Precalculus for Team-Based Inquiry Learning

2024 Development Edition

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Website: Precalculus for Team-Based Inquiry Learning[1]

[1]teambasedinquirylearning.github.io/precalculus/

[2]github.com/TeamBasedInquiryLearning/precalculus/blob/main/LICENSE.md

Contents

Chapter 1

Equations, Inequalities, and Applications (EQ)

Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx
By the end of this chapter, you should be able to...

1. Solve a linear equation in one variable. Solve a linear inequality in one variable and express the solution graphically and using interval notation.

2. Solve application problems involving linear equations.

3. Given two points, determine the distance between them and the midpoint of the line segment connecting them.

4. Solve a linear equation involving an absolute value. Solve a linear inequality involving absolute values and express the answers graphically and using interval notation.

5. Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

6. Solve a rational equation.

7. Solve quadratic inequalities and express the solution graphically and

with interval notation. Solve rational inequalities and express the solution graphically and using interval notation.

1.1 Linear Equations and Inequalities (EQ1)

Objectives

- Solve a linear equation in one variable. Solve a linear inequality in one variable and express the solution graphically and using interval notation.

Remark 1.1.1 Recall that when solving a linear equation, you use addition, subtraction, multiplication and division to isolate the variable.

Activity 1.1.2 Solve the linear equations.

(a) $3x - 8 = 5x + 2$

A. $x = 2$

B. $x = 5$

C. $x = -5$

D. $x = -2$

(b) $5(3x - 4) = 2x - (x + 3)$

A. $x = \dfrac{17}{14}$

B. $x = \dfrac{14}{17}$

C. $x = \dfrac{23}{14}$

D. $x = \dfrac{14}{23}$

(c) $\frac{2}{3}x - 8 = \frac{5x+1}{6}$

Activity 1.1.3 The following linear equations appear similar, but their solutions are very different.

(a) Which of these equations has one unique solution?

A. $4(x-2) = 4x+6$

B. $4(x-1) = 4x+4$

C. $4(x-1) = x+4$

(b) Which of these equations has many solutions?

A. $4(x-2) = 4x+6$

B. $4(x-1) = 4x+4$

C. $4(x-1) = x+4$

(c) Which of these equations has no solutions?

A. $4(x-2) = 4x+6$

B. $4(x-1) = 4x+4$

C. $4(x-1) = x+4$

(d) What happens when solving a linear equation when it has no solution? when it has many solutions?

Definition 1.1.4 A linear equation with one unique solution is a **conditional equation.** A linear equation that is true for all values of the variable is an **identity equation.**. A linear equation with no solutions is an **inconsistent equation.** $\Diamond$

Activity 1.1.5 An inequality is a relationship between two values that are not equal.

(a) What is the solution to the linear equation $3x - 1 = 5$?

(b) Which of these values is a solution of the inequality $3x - 1 \geq 5$?

A. $x = 0$

B. $x = 2$

C. $x = 4$

D. $x = 10$

(c) Express the solution of the inequality $3x - 1 \geq 5$ in interval notation.

A. $(-\infty, 2]$

B. $(-\infty, 2)$

C. $(2, \infty)$

D. $[2, \infty)$

(d) Draw the solution to the inequality on a number line.

Activity 1.1.6

(a) Which of these values is a solution of the inequality $-x < 8$?

A. $x = -10$

B. $x = -8$

C. $x = 4$

D. $x = 10$

(b) Solve the linear inequality $-x < 8$. How does your solution compare to the values chosen in part (a)?

(c) Expression the solution of the inequality $-x < 8$ in interval notation.

A. $(-\infty, -8)$

B. $(-\infty, -8]$

C. $(-8, \infty)$

D. $[-8, \infty)$

Remark 1.1.7 You can treat solving linear inequalities, just like solving an equation. The one exception is when you multiply or divide by a negative value, reverse the inequality symbol.

Activity 1.1.8 Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a) $-3x - 1 \leq 5$

(b) $3(x + 4) > 2x - 1$

(c) $-\frac{1}{2}x \geq -\frac{2}{4} + \frac{5}{4}x$

Definition 1.1.9 A **compound inequality** includes multiple inequalities in one statement. $\Diamond$

Activity 1.1.10 Consider the statement $3 \leq x < 8$. This really means that $3 \leq x$ and $x < 8$.

(a) Which of the following inequalities are equivalent to the compound inequality $3 \leq 2x - 3 < 8$?

A. $3 \leq 2x - 3$

B. $3 \geq 2x - 3$

C. $2x - 3 < 8$

D. $2x - 3 > 8$

(b) How do the solutions overlap?

Remark 1.1.11 Solving a compound linear inequality, uses the same methods as a single linear inequality ensuring that you perform the same operations on all three parts. Alternatively, you can break the compound inquality up into two and solve separately.

1.2 Applications of Linear Equations (EQ2)

Objectives

- Solve application problems involving linear equations.

1.3 Distance and Midpoint (EQ3)

Objectives

- Given two points, determine the distance between them and the midpoint of the line segment connecting them.

1.4 Absolute Value Equations and Inequalities (EQ4)

Objectives

- Solve a linear equation involving an absolute value. Solve a linear inequality involving absolute values and express the answers graphically and using interval notation.

1.5 Quadratic Equations (EQ5)

Objectives

- Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

1.6 Rational Equations (EQ6)

Objectives

- Solve a rational equation.

1.7 Quadratic and Rational Inequalities (EQ7)

Objectives

- Solve a rational equation.

Chapter 2

Functions (FN)

Chapter 3

Linear Functions (LF)

Chapter 4

Polynomial and Rational Functions (PR)

Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx
By the end of this chapter, you should be able to...

1. Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.
2. Use quadratic models to solve an application problem and establish conclusions.
3. Rewrite a rational function as a polynomial plus a proper rational function.
4. Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.
5. Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.
6. Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

4.1 Graphing Quadratic Functions (PR1)

Objectives

- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

Observation 4.1.1 Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.

Activity 4.1.2 Use the graph of the quadratic function $f(x) = 3(x-2)^2 - 4$ to answer the questions below.

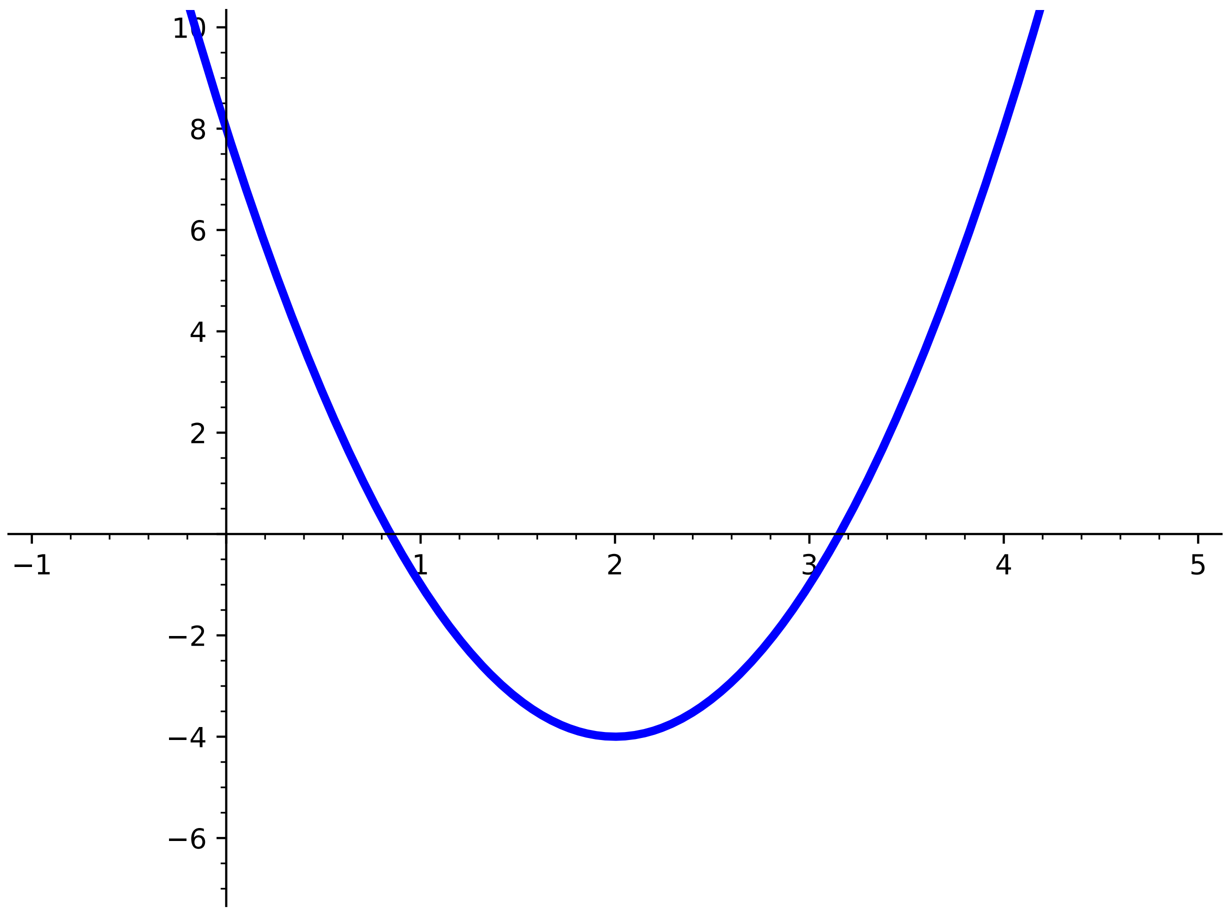


Figure 4.1.3

(a) Make a table for values of $f(x)$ corresponding to the given x-values. What is happening to the y-values as the x-values increase? Do you notice any other patterns of the y-values of the table?

