

# Precalculus for Team-Based Inquiry Learning

2024 Development Edition

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## Back Matter

# Chapter 1

## Equations, Inequalities, and Applications (EQ)

### Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx

By the end of this chapter, you should be able to...

1. Solve a linear equation in one variable. Solve a linear inequality in one variable and express the solution graphically and using interval notation.
2. Solve application problems involving linear equations.
3. Given two points, determine the distance between them and the midpoint of the line segment connecting them.
4. Solve a linear equation involving an absolute value. Solve a linear inequality involving absolute values and express the answers graphically and using interval notation.
5. Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.
6. Solve a rational equation.
7. Solve quadratic inequalities and express the solution graphically and

with interval notation. Solve rational inequalities and express the solution graphically and using interval notation.



## 1.1 Linear Equations and Inequalities (EQ1)

### Objectives

- Solve a linear equation in one variable. Solve a linear inequality in one variable and express the solution graphically and using interval notation.

## Linear Equations and Inequalities (EQ1)

**Remark 1.1.1** Recall that when solving a linear equation, you use addition, subtraction, multiplication and division to isolate the variable.

## Linear Equations and Inequalities (EQ1)

**Activity 1.1.2** Solve the linear equations.

(a)  $3x - 8 = 5x + 2$

A.  $x = 2$

C.  $x = -5$

B.  $x = 5$

D.  $x = -2$

(b)  $5(3x - 4) = 2x - (x + 3)$

A.  $x = \frac{17}{14}$

C.  $x = \frac{23}{14}$

B.  $x = \frac{14}{17}$

D.  $x = \frac{14}{23}$

## Linear Equations and Inequalities (EQ1)

**Activity 1.1.3** Solve the linear equation.

$$\frac{2}{3}x - 8 = \frac{5x + 1}{6}$$

(a) Which equation is equivalent to  $\frac{2}{3}x - 8 = \frac{5x+1}{6}$  but does not contain any fractions?

A.  $12x - 48 = 15x + 3$

C.  $4x - 8 = 5x + 1$

B.  $3x - 24 = 10x + 2$

D.  $4x - 48 = 5x + 1$

(b) Use the simplified equation from part (a) to solve  $\frac{2}{3}x - 8 = \frac{5x+1}{6}$ .

A.  $x = -17$

C.  $x = -9$

B.  $x = -\frac{26}{7}$

D.  $x = -49$

## Linear Equations and Inequalities (EQ1)

**Activity 1.1.4** It is not always the case that a linear equation has exactly one solution. Consider the following linear equations which appear similar, but their solutions are very different.

(a) Which of these equations has one unique solution?

A.  $4(x - 2) = 4x + 6$

C.  $4(x - 1) = x + 4$

B.  $4(x - 1) = 4x + 4$

(b) Which of these equations has no solutions?

A.  $4(x - 2) = 4x + 6$

C.  $4(x - 1) = x + 4$

B.  $4(x - 1) = 4x + 4$

(c) Which of these equations has many solutions?

A.  $4(x - 2) = 4x + 6$

C.  $4(x - 1) = x + 4$

B.  $4(x - 1) = 4x + 4$

(d) What happens to the  $x$  variable when a linear equation has no solution or many solutions?

## Linear Equations and Inequalities (EQ1)

**Definition 1.1.5** A linear equation with one unique solution is a **conditional equation**. A linear equation that is true for all values of the variable is an **identity equation**. A linear equation with no solutions is an **inconsistent equation**.  $\diamond$

## Linear Equations and Inequalities (EQ1)

**Activity 1.1.6** An inequality is a relationship between two values that are not equal.

(a) What is the solution to the linear equation  $3x - 1 = 5$ ?

(b) Which of these values is a solution of the inequality  $3x - 1 \geq 5$ ?

A.  $x = 0$

C.  $x = 4$

B.  $x = 2$

D.  $x = 10$

(c) Express the solution of the inequality  $3x - 1 \geq 5$  in interval notation.

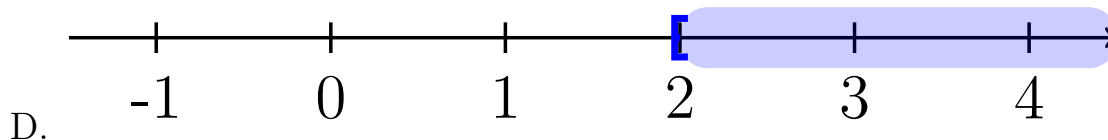
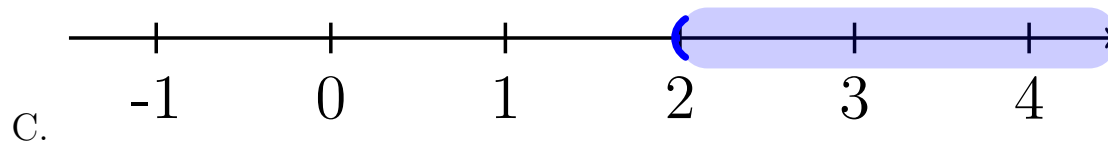
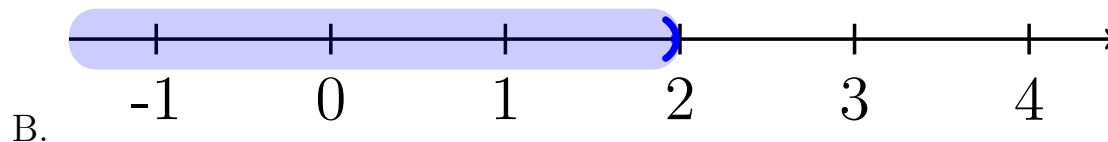
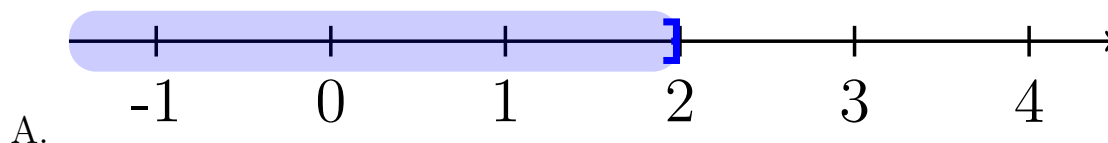
A.  $(-\infty, 2]$

C.  $(2, \infty)$

B.  $(-\infty, 2)$

D.  $[2, \infty)$

(d) Draw the solution to the inequality on a number line.



## Linear Equations and Inequalities (EQ1)

### Activity 1.1.7

(a) Which of these values is a solution of the inequality  $-x < 8$ ?

A.  $x = -10$

C.  $x = 4$

B.  $x = -8$

D.  $x = 10$

(b) Solve the linear inequality  $-x < 8$ . How does your solution compare to the values chosen in part (a)?

(c) Expression the solution of the inequality  $-x < 8$  in interval notation.

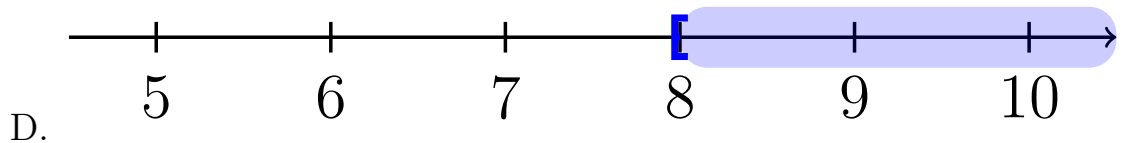
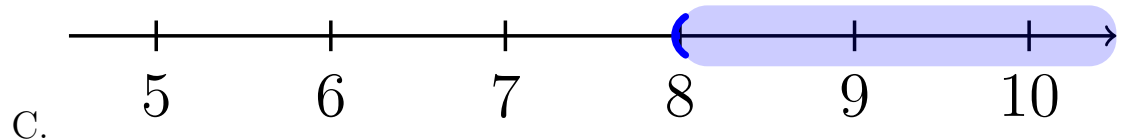
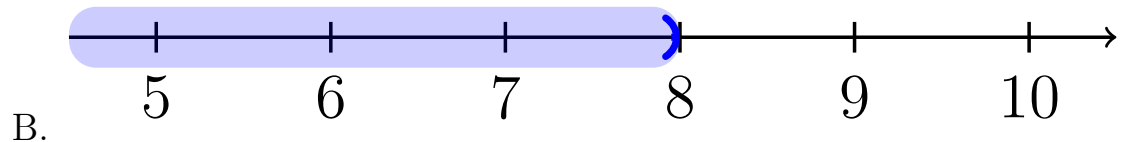
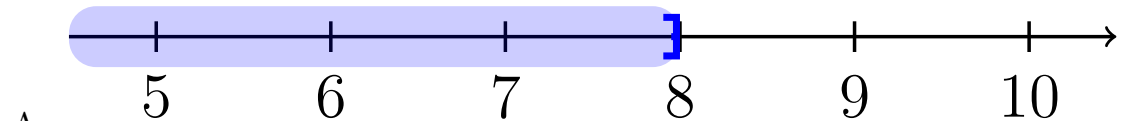
A.  $(-\infty, -8]$

C.  $(-8, \infty)$

B.  $(-\infty, -8)$

D.  $[-8, \infty)$

(d) Draw the solution to the inequality on a number line.





## Linear Equations and Inequalities (EQ1)

**Remark 1.1.8** You can treat solving linear inequalities, just like solving an equation. The one exception is when you multiply or divide by a negative value, reverse the inequality symbol.

## Linear Equations and Inequalities (EQ1)

**Activity 1.1.9** Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a)  $-3x - 1 \leq 5$

(b)  $3(x + 4) > 2x - 1$

(c)  $-\frac{1}{2}x \geq -\frac{2}{4} + \frac{5}{4}x$

## Linear Equations and Inequalities (EQ1)

**Definition 1.1.10** A **compound inequality** includes multiple inequalities in one statement.  $\diamond$

## Linear Equations and Inequalities (EQ1)

**Activity 1.1.11** Consider the statement  $3 \leq x < 8$ . This really means that  $3 \leq x$  and  $x < 8$ .

(a) Which of the following inequalities are equivalent to the compound inequality  $3 \leq 2x - 3 < 8$ ?

