

# Precalculus for Team-Based Inquiry Learning

2024 Development Edition

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## Back Matter

# Chapter 1

## Polynomial and Rational Functions (PR)

### Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx

By the end of this chapter, you should be able to...

1. Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.
2. Use quadratic models to solve an application problem and establish conclusions.
3. Rewrite a rational function as a polynomial plus a proper rational function.
4. Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.
5. Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.
6. Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

## 1.1 Graphing Quadratic Functions (PR1)

### Objectives

- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

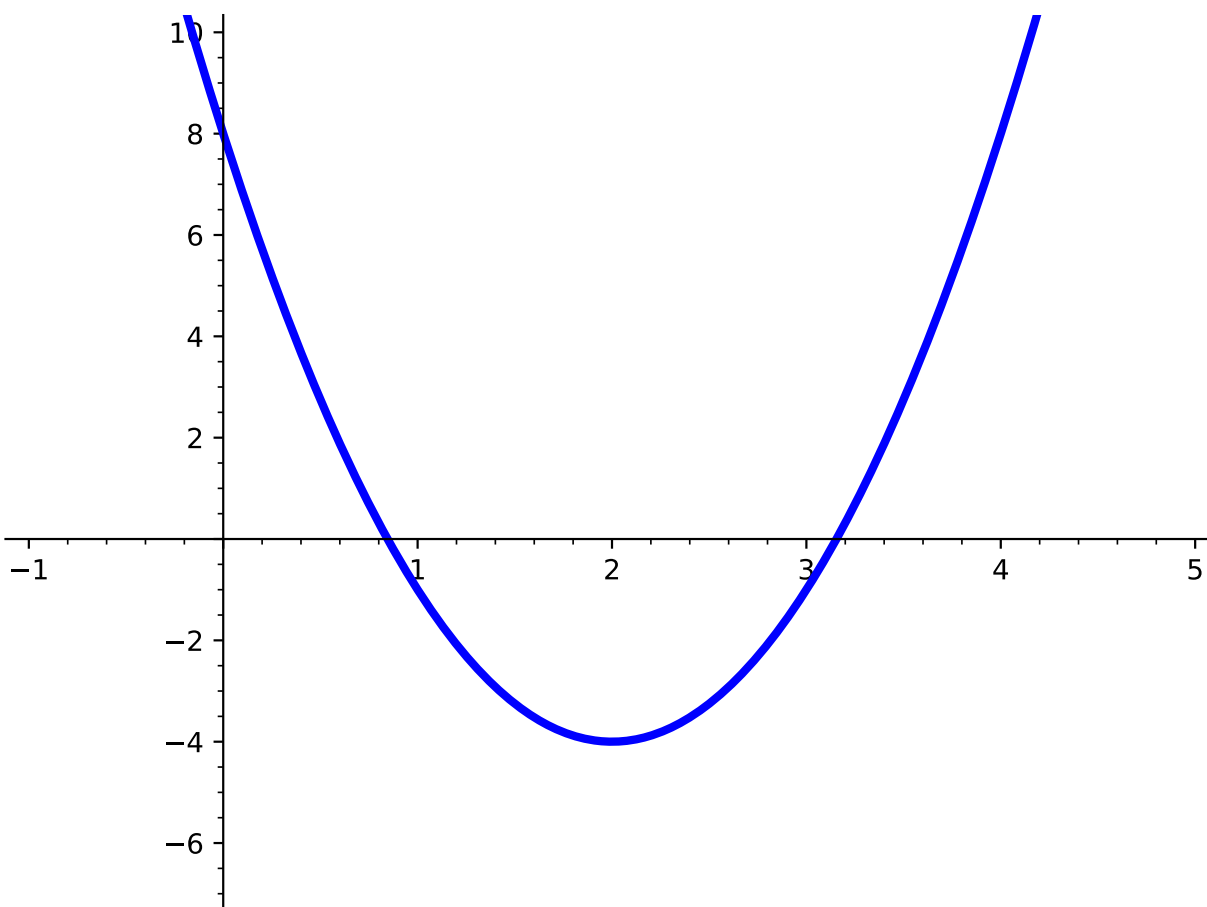
## Graphing Quadratic Functions (PR1)

**Observation 1.1.1** Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.



## Graphing Quadratic Functions (PR1)

**Activity 1.1.2** Use the graph of the quadratic function  $f(x) = 3(x - 2)^2 - 4$  to answer the questions below.



**Figure 1.1.3**

- (a) Make a table for values of  $f(x)$  corresponding to the given  $x$ -values. What is happening to the  $y$ -values as the  $x$ -values increase? Do you notice any other patterns of the  $y$ -values of the table?

## Graphing Quadratic Functions (PR1)

Table 1.1.4

$x$	$f(x)$
-2	
-1	
0	
1	
2	
3	
4	
5	

- (b) At which point  $(x, y)$  does  $f(x)$  have a minimum value? That is, is there a point on the graph that is lower than all other points?
- A. The minimum value appears to occur near  $(0, 8)$ .
  - B. The minimum value appears to occur near  $(-\frac{1}{5}, 10)$ .
  - C. The minimum value appears to occur near  $(2, -4)$ .
  - D. There is no minimum value of this function.
- (c) At which point  $(x, y)$  does  $f(x)$  have a maximum value? That is, is there a point on the graph that is higher than all other points?
- A. The maximum value appears to occur near  $(-2, 44)$ .
  - B. The maximum value appears to occur near  $(-\frac{1}{5}, 10)$ .
  - C. The maximum value appears to occur near  $(2, -4)$ .
  - D. There is no maximum value of this function.