Table 4.1.4

x	$f(x)$
-2	
-1	
0	
1	
2	
3	
4	
5	

(b) At which point (x, y) does $f(x)$ have a minimum value? That is, is there a point on the graph that is lower than all other points?

A. The minimum value appears to occur near $(0, 8)$.

B. The minimum value appears to occur near $(-\frac{1}{5}, 10)$.

C. The minimum value appears to occur near $(2, -4)$.

D. There is no minimum value of this function.

(c) At which point (x, y) does $f(x)$ have a maximum value? That is, is there a point on the graph that is higher than all other points?

A. The maximum value appears to occur near $(-2, 44)$.

B. The maximum value appears to occur near $(-\frac{1}{5}, 10)$.

C. The maximum value appears to occur near $(2, -4)$.

D. There is no maximum value of this function.

Definition 4.1.5 The maximum or minimum of a quadratic function is also known as its **vertex**. The **vertex form** of a quadratic function is given by $f(x) = a(x-h)^2 + k$, where (h, k) is the **vertex** of the parabola and $x = h$ is the **axis of symmetry.** $\Diamond$

Activity 4.1.6 Use the given the quadratic function, $f(x) = 3(x-2)^2 - 4$, to answer the following:

(a) Applying Definition 4.1.5, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -4)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 4)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -4)$; axis of symmetry: $x = -2$

D. vertex: $(2, 4)$; axis of symmetry: $x = 2$

(b) Compare what you got in part a with the values you found in Activity 4.1.2. What do you notice?

Graphing Quadratic Functions (PR1)

Definition 4.1.7 Given the **standard form** of a quadratic function, $f(x) = ax^2 + bx + c$, with real coefficients a, b, and c, the **axis of symmetry** is defined as $x = \frac{-b}{2a}$ and has a **vertex** at the point $(\frac{-b}{2a}, f(\frac{-b}{2a}))$. $\diamondsuit$

Activity 4.1.8 Use the graph of the quadratic function to answer the questions below.

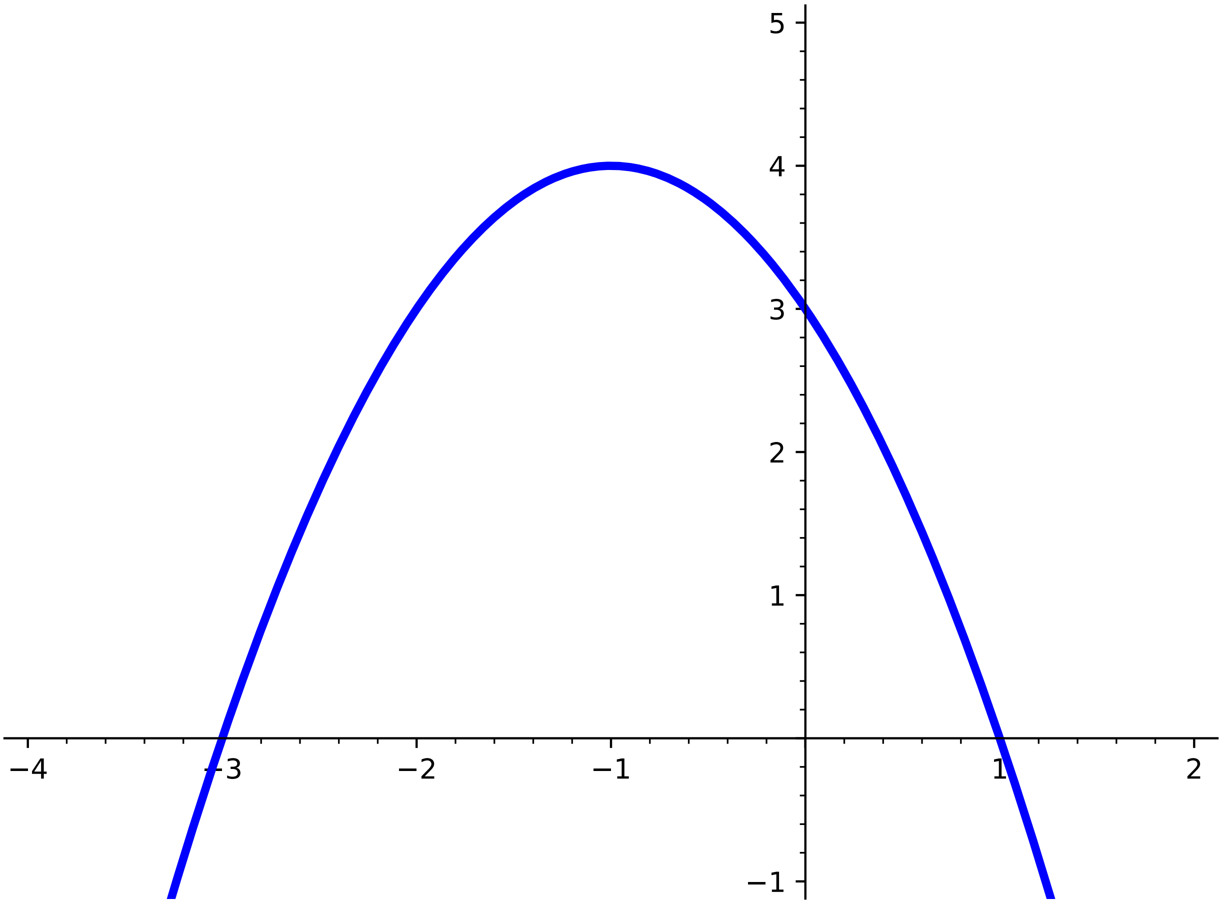


Figure 4.1.9

(a) Which of the following quadratic functions could be the graph shown in the figure?

A. $f(x) = x^2 + 2x + 3$

B. $f(x) = -(x + 1)^2 + 4$

C. $f(x) = -x^2 - 2x + 3$

D. $f(x) = (x + 1)^2 + 4$

(b) What is the maximum or minimum value?

