

(a) Let $\vec{u} = \left[\begin{array}{c} 1 \\ 4 \\ 1 \\ 1 \end{array}\right], \vec{v} = \left[\begin{array}{c} 2 \\ -1 \\ 1 \\ -1 \end{array}\right]$. Then both $\vec{u}, \vec{w}$ are elements of W. Their sum is

$$\vec{w} = \left[\begin{array}{c} 3 \\ 3 \\ 2 \\ 0 \end{array}\right]$$

and since

$$3 + 3 = 3 \cdot (2) + 2 \cdot (0),$$

it follows that $\vec{w}$ is also in W and so W is closed under vector addition.

(b) If $\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right], \left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right]$ are in W, we need to show that $\left[\begin{array}{c} x+a \\ y+b \\ z+c \\ w+d \end{array}\right]$ is also in W. To be in W, we need

$$(x + a) + (y + b) = 3(z + c) + 2(w + d).$$

Well, if

$$(x + a) + (y + b) = 3(z + c) + 2(w + d),$$

then we know that

$$x + y - 3z - 2w + a + b - 3c - 2d = 0$$

by moving everything over to the left hand side. Since we are assumming that $x + y - 3z - 2w = 0$ and $a + b - 3c - 2d = 0$, it follows that $0 = 0$, which is true, which proves that vector addition is closed.

2.4 Linear Independence (EV4)

Activity 2.4.1 Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}.$$

(a) Decide which of $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution vector.

(b) Consider now the following vector equation:

$$y_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix} = \vec{0}.$$

How is this vector equation related to the original one?

(c) Use the solution vector you found in part (a) to construct a solution vector to this new equation.

Linear Independence (EV4)

Activity 2.4.2 Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}.$$

Which of the following is true?

A. $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

B. $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

C. $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Activity 2.4.4 Consider the following three vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}.$$

(a) Let $\vec{w} = 3\vec{v}_1 - \vec{v}_2 - 5\vec{v}_3 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ is...

A. linearly dependent: at least one vector is a linear combination of others

B. linearly independent: no vector is a linear combination of others

(b) Find

$$\operatorname{RREF} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{bmatrix} = \operatorname{RREF} \begin{bmatrix} -2 & 1 & -2 & ? \\ 0 & 3 & 5 & ? \\ 0 & 0 & 4 & ? \end{bmatrix} = ?.$$

What does this tell you about solution set for the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{w} = \vec{0}$?

A. It is inconsistent.

B. It is consistent with one solution.

C. It is consistent with infinitely many solutions.

(c) Which of these might explain the connection?

A. A pivot column establishes linear independence and creates a contradiction.

B. A non-pivot column both describes a linear combination and reveals the number of solutions.

C. A pivot row describes the bound variables and prevents a contradiction.

D. A non-pivot row prevents contradictions and makes the vector equation solvable.

Linear Independence (EV4)

Activity 2.4.6 Find

$$\text{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Activity 2.4.8

(a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

(i) "The set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 6 \\ 3 \end{bmatrix} \right\}$ is linearly *independent*."

(ii) "The set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 6 \\ 3 \end{bmatrix} \right\}$ is linearly *dependent*."

(b) Explain how to determine which of these statements is true.

Activity 2.4.9 What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

A. 3

B. 4

C. 5

D. You can have infinitely many vectors and still be linearly independent.

Activity 2.4.10 Is it possible for the set of Euclidean vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ to be linearly independent?

A. Yes

B. No

Linear Independence (EV4)

Activity 2.4.12 Consider the statement: The set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent because the vector $\vec{v}_3$ is a linear combination of $\vec{v_1}$ and $\vec{v}_2$. Construct an analogous statement involving ingredients, meals, and recipes, using the terms *linearly independent* and *linear combination*.

Activity 2.4.13 The following exercises are designed to help develop your geometric intution around linear dependence.

(a) Draw sketches that depict the following:

- Three linearly independent vectors in $\mathbb{R}^3$.
- Three linearly dependent vectors in $\mathbb{R}^3$.

(b) If you have three linearly dependent vectors, is it necessarily the case that one of the vectors is a multiple of the other?

2.5 Identifying a Basis (EV5)

Activity 2.5.2 Consider the following set of ingredients:

$$S = \{\text{tomato}, \text{olive oil}, \text{dough}, \text{cheese}, \text{pizza sauce}, \text{garlic}\}$$

(a) Does ”pizza” live inside of $\operatorname{span}(S)$?

(b) Identify which ingredients in S make the set linearly dependent.

(c) Can you think of a subset S' of S that is linearly independent and for which ”pizza” is still in $\operatorname{span} S'$?

Identifying a Basis (EV5)

Activity 2.5.3 Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S, i.e. find scalars such that

$$\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} = ? \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} + ? \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix} + ? \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Find a *different* way to express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S.

(c) Consider another vector $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \end{bmatrix}$. Without computing the RREF of another matrix, how many ways can this vector be written as a linear combination of the vectors in S?

A. Zero.

B. One.

C. Infinitely-many.

D. Computing a new matrix RREF is necessary.

Activity 2.5.4 Let's review some of the terminology we've been dealing with...

(a) If every vector in a vector space can be constructed as one or more linear combinations of vectors in a set S, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(b) If the zero vector $\vec{0}$ can be constructed as a *unique* linear combination of vectors in a set S (the combination multiplying every vector by the scalar value 0), we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(c) If every vector of a vector space can either be constructed as a *unique* linear combination of vectors in a set S, or not at all, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

Activity 2.5.7 Let S be a basis (Definition 2.5.5) for a vector space. Then...

A. the set S must both span the vector space and be linearly independent.

B. the set S must span the vector space but could be linearly dependent.

C. the set S must be linearly independent but could fail to span the vector space.

D. the set S could fail to span the vector space and could be linearly dependent.

Activity 2.5.8 The vectors

$$\hat{\imath} = (1,0,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ used frequently in multivariable calculus.

Find the unique linear combination of these vectors

$$?\,\hat{\imath} + ?\,\hat{\jmath} + ?\,\hat{k}$$

that equals the vector

$$(3,-2,4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

in xyz space.

Activity 2.5.10 Take the RREF of an appropriate matrix to determine if each of the following sets is a basis for $\mathbb{R}^4$.

Identifying a Basis (EV5)

(a)

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(b)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(c)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(d)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(e)

$$\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

Activity 2.5.11 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ has a pivot in every row (because it spans), and has a pivot in every column (because it's linearly independent).

What is $\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix}?&?&?&?\\?&?&?&?\\?&?&?&?\\?&?&?&?\end{bmatrix}$$

Identifying a Basis (EV5)

Activity 2.5.13 Let S denote a set of vectors in $\mathbb{R}^n$. Without referring to your Activity Book, write down:

(a) The definition of what it means for S to be linearly independent.

(b) The definition of what it means for S to span $\mathbb{R}^n$.

(c) The definition of what it means for S to be a basis for $\mathbb{R}^n$.

Activity 2.5.14 You are going on a trip and need to pack. Let S denote the set of items that you are packing in your suitcase.

(a) Give an example of such a set of items S that you would say "spans" everything you need, but is linearly dependent.

(b) Give an example of such a set of items S that is linearly independent, but does not "span" everything you need.

(c) Give an example of such a set S that you might reasonably consider to be a "basis" for what you need?

2.6 Subspace Basis and Dimension (EV6)

Activity 2.6.1 Consider the set S of vectors in $\mathbb{R}^4$ given by

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(a) Is the set S linearly independent or linearly dependent?

(b) How would you describe the subspace $\operatorname{span} S$ geometrically?

(c) What do the spaces $\operatorname{span} S$ and $\mathbb{R}^2$ have in common? In what ways do they differ?

Subspace Basis and Dimension (EV6)

Activity 2.6.3 Consider the subspace of $\mathbb{R}^4$ given by $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$.

(a) Mark the column of RREF $\begin{bmatrix}2&2&2&1\\3&0&-3&5\\0&1&2&-1\\1&-1&-3&0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

(b) What would be the result of removing the vector that gave us this column?

A. The set still spans W, and remains linearly dependent.

B. The set still spans W, but is now also linearly independent.

C. The set no longer spans W, and remains linearly dependent.

D. The set no longer spans W, but is now linearly independent.

Activity 2.6.6

(a) Find a basis for $\operatorname{span} S$ where

$$S = \left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

(b) Find a basis for $\operatorname{span} T$ where

$$T = \left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Activity 2.6.10 Consider the following subspace W of $\mathbb{R}^4$:

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\0\\0\\2\end{bmatrix},\begin{bmatrix}-3\\1\\-5\\5\end{bmatrix},\begin{bmatrix}12\\-3\\15\\-18\end{bmatrix}\right\}.$$

(a) Explain and demonstrate how to find a basis of W.

(b) Explain and demonstrate how to find the dimension of W.

Subspace Basis and Dimension (EV6)

Activity 2.6.11 The dimension of a subspace may be found by doing what with an appropriate RREF matrix?

A. Count the rows.

B. Count the non-pivot columns.

C. Count the pivots.

D. Add the number of pivot rows and pivot columns.

Subspace Basis and Dimension (EV6)

Activity 2.6.12 In Observation 2.6.5, we found a basis for the subspace

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}0\\-2\\-2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}.$$

To do so, we use the results of the calculation:

$$\operatorname{RREF}\begin{bmatrix}1 & 2 & 0 & 1\\2 & 4 & -2 & 2\\3 & 6 & -2 & 1\end{bmatrix} = \begin{bmatrix}\boxed{1} & 2 & 0 & 1\\0 & 0 & \boxed{1} & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

to conclude that the set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\-2\\-2\end{bmatrix}\right\}$, the set of vectors *corresponding* to the pivot columns of the RREF, is a basis for W.

(a) Explain why neither of the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ are elements of W.

(b) Explain why this shows that, in general, when we calculate a basis for $W = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$, the pivot columns of $\operatorname{RREF}[\vec{v}_1 \dots \vec{v}_n]$ themselves do not form a basis for W.

2.7 Homogeneous Linear Systems (EV7)

Activity 2.7.2 In Section 2.3, we observed that if

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

is a homogenous vector equation, then:

- The zero vector $\vec{0}$ is a solution;
- The sum of any two solutions is again a solution;
- Multiplying a solution by a scalar produces another solution.

Based on this recollection, which of the following best describes the solution set to the homogenous equation?

A. A basis for $\mathbb{R}^n$.

B. A subspace of $\mathbb{R}^n$.

C. All of $\mathbb{R}^n$.

D. The empty set.

Homogeneous Linear Systems (EV7)

Activity 2.7.3 Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \qquad + \ x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \ x_4 &= 0 \end{aligned}$$

(a) Find its solution set (a subspace of $\mathbb{R}^4$).

(b) Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

(c) Rewrite this solution space in the form

$$\text{span}\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}.$$

(d) Which of these choices best describes the set of two vectors $\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}$ used in this span?

