Precalculus for Team-Based Inquiry Learning

2024 Development Edition

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[1]teambasedinquirylearning.github.io/precalculus/

[2]github.com/TeamBasedInquiryLearning/precalculus/blob/main/LICENSE.md

Contents

Chapter 1

Equations, Inequalities, and Applications (EQ)

Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx
By the end of this chapter, you should be able to...

1. Solve a linear equation in one variable. Solve a linear inequality in one variable and express the solution graphically and using interval notation.
2. Solve application problems involving linear equations.
3. Given two points, determine the distance between them and the midpoint of the line segment connecting them.
4. Solve a linear equation involving an absolute value. Solve a linear inequality involving absolute values and express the answers graphically and using interval notation.
5. Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.
6. Solve a rational equation.
7. Solve quadratic inequalities and express the solution graphically and

with interval notation. Solve rational inequalities and express the solution graphically and using interval notation.

1.1 Linear Equations and Inequalities (EQ1)

Objectives

- Solve a linear equation in one variable. Solve a linear inequality in one variable and express the solution graphically and using interval notation.

Remark 1.1.1 Recall that when solving a linear equation, you use addition, subtraction, multiplication and division to isolate the variable.

Activity 1.1.2 Solve the linear equations.

(a) $3x - 8 = 5x + 2$

A. $x = 2$

B. $x = 5$

C. $x = -5$

D. $x = -2$

(b) $5(3x - 4) = 2x - (x + 3)$

A. $x = \frac{17}{14}$

B. $x = \frac{14}{17}$

C. $x = \frac{23}{14}$

D. $x = \frac{14}{23}$

Activity 1.1.3 Solve the linear equation.

$$\frac{2}{3}x - 8 = \frac{5x+1}{6}$$

(a) Which equation is equivalent to $\frac{2}{3}x - 8 = \frac{5x+1}{6}$ but does not contain any fractions?

A. $12x - 48 = 15x + 3$

B. $3x - 24 = 10x + 2$

C. $4x - 8 = 5x + 1$

D. $4x - 48 = 5x + 1$

(b) Use the simplified equation from part (a) to solve $\frac{2}{3}x - 8 = \frac{5x+1}{6}$.

A. $x = -17$

B. $x = -\dfrac{26}{7}$

C. $x = -9$

D. $x = -49$

Activity 1.1.4 It is not always the case that a linear equation has exactly one solution. Consider the following linear equations which appear similar, but their solutions are very different.

(a) Which of these equations has one unique solution?

A. $4(x-2)=4x+6$

B. $4(x-1)=4x+4$

C. $4(x-1)=x+4$

(b) Which of these equations has no solutions?

A. $4(x-2)=4x+6$

B. $4(x-1)=4x+4$

C. $4(x-1)=x+4$

(c) Which of these equations has many solutions?

A. $4(x-2)=4x+6$

B. $4(x-1)=4x+4$

C. $4(x-1)=x+4$

(d) What happens to the x variable when a linear equation has no solution or many solutions?

Definition 1.1.5 A linear equation with one unique solution is a **conditional equation.** A linear equation that is true for all values of the variable is an **identity equation**. A linear equation with no solutions is an **inconsistent equation.** $\Diamond$

Activity 1.1.6 An inequality is a relationship between two values that are not equal.

(a) What is the solution to the linear equation $3x - 1 = 5$?

(b) Which of these values is a solution of the inequality $3x - 1 \geq 5$?

A. $x = 0$

B. $x = 2$

C. $x = 4$

D. $x = 10$

(c) Express the solution of the inequality $3x - 1 \geq 5$ in interval notation.

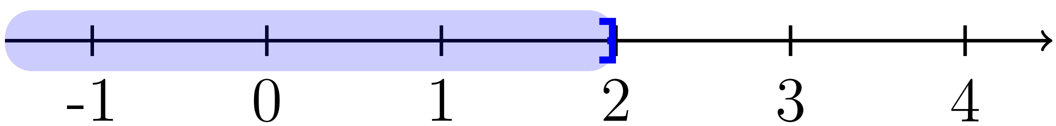
A. $(-\infty, 2]$

B. $(-\infty, 2)$

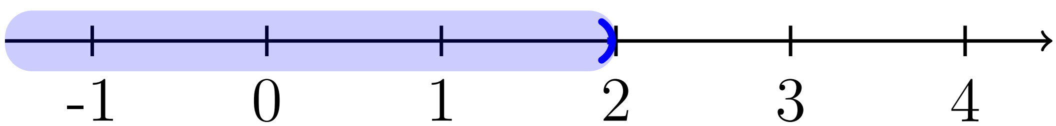
C. $(2, \infty)$

D. $[2, \infty)$

(d) Draw the solution to the inequality on a number line.



A.



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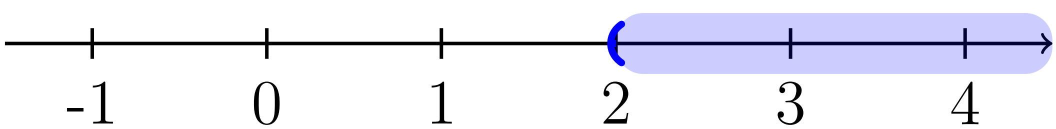

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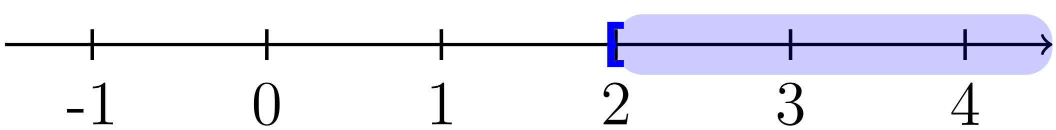


D.

Activity 1.1.7

(a) Which of these values is a solution of the inequality $-x < 8$?

A. $x = -10$

B. $x = -8$

C. $x = 4$

D. $x = 10$

(b) Solve the linear inequality $-x < 8$. How does your solution compare to the values chosen in part (a)?

(c) Expression the solution of the inequality $-x < 8$ in interval notation.

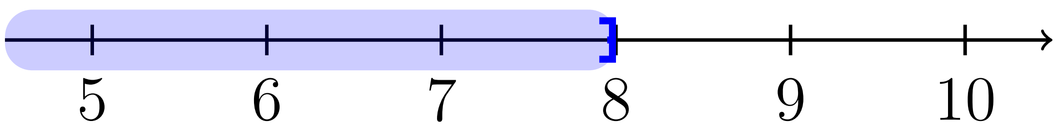
A. $(-\infty, -8]$

B. $(-\infty, -8)$

C. $(-8, \infty)$

D. $[-8, \infty)$

(d) Draw the solution to the inequality on a number line.

A.


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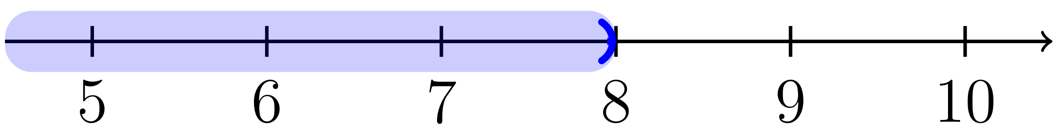

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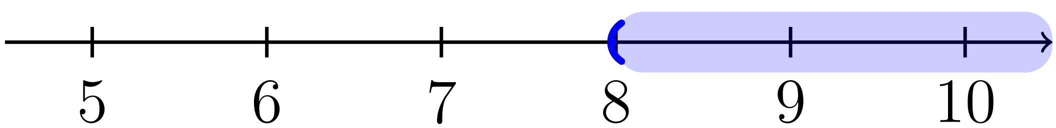

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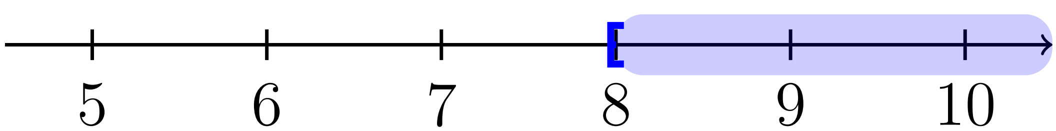


Remark 1.1.8 You can treat solving linear inequalities, just like solving an equation. The one exception is when you multiply or divide by a negative value, reverse the inequality symbol.

Activity 1.1.9 Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a) $-3x - 1 \leq 5$

(b) $3(x + 4) > 2x - 1$

(c) $-\frac{1}{2}x \geq -\frac{2}{4} + \frac{5}{4}x$

Definition 1.1.10 A **compound inequality** includes multiple inequalities in one statement. $\Diamond$

Activity 1.1.11 Consider the statement $3 \leq x < 8$. This really means that $3 \leq x$ and $x < 8$.

(a) Which of the following inequalities are equivalent to the compound inequality $3 \leq 2x - 3 < 8$?

A. $3 \leq 2x - 3$

B. $3 \geq 2x - 3$

C. $2x - 3 < 8$

D. $2x - 3 > 8$

(b) Solve the inequality $3 \leq 2x - 3$.

A. $x \leq 0$

B. $x \geq 0$

C. $x \leq 3$

D. $x \geq 3$

(c) Solve the inequality $2x - 3 < 8$.

A. $x > \frac{11}{2}$

B. $x < \frac{11}{2}$

C. $x > \frac{5}{2}$

D. $x < \frac{5}{2}$

(d) Which compound inequality describes how the two solutions overlap?

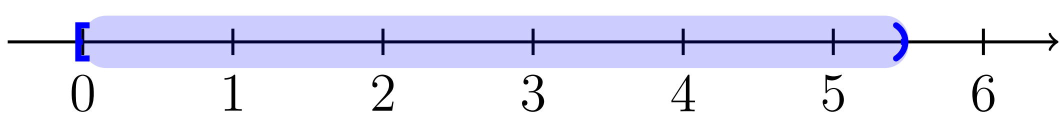
A. $0 \leq x < \frac{11}{2}$

B. $0 \leq x < \frac{5}{2}$

C. $\frac{5}{2} < x \leq 3$

D. $3 \leq x < \frac{11}{2}$

(e) Draw the solution to the inequality on a number line.

A.


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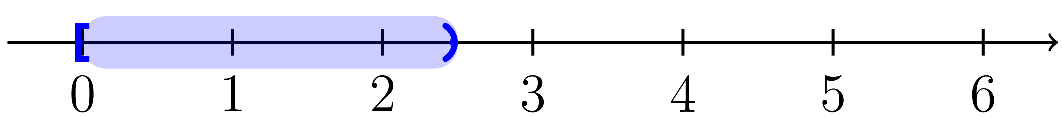

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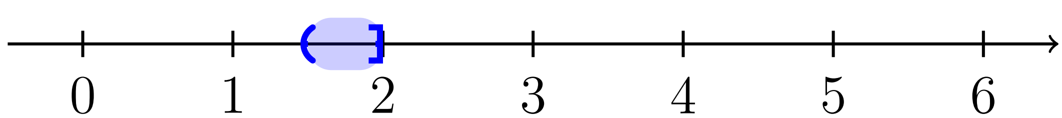

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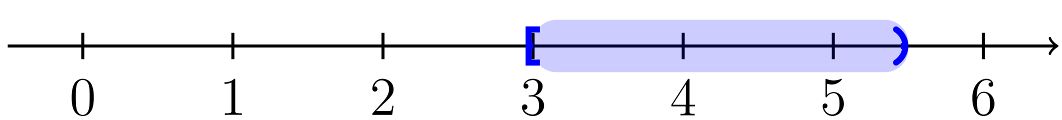


Remark 1.1.12 Solving a compound linear inequality, uses the same methods as a single linear inequality ensuring that you perform the same operations on all three parts. Alternatively, you can break the compound inquality up into two and solve separately.

1.2 Applications of Linear Equations (EQ2)

Objectives

- Solve application problems involving linear equations.

Applications of Linear Equations (EQ2)

Observation 1.2.1 Linear equations can be used to solve many types of real-world applications. We'll investigate some of those in this section.

Remark 1.2.2 Distance, rate, and time problems are a standard example of an application of a linear equation. For these, it's important to remember that

$$d = rt$$

where d is distance, r is the rate (or speed), and t is time.

Often we will have more than one moving object, so it is helpful to denote which object's distance, rate, or time we are referring to. One way we can do this is by using a subscript. For example, if we are describing an eastbound train (as we will in the first example), it may be helpful to denote its distance, rate, and time as d_E, r_E, and t_E respectively. Notice that the subscript E is a label reminding us that we are referring to the eastbound train.

Activity 1.2.3 Two trains leave a station at the same time. One is heading east at a speed of 75 mph, while the other is heading west at a speed of 85 mph. After how long will the trains be 400 miles apart?

(a) How fast is each train traveling?

A. $r_E = 85$ mph, $r_W = 75$ mph

B. $r_E = 75$ mph, $r_W = 85$ mph

C. $r_E = 400$ mph, $r_W = 400$ mph

D. $r_E = 75$ mph, $r_W = 400$ mph

E. $r_E = 400$ mph, $r_W = 85$ mph

(b) Which of the statements describes how the times of the eastbound and westbound train are related?

A. The eastbound train is slower than the westbound train, so $75 + t_E = 85 + t_W$.

B. The eastbound train left an hour before the westbound train, so if we let $t_E = t$, then $t_W = t - 1$.

C. Both trains have been traveling the same amount of time, so $t_E = t_W$. Since they are the same, we can just call them both t.

D. We don't know how the times relate to each other, so we must denote them separately as t_E and t_W.

E. Since the trains are traveling at different speeds, we need the proportion $\frac{r_E}{r_W} = \frac{t_E}{t_W}$.

(c) Fill in the following table using the information you've just determined about the trains' rates and times since they left the station. Some values are there to help you get started.

Table 1.2.4

	rate	**time**	**distance from station**
eastbound train			$75t$
westbound train		t	

(d) At the moment in question, the trains are 400 miles apart. How does that total distance relate to the distance each train has traveled?

A. The 400 miles is irrelevant. They've been traveling the same amount of time so they must be the same distance away from the station. That tells us $d_E = d_W$.

B. The 400 miles is the difference between the distance each train traveled, so $d_E - d_W = 400$.

C. The 400 miles represents the sum of the distances that each train has traveled, so $d_E + d_W = 400$.

D. The 400 miles is the product of the distance each train traveled, so $(d_E)(d_W) = 400$.

(e) Now plug in the expressions from your table for d_E and d_w. What equation do you get?

A. $75t = 85t$

B. $75t - 85t = 400$

C. $75t + 85t = 400$

D. $(75t)(85t) = 400$

(f) Notice that we now have a linear equation in one variable, t. Solve for t, and put that answer in context of the problem.

A. The trains are 400 miles apart after 2 hours.

B. The trains are 400 miles apart after 2.5 hours.

C. The trains are 400 miles apart after 3 hours.

D. The trains are 400 miles apart after 3.5 hours.

E. The trains are 400 miles apart after 4 hours.

Remark 1.2.5 In Activity 1.2.3 we examined the motion of two objects moving at the same time in opposite directions. In Activity 1.2.6 we will examine a different perspective, but still apply $d = rt$ to solve.

Activity 1.2.6 Jalen needs groceries, so decides to ride his bike to the store. It takes him half an hour to get there. After finishing his shopping, he sees his friend Alex who offers him a ride home. He takes the same route home as he did to the store, but this time it only takes one-fifth of an hour. If his average speed was 18 mph faster on the way home, how far away does Jalen live from the grocery store?

We'll use the subscript b to refer to variables relating to Jalen's trip to the store while riding his *b*ike and the subscript c to refer to variables relating to Jalen's trip home while riding in his friend's *c*ar.

(a) How long does his bike trip from home to the store and his car trip from the store back home take?

A. $t_b = 18$ hours, $t_c = 18$ hours

B. $t_b = \frac{1}{5}$ of an hour, $t_c = \frac{1}{2}$ of an hour

C. $t_b = \frac{1}{2}$ of an hour, $t_c = \frac{1}{5}$ of an hour

D. $t_b = 2$ hours, $t_c = 5$ hours

E. $t_b = 5$ hours, $t_c = 2$ hours

(b) Which of the statements describes how the speed (rate) of the bike trip and the car trip are related?

A. Both the trip to the store and the trip home covered the same distance, so $r_b = r_c$. Since they are the same, we can just call them both r.

B. We don't know how the two rates relate to each other, so cannot write an equation comparing them and must leave them as separate variables r_b and r_c.

C. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let $r_b = r$, then $r_c = r - 18$.

D. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let $r_b = r$, then $r_c = r + 18$.

(c) Fill in the following table using the information you've just determined about the Jalen's rates and times on each leg of his grocery store trip. Then fill in the distance column based on how distance relates to rate and time in each case.

Table 1.2.7

	rate	time	distance covered
bike trip (to the store)			
car trip (going back home)			

(d) Our goal is to figure out how far away Jalen lives from the store. To help us get there, write an equation relating d_b and d_c.

A. The distance he traveled by bike is the same as the distance he traveled by car, so $d_b = d_c$

B. The distance he traveled by bike took longer than the distance he traveled by car, so $d_b + \frac{1}{2} = d_c + \frac{1}{5}$

C. The distance, d, between his house and the grocery store is sum of the distance he traveled on his bike and the distance he traveled in the car, so $d_b + d_c = d$.

D. The distance, d, between his house and the grocery store is sum of the difference he traveled on his bike and the distance he traveled in the car, so $d_b - d_c = d$.

(e) Now plug in the expressions from your table for d_b and d_c into the equation you just found. Notice that it is a linear equation in one variable, r. Solve for r.

(f) Our goal was to determine the distance between Jalen's house and the grocery store. Solving for r did not tell us that distance, but it did get us one step closer. Use that value to help you determine the distance between his house and the store, and write your answer using the context of the problem. (Hint: can you find an expression involving r that we made that represents that distance?)

A. The grocery store is 6 miles away from Jalen's house.

B. The grocery store is 8 miles away from Jalen's house.

C. The grocery store is 10 miles away from Jalen's house.

D. The grocery store is 12 miles away from Jalen's house.

E. The grocery store is 14 miles away from Jalen's house.

Remark 1.2.8 Another type of application of linear equations is called a mixture problem. In these we will mix together two things, like two types of candy in a candy store or two solutions of different concentrations of alcohol.

Activity 1.2.9 Ammie's favorite snack to share with friends is candy salad, which is a mixture of different types of candy. Today she chooses to mix Nerds Gummy Clusters, which cost \$8.38 per pound, and Starburst Jelly Beans, which cost \$7.16 per pound. If she makes seven pounds of candy salad and spends a total of \$55.61, how many pounds of each candy did she buy?

(a) There are two "totals" in this situation: the total weight (in pounds) of candy Ammie bought and the total amount of money (in dollars) Ammie spent. Let's begin with the total weight. If we let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans, which of the following equations can represent the total weight?

A. $N - S = 7$

B. $NS = 7$

C. $N + S = 7$

D. $\dfrac{N}{S} = 7$

(b) Which expressions represent the amount she spent on each candy? Again, we will let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans.

A. N spent on Nerds Gummy Clusters; S spent on Starburst Jelly Beans

B. $8.38N$ spent on Nerds Gummy Clusters; $7.16S$ spent on Starburst Jelly Beans

C. $8.38 + N$ spent on Nerds Gummy Clusters; $7.16 + S$ spent on Starburst Jelly Beans

D. $8.38 - N$ spent on Nerds Gummy Clusters; $7.16 - S$ spent on Starburst Jelly Beans

(c) Now we focus on the total cost. Which of the following equations can represent the total amount she spent?

A. $N + S = 55.61$

B. $8.38N + 7.16S = 55.61$

C. $8.38 + N + 7.16 + S = 55.61$

D. $8.38 - N + 7.16 - S = 55.61$

(d) We are almost ready to solve, but we have two variables in our weight equation and our cost equation. We will get the cost equation to one variable by using the weight equation as a substitution. Which of the following is a way to express one variable in terms of the other? (Hint: More than one answer may be correct here!)

A. If N is the total weight of the Nerds Gummy Clusters, then $7-N$ could represent the weight of the Starburst Jelly Beans.

B. If N is the total weight of the Nerds Gummy Clusters, then $7+N$ could represent the weight of the Starburst Jelly Beans.