## Graphing Quadratic Functions (PR1)

**Definition 1.1.5** The **vertex form** of a quadratic function is given by  $f(x) = a(x - h)^2 + k$ , where  $(h, k)$  is the **vertex** of the parabola and  $x = h$  is the **axis of symmetry**.  $\diamond$

## Graphing Quadratic Functions (PR1)

**Activity 1.1.6** Use the given the quadratic function,  $f(x) = 3(x - 2)^2 - 4$ , to answer the following:

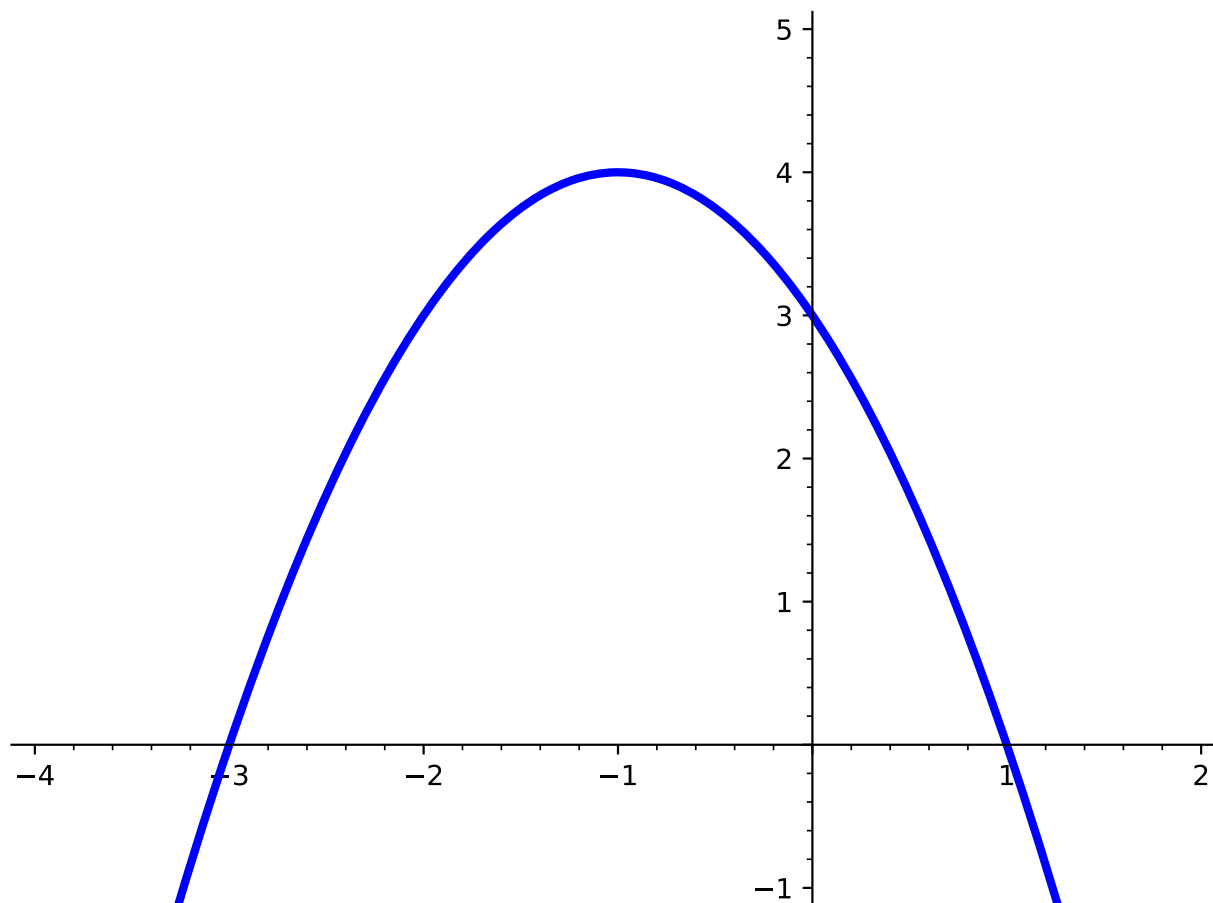
- (a) Applying [Definition 1.1.5](#), what is the vertex and axis of symmetry of  $f(x)$ ?
- A. vertex:  $(2, -4)$ ; axis of symmetry:  $x = 2$
  - B. vertex:  $(-2, 4)$ ; axis of symmetry:  $x = -2$
  - C. vertex:  $(-2, -4)$ ; axis of symmetry:  $x = -2$
  - D. vertex:  $(2, 4)$ ; axis of symmetry:  $x = 2$
- (b) Compare what you got in part *a* with the values you found in [Activity 1.1.2](#). What do you notice?

## Graphing Quadratic Functions (PR1)

**Definition 1.1.7** Given the **standard form** of a quadratic function,  $f(x) = ax^2 + bx + c$ , with real coefficients  $a, b$ , and  $c$ , the **axis of symmetry** is defined as  $x = \frac{-b}{2a}$  and has a **vertex** at the point  $(\frac{-b}{2a}, f(\frac{-b}{2a}))$ .  $\diamond$

## Graphing Quadratic Functions (PR1)

**Activity 1.1.8** Use the graph of the quadratic function to answer the questions below.



**Figure 1.1.9**

(a) Which of the following quadratic functions could be the graph shown in the figure?