A. The set is linearly dependent.

B. The set is linearly independent.

C. The set spans all of $\mathbb{R}^4$.

D. The set fails to span the solution space.

Activity 2.7.5 Consider the homogeneous system of equations

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 - 4x_4 &= 0 \\ -2x_1 - 4x_2 + \ x_3 + \ x_4 &= 0 \\ 3x_1 + 6x_2 - \ x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity 2.7.6 Consider the homogeneous vector equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis for its solution space.

Activity 2.7.7 Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 + 6x_2 + 4x_3 &= 0 \\ x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

(a) Find its solution space.

(b) Which of these is the best choice of basis for this solution space?

A $\{\}$

B $\{\vec{0}\}$

C The basis does not exist

Activity 2.7.8 To create a computer-animated film, an animator first models a scene as a subset of $\mathbb{R}^3$. Then to transform this three-dimensional visual data for display on a two-dimensional movie screen or television set, the computer could apply a linear tranformation that maps visual information at the point $(x, y, z) \in \mathbb{R}^3$ onto the pixel located at $(x + y, y - z) \in \mathbb{R}^2$.

(a) What homoegeneous linear system describes the positions (x, y, z) within the original scene that would be aligned with the pixel $(0, 0)$ on the screen?

(b) Solve this system to describe these locations.

Activity 2.7.9 Let $S = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & 1 & 3 \end{bmatrix}$; note that

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following statements are all *invalid* for at least one reason. Determine what makes them invalid and, suggest alternative *valid* statements that the author may have meant instead.

(a) The matrix A is linearly independent because $\mathrm{RREF}(A)$ has a pivot in each column.

(b) The matrix A does not span $\mathbb{R}^4$ because $\mathrm{RREF}(A)$ has a row of zeroes.

(c) The set of vectors S spans.

(d) The set of vectors S is a basis.

Chapter 3: Algebraic Properties of Linear Maps (AT)

3.1 Linear Transformations (AT1)

Activity 3.1.1

(a) What is our definition for a set S of vectors to be linearly independent?

(b) What specific calculation would you perform to test is a set S of Euclidean vectors is linearly independent?

Activity 3.1.2

(a) What is our definition for a set S of vectors in $\mathbb{R}^n$ to span $\mathbb{R}^n$?

(b) What specific calculation would you perform to test is a set S of Euclidean vectors spans all of $\mathbb{R}^n$?

Linear Transformations (AT1)

Activity 3.1.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}.$$

(a) Compute the result of adding vectors before a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right)$$

A. $\begin{bmatrix} x - u + z - w \\ 3y - 3v \end{bmatrix}$

B. $\begin{bmatrix} x + u - z - w \\ 3y + 3v \end{bmatrix}$

C. $\begin{bmatrix} x + u \\ 3y + 3v \\ z + w \end{bmatrix}$

D. $\begin{bmatrix} x - u \\ 3y - 3v \\ z - w \end{bmatrix}$

(b) Compute the result of adding vectors after a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix}$$

A. $\begin{bmatrix} x - u + z - w \\ 3y - 3v \end{bmatrix}$

B. $\begin{bmatrix} x + u - z - w \\ 3y + 3v \end{bmatrix}$

C. $\begin{bmatrix} x + u \\ 3y + 3v \\ z + w \end{bmatrix}$

D. $\begin{bmatrix} x - u \\ 3y - 3v \\ z - w \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

(d) Compute the result of scalar multiplcation before a T transformation:

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right)$$

A. $\begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$

B. $\begin{bmatrix} cx + cz \\ -3cy \end{bmatrix}$

C. $\begin{bmatrix} x + c \\ 3y + c \\ z + c \end{bmatrix}$

D. $\begin{bmatrix} x - c \\ 3y - c \\ z - c \end{bmatrix}$

(e) Compute the result of scalar multiplcation after a T transformation:

$$cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x-z \\ 3y \end{bmatrix}$$

A. $\begin{bmatrix} cx-cz \\ 3cy \end{bmatrix}$

B. $\begin{bmatrix} cx+cz \\ -3cy \end{bmatrix}$

C. $\begin{bmatrix} x+c \\ 3y+c \\ z+c \end{bmatrix}$

D. $\begin{bmatrix} x-c \\ 3y-c \\ z-c \end{bmatrix}$

(f) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Activity 3.1.7 Let $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

(a) Compute

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = S\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(b) Compute

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + S\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0^2 \\ 1+3 \\ 1-2^0 \end{bmatrix} + \begin{bmatrix} 2+3 \\ 2^2 \\ 3+3 \\ 3-2^2 \end{bmatrix}$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Activity 3.1.8 Fill in the ?s, assuming $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear:

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(? \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = ?\, T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Activity 3.1.10

(a) Consider the following maps of Euclidean vectors $P : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2\,x - 3\,y - 3\,z \\ 3\,x + 4\,y + 4\,z \\ 3\,x + 4\,y + 5\,z \end{bmatrix} \quad \text{and} \quad Q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 4\,y + 9\,z \\ y - 2\,z \\ 8\,y^2 - 3\,xz \end{bmatrix}.$$

Which do you *suspect*?

A. P is linear, but Q is not.

B. Q is linear, but P is not.

C. Both maps are linear.

D. Neither map is linear.

(b) Consider the following map of Euclidean vectors $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2\,y \\ 9\,xy \end{bmatrix}.$$

Prove that S *is not* a linear transformation.

(c) Consider the following map of Euclidean vectors $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 8\,x - 6\,y \\ 6\,x - 4\,y \end{bmatrix}.$$

Prove that T *is* a linear transformation.

Linear Transformations (AT1)

Activity 3.1.11 Let $f(x) = x^3 - 1$. Then, $f\colon \mathbb{R} \to \mathbb{R}$ is a function with domain and codomain equal to $\mathbb{R}$. Is $f(x)$ is a linear transformation?

Activity 3.1.12

(a) Is it the case that rotating $\vec{u} + \vec{v}$ about the origin by $\frac{\pi}{2} = 90°$ is the same as first rotating each of $\vec{u}, \vec{v}$ and then adding them together?

(b) Is it the case that rotating $5\vec{u}$ about the origin by $\frac{\pi}{2} = 90°$ is the same as first rotating $\vec{u}$ by $\frac{\pi}{2} = 90°$ and then scaling by 5?

(c) Based on this, do you suspect that the transformation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating vectors about the origin through an angle of $\frac{\pi}{2} = 90°$ is linear? Do you think there is anything special about the angle $\frac{\pi}{2} = 90°$?

Activity 3.1.13 In Activity 2.2.1, we made an analogy between vectors and linear combinations with ingredients and recipes. Let us think of *cooking* as a transformation of ingredients. In this analogy, would it be appropriate for us to consider ”cooking” to be a linear transformation or not? Describe your reasoning.

3.2 Standard Matrices (AT2)

Activity 3.2.2 Can you recall the following?

(a) Given a transformation, what do the terms *domain* and *codomain* mean?

(b) What does the notation $T\colon V \to W$ mean?

Standard Matrices (AT2)

Activity 3.2.3 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

D. $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Activity 3.2.4 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity 3.2.5 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity 3.2.6 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What piece of information would help you compute $T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right)$?

A. The value of $T\left(\begin{bmatrix}0\\-4\\0\end{bmatrix}\right)$.

B. The value of $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

C. The value of $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

D. Any of the above.

Activity 3.2.9 Let $T:\mathbb{R}^4\to\mathbb{R}^3$ be the linear transformation given by

$$T\left(\vec{e}_1\right)=\begin{bmatrix}0\\3\\-2\end{bmatrix}\qquad T\left(\vec{e}_2\right)=\begin{bmatrix}-3\\0\\1\end{bmatrix}\qquad T\left(\vec{e}_3\right)=\begin{bmatrix}4\\-2\\1\end{bmatrix}\qquad T\left(\vec{e}_4\right)=\begin{bmatrix}2\\0\\0\end{bmatrix}$$

Write the standard matrix $[T(\vec{e}_1)\ \cdots\ T(\vec{e}_n)]$ for T.

Activity 3.2.10 Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+3z\\2x-y-4z\end{bmatrix}$$

(a) Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

(b) Find the standard matrix for T.

Activity 3.2.12 Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}3&-2&-1\\4&5&2\\0&-2&1\end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$.

(b) Compute $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$.

Activity 3.2.13 Compute the following linear transformations of vectors given their standard matrices.

(a)

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\text{ for the standard matrix }A_1=\begin{bmatrix}4&3\\0&-1\\1&1\\3&0\end{bmatrix}$$

Standard Matrices (AT2)

(b)

$$T_2\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

(c)

$$T_3\left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Standard Matrices (AT2)

Activity 3.2.14 Consider the linear transformation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating vectors about the origin through an angle of $\frac{\pi}{4} = 45°$.

- **(a)** If $\vec{e}_1, \vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$, calculate $R(\vec{e}_1), R(\vec{e}_2)$.
- **(b)** What is the standard matrix representing R?

Activity 3.2.15 Consider the linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by reflecting vectors across the line $x_1 = x_2$.

- **(a)** If $\vec{e}_1, \vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$, calculate $S(\vec{e}_1), S(\vec{e}_2)$.
- **(b)** What is the standard matrix representing S?

3.3 Image and Kernel (AT3)

Activity 3.3.1 Consider the matrix $A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$.

(a) The matrix A is the standard matrix of a linear transformation T. What is the domain and the codomain of the transformation T?

(b) Describe how T transforms the standard basis vectors of the domain that you found above.

Image and Kernel (AT3)

Activity 3.3.2 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes the set of all vectors that transform into $\vec{0}$?

A. $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

Activity 3.3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

Activity 3.3.5 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

(a) Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

(b) Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

Activity 3.3.6 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

Image and Kernel (AT3)

Find a basis for the kernel of T.

Activity 3.3.7 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

Activity 3.3.9 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

A. $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

Activity 3.3.10 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix}.$$

Consider the question: Which vectors $\vec{w}$ in $\mathbb{R}^3$ belong to $\operatorname{Im} T$?

(a) Determine if $\begin{bmatrix} 12 \\ 3 \\ 3 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(b) Determine if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(c) An arbitrary vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ belongs to $\operatorname{Im} T$ provided the equation

$$x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) + x_4 T(\vec{e}_4) = \vec{w}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) Based on this, how do $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ relate to each other?

A. The set $\operatorname{Im} T$ contains $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ but is not equal to it.

B. The set $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ contains $\operatorname{Im} T$ but is not equal to it.

C. The set $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ are equal to each other.

D. There is no relation between these two sets.

Activity 3.3.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Activity 3.3.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the kernel of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Activity 3.3.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the image of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Activity 3.3.17 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x - y + 5\,z + 3\,w \\ -x - 4\,z - 2\,w \\ y - 2\,z - w \end{bmatrix}.$$

(a) Explain and demonstrate how to find the image of T and a basis for that image.

(b) Explain and demonstrate how to find the kernel of T and a basis for that kernel.

(c) Explain and demonstrate how to find the rank and nullity of T, and why the rank-nullity theorem holds for T.