A.  $3 \leq 2x - 3$

C.  $2x - 3 < 8$

B.  $3 \geq 2x - 3$

D.  $2x - 3 > 8$

(b) Solve the inequality  $3 \leq 2x - 3$ .

A.  $x \leq 0$

C.  $x \leq 3$

B.  $x \geq 0$

D.  $x \geq 3$

(c) Solve the inequality  $2x - 3 < 8$ .

A.  $x > \frac{11}{2}$

C.  $x > \frac{5}{2}$

B.  $x < \frac{11}{2}$

D.  $x < \frac{5}{2}$

(d) Which compound inequality describes how the two solutions overlap?

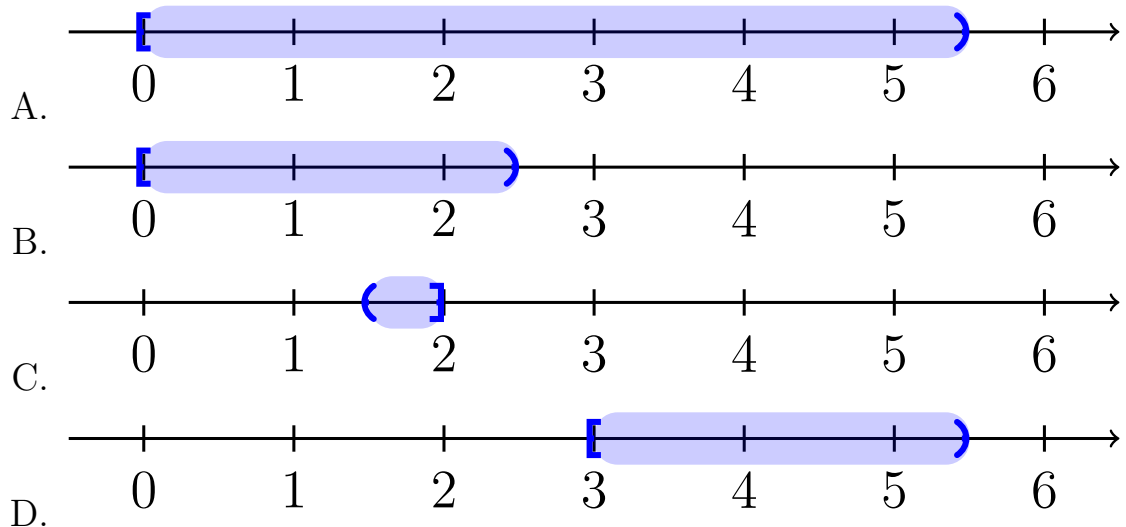
A.  $0 \leq x < \frac{11}{2}$

C.  $\frac{5}{2} < x \leq 3$

B.  $0 \leq x < \frac{5}{2}$

D.  $3 \leq x < \frac{11}{2}$

(e) Draw the solution to the inequality on a number line.



## Linear Equations and Inequalities (EQ1)

**Remark 1.1.12** Solving a compound linear inequality, uses the same methods as a single linear inequality ensuring that you perform the same operations on all three parts. Alternatively, you can break the compound inequality up into two and solve separately.

## 1.2 Applications of Linear Equations (EQ2)

### Objectives

- Solve application problems involving linear equations.

## Applications of Linear Equations (EQ2)

**Observation 1.2.1** Linear equations can be used to solve many types of real-world applications. We'll investigate some of those in this section.

## Applications of Linear Equations (EQ2)

**Remark 1.2.2** Distance, rate, and time problems are a standard example of an application of a linear equation. For these, it's important to remember that

$$d = rt$$

where  $d$  is distance,  $r$  is the rate (or speed), and  $t$  is time.

Often we will have more than one moving object, so it is helpful to denote which object's distance, rate, or time we are referring to. One way we can do this is by using a subscript. For example, if we are describing an eastbound train (as we will in the first example), it may be helpful to denote its distance, rate, and time as  $d_E$ ,  $r_E$ , and  $t_E$  respectively. Notice that the subscript  $E$  is a label reminding us that we are referring to the eastbound train.



## Applications of Linear Equations (EQ2)

**Activity 1.2.3** Two trains leave a station at the same time. One is heading east at a speed of 75 mph, while the other is heading west at a speed of 85 mph. After how long will the trains be 400 miles apart?

- (a) How fast is each train traveling?
- A.  $r_E = 85$  mph,  $r_W = 75$  mph
  - B.  $r_E = 75$  mph,  $r_W = 85$  mph
  - C.  $r_E = 400$  mph,  $r_W = 400$  mph
  - D.  $r_E = 75$  mph,  $r_W = 400$  mph
  - E.  $r_E = 400$  mph,  $r_W = 85$  mph
- (b) Which of the statements describes how the times of the eastbound and westbound train are related?
- A. The eastbound train is slower than the westbound train, so  $75 + t_E = 85 + t_W$ .
  - B. The eastbound train left an hour before the westbound train, so if we let  $t_E = t$ , then  $t_W = t - 1$ .
  - C. Both trains have been traveling the same amount of time, so  $t_E = t_W$ . Since they are the same, we can just call them both  $t$ .
  - D. We don't know how the times relate to each other, so we must denote them separately as  $t_E$  and  $t_W$ .
  - E. Since the trains are traveling at different speeds, we need the proportion  $\frac{r_E}{r_W} = \frac{t_E}{t_W}$ .
- (c) Fill in the following table using the information you've just determined about the trains' rates and times since they left the station. Some values are there to help you get started.

**Table 1.2.4**

	rate	time	distance from station
eastbound train			$75t$
westbound train		$t$	

- (d) At the moment in question, the trains are 400 miles apart. How does that total distance relate to the distance each train has traveled?

## Applications of Linear Equations (EQ2)

- A. The 400 miles is irrelevant. They've been traveling the same amount of time so they must be the same distance away from the station. That tells us  $d_E = d_W$ .
  - B. The 400 miles is the difference between the distance each train traveled, so  $d_E - d_W = 400$ .
  - C. The 400 miles represents the sum of the distances that each train has traveled, so  $d_E + d_W = 400$ .
  - D. The 400 miles is the product of the distance each train traveled, so  $(d_E)(d_W) = 400$ .
- (e) Now plug in the expressions from your table for  $d_E$  and  $d_w$ . What equation do you get?
- A.  $75t = 85t$
  - B.  $75t - 85t = 400$
  - C.  $75t + 85t = 400$
  - D.  $(75t)(85t) = 400$
- (f) Notice that we now have a linear equation in one variable,  $t$ . Solve for  $t$ , and put that answer in context of the problem.
- A. The trains are 400 miles apart after 2 hours.
  - B. The trains are 400 miles apart after 2.5 hours.
  - C. The trains are 400 miles apart after 3 hours.
  - D. The trains are 400 miles apart after 3.5 hours.
  - E. The trains are 400 miles apart after 4 hours.

## Applications of Linear Equations (EQ2)

**Remark 1.2.5** In [Activity 1.2.3](#) we examined the motion of two objects moving at the same time in opposite directions. In [Activity 1.2.6](#) we will examine a different perspective, but still apply  $d = rt$  to solve.

## Applications of Linear Equations (EQ2)

**Activity 1.2.6** Jalen needs groceries, so decides to ride his bike to the store. It takes him half an hour to get there. After finishing his shopping, he sees his friend Alex who offers him a ride home. He takes the same route home as he did to the store, but this time it only takes one-fifth of an hour. If his average speed was 18 mph faster on the way home, how far away does Jalen live from the grocery store?

We'll use the subscript  $b$  to refer to variables relating to Jalen's trip to the store while riding his bike and the subscript  $c$  to refer to variables relating to Jalen's trip home while riding in his friend's car.

- (a) How long does his bike trip from home to the store and his car trip from the store back home take?
- A.  $t_b = 18$  hours,  $t_c = 18$  hours
  - B.  $t_b = \frac{1}{5}$  of an hour,  $t_c = \frac{1}{2}$  of an hour
  - C.  $t_b = \frac{1}{2}$  of an hour,  $t_c = \frac{1}{5}$  of an hour
  - D.  $t_b = 2$  hours,  $t_c = 5$  hours
  - E.  $t_b = 5$  hours,  $t_c = 2$  hours
- (b) Which of the statements describes how the speed (rate) of the bike trip and the car trip are related?
- A. Both the trip to the store and the trip home covered the same distance, so  $r_b = r_c$ . Since they are the same, we can just call them both  $r$ .
  - B. We don't know how the two rates relate to each other, so cannot write an equation comparing them and must leave them as separate variables  $r_b$  and  $r_c$ .
  - C. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let  $r_b = r$ , then  $r_c = r + 18$ .
  - D. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let  $r_b = r$ , then  $r_c = r - 18$ .
- (c) Fill in the following table using the information you've just determined about the Jalen's rates and times on each leg of his grocery store trip. Then fill in the distance column based on how distance relates to rate and time in each case.

## Applications of Linear Equations (EQ2)

**Table 1.2.7**

	rate	time	distance covered
bike trip (to the store)			
car trip (going back home)			

- (d) Our goal is to figure out how far away Jalen lives from the store. To help us get there, write an equation relating  $d_b$  and  $d_c$ .
- The distance he traveled by bike is the same as the distance he traveled by car, so  $d_b = d_c$
  - The distance he traveled by bike took longer than the distance he traveled by car, so  $d_b + \frac{1}{2} = d_c + \frac{1}{5}$
  - The distance,  $d$ , between his house and the grocery store is sum of the distance he traveled on his bike and the distance he traveled in the car, so  $d_b + d_c = d$ .
  - The distance,  $d$ , between his house and the grocery store is sum of the difference he traveled on his bike and the distance he traveled in the car, so  $d_b - d_c = d$ .
- (e) Now plug in the expressions from your table for  $d_b$  and  $d_c$  into the equation you just found. Notice that it is a linear equation in one variable,  $r$ . Solve for  $r$ .
- (f) Our goal was to determine the distance between Jalen's house and the grocery store. Solving for  $r$  did not tell us that distance, but it did get us one step closer. Use that value to help you determine the distance between his house and the store, and write your answer using the context of the problem. (Hint: can you find an expression involving  $r$  that we made that represents that distance? )
- The grocery store is 6 miles away from Jalen's house.
  - The grocery store is 8 miles away from Jalen's house.
  - The grocery store is 10 miles away from Jalen's house.
  - The grocery store is 12 miles away from Jalen's house.
  - The grocery store is 14 miles away from Jalen's house.