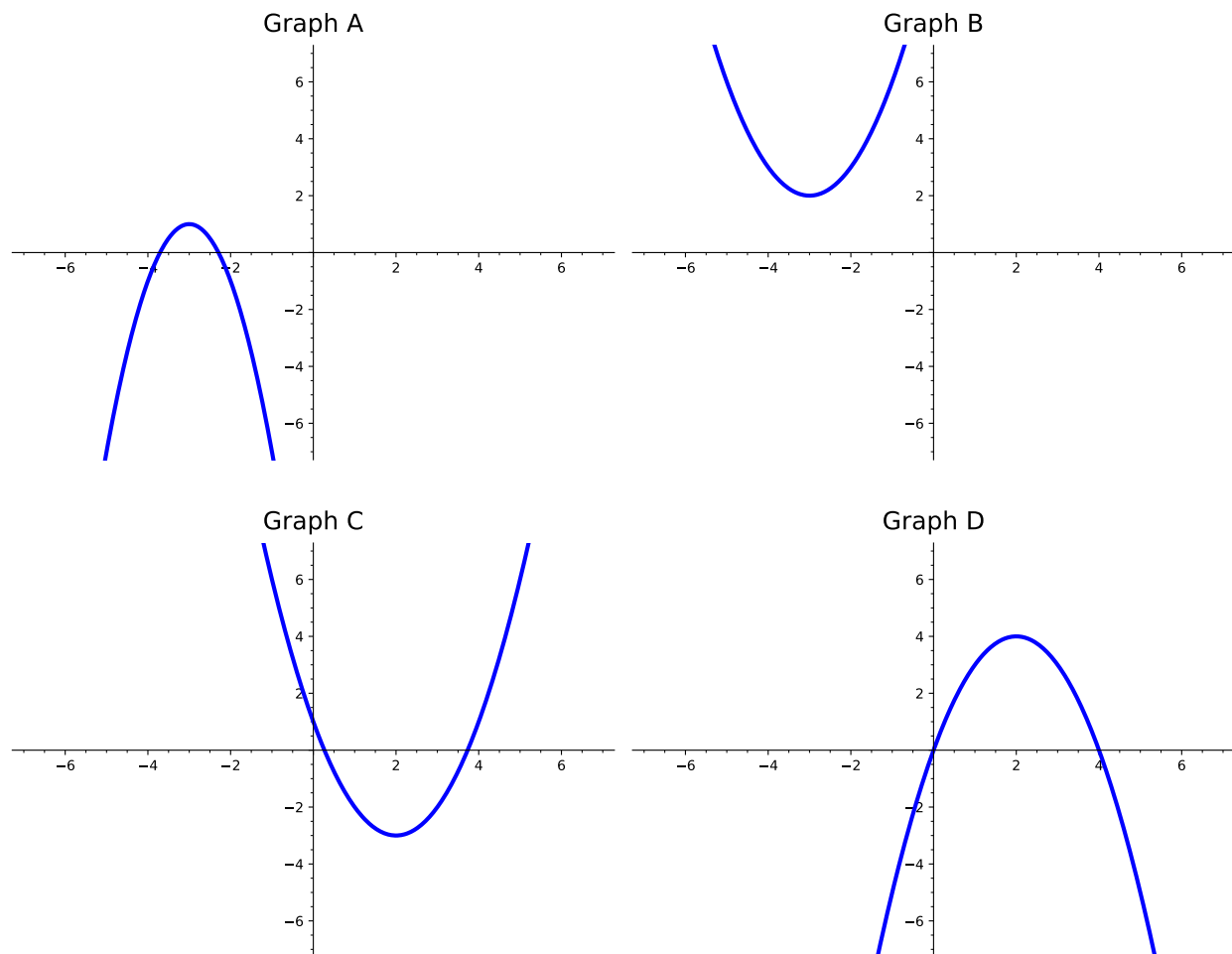
- A.  $f(x) = x^2 + 2x + 3$
- B.  $f(x) = -(x + 1)^2 + 4$
- C.  $f(x) = -x^2 - 2x + 3$
- D.  $f(x) = (x + 1)^2 + 4$

(b) What is the maximum or minimum value?

- |       |       |
|-------|-------|
| A. -1 | C. -3 |
| B. 4  | D. 1  |

## Graphing Quadratic Functions (PR1)

**Activity 1.1.10** Consider the following four graphs of quadratic functions:



(a) Which of the graphs above have a maximum?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

(b) Which of the graphs above have a minimum?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

(c) Which of the graphs above have an axis of symmetry at  $x = 2$ ?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

**Graphing Quadratic Functions (PR1)**

(d) Which of the graphs above represents the function  $f(x) = -(x-2)^2 + 4$ ?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

(e) Which of the graphs above represents the function  $f(x) = x^2 - 4x + 1$ ?

A. Graph A

C. Graph C

B. Graph B

D. Graph D



## Graphing Quadratic Functions (PR1)

**Remark 1.1.11** Notice that the maximum or minimum value of the quadratic function is the vertex. How can you determine if the vertex is a maximum or minimum?

### Graphing Quadratic Functions (PR1)

**Activity 1.1.12** A function  $f(x)$  has a maximum value at 7 and its axis of symmetry at  $x = -2$ .

- (a) Sketch a graph of a function that meets the criteria for  $f(x)$ .
- (b) Was your graph the only possible answer? Try to sketch another graph that meets this criteria.

## Graphing Quadratic Functions (PR1)

**Remark 1.1.13** Other points, such as zeros (i.e.,  $x$ -intercepts), may be helpful in sketching a more accurate graph of a quadratic function.

## Graphing Quadratic Functions (PR1)

**Activity 1.1.14** Consider the following two quadratic functions  $f(x) = x^2 - 4x + 12$  and  $g(x) = 2x^2 + 8x - 10$  and answer the following questions:

- (a) Applying [Definition 1.1.7](#), what is the vertex and axis of symmetry of  $f(x)$ ?
- A. vertex:  $(2, -16)$ ; axis of symmetry:  $x = 2$
  - B. vertex:  $(-2, 16)$ ; axis of symmetry:  $x = -2$
  - C. vertex:  $(-2, -16)$ ; axis of symmetry:  $x = -2$
  - D. vertex:  $(2, 16)$ ; axis of symmetry:  $x = 2$
- (b) Applying [Definition 1.1.7](#), what is the vertex and axis of symmetry of  $g(x)$ ?
- A. vertex:  $(2, -16)$ ; axis of symmetry:  $x = 2$
  - B. vertex:  $(-2, 16)$ ; axis of symmetry:  $x = -2$
  - C. vertex:  $(-2, -16)$ ; axis of symmetry:  $x = -2$
  - D. vertex:  $(2, 16)$ ; axis of symmetry:  $x = 2$
- (c) What do you notice about  $f(x)$  and  $g(x)$ ?
- (d) Now graph both  $f(x)$  and  $g(x)$  and draw a sketch of each graph on one coordinate plane. How are they similar/different?