Image and Kernel (AT3)

Activity 3.3.18 In this section, we've introduced two important subspaces that are associated with a linear transformation $T\colon V \to W$, namely: $\operatorname{Im} T$, the image of T, and $\ker T$, the kernel of T. The following sequence is designed to help you internalize these definitions. Try to complete them without referring to your Activity Book, and then check your answers.

(a) One of $\ker T$ and $\operatorname{Im} T$ is a subspace of the domain and the other is a subspace of the codomain. Which is which?

(b) Write down the precise definitions of these subspaces.

(c) How would you describe these definitions to a layperson?

(d) What picture, or other study strategy would be helpful to you in conceptualizing how these defintions fit together?

3.4 Injective and Surjective Linear Maps (AT4)

Activity 3.4.1 Consider the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ that is represented by the standard matrix $A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$. Which of the following processes helps us compute a basis for $\operatorname{Im} T$ and which helps us compute a basis for $\ker T$?

A. Compute $\operatorname{RREF}(A)$ and consider the set of columns of A that correspond to columns in $\operatorname{RREF}(A)$ with pivots.

B. Calculate a basis for the solution space to the homogenous system of equations for which A is the coefficient matrix.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

Activity 3.4.4 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

Activity 3.4.6 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

B. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity 3.4.7 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

B. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Activity 3.4.8 Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Activity 3.4.10 Let $T : V \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T$ may be spanned by only two vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Activity 3.4.13 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The kernel of T is trivial, i.e. $\ker T = \{\vec{0}\}$.
2. The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.

3. For every $\vec{w} \in \mathbb{R}^m$, the set $\{\vec{v} \in \mathbb{R}^n | T(\vec{v}) = \vec{w}\}$ contains exactly one vector.

Activity 3.4.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The columns of A span $\mathbb{R}^m$.
2. The columns of A form a basis for $\mathbb{R}^m$.
3. The columns of A are linearly independent.

Activity 3.4.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. $\operatorname{RREF}(A)$ is the identity matrix.
2. Every column of $\operatorname{RREF}(A)$ has a pivot.
3. Every row of $\operatorname{RREF}(A)$ has a pivot.

Activity 3.4.16 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$.
2. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for all $\vec{b} \in \mathbb{R}^m$.
3. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has exactly one solution.

Activity 3.4.18 What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Activity 3.4.19 What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\left[\begin{array}{ccc} a & b & c \\ d & e & f \end{array}\right]$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Activity 3.4.21 Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{array}\right]$ is bijective.

(a) How many pivot rows must RREF A have?

(b) How many pivot columns must RREF A have?

(c) What is RREF A?

Activity 3.4.22 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

A. RREF(A) is the identity matrix.

B. The columns of A form a basis for $\mathbb{R}^n$

C. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.

Activity 3.4.24 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \left[\begin{array}{ccc} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{array}\right].$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.25 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\left[\begin{array}{c} x \\ y \\ z \end{array}\right]\right) = \left[\begin{array}{c} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{array}\right].$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.26 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.27 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y-z \\ 4x+y+z \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

3.4.0.1 Cool Down

Activity 3.4.28 Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. We reasoned during class that the following statements are logically equivalent:

1. The columns of A are linearly independent.
2. RREF(A) has a pivot in each column.
3. The transformation T is injective.
4. The system of equations given by $[A|\vec{0}]$ has a unique solution.

While they are all logically equivalent, they are different statements that offer varied perspectives on our growing conceptual knowledge of linear algebra.

(a) If you are asked to decide if a transformation T is injective, which of the above statements do you think is the most useful?

(b) Can you think of some situations in which translating between these four statements might be useful to you?

Activity 3.4.29 Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. We reasoned during class that the following statements are logically equivalent:

1. The columns of A span all of $\mathbb{R}^m$.
2. RREF(A) has a pivot in each row.
3. The transformation T is surjective.
4. The system of equations given by $[A|\vec{b}]$ is always consistent.

While they are all logically equivalent, they are different statements that offer varied perspectives on our growing conceptual knowledge of linear algebra.

(a) If you are asked to decide if a transformation T is surjective, which of the above statements do you think is the most useful?

(b) Can you think of some situations in which translating between these four statements might be useful to you?

3.5 Vector Spaces (AT5)

Activity 3.5.1

(a) How would you describe a sandwich to someone who has never seen a sandwich before?

(b) How would you describe to someone what a vector is?

Vector Spaces (AT5)

Activity 3.5.3 Which of the following properites of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

B. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

C. There exists some $\begin{bmatrix} ? \\ ? \end{bmatrix}$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

D. There exists some $\begin{bmatrix} ? \\ ? \end{bmatrix}$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

E. $\frac{1}{2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$ is the only vector whose endpoint is equally distant from the endpoints of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Activity 3.5.5 Which of the following properites of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $a\left(b \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ab \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

B. $1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

C. There exists some $?$ such that $? \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

D. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.

E. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

Activity 3.5.10 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$.
Which of the following vectors is not in V?

A. $(0, 0)$

B. $(1, 2)$

C. $(2, 4)$

D. $(3, 8)$

Activity 3.5.11 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$ with the operation $\oplus$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2).$$

Let $\vec{u}, \vec{v}$ be in V with $\vec{u} = (1, 2)$ and $\vec{v} = (2, 4)$. Using the operations defined for V, which of the following is $\vec{u} \oplus \vec{v}$?

A. $(2, 6)$

B. $(2, 8)$

C. $(3, 6)$

D. $(3, 8)$

Vector Spaces (AT5)

Activity 3.5.12 Consider the set $V = \{(x, y) \mid y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c).$$

Let $a = 2, b = -3$ be scalars and $\vec{u} = (1, 2) \in V$.

(a) Verify that

$$(a + b) \odot \vec{u} = \left(-1, \frac{1}{2}\right).$$

(b) Compute the value of

$$(a \odot \vec{u}) \oplus (b \odot \vec{u}).$$

Activity 3.5.13 Consider the set $V = \{(x, y) \mid y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c).$$

Let a, b be unspecified scalars in $\mathbb{R}$ and $\vec{u} = (x, y)$ be an unspecified vector in V.

(a) Show that both sides of the equation

$$(a + b) \odot (x, y) = (a \odot (x, y)) \oplus (b \odot (x, y))$$

simplify to the expression $(ax + bx, y^a y^b)$.

(b) Show that V contains an additive identity element $\vec{z} = (\,?\,,\,?\,)$ satisfying

$$(x, y) \oplus (\,?\,,\,?\,) = (x, y)$$

for all $(x, y) \in V$.

That is, pick appropriate values for $\vec{z} = (\,?\,,\,?\,)$ and then simplify $(x, y) \oplus (\,?\,,\,?\,)$ into just (x, y).

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Activity 3.5.15 Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2)$$

$$c \odot (x, y) = (x^c, y + c - 1).$$

(a) Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

(b) Show that V does not have an additive identity element $\vec{z} = (z, w)$ by showing that $(0, -1) \oplus (z, w) \neq (0, -1)$ no matter what the values of z, w are.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Activity 3.5.16 Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x, y) = (cx, cy).$$

(a) Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for *all* $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

(b) Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for *some* vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Vector Spaces (AT5)

Activity 3.5.17

(a) What are some objects that are important to you personally, academically, or otherwise that appear vector-like to you? What makes them feel vector-like? Which axiom for vector spaces does not hold for these objects, if any.

(b) Our vector space axioms have eight properties. While these eight properties are enough to capture vectors, the objects that we study in the real-world often have additional structures not captured by these axioms. What are some structures that you have encountered in other classes, or in previous experiences, that are not captured by these eight axioms?

3.6 Polynomial and Matrix Spaces (AT6)

Activity 3.6.1 Consider the following vector equation and statements about it:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{w}$$

1. The above vector equation is consistent for every choice of $\vec{w}$.

2. When the right hand is equal to $\vec{0}$, the equation has a unique solution.

3. The given equation always has a unique solution, no matter what $\vec{w}$ is.

Which, if any, of these statements make sense if we no longer assume that the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are Euclidean vectors, but rather elements of a vector space?

Polynomial and Matrix Spaces (AT6)

Activity 3.6.3 Let V be a vector space with the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Which of these completes the following definition for a bijective linear map $T : V \to \mathbb{R}^3$?

$$T(\vec{v}) = T(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} a+b+c \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Activity 3.6.5 The matrix space $M_{2,2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ has the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) What is the dimension of $M_{2,2}$?

A. 2 C. 4

B. 3 D. 5

(b) Which Euclidean space is $M_{2,2}$ isomorphic to?

A. $\mathbb{R}^2$ C. $\mathbb{R}^4$

B. $\mathbb{R}^3$ D. $\mathbb{R}^5$

(c) Describe an isomorphism $T : M_{2,2} \to \mathbb{R}^{?}$:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} ? \\ \vdots \\ ? \end{bmatrix}$$

Activity 3.6.6 The polynomial space $\mathcal{P}^4 = \{a + bx + cx^2 + dx^3 + ex^4 | a, b, c, d, e \in \mathbb{R}\}$ has the basis

$$\{1, x, x^2, x^3, x^4\}.$$

(a) What is the dimension of $\mathcal{P}^4$?

A. 2 C. 4

B. 3 D. 5

(b) Which Euclidean space is $\mathcal{P}^4$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

(c) Describe an isomorphism $T : \mathcal{P}^4 \to \mathbb{R}^{?}$:

$$T\left(a + bx + cx^2 + dx^3 + ex^4\right) = \begin{bmatrix} ? \\ \vdots \\ ? \end{bmatrix}$$

Activity 3.6.8 Consider how to construct the polynomial $x^3 + x^2 + 5x + 1$ as a linear combination of polynomials from the set

$$\left\{x^3 - 2\,x^2 + x + 2, 2\,x^2 - 1, -x^3 + 3\,x^2 + 3\,x - 2, x^3 - 6\,x^2 + 9\,x + 5\right\}.$$

(a) Describe the vector space involved in this problem, and an isomorphic Euclidean space and relevant Eucldean vectors that can be used to solve this problem.

(b) Show how to construct an appropriate Euclidean vector from an approriate set of Euclidean vectors.

(c) Use this result to answer the original question.

Activity 3.6.10 Let $A = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ and let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ denote the corresponding linear transformation. Note that

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following statements are all *invalid* for at least one reason. Determine what makes them invalid and, suggest alternative *valid* statements that the author may have meant instead.

(a) The matrix A is injective because $\mathrm{RREF}(A)$ has a pivot in each column.

(b) The matrix A does not span $\mathbb{R}^4$ because $\mathrm{RREF}(A)$ has a row of zeroes.

(c) The transformation T does not span $\mathbb{R}^4$.

(d) The transformation T is linearly independent.

Chapter 4: Matrices (MX)

4.1 Matrices and Multiplication (MX1)

Activity 4.1.1 Suppose that $T\colon V \to W$ is a linear transformation.

(a) What is the definition of $\ker T$? How does it relate to the codomain of T?

(b) What is definition of $\operatorname{Im} T$? How does it relate to the codomain of T?

Matrices and Multiplication (MX1)

Activity 4.1.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by the 2×3 standard matrix B and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by the 4×2 standard matrix A:

$$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

(a) What are the domain and codomain of the composition map $S \circ T$?

A. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

B. The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$

C. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$

D. The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$

(b) What size will the standard matrix of $S \circ T$ be?

A. 4 (rows) $\times$ 3 (columns)

B. 3 (rows) $\times$ 4 (columns)

C. 3 (rows) $\times$ 2 (columns)

D. 2 (rows) $\times$ 4 (columns)

(c) Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

(d) Compute $(S \circ T)(\vec{e}_2)$.

(e) Compute $(S \circ T)(\vec{e}_3)$.

(f) Use $(S \circ T)(\vec{e}_1), (S \circ T)(\vec{e}_2), (S \circ T)(\vec{e}_3)$ to write the standard matrix for $S \circ T$.

Activity 4.1.5 Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

(a) Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

(b) Find the standard matrix AB of $S \circ T$.

(c) Find the standard matrix BA of $T \circ S$.

Matrices and Multiplication (MX1)

Activity 4.1.6 Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

(a) Find the domain and codomain of each of the three linear maps corresponding to A, B, and C.

(b) Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Activity 4.1.7 Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) Compute the product BA by hand.

(b) Check your work using technology. Using Octave:

```
B = [3 -4 0 ; 2 0 -1 ; 0 -3 3]
A = [2 7 -1 ; 0 3 2  ; 1 1 -1]
B*A
```

Activity 4.1.8 Of the following three matrices, only two may be multiplied.

$$A = \begin{bmatrix} -1 & 3 & -2 & -3 \\ 1 & -4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ -2 & 4 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

Explain which two can be multiplied and why. Then show how to find their product.

Activity 4.1.9 Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ In Fact 3.2.11 we adopted the notation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Matrices and Multiplication (MX1)

Verify that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ in terms of matrix multiplication.

Matrices and Multiplication (MX1)

Activity 4.1.10 Given two $n \times n$ matrices A and B, explain why the sentence ”Multiply the matrices A and B together.” is ambiguous. How could you re-write the sentence in order to eliminate the ambiguity?

4.2 The Inverse of a Matrix (MX2)

Activity 4.2.1 Consider the matrices:

$$A = \begin{bmatrix} 1 & 5 & -1 \\ 0 & 3 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 3 & 2 & -2 \\ 1 & 1 & -1 & -3 \end{bmatrix}.$$

Without using technology, what is the third column of the product AB?

The Inverse of a Matrix (MX2)

Activity 4.2.2 Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Activity 4.2.5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is *injective*, a group that means T is *surjective*, and a group that means T is *bijective*.

A. $T(\vec{x}) = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

B. $T(\vec{x}) = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

C. $T(\vec{x}) = \vec{0}$ has a unique solution.

D. The columns of A span $\mathbb{R}^m$

E. The columns of A are linearly independent

F. The columns of A are a basis of $\mathbb{R}^m$

G. Every column of $\text{RREF}(A)$ has a pivot

H. Every row of $\text{RREF}(A)$ has a pivot

I. $m = n$ and $\text{RREF}(A) = I$

Activity 4.2.7 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear bijection given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

(a) To find $\vec{x} = T^{-1}(\vec{e}_1)$, we need to find the unique solution for $T(\vec{x}) = \vec{e}_1$. Which of these linear systems can be used to find this solution?

A. $\begin{array}{llll} 2x_1 & -1x_2 & -6x_3 & = x_1 \\ 2x_1 & +1x_2 & +3x_3 & = 0 \\ 1x_1 & +1x_2 & +4x_3 & = 0 \end{array}$

B. $\begin{array}{llll} 2x_1 & -1x_2 & -6x_3 & = x_1 \\ 2x_1 & +1x_2 & +3x_3 & = x_2 \\ 1x_1 & +1x_2 & +4x_3 & = x_3 \end{array}$

C. $\begin{array}{llll} 2x_1 & -1x_2 & -6x_3 & = 1 \\ 2x_1 & +1x_2 & +3x_3 & = 0 \\ 1x_1 & +1x_2 & +4x_3 & = 0 \end{array}$

D. $\begin{array}{llll} 2x_1 & -1x_2 & -6x_3 & = 1 \\ 2x_1 & +1x_2 & +3x_3 & = 1 \\ 1x_1 & +1x_2 & +4x_3 & = 1 \end{array}$

(b) Use that system to find the solution $\vec{x} = T^{-1}(\vec{e}_1)$ for $T(\vec{x}) = \vec{e}_1$.

(c) Similarly, solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$, and solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

(d) Use these to write

$$A^{-1} = [T^{-1}(\vec{e}_1) \quad T^{-1}(\vec{e}_2) \quad T^{-1}(\vec{e}_3)],$$

the standard matrix for T^{-1}.

Activity 4.2.8 Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & 0 & -4 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

by computing how it transforms each of the standard basis vectors for $\mathbb{R}^4$: $T^{-1}(\vec{e}_1)$, $T^{-1}(\vec{e}_2)$, $T^{-1}(\vec{e}_3)$, and $T^{-1}(\vec{e}_4)$.

Activity 4.2.9 Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible?

A. Yes, because its transformation is a bijection.

B. Yes, because its transformation is not a bijection.

C. No, because its transformation is a bijection.

D. No, because its transformation is not a bijection.

Activity 4.2.11 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

(a) Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

(b) If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

The Inverse of a Matrix (MX2)

Activity 4.2.13 Now that we have defined the inverse of a matrix, we have the ability to solve matrix equations. In the following equations, A, B all denote square matrices of the same size and I denotes the identity matrix. For each equation, solve for X.

(a) $A^{-1}XA = B$

(b) $AXA^{-1} = B$

(c) $ABX = I$

(d) $BAX = I$

4.3 Solving Systems with Matrix Inverses (MX3)

Activity 4.3.1 Which of the following matrices is invertible? Find the inverse for the one that is invertible.

A. $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix}$

Solving Systems with Matrix Inverses (MX3)

Activity 4.3.2 Consider the following linear system with a unique solution:

$$\begin{array}{rcrcrcrcr} 3x_1 & - & 2x_2 & - & 2x_3 & - & 4x_4 & = & -7 \\ 2x_1 & - & x_2 & - & x_3 & - & x_4 & = & -1 \\ -x_1 & & & + & x_3 & & & = & -1 \\ & - & x_2 & & & - & 2x_4 & = & -5 \end{array}$$

(a) Suppose we let

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 & - & 2x_2 & - & 2x_3 & - & 4x_4 \\ 2x_1 & - & x_2 & - & x_3 & - & x_4 \\ -x_1 & & & + & x_3 & & \\ & - & x_2 & & & - & 2x_4 \end{bmatrix}.$$

Which of these choices would help us solve the given system?

A. Compute $T\left(\begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}\right)$

B. Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

(b) How can we express this in terms of matrix multiplication?

A. $\begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

B. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

C. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

D. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$

(c) How could a matrix equation of the form $A\vec{x} = \vec{b}$ be solved for $\vec{x}$?

A. Multiply: $(\text{RREF } A)(A\vec{x}) = (\text{RREF } A)\vec{b}$

B. Add: $(\text{RREF } A) + A\vec{x} = (\text{RREF } A) + \vec{b}$

C. Multiply: $(A^{-1})(A\vec{x}) = (A^{-1})\vec{b}$

D. Add: $(A^{-1}) + A\vec{x} = (A^{-1}) + \vec{b}$

(d) Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ using the method you chose in (c).

Activity 4.3.4 Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$$

with a unique solution.

(a) Explain and demonstrate how this problem can be restated using matrix multiplication.

(b) Use the properties of matrix multiplication to find the unique solution.

Solving Systems with Matrix Inverses (MX3)

Activity 4.3.5 Solving linear systems using matrix multiplication is most useful when we are working with one common coefficient matrix, and varying the right-hand side. That is, when we have $A\vec{x} = \vec{b}$ for several different values of $\vec{b}$.

In the following, let $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ and consider the following questions about various equations of the form $A\vec{x} = \vec{b}$?

(a) Suppose that $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. If asked to solve the equation $A\vec{x} = \vec{b}$, which of the following approaches do you prefer?

A. Calculate RREF$[A|\vec{b}]$.

B. Calculate A^{-1} and then compute $\vec{x} = A^{-1}\vec{b}$

(b) Suppose that $\vec{b}_1, \vec{b}_2, \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$. If asked to solve each of the equations $A\vec{x} = \vec{b}_1, A\vec{x} = \vec{b}_2, A\vec{x} = \vec{b}_3$, which of the following approaches do you prefer?

A. Calculate RREF$[A|\vec{b}_1]$, RREF$[A|\vec{b}_2]$, and RREF$[A|\vec{b}_3]$

B. Calculate A^{-1} and then compute $\vec{x} = A^{-1}\vec{b}_1$, $\vec{x} = A^{-1}\vec{b}_2$, and $\vec{x} = A^{-1}\vec{b}_3$

(c) Suppose that $\vec{b}_1, \ldots, \vec{b}_{10}$ are 10 distinct vectors. If asked to solve each of the equations $A\vec{x} = \vec{b}_1, \ldots, A\vec{x} = \vec{b}_{10}$, which of the following approaches do you prefer?

A. Calculate RREF$[A|\vec{b}_1]$, ... RREF$[A|\vec{b}_{10}]$.

B. Calculate A^{-1} and then compute $\vec{x} = A^{-1}\vec{b}_1$, ... $\vec{x} = A^{-1}\vec{b}_{10}$.

4.4 Row Operations as Matrix Multiplication (MX4)

Activity 4.4.1 Given a linear transformation T, how did we define its standard matrix A? How do we compute the standard matrix A from T?

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.2 Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

(a) Which of these tweaks of the identity matrix yields a matrix that doubles the third row of A when left-multiplying? $(2R_3 \to R_3)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

A. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) Which of these tweaks of the identity matrix yields a matrix that swaps the first and third rows of A when left-multiplying? $(R_1 \leftrightarrow R_3)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Which of these tweaks of the identity matrix yields a matrix that adds 5 times the third row of A to the first row when left-multiplying? $(R_1 + 5R_3 \to R_1)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 5 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.4 What would happen if you *right*-multiplied by the tweaked identity matrix rather than left-multiplied?

A. The manipulated rows would be reversed.

B. Columns would be manipulated instead of rows.

C. The entries of the resulting matrix would be rotated 180 degrees.

Activity 4.4.5 Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \to R_1$ applied as follows to show $A \sim B$:

$$\begin{aligned} A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} &\sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B \end{aligned}$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Activity 4.4.6 Let A be *any* 4×4 matrix.

(a) Give a 4×4 matrix M that may be used to perform the row operation $-5R_2 \to R_2$.

(b) Give a 4×4 matrix Y that may be used to perform the row operation $R_2 \leftrightarrow R_3$.

(c) Use matrix multiplication to describe the matrix obtained by applying $-5R_2 \to R_2$ and then $R_2 \leftrightarrow R_3$ to A (note the order).

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.7 Consider the matrix $A = \begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}$. Illustrate Fact 4.4.3 by finding row operation matrices $R_1, \dots, R_k$ for which

$$\mathrm{RREF}(A) = R_k \cdots R_2 R_1 A.$$

If you and a teammate were to do this independently, would you necessarily come up with the same sequence of matrices $R_1, \dots, R_k$?