## Applications of Linear Equations (EQ2)

**Remark 1.2.8** Another type of application of linear equations is called a mixture problem. In these we will mix together two things, like two types of candy in a candy store or two solutions of different concentrations of alcohol.

## Applications of Linear Equations (EQ2)

**Activity 1.2.9** Ammie's favorite snack to share with friends is candy salad, which is a mixture of different types of candy. Today she chooses to mix Nerds Gummy Clusters, which cost \$8.38 per pound, and Starburst Jelly Beans, which cost \$7.16 per pound. If she makes seven pounds of candy salad and spends a total of \$55.61, how many pounds of each candy did she buy?

We'll use the subscript  $n$  to refer to variables relating to the Nerds and the subscript  $s$  to refer to variables relating to the Starburst jelly beans.

(a) question

A.

## Applications of Linear Equations (EQ2)

### Activity 1.2.10

A chemist needs to mix two solutions to create a mixture consisting of 30% alcohol. She uses 20 liters of the first solution, which has a concentration of 21% alcohol. How many liters of the second solution (that is 45% alcohol) should she add to the first solution to create the mixture that is 30% alcohol?



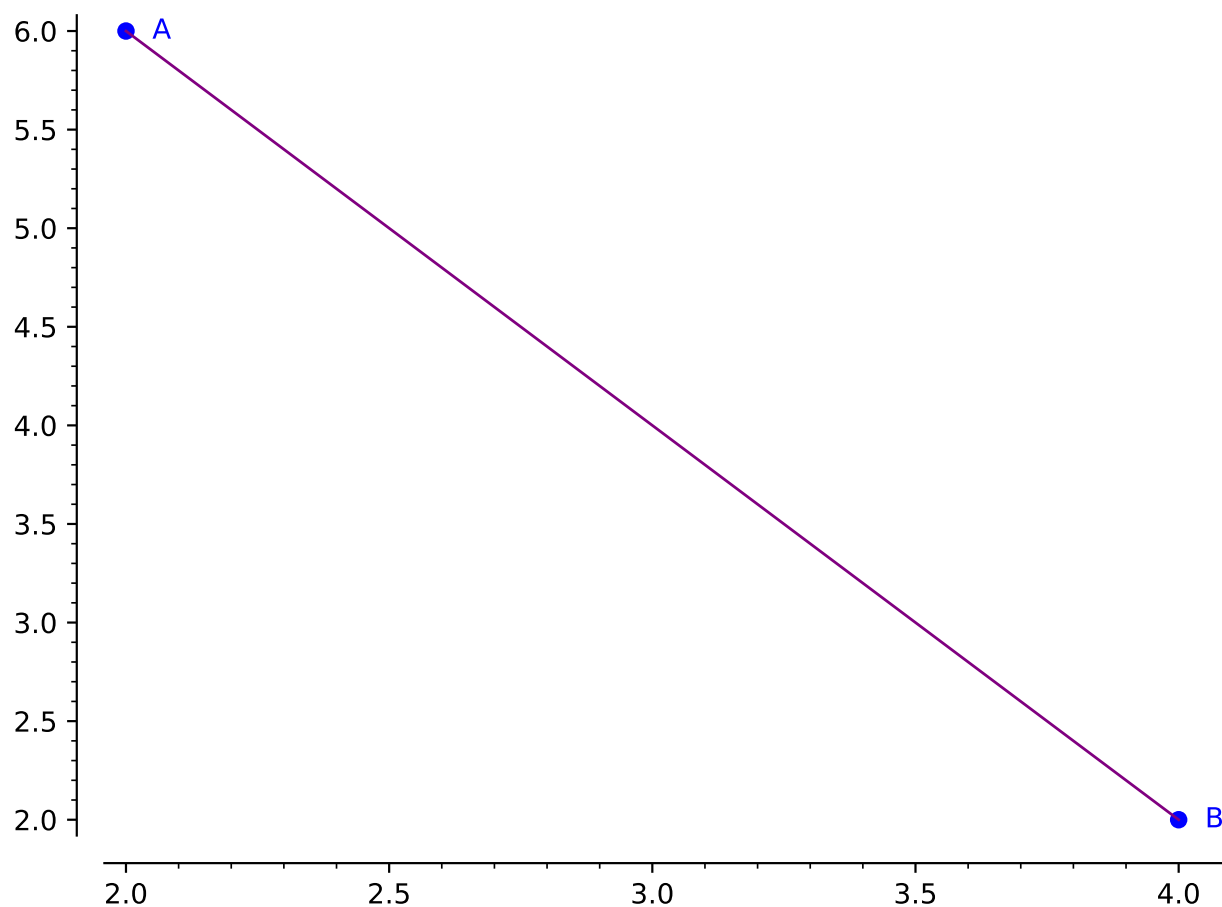
## 1.3 Distance and Midpoint (EQ3)

### Objectives

- Given two points, determine the distance between them and the midpoint of the line segment connecting them.

### Distance and Midpoint (EQ3)

**Activity 1.3.1** The points  $A$  and  $B$  are shown in the graph below. Use the graph to answer the following questions:



**Figure 1.3.2**

- (a) Draw a right triangle so that the hypotenuse is the line segment between points  $A$  and  $B$ . Label the third point of the triangle  $C$ .
- (b) Find the lengths of line segments  $AC$  and  $BC$ .
- (c) Now that you know the lengths of  $AC$  and  $BC$ , how can you find the length of  $AB$ ? Find the length of  $AB$ .

### Distance and Midpoint (EQ3)

**Remark 1.3.3** Using the **Pythagorean Theorem**( $a^2 + b^2 = c^2$ ) can be helpful in finding the distance of a line segment (as long as you create a right triangle!).

### Distance and Midpoint (EQ3)

**Activity 1.3.4** Suppose you are given two points  $(x_1, y_1)$  and  $(x_2, y_2)$ . Let's investigate how to find the length of the line segment that connects these two points!

- (a) Draw a sketch of a right triangle so that the hypotenuse is the line segment between the two points.
- (b) Find the lengths of the legs of the right triangle in terms of  $x$  and  $y$ .
- (c) Find the length of the line segment that connects the two original points in terms of  $x$  and  $y$ .

### Distance and Midpoint (EQ3)

**Definition 1.3.5** The distance,  $d$ , between two points,  $(x_1, y_1)$  and  $(x_2, y_2)$ , can be found by using the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that the distance formula is an application of the Pythagorean Theorem! ◇

### Distance and Midpoint (EQ3)

**Activity 1.3.6** Apply the [Definition 1.3.5](#) to calculate the distance between the given points.

(a) What is the distance between  $(4, 6)$  and  $(9, 15)$ ?

A. 10.2

C.  $\sqrt{106}$

B. 10.3

D.  $\sqrt{56}$

(b) What is the distance between  $(-2, 5)$  and  $(-7, -1)$ ?

A.  $\sqrt{11}$

C. 3.3

B. 7.8

D.  $\sqrt{61}$

(c) Suppose the line segment  $AB$  has one endpoint,  $A$ , at the origin. For which coordinate of  $B$  would make the line segment  $AB$  the longest?

A.  $(3, 7)$

C.  $(-6, 4)$

B.  $(2, -8)$

D.  $(-5, -5)$

### Distance and Midpoint (EQ3)

**Remark 1.3.7** Notice in [Activity 1.3.6](#), you can give a distance in either exact form (leaving it with a square root) or as an approximation (as a decimal). Make sure you can give either form as sometimes one form is better than another!

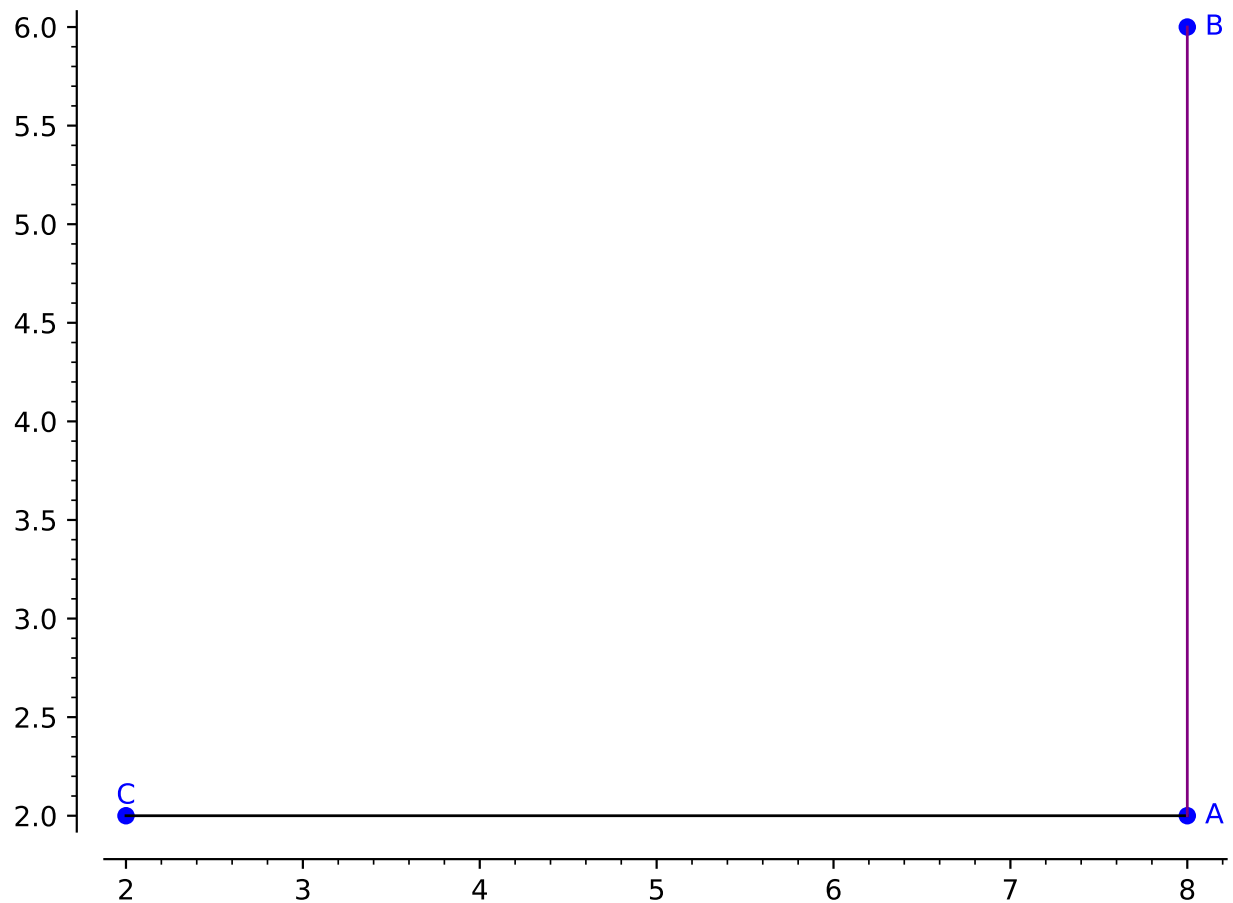
### Distance and Midpoint (EQ3)