## 1.2 Quadratic Models and Meanings (PR2)

### Objectives

- Use quadratic models to solve an application problem and establish conclusions.

## Quadratic Models and Meanings (PR2)

**Activity 1.2.1** A water balloon is tossed vertically from a fifth story window. It's height  $h(t)$ , in meters, at a time  $t$ , in seconds, is modeled by the function

$$h(t) = -5t^2 + 20t + 25$$

(a) Complete the following table.

**Table 1.2.2**

$t$	$h(t)$
0	
1	
2	
3	
4	
5	

(b) Explain why  $h(t)$  is not a linear function.

(c) What is the meaning of  $h(0) = 25$ ?

- A. the initial height of the water balloon is 25 meters.
- B. the water balloon reaches a maximum height of 25 meters.
- C. the water balloon hits the ground after 25 seconds.
- D. the water balloon travels 25 meters before hitting the ground.

(d) Find the vertex of the quadratic function.

- A. (0, 25)
- B. (2, 45)
- C. (5, 0)
- D. (1, 40)

(e) What is the meaning of the vertex?

- A. The water balloon reaches a maximum height of 25 meters at the start.
- B. After 2 seconds, the water balloon reaches a maximum height of 45 meters.
- C. After 5 seconds, the water balloon reaches a maximum height.
- D. After 1 second, the water balloon reaches a maximum height of 40 meters.

### Quadratic Models and Meanings (PR2)

**Activity 1.2.3** The population of a small city is given by the function  $P(t) = -50t^2 + 1200t + 32000$ , where  $t$  is the number of years after 2015.

(a) When will the population of the city reach a maximum?

A. 2020

C. 2025

B. 2022

D. 2027

(b) Determine when the population of the city is increasing and when it is decreasing.

(c) When will the population of the city reach 36,000 people?

A. 2019

C. 2027

B. 2025

D. 2035

## Quadratic Models and Meanings (PR2)

**Activity 1.2.4** The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of \$6. Market research has suggested that if the owners raise the price to \$8, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

- (a) Which of the following represents a linear function which relates the price of the streaming service  $p$  to the number of subscribers  $Q$ ?

A.  $Q(p) = -2p$

C.  $Q(p) = -2p - 4$

B.  $Q(p) = -2p + 84$

D.  $Q(p) = -2p + 96$

- (b) Using the fact that Revenue =  $pQ$ , which of the following represents the Revenue  $R$  in terms of the price  $p$ .

A.  $R(p) = -2p^2$

C.  $R(p) = -2p^2 - 4p$

B.  $R(p) = -2p^2 + 84p$

D.  $R(p) = -2p^2 + 96p$

- (c) What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. \$10

C. \$24

B. \$19.50

D. \$28.25

- (d) How many subscribers would the company have at this price?

A. 39.5 million

C. 57 million

B. 48 million

D. 76 million



## 1.3 Polynomial Long Division (PR3)

### Objectives

- Rewrite a rational function as a polynomial plus a proper rational function.

### Polynomial Long Division (PR3)

**Activity 1.3.1** Using long division, find the quotient and remainder for the given rational function. Rewrite the function as a polynomial plus a proper rational function, given  $f(x) = \frac{3x^5 - 5x^2 + 2}{x^2 + x - 1}$ .

- (a) What is the quotient?
- (b) What is the remainder?
- (c) What is the divisor?
- (d) Write the rational function as a polynomial plus a proper rational function.
- (e) How can you check your answer? (Hint: Think of regular long division with positive integers.)