Chapter 5: Geometric Properties of Linear Maps (GT)

5.1 Row Operations and Determinants (GT1)

Activity 5.1.1 Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the standard matrix $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$.

(a) Draw a figure that depicts how T transforms the unit square.

(b) What geometric features of the unit square were preserved by the transformation? Which geometric features changed?

Activity 5.1.2 The image in Figure 46 illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

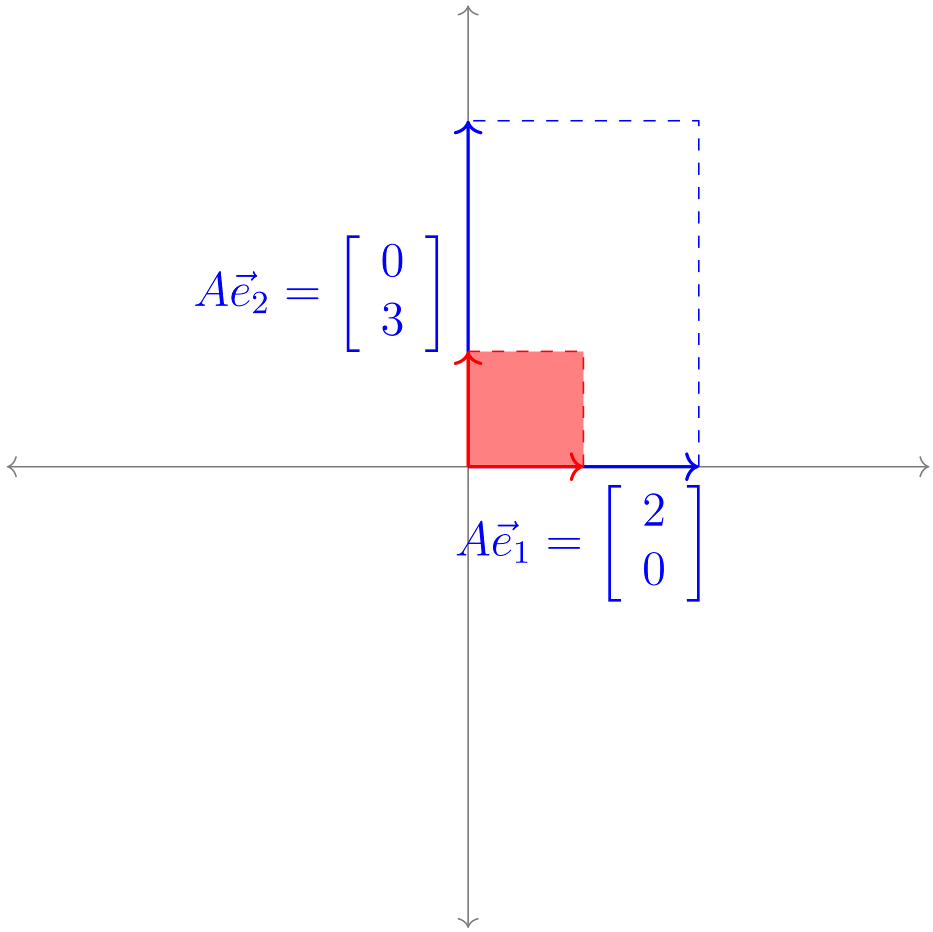


Figure 46 Transformation of the unit square by the matrix A.

(a) What are the lengths of $A\vec{e}_1$ and $A\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity 5.1.3 The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ transforms the unit square.

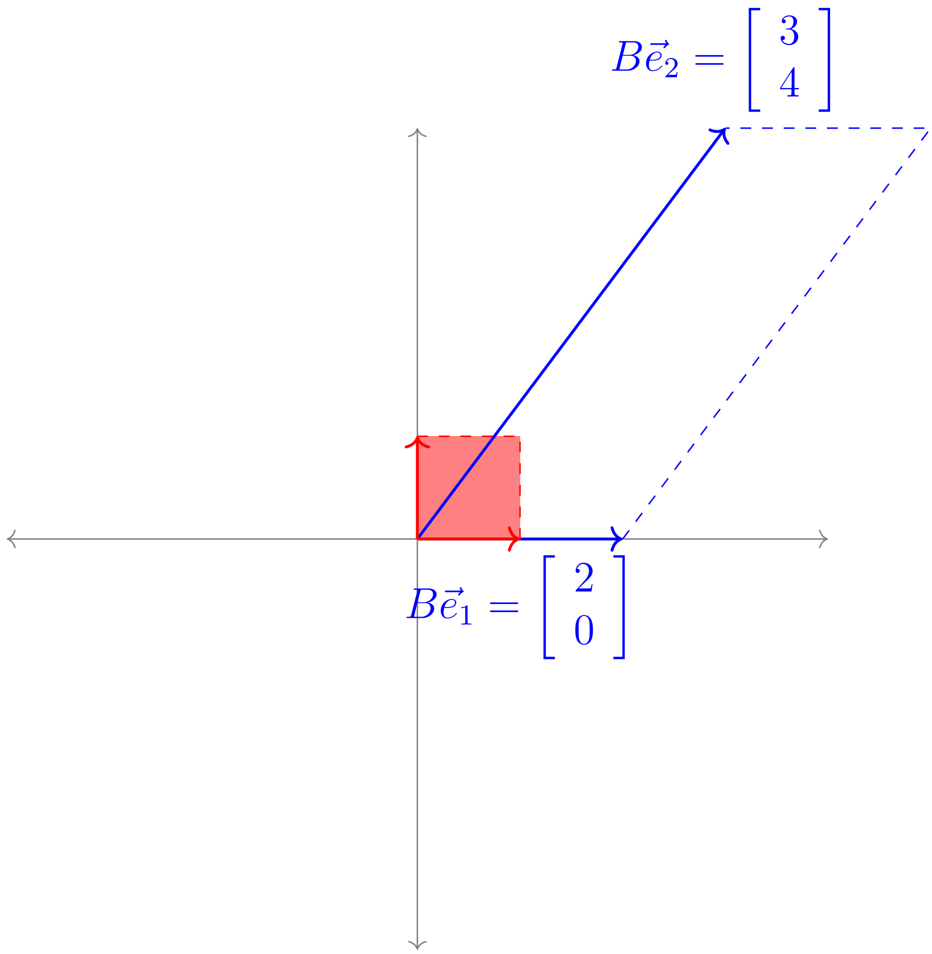


Figure 47 Transformation of the unit square by the matrix B

(a) What are the lengths of $B\vec{e}_1$ and $B\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity 5.1.7 The transformation of the unit square by the standard matrix $[\vec{e}_1 \ \ \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. If $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$ is the area of resulting parallelogram, what is the value of $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$?

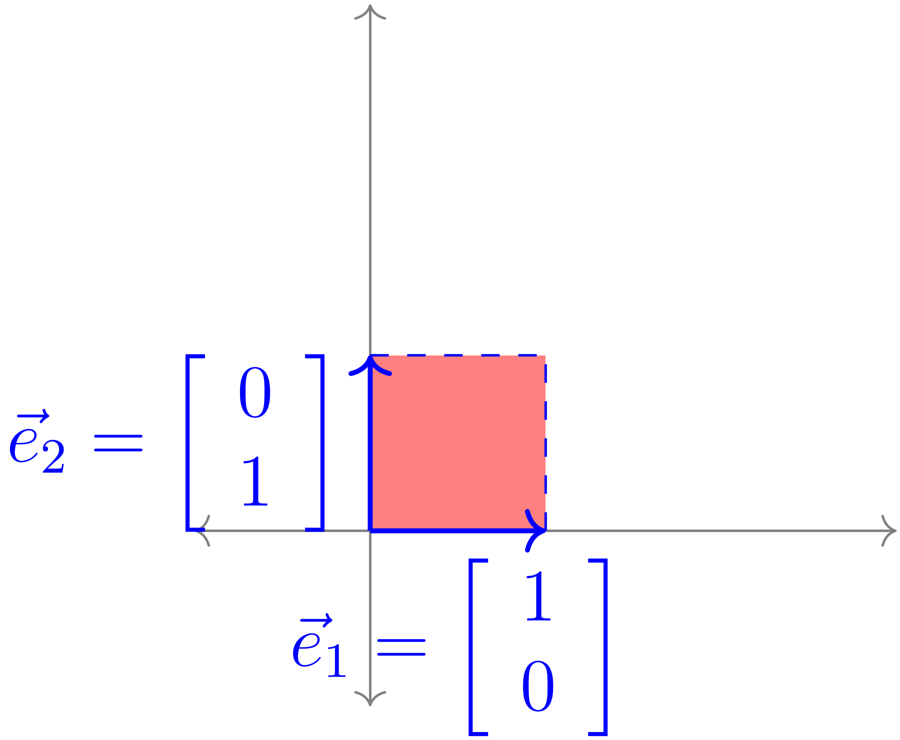


Figure 51 The transformation of the unit square by the identity matrix.

The value for $\det([\vec{e}_1\ \ \vec{e}_2]) = \det(I)$ is:

A. 0

B. 1

C. 2

D. 4

Activity 5.1.8 The transformation of the unit square by the standard matrix $[\vec{v}\ \ \vec{v}]$ is illustrated below: both $T(\vec{e}_1) = T(\vec{e}_2) = \vec{v}$. If $\det([\vec{v}\ \ \vec{v}])$ is the area of the generated parallelogram, what is the value of $\det([\vec{v}\ \ \vec{v}])$?

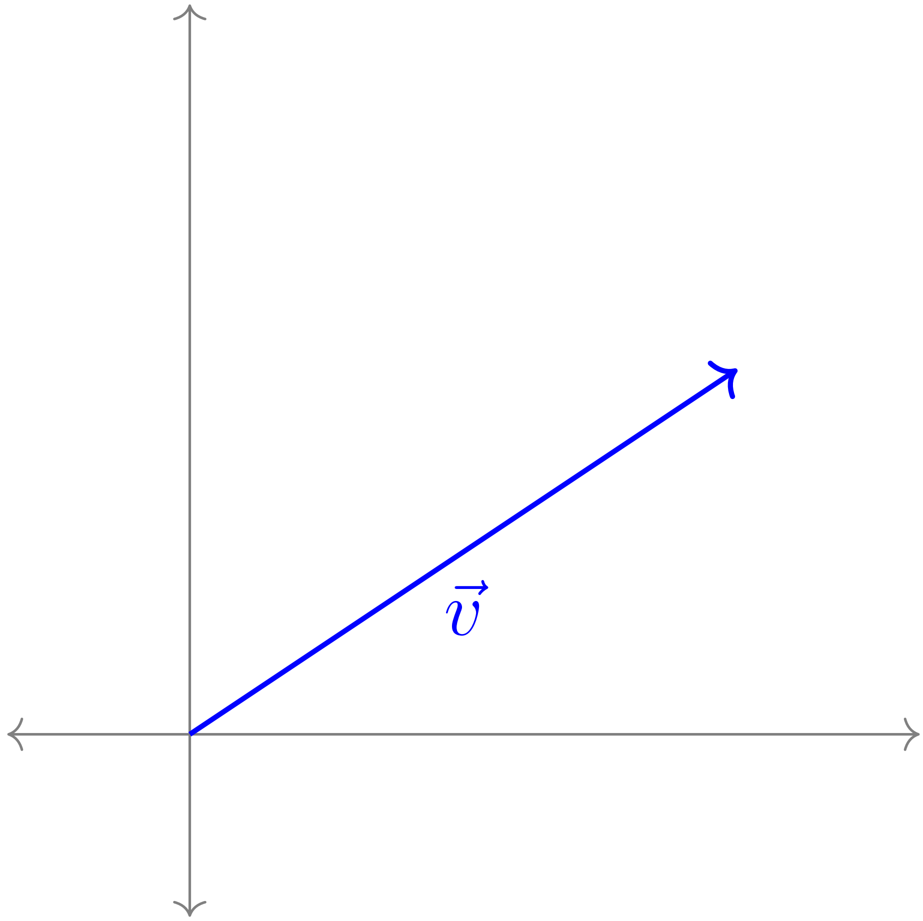


Figure 52 Transformation of the unit square by a matrix with identical columns.