C. If S is the total weight of the Starburst Jelly Beans, then $7 - S$ could represent the weight of the Nerds Gummy Clusters.

D. If S is the total weight of the Starburst Jelly Beans, then $7 + S$ could represent the weight of the Nerds Gummy Clusters.

(e) Plug your expressions in to the total cost equation. (Hint: More than one of these may be correct!)

A. $8.38N + 7.16(7 - N) = 55.61$

B. $8.38S + 7.16(7 - S) = 55.61$

C. $8.38(7 - N) + 7.16N = 55.61$

D. $8.38(7 - S) + 7.16S = 55.61$

(f) Now solve for N and S, and put your answer in the context of the problem.

A. Ammie bought 2.5 lbs of Nerds Gummy Clusters and 4.5 lbs of Starburst Jelly Beans.

B. Ammie bought 3.5 lbs of Nerds Gummy Clusters and 3.5 lbs of Starburst Jelly Beans.

C. Ammie bought 4.5 lbs of Nerds Gummy Clusters and 2.5 lbs of Starburst Jelly Beans.

D. Ammie bought 5.5 lbs of Nerds Gummy Clusters and 1.5 lbs of Starburst Jelly Beans.

Activity 1.2.10 A chemist needs to mix two solutions to create a mixture consisting of 30% alcohol. She uses 20 liters of the first solution, which has a concentration of 21% alcohol. How many liters of the second solution (that is 45% alcohol) should she add to the first solution to create the mixture that is 30% alcohol?

1.3 Distance and Midpoint (EQ3)

Objectives

- Given two points, determine the distance between them and the midpoint of the line segment connecting them.

Activity 1.3.1 The points A and B are shown in the graph below. Use the graph to answer the following questions:

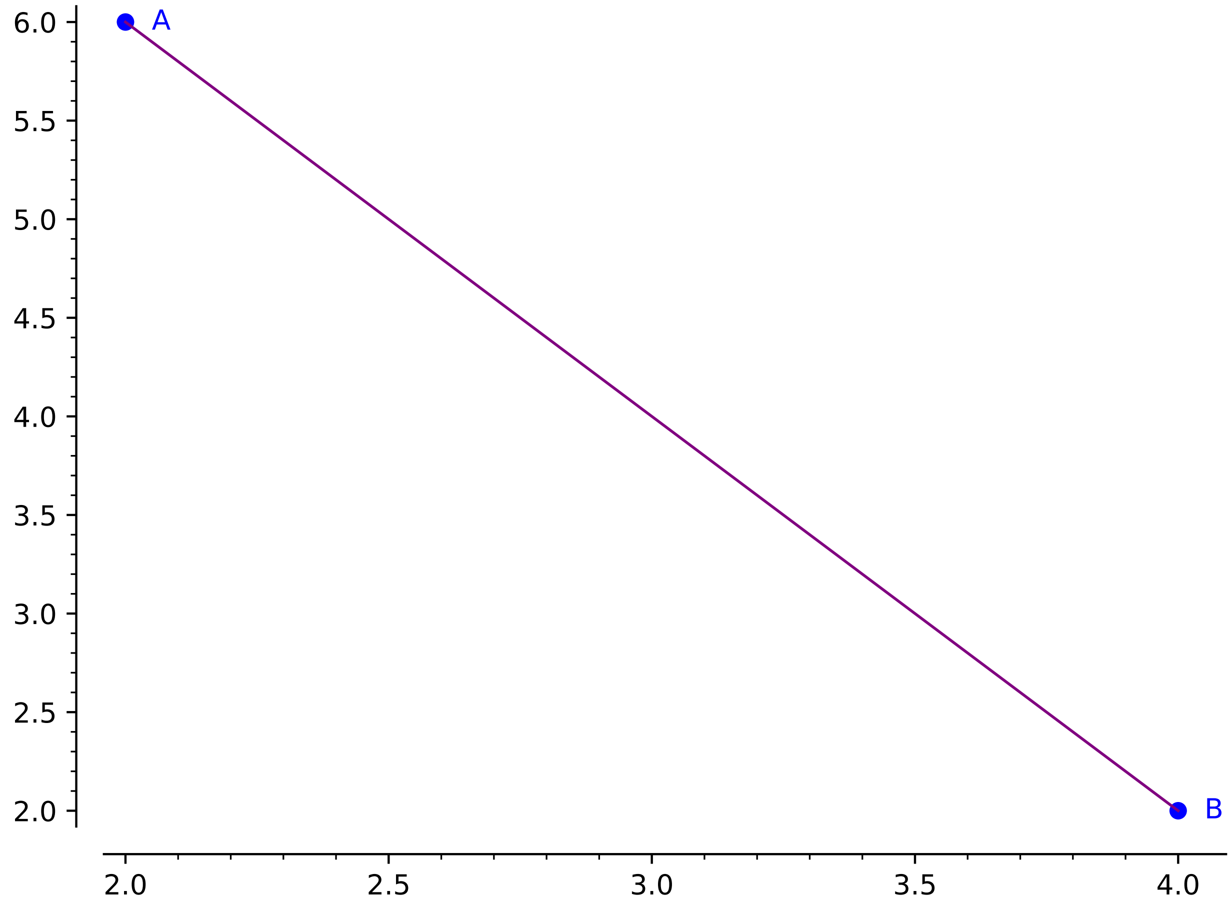


Figure 1.3.2

(a) Draw a right triangle so that the hypotenuse is the line segment between points A and B. Label the third point of the triangle C.

(b) Find the lengths of line segments AC and BC.

(c) Now that you know the lengths of AC and BC, how can you find the length of AB? Find the length of AB.

Remark 1.3.3 Using the **Pythagorean Theorem**($a^2 + b^2 = c^2$) can be helpful in finding the distance of a line segment (as long as you create a right triangle!).

Activity 1.3.4 Suppose you are given two points (x_1, y_1) and (x_2, y_2). Let's investigate how to find the length of the line segment that connects these two points!

(a) Draw a sketch of a right triangle so that the hypotenuse is the line segment between the two points.

(b) Find the lengths of the legs of the right triangle in terms of x and y.

(c) Find the length of the line segment that connects the two original points in terms of x and y.

Definition 1.3.5 The distance, d, between two points, (x_1, y_1) and (x_2, y_2), can be found by using the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that the distance formula is an application of the Pythagorean Theorem! $\Diamond$

Activity 1.3.6 Apply the Definition 1.3.5 to calculate the distance between the given points.

(a) What is the distance between $(4, 6)$ and $(9, 15)$?

A. 10.2

B. 10.3

C. $\sqrt{106}$

D. $\sqrt{56}$

(b) What is the distance between $(-2, 5)$ and $(-7, -1)$?

A. $\sqrt{11}$

B. 7.8

C. 3.3

D. $\sqrt{61}$

(c) Suppose the line segment AB has one endpoint, A, at the origin. For which coordinate of B would make the line segment AB the longest?

A. $(3, 7)$

B. $(2, -8)$

C. $(-6, 4)$

D. $(-5, -5)$

Remark 1.3.7 Notice in Activity 1.3.6, you can give a distance in either exact form (leaving it with a square root) or as an approximation (as a decimal). Make sure you can give either form as sometimes one form is better than another!

Remark 1.3.8 A **midpoint** refers to the point that is located in the middle of a line segment. In other words, the midpoint is the point that is halfway between the two endpoints of a given line segment.

Activity 1.3.9 Two line segments are shown in the graph below. Use the graph to answer the following questions:

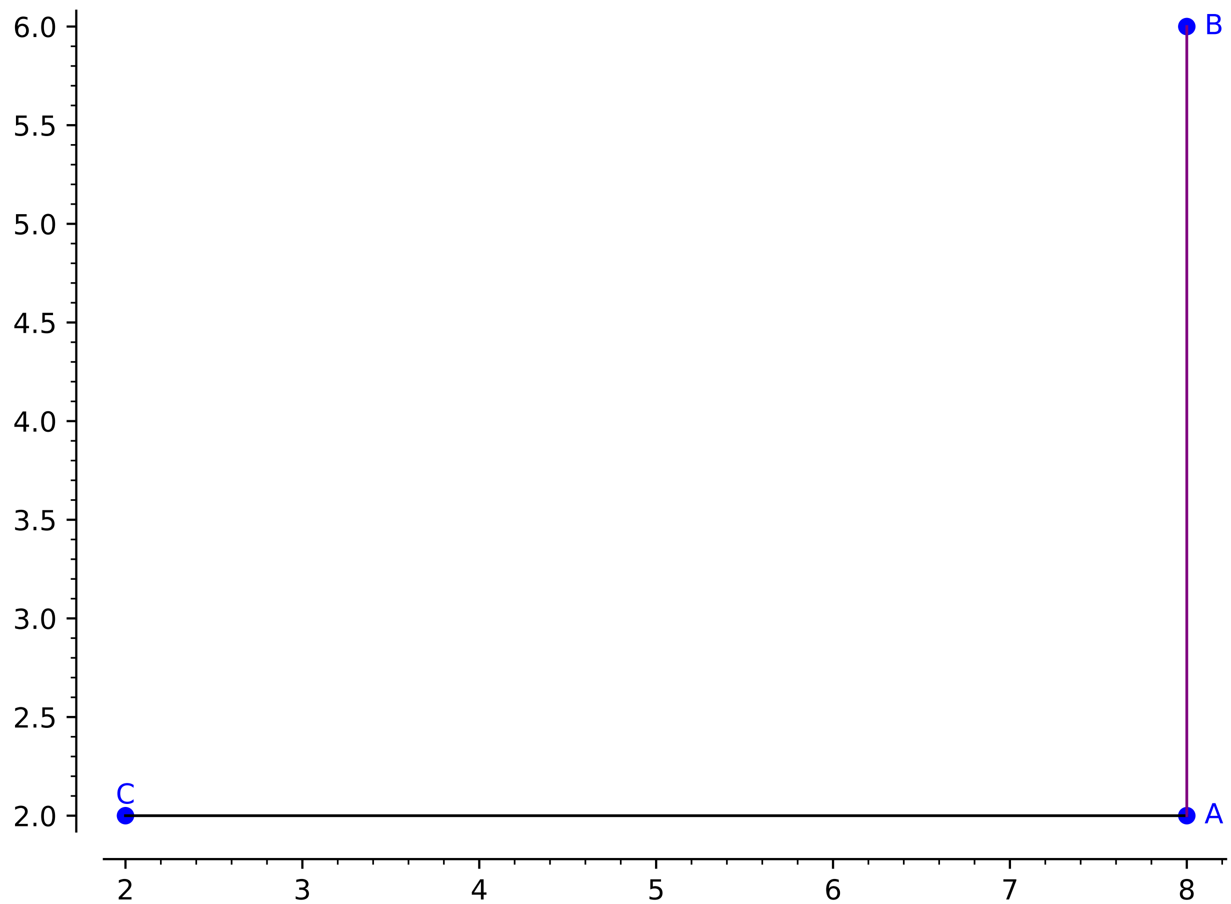


Figure 1.3.10

(a) What is the midpoint of the line segment AB?

A. $(16, 4)$
B. $(8, 4)$
C. $(8, 8)$
D. $(10, 2)$

(b) What is the midpoint of the line segment AC?

A. $(6, 0)$
B. $(4, 4)$
C. $(6, 4)$
D. $(5, 2)$

(c) Suppose we connect the two endpoints of the two line segments together, to create the new line segment, BC. Can you make an educated guess to where the midpoint of BC is?

A. $(10, 8)$
B. $(6, 4)$
C. $(5, 4)$
D. $(5, 2)$

(d) How can you test your conjecture? Is there a mathematical way to find the midpoint of any line segment?

Definition 1.3.11 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2), can be found by taking the average of the x and y values. Mathematically, the **midpoint formula** states that the midpoint of a line segment can be found by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$\diamondsuit$

Activity 1.3.12 Apply the Definition 1.3.11 to calculate the midpoint of the following line segments.

(a) What is the midpoint of the line segment with endpoints $(-4, 5)$ and $(-2, -3)$?

A. $(3, 1)$
B. $(-3, 1)$
C. $(1, 1)$
D. $(1, 4)$

(b) What is the midpoint of the line segment with endpoints $(2, 6)$ and $(-6, -8)$?

A. $(-3, -1)$
B. $(-2, 0)$
C. $(-2, -1)$
D. $(4, 7)$

(c) Suppose C is the midpoint of AB and is located at $(9, 8)$. The coordinates of A are $(10, 10)$. What are the coordinates of B?

A. $(9.5, 9)$
B. $(11, 12)$
C. $(18, 16)$
D. $(8, 6)$

Activity 1.3.13 On a map, your friend Sarah's house is located at $(-2, 5)$ and your other friend Austin's house is at $(6, -2)$.

(a) How long is the direct path from Sarah's house to Austin's house?

(b) Suppose your other friend, Micah, lives in the middle between Sarah and Austin. What is the location of Micah's house on the map?

1.4 Absolute Value Equations and Inequalities (EQ4)

Objectives

- Solve a linear equation involving an absolute value. Solve a linear inequality involving absolute values and express the answers graphically and using interval notation.

Absolute Value Equations and Inequalities (EQ4)

Remark 1.4.1 An absolute value, written $|x|$, is the non-negative value of x. If x is a positive number, then $|x| = x$. If x is a negative number, then $|x| = -x$.

Activity 1.4.2 Let's consider how to solve an equation when an absolute value is involved.

(a) Which values are solutions to the absolute value equation $|x| = 2$?

A. $x = 2$

B. $x = 0$

C. $x = -1$

D. $x = -2$

(b) Which values are solutions to the absolute value equation $|x - 7| = 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(c) Which values are solutions to the absolute value equation $3|x - 7| + 5 = 11$? It may be helpful to rewrite the equation to isolate the absolute value.

A. $x = 7$

B. $x = -9$

C. $x = 5$

D. $x = 9$

Activity 1.4.3 Absolute value represents the distance a value is from 0 on the number line. So, $|x-7|=2$ means that the expression $x-7$ is 2 units away from 0.

(a) What values on the number line could $x-7$ equal?

A. $x=-7$
B. $x=-2$
C. $x=0$
D. $x=2$
E. $x=7$

(b) This gives us two separate equations to solve. What are those two equations?

A. $x-7=-7$
B. $x-7=-2$
C. $x-7=0$
D. $x-7=2$
E. $x-7=7$

(c) Solve each equation for x.

Remark 1.4.4 When solving an absolute value equation, begin by isolating the absolute value expression. Then rewrite the equation into two linear equations and solve. If $c > 0$,

$$|ax + b| = c$$

becomes the following two equations

$$ax + b = c \quad \text{and} \quad ax + b = -c$$

Activity 1.4.5 Solve the following absolute value equations.

(a) $|3x+4|=10$

A. $\{-2,2\}$

B. $\left\{-\frac{14}{3},2\right\}$

C. $\{-10,10\}$

D. No solution

(b) $3|x-7|+5=11$

A. $\{-2,2\}$

B. $\{-9,9\}$

C. $\{5,9\}$

D. No solution

(c) $2|x+1|+8=4$

A. $\{-4,4\}$

B. $\{-6,6\}$

C. $\{5,7\}$

D. No solution

Remark 1.4.6 Since the absolute value represents a distance, it is always a positive number. Whenever you encounter an isolated absolute value equation equal to a negative value, there will be no solution.

Activity 1.4.7 Just as with linear equations and inequalities, we can consider absolute value inequalities from equations.

(a) Which values are solutions to the absolute value inequality $|x - 7| \leq 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(b) Rewrite the absolute value inequality $|x - 7| \leq 2$ as a compound inequality.

A. $0 \leq x - 7 \leq 2$

B. $-2 \leq x - 7 \leq 2$

C. $-2 \leq x - 7 \leq 0$

D. $2 \leq x \leq 7$

(c) Solve the compound inequality that is equivalent to $|x - 7| \leq 2$ found in part (b). Write the solution in interval notation.

A. $[7, 9]$

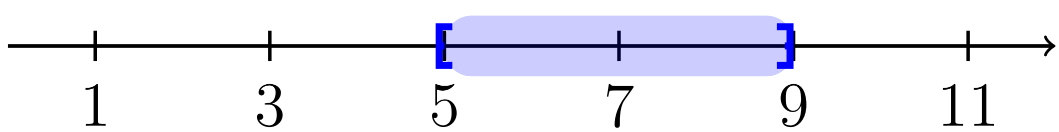
B. $[5, 9]$

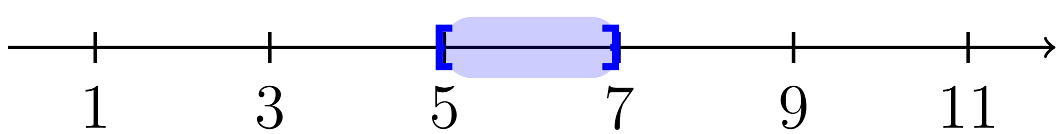
C. $[5, 7]$

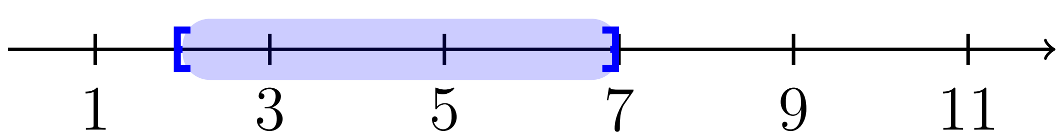
D. $[2, 7]$

(d) Draw the solution to $|x - 7| \leq 2$ on the number line.

A. 

B. 

C. 

D. 

Activity 1.4.8 Now let's consider another type of absolute value inequality.

(a) Which values are solutions to the absolute value inequality $|x - 7| \geq 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(b) Which two of the following inequalities are equivalent to $|x - 7| \geq 2$.

A. $x - 7 \leq 2$

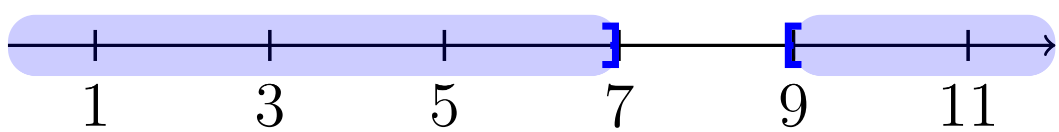
B. $x - 7 \leq -2$

C. $x - 7 \geq 2$

D. $x - 7 \geq -2$

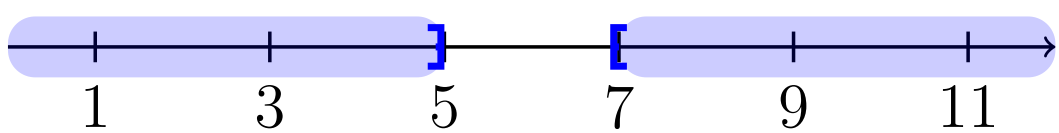
(c) Solve the two inequalities found in part (b). Write the solution in interval notation and graph on the number line.

A. $(-\infty, 7] \cup [9, \infty)$

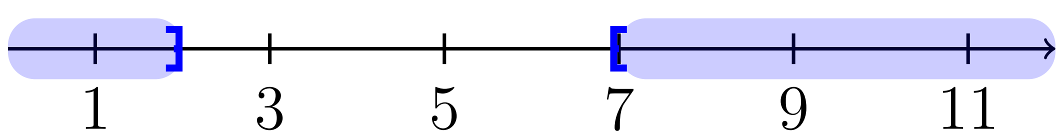


B. $(-\infty, 5] \cup [9, \infty)$

C. $(-\infty, 5] \cup [7, \infty)$



D. $(-\infty, 2] \cup [7, \infty)$



Definition 1.4.9 When solving an absolute value inequality, rewrite it as compound inequalities. Assume k is positive. $|x| < k$ becomes $-k < x < k$. $|x| > k$ becomes $x > k$ and $x < -k$. $\Diamond$

Activity 1.4.10 Solve the following absolute value inequalities. Write your solution in interval notation and graph on a number line.

(a) $|3x + 4| < 10$

(b) $3|x - 7| + 5 > 11$

1.5 Quadratic Equations (EQ5)

Objectives

- Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

Definition 1.5.1 A **quadratic equation** is of the form:

$$ax^2 + bx + c = 0$$

where a and b are coefficients (and $a \neq 0$), x is the variable, and c is the constant term. $\Diamond$

Quadratic Equations (EQ5)

Activity 1.5.2 Before beginning to solve quadratic equations, we need to be able to identify all various forms of quadratics.Which of the following is a quadratic equation?