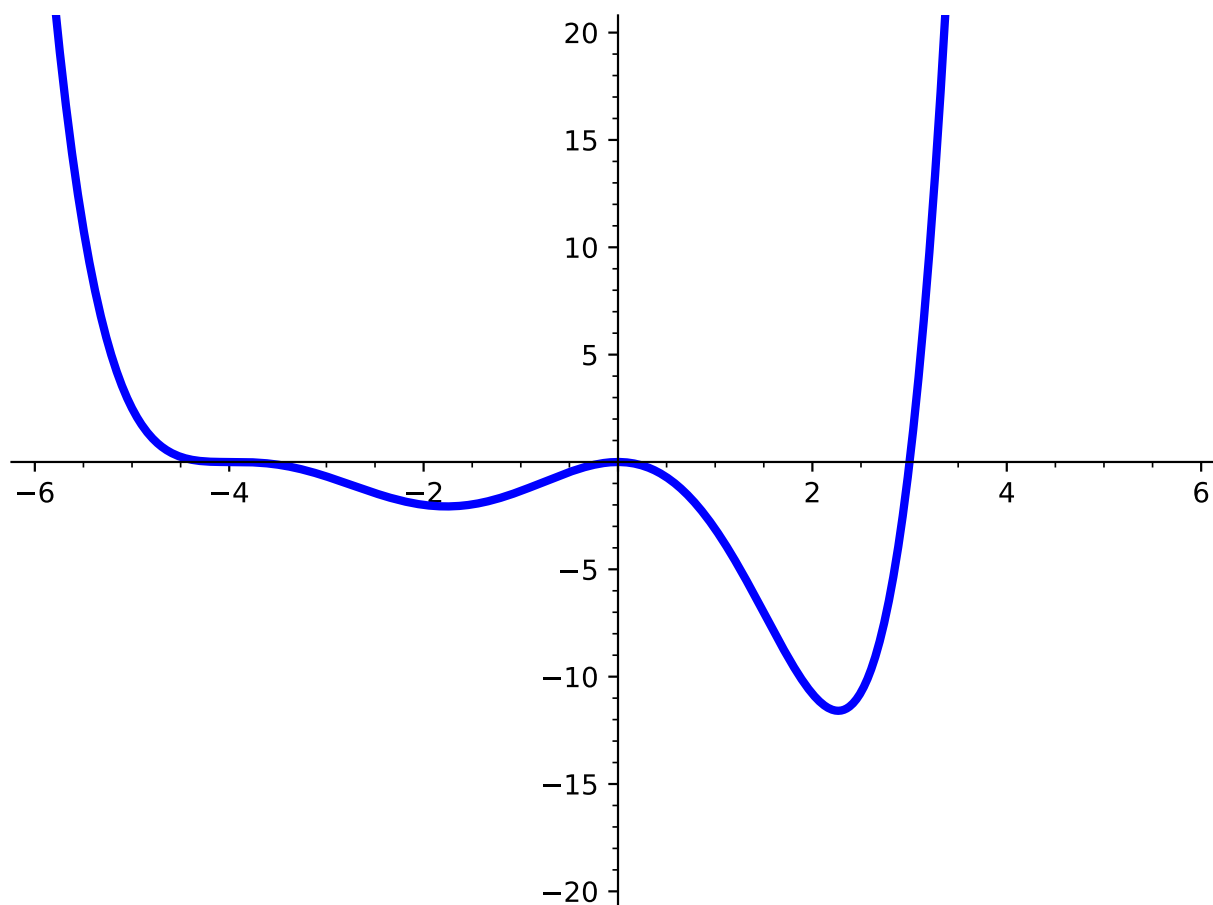
## 1.4 Zeroes of Polynomial Functions (PR4)

### Objectives

- Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.

### Zeroes of Polynomial Functions (PR4)

**Activity 1.4.1** Write the polynomial function in factored form using information from the graph below.



**Figure 1.4.2**

- (a) Using the given graph, what are the real zeros of this function? Select all that apply.
- A. 0
  - B. 1
  - C. -3
  - D. 3
  - E. 4
  - F. -4
- (b) What are the least possible multiplicities for each zero?

### Zeroes of Polynomial Functions (PR4)

- (c) What is the least degree of the function?
- A. 3
  - B. 4
  - C. 5
  - D. 6
- (d) Describe the end behavior of the graph.
- A. As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$
  - B. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$
  - C. As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$
  - D. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$
- (e) Combining the information from the end behavior with the degree of the function, will the leading coefficient be positive or negative?
- A. positive
  - B. negative
- (f) Given the point  $(2, \frac{-54}{5})$  is on the curve, and using the information in parts (a) through (e), write the function for the graph above in factored form.

### Zeroes of Polynomial Functions (PR4)

**Activity 1.4.3** Given the function  $f(x) = x^6 - 3x^4 - 2x^3$ ,

- (a) Find all the zeros and their corresponding multiplicities.
- (b) Write the function  $f(x)$  in factored form using linear factors.

### Zeroes of Polynomial Functions (PR4)

**Activity 1.4.4** Given the function  $f(x) = 2x^4 - 4x^3 + 10x^2 - 16x + 8$ ,

- (a) Find all the zeros and their corresponding multiplicities.
- (b) Write the function  $f(x)$  in factored form using linear factors.

### Zeroes of Polynomial Functions (PR4)

**Activity 1.4.5** The zeros of a function are  $x=2$ , with multiplicity 1,  $x=-1$ , with multiplicity 2 and  $x=i$ .

- (a) Given the information above, find a polynomial function with real coefficients of least degree.