**Remark 1.3.8** A **midpoint** refers to the point that is located in the middle of a line segment. In other words, the midpoint is the point that is halfway between the two endpoints of a given line segment.



### Distance and Midpoint (EQ3)

**Activity 1.3.9** Two line segments are shown in the graph below. Use the graph to answer the following questions:



**Figure 1.3.10**

- (a) What is the midpoint of the line segment  $AB$ ?
- |              |              |
|--------------|--------------|
| A. $(16, 4)$ | C. $(8, 8)$  |
| B. $(8, 4)$  | D. $(10, 2)$ |
- (b) What is the midpoint of the line segment  $AC$ ?
- |             |             |
|-------------|-------------|
| A. $(6, 0)$ | C. $(6, 4)$ |
| B. $(4, 4)$ | D. $(5, 2)$ |
- (c) Suppose we connect the two endpoints of the two line segments together, to create the new line segment,  $BC$ . Can you make an educated guess to where the midpoint of  $BC$  is?

**Distance and Midpoint (EQ3)**

A.  $(10, 8)$

C.  $(5, 4)$

B.  $(6, 4)$

D.  $(5, 2)$

(d) How can you test your conjecture? Is there a mathematical way to find the midpoint of any line segment?

### Distance and Midpoint (EQ3)

**Definition 1.3.11** The midpoint of a line segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$ , can be found by taking the average of the  $x$  and  $y$  values. Mathematically, the **midpoint formula** states that the midpoint of a line segment can be found by:

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



### Distance and Midpoint (EQ3)

**Activity 1.3.12** Apply the [Definition 1.3.11](#) to calculate the midpoint of the following line segments.

- (a) What is the midpoint of the line segment with endpoints  $(-4, 5)$  and  $(-2, -3)$ ?

A.  $(3, 1)$

C.  $(1, 1)$

B.  $(-3, 1)$

D.  $(1, 4)$

- (b) What is the midpoint of the line segment with endpoints  $(2, 6)$  and  $(-6, -8)$ ?

A.  $(-3, -1)$

C.  $(-2, -1)$

B.  $(-2, 0)$

D.  $(4, 7)$

- (c) Suppose  $C$  is the midpoint of  $AB$  and is located at  $(9, 8)$ . The coordinates of  $A$  are  $(10, 10)$ . What are the coordinates of  $B$ ?

A.  $(9.5, 9)$

C.  $(18, 16)$

B.  $(11, 12)$

D.  $(8, 6)$

### Distance and Midpoint (EQ3)

**Activity 1.3.13** On a map, your friend Sarah's house is located at  $(-2, 5)$  and your other friend Austin's house is at  $(6, -2)$ .

- (a) How long is the direct path from Sarah's house to Austin's house?
- (b) Suppose your other friend, Micah, lives in the middle between Sarah and Austin. What is the location of Micah's house on the map?

## 1.4 Absolute Value Equations and Inequalities (EQ4)

### Objectives

- Solve a linear equation involving an absolute value. Solve a linear inequality involving absolute values and express the answers graphically and using interval notation.

## Absolute Value Equations and Inequalities (EQ4)

**Remark 1.4.1** An absolute value, written  $|x|$ , is the non-negative value of  $x$ . If  $x$  is a positive number, then  $|x| = x$ . If  $x$  is a negative number, then  $|x| = -x$ .

## Absolute Value Equations and Inequalities (EQ4)

**Activity 1.4.2** Let's consider how to solve an equation when an absolute value is involved.

(a) Which values are solutions to the absolute value equation  $|x| = 2$ ?

A.  $x = 2$

C.  $x = -1$

B.  $x = 0$

D.  $x = -2$

(b) Which values are solutions to the absolute value equation  $|x - 7| = 2$ ?

A.  $x = 9$

C.  $x = 5$

B.  $x = 7$

D.  $x = -9$

(c) Which values are solutions to the absolute value equation  $3|x - 7| + 5 = 11$ ? It may be helpful to rewrite the equation to isolate the absolute value.

A.  $x = 7$

C.  $x = 5$

B.  $x = -9$

D.  $x = 9$



### Absolute Value Equations and Inequalities (EQ4)

**Activity 1.4.3** Absolute value represents the distance a value is from 0 on the number line. So,  $|x - 7| = 2$  means that the expression  $x - 7$  is 2 units away from 0.

(a) What values on the number line could  $x - 7$  equal?

A.  $x = -7$

D.  $x = 2$

B.  $x = -2$

E.  $x = 7$

C.  $x = 0$

(b) This gives us two separate equations to solve. What are those two equations?

A.  $x - 7 = -7$

B.  $x - 7 = -2$

C.  $x - 7 = 0$

D.  $x - 7 = 2$

E.  $x - 7 = 7$

(c) Solve each equation for  $x$ .

### Absolute Value Equations and Inequalities (EQ4)

**Remark 1.4.4** When solving an absolute value equation, begin by isolating the absolute value expression. Then rewrite the equation into two linear equations and solve. If  $c > 0$ ,

$$|ax + b| = c$$

becomes the following two equations

$$ax + b = c \quad \text{and} \quad ax + b = -c$$

**Absolute Value Equations and Inequalities (EQ4)**

**Activity 1.4.5** Solve the following absolute value equations.

(a)  $|3x + 4| = 10$

A.  $\{-2, 2\}$

C.  $\{-10, 10\}$

B.  $\left\{-\frac{14}{3}, 2\right\}$

D. No solution

(b)  $3|x - 7| + 5 = 11$

A.  $\{-2, 2\}$

C.  $\{5, 9\}$

B.  $\{-9, 9\}$

D. No solution

(c)  $2|x + 1| + 8 = 4$

A.  $\{-4, 4\}$

C.  $\{5, 7\}$

B.  $\{-6, 6\}$

D. No solution

## Absolute Value Equations and Inequalities (EQ4)

**Remark 1.4.6** Since the absolute value represents a distance, it is always a positive number. Whenever you encounter an isolated absolute value equation equal to a negative value, there will be no solution.

# Absolute Value Equations and Inequalities (EQ4)

**Activity 1.4.7** Just as with linear equations and inequalities, we can consider absolute value inequalities from equations.

(a) Which values are solutions to the absolute value inequality  $|x - 7| \leq 2$ ?

A.  $x = 9$

C.  $x = 5$

B.  $x = 7$

D.  $x = -9$

(b) Rewrite the absolute value inequality  $|x - 7| \leq 2$  as a compound inequality.

A.  $0 \leq x - 7 \leq 2$

C.  $-2 \leq x - 7 \leq 0$

B.  $-2 \leq x - 7 \leq 2$

D.  $2 \leq x \leq 7$

(c) Solve the compound inequality that is equivalent to  $|x - 7| \leq 2$  found in part (b). Write the solution in interval notation.

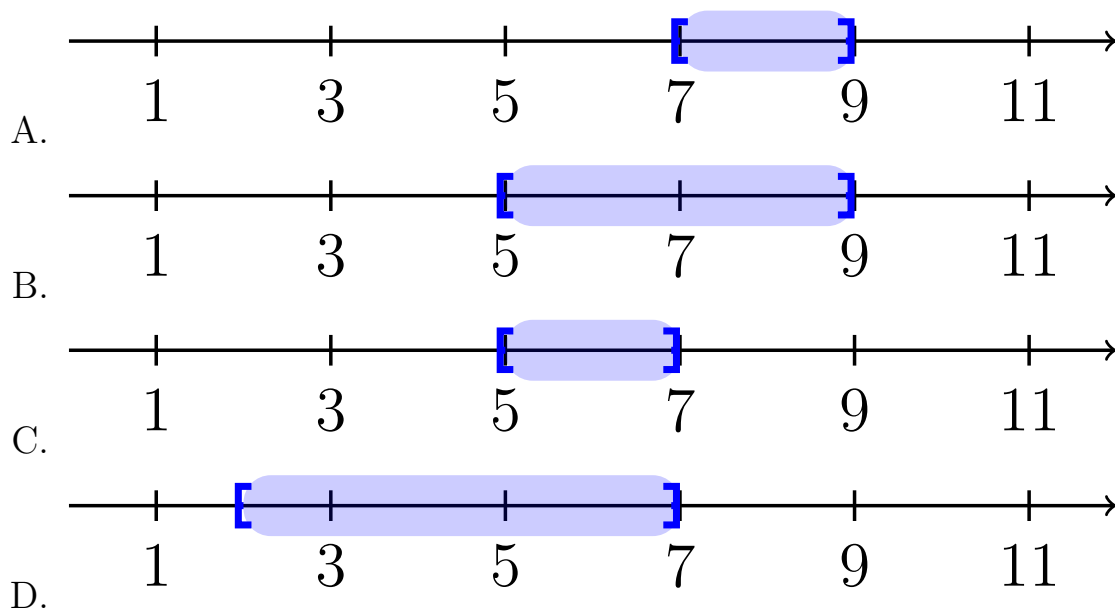
A.  $[7, 9]$

C.  $[5, 7]$

B.  $[5, 9]$

D.  $[2, 7]$

(d) Draw the solution to  $|x - 7| \leq 2$  on the number line.