A. $6 - x^2 = 3x$

B. $(2x - 1)(x + 3) = 0$

C. $4(x - 3) + 7 = 0$

D. $(x - 4)^2 + 1 = 0$

E. $5x^2 - 3x = 17 - 4x$

Definition 1.5.3 To solve a quadratic equation, we will need to apply the **zero product property**, which states that if $a * b = 0$, then either $a = 0$ or $b = 0$. In other words, you can only have a product of 0 if one (or both!) of the factors is 0. $\Diamond$

Activity 1.5.4 In this activity, we will look at how to apply the zero product property when solving quadratic equations.

(a) Which of the following equations can you apply Definition 1.5.3 as your first step in solving?

A. $2x^2 - 3x + 1 = 0$

B. $(2x + 1)(x + 1) = 0$

C. $3x^2 - 4 = 6x$

D. $x(3x + 5) = 0$

(b) Suppose you are given the quadratic equation, $(2x - 1)(x - 1) = 0$. Applying Definition 1.5.3, would give you:

A. $2x^2 - 3x + 1 = 0$

B. $(2x + 1) = 0$ and $(x + 1) = 0$

C. $(2x - 1) = 0$ and $(x - 1) = 0$

(c) After applying the zero product property, what are the solutions to the quadratic equation $(2x - 1)(x - 1) = 0$?

A. $x = -\frac{1}{2}$ and $x = 1$

B. $x = \frac{1}{2}$ and $x = -1$

C. $x = -\frac{1}{2}$ and $x = -1$

D. $x = \frac{1}{2}$ and $x = 1$

Remark 1.5.5 Notice in Activity 1.5.2 and Activity 1.5.4, that not all equations are set up ”nicely.” You will need to do some manipulation to get everything on one side (AND in factored form!) and 0 on the other ∗before∗ applying the zero product property.

Activity 1.5.6 Suppose you want to solve the equation $2x^2 + 5x - 12 = 0$, which is NOT in factored form.

(a) Which of the following is the correct factored form of $2x^2 + 5x - 12 = 0$?

A. $(2x - 3)(x - 4) = 0$

B. $(2x + 3)(x - 4) = 0$

C. $(2x + 3)(x + 4) = 0$

D. $(2x - 3)(x + 4) = 0$

(b) After applying Definition 1.5.3, which of the following will be a solution to $2x^2 + 5x - 12 = 0$?

A. $x = -\frac{3}{2}$ and $x = -4$

B. $x = \frac{3}{2}$ and $x = 4$

C. $x = -\frac{3}{2}$ and $x = 4$

D. $x = \frac{3}{2}$ and $x = -4$

Activity 1.5.7 Solve each of the following quadratic equations:

(a) $(2x-5)(x+7)=0$

(b) $3x(4x-1)=0$

(c) $3x^2-14x-5=0$

(d) $6-x^2=3x$

Activity 1.5.8 Suppose you are given the equation, $x^2 = 9$:

(a) How many solutions does this equation have?

A. 0

B. 1

C. 2

D. 3

(b) What are the solutions to this equation?

A. $x = 0$

B. $x = 3$

C. $x = 3, -3$

D. $x = 9, -9$

(c) How is this quadratic equation different than the equations we've solved thus far?

Definition 1.5.9 The **square root property** states that a quadratic equation of the form $x^2 = k^2$ (where k is a nonzero number) will give solutions $x = k$ and $x = -k$. In other words, if we have an equation with a perfect square on one side and a number on the other side, we can take the square root of both sides to solve the equation. $\Diamond$

Activity 1.5.10 Suppose you are given the equation, $3x^2 - 8 = 4$:

(a) What would be the first step in solving $3x^2 - 8 = 4$?

A. Divide by 3 on both sides

B. Subtract 4 on both sides

C. Add 8 on both sides

D. Multiply by 3 on both sides

(b) Isolate the x^2 term and apply Definition 1.5.9 solve for x.

(c) What are the solution(s) to $3x^2 - 8 = 4$?

A. $x = 6, -6$

B. $x = 2, -2$

C. $x = 0$

D. $x = 2$

Activity 1.5.11 Solve the following quadratic equations by applying the square root property (Definition 1.5.9).

(a) $5x^2 + 7 = 47$

(b) $2x^2 = -144$

(c) $3x^2 + 1 = 25$

(d) $(x+2)^2 + 3 = 19$

(e) $3(x-4)^2 = 15$

Remark 1.5.12 Not all quadratic equations can be factored or can be solved by using the square root property. In the next few activities, we will learn two additional methods in solving quadratics.

Definition 1.5.13 Another method for solving a quadratic equation is known as **completing the square**. With this method, we add or subtract terms to both sides of an equation until we have a perfect square trinomial on one side of the equal sign and a constant on the other side. We then apply the square root property. Note: A perfect square trinomial is a trinomial that can be factored into a binomial squared. For example, $x^2 + 4x + 4$ is a perfect square trinomial because it can be factored into $(x+2)(x+2)$ or $(x+2)^2$. $\diamondsuit$

Activity 1.5.14 Let's work through an example together to solve $x^2+6x=4$. (Notice that the methods of factoring and the square root property do not work with this equation.)

(a) In order to apply Definition 1.5.13, we first need to have a perfect square trinomial on one side of the equals sign. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

B. 9

C. -9

D. 2

(b) Add your answer from part a to the right side of the equation as well (i.e. whatever you do to one side of an equation you must do to the other side too!) and then factor the perfect square trinomial on the left side. Which equation best represents the equation now?

A. $(x+3)^2=-5$

B. $(x-3)^2=13$

C. $(x+3)^2=13$

D. $(x-3)^2=-5$

(c) Apply the square root property (Definition 1.5.9) to both sides of the equation to determine the solution(s). Which of the following is the solution(s) of $x^2+6x=4$?

A. $3+\sqrt{13}$ and $3-\sqrt{13}$

B. $-3+\sqrt{13}$ and $-3-\sqrt{13}$

C. $3+\sqrt{5}$ and $3-\sqrt{5}$

D. $-3+\sqrt{5}$ and $-3-\sqrt{5}$

Remark 1.5.15 To complete the square, the leading coefficient, a (i.e., the coefficient of the x^2 term), must equal 1. If it does not, then factor the entire equation by a and then follow similar steps as in Activity 1.5.14.

Activity 1.5.16 Let's solve the equation $2x^2 + 8x - 5 = 0$ by completing the square.

(a) Rewrite the equation so that all the terms with the variable x is on one side of the equation and a constant is on the other.

(b) Notice that the coefficient of the x^2 term is not 1. What could we factor the left side of the equation by so that the coefficient of the x^2 is 1?

(c) Once you factor the left side, what equation represents the equation you now have?

A. $2(x^2 - 8x) = -5$

B. $2(x^2 - 4x) = -5$

C. $2(x^2 + 4x) = 5$

D. $2(x^2 + 8x) = 5$

(d) Just like in Activity 1.5.14, let's now try and create the perfect square trinomial (inside the parentheses) on the left side of the equation. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

B. 8

C. -8

D. 2

(e) What would we need to add to the right-hand side of the equation to keep the equation balanced?

A. 4

B. 8

C. -8

D. 2

(f) Which of the following equation represents the quadratic equation you have now?

A. $2(x + 2)^2 = 9$

B. $2(x - 2)^2 = 9$

C. $2(x + 2)^2 = 13$

D. $2(x - 2)^2 = 13$

(g) Apply the square root property and solve the quadratic equation.

Activity 1.5.17 Solve the following quadratic equations by completing the square.

(a) $x^2 - 12x = -11$

(b) $x^2 + 2x - 33 = 0$

(c) $4x^2 + 16x = 65$

Definition 1.5.18 The last method for solving quadratic equations is the **quadratic formula** - a formula that will solve all quadratic equations! A quadratic equation of the form $ax^2+bx+c=0$ can be solved by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b, and c are real numbers and $a \neq 0$. $\Diamond$

Activity 1.5.19 Use the quadratic formula (Definition 1.5.18) to solve $x^2 + 4x = -3$.

(a) When written in standard form, what are the values of a, b, and c?

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$

B. $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

C. $x = \dfrac{-4 \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

D. $x = \dfrac{4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$

(c) What is the solution(s) to $x^2 + 4x = -3$?

A. $x = -1, 3$

B. $x = 1, 3$

C. $x = -1, -3$

D. $x = 1, -3$

Activity 1.5.20 Use the quadratic formula (Definition 1.5.18) to solve $2x^2 - 13 = 7x$.

(a) When written in standard form, what are the values of a, b, and c?

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-7 \pm \sqrt{7^2 - 4(2)(13)}}{2(1)}$

B. $x = \dfrac{7 \pm \sqrt{(-7)^2 - 4(2)(-13)}}{2(2)}$

C. $x = \dfrac{-7 \pm \sqrt{(-7)^2 - 4(1)(-13)}}{2(1)}$

D. $x = \dfrac{7 \pm \sqrt{7^2 - 4(2)(-13)}}{2(2)}$

(c) What is the solution(s) to $2x^2 - 13 = 7x$?

A. $x = \frac{7+\sqrt{73}}{4}$ and $\frac{7-\sqrt{73}}{4}$

B. $x = \frac{7+\sqrt{101}}{4}$ and $\frac{-7-\sqrt{101}}{4}$

C. $x = \frac{-7+\sqrt{55}}{4}$ and $\frac{7-\sqrt{55}}{4}$

D. $x = \frac{-7+\sqrt{155}}{4}$ and $\frac{-7-\sqrt{155}}{4}$

Activity 1.5.21 Use the quadratic formula (Definition 1.5.18) to solve $x^2 = 6x - 12$.

(a) When written in standard form, what are the values of a, b, and c?

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(12)}}{2(1)}$

B. $x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$

C. $x = \dfrac{-6 \pm \sqrt{(-6)^2 - 4(1)(-12)}}{2(1)}$

D. $x = \dfrac{6 \pm \sqrt{6^2 - 4(1)(-12)}}{2(1)}$

(c) Notice that the number under the square root is a negative. Recall that when you have a negative number under a square root, that gives an imaginary number ($\sqrt{-1} = i$. What is the solution(s) to $x^2 = 6x - 12$?

A. $x = 3 + i\sqrt{3}$ and $3 - i\sqrt{3}$

B. $x = 6 + i\sqrt{12}$ and $6 - i\sqrt{12}$

C. $x = -3 + i\sqrt{3}$ and $-3 - i\sqrt{3}$

D. $x = -6 + i\sqrt{12}$ and $-6 - i\sqrt{12}$

Activity 1.5.22 Solve the following quadratic equations by applying the quadratic formula (Definition 1.5.18).

(a) $2x^2 - 3x = 5$

(b) $4x^2 - 1 = -8x$

(c) $2x^2 - 7x - 13 = -10$

(d) $x^2 - 6x + 12 = 0$

Activity 1.5.23 Now that you have seen all the different ways to solve a quadratic equation, you will need to know WHEN to use which method. Are some methods better than others?

(a) Which is the best method to use to solve $5x^2 = 80$?

A. Factoring and Zero Product Property

B. Square Root Property

C. Completing the Square

D. Quadratic Formula

(b) Which is the best method to use to solve $5x^2 + 9x = -4$?

A. Factoring and Zero Product Property

B. Square Root Property

C. Completing the Square

D. Quadratic Formula

(c) Which is the best method to use to solve $3x^2 + 9x = 0$?

A. Factoring and Zero Product Property

B. Square Root Property

C. Completing the Square

D. Quadratic Formula

(d) Go back to parts a, b, and c and solve each of the quadratic equations. Would you still use the same method?

1.6 Rational Equations (EQ6)

Objectives

- Solve a rational equation.

Rational Equations (EQ6)

Definition 1.6.1 An algebraic expression is called a **rational expression** if it can be written as the ratio of two polynomials, p and q.

An equation is called a **rational equation** if it consists of only rational expressions and constants. $\diamondsuit$

Observation 1.6.2 Technically, linear and quadratic equations are also rational equations. They are a special case where the denominator of the rational expressions is 1. We will focus in this section on cases where the denominator is not a constant; that is, rational equations where there are variables in the denominator.

Activity 1.6.3 Because some of the denominators in a rational equation will have variables, we need to be sure to exclude any values that would make those denominators equal to zero. Which value(s) should be exclude as possible solutions to the following rational equations? Select all that apply.

(a)

$$\frac{2}{x+5} = \frac{x-3}{x-8} - 7$$

A. -7

B. -5

C. 2

D. 3

E. 8

(b)

$$\frac{x^2-6x+8}{x^2-4x+3} = 0$$

A. 0

B. 1

C. 2

D. 3

E. 4

Activity 1.6.4 Consider the rational equation

$$5 = -\frac{6}{x-2}$$

(a) What value should be excluded as a possible solution?

A. 5

B. 6

C. -6

D. 2

E. -2

(b) To solve, we begin by clearing out the fraction involved. What can we multiply each term by that will clear the fraction?

A. $x-5$

B. $x-6$

C. $x+6$

D. $x-2$

E. $x+2$

(c) Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $5(x-5) = -6$

B. $5(x-6) = -6$

C. $5(x+6) = -6$

D. $5(x-2) = -6$

E. $5(x+2) = -6$

(d) Solve the linear equation. Check your answer using the original rational equation.

Activity 1.6.5 Consider the rational equation

$$\frac{4}{x+1} = -\frac{2}{x+6}$$

(a) What values should be excluded as possible solutions?

A. 2 and 4

B. 1 and 6

C. -1 and -6

D. 1 and 4

E. 2 and 6

(b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x-2$ and $x-4$

B. $x-1$ and $x-6$

C. $x+1$ and $x+6$

D. $x-1$ and $x-4$

E. $x-2$ and $x-6$

(c) Multiply each term by the expressions you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $4(x+1) = -2(x+6)$

B. $4(x+6) = -2(x+1)$

C. $4(x+1)(x+6) = -2(x+1)(x+6)$

D. $4(x+1) = -2(-x-6)$

E. $4(x+6) = -2(-x-1)$

(d) Solve the linear equation. Check your answer using the original rational equation.

Observation 1.6.6 In Activity 1.6.5, you may have noticed that the resulting linear equation looked like the result of cross-multiplying. This is no coincidence! Cross-multiplying is a method of clearing out fractions that works specifically when the equation is in proportional form: $\frac{a}{b} = \frac{c}{d}$.

Activity 1.6.7 Consider the rational equation

$$\frac{x}{x+2} = -\frac{2}{x+2} - \frac{2}{5}$$

(a) What value(s) should be excluded as possible solutions?

(b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x+2$, $x+2$, and 5

B. $x+2$ and 5

C. $x+2$

D. 5

(c) Multiply each term by the expressions you chose and simplify. You should end up with a linear equation.

(d) Solve the linear equation. Check your answer using the original rational equation.

Observation 1.6.8 Activity 1.6.7 demonstrates why it is so important to determine excluded values and check our answers when solving rational equations. Just because a number is a solution to the *linear* equation we found, it doesn't mean it is automatically a solution to the *rational* equation we started with.

Rational Equations (EQ6)

Activity 1.6.9 Consider the rational equation

$$\frac{2x}{x-1} - \frac{3}{x-3} = \frac{x^2 - 11x + 18}{x^2 - 4x + 3}$$

(a) What values should be excluded as possible solutions? Select all that apply.

A. 0

B. 1

C. 2

D. 3

E. 9

(b) To solve, we'll begin by clearing out any fractions involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x - 1$

B. $x - 1$ and $x - 3$

C. $x - 1$, $x - 3$, and $x^2 - 4x + 3$

D. $x - 1$ and $x^2 - 4x + 3$

E. $x - 3$ and $x^2 - 4x + 3$

(c) Multiply each term by the expressions you chose and simplify. Notice that the result is a quadratic equation. Which of the following quadratic equations does the rational equation simplify to?

A. $x^2 + 2x - 15 = 0$

B. $x^2 - 11x + 18 = 0$

C. $x^2 - 9x - 9 = 0$

D. $x^2 - 13x + 21 = 0$

(d) Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

A. $x = 3$ and $x = -5$

B. $x = -3$ and $x = 5$

C. $x = 3$

D. $x = -5$

E. $x = -3$

F. $x = 5$

Activity 1.6.10 Consider the rational equation

$$\frac{2x}{x-2} - \frac{x^2+21x-15}{x^2+3x-10} = \frac{-6}{x+5}$$

(a) What values should be excluded as possible solutions?

(b) What expression(s) should we multiply by to clear out all of the fractions?

(c) Multiply each term by the expressions you chose and simplify. Your result should be a quadratic equation.

(d) Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

Activity 1.6.11 Solve the following rational equations.

(a) $\dfrac{4}{x}+9=16$

(b) $-5=\dfrac{2}{x-4}$

(c) $\dfrac{-3}{x-10}=\dfrac{x}{x-6}$

(d) $\dfrac{x+2}{x-3}+\dfrac{x}{2x-1}=6$

1.7 Quadratic and Rational Inequalities (EQ7)

Objectives

- Solve a rational equation.

Chapter 2

Functions (FN)

Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx
By the end of this chapter, you should be able to...

1. Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.
2. Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.
3. Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maximums and minimums, and where it is increasing and decreasing in interval notation.
4. Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.
5. Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.
6. Find the inverse of a one-to-one function.

2.1 Introduction to Functions (FN1)

Objectives

- Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.

Definition 2.1.1 A **relation** is a relationship between sets of values. Relations in mathematics are usually represented as ordered pairs: (input, output) or (x, y). When observing relations, we often refer the x-values as the **domain** and the y-values as the **range**. $\Diamond$

Definition 2.1.2 Mapping Notation (also known as an arrow diagram) is a way to show relationships visually between sets. For example, suppose you are given the following ordered pairs: $(3,-8),(4,6),(2,-1)$. Each of the x-values "map onto" the y-values and can be represented in the following way:

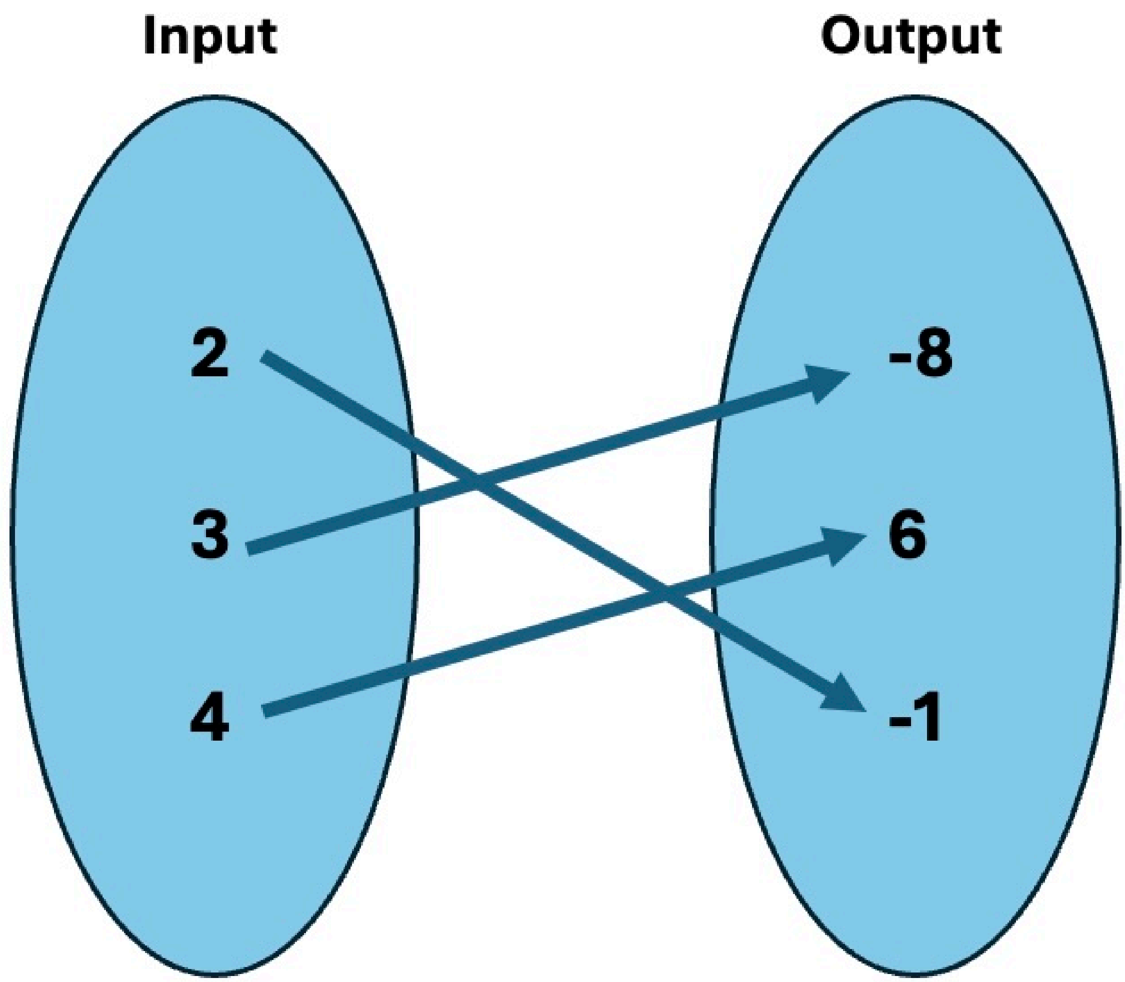


Figure 2.1.3 Every x-value from the ordered pair list is listed in the input set and every y-value is listed in the output set. An arrow is drawn from every x-value to its corresponding y-value.