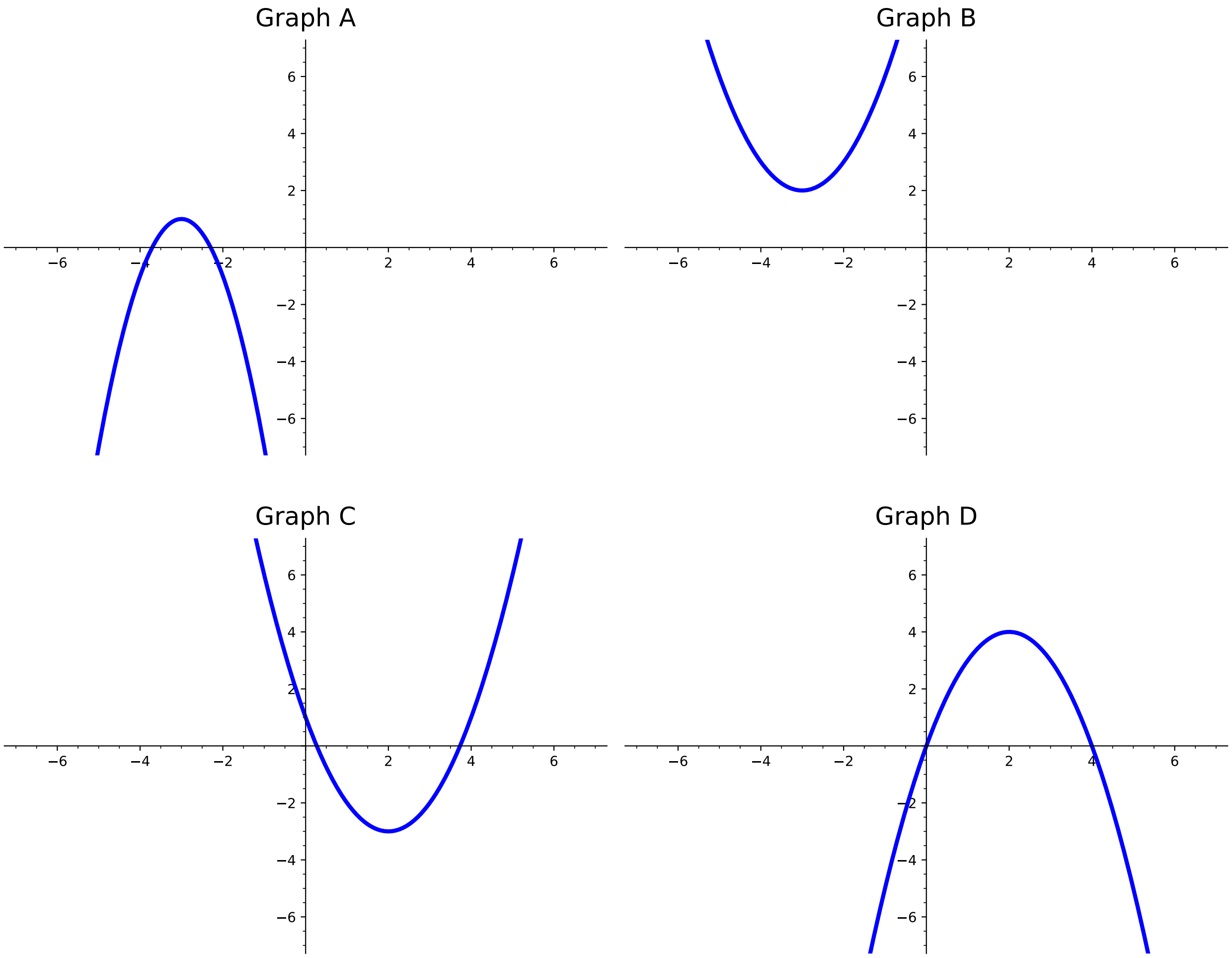
A. -1

B. 4

C. -3

D. 1

Activity 4.1.10 Consider the following four graphs of quadratic functions:



(a) Which of the graphs above have a maximum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(b) Which of the graphs above have a minimum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(c) Which of the graphs above have an axis of symmetry at $x = 2$?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(d) Which of the graphs above represents the function $f(x) = -(x-2)^2+4$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

(e) Which of the graphs above represents the function $f(x) = x^2 - 4x + 1$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Remark 4.1.11 Notice that the maximum or minimum value of the quadratic function is the y-value of the vertex. How can you determine if the vertex is a maximum or minimum?

Activity 4.1.12 A function $f(x)$ has a maximum value at 7 and its axis of symmetry at $x = -2$.

(a) Sketch a graph of a function that meets the criteria for $f(x)$.

(b) Was your graph the only possible answer? Try to sketch another graph that meets this criteria.

Remark 4.1.13 Other points, such as x- and y-intercepts, may be helpful in sketching a more accurate graph of a quadratic function.

Activity 4.1.14 Consider the following two quadratic functions $f(x) = x^2 - 4x + 20$ and $g(x) = 2x^2 - 8x + 24$ and answer the following questions:

(a) Applying Definition 4.1.7, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(b) Applying Definition 4.1.7, what is the vertex and axis of symmetry of $g(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(c) What do you notice about $f(x)$ and $g(x)$?

(d) Now graph both $f(x)$ and $g(x)$ and draw a sketch of each graph on one coordinate plane. How are they similar/different?

4.2 Quadratic Models and Meanings (PR2)

Objectives

- Use quadratic models to solve an application problem and establish conclusions.

Activity 4.2.1 A water balloon is tossed vertically from a fifth story window. It's height $h(t)$, in feet, at a time t, in seconds, is modeled by the function

$$h(t) = -16t^2 + 40t + 50$$

(a) Complete the following table. Do all the values have meaning in terms of the model?

Table 4.2.2

t	$h(t)$
0	
1	
2	
3	
4	
5	

(b) Compute the slope of the line joining $t = 0$ and $t = 1$. Then, compute the slope of the line joining $t = 1$ and $t = 2$. What do you notice about the slopes?

(c) What is the meaning of $h(0) = 50$?

A. the initial height of the water balloon is 50 feet.

B. the water balloon reaches a maximum height of 50 feet.

C. the water balloon hits the ground after 50 seconds.

D. the water balloon travels 50 feet before hitting the ground.

(d) Find the vertex of the quadratic function.

A. $(0, 50)$

B. $(1, 74)$

C. $(1.25, 75)$

D. $(3.4, 0)$

(e) What is the meaning of the vertex?

A. The water balloon reaches a maximum height of 50 feet at the start.

B. After 1 second, the water balloon reaches a maximum height of 74 feet.

C. After 1.25 seconds, the water balloon reaches the maximum height.

D. After 3.5 seconds, the water balloon hits the ground.

Activity 4.2.3 The population of a small city is given by the function $P(t) = -50t^2 + 1200t + 32000$, where t is the number of years after 2015.

(a) When will the population of the city reach a maximum?

A. 2020

B. 2022

C. 2025

D. 2027

(b) Determine when the population of the city is increasing and when it is decreasing.

(c) When will the population of the city reach 36,000 people?

A. 2019

B. 2025

C. 2027

D. 2035

Activity 4.2.4 The unit price of an item affects its supply and demand. That is, if the unit price increases, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of \$6. Market research has suggested that if the owners raise the price to \$8, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

(a) Which of the following represents a linear function which relates the price of the streaming service p to the number of subscribers Q?

A. $Q(p) = -2p$

B. $Q(p) = -2p + 84$

C. $Q(p) = -2p - 4$

D. $Q(p) = -2p + 96$

(b) Using the fact that Revenue $= pQ$, which of the following represents the Revenue R in terms of the price p.

A. $R(p) = -2p^2$

B. $R(p) = -2p^2 + 84p$

C. $R(p) = -2p^2 - 4p$

D. $R(p) = -2p^2 + 96p$

(c) What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. \$10

B. \$19.50

C. \$24

D. \$28.25

(d) How many subscribers would the company have at this price?

A. 39.5 million

B. 48 million

C. 57 million

D. 76 million

(e) What is the maximum revenue?

A. 760 million

B. 1112 million

C. 1152 million

D. 1116 million

Activity 4.2.5 The owner of a ranch decides to enclose a rectangular region with 240 feet of fencing. To help the fencing cover more land, he plans to use one side of his barn as part of the enclosed region. What is the maximum area the rancher can enclose?

4.3 Polynomial Long Division (PR3)

Objectives

- Rewrite a rational function as a polynomial plus a proper rational function.

Activity 4.3.1

(a) Use long division to find the quotient and remainder when 346 is divided by 17.

A. quotient 20, remainder 3
B. quotient 20, remainder 6
C. quotient 6, remainder 20
D. quotient 6, remainder 3

(b) What is the divisor?

A. 3
B. 17
C. 20
D. 6

(c) Now write the answer as a mixed number.

A. $20 + \frac{3}{17}$
B. $20 + \frac{6}{17}$
C. $6 + \frac{20}{17}$
D. $6 + \frac{3}{17}$

(d) How can you check your answer?

Activity 4.3.2 Now use long division to find the quotient and remainder.

(a) Simplify the following rational expression using factoring. $f(x) = \frac{x^2+2x-8}{x+4}$

A. $x + 4$

B. $x - 4$

C. $x - 2$

D. $x + 2$

(b) What is the quotient?

A. $x + 4$

B. $x - 4$

C. $x - 2$

D. $x + 2$

(c) What is the remainder?

A. $x + 4$

B. $x - 4$

C. $x - 2$

D. 0

Polynomial Long Division (PR3)

Activity 4.3.3 Repeat this process, using long division, given $f(x) = \frac{6x^2+5x-10}{2x+3}$.

(a) What is the result when $6x^2$ is divided by $2x$? Place this number at the top left when doing long division.

A. 3

B. $3x$

C. $3x^2$

(b) Next, multiply this result by $2x+3$ and write this below $6x^2+5x-10$. Be sure to line up like terms in the same column. What is the result?