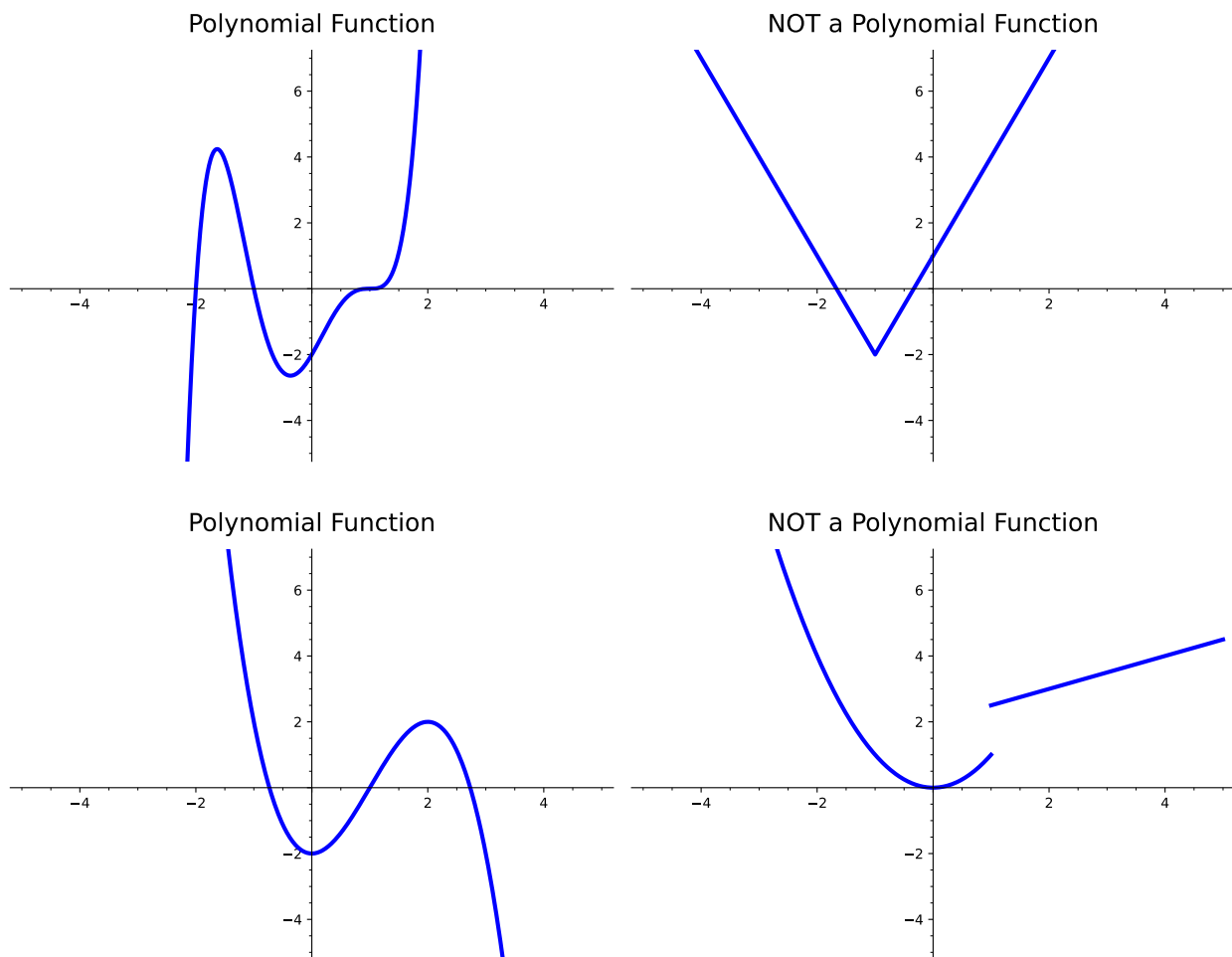
## 1.5 Graphs of Polynomial Functions (PR5)

### Objectives

- Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.

## Graphs of Polynomial Functions (PR5)

**Activity 1.5.1** Some of the graphs shown below are polynomial functions and some are not. Use the following graphs to explore the characteristics of polynomials.



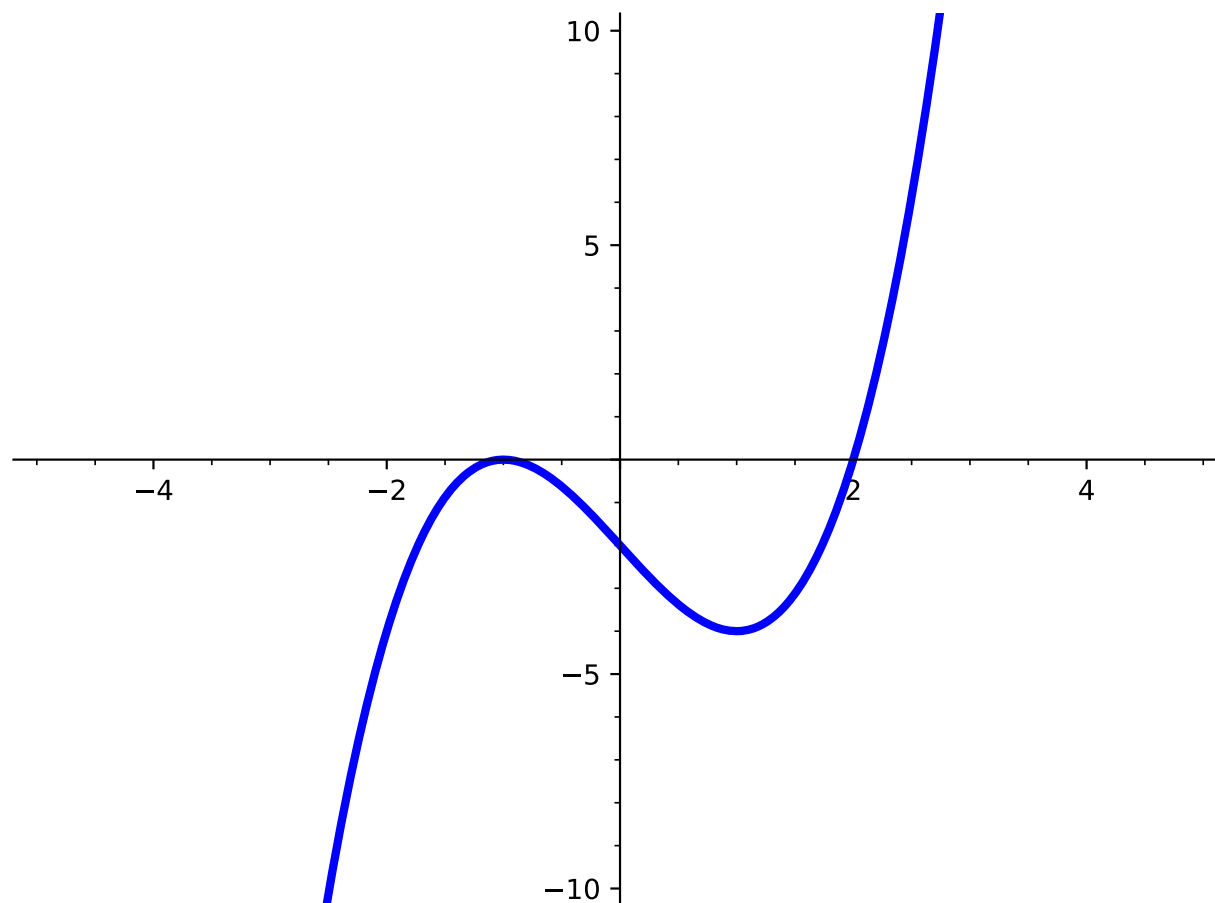
- (a) By looking at the graphs that are labeled "NOT a Polynomial Function," what type of characteristics do you notice? How are these different from the graphs labeled as being polynomial functions?
- (b) Can you make any conjectures about what characteristics all polynomials have based on what you see on these graphs?