# Absolute Value Equations and Inequalities (EQ4)

**Activity 1.4.8** Now let's consider another type of absolute value inequality.

(a) Which values are solutions to the absolute value inequality  $|x - 7| \geq 2$ ?

A.  $x = 9$

C.  $x = 5$

B.  $x = 7$

D.  $x = -9$

(b) Which two of the following inequalities are equivalent to  $|x - 7| \geq 2$ .

A.  $x - 7 \leq 2$

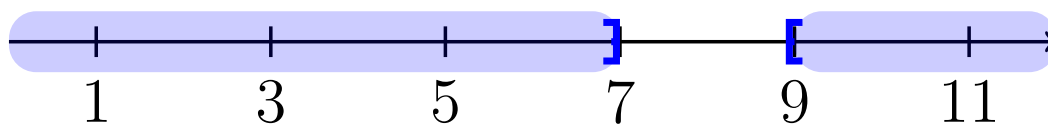
B.  $x - 7 \leq -2$

C.  $x - 7 \geq 2$

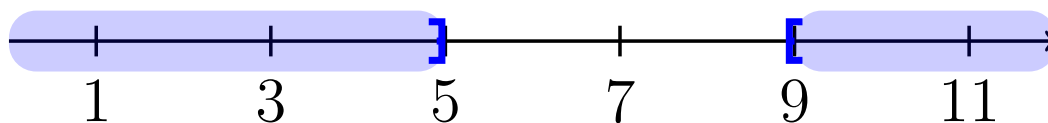
D.  $x - 7 \geq -2$

(c) Solve the two inequalities found in part (b). Write the solution in interval notation and graph on the number line.

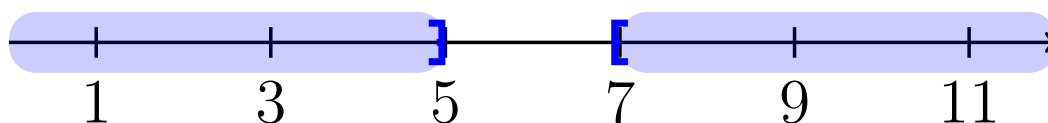
A.  $(-\infty, 7] \cup [9, \infty)$



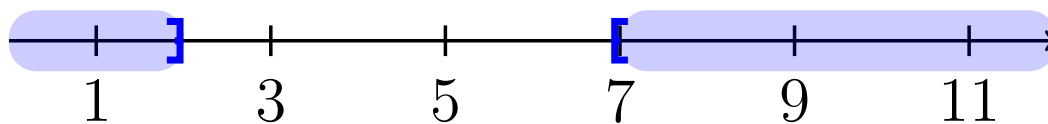
B.  $(-\infty, 5] \cup [9, \infty)$



C.  $(-\infty, 5] \cup [7, \infty)$



D.  $(-\infty, 2] \cup [7, \infty)$



## Absolute Value Equations and Inequalities (EQ4)

**Definition 1.4.9** When solving an absolute value inequality, rewrite it as compound inequalities. Assume  $k$  is positive.  $|x| < k$  becomes  $-k < x < k$ .  $|x| > k$  becomes  $x > k$  and  $x < -k$ .  $\diamond$

### Absolute Value Equations and Inequalities (EQ4)

**Activity 1.4.10** Solve the following absolute value inequalities. Write your solution in interval notation and graph on a number line.

(a)  $|3x + 4| < 10$

(b)  $3|x - 7| + 5 > 11$



## 1.5 Quadratic Equations (EQ5)

### Objectives

- Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

## Quadratic Equations (EQ5)

**Definition 1.5.1** A **quadratic equation** is of the form:

$$ax^2 + bx + c = 0$$

where  $a$  and  $b$  are coefficients (and  $a \neq 0$ ),  $x$  is the variable, and  $c$  is the constant term.  $\diamond$

### Quadratic Equations (EQ5)

**Activity 1.5.2** Before beginning to solve quadratic equations, we need to be able to identify all various forms of quadratics. Which of the following is a quadratic equation?

A.  $6 - x^2 = 3x$

D.  $(x - 4)^2 + 1 = 0$

B.  $(2x - 1)(x + 3) = 0$

C.  $4(x - 3) + 7 = 0$

E.  $5x^2 - 3x = 17 - 4x$

## Quadratic Equations (EQ5)

**Definition 1.5.3** To solve a quadratic equation, we will need to apply the **zero product property**, which states that if  $a * b = 0$ , then either  $a = 0$  or  $b = 0$ . In other words, you can only have a product of 0 if one (or both!) of the factors is 0.  $\diamond$

## Quadratic Equations (EQ5)

**Activity 1.5.4** In this activity, we will look at how to apply the zero product property when solving quadratic equations.

- (a) Which of the following equations can you apply [Definition 1.5.3](#) as your first step in solving?

A.  $2x^2 - 3x + 1 = 0$

B.  $(2x + 1)(x + 1) = 0$

C.  $3x^2 - 4 = 6x$

D.  $x(3x + 5) = 0$

- (b) Suppose you are given the quadratic equation,  $(2x - 1)(x - 1) = 0$ . Applying [Definition 1.5.3](#), would give you:

A.  $2x^2 - 3x + 1 = 0$

B.  $(2x + 1) = 0$  and  $(x + 1) = 0$

C.  $(2x - 1) = 0$  and  $(x - 1) = 0$

- (c) After applying the zero product property, what are the solutions to the quadratic equation  $(2x - 1)(x - 1) = 0$ ?

A.  $x = -\frac{1}{2}$  and  $x = 1$

B.  $x = \frac{1}{2}$  and  $x = -1$

C.  $x = -\frac{1}{2}$  and  $x = -1$

D.  $x = \frac{1}{2}$  and  $x = 1$

## Quadratic Equations (EQ5)

**Remark 1.5.5** Notice in [Activity 1.5.2](#) and [Activity 1.5.4](#), that not all equations are set up "nicely." You will need to do some manipulation to get everything on one side (AND in factored form!) and 0 on the other \*before\* applying the zero product property.

## Quadratic Equations (EQ5)

**Activity 1.5.6** Suppose you want to solve the equation  $2x^2 + 5x - 12 = 0$ , which is NOT in factored form.

(a) Which of the following is the correct factored form of  $2x^2 + 5x - 12 = 0$ ?

A.  $(2x - 3)(x - 4) = 0$

B.  $(2x + 3)(x - 4) = 0$

C.  $(2x + 3)(x + 4) = 0$

D.  $(2x - 3)(x + 4) = 0$

(b) After applying [Definition 1.5.3](#), which of the following will be a solution to  $2x^2 + 5x - 12 = 0$ ?

A.  $x = -\frac{3}{2}$  and  $x = -4$

B.  $x = \frac{3}{2}$  and  $x = 4$

C.  $x = -\frac{3}{2}$  and  $x = 4$

D.  $x = \frac{3}{2}$  and  $x = -4$

## Quadratic Equations (EQ5)

**Activity 1.5.7** Solve each of the following quadratic equations:

(a)  $(2x - 5)(x + 7) = 0$

(b)  $3x(4x - 1) = 0$

(c)  $3x^2 - 14x - 5 = 0$

(d)  $6 - x^2 = 3x$



## Quadratic Equations (EQ5)

**Activity 1.5.8** Suppose you are given the equation,  $x^2 = 9$ :

(a) How many solutions does this equation have?

A. 0

C. 2

B. 1

D. 3

(b) What are the solutions to this equation?

A.  $x = 0$

C.  $x = 3, -3$

B.  $x = 3$

D.  $x = 9, -9$

(c) How is this quadratic equation different than the equations we've solved thus far?

## Quadratic Equations (EQ5)

**Definition 1.5.9** The **square root property** states that a quadratic equation of the form  $x^2 = k^2$  (where  $k$  is a nonzero number) will give solutions  $x = k$  and  $x = -k$ . In other words, if we have an equation with a perfect square on one side and a number on the other side, we can take the square root of both sides to solve the equation.  $\diamond$

## Quadratic Equations (EQ5)

**Activity 1.5.10** Suppose you are given the equation,  $3x^2 - 8 = 4$ :

- (a) What would be the first step in solving  $3x^2 - 8 = 4$ ?
- A. Divide by 3 on both sides
  - B. Subtract 4 on both sides
  - C. Add 8 on both sides
  - D. Multiply by 3 on both sides
- (b) Isolate the  $x^2$  term and apply [Definition 1.5.9](#) solve for  $x$ .
- (c) What are the solution(s) to  $3x^2 - 8 = 4$ ?
- |                |            |
|----------------|------------|
| A. $x = 6, -6$ | C. $x = 0$ |
| B. $x = 2, -2$ | D. $x = 2$ |

## Quadratic Equations (EQ5)

**Activity 1.5.11** Solve the following quadratic equations by applying the square root property ([Definition 1.5.9](#)).

(a)  $5x^2 + 7 = 47$

(b)  $2x^2 = -144$

(c)  $3x^2 + 1 = 25$

(d)  $(x + 2)^2 + 3 = 19$

(e)  $3(x - 4)^2 = 15$

## Quadratic Equations (EQ5)

**Remark 1.5.12** Not all quadratic equations can be factored or can be solved by using the square root property. In the next few activities, we will learn two additional methods in solving quadratics.