A. $6x+9$

B. $6x^2+9x$

C. $6x^3+9x^2$

(c) Now subtract the like terms by placing them in the corresponding columns.

A. $6x^2-x-19$

B. $-4x-10$

C. $-6x^3+5x-10$

D. $14x-10$

(d) Next, divide the first term in the previous result by $2x$. Write the answer at the top beside the first answer. What is the expression at the top now?

A. $3-3x^2$

B. $3x-2$

C. $3x^2-3x^2$

D. $3x+7$

(e) Multiply this resulting term by $2x+3$ and write that answer in the proper columns of like terms. What is this result?

A. $-6x^3-9x^2$

B. $-4x - 6$

C. $14x + 21$

(f) Subtract the like terms. This is the remainder. What is the remainder?

A. $-6x^3 - 9x^2 - 4x - 10$

B. -4

C. $-18x - 31$

(g) What is the quotient?

A. $3 - 3x^2$

B. $3x - 2$

C. 0

D. $3x + 7$

(h) How can you check your answer? (Hint: Think of regular long division with positive integers.)

Polynomial Long Division (PR3)

Activity 4.3.4 Using long division, find the quotient and remainder for the given rational function. Rewrite the function as a polynomial plus a proper rational function, given $f(x) = \frac{3x^5-5x^2+2}{x^2+x-1}$.

(a) What is the quotient?

(b) What is the remainder?

(c) What is the divisor?

(d) Write the rational function as a polynomial plus a proper rational function.

(e) How can you check your answer? (Hint: Think of regular long division with positive integers.)

Polynomial Long Division (PR3)

Activity 4.3.5 Using long division, find the quotient and remainder for the given rational function, $f(x) = \frac{3x^4-5x^2+2}{x-1}$.

(a) What is the quotient?

(b) What is the remainder?

(c) How can you check your answer? (Hint: Think of regular long division with positive integers.)

Activity 4.3.6 Using synthetic division, find the quotient and remainder for the given rational function. $f(x) = \frac{x^2+2x-8}{x+4}$

(a) Write only the coefficients of the numerator in a row.

(b) Set the denominator equal to zero and solve for x. Place this number out in front, in the far left corner.

(c) Make 3 rows and drop the 1st coefficient down from the top row to the bottom row.

(d) Now multiply the number in the top left corner by the first coefficient and place this number under the 2nd coefficient.

(e) Add the numbers in the 2nd column and write the result in the same column, third row.

(f) Repeat the process with the resulting number. Multiply this number by the number in the upper left corner. Place it in the 2nd row, third column. Add the numbers in the third column and place this in the third row and column. What number did you get?

A. 1

B. -2

C. 0

D. -4

How do the numbers in the third row relate to Activity 4.3.2?

Activity 4.3.7 Now use synthetic division, to find the quotient and remainder for the given rational function. $f(x) = \frac{x^3+2x^2-3x+4}{x-2}$

(a) What is the quotient?

A. $x^2 - 3$

B. $x^3 + 2x - 3x + 4$

C. $x^2 + 4x + 5$

D. 5

(b) What is the remainder?

A. 14

B. -10

C. 5

D. 6

4.4 Zeroes of Polynomial Functions (PR4)

Objectives

- Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.

Activity 4.4.1 Write the polynomial function in factored form using information from the graph below.

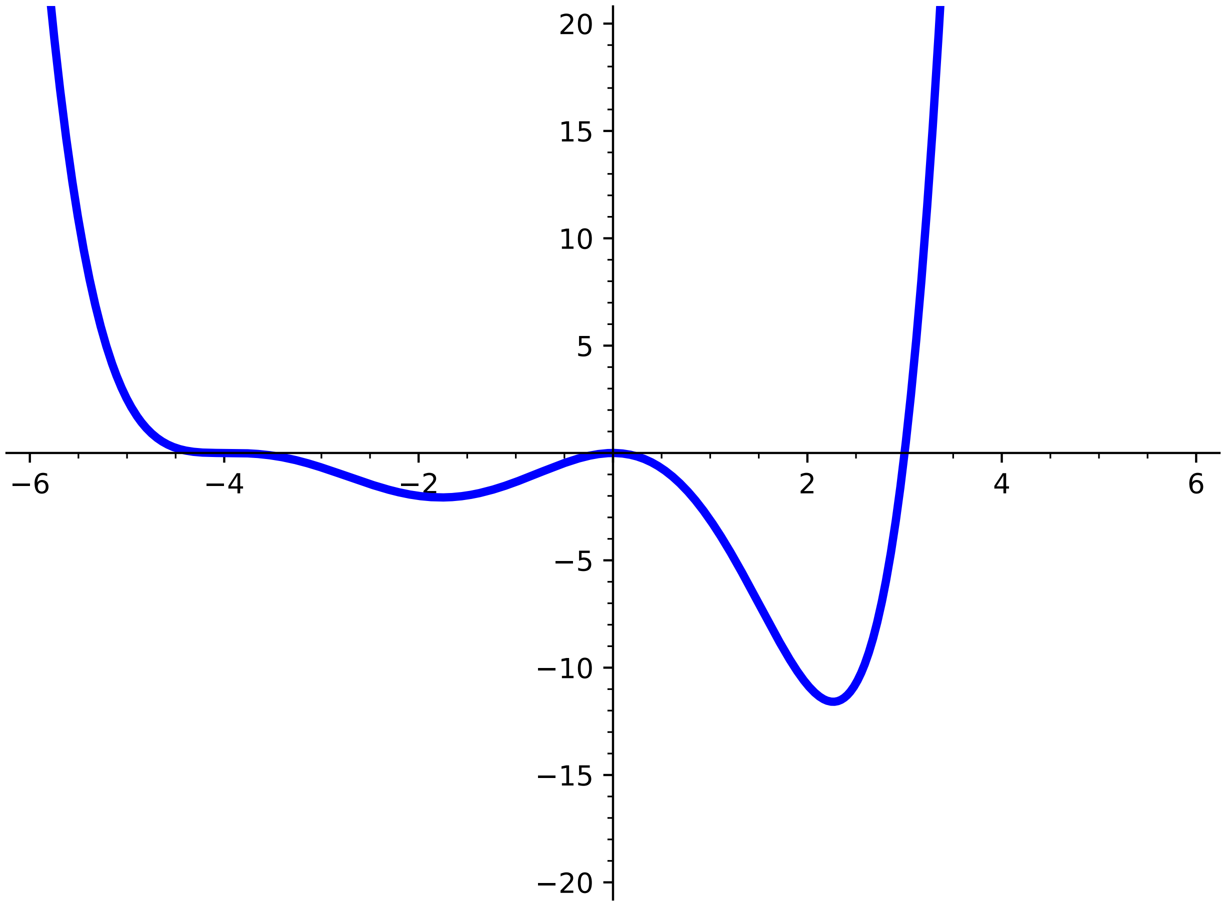


Figure 4.4.2

(a) Using the given graph, what are the real zeros of this function? Select all that apply.

A. 0

B. 1

C. -3

D. 3

E. 4

F. -4

(b) What are the least possible multiplicities for each zero?

(c) What is the least degree of the function?

A. 3

B. 4

C. 5

D. 6

(d) Describe the end behavior of the graph.

A. As $x \to \infty$, $f(x) \to \infty$

B. As $x \to -\infty$, $f(x) \to \infty$

C. As $x \to \infty$, $f(x) \to -\infty$

D. As $x \to -\infty$, $f(x) \to -\infty$

(e) Combining the information from the end behavior with the degree of the function, will the leading coefficient be positive or negative?

A. positive

B. negative

(f) Given the point $(2, \frac{-54}{5})$ is on the curve, and using the information in parts (a) through (e), write the function for the graph above in factored form.

Activity 4.4.3 Given the function $f(x) = x^6 - 3x^4 - 2x^3$,

(a) Find all the zeros and their corresponding multiplicities.

(b) Write the function $f(x)$ in factored form using linear factors.

Activity 4.4.4 Given the function $f(x) = 2x^4 - 4x^3 + 10x^2 - 16x + 8$,

(a) Find all the zeros and their corresponding multiplicities.

(b) Write the function $f(x)$ in factored form using linear factors.

Activity 4.4.5 The zeros of a function are x=2, with multiplicity 1, x=-1, with multiplicity 2 and x=i.

(a) Given the information above, find a polynomial function with real coefficients of least degree.