Notice that an arrow is used to indicate which x-value is mapped onto its corresponding y-value. $\Diamond$

Activity 2.1.4 Use mapping notation to create a visual representation of each relation.

(a) $(-1,5),(2,6),(4,-2)$

(b) $(6,4),(3,4),(6,5)$

(c) $(1,2),(-5,2),(-7,2)$

(d) Determine the domain and range of each relation.

(e) What kinds of relationships do you notice?

Remark 2.1.5 Notice that in Activity 2.1.4, each set represents a very different relationship. Many concepts in mathematics will depend on particular relationships, so it is important to be able to visualize relationships and compare them.

Definition 2.1.6 A **function** is a relation where every input (or x-value) is mapped onto *exactly one* output (or y-value).

Note that all functions are relations but not all relations are functions!

$\Diamond$

Activity 2.1.7 Relations can be expressed in multiple ways (ordered pairs, tables, and verbal descriptions).

(a) Determine whether each of the following sets of ordered pairs represent a function.

- $(-1, 5), (2, 6), (4, -2)$
- $(6, 4), (3, 4), (6, 5)$
- $(1, 2), (-5, 2), (-7, 2)$

(b) Note that relations can be expressed in a table. A table of values is shown below. Is this an example of a function? Why or why not?

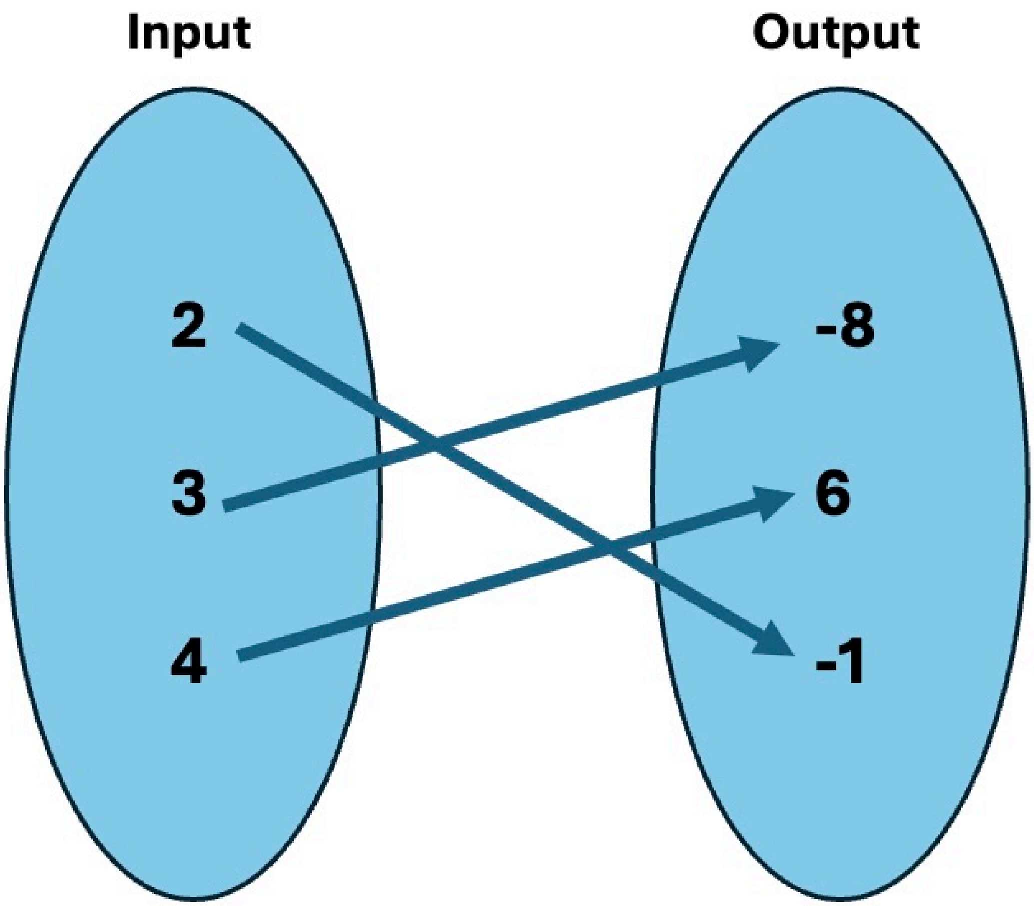
x	y
-5	-2
-4	-5
-2	8
8	-4
8	1

(c) Relations can also be expressed in words. Suppose you are looking at the amount of time you spend studying versus the grade you earn in your Algebra class. Is this an example of a function? Why or why not?

Remark 2.1.8 Notice that when trying to determine if a relation is a function, we often have to rely on looking at the domain and range values. Thus, it is important to be able to idenfity the domain and range of any relation!

Activity 2.1.9 For each of the given functions, determine the domain and range.

(a) $(-4, 3), (-1, 8), (7, 4), (1, 9)$



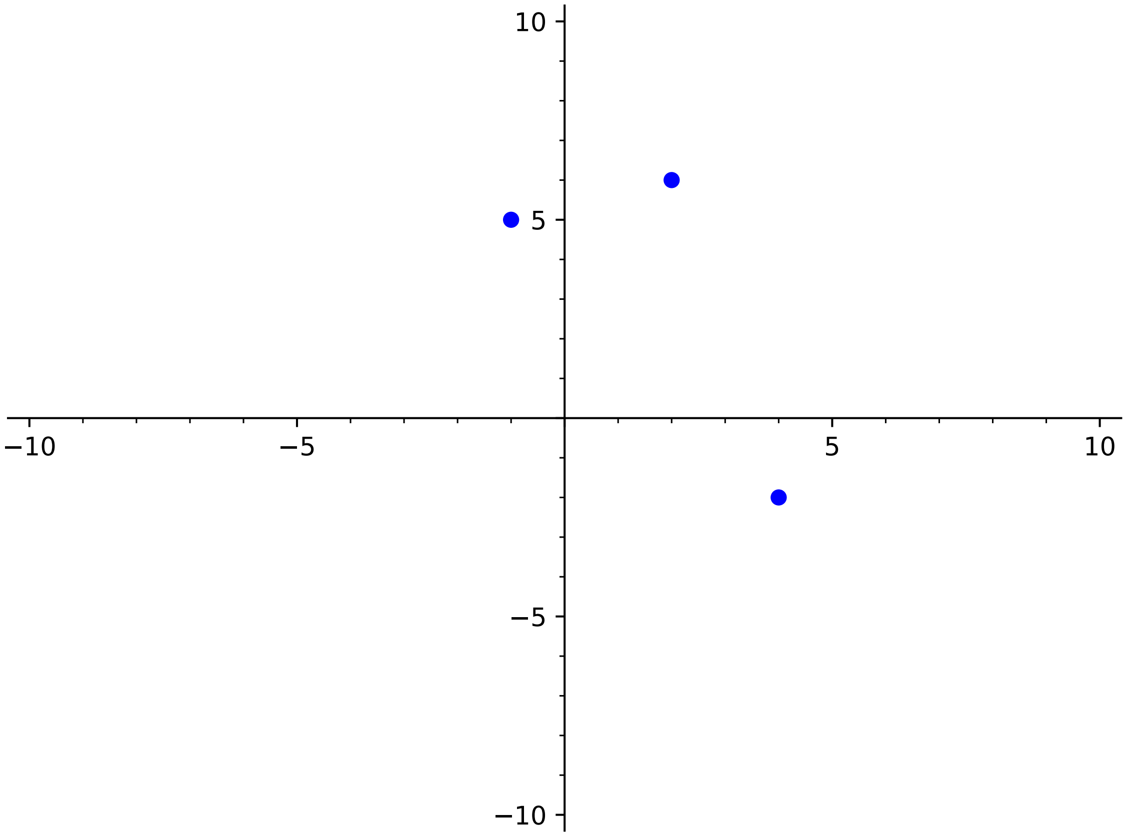
(b)

(c)

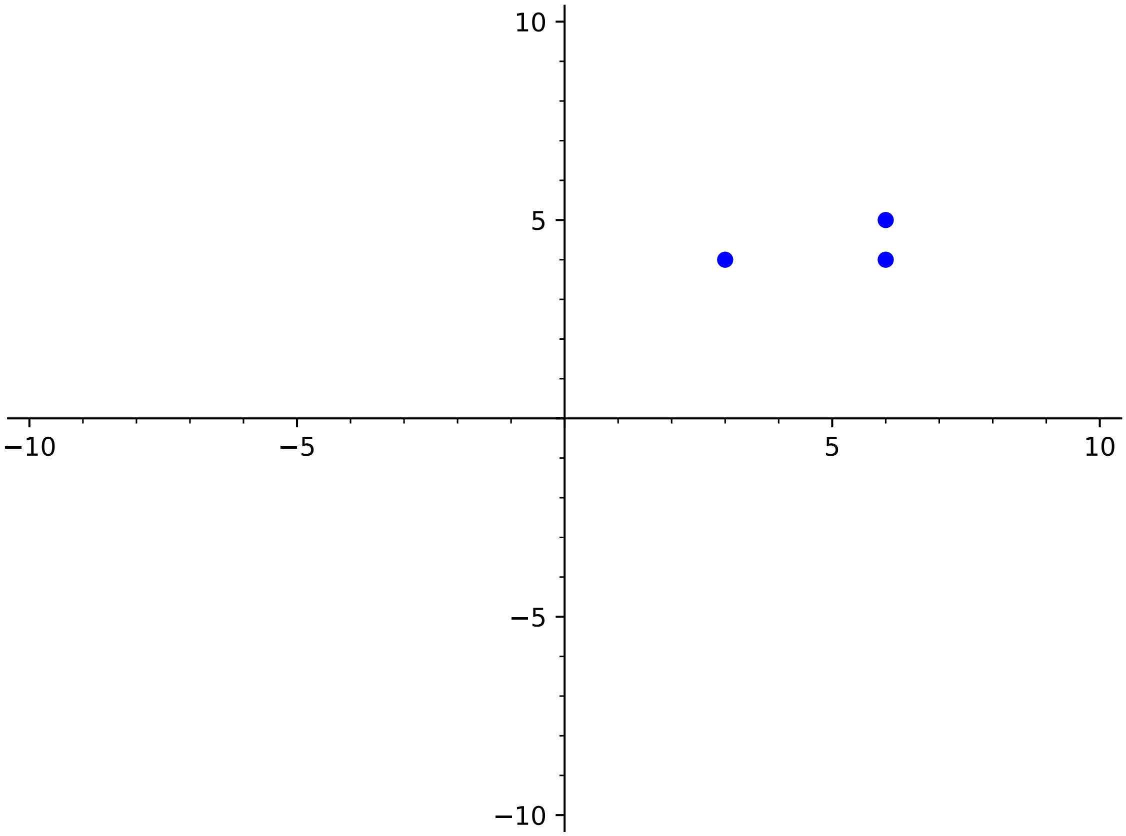
x	y
-2	5
0	4
3	6
8	1

(d) The amount of time you spend studying versus the grade you earn in your Algebra class.

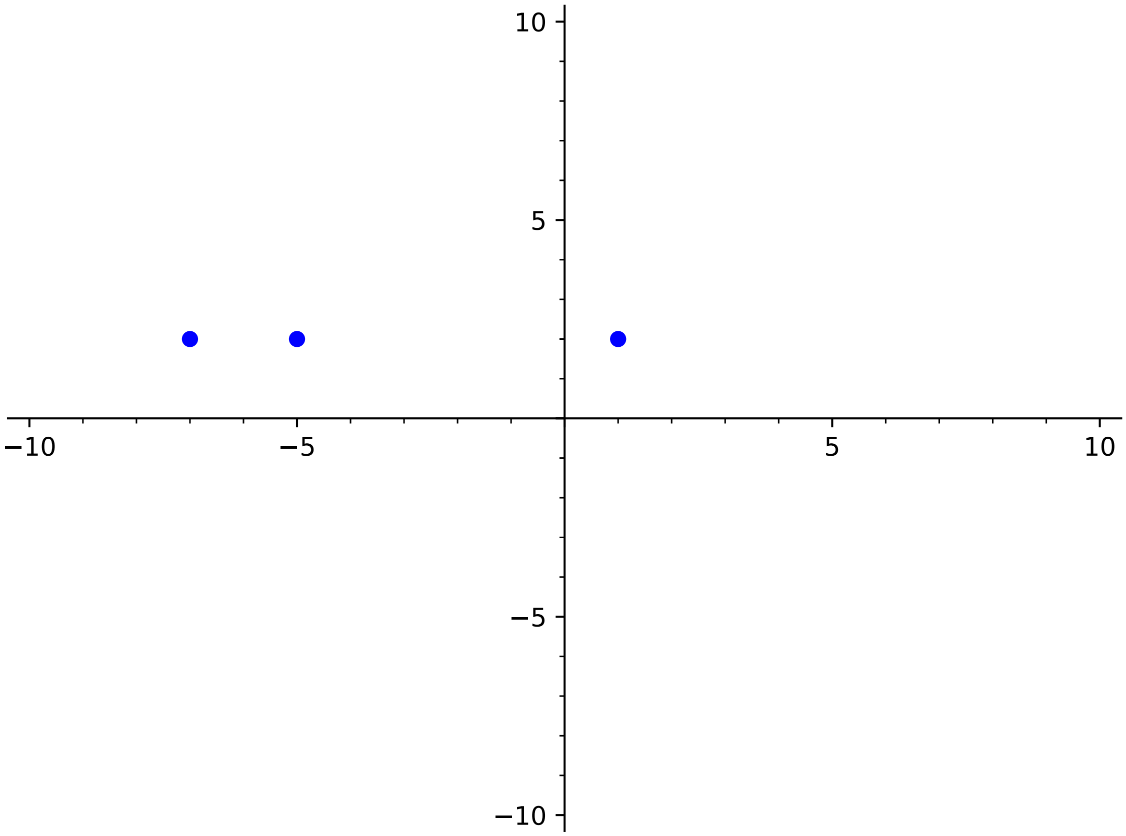
Activity 2.1.10 Determine whether each of the following relations is a function.



(a)



(b)



(c)

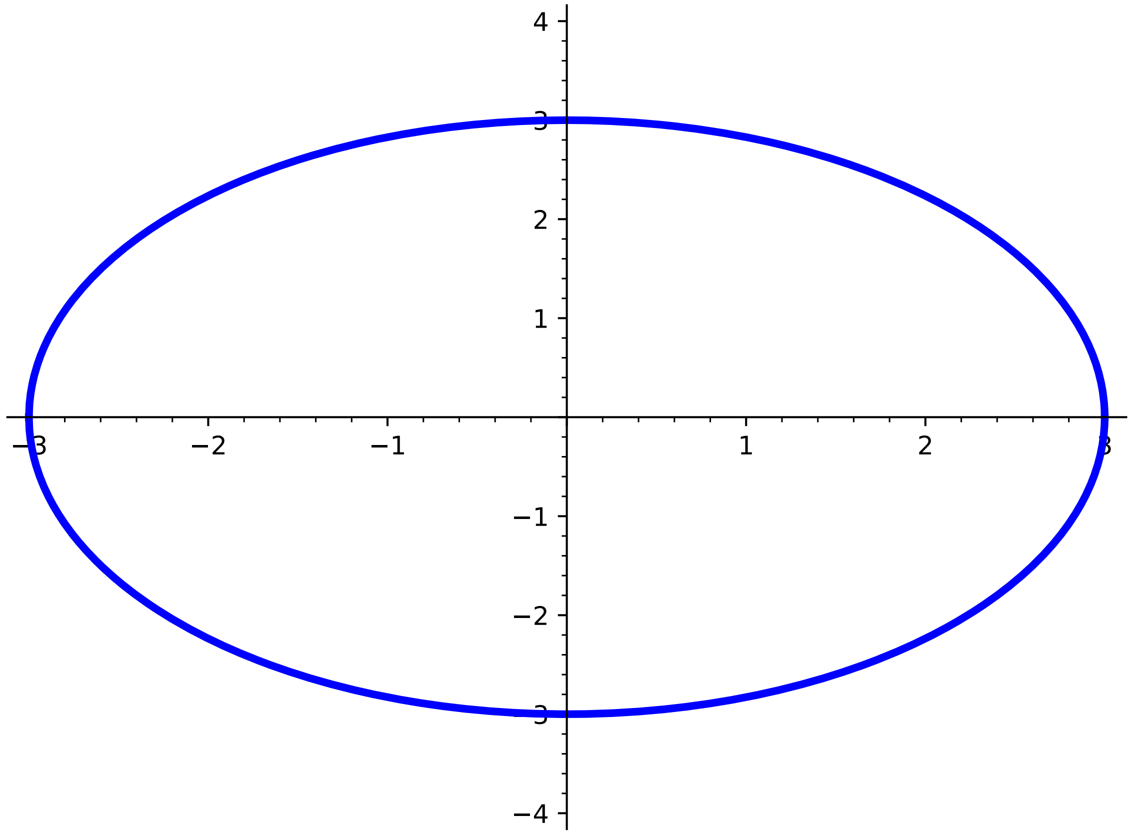
Definition 2.1.11 The **vertical line test** is a method used to determine whether a relation on a graph is a function.

Start by drawing a vertical line anywhere on the graph and observe the number of times the relation on the graph intersects with the vertical line. If the vertical line intersects the graph at exactly one point, then the relation is a function. If, however, the graph of the relation intersects the vertical line more than once (anywhere on the graph), then the relations is not a function.