## Quadratic Equations (EQ5)

**Definition 1.5.13** Another method for solving a quadratic equation is known as **completing the square**. With this method, we add or subtract terms to both sides of an equation until we have a perfect square trinomial on one side of the equal sign and a constant on the other side. We then apply the square root property. Note: A perfect square trinomial is a trinomial that can be factored into a binomial squared. For example,  $x^2 + 4x + 4$  is a perfect square trinomial because it can be factored into  $(x + 2)(x + 2)$  or  $(x + 2)^2$ .  $\diamond$

## Quadratic Equations (EQ5)

**Activity 1.5.14** Let's work through an example together to solve  $x^2 + 6x = 4$ . (Notice that the methods of factoring and the square root property do not work with this equation.)

- (a) In order to apply [Definition 1.5.13](#), we first need to have a perfect square trinomial on one side of the equals sign. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

C.  $-9$

B. 9

D. 2

- (b) Add your answer from part *a* to the right side of the equation as well (i.e. whatever you do to one side of an equation you must do to the other side too!) and then factor the perfect square trinomial on the left side. Which equation best represents the equation now?

A.  $(x + 3)^2 = -5$

B.  $(x - 3)^2 = 13$

C.  $(x + 3)^2 = 13$

D.  $(x - 3)^2 = -5$

- (c) Apply the square root property ([Definition 1.5.9](#)) to both sides of the equation to determine the solution(s). Which of the following is the solution(s) of  $x^2 + 6x = 4$ ?

A.  $3 + \sqrt{13}$  and  $3 - \sqrt{13}$

B.  $-3 + \sqrt{13}$  and  $-3 - \sqrt{13}$

C.  $3 + \sqrt{5}$  and  $3 - \sqrt{5}$

D.  $-3 + \sqrt{5}$  and  $-3 - \sqrt{5}$

## Quadratic Equations (EQ5)

**Remark 1.5.15** To complete the square, the leading coefficient,  $a$  (i.e., the coefficient of the  $x^2$  term), must equal 1. If it does not, then factor the entire equation by  $a$  and then follow similar steps as in [Activity 1.5.14](#).



## Quadratic Equations (EQ5)

**Activity 1.5.16** Let's solve the equation  $2x^2 + 8x - 5 = 0$  by completing the square.

- (a) Rewrite the equation so that all the terms with the variable  $x$  is on one side of the equation and a constant is on the other.
- (b) Notice that the coefficient of the  $x^2$  term is not 1. What could we factor the left side of the equation by so that the coefficient of the  $x^2$  is 1?
- (c) Once you factor the left side, what equation represents the equation you now have?

A.  $2(x^2 - 8x) = -5$

C.  $2(x^2 + 4x) = 5$

B.  $2(x^2 - 4x) = -5$

D.  $2(x^2 + 8x) = 5$

- (d) Just like in [Activity 1.5.14](#), let's now try and create the perfect square trinomial (inside the parentheses) on the left side of the equation. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

C.  $-8$

B. 8

D. 2

- (e) What would we need to add to the right-hand side of the equation to keep the equation balanced?

A. 4

C.  $-8$

B. 8

D. 2

- (f) Which of the following equation represents the quadratic equation you have now?

A.  $2(x + 2)^2 = 9$

C.  $2(x + 2)^2 = 13$

B.  $2(x - 2)^2 = 9$

D.  $2(x - 2)^2 = 13$

- (g) Apply the square root property and solve the quadratic equation.

### Quadratic Equations (EQ5)

**Activity 1.5.17** Solve the following quadratic equations by completing the square.

(a)  $x^2 - 12x = -11$

(b)  $x^2 + 2x - 33 = 0$

(c)  $4x^2 + 16x = 65$

## Quadratic Equations (EQ5)

**Definition 1.5.18** The last method for solving quadratic equations is the **quadratic formula** - a formula that will solve all quadratic equations! A quadratic equation of the form  $ax^2 + bx + c = 0$  can be solved by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .

◇

## Quadratic Equations (EQ5)

**Activity 1.5.19** Use the quadratic formula ([Definition 1.5.18](#)) to solve  $x^2 + 4x = -3$ .

- (a) When written in standard form, what are the values of  $a$ ,  $b$ , and  $c$ ?
- (b) When applying the quadratic formula, what would you get when you substitute  $a$ ,  $b$ , and  $c$ ?

A.  $x = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$

B.  $x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

C.  $x = \frac{-4 \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

D.  $x = \frac{4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$

- (c) What is the solution(s) to  $x^2 + 4x = -3$ ?

A.  $x = -1, 3$

C.  $x = -1, -3$

B.  $x = 1, 3$

D.  $x = 1, -3$

## Quadratic Equations (EQ5)

**Activity 1.5.20** Use the quadratic formula ([Definition 1.5.18](#)) to solve  $2x^2 - 13 = 7x$ .

- (a) When written in standard form, what are the values of  $a$ ,  $b$ , and  $c$ ?
- (b) When applying the quadratic formula, what would you get when you substitute  $a$ ,  $b$ , and  $c$ ?

A.  $x = \frac{-7 \pm \sqrt{7^2 - 4(2)(13)}}{2(1)}$

B.  $x = \frac{7 \pm \sqrt{(-7)^2 - 4(2)(-13)}}{2(2)}$

C.  $x = \frac{-7 \pm \sqrt{(-7)^2 - 4(1)(-13)}}{2(1)}$

D.  $x = \frac{7 \pm \sqrt{7^2 - 4(2)(-13)}}{2(2)}$

- (c) What is the solution(s) to  $2x^2 - 13 = 7x$ ?

A.  $x = \frac{7+\sqrt{73}}{4}$  and  $\frac{7-\sqrt{73}}{4}$

B.  $x = \frac{7+\sqrt{101}}{4}$  and  $\frac{-7-\sqrt{101}}{4}$

C.  $x = \frac{-7+\sqrt{55}}{4}$  and  $\frac{7-\sqrt{55}}{4}$

D.  $x = \frac{-7+\sqrt{155}}{4}$  and  $\frac{-7-\sqrt{155}}{4}$

## Quadratic Equations (EQ5)

**Activity 1.5.21** Use the quadratic formula ([Definition 1.5.18](#)) to solve  $x^2 = 6x - 12$ .

- (a) When written in standard form, what are the values of  $a$ ,  $b$ , and  $c$ ?
- (b) When applying the quadratic formula, what would you get when you substitute  $a$ ,  $b$ , and  $c$ ?

A.  $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(12)}}{2(1)}$

B.  $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$

C.  $x = \frac{-6 \pm \sqrt{(-6)^2 - 4(1)(-12)}}{2(1)}$

D.  $x = \frac{6 \pm \sqrt{6^2 - 4(1)(-12)}}{2(1)}$

- (c) Notice that the number under the square root is a negative. Recall that when you have a negative number under a square root, that gives an imaginary number ( $\sqrt{-1} = i$ ). What is the solution(s) to  $x^2 = 6x - 12$ ?

A.  $x = 3 + i\sqrt{3}$  and  $3 - i\sqrt{3}$

B.  $x = 6 + i\sqrt{12}$  and  $6 - i\sqrt{12}$

C.  $x = -3 + i\sqrt{3}$  and  $-3 - i\sqrt{3}$

D.  $x = -6 + i\sqrt{12}$  and  $-6 - i\sqrt{12}$

## Quadratic Equations (EQ5)

**Activity 1.5.22** Solve the following quadratic equations by applying the quadratic formula ([Definition 1.5.18](#)).

(a)  $2x^2 - 3x = 5$

(b)  $4x^2 - 1 = -8x$

(c)  $2x^2 - 7x - 13 = -10$

(d)  $x^2 - 6x + 12 = 0$

### Quadratic Equations (EQ5)

**Activity 1.5.23** Now that you have seen all the different ways to solve a quadratic equation, you will need to know WHEN to use which method. Are some methods better than others?

(a) Which is the best method to use to solve  $5x^2 = 80$ ?

- |  |                          |
|--|--------------------------|
| A. Factoring and Zero Product Property | C. Completing the Square |
| B. Square Root Property                | D. Quadratic Formula     |

(b) Which is the best method to use to solve  $5x^2 + 9x = -4$ ?

- |  |                          |
|--|--------------------------|
| A. Factoring and Zero Product Property | C. Completing the Square |
| B. Square Root Property                | D. Quadratic Formula     |

(c) Which is the best method to use to solve  $3x^2 + 9x = 0$ ?

- |  |                          |
|--|--------------------------|
| A. Factoring and Zero Product Property | C. Completing the Square |
| B. Square Root Property                | D. Quadratic Formula     |

(d) Go back to parts *a*, *b*, and *c* and solve each of the quadratic equations. Would you still use the same method?



## 1.6 Rational Equations (EQ6)

### Objectives

- Solve a rational equation.

## Rational Equations (EQ6)

**Definition 1.6.1** An algebraic expression is called a **rational expression** if it can be written as the ratio of two polynomials,  $p$  and  $q$ .

An equation is called a **rational equation** if it consists of only rational expressions and constants.  $\diamond$

## Rational Equations (EQ6)

**Observation 1.6.2** Technically, linear and quadratic equations are also rational equations. They are a special case where the denominator of the rational expressions is 1. We will focus in this section on cases where the denominator is not a constant; that is, rational equations where there are variables in the denominator.

### Rational Equations (EQ6)

**Activity 1.6.3** Because some of the denominators in a rational equation will have variables, we need to be sure to exclude any values that would make those denominator equal to zero. Which value(s) should be exclude as possible solutions to the following rational equations? Select all that apply.