## Graphs of Polynomial Functions (PR5)

**Remark 1.5.2** There are two primary characteristics we use to distinguish polynomial functions from other functions. Polynomial functions have graphs that are smooth and continuous. Smooth functions are functions that contain only rounded curves (no sharp corners). Continuous functions are functions that can be drawn without lifting your pencil (no breaks).

### Graphs of Polynomial Functions (PR5)

**Activity 1.5.3** Now that we know what polynomial functions look like, we should be able to determine some characteristics. Use the graph below to find the given characteristics.



**Figure 1.5.4**

- (a) What are the  $x$ -intercept(s) of the polynomial function? Select all that apply.
- |              |              |
|--------------|--------------|
| A. $(1, 0)$  | C. $(2, 0)$  |
| B. $(-1, 0)$ | D. $(0, -2)$ |
- (b) What are the  $y$ -intercept(s) of the polynomial function?
- |              |              |
|--------------|--------------|
| A. $(1, 0)$  | C. $(2, 0)$  |
| B. $(-1, 0)$ | D. $(0, -2)$ |
- (c) How many zeros does this polynomial function have?

Graphs of Polynomial Functions (PR5)

A. 0

C. 2

B. 1

D. 3

(d) At what point is the local minimum located?

A.  $(2, -4)$

D.  $(1, -4)$

B.  $(-1, 0)$

E.  $(2, 0)$

C.  $(-2, 0)$

(e) At what point is the local maximum located?

A.  $(2, -4)$

D.  $(1, -4)$

B.  $(-1, 0)$

E.  $(2, 0)$

C.  $(-2, 0)$

(f) How do you describe the behavior of the polynomial function as  $x \rightarrow \infty$ ?

A. the  $y$ -values go to negative infinity

C. the  $y$ -values go to positive infinity

B.  $f(x) \rightarrow \infty$

D.  $f(x) \rightarrow -\infty$

(g) How do you describe the behavior of the polynomial function as  $x \rightarrow -\infty$ ?

A. the  $y$ -values go to negative infinity

C. the  $y$ -values go to positive infinity

B.  $f(x) \rightarrow \infty$

D.  $f(x) \rightarrow -\infty$

### Graphs of Polynomial Functions (PR5)

**Activity 1.5.5** Sketch the graph of a function  $f(x)$  that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

- (a) The  $x$ -intercepts of  $f(x)$  are 0, 2, and 5.
- (b)  $f(x)$  has one maximum at 0.  $f(x)$  has one minimum at -5 and another at -16.
- (c) The end behavior of  $f(x)$  is given as:
  - As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$
  - As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$

## 1.6 Properties and Graphs of Rational Functions (PR6)

### Objectives

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

## Properties and Graphs of Rational Functions (PR6)

**Definition 1.6.1** A function  $r$  is **rational** provided that it is possible to write  $r$  as the ratio of two polynomials,  $p$  and  $q$ . That is,  $r$  is rational provided that for some polynomial functions  $p$  and  $q$ , we have

$$r(x) = \frac{p(x)}{q(x)}.$$





## Properties and Graphs of Rational Functions (PR6)

**Activity 1.6.2** Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

- (a) Find  $r(1)$ ,  $r(2)$ ,  $r(3)$ , and  $r(4)$ .
- (b) Some of these answers gave us information about the domain of  $r(x)$ . Which one(s)? What did they tell us?
- |           |           |
|-----------|-----------|
| A. $r(1)$ | C. $r(3)$ |
| B. $r(2)$ | D. $r(4)$ |
- (c) Another answer gave us some information about the zeros of  $r(x)$ . Which one? Why?
- |           |           |
|-----------|-----------|
| A. $r(1)$ | C. $r(3)$ |
| B. $r(2)$ | D. $r(4)$ |
- (d) Another answer gave us some information about a point on the graph of  $r(x)$  that is not a zero. Which one? How do you know?
- |           |           |
|-----------|-----------|
| A. $r(1)$ | C. $r(3)$ |
| B. $r(2)$ | D. $r(4)$ |

## Properties and Graphs of Rational Functions (PR6)

**Definition 1.6.3** Let  $p$  and  $q$  be polynomial functions so that  $r(x) = \frac{p(x)}{q(x)}$  is a rational function. The **domain** of  $r$  is the set of all real numbers except those for which  $q(x) = 0$ .  $\diamond$

## Properties and Graphs of Rational Functions (PR6)

**Activity 1.6.4** Let's investigate the domain of  $r(x)$  more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

- (a) Rewrite  $r(x)$  by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?
- (b) The function was not defined for  $x = 3$ . Make a table for values of  $r(x)$  near  $x = 3$ .

**Table 1.6.5**

$x$	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

- (c) Which of the following describe the behavior of the graph near  $x = 3$ ?
- A. As  $x \rightarrow 3$ ,  $r(x)$  approaches a finite number
  - B. As  $x \rightarrow 3$  from the left,  $r(x) \rightarrow \infty$
  - C. As  $x \rightarrow 3$  from the left,  $r(x) \rightarrow -\infty$
  - D. As  $x \rightarrow 3$  from the right,  $r(x) \rightarrow \infty$
  - E. As  $x \rightarrow 3$  from the right,  $r(x) \rightarrow -\infty$
- (d) The function was also not defined for  $x = 1$ . Make a table for values of  $r(x)$  near  $x = 1$ .