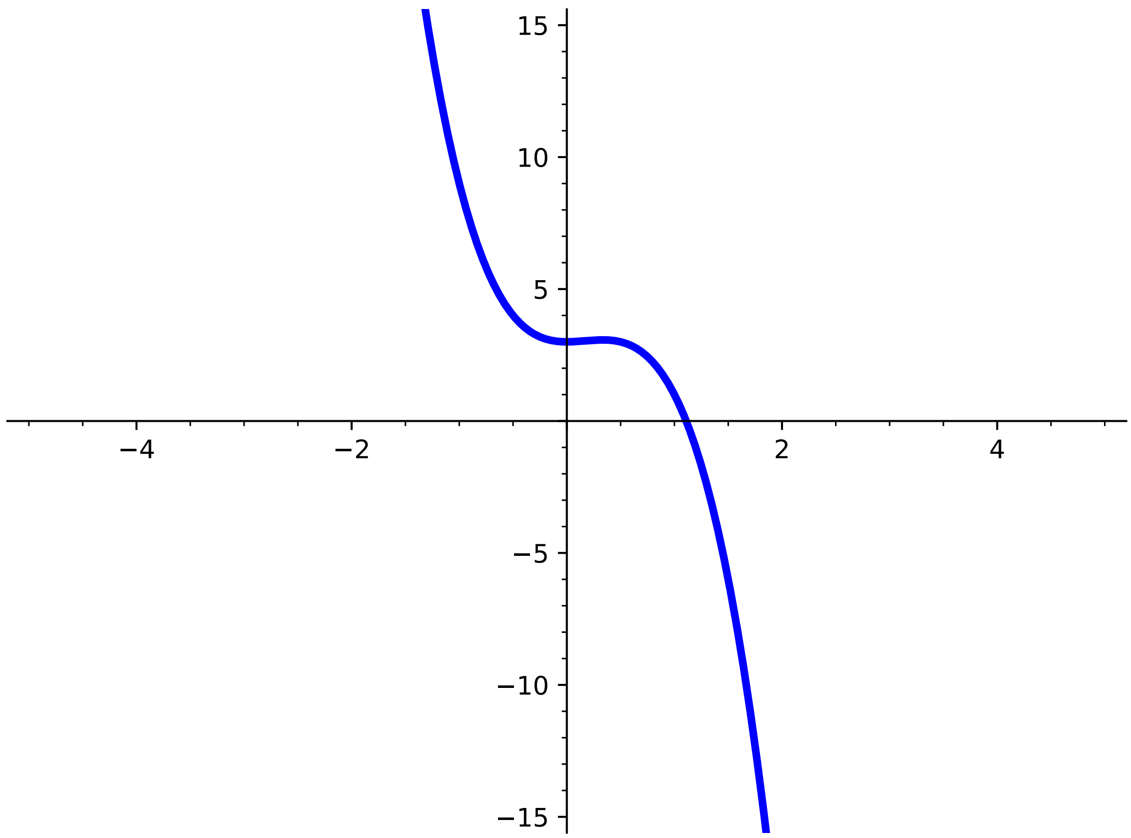
$\Diamond$

Activity 2.1.12 Determine whether each of the following graphs are functions by using the vertical line test (Definition 2.1.11).

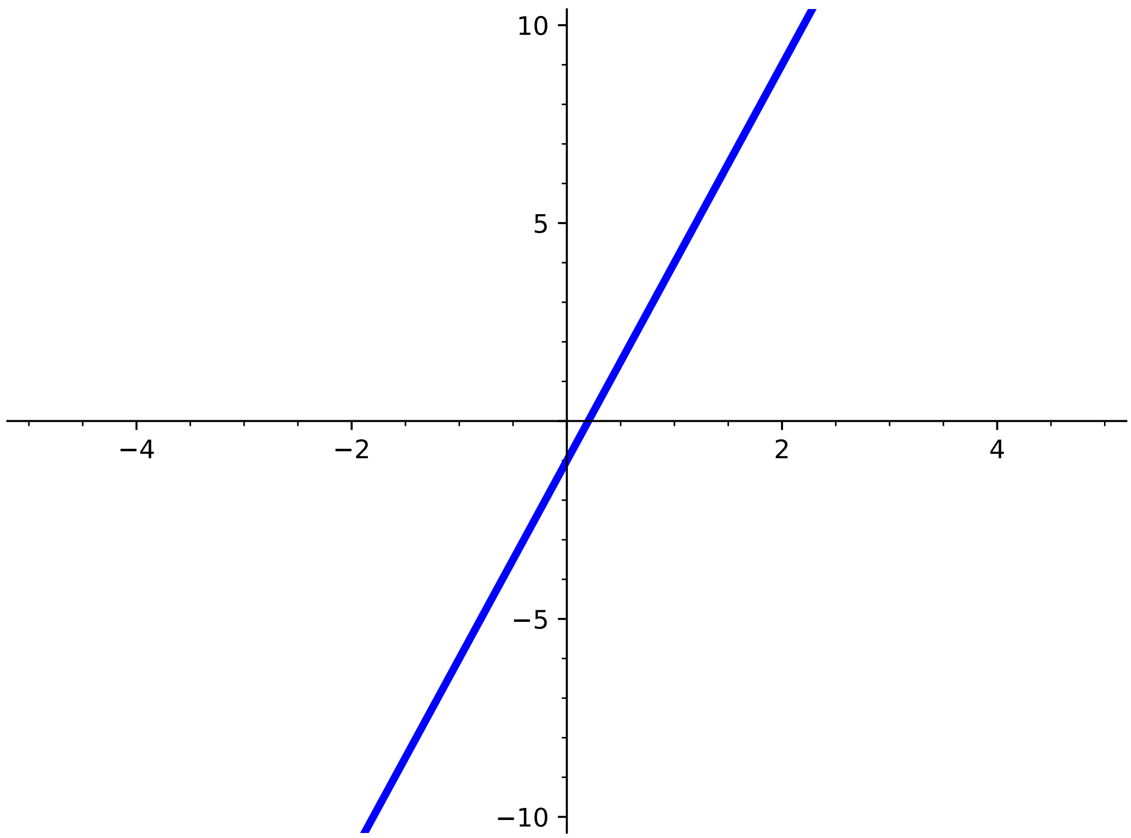
(a)



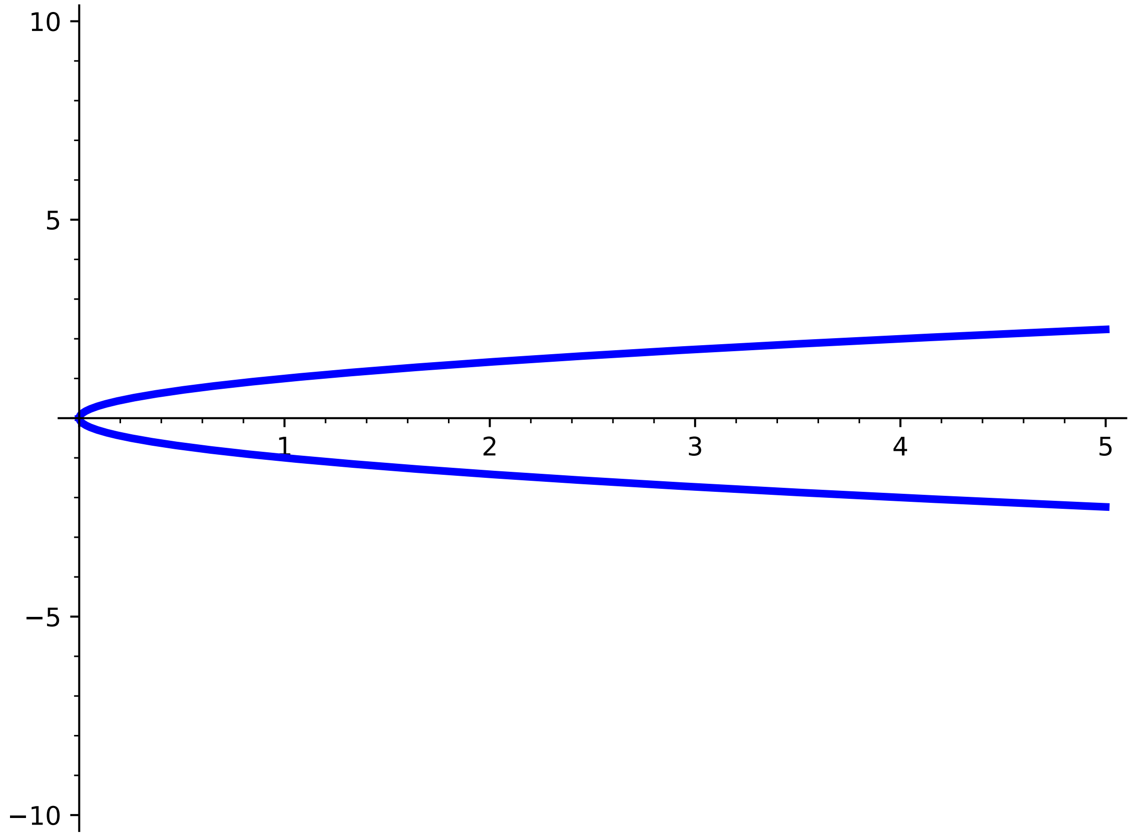
(b)



(c)



(d)



Activity 2.1.13 Let's explore how to look at an equation to determine if it is a function.

(a) Suppose you are given the equation $x^2 + y^2 = 9$.

- If $x = 4$, what kind of y-values would you get for $x^2 + y^2 = 9$?
- Based on this information, do you think $x^2 + y^2 = 9$ is a function?

(b) Suppose you are given the equation $y = 3x^2 + 2$.

- If $x = 4$, what kind of y-values would you get for $y = 3x^2 + 2$?
- Based on this information, do you think $y = 3x^2 + 2$ is a function?

(c) Suppose you are given the equation $x = y^2$.

- If $x = 4$, what kind of y-values would you get for $x = y^2$?
- Based on this information, do you think $x = y^2$ is a function?

(d) Suppose you are given the equation $y = -4x - 3$.

- If $x = 4$, what kind of y-values would you get for $y = -4x - 3$?
- Based on this information, do you think $y = -4x - 3$ is a function?

(e) How can you look at an equation to determine whether or not it is a function?

Remark 2.1.14 Notice that Activity 2.1.13 shows that equations with a y^2 term generally do not define functions.

Activity 2.1.15 It's important to be able to determine the domain of any equation, especially when thinking about functions. Answer the following questions given the equation $y = \sqrt{x}$.

(a) What values of x would give an error (if any)?

A. -2

B. 0

C. 4

D. -5

(b) Based on this information, what values of x would the equation exist?

A. -2

B. 0

C. 4

D. -5

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 0)$

B. $(0, \infty)$

C. $(0, 0)$

D. $(-\infty, \infty)$

Activity 2.1.16 Answer the following questions given the equation $y = -5x + 1$.

(a) What values of x would give an error (if any)?

A. -2

B. 0

C. 4

D. -5

(b) Based on this information, what values of x would the equation exist?

A. -2

B. 0

C. 4

D. -5

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 0)$

B. $(0, \infty)$

C. $(-5, 1)$

D. $(-\infty, \infty)$

Activity 2.1.17 Answer the following questions given the equation $y = \frac{3}{x-5}$.

(a) What values of x would give an error (if any)?

A. -3

B. 0

C. -4

D. 5

(b) Based on this information, what values of x would the equation exist?

A. -3

B. 0

C. -4

D. 5

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 5)$

B. $(5, \infty)$

C. $(-5, 5)$

D. $(-\infty, 5)\mathrm{U}(5, \infty)$

Remark 2.1.18 When determining the domain of an equation, it is often easier to first find values of x that make the function undefined. Once you have those values, then you know that x can be any value but those.

2.2 Function Notation (FN2)

Objectives

- Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.

2.3 Characteristics of a Function's Graph (FN3)

Objectives

- Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maximums and minimums, and where it is increasing and decreasing in interval notation.

2.4 Transformation of Functions (FN4)

Objectives

- Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.

2.5 Combining and Composing Functions (FN5)

Objectives

- Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.

2.6 Finding the Inverse Function (FN6)

Objectives

- Find the inverse of a one-to-one function.

Remark 2.6.1 A function is a process that converts a collection of inputs to a corresponding collection of outputs. One question we can ask is: for a particular function, can we reverse the process and think of the original function's outputs as the inputs?

Activity 2.6.2 Temperature can be measured using many different units such as Fahrenheit, Celsius, and Kelvin. Fahrenheit is what is usually reported on the news each night in the United States, while Celsius is commonly used for scientific work. We will begin by converting between these two units. To convert from degrees Fahrenheit to Celsius use the following formula.

$$C = \frac{5}{9}(F - 32)$$

(a) Room temperature is around 68 degrees Fahrenheit. Use the above equation to convert this temperature to Celsius.

A. 5.8

B. 20

C. 155.4

D. 293

(b) Solve the equation $C = \frac{5}{9}(F - 32)$ for F in terms of C.

A. $F = \dfrac{5}{9}C + 32$

B. $F = \dfrac{5}{9}C - 32$

C. $F = \dfrac{9}{5}(C + 32)$

D. $F = \dfrac{9}{5}C + 32$

(c) Alternatively, 20 degrees Celsius is a fairly comfortable temperate. Use your solution for F in terms of C to convert this temperature to Fahrenheit.

A. 43.1

B. -20.9

C. 93.6

D. 68

(d) Something here to sum this up... like find the composition of the two functions what do you get?

Finding the Inverse Function (FN6)

Definition 2.6.3 Let f be a function. If there exists a function g such that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

for all x, then we say f has an **inverse function**, or that g is the **inverse of** f. When a given function f has an inverse function, we usually denote it as f^{-1}, which is read as "f inverse". $\Diamond$

Remark 2.6.4 An inverse is a function that "undoes" another function. For any input in the domain, the function g will reverse the process of f.

Activity 2.6.5 It is important to note that in Definition 2.6.3 we say ”if there exists a function,” but we don’t guarantee that this is always the case. How can we determine whether a function has a corresponding inverse or not? Consider the following two functions f and g represented by the tables.

Table 2.6.6

x	$f(x)$
0	6
1	4
2	3
3	4
4	6

Table 2.6.7

x	$g(x)$
0	3
1	1
2	4
3	2
4	0

(a) Use the definition of $g(x)$ in Table 2.6.7 to find an x such that $g(x) = 4$.

A. $x = 0$

B. $x = 1$

C. $x = 2$

D. $x = 3$

E. $x = 4$

(b) Is it possible to reverse the input and output rows of the function $g(x)$ and have the new table result in a function?

(c) Use the definition of $f(x)$ in Table 2.6.6 to find an x such that $f(x) = 4$.

A. $x = 0$

B. $x = 1$

C. $x = 2$

D. $x = 3$

E. $x = 4$

(d) Is it possible to reverse the input and output rows of the function $f(x)$ and have the new table result in a function?

Remark 2.6.8 Some functions, like $f(x)$ in Table 2.6.6, have a given output value that corresponds to two or more input values: $f(0) = 6$ and $f(4) = 6$. If we attempt to reverse the process of this function, we have a situation where the new input 6 would correspond to two potential outputs.

Finding the Inverse Function (FN6)

Definition 2.6.9 A **one-to-one function** is a function in which each output value corresponds to exactly one input. $\Diamond$

Finding the Inverse Function (FN6)

Remark 2.6.10 A function must be one-to-one in order to have an inverse.

Finding the Inverse Function (FN6)

Activity 2.6.11 Consider the function $f(x) = \dfrac{x-5}{3}$.

(a) When you evaluate this expression for a given input value of x, what operations do you perform and in what order?

A. divide by 3, subtract 5

B. subtract 5, divide by 3

C. add 5, multiply by 3

D. multiply by 3, add 5

(b) When you "undo" this expression to solve for a given ouput value of y, what operations do you perform and in what order?

A. divide by 3, subtract 5

B. subtract 5, divide by 3

C. add 5, multiply by 3

D. multiply by 3, add 5

(c) This set of operations, reverses the process for the original function, so can be considered the inverse function. Write an equation to express the inverse function f^{-1}.

A. $f^{-1}(x) = \dfrac{x}{3} - 5$

B. $f^{-1}(x) = \dfrac{x-5}{3}$

C. $f^{-1}(x) = 5(x+3)$

D. $f^{-1}(x) = 3x+5$

(d) Check your answer to the previous question by finding $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

Finding the Inverse Function (FN6)

Observation 2.6.12 To find the inverse of a one-to-one function, perform the reverse operations in the opposite order. Alternatively, solve the function for x and then interchange the x and y.

Finding the Inverse Function (FN6)

Activity 2.6.13 Find the inverse of each function. Check your answer using function composition.

(a) $f(x) = 3 - \sqrt{x+5}$

A. $f^{-1}(x) = 3 + \sqrt{x-5}$

B. $f^{-1}(x) = (x-3)^2 + 5$

C. $f^{-1}(x) = \dfrac{1}{3 - \sqrt{x+5}}$

D. $f^{-1}(x) = (3-x)^2 - 5$

(b) $g(x) = \dfrac{4x-1}{7}$

A. $g^{-1}(x) = \dfrac{7x+1}{4}$

B. $g^{-1}(x) = \dfrac{7x}{4} + 1$

C. $g^{-1}(x) = \dfrac{4x+1}{7}$

D. $g^{-1}(x) = \dfrac{7}{4x-1}$

(c) $h(x) = \dfrac{x}{x+1}$

Chapter 3

Linear Functions (LF)

Chapter 4

Polynomial and Rational Functions (PR)

Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx
By the end of this chapter, you should be able to...

1. Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

2. Use quadratic models to solve an application problem and establish conclusions.

3. Rewrite a rational function as a polynomial plus a proper rational function.

4. Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.

5. Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.

6. Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

4.1 Graphing Quadratic Functions (PR1)

Objectives

- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

Observation 4.1.1 Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.

Activity 4.1.2 Use the graph of the quadratic function $f(x) = 3(x-2)^2 - 4$ to answer the questions below.

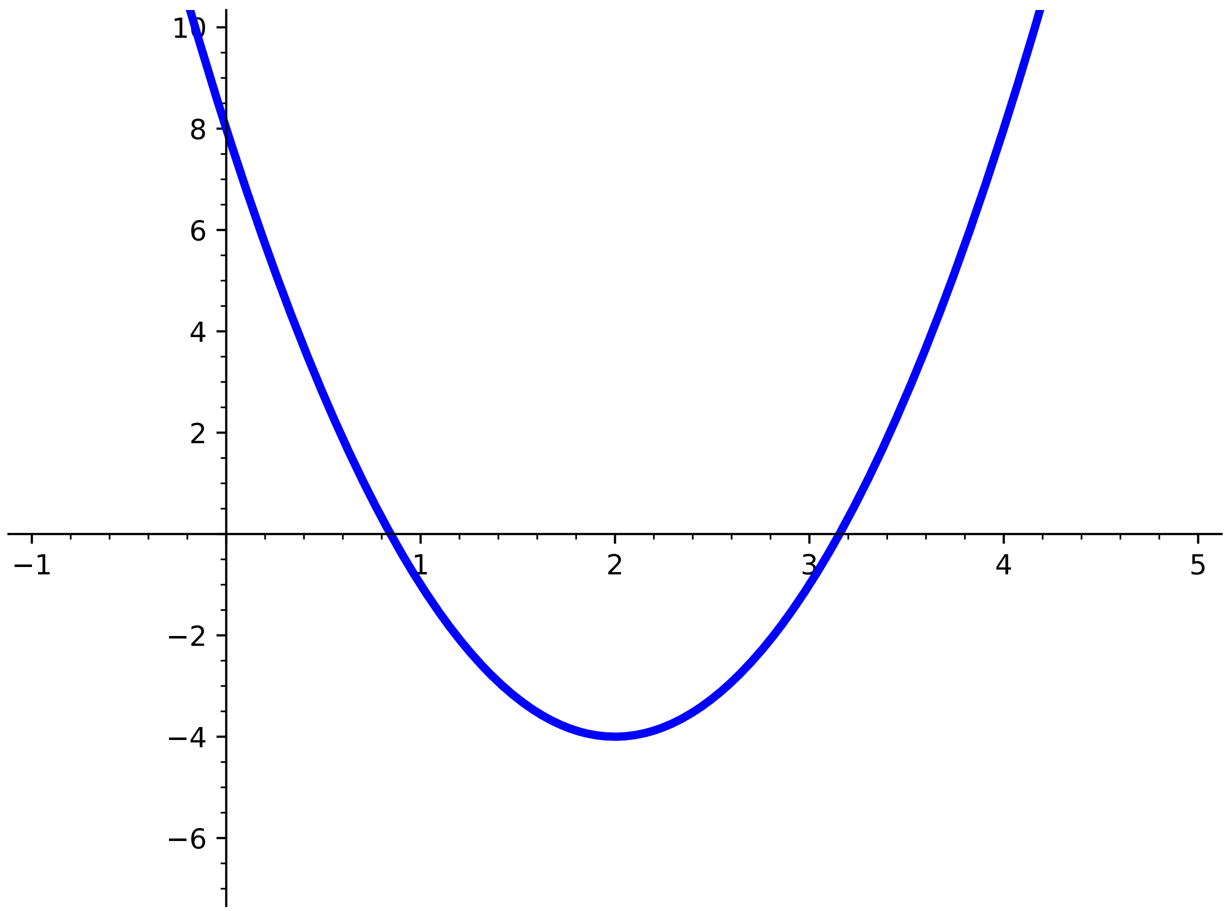


Figure 4.1.3

(a) Make a table for values of $f(x)$ corresponding to the given x-values. What is happening to the y-values as the x-values increase? Do you notice any other patterns of the y-values of the table?