4.5 Graphs of Polynomial Functions (PR5)

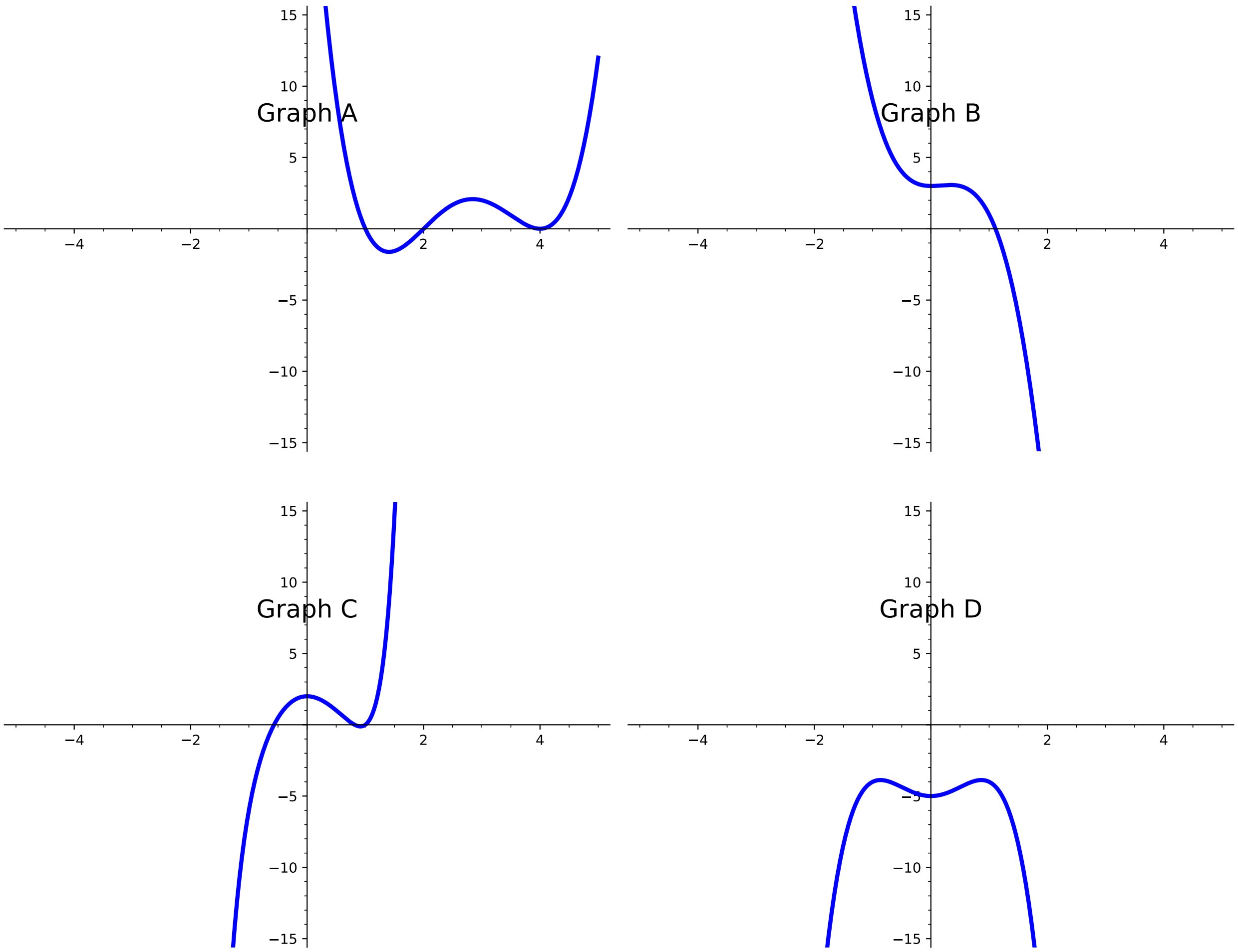
Objectives

- Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.

Remark 4.5.1 Just like with quadratic functions, we should be able to determine key characteristics that will help guide us in creating a sketch of any polynomial function. We can start by finding both x and y-intercepts and then explore other characteristics polynomial functions can have. Recall that the **zeros** of a function are the x**-intercepts** - i.e., the values of x that cross or touch the x-axis. Just like with quadratic functions, we can find the zeros of a function by setting the function equal to 0 and solving for x.

Definition 4.5.2 The **end behavior** of a polynomial function describes the behavior of the graph at the "ends" of the function. In other words, as we move to the right of the graph (as the x values increase), what happens to the y values? Similarly, as we move to the left of the graph (as the x values decrease), what happens to the y values? $\Diamond$

Activity 4.5.3 Use the graphs of the following polynomial functions to answer the questions below.



(a) How would you describe the behavior of Graph A as you approach the ends?

A. Graph A rises on the left and on the right.

B. Graph A rises on the left, but falls on the right.

C. Graph A rises on the right, but falls on the left.

D. Graph A falls on the left and on the right.

(b) How would you describe the behavior of Graph B as you approach the ends?

A. Graph B rises on the left and on the right.

B. Graph B rises on the left, but falls on the right.

C. Graph B rises on the right, but falls on the left.

D. Graph B falls on the left and on the right.

(c) How would you describe the behavior of Graph C as you approach the ends?

A. Graph C rises on the left and on the right.

B. Graph C rises on the left, but falls on the right.

C. Graph C rises on the right, but falls on the left.

D. Graph C falls on the left and on the right.

(d) How would you describe the behavior of Graph D as you approach the ends?

A. Graph D rises on the left and on the right.

B. Graph D rises on the left, but falls on the right.

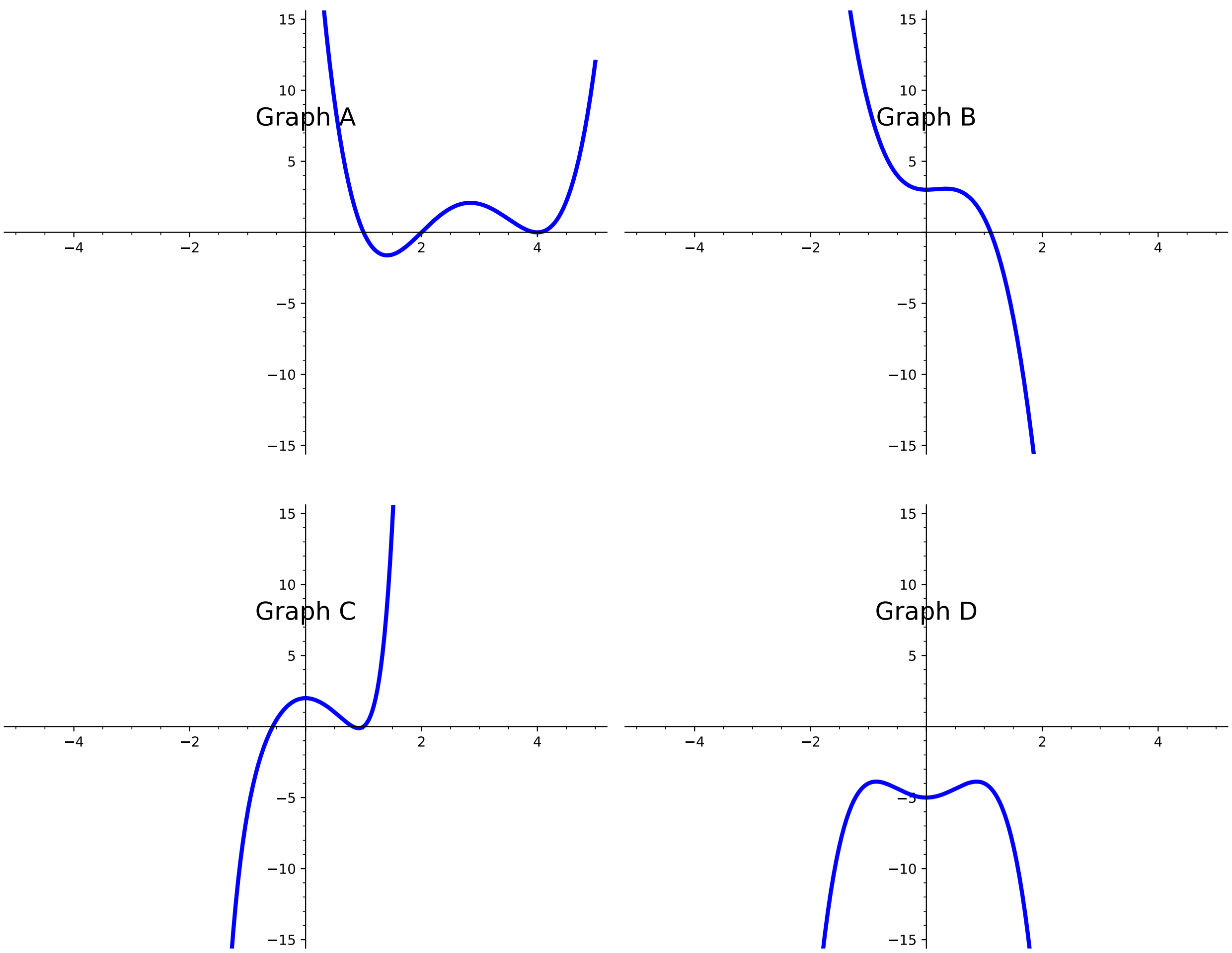
C. Graph D rises on the right, but falls on the left.