# Properties and Graphs of Rational Functions (PR6)

**Table 1.6.6**

$x$	$r(x)$
0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

- (e) Which of the following describe the behavior of the graph near  $x = 1$ ?
- A. As  $x \rightarrow 1$ ,  $r(x)$  approaches a finite number
  - B. As  $x \rightarrow 1$  from the left,  $r(x) \rightarrow \infty$
  - C. As  $x \rightarrow 1$  from the left,  $r(x) \rightarrow -\infty$
  - D. As  $x \rightarrow 1$  from the right,  $r(x) \rightarrow \infty$
  - E. As  $x \rightarrow 1$  from the right,  $r(x) \rightarrow -\infty$
- (f) The function is behaving differently near  $x = 1$  than it is near  $x = 3$ . Can you see anything in the factored form of  $r(x)$  that may help you account for the difference?

**Definition 1.6.7** DEFINITION OF VA WILL GO HERE



**Definition 1.6.8** DEFINITION OF HOLE WILL GO HERE



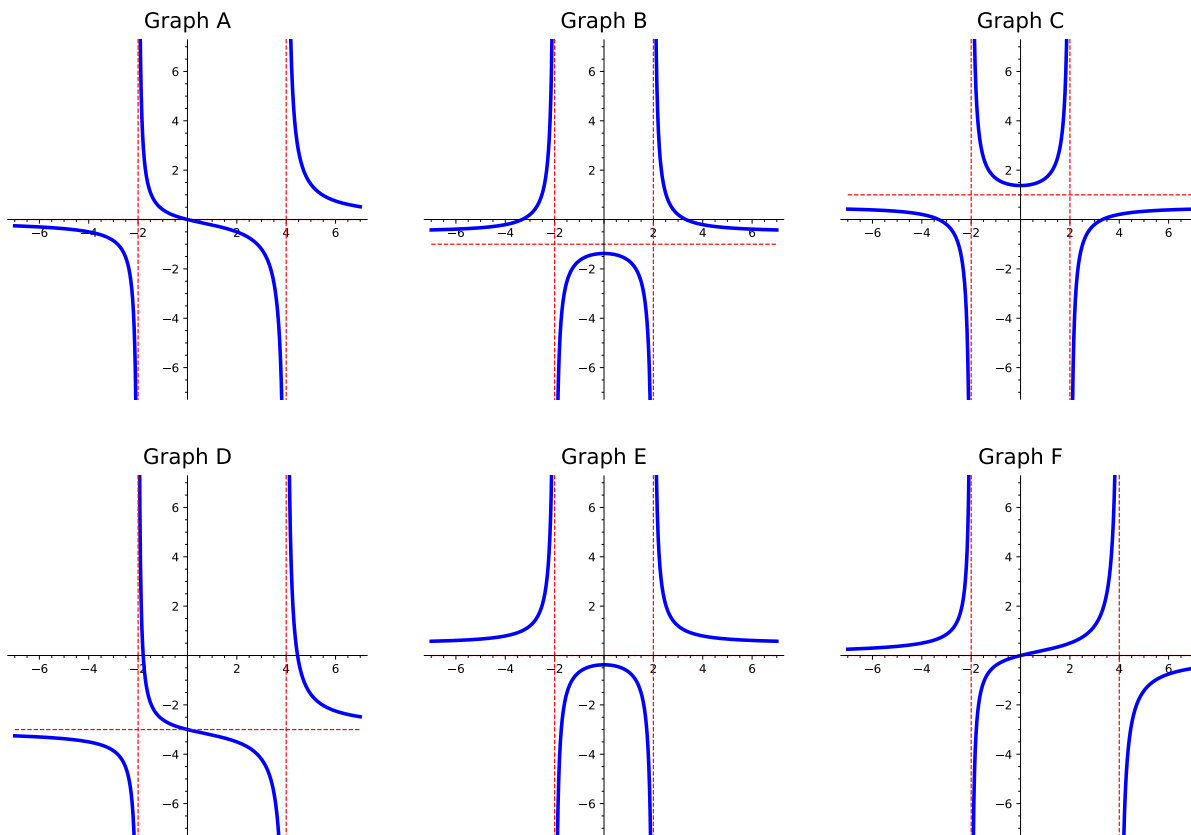
## Properties and Graphs of Rational Functions (PR6)

**Remark 1.6.9 Features of a rational function.** Let  $r(x) = \frac{p(x)}{q(x)}$  be a rational function.

- If  $p(a) = 0$  and  $q(a) \neq 0$ , then  $r(a) = 0$ , so  $r$  has a zero at  $x = a$ .
- If  $q(a) = 0$  and  $p(a) \neq 0$ , then  $r(a)$  is undefined and  $r$  has a vertical asymptote at  $x = a$ .
- If  $p(a) = 0$  and  $q(a) = 0$  and we can show that there is a finite number  $L$  such that  $r(x) \rightarrow L$ , then  $r(a)$  is not defined and  $r$  has a hole at the point  $(a, L)$ .

## Properties and Graphs of Rational Functions (PR6)

**Activity 1.6.10** Consider the following six graphs of rational functions:



- (a) Which of the graphs above represents the function  $f(x) = \frac{2x}{x^2 - 2x - 8}$ ?
- (b) Which of the graphs above represents the function  $g(x) = \frac{x^2 + 3}{2x^2 - 8}$ ?



**Properties and Graphs of Rational Functions (PR6)**

**Activity 1.6.11 FLUENCY ACTIVITY TO MATCH CHECKIT**

**(a) QUESTIONS HERE!**

## **Colophon**

This book was authored in PreTeXt.