Table 4.1.4

x	$f(x)$
-2	
-1	
0	
1	
2	
3	
4	
5	

(b) At which point (x, y) does $f(x)$ have a minimum value? That is, is there a point on the graph that is lower than all other points?

A. The minimum value appears to occur near $(0, 8)$.
B. The minimum value appears to occur near $(-\frac{1}{5}, 10)$.
C. The minimum value appears to occur near $(2, -4)$.
D. There is no minimum value of this function.

(c) At which point (x, y) does $f(x)$ have a maximum value? That is, is there a point on the graph that is higher than all other points?

A. The maximum value appears to occur near $(-2, 44)$.
B. The maximum value appears to occur near $(-\frac{1}{5}, 10)$.
C. The maximum value appears to occur near $(2, -4)$.
D. There is no maximum value of this function.

Definition 4.1.5 The maximum or minimum of a quadratic function is also known as its **vertex**. The **vertex form** of a quadratic function is given by $f(x) = a(x - h)^2 + k$, where (h, k) is the **vertex** of the parabola.

Vertex form can be used to identify the **axis of symmetry**, also known as the line of symmetry (the line that makes the shape of an object symmetrical). For a quadratic function, the axis of symmetry always passes through the vertex (h, k) and so $x = h$ is the axis of symmetry. $\diamondsuit$

Activity 4.1.6 Use the given quadratic function, $f(x) = 3(x-2)^2 - 4$, to answer the following:

(a) Applying Definition 4.1.5, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -4)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 4)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -4)$; axis of symmetry: $x = -2$

D. vertex: $(2, 4)$; axis of symmetry: $x = 2$

(b) Compare what you got in part a with the values you found in Activity 4.1.2. What do you notice?

Definition 4.1.7 The **standard form** of a quadratic function is given by $f(x) = ax^2 + bx + c$, where a, b, and c are real coefficients.

Just as with the vertex form of a quadratic, we can use the standard form of a quadratic to find the **axis of symmetry** and the **vertex** by using the values of a, b, and c. Given the standard form of a quadratic, the axis of symmetry is $x = \frac{-b}{2a}$ and has a vertex at the point $(\frac{-b}{2a}, f(\frac{-b}{2a}))$. $\Diamond$

Activity 4.1.8 Use the graph of the quadratic function to answer the questions below.

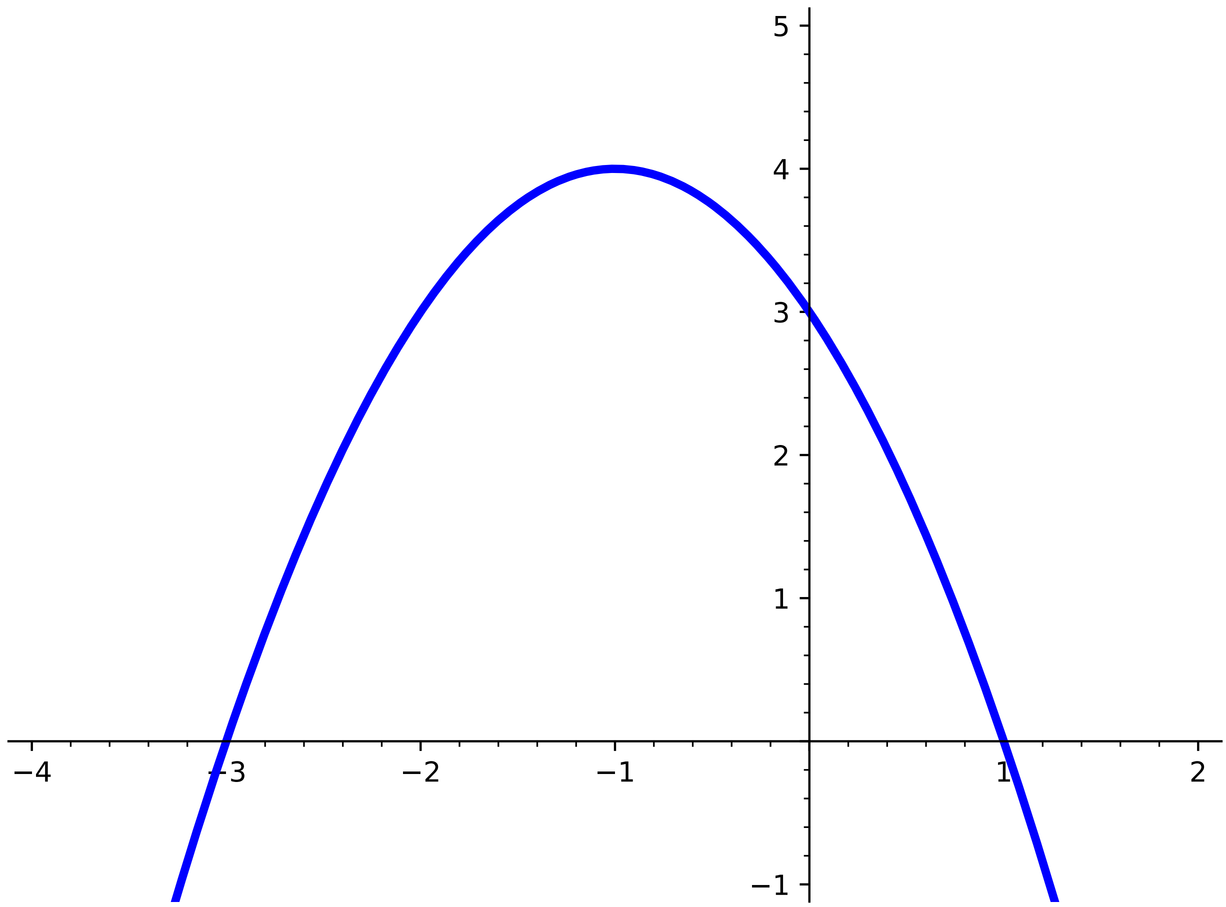


Figure 4.1.9

(a) Which of the following quadratic functions could be the graph shown in the figure?

A. $f(x) = x^2 + 2x + 3$

B. $f(x) = -(x + 1)^2 + 4$

C. $f(x) = -x^2 - 2x + 3$

D. $f(x) = (x + 1)^2 + 4$

(b) What is the maximum or minimum value?

A. -1

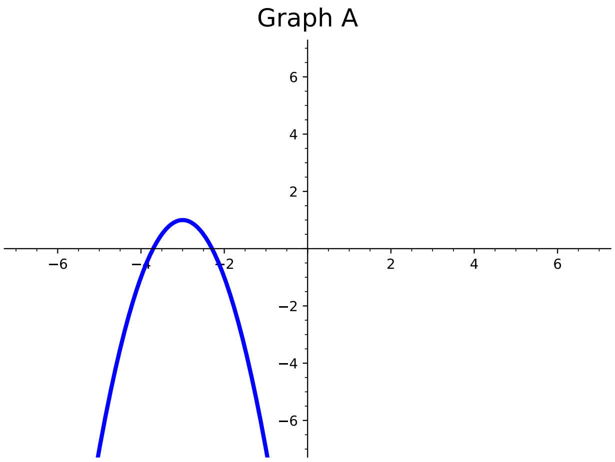
B. 4

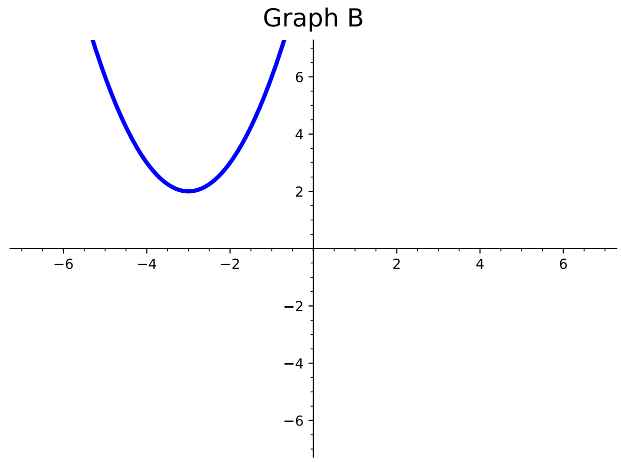
C. -3

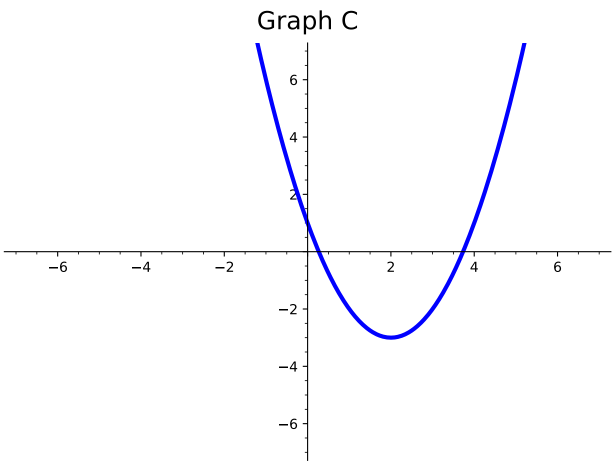
D. 1

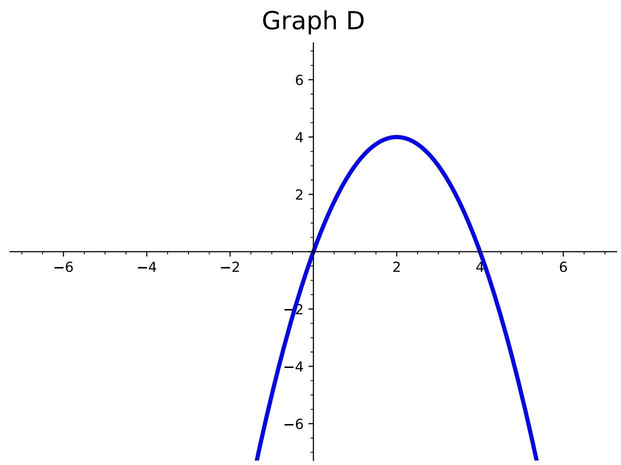
Graphing Quadratic Functions (PR1)

Activity 4.1.10 Consider the following four graphs of quadratic functions:









(a) Which of the graphs above have a maximum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(b) Which of the graphs above have a minimum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(c) Which of the graphs above have an axis of symmetry at $x = 2$?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(d) Which of the graphs above represents the function $f(x) = -(x-2)^2+4$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

(e) Which of the graphs above represents the function $f(x) = x^2 - 4x + 1$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Remark 4.1.11 Notice that the maximum or minimum value of the quadratic function is the y-value of the vertex.

Activity 4.1.12 A function $f(x)$ has a maximum value at 7 and its axis of symmetry at $x = -2$.

(a) Sketch a graph of a function that meets the criteria for $f(x)$.

(b) Was your graph the only possible answer? Try to sketch another graph that meets this criteria.

Remark 4.1.13 Other points, such as x- and y-intercepts, may be helpful in sketching a more accurate graph of a quadratic function.

Graphing Quadratic Functions (PR1)

Activity 4.1.14 Consider the following two quadratic functions $f(x) = x^2 - 4x + 20$ and $g(x) = 2x^2 - 8x + 24$ and answer the following questions:

(a) Applying Definition 4.1.7, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(b) Applying Definition 4.1.7, what is the vertex and axis of symmetry of $g(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(c) What do you notice about $f(x)$ and $g(x)$?

(d) Now graph both $f(x)$ and $g(x)$ and draw a sketch of each graph on one coordinate plane. How are they similar/different?

4.2 Quadratic Models and Meanings (PR2)

Objectives

- Use quadratic models to solve an application problem and establish conclusions.

Activity 4.2.1 A water balloon is tossed vertically from a fifth story window. It's height $h(t)$, in feet, at a time t, in seconds, is modeled by the function

$$h(t) = -16t^2 + 40t + 50$$

(a) Complete the following table. Do all the values have meaning in terms of the model?

Table 4.2.2

t	$h(t)$
0	
1	
2	
3	
4	
5	

(b) Compute the slope of the line joining $t = 0$ and $t = 1$. Then, compute the slope of the line joining $t = 1$ and $t = 2$. What do you notice about the slopes?

(c) What is the meaning of $h(0) = 50$?

A. The initial height of the water balloon is 50 feet.

B. The water balloon reaches a maximum height of 50 feet.

C. The water balloon hits the ground after 50 seconds.

D. The water balloon travels 50 feet before hitting the ground.

(d) Find the vertex of the quadratic function $h(t)$.

A. $(0, 50)$

B. $(1, 74)$

C. $(1.25, 75)$

D. $(3.4, 0)$

(e) What is the meaning of the vertex?

A. The water balloon reaches a maximum height of 50 feet at the start.

B. After 1 second, the water balloon reaches a maximum height of 74 feet.

C. After 1.25 seconds, the water balloon reaches the maximum height.

D. After 3.4 seconds, the water balloon hits the ground.

Activity 4.2.3 The population of a small city is given by the function $P(t) = -50t^2 + 1200t + 32000$, where t is the number of years after 2015.

(a) When will the population of the city reach a maximum?

A. 2020

B. 2022

C. 2025

D. 2027

(b) Determine when the population of the city is increasing and when it is decreasing.

(c) When will the population of the city reach 36,000 people?

A. 2019

B. 2025

C. 2027

D. 2035

Activity 4.2.4 The unit price of an item affects its supply and demand. That is, if the unit price increases, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of \$6. Market research has suggested that if the owners raise the price to \$8, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

(a) Which of the following represents a linear function which relates the price of the streaming service p to the number of subscribers Q?

A. $Q(p) = -2p$

B. $Q(p) = -2p + 84$

C. $Q(p) = -2p - 4$

D. $Q(p) = -2p + 96$

(b) Using the fact that revenue R is price times the number of items sold, $R = pQ$, which of the following represents the revenue in terms of the price?

A. $R(p) = -2p^2$

B. $R(p) = -2p^2 + 84p$

C. $R(p) = -2p^2 - 4p$

D. $R(p) = -2p^2 + 96p$

(c) What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. \$10

B. \$19.50

C. \$24

D. \$28.25

(d) How many subscribers would the company have at this price?

A. 39.5 million

B. 48 million

C. 57 million

D. 76 million

(e) What is the maximum revenue?

A. 760 million

B. 1112 million

C. 1152 million

D. 1116 million

Activity 4.2.5 The owner of a ranch decides to enclose a rectangular region with 240 feet of fencing. To help the fencing cover more land, he plans to use one side of his barn as part of the enclosed region. What is the maximum area the rancher can enclose?

(a) Draw a picture to represent the fenced area against the barn. Use l to represent the length of fence parallel to the barn and w to represent the two sides perpendicular to the barn.

(b) Find an equation for the area of the fence in terms of the length l. It may be useful to find an equation for the total amount of fencing in terms of the length l and width w.

A. $A = lw$

B. $A = l^2$

C. $A = l(240 - l)$

D. $A = l\left(120 - \frac{l}{2}\right)$

(c) Use the area equation to find the maximum area the rancher can enclose.

4.3 Polynomial Long Division (PR3)

Objectives

- Rewrite a rational function as a polynomial plus a proper rational function.

Activity 4.3.1

(a) Use long division to find the quotient and remainder when 346 is divided by 17.

A. quotient 20, remainder 3

B. quotient 6, remainder 3

C. quotient 6, remainder 20

D. quotient 20, remainder 6

(b) What is the divisor?

A. 3

B. 17

C. 20

D. 6

(c) Now write the answer as a mixed number.

A. $20 + \frac{3}{17}$

B. $6 + \frac{20}{17}$

C. $20 + \frac{6}{17}$

D. $6 + \frac{3}{17}$

(d) How can you check your answer?

Activity 4.3.2

(a) Simplify the following rational expression using factoring. $f(x) = \frac{3x^2+11x-4}{x+4}$

A. $x + 4$

B. $3x - 1$

C. $x - 4$

D. $3x + 1$

Polynomial Long Division (PR3)

Activity 4.3.3 Now we will simplify using long division, given $f(x) = \frac{3x^2+11x-4}{x+4}$.

(a) What is the result when $3x^2$ is divided by x? Place this number at the top left when doing long division.

A. 3

B. $3x$

C. $3x^3$

(b) Next, multiply this result by $x + 4$ and write this below $3x^2 + 11x - 4$. Be sure to line up like terms in the same column. What is the result?

A. $3x^2 + 12x$

B. $-3x^2 - 12x$

C. $3x^3 + 12x$

(c) Now subtract the like terms by placing them in the corresponding columns.

A. $-3x^2 - x - 4$

B. $x + 4$

C. $23x - 4$

D. $-x - 4$

(d) Next, divide the first term in the previous result by x. Write the answer on the top beside the first answer. What is the expression at the top now?

A. $3x^2 - 1$

B. $-3x + 1$

C. $3x - 1$

D. $3x + 1$

(e) Multiply this resulting term by $x + 4$ and write that answer in the proper columns of like terms. What is this result?

A. $-x - 4$

B. $x + 4$

(f) Subtract the like terms. This is the remainder. What is the remainder?

A. $2x + 8$
B. $-x - 4$
C. 0

(g) What is the quotient?

A. $3x^2 + 11x - 4$
B. $3x - 1$
C. 0
D. $x + 4$

(h) How can you check your answer? (Hint: Think of regular long division with positive integers.)

(i) How does your answer for the quotient and remainder compare to the simplifying by factoring method?

Polynomial Long Division (PR3)

Activity 4.3.4 Using long division, find the quotient and remainder for the given rational function. If there are any missing terms, use 0 as a place holder. Rewrite the function as a polynomial plus a proper rational function, given $f(x) = \frac{3x^5 - 5x^2 + 2}{x^2 + x - 1}$.

(a) What is the quotient?

(b) What is the remainder?

(c) What is the divisor?

(d) Write the rational function as a polynomial plus a proper rational function.

(e) How can you check your answer? (Hint: Think of regular long division with positive integers.)

Activity 4.3.5 Using long division, find the quotient and remainder for the given rational function, $f(x) = \frac{3x^4 - 5x^2 + 2}{x - 1}$.

- **(a)** What is the quotient?
- **(b)** What is the remainder?
- **(c)** How can you check your answer? (Hint: Think of regular long division with positive integers.)

Activity 4.3.6 Using synthetic division, find the quotient and remainder for the given rational function. $(x) = \frac{3x^2+11x-4}{x+4}$

(a) Write only the coefficients of the numerator in a row.

(b) Set the denominator equal to zero and solve for x. Place this number out in front, in the far left corner.

(c) Make 3 rows and copy the 1st coefficient from the top row to the bottom row.

(d) Now multiply the number in the top left corner by the first coefficient and place this number under the 2nd coefficient.

(e) Add the numbers in the 2nd column and write the result in the same column, third row.

(f) Repeat the process with the resulting number. Multiply this number by the number in the upper left corner. Place it in the 2nd row, third column. Add the numbers in the third column and place this in the third row and column. What number did you get?

A. 8

B. -1

C. 0

D. 148

How do the numbers in the third row relate to Activity 4.3.2 and Activity 4.3.3?

Activity 4.3.7 Now use synthetic division, to find the quotient and remainder for the given rational function. $f(x) = \frac{x^3+2x^2-3x+4}{x-2}$

(a) What is the quotient?

A. $x^2 - 3$

B. $x^3 + 2x - 3x + 4$

C. $x^2 + 4x + 5$

D. 5

(b) What is the remainder?

A. 14

B. -10

C. 5

D. 6

4.4 Zeros of Polynomial Functions (PR4)

Objectives

- Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.

Activity 4.4.1

Remark 4.4.2 Recall that to find the x intercepts of a linear or quadratic function, let $y = 0$ and solve for x.

(a) What is the factored form of the function, $f(x) = x^2 + 2x - 15$?

A. 0

B. $(x - 3)(x + 5)$

C. $(x + 3)(x - 5)$

D. $(x - 3)(x - 5)$

E. $(x + 3)(x + 5)$

(b) What are the x intercepts of the function above?

A. 0

B. -3 and 5

C. 3 and -5

D. -3 and -5

E. 3 and 5

(c) How do the x intercepts relate to the factored form of the function?