(a)

$$\frac{2}{x+5} = \frac{x-3}{x-8} = -7$$

- A.  $-7$
- B.  $-5$
- C.  $2$
- D.  $3$
- E.  $8$

(b)

$$\frac{x^2 - 6x + 8}{x^2 - 4x + 3} = 0$$

- A.  $0$
- B.  $1$
- C.  $2$
- D.  $3$
- E.  $4$

### Rational Equations (EQ6)

**Activity 1.6.4** Consider the rational equation

$$5 = -\frac{6}{x-2}$$

- (a) What value should be excluded as a possible solution?
- A. 5
  - B. 6
  - C.  $-6$
  - D. 2
  - E.  $-2$
- (b) To solve, we begin by clearing out the fraction involved. What can we multiply each term by that will clear the fraction?
- A.  $x - 5$
  - B.  $x - 6$
  - C.  $x + 6$
  - D.  $x - 2$
  - E.  $x + 2$
- (c) Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?
- A.  $5(x - 5) = -6$
  - B.  $5(x - 6) = -6$
  - C.  $5(x + 6) = -6$
  - D.  $5(x - 2) = -6$
  - E.  $5(x + 2) = -6$
- (d) Solve the linear equation. Check your answer using the original rational equation.

### Rational Equations (EQ6)

**Activity 1.6.5** Consider the rational equation

$$\frac{4}{x+1} = -\frac{2}{x+6}$$

- (a) What values should be excluded as possible solutions?
- A. 2 and 4
  - B. 1 and 6
  - C.  $-1$  and  $-6$
  - D. 1 and 4
  - E. 2 and 6
- (b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?
- A.  $x - 2$  and  $x - 4$
  - B.  $x - 1$  and  $x - 6$
  - C.  $x + 1$  and  $x + 6$
  - D.  $x - 1$  and  $x - 4$
  - E.  $x - 2$  and  $x - 6$
- (c) Multiply each term by the expressions you chose and simplify. Which of the following linear equations does the rational equation simplify to?
- A.  $4(x + 1) = -2(x + 6)$
  - B.  $4(x + 6) = -2(x + 1)$
  - C.  $4(x + 1)(x + 6) = -2(x + 1)(x + 6)$
  - D.  $4(x + 1) = -2(-x - 6)$
  - E.  $4(x + 6) = -2(-x - 1)$
- (d) Solve the linear equation. Check your answer using the original rational equation.

## Rational Equations (EQ6)

**Observation 1.6.6** In [Activity 1.6.5](#), you may have noticed that the resulting linear equation looked like the result of cross-multiplying. This is no coincidence! Cross-multiplying is a method of clearing out fractions that works specifically when the equation is in proportional form:  $\frac{a}{b} = \frac{c}{d}$ .

### Rational Equations (EQ6)

**Activity 1.6.7** Consider the rational equation

$$\frac{x}{x+2} = -\frac{2}{x+2} - \frac{2}{5}$$

- (a) What value(s) should be excluded as possible solutions?
- (b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?
  - A.  $x + 2$ ,  $x + 2$ , and 5
  - B.  $x + 2$  and 5
  - C.  $x + 2$
  - D. 5
- (c) Multiply each term by the expressions you chose and simplify. You should end up with a linear equation.
- (d) Solve the linear equation. Check your answer using the original rational equation.



## Rational Equations (EQ6)

**Observation 1.6.8** [Activity 1.6.7](#) demonstrates why it is so important to determine excluded values and check our answers when solving rational equations. Just because a number is a solution to the *linear* equation we found, it doesn't mean it is automatically a solution to the *rational* equation we started with.

### Rational Equations (EQ6)

**Activity 1.6.9** Consider the rational equation

$$\frac{2x}{x-1} - \frac{3}{x-3} = \frac{x^2 - 11x + 18}{x^2 - 4x + 3}$$

- (a) What values should be excluded as possible solutions? Select all that apply.
- A. 1
  - B. 2
  - C. 3
  - D. 4
  - E. 5
- (b) To solve, we'll begin by clearing out any fractions involved. Which of the following should we multiply each term by to clear out all of the fractions?
- A.  $x - 1$
  - B.  $x - 1$  and  $x - 3$
  - C.  $x - 1$ ,  $x - 3$ , and  $x^2 - 4x + 3$
  - D.  $x - 1$  and  $x^2 - 4x + 3$
  - E.  $x - 3$ , and  $x^2 - 4x + 3$
- (c) Multiply each term by the expressions you chose and simplify. Notice that the result is a quadratic equation. Which of the following quadratic equations does the rational equation simplify to?
- A. add in choices
- (d) Solve the quadratic equation. Check your answer using the original rational equation.

## 1.7 Quadratic and Rational Inequalities (EQ7)

### Objectives

- Solve a rational equation.

# Chapter 2

## Functions (FN)

### Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx  
By the end of this chapter, you should be able to...

1. Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.
2. Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.
3. Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maximums and minimums, and where it is increasing and decreasing in interval notation.
4. Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.
5. Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.
6. Find the inverse of a one-to-one function.

## 2.1 Introduction to Functions (FN1)

### Objectives

- Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.

## 2.2 Function Notation (FN2)

### Objectives

- Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.

## 2.3 Characteristics of a Function's Graph (FN3)

### Objectives

- Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maximums and minimums, and where it is increasing and decreasing in interval notation.

## 2.4 Transformation of Functions (FN4)

### Objectives

- Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.



## 2.5 Combining and Composing Functions (FN5)

### Objectives

- Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.

## 2.6 Finding the Inverse Function (FN6)

### Objectives

- Find the inverse of a one-to-one function.

## Chapter 3

# Linear Functions (LF)

# Chapter 4

## Polynomial and Rational Functions (PR)

### Objectives

BIG IDEA for the chapter goes here, in outcomes/main.ptx

By the end of this chapter, you should be able to...

1. Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.
2. Use quadratic models to solve an application problem and establish conclusions.
3. Rewrite a rational function as a polynomial plus a proper rational function.
4. Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.
5. Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.
6. Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

## 4.1 Graphing Quadratic Functions (PR1)

### Objectives

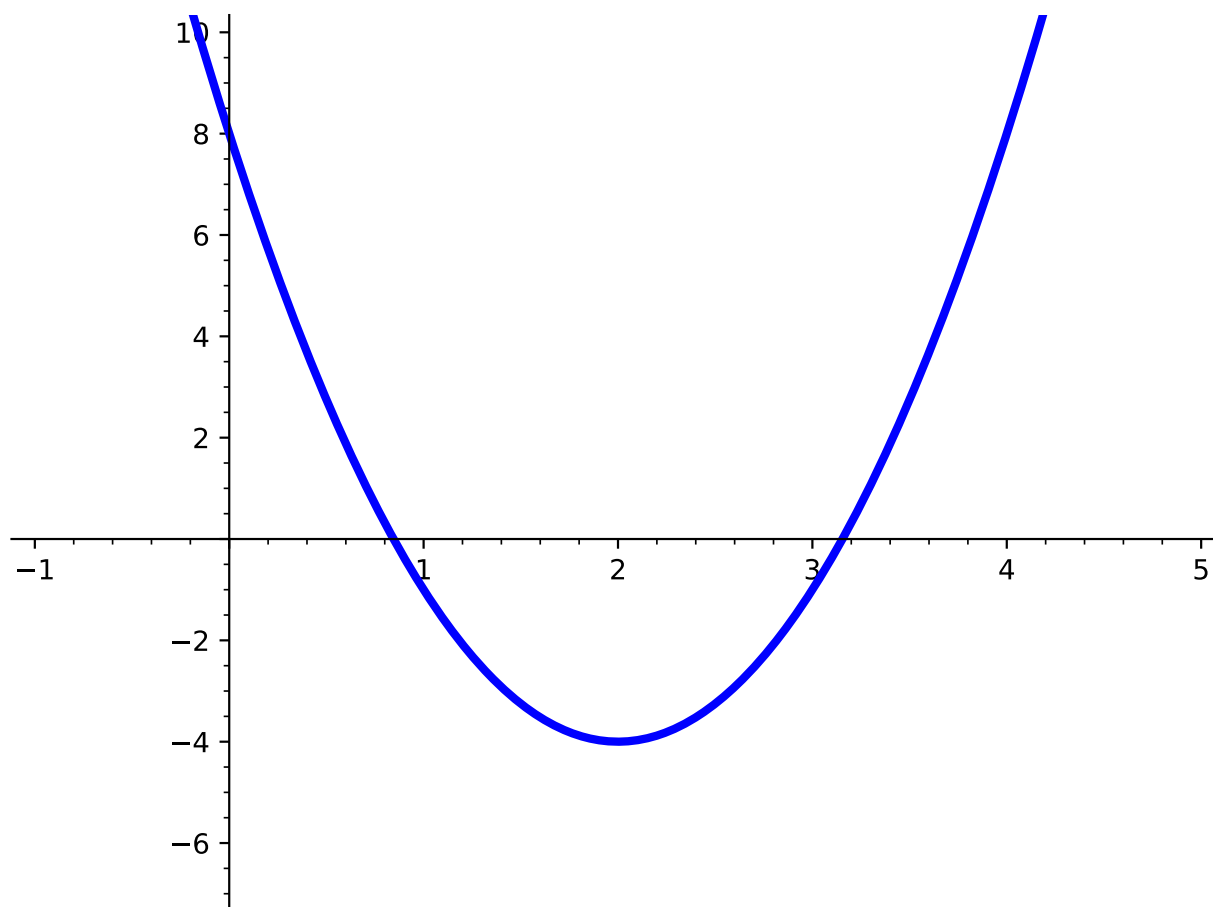
- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

## Graphing Quadratic Functions (PR1)

**Observation 4.1.1** Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.

### Graphing Quadratic Functions (PR1)

**Activity 4.1.2** Use the graph of the quadratic function  $f(x) = 3(x - 2)^2 - 4$  to answer the questions below.



**Figure 4.1.3**

- (a) Make a table for values of  $f(x)$  corresponding to the given  $x$ -values. What is happening to the  $y$ -values as the  $x$ -values increase? Do you notice any other patterns of the  $y$ -values of the table?