D. Graph D falls on the left and on the right.

Definition 4.5.4 Typically, when given an equation of a polynomial function, we look at the **degree** and **leading coefficient** to help us determine the behavior of the ends. The **degree** is the highest exponential power in the polynomial. The **leading coefficient** is the number written in front of the variable with the highest exponential power. $\Diamond$

Activity 4.5.5 Let's refer back to the graphs in Activity 4.5.3 and look at the equations of those polynomial functions. Let's apply Definition 4.5.4 to see if we can determine how the degree and leading coefficients of those graphs affect their end behavior.

- Graph A: $f(x) = -11x^3 + 32 + 42x^2 + x^4 - 64x$
- Graph B: $g(x) = 2x^2 + 3 - 4x^3$
- Graph C: $h(x) = x^7 + 2x^3 - 5x^2 + 2$
- Graph D: $j(x) = 3x^2 - 2x^4 - 5$



(a) What is the degree and leading coefficient of Graph A?

A. Degree: -64; Leading Coefficient: 4

B. Degree: 4; Leading Coefficient: 0

C. Degree: 1; Leading Coefficient: -64

D. Degree: 4; Leading Coefficient: 1

(b) What is the degree and leading coefficient of Graph B?

A. Degree: 3; Leading Coefficient: -4
B. Degree: -4; Leading Coefficient: 3
C. Degree: 2; Leading Coefficient: 3
D. Degree: 3; Leading Coefficient: 4

(c) What is the degree and leading coefficient of Graph C?

A. Degree: -5; Leading Coefficient: 2
B. Degree: 0; Leading Coefficient: 7
C. Degree: -5; Leading Coefficient: 3
D. Degree: 7; Leading Coefficient: 1

(d) What is the degree and leading coefficient of Graph D?

A. Degree: -2; Leading Coefficient: 4
B. Degree: 3; Leading Coefficient: 2
C. Degree: -2; Leading Coefficient: 4
D. Degree: -5; Leading Coefficient: 4

(e) Notice that Graph A and Graph D have their ends going in the same direction. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

(f) Notice that Graph B and Graph C have their ends going in opposite directions. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

Remark 4.5.6 From Activity 4.5.5, we saw that the degree and leading coefficient of a polynomial function can give us more clues about the behavior of the function. In summary, we know:

- If the degree is even, the ends of the polynomial function will be going in the same direction. If the leading coefficient is positive, both ends will be pointing up. If the leading coefficient is negative, both ends will be pointing down.

- If the degree is odd, the ends of the polynomial function will be going in opposite directions. If the leading coefficient is positive, the left end will fall and the right end will rise. If the leading coefficient is negative, the left end will rise and the right end will fall.

Definition 4.5.7 When describing end behavior, mathematicians typically use **arrow notation**. Just as the name suggests, arrows are used to indicate the behavior of certain values on a graph.

For end behavior, students are often asked to determine the behavior of y-values as x-values either increase or decrease. The statement "As $x \to \infty$, $f(x) \to -\infty$" can be translated to "As x approaches infinity (or as x increases), $f(x)$ (or the y-values) go to negative infinity (i.e., it decreases)."

$\Diamond$

Activity 4.5.8 Use the graph of $f(x)$ to answer the questions below.

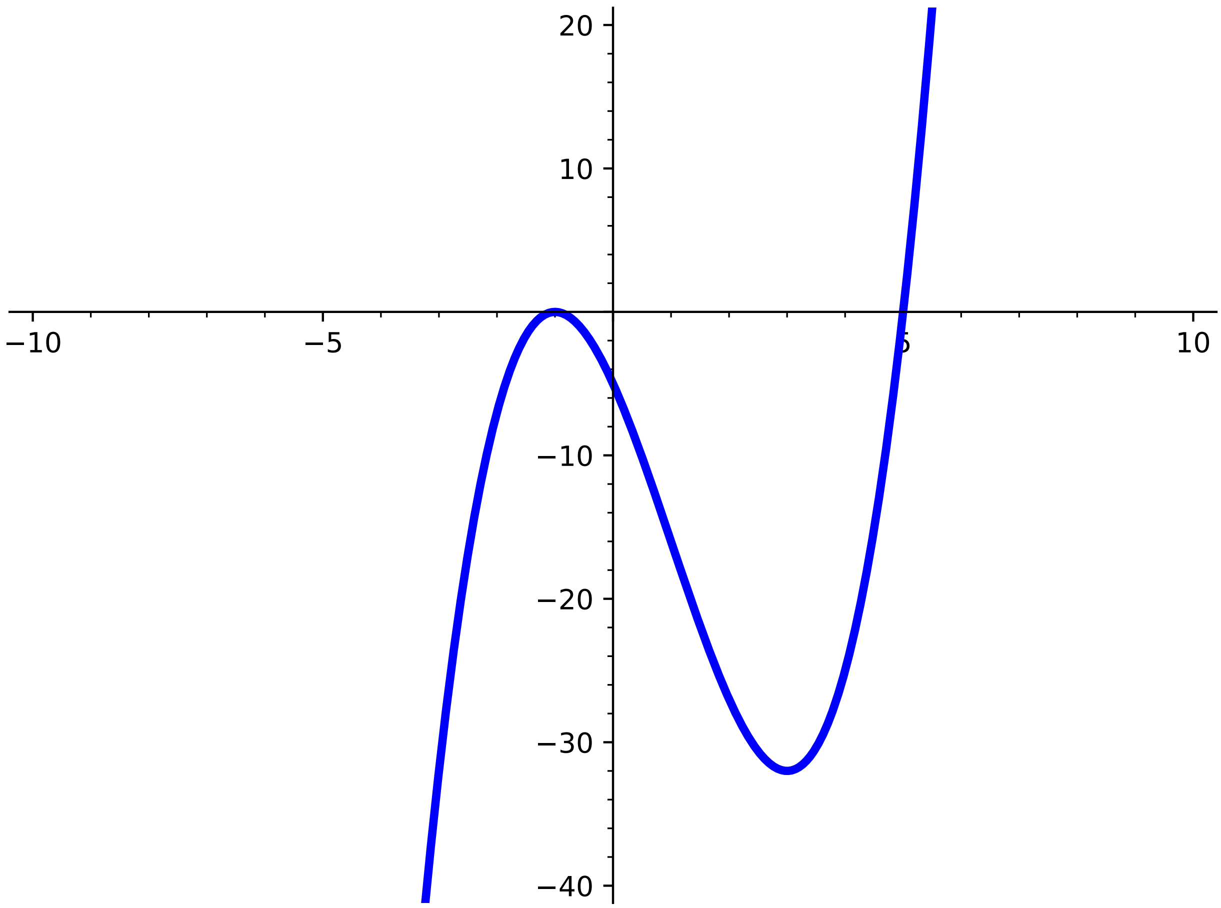


Figure 4.5.9

(a) How would you describe the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.

B. $f(x)$ rises on the left, but falls on the right.

C. $f(x)$ rises on the right, but falls on the left.

D. $f(x)$ falls on the left and on the right.

(b) How would you describe the end behavior of $f(x)$ using arrow notation?

A. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

B. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$

C. As $x \to -\infty$, $f(x) \to \infty$
   As $x \to \infty$, $f(x) \to -\infty$

D. As $x \to -\infty$, $f(x) \to \infty$
   As $x \to \infty$, $f(x) \to \infty$

Definition 4.5.10 When graphing polynomial functions, you may notice that these functions have some ”hills” and ”valleys.” These characteristics of the graph are known as the **local maxima** and **local minima** of the graph - similar to what we’ve already seen with quadratic functions. Unlike quadratic functions, however, a polynomial graph can have many local maxima/minima (quadratic functions only have one). $\Diamond$

Activity 4.5.11 Sketch the function, $f(x) = (x-2)(x+1)(x-3)^2$, by first finding the given characteristics.