Definition 4.4.3 Real zeros of a polynomial function are the same as the x intercepts. $\Diamond$

Activity 4.4.4 Find the zeros of the polynomial by factoring $f(x) = x^3 - x^2 - 56x$.

A. 0

B. -7

C. 8

D. $-7, 8$

E. $0, -7, 8$

F. $7, -8$

Activity 4.4.5

(a) Given $x-3$ is a factor of the polynomial $f(x)=2x^3-7x^2-33x+108$, find the remaining factors using division.

A. $(x+4),(2x-9)$

B. $(x-4)(2x+9))$

C. $(2x+3)(x-9))$

D. $(2x-3)(x+9))$

(b) How many zeros does the function have?

A. 1

B. 2

C. 3

D. 4

Definition 4.4.6 The **multiplicity** of a zero is the number of times the corresponding linear factor appears in the factored form of the polynomial function. $\Diamond$

Definition 4.4.7 The **degree** of a polynomial function is the sum of the multiplicities of the zeros. $\Diamond$

Activity 4.4.8 Given the function $5y - 4 = x$, what is the degree of the function?

A. 0

B. 1

C. 5

Activity 4.4.9

(a) Given $y = 5x^2 + 7x - 6$, what is the degree of the function?

A. 0

B. 1

C. 2

D. 5

(b) Find the zeros of the quadratic function above. How many are there?

A. 0

B. 1

C. 2

D. 5

Activity 4.4.10

(a) Given the polynomial, $f(x) = (3x-2)(x+1)^2(x-6)^3$, find all the zeros of the function with their corresponding multiplicities. What is the degree of the function?

A. 3

B. 5

C. 6

Activity 4.4.11

(a) Given $f(x) = x^4 - 5x^3 + x^2 + 21x - 18$, find all the zeros of the function with their corresponding multiplicities. What is the degree of the function?

A. 5

B. 4

C. 3

(b) How many zeros are there in total?

A. 5

B. 4

C. 3

Activity 4.4.12

(a) Given -1 is a zero with multiplicity 2, 4 is a zero with multiplicity 1 and 7 is a zero with multiplicity 3, write a polynomial function in factored form.

Activity 4.4.13

(a) Find the complex zeros for the function, $f(x) = x^2 + 16$.

Activity 4.4.14 Write the polynomial function in factored form using information from the graph below.

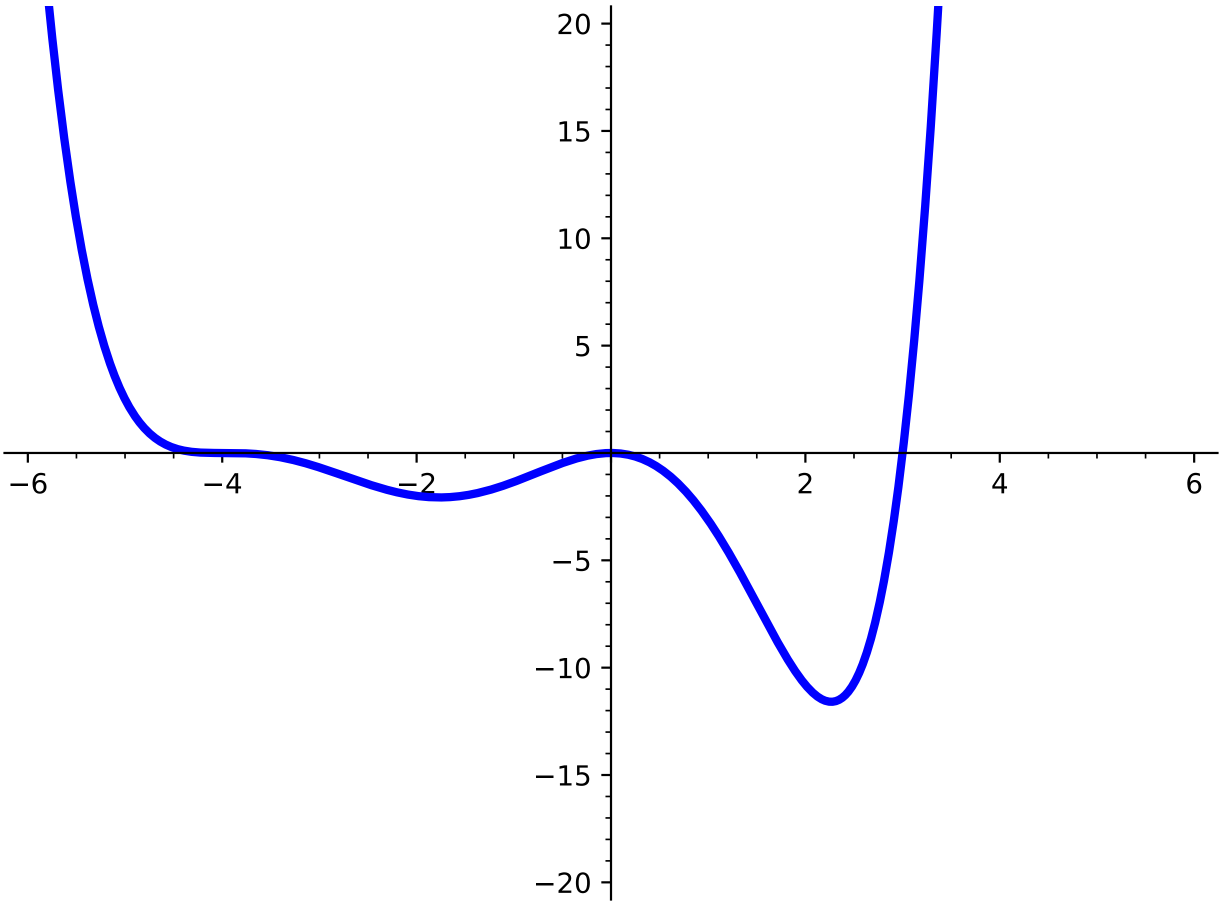


Figure 4.4.15

(a) Using the given graph, what are the real zeros of this function? Select all that apply.

A. 0

B. 1

C. -3

D. 3

E. 4

F. -4

(b) What are the least possible multiplicities for each zero?

(c) What is the least degree of the function?

A. 3

B. 4

C. 5

D. 6

(d) Write the function $f(x)$ in factored form using linear factors.

Activity 4.4.16 Given the function $f(x) = 2x^4 - 4x^3 + 10x^2 - 16x + 8$,

(a) Find all the zeros and their corresponding multiplicities.

(b) Write a function for the graph $f(x)$ in factored form using linear factors.

Activity 4.4.17 The zeros of a function are $x = 2$, with multiplicity 1, $x = -1$, with multiplicity 2 and $x = i$.

(a) Given the information above, find a polynomial function with real coefficients of least degree.

4.5 Graphs of Polynomial Functions (PR5)

Objectives

- Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.

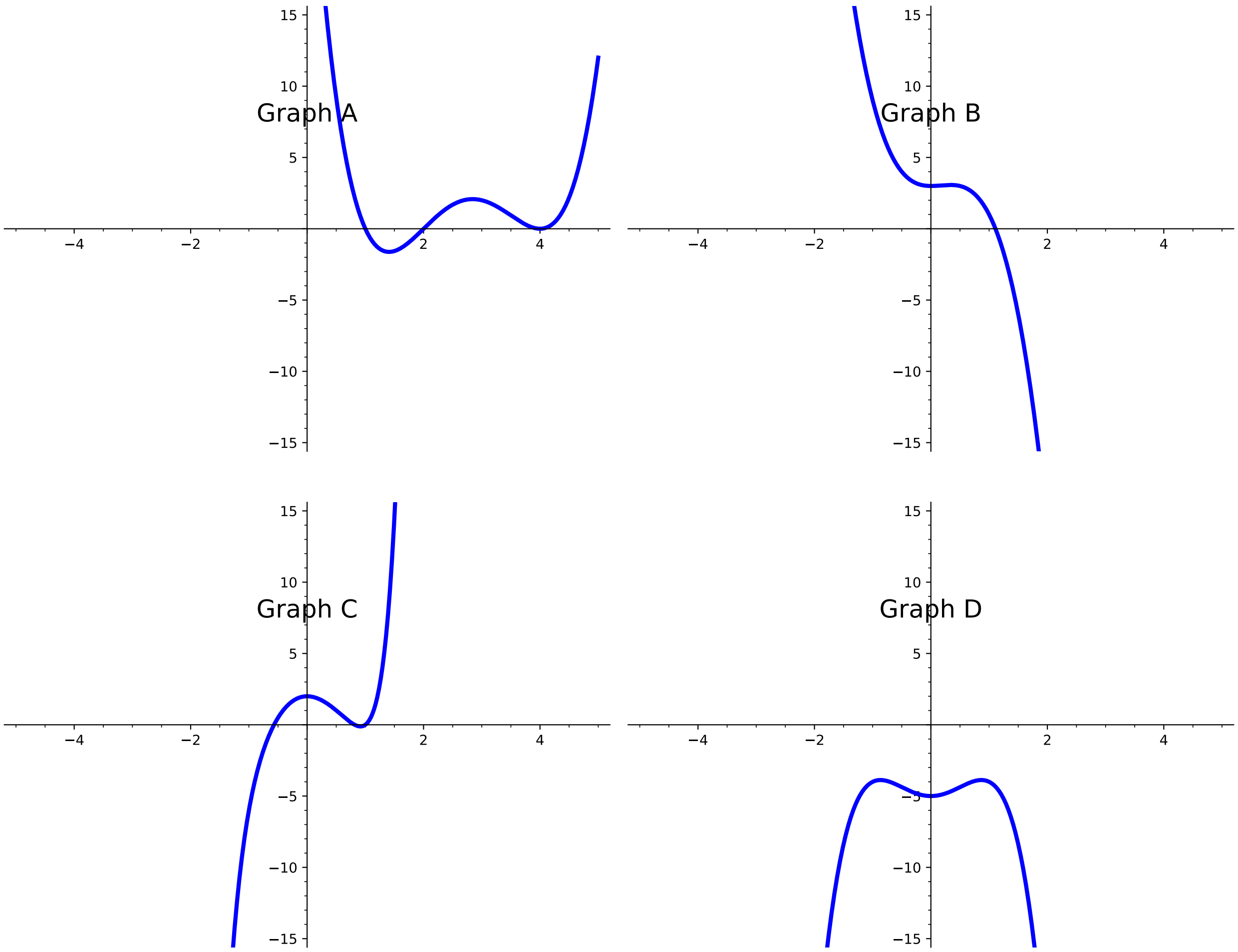
Remark 4.5.1 Just like with quadratic functions, we should be able to determine key characteristics that will help guide us in creating a sketch of any polynomial function. We can start by finding both x and y-intercepts and then explore other characteristics polynomial functions can have. Recall that the **zeros** of a function are the x**-intercepts** - i.e., the values of x that cross or touch the x-axis. Just like with quadratic functions, we can find the zeros of a function by setting the function equal to 0 and solving for x.

Definition 4.5.2 The **end behavior** of a polynomial function describes the behavior of the graph at the "ends" of the function. In other words, as we move to the right of the graph (as the x values increase), what happens to the y values? Similarly, as we move to the left of the graph (as the x values decrease), what happens to the y values?

Although we are looking at the "ends" to determine the end behavior, note that polynomials do not actually have ends. In other words, polynomial functions have y-values that exist for every x-value.

$\Diamond$

Activity 4.5.3 Use the graphs of the following polynomial functions to answer the questions below.



(a) How would you describe the behavior of Graph A as you approach the ends?

A. Graph A rises on the left and on the right.

B. Graph A rises on the left, but falls on the right.

C. Graph A rises on the right, but falls on the left.

D. Graph A falls on the left and on the right.

(b) How would you describe the behavior of Graph B as you approach the ends?

A. Graph B rises on the left and on the right.

B. Graph B rises on the left, but falls on the right.

C. Graph B rises on the right, but falls on the left.

D. Graph B falls on the left and on the right.

(c) How would you describe the behavior of Graph C as you approach the ends?

A. Graph C rises on the left and on the right.

B. Graph C rises on the left, but falls on the right.

C. Graph C rises on the right, but falls on the left.

D. Graph C falls on the left and on the right.

(d) How would you describe the behavior of Graph D as you approach the ends?

A. Graph D rises on the left and on the right.

B. Graph D rises on the left, but falls on the right.

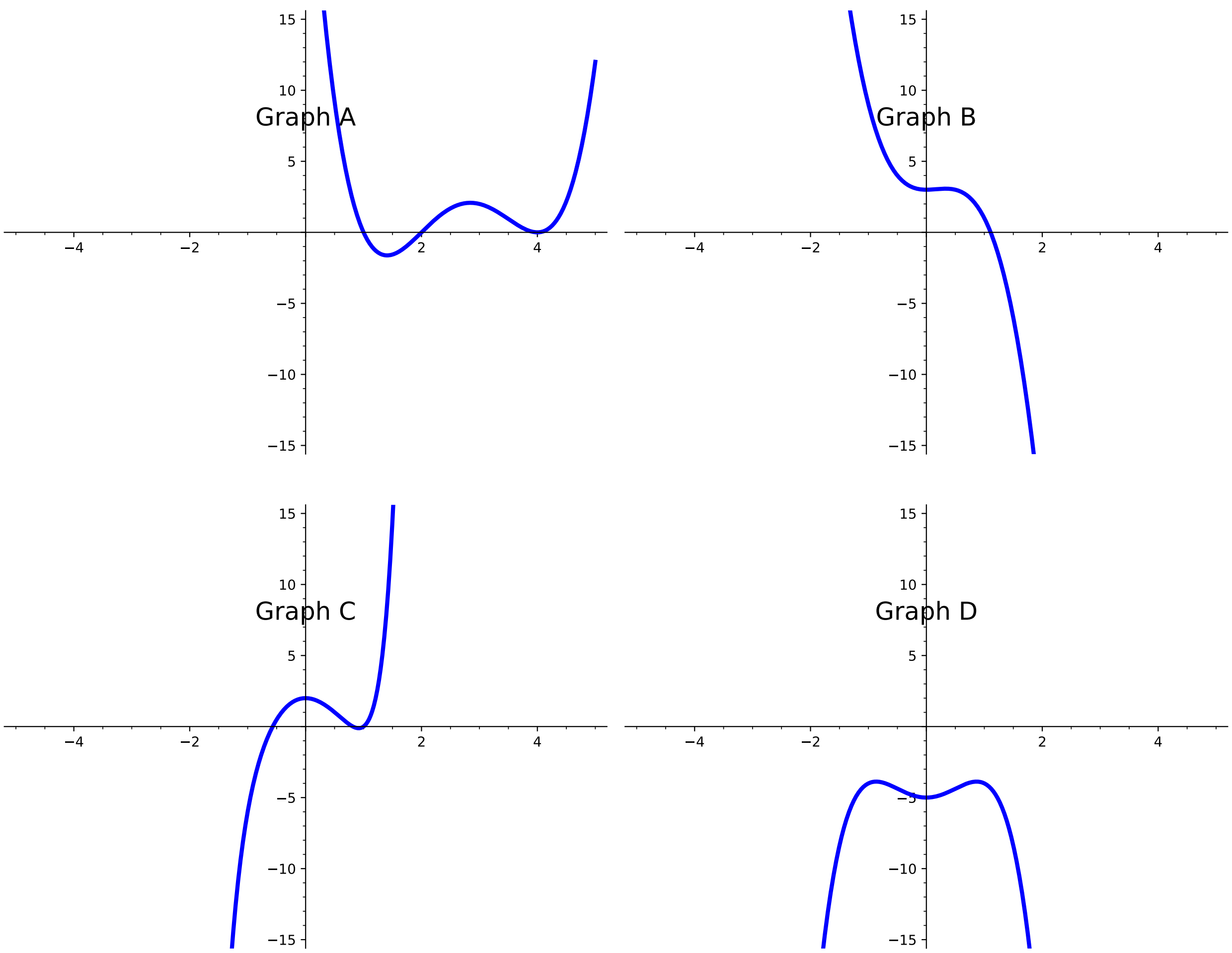
C. Graph D rises on the right, but falls on the left.

D. Graph D falls on the left and on the right.

Definition 4.5.4 Typically, when given an equation of a polynomial function, we look at the **degree** and **leading coefficient** to help us determine the behavior of the ends. The **degree** is the highest exponential power in the polynomial. The **leading coefficient** is the number written in front of the variable with the highest exponential power. $\Diamond$

Activity 4.5.5 Let's refer back to the graphs in Activity 4.5.3 and look at the equations of those polynomial functions. Let's apply Definition 4.5.4 to see if we can determine how the degree and leading coefficients of those graphs affect their end behavior.

- Graph A: $f(x) = -11x^3 + 32 + 42x^2 + x^4 - 64x$
- Graph B: $g(x) = 2x^2 + 3 - 4x^3$
- Graph C: $h(x) = x^7 + 2x^3 - 5x^2 + 2$
- Graph D: $j(x) = 3x^2 - 2x^4 - 5$



(a) What is the degree and leading coefficient of Graph A?

A. Degree: -64; Leading Coefficient: 4

B. Degree: 4; Leading Coefficient: 0

C. Degree: 1; Leading Coefficient: -64

D. Degree: 4; Leading Coefficient: 1

(b) What is the degree and leading coefficient of Graph B?

A. Degree: 3; Leading Coefficient: -4
B. Degree: -4; Leading Coefficient: 3
C. Degree: 2; Leading Coefficient: 3
D. Degree: 3; Leading Coefficient: 4

(c) What is the degree and leading coefficient of Graph C?

A. Degree: -5; Leading Coefficient: 2
B. Degree: 0; Leading Coefficient: 7
C. Degree: -5; Leading Coefficient: 3
D. Degree: 7; Leading Coefficient: 1

(d) What is the degree and leading coefficient of Graph D?

A. Degree: -2; Leading Coefficient: 4
B. Degree: 3; Leading Coefficient: 2
C. Degree: -2; Leading Coefficient: 4
D. Degree: -5; Leading Coefficient: 4

(e) Notice that Graph A and Graph D have their ends going in the same direction. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

(f) Notice that Graph B and Graph C have their ends going in opposite directions. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

Remark 4.5.6 From Activity 4.5.5, we saw that the degree and leading coefficient of a polynomial function can give us more clues about the behavior of the function. In summary, we know:

- If the degree is even, the ends of the polynomial function will be going in the same direction. If the leading coefficient is positive, both ends will be pointing up. If the leading coefficient is negative, both ends will be pointing down.
- If the degree is odd, the ends of the polynomial function will be going in opposite directions. If the leading coefficient is positive, the left end will fall and the right end will rise. If the leading coefficient is negative, the left end will rise and the right end will fall.

Definition 4.5.7 When describing end behavior, mathematicians typically use **arrow notation**. Just as the name suggests, arrows are used to indicate the behavior of certain values on a graph.

For end behavior, students are often asked to determine the behavior of y-values as x-values either increase or decrease. The statement "As $x \to \infty$, $f(x) \to -\infty$" can be translated to "As x approaches infinity (or as x increases), $f(x)$ (or the y-values) go to negative infinity (i.e., it decreases)."

$\Diamond$

Activity 4.5.8 Use the graph of $f(x)$ to answer the questions below.