The value of $\det([\vec{v}\ \ \vec{v}])$ is:

A. 0

B. 1

C. 2

D. 4

Activity 5.1.9 The transformations of the unit square by the standard matrices $[\vec{v}\ \ \vec{w}]$ and $[c\vec{v}\ \ \vec{w}]$ are illustrated below. Describe the value of $\det([c\vec{v}\ \ \vec{w}])$.

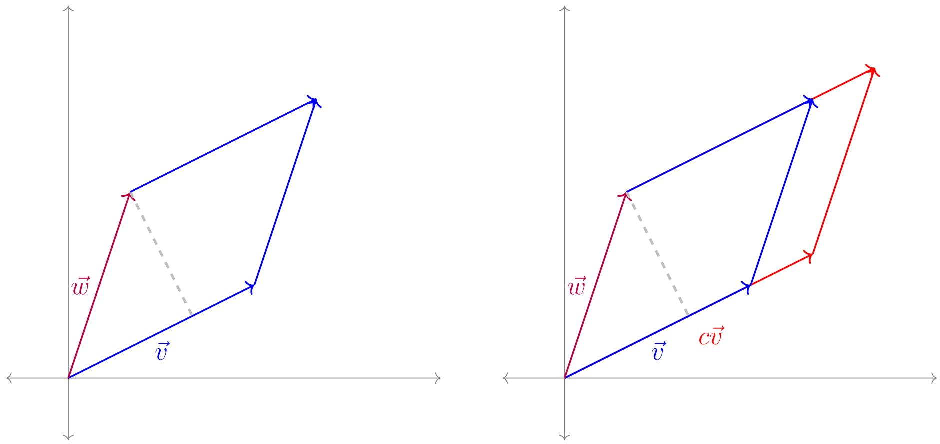


Figure 53 The parallelograms generated by $\vec{v}$ and $\vec{w}$, and by c$\vec{v}$ and $\vec{w}$

Describe the value of $\det([c\vec{v}\ \ \vec{w}])$:

A. $\det([\vec{v}\ \ \vec{w}])$

B. $c\det([\vec{v}\ \ \vec{w}])$

C. $c^2\det([\vec{v}\ \ \vec{w}])$

D. Cannot be determined from this information.

Activity 5.1.13 The parallelograms generated by the standard matrices $[\vec{u}\ \ \vec{w}]$, $[\vec{v}\ \ \vec{w}]$ and $[\vec{u}+\vec{v}\ \ \vec{w}]$ are illustrated below.

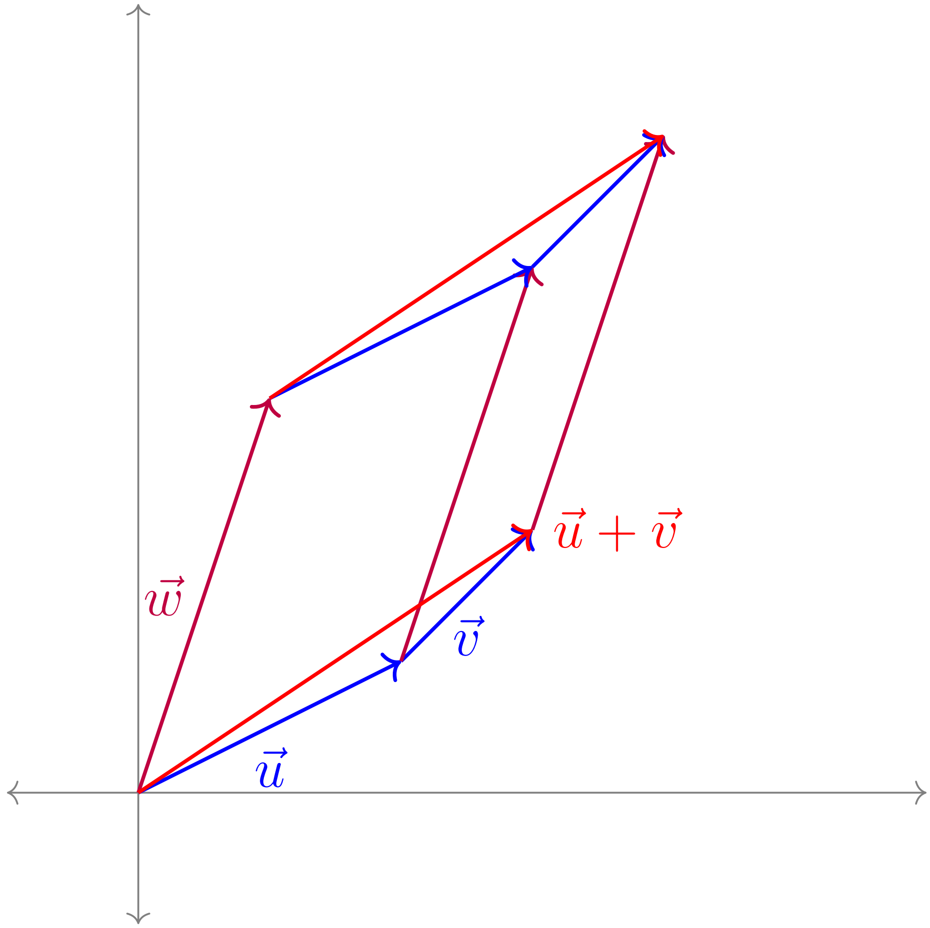


Figure 57 Parallelogram generated by $\vec{u}+\vec{v}$ and $\vec{w}$

Describe the value of $\det([\vec{u}+\vec{v}\ \ \vec{w}])$.

A. $\det([\vec{u}\ \ \vec{w}]) = \det([\vec{v}\ \ \vec{w}])$

B. $\det([\vec{u}\ \ \vec{w}]) + \det([\vec{v}\ \ \vec{w}])$

C. $\det([\vec{u}\ \ \vec{w}])\det([\vec{v}\ \ \vec{w}])$

D. Cannot be determined from this information.

Activity 5.1.19 The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?

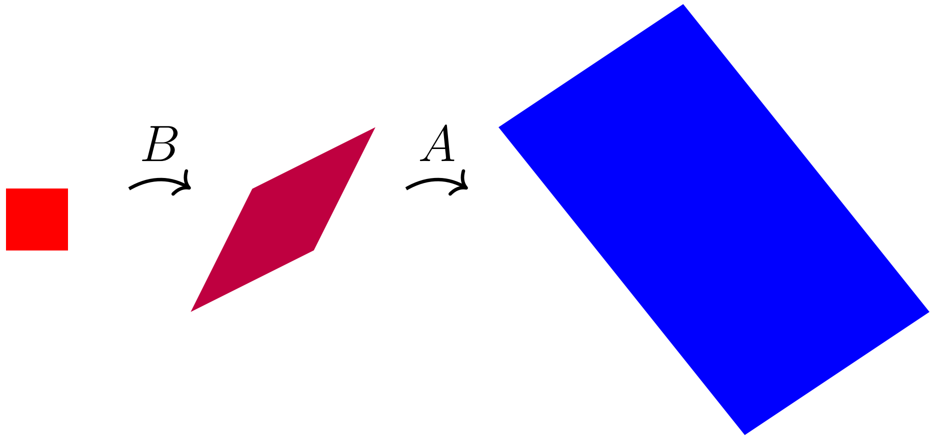


Figure 60 Area changing under the composition of two linear maps

A. 1

B. 7

C. 12

D. Cannot be determined

Activity 5.1.23 Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1+4(9) & 2+4(10) & 3+4(11) & 4+4(12) \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{4,4}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity 5.1.24 Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{4,4}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Row Operations and Determinants (GT1)

Activity 5.1.25 Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3(5) & 3(6) & 3(7) & 3(8) \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{4,4}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Activity 5.1.26 Let A be *any* 4×4 matrix with determinant 2.

(a) Let B be the matrix obtained from A by applying the row operation $R_1 - 5R_3 \to R_1$. What is $\det B$?

A -4 B -2 C 2 D 10

(b) Let M be the matrix obtained from A by applying the row operation $R_3 \leftrightarrow R_1$. What is $\det M$?

A -4 B -2 C 2 D 10

(c) Let P be the matrix obtained from A by applying the row operation $2R_4 \to R_4$. What is $\det P$?

A -4 B -2 C 2 D 10

Activity 5.1.30 Complete the following derivation for a formula calculating 2×2 determinants:

$$\begin{aligned}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ?\det \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \\
&= ?\det \begin{bmatrix} 1 & b/a \\ c-c & d-bc/a \end{bmatrix} \\
&= ?\det \begin{bmatrix} 1 & b/a \\ 0 & d-bc/a \end{bmatrix} \\
&= ?\det \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \\
&= ?\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= ?\det I \\
&= ?
\end{aligned}$$

Row Operations and Determinants (GT1)

Activity 5.1.32 Suppose we have a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$. Given some shape S in the plane $\mathbb{R}^2$, we can use to T to transform it into some new shape $T(S)$. Consider the following questions about properties that may or may not be preserved by T.

(a) If S is a straight line segment, explain why $T(S)$ is also a straight line segment.

(b) If S is a straight line segment, does $T(S)$ necessarily have to have the same length as that of S?

(c) If S is a triangle, explain why $T(S)$ is also a triangle.

(d) Continuing as above, do the angles of $T(S)$ necessarily have to be the same as those of S?

5.2 Computing Determinants (GT2)

Activity 5.2.1 Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Use a combination of row and column operations to transform A into the identity matrix. Use this to calculuate the determinant of A.

(b) Check your work using the formula for the determinant of a 2×2 matrix.