- Find the zeros of $f(x)$.
- Find the multiplicities at each zero.
- Find the y-intercept of $f(x)$.
- Describe the end behavior of $f(x)$.
- Estimate where any local maximums and minimums may occur.

Activity 4.5.12 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

(a) The x-intercepts of $f(x)$ are $0, 2,$ and 5.

(b) $f(x)$ has one maximum at 0. $f(x)$ has one minimum at -5 and another at -16.

(c) The end behavior of $f(x)$ is given as:

- As $x \to \infty$, $f(x) \to \infty$
- As $x \to -\infty$, $f(x) \to -\infty$

Activity 4.5.13 Now that we know all the different characterisitics of polynomials, we should also be able to identify them from a graph. Use the graph below to find the given charactertistics.

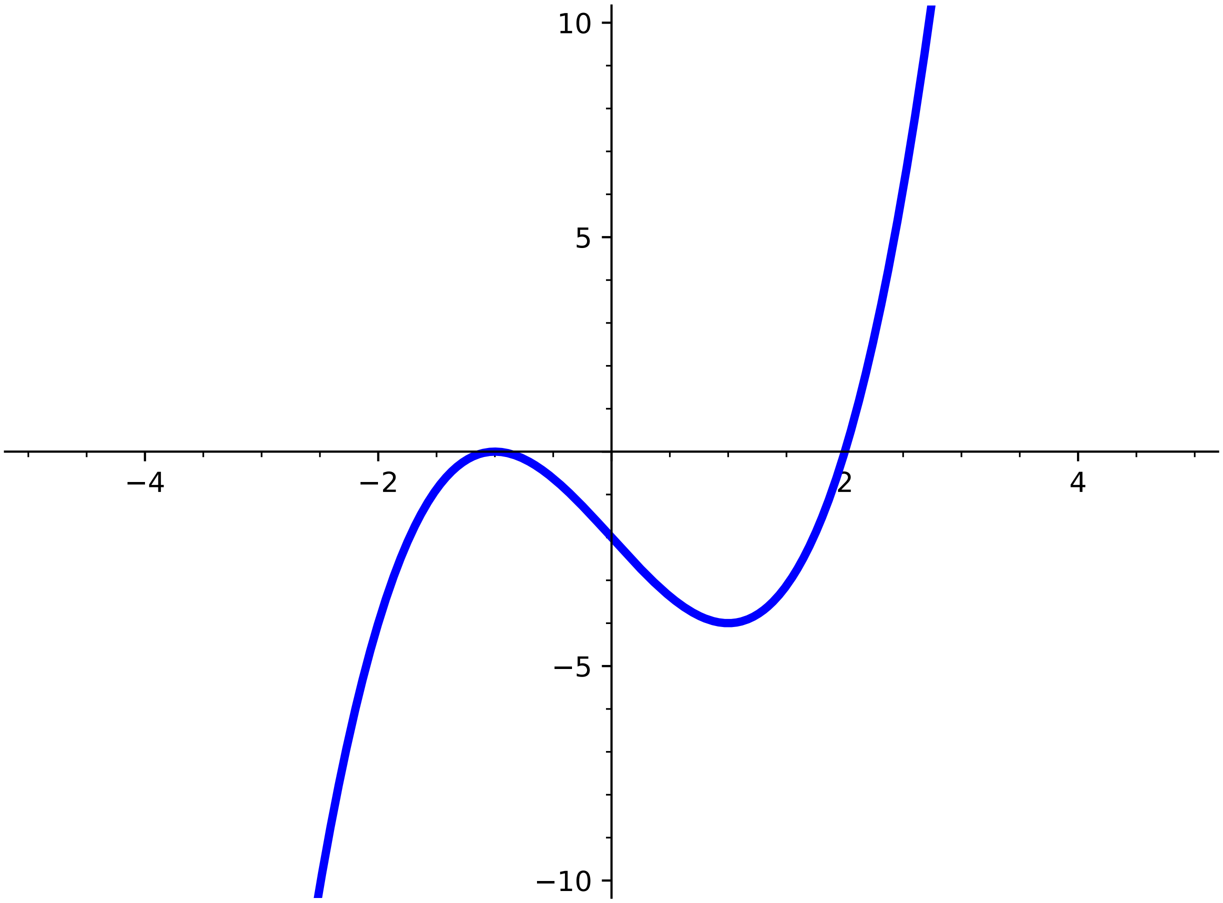


Figure 4.5.14

(a) What are the x-intercept(s) of the polynomial function? Select all that apply.

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(b) What are the y-intercept(s) of the polynomial function?

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(c) How many zeros does this polynomial function have?

A. 0
B. 1
C. 2
D. 3

(d) At what point is the local minimum located?

A. $(2, -4)$
B. $(-1, 0)$
C. $(-2, 0)$
D. $(1, -4)$
E. $(2, 0)$

(e) At what point is the local maximum located?

A. $(2, -4)$
B. $(-1, 0)$
C. $(-2, 0)$
D. $(1, -4)$
E. $(2, 0)$

(f) How do you describe the behavior of the polynomial function as $x \to \infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

(g) How do you describe the behavior of the polynomial function as $x \to -\infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

Activity 4.5.15 Use the given function, $f(x) = (x+1)^2(x-5)$, to answer the following questions.

(a) What are the zeros of $f(x)$?

A. $-1, -5$

B. $-1, 5$

C. $1, -5$

D. $1, 5$

(b) What are the multiplicities at each zero?

A. At $x = -1$, the mulitplicity is even.
At $x = 5$, the mulitplicity is even.

B. At $x = -1$, the mulitplicity is even.
At $x = 5$, the mulitplicity is odd.

C. At $x = -1$, the mulitplicity is odd.
At $x = 5$, the mulitplicity is even.

D. At $x = -1$, the mulitplicity is odd.
At $x = 5$, the mulitplicity is odd.

(c) What is the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.

B. $f(x)$ rises on the left, but falls on the right.

C. $f(x)$ rises on the right, but falls on the left.

D. $f(x)$ falls on the left and on the right.

(d) Using what you know about the zeros, multiplicities, and end behavior, where on the graph can we estimate the local maxima and minima to be?

(e) Now look at the graph of $f(x)$. At which zero does a local maximum or local minimum occur? Explain how you know.

Remark 4.5.16 We can estimate where these local maxima and minima occur by looking at other characteristics, such as multiplicities and end behavior.

From Activity 4.5.15, we saw that when the function touches the x-axis at a zero, then that zero could be either a local maximum or a local minimum of the graph. When the function crosses the x-axis, however, the local maximum or local minimum occurs between the zeros.

4.6 Properties and Graphs of Rational Functions (PR6)

Objectives

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

Properties and Graphs of Rational Functions (PR6)

Definition 4.6.1 A function r is **rational** provided that it is possible to write r as the ratio of two polynomials, p and q. That is, r is rational provided that for some polynomial functions p and q, we have

$$r(x) = \frac{p(x)}{q(x)}.$$

$\Diamond$

Observation 4.6.2 Rational functions occur in many applications, so our goal in this lesson is to learn about their properties and be able to graph them. In particular we want to investigate the domain, end behavior, and zeros of rational functions.

Activity 4.6.3 Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Find $r(1)$, $r(2)$, $r(3)$, and $r(4)$.

(b) Label each of these four points as giving us information about the DOMAIN of $r(x)$, information about the ZEROES of $r(x)$, or NEITHER.

Properties and Graphs of Rational Functions (PR6)

Definition 4.6.4 Let p and q be polynomial functions so that $r(x) = \frac{p(x)}{q(x)}$ is a rational function. The **domain** of r is the set of all real numbers except those for which $q(x) = 0$. $\Diamond$

Activity 4.6.5 Let's investigate the domain of $r(x)$ more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Rewrite $r(x)$ by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?

(b) The function was not defined for $x = 3$. Make a table for values of $r(x)$ near $x = 3$.

Table 4.6.6

x	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

(c) Which of the following describe the behavior of the graph near $x = 3$?

A. As $x \to 3$, $r(x)$ approaches a finite number

B. As $x \to 3$ from the left, $r(x) \to \infty$

C. As $x \to 3$ from the left, $r(x) \to -\infty$

D. As $x \to 3$ from the right, $r(x) \to \infty$

E. As $x \to 3$ from the right, $r(x) \to -\infty$

(d) The function was also not defined for $x = 1$. Make a table for values of $r(x)$ near $x = 1$.

Table 4.6.7

x	$r(x)$
0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

(e) Which of the following describe the behavior of the graph near $x = 1$?