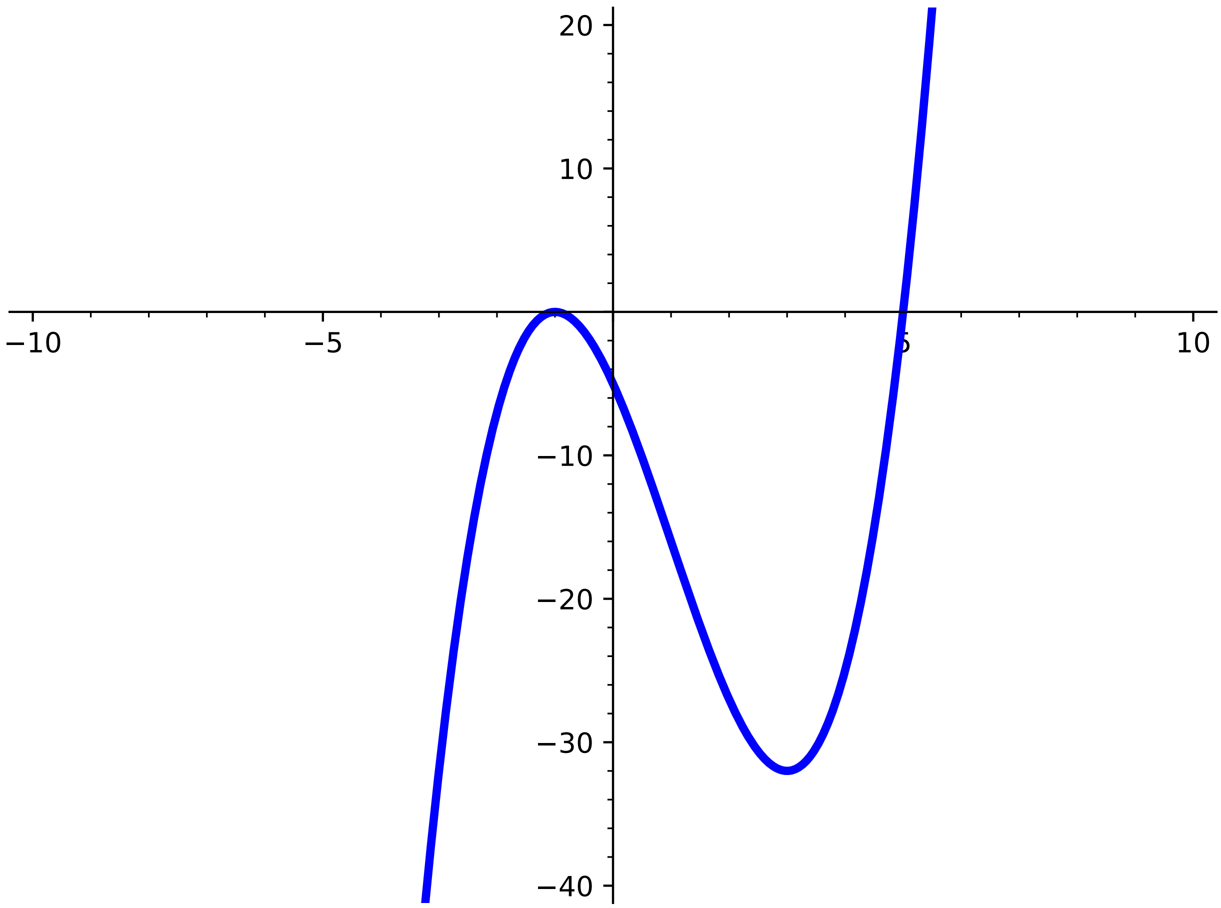


Figure 4.5.9

(a) How would you describe the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.

B. $f(x)$ rises on the left, but falls on the right.

C. $f(x)$ rises on the right, but falls on the left.

D. $f(x)$ falls on the left and on the right.

(b) How would you describe the end behavior of $f(x)$ using arrow notation?

A. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

B. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$

C. As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$

D. As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to \infty$

Definition 4.5.10 When graphing polynomial functions, you may notice that these functions have some ”hills” and ”valleys.” These characteristics of the graph are known as the **local maxima** and **local minima** of the graph - similar to what we’ve already seen with quadratic functions. Unlike quadratic functions, however, a polynomial graph can have many local maxima/minima (quadratic functions only have one). $\Diamond$

Activity 4.5.11 Sketch the function, $f(x) = (x-2)(x+1)(x-3)^2$, by first finding the given characteristics.

(a) Find the zeros of $f(x)$.

(b) Find the multiplicities at each zero.

(c) Find the y-intercept of $f(x)$.

(d) Describe the end behavior of $f(x)$.

(e) Estimate where any local maximums and minimums may occur.

Activity 4.5.12 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

- The x-intercepts of $f(x)$ are $0, 2$, and 5.
- $f(x)$ has one maximum at 0. $f(x)$ has one minimum at -5 and another at -16.
- The end behavior of $f(x)$ is given as: As $x \to \infty$, $f(x) \to \infty$ and As $x \to -\infty$, $f(x) \to -\infty$

Activity 4.5.13 Now that we know all the different characterisitics of polynomials, we should also be able to identify them from a graph. Use the graph below to find the given charactertistics.

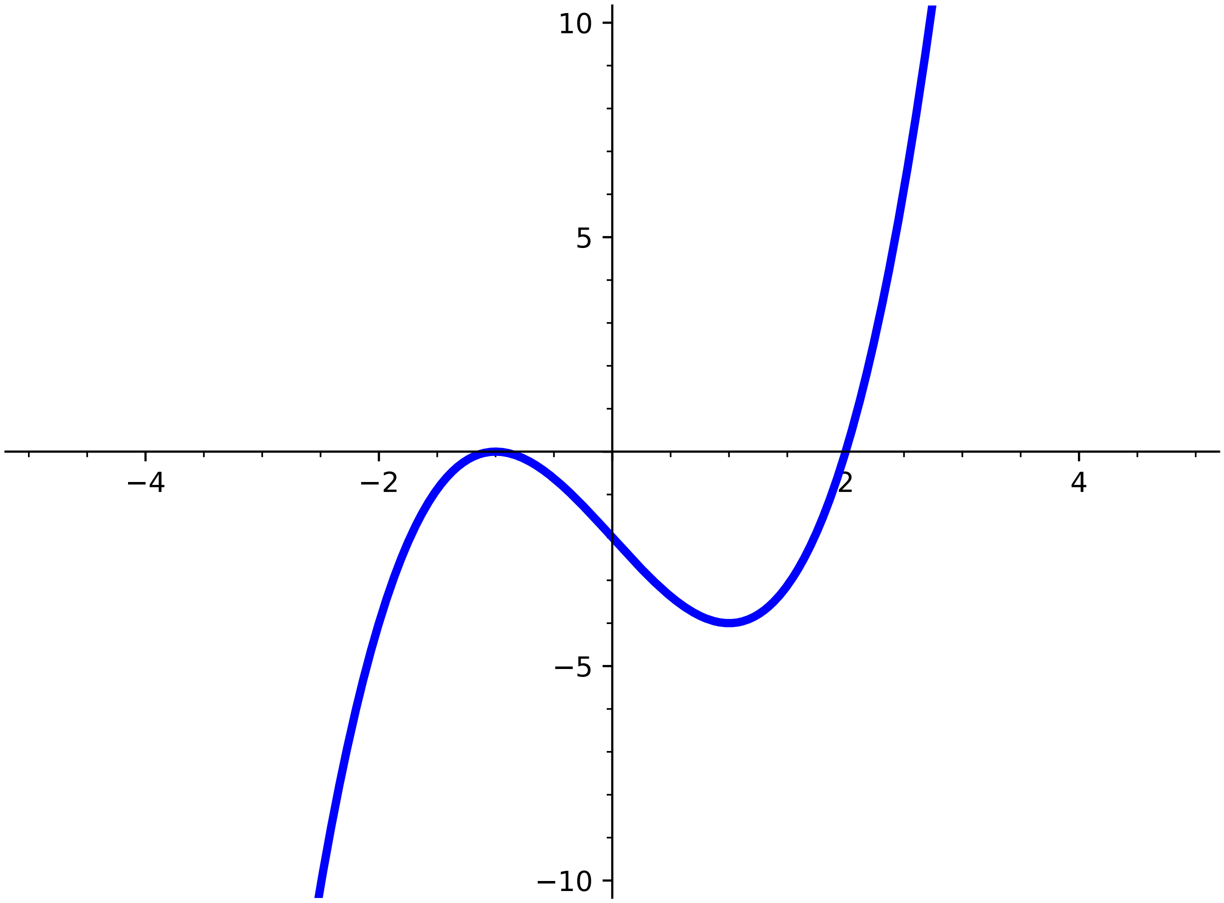


Figure 4.5.14

(a) What are the x-intercept(s) of the polynomial function? Select all that apply.

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(b) What are the y-intercept(s) of the polynomial function?

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(c) How many zeros does this polynomial function have?

A. 0
B. 1
C. 2
D. 3

(d) At what point is the local minimum located?

A. $(2,-4)$
B. $(-1,0)$
C. $(-2,0)$
D. $(1,-4)$
E. $(2,0)$

(e) At what point is the local maximum located?

A. $(2,-4)$
B. $(-1,0)$
C. $(-2,0)$
D. $(1,-4)$
E. $(2,0)$

(f) How do you describe the behavior of the polynomial function as $x \to \infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

(g) How do you describe the behavior of the polynomial function as $x \to -\infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

Activity 4.5.15 Use the given function, $f(x) = (x+1)^2(x-5)$, to answer the following questions.

(a) What are the zeros of $f(x)$?

A. $-1, -5$

B. $-1, 5$

C. $1, -5$

D. $1, 5$

(b) What are the multiplicities at each zero?

A. At $x = -1$, the mulitplicity is even.
At $x = 5$, the mulitplicity is even.

B. At $x = -1$, the mulitplicity is even.
At $x = 5$, the mulitplicity is odd.

C. At $x = -1$, the mulitplicity is odd.
At $x = 5$, the mulitplicity is even.

D. At $x = -1$, the mulitplicity is odd.
At $x = 5$, the mulitplicity is odd.

(c) What is the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.

B. $f(x)$ rises on the left, but falls on the right.

C. $f(x)$ rises on the right, but falls on the left.

D. $f(x)$ falls on the left and on the right.

(d) Using what you know about the zeros, multiplicities, and end behavior, where on the graph can we estimate the local maxima and minima to be?

(e) Now look at the graph of $f(x)$. At which zero does a local maximum or local minimum occur? Explain how you know.

Remark 4.5.16 We can estimate where these local maxima and minima occur by looking at other characteristics, such as multiplicities and end behavior.

From Activity 4.5.15, we saw that when the function touches the x-axis at a zero, then that zero could be either a local maximum or a local minimum of the graph. When the function crosses the x-axis, however, the local maximum or local minimum occurs between the zeros.

4.6 Properties and Graphs of Rational Functions (PR6)

Objectives

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

Properties and Graphs of Rational Functions (PR6)

Definition 4.6.1 A function r is **rational** provided that it is possible to write r as the ratio of two polynomials, p and q. That is, r is rational provided that for some polynomial functions p and q, we have

$$r(x) = \frac{p(x)}{q(x)}.$$

$\Diamond$

Observation 4.6.2 Rational functions occur in many applications, so our goal in this lesson is to learn about their properties and be able to graph them. In particular we want to investigate the domain, end behavior, and zeros of rational functions.

Activity 4.6.3 Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Find $r(1)$, $r(2)$, $r(3)$, and $r(4)$.

(b) Label each of these four points as giving us information about the DOMAIN of $r(x)$, information about the ZEROES of $r(x)$, or NEITHER.

Properties and Graphs of Rational Functions (PR6)

Definition 4.6.4 Let p and q be polynomial functions so that $r(x) = \frac{p(x)}{q(x)}$ is a rational function. The **domain** of r is the set of all real numbers except those for which $q(x) = 0$. $\Diamond$

Activity 4.6.5 Let's investigate the domain of $r(x)$ more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Rewrite $r(x)$ by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?

(b) The function was not defined for $x = 3$. Make a table for values of $r(x)$ near $x = 3$.

Table 4.6.6

x	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

(c) Which of the following describe the behavior of the graph near $x = 3$?

A. As $x \to 3$, $r(x)$ approaches a finite number

B. As $x \to 3$ from the left, $r(x) \to \infty$

C. As $x \to 3$ from the left, $r(x) \to -\infty$

D. As $x \to 3$ from the right, $r(x) \to \infty$

E. As $x \to 3$ from the right, $r(x) \to -\infty$

(d) The function was also not defined for $x = 1$. Make a table for values of $r(x)$ near $x = 1$.

Table 4.6.7

x	$r(x)$
0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

(e) Which of the following describe the behavior of the graph near $x = 1$?

A. As $x \to 1$, $r(x)$ approaches a finite number

B. As $x \to 1$ from the left, $r(x) \to \infty$

C. As $x \to 1$ from the left, $r(x) \to -\infty$

D. As $x \to 1$ from the right, $r(x) \to \infty$

E. As $x \to 1$ from the right, $r(x) \to -\infty$

(f) The function is behaving differently near $x = 1$ than it is near $x = 3$. Can you see anything in the factored form of $r(x)$ that may help you account for the difference?

Remark 4.6.8 Features of a rational function. Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function.

- If $p(a) = 0$ and $q(a) \neq 0$, then $r(a) = 0$, so r has a **zero** at $x = a$.
- If $q(a) = 0$ and $p(a) \neq 0$, then $r(a)$ is undefined and r has a **vertical asymptote** at $x = a$.
- If $p(a) = 0$ and $q(a) = 0$ and we can show that there is a finite number L such that $r(x) \to L$, then $r(a)$ is not defined and r has a **hole** at the point (a, L).

Activity 4.6.9 Another property of rational functions we want to explore is the end behavior. This means we want to explore what happens to a given rational function $r(x)$ when x goes toward positive infinity or negative infinity.

(a) Consider the rational function $r(x) = \dfrac{1}{x^3}$. Plug in some very large positive numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches positive infinity?

A. As $x \to \infty$, $r(x) \to \infty$.

B. As $x \to \infty$, $r(x) \to -\infty$.

C. As $x \to \infty$, $r(x) \to 0$.

D. As $x \to \infty$, $r(x) \to 1$.

(b) Now let's look at $r(x)$ as x tends toward negative infinity. Plug in some very large negative numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches negative infinity?

A. As $x \to -\infty$, $r(x) \to \infty$.

B. As $x \to -\infty$, $r(x) \to -\infty$.

C. As $x \to -\infty$, $r(x) \to 0$.

D. As $x \to -\infty$, $r(x) \to 1$.

Observation 4.6.10 We can generalize what we have just found to any function of the form $\frac{1}{x^n}$, where $n > 0$. Since x^n increases without bound as $x \to \infty$, we find that $\frac{1}{x^n}$ will tend to 0. In fact, the numerator can be any constant and the function will still tend to 0!

Similarly, as $x \to -\infty$, we find that $\frac{1}{x^n}$ will tend to 0 too.

Activity 4.6.11 Consider the rational function $r(x) = \frac{3x^2-5x+1}{7x^2+2x-11}$.

Observe that the largest power of x that's present in $r(x)$ is x^2. In addition, because of the dominant terms of $3x^2$ in the numerator and $7x^2$ in the denominator, both the numerator and denominator of r increase without bound as x increases without bound.

(a) In order to understand the end behavior of r, we will start by writing the function in a different algebraic form.

Multiply the numerator and denominator of r by $\frac{1}{x^2}$. Then distribute and simplify as much as possible in both the numerator and denominator to write r in a different algebraic form. Which of the following is that new form?

A. $\dfrac{3x^4 - 5x^3 + x^2}{7x^4 + 2x^3 - 11x^2}$

B. $\dfrac{3 - \frac{5}{x} + \frac{1}{x^2}}{7 + \frac{2}{x} - \frac{11}{x^2}}$

C. $\dfrac{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}{\frac{7x^2}{x^2} + \frac{2x}{x^2} - \frac{11}{x^2}}$

D. another wrong answer?

(b) Now determine the end behavior of each piece of the numerator and each piece of the denominator. Hint: Use Observation 4.6.10 to help!

(c) Simplify your work from the previous step. Which of the following best describes the end behavior of $r(x)$?

A. As $x \to \pm\infty$, $r(x)$ goes to 0.

B. As $x \to \pm\infty$, $r(x)$ goes to $\frac{3}{7}$.

C. As $x \to \pm\infty$, $r(x)$ goes to ∞.

D. As $x \to \pm\infty$, $r(x)$ goes to $-\infty$.

Observation 4.6.12 If the end behavior of a function tends toward a specific value a, then we say that the function has a **horizontal asymptote** at $y = a$.

Activity 4.6.13 Find the horizontal asymptote (if one exists) of the following rational functions. Follow the same method we used in Activity 4.6.11.

(a) $f(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(b) $g(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^5 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(c) $h(x) = \dfrac{4x^5 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

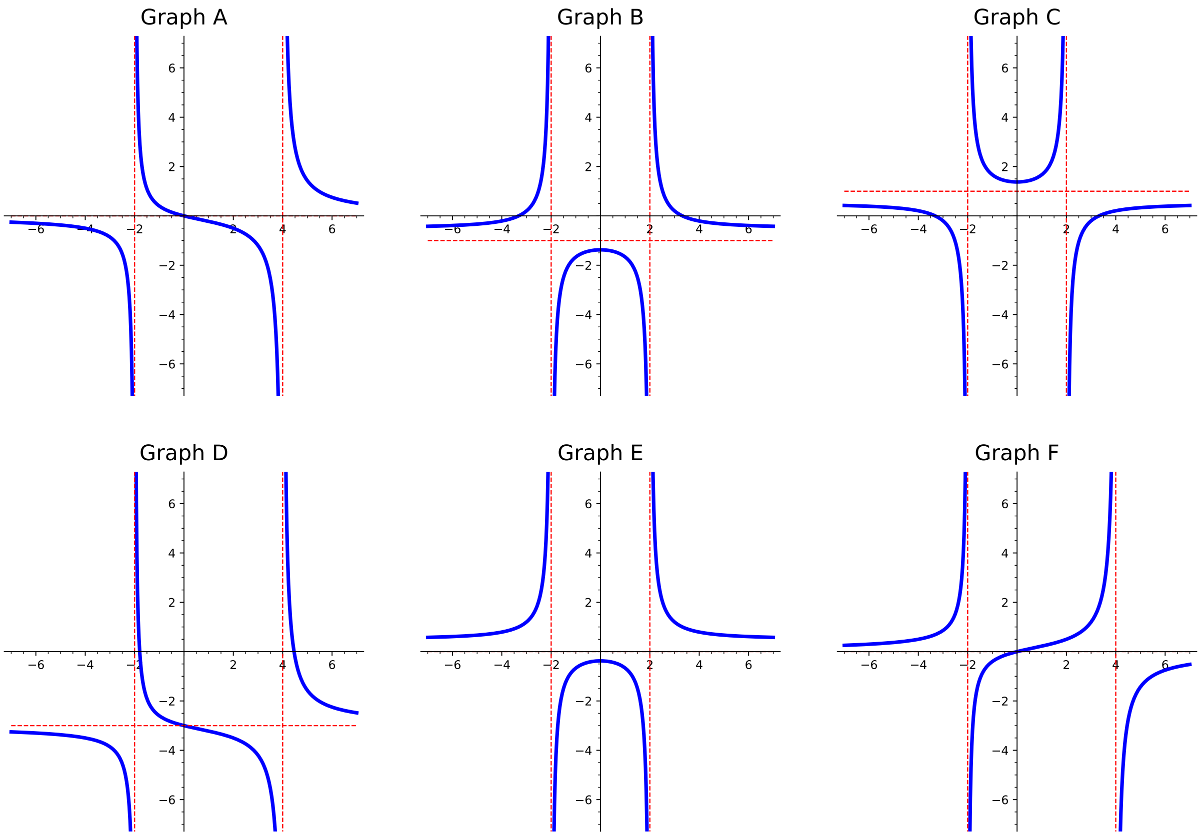
Activity 4.6.14 Some patterns have emerged from the previous problem. Fill in the rest of the sentences below to describe how to find horizontal asymptotes of rational functions.

(a) If the degree of the numerator is the same as the degree of the denominator, then...

(b) If the degree of the numerator is less than the degree of the denominator, then...

(c) If the degree of the numerator is greater than the degree of the denominator, then...

Activity 4.6.15 Consider the following six graphs of rational functions:



(a) Which of the graphs above represents the function $f(x) = \frac{2x}{x^2-2x-8}$?

(b) Which of the graphs above represents the function $g(x) = \frac{x^2+3}{2x^2-8}$?

Activity 4.6.16 MATCHING ACTIVITY INCLUDING GRAPHS WITH HOLES

(a) QUESTIONS HERE!

Activity 4.6.17

(a) Find the roots of the rational function $f(x) = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(b) Find the y-intercept of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(c) Find any horizontal asymptotes of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(d) Find any vertical asymptotes of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(e) Find any holes of $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(f) Sketch $y = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

Activity 4.6.18 EXTENSION ACTIVITY ABOUT HOLES

(a) QUESTIONS HERE!

Colophon

This book was authored in PreTeXt.