Activity 5.2.3 The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

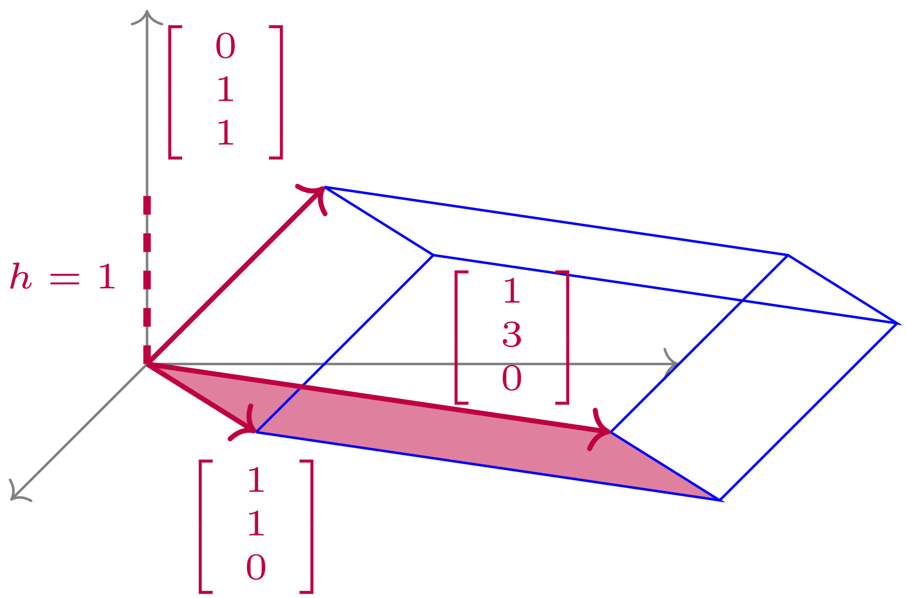


Figure 62 Transformation of the unit cube by the linear transformation.

Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

A. $\det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

B. $\det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

C. $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

D. $\det \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Activity 5.2.6 Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity 5.2.7 Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Factor out a 2 from a column.

(b) Swap rows or columns to put a 1 on the main diagonal.

Computing Determinants (GT2)

Activity 5.2.8 Simplify $\det\begin{bmatrix}4 & -2 & 2\\ 3 & 1 & 4\\ 1 & -1 & 3\end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Use row/column operations to create two zeroes in the same row or column.

(b) Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Activity 5.2.10 Rewrite

$$\det\begin{bmatrix}2 & 1 & -2 & 1\\ 3 & 0 & 1 & 4\\ -2 & 2 & 3 & 0\\ -2 & 0 & -3 & -3\end{bmatrix}$$

as a multiple of a determinant of a 3×3 matrix.

Activity 5.2.11 Compute $\det\begin{bmatrix}2 & 3 & 5 & 0\\ 0 & 3 & 2 & 0\\ 1 & 2 & 0 & 3\\ -1 & -1 & 2 & 2\end{bmatrix}$ by using any combination of row/column operations.

Activity 5.2.14 Based on the previous activities, which technique is easier for computing determinants?

A. Memorizing formulas.

B. Using row/column operations.

C. Laplace expansion.

D. Some other technique.

Activity 5.2.15 Use your preferred technique to compute $\det\begin{bmatrix}4 & -3 & 0 & 0\\ 1 & -3 & 2 & -1\\ 3 & 2 & 0 & 3\\ 0 & -3 & 2 & -2\end{bmatrix}$.

Computing Determinants (GT2)

Activity 5.2.17 A *diagonal* matrix is a matrix that has zeroes in every position except (possibly) the main upper-left to lower-right diagonal. A matrix is *upper (resp. lower) triangular* if every entry below (resp. above) the main diagonal is zero.

(a) Explain why the determinant of a diagonal matrix is always equal to the product of the entries on the main diagonal.

(b) Explain why the determinant of an upper (or lower) triangular matrix is always equal to the product of the entries on the main diagonal.

5.3 Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.1 Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by rotating vectors about the origin through and angle of $45°$, and let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation that reflects vectors about the line $x_1 = x_2$.

(a) If L is a line, let $R(L)$ denote the line obtained by applying R to it. Are there any lines L for which $R(L)$ is parallel to L?

(b) Now consider the transformation S. Are there any lines L for which $S(L)$ is parallel to L?

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.2 An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to $\det(M)\det(M^{-1})$?

A. -1

B. 0

C. 1

D. 4

Activity 5.3.6 Finding the eigenvalues λ that satisfy

$$A\vec{x} = \lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$$

for some nontrivial eigenvector $\vec{x}$ is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

(a) If λ is an eigenvalue, and T is the transformation with standard matrix $A - \lambda I$, which of these must contain a non-zero vector?

A. The kernel of T

B. The image of T

C. The domain of T

D. The codomain of T

(b) Therefore, what can we conclude?

A. A is invertible

B. A is not invertible

C. $A - \lambda I$ is invertible

D. $A - \lambda I$ is not invertible

(c) And what else?

A. $\det A = 0$

B. $\det A = 1$

C. $\det(A - \lambda I) = 0$

D. $\det(A - \lambda I) = 1$

Activity 5.3.9 Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

(a) Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

(b) Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of A.

Activity 5.3.10 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Activity 5.3.11 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.12 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.13 Let $A \in M_{n,n}$ and $\lambda \in \mathbb{R}$. The eigenvalues of A that correspond to λ are the vectors that get stretched by a factor of λ. Consider the following special cases for which we can make more geometric meaning.

(a) What are some other ways we can think of the eigenvalues corresponding to eigenvalue $\lambda = 0$?

(b) What are some other ways we can think of the eigenvalues corresponding to eigenvalue $\lambda = 1$?

(c) What are some other ways we can think of the eigenvalues corresponding to eigenvalue $\lambda = -1$?

(d) How might we interpret a matrix that has no (real) eigenvectors/values?

5.4 Eigenvectors and Eigenspaces (GT4)

Activity 5.4.1 Which of the following vectors is an eigenvector for $A = \begin{bmatrix} 2 & 4 & -1 & -5 \\ 0 & 0 & -3 & -9 \\ 1 & 1 & 0 & 2 \\ -2 & -2 & 3 & 5 \end{bmatrix}$?

A. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} -3 \\ 3 \\ -2 \\ 1 \end{bmatrix}$

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.2 It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{x}$ such that $A\vec{x} = -2\vec{x}$.

Activity 5.4.4 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ associated with the eigenvalue 3.

Activity 5.4.5 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 5 & -2 & 0 & 4 \\ 6 & -2 & 1 & 5 \\ -2 & 1 & 2 & -3 \\ 4 & 5 & -3 & 6 \end{bmatrix}$ associated with the eigenvalue 1.

Activity 5.4.6 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ associated with the eigenvalue 2.

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.7 Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix A. Further, suppose that we know that $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are eigenvectors corresponding to eigenvalues 2 and -3 respectively.

(a) Express the vector $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}$.

(b) Determine $T(\vec{w})$.

Chapter A: Applications

A.1 Civil Engineering: Trusses and Struts

Activity A.1.2 Consider the representation of a simple truss pictured below. All of the seven struts are of equal length, affixed to two anchor points applying a normal force to nodes C and E, and with a $10000N$ load applied to the node given by D.

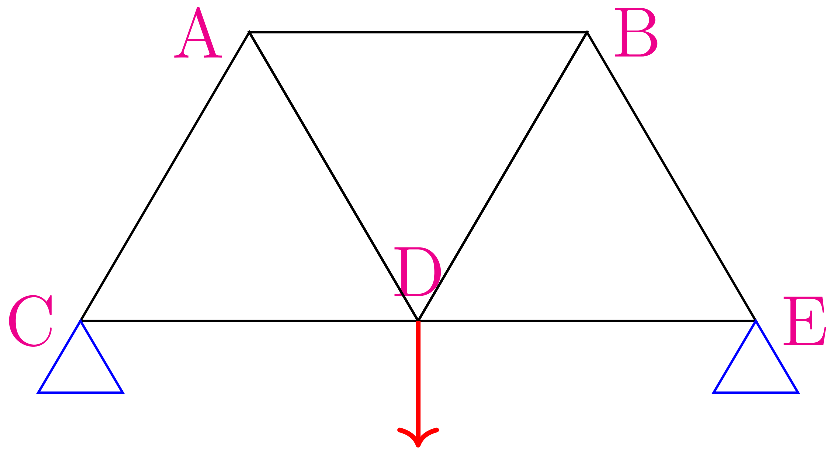


Figure 71 A simple truss

Which of the following must hold for the truss to be stable?

1. All of the struts will experience compression.
2. All of the struts will experience tension.
3. Some of the struts will be compressed, but others will be tensioned.

Activity A.1.5 Using the conventions of the previous remark, and where $\vec{L}$ represents the load vector on node D, find four more vector equations that must be satisfied for each of the other four nodes of the truss.

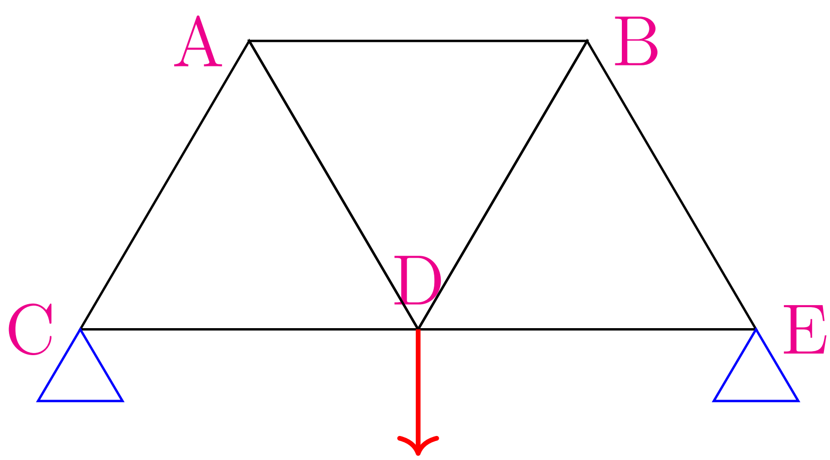


Figure 74 A simple truss

$$A : ?$$

$$B : ?$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$D : ?$$

$$E : ?$$

Activity A.1.8 To write a linear system that models the truss under consideration with constant load 10000 newtons, how many scalar variables will be required?

- 7: 5 from the nodes, 2 from the anchors
- 9: 7 from the struts, 2 from the anchors
- 11: 7 from the struts, 4 from the anchors
- 12: 7 from the struts, 4 from the anchors, 1 from the load
- 13: 5 from the nodes, 7 from the struts, 1 from the load

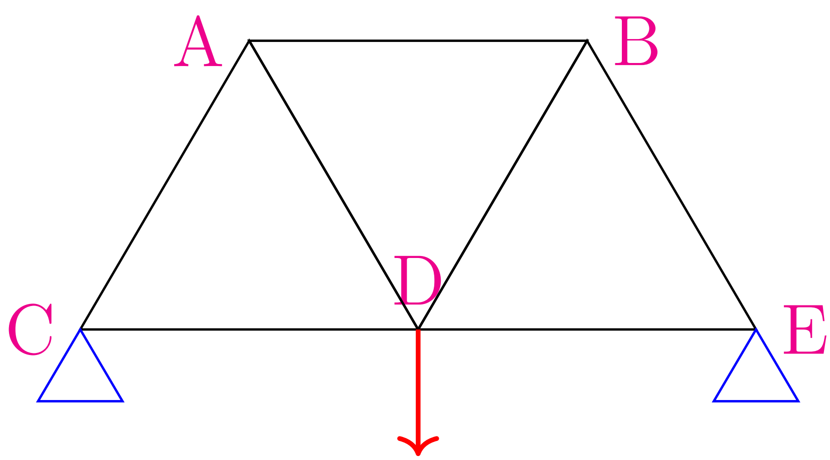


Figure 75 A simple truss

Activity A.1.12 Expand the vector equation given below using sine and cosine of appropriate angles, then compute each component (approximating $\sqrt{3}/2 \approx 0.866$).