A. As $x \to 1$, $r(x)$ approaches a finite number

B. As $x \to 1$ from the left, $r(x) \to \infty$

C. As $x \to 1$ from the left, $r(x) \to -\infty$

D. As $x \to 1$ from the right, $r(x) \to \infty$

E. As $x \to 1$ from the right, $r(x) \to -\infty$

(f) The function is behaving differently near $x = 1$ than it is near $x = 3$. Can you see anything in the factored form of $r(x)$ that may help you account for the difference?

Remark 4.6.8 Features of a rational function. Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function.

- If $p(a) = 0$ and $q(a) \neq 0$, then $r(a) = 0$, so r has a **zero** at $x = a$.
- If $q(a) = 0$ and $p(a) \neq 0$, then $r(a)$ is undefined and r has a **vertical asymptote** at $x = a$.
- If $p(a) = 0$ and $q(a) = 0$ and we can show that there is a finite number L such that $r(x) \to L$, then $r(a)$ is not defined and r has a **hole** at the point (a, L).

Activity 4.6.9 Another property of rational functions we want to explore is the end behavior. This means we want to explore what happens to a given rational function $r(x)$ when x goes toward positive infinity or negative infinity.

(a) Consider the rational function $r(x) = \frac{1}{x^3}$. Plug in some very large positive numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches positive infinity?

A. As $x \to \infty$, $r(x) \to \infty$.

B. As $x \to \infty$, $r(x) \to -\infty$.

C. As $x \to \infty$, $r(x) \to 0$.

D. As $x \to \infty$, $r(x) \to 1$.

(b) Now let's look at $r(x)$ as x tends toward negative infinity. Plug in some very large negative numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches negative infinity?

A. As $x \to -\infty$, $r(x) \to \infty$.

B. As $x \to -\infty$, $r(x) \to -\infty$.

C. As $x \to -\infty$, $r(x) \to 0$.

D. As $x \to -\infty$, $r(x) \to 1$.

Observation 4.6.10 We can generalize what we have just found to any function of the form $\frac{1}{x^n}$, where $n > 0$. Since x^n increases without bound as $x \to \infty$, we find that $\frac{1}{x^n}$ will tend to 0. In fact, the numerator can be any constant and the function will still tend to 0!

Similarly, as $x \to -\infty$, we find that $\frac{1}{x^n}$ will tend to 0 too.

Activity 4.6.11 Consider the rational function $r(x) = \frac{3x^2-5x+1}{7x^2+2x-11}$.

Observe that the largest power of x that's present in $r(x)$ is x^2. In addition, because of the dominant terms of $3x^2$ in the numerator and $7x^2$ in the denominator, both the numerator and denominator of r increase without bound as x increases without bound.

(a) In order to understand the end behavior of r, we will start by writing the function in a different algebraic form.

Multiply the numerator and denominator of r by $\frac{1}{x^2}$. Then distribute and simplify as much as possible in both the numerator and denominator to write r in a different algebraic form. Which of the following is that new form?

A. $\dfrac{3x^4-5x^3+x^2}{7x^4+2x^3-11x^2}$

B. $\dfrac{3-\frac{5}{x}+\frac{1}{x^2}}{7+\frac{2}{x}-\frac{11}{x^2}}$

C. $\dfrac{\frac{3x^2}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}}{\frac{7x^2}{x^2}+\frac{2x}{x^2}-\frac{11}{x^2}}$

D. another wrong answer?

(b) Now determine the end behavior of each piece of the numerator and each piece of the denominator. Hint: Use Observation 4.6.10 to help!

(c) Simplify your work from the previous step. Which of the following best describes the end behavior of $r(x)$?

A. As $x \to \pm\infty$, $r(x)$ goes to 0.

B. As $x \to \pm\infty$, $r(x)$ goes to $\frac{3}{7}$.

C. As $x \to \pm\infty$, $r(x)$ goes to ∞.

D. As $x \to \pm\infty$, $r(x)$ goes to $-\infty$.

Observation 4.6.12 If the end behavior of a function tends toward a specific value a, then we say that the function has a **horizontal asymptote** at $y = a$.

Activity 4.6.13 Find the horizontal asymptote (if one exists) of the following rational functions. Follow the same method we used in Activity 4.6.11.

(a) $f(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(b) $g(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^5 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(c) $h(x) = \dfrac{4x^5 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

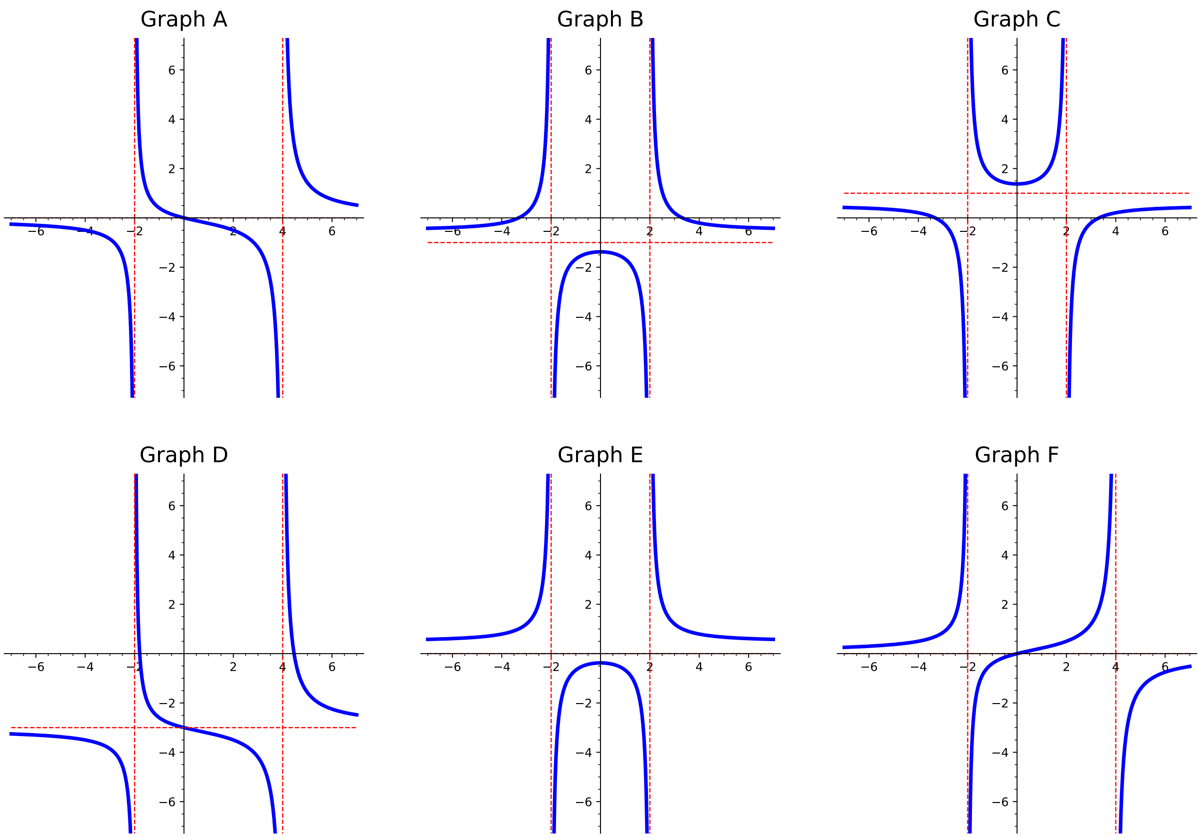
Activity 4.6.14 Some patterns have emerged from the previous problem. Fill in the rest of the sentences below to describe how to find horizontal asymptotes of rational functions.

(a) If the degree of the numerator is the same as the degree of the denominator, then...

(b) If the degree of the numerator is less than the degree of the denominator, then...

(c) If the degree of the numerator is greater than the degree of the denominator, then...

Activity 4.6.15 Consider the following six graphs of rational functions:



(a) Which of the graphs above represents the function $f(x) = \frac{2x}{x^2-2x-8}$?

(b) Which of the graphs above represents the function $g(x) = \frac{x^2+3}{2x^2-8}$?

Activity 4.6.16 MATCHING ACTIVITY INCLUDING GRAPHS WITH HOLES

(a) QUESTIONS HERE!

Activity 4.6.17

(a) Find the roots of the rational function $f(x) = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(b) Find the y-intercept of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(c) Find any horizontal asymptotes of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(d) Find any vertical asymptotes of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(e) Find any holes of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(f) Sketch $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

Activity 4.6.18 EXTENSION ACTIVITY ABOUT HOLES

(a) QUESTIONS HERE!

Colophon

This book was authored in PreTeXt.