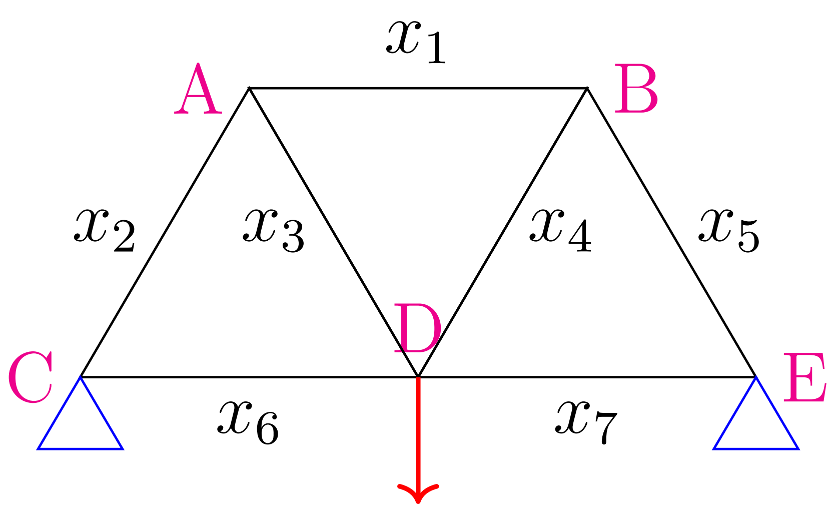


Figure 79 Variables for the truss

$$D : \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DC} + \vec{F}_{DE} = -\vec{L}$$

$$\Leftrightarrow x_3 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_4 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_6 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_7 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\Leftrightarrow x_3 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_4 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_6 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_7 \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A.2 Computer Science: PageRank

Activity A.2.1

In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.

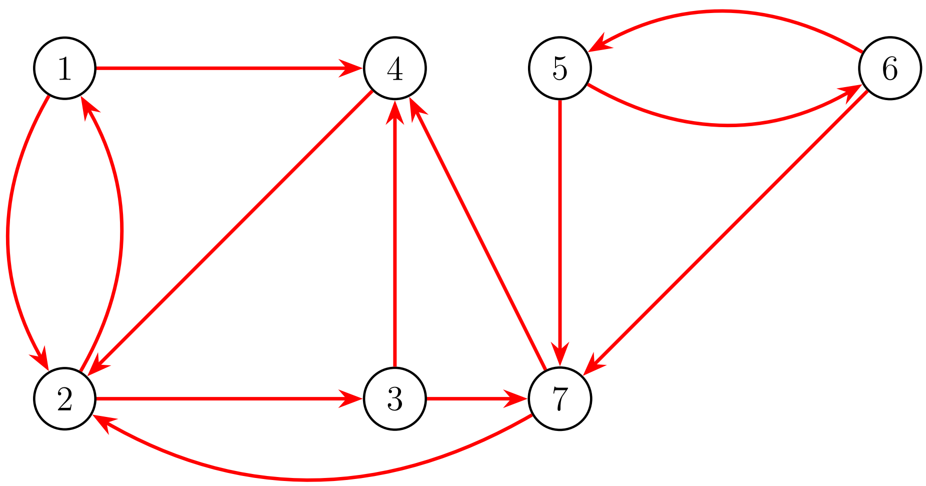


Figure 81 A seven-webpage network

Based on how these pages link to each other, write a list of the 7 webpages in order from most important to least important.

Activity A.2.5 Thus, our \$978,000,000,000 problem is what kind of problem?

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A. An antiderivative problem

B. A bijection problem

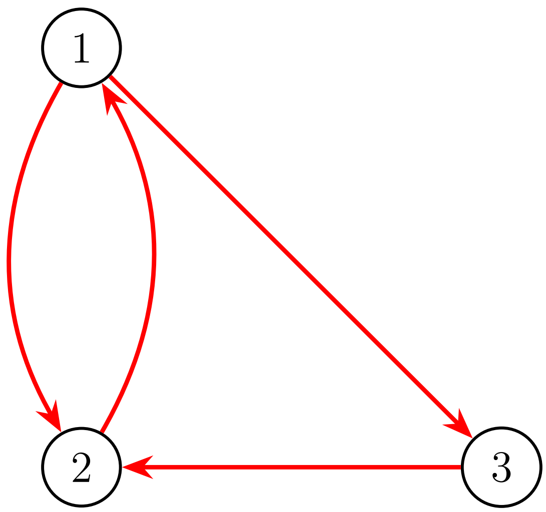
C. A cofactoring problem

D. A determinant problem

E. An eigenvector problem

Activity A.2.6 Find a page rank vector $\vec{x}$ satisfying $A\vec{x} = 1\vec{x}$ for the following network's page rank matrix A.

That is, find the eigenspace associated with $\lambda = 1$ for the matrix A, and choose a vector from that eigenspace.



$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Figure 84 A three-webpage network

Activity A.2.8 Compute the 7×7 page rank matrix for the following network.

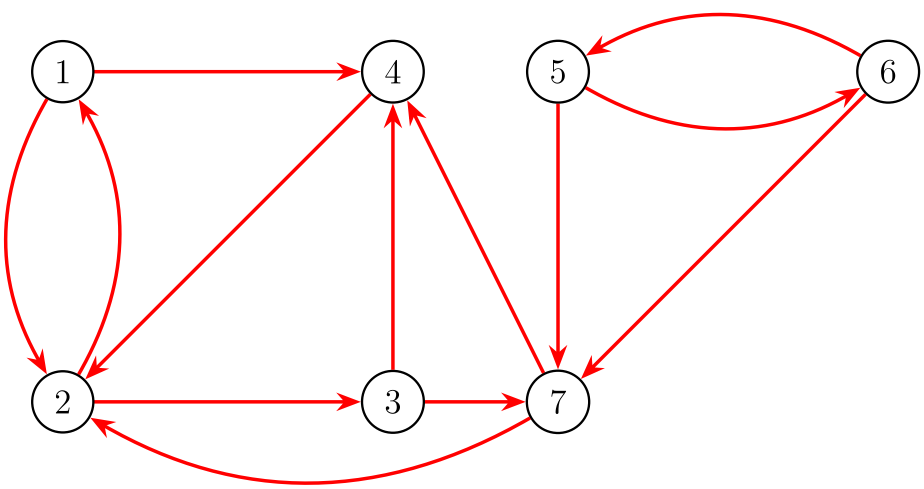


Figure 85 A seven-webpage network

For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.2.9 Find a page rank vector for the given page rank matrix.

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

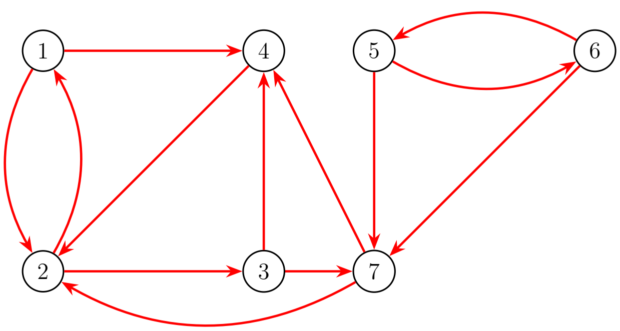


Figure 86 A seven-webpage network

Which webpage is most important?

Activity A.2.11 Given the following diagram, use a page rank vector to rank the pages 1 through 7 in order from most important to least important.

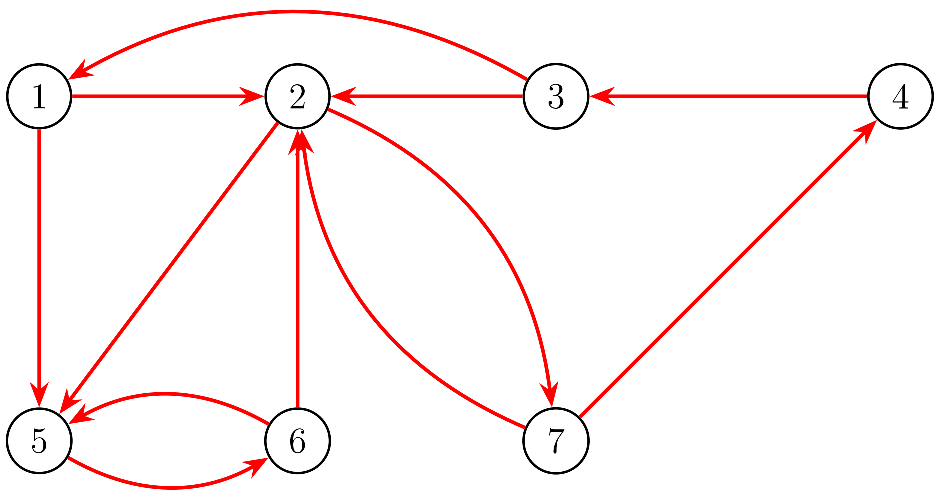


Figure 88 Another seven-webpage network

A.3 Geology: Phases and Components

Activity A.3.3 To study this vector space, each of the three components $\vec{c}_1, \vec{c}_2, \vec{c}_3$ may be considered as the three components of a Euclidean vector.

$$\vec{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Determine if the set of phases is linearly dependent or linearly independent.

Activity A.3.4 Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_2 = \mathrm{CaMgSiO_4} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{p}_3 = \mathrm{CaSiO_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

(a) Set up a system of equations equivalent to this vector equation.

(b) Find a basis for its solution space.

(c) Interpret each basis vector as a vector equation and a chemical equation.

Activity A.3.5 We found two basis vectors $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, corresponding to the vector and chemical equations

$$\begin{aligned} 2\vec{p}_2 + 2\vec{p}_3 &= \vec{p}_1 + \vec{p}_4 & \qquad 2\mathrm{CaMgSiO_4} + 2\mathrm{CaSiO_3} &= \mathrm{Ca_3MgSi_2O_8} + \mathrm{CaMgSi_2O_6} \\ \vec{p}_2 + \vec{p}_3 &= \vec{p}_5 & \mathrm{CaMgSiO_4} + \mathrm{CaSiO_3} &= \mathrm{Ca_2MgSi_2O_7} \end{aligned}$$

Combine the basis vectors to produce a chemical equation among the five phases that does not involve $\vec{p}_2 = \mathrm{CaMgSiO_4}$.