## Graphing Quadratic Functions (PR1)

Table 4.1.4

$x$	$f(x)$
-2	
-1	
0	
1	
2	
3	
4	
5	

- (b) At which point  $(x, y)$  does  $f(x)$  have a minimum value? That is, is there a point on the graph that is lower than all other points?
- A. The minimum value appears to occur near  $(0, 8)$ .
  - B. The minimum value appears to occur near  $(-\frac{1}{5}, 10)$ .
  - C. The minimum value appears to occur near  $(2, -4)$ .
  - D. There is no minimum value of this function.
- (c) At which point  $(x, y)$  does  $f(x)$  have a maximum value? That is, is there a point on the graph that is higher than all other points?
- A. The maximum value appears to occur near  $(-2, 44)$ .
  - B. The maximum value appears to occur near  $(-\frac{1}{5}, 10)$ .
  - C. The maximum value appears to occur near  $(2, -4)$ .
  - D. There is no maximum value of this function.



## Graphing Quadratic Functions (PR1)

**Definition 4.1.5** The maximum or minimum of a quadratic function is also known as its **vertex**. The **vertex form** of a quadratic function is given by  $f(x) = a(x - h)^2 + k$ , where  $(h, k)$  is the **vertex** of the parabola and  $x = h$  is the **axis of symmetry**.  $\diamond$

## Graphing Quadratic Functions (PR1)

**Activity 4.1.6** Use the given the quadratic function,  $f(x) = 3(x - 2)^2 - 4$ , to answer the following:

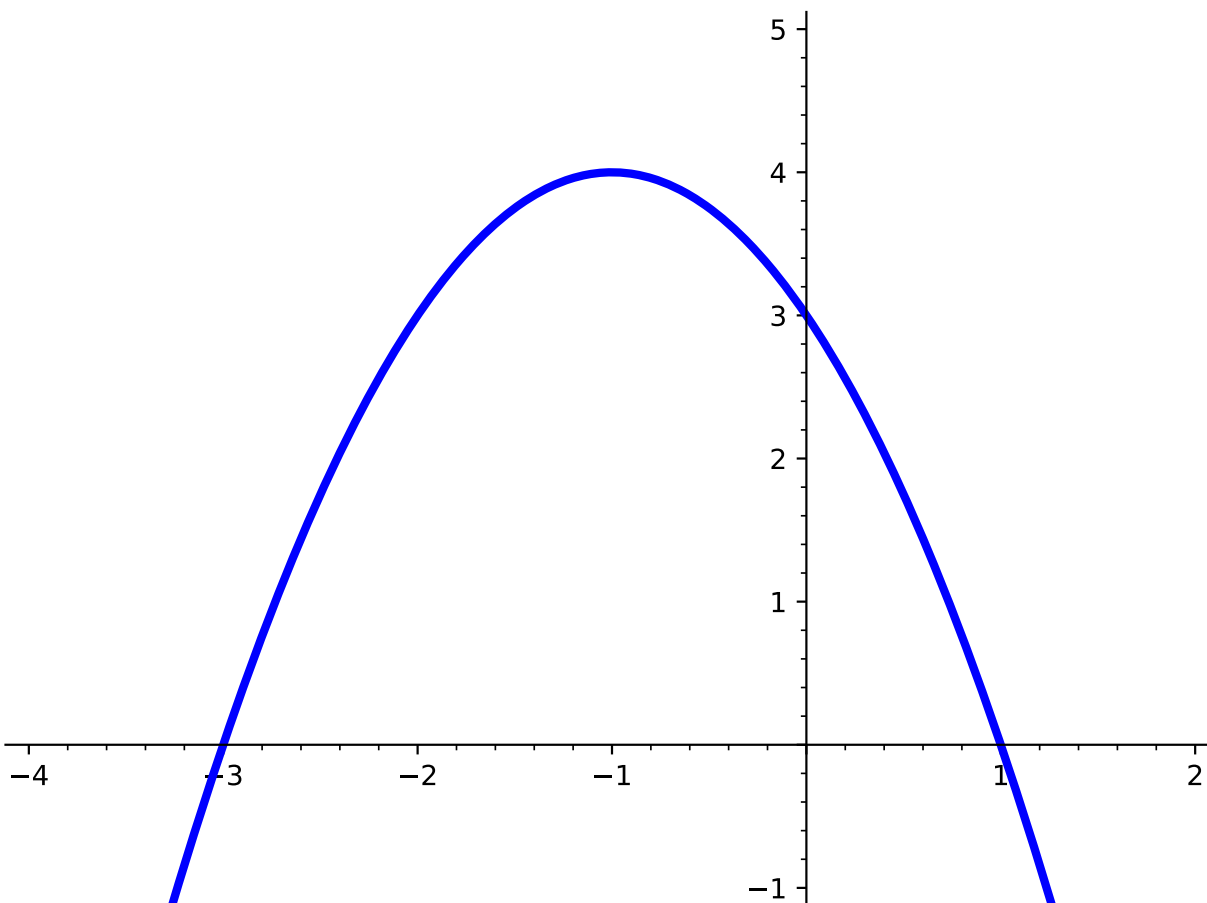
- (a) Applying [Definition 4.1.5](#), what is the vertex and axis of symmetry of  $f(x)$ ?
- A. vertex:  $(2, -4)$ ; axis of symmetry:  $x = 2$
  - B. vertex:  $(-2, 4)$ ; axis of symmetry:  $x = -2$
  - C. vertex:  $(-2, -4)$ ; axis of symmetry:  $x = -2$
  - D. vertex:  $(2, 4)$ ; axis of symmetry:  $x = 2$
- (b) Compare what you got in part *a* with the values you found in [Activity 4.1.2](#). What do you notice?

## Graphing Quadratic Functions (PR1)

**Definition 4.1.7** Given the **standard form** of a quadratic function,  $f(x) = ax^2 + bx + c$ , with real coefficients  $a, b$ , and  $c$ , the **axis of symmetry** is defined as  $x = \frac{-b}{2a}$  and has a **vertex** at the point  $(\frac{-b}{2a}, f(\frac{-b}{2a}))$ .  $\diamond$

### Graphing Quadratic Functions (PR1)

**Activity 4.1.8** Use the graph of the quadratic function to answer the questions below.



**Figure 4.1.9**

(a) Which of the following quadratic functions could be the graph shown in the figure?

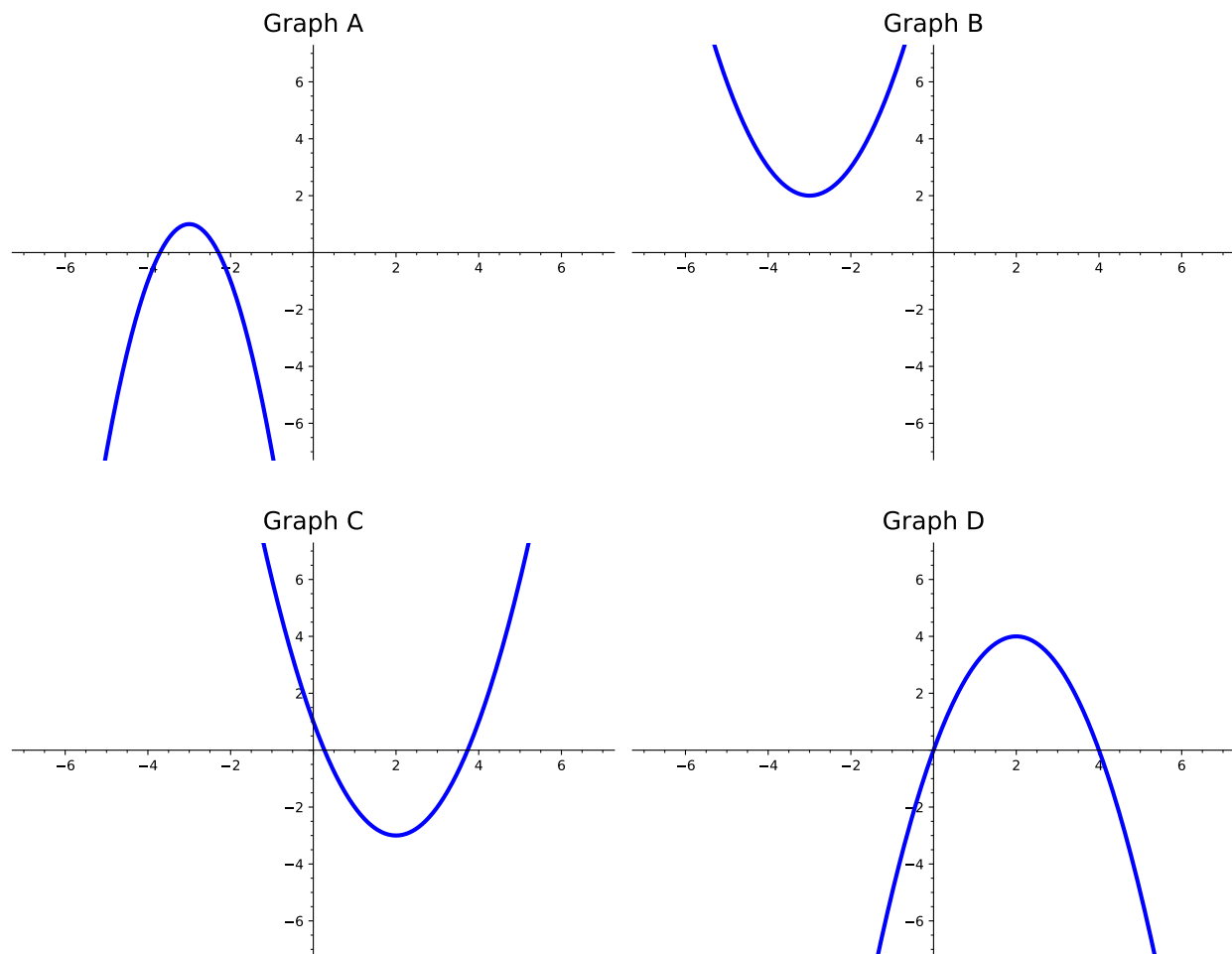
- A.  $f(x) = x^2 + 2x + 3$
- B.  $f(x) = -(x + 1)^2 + 4$
- C.  $f(x) = -x^2 - 2x + 3$
- D.  $f(x) = (x + 1)^2 + 4$

(b) What is the maximum or minimum value?

- |         |         |
|---------|---------|
| A. $-1$ | C. $-3$ |
| B. $4$  | D. $1$  |

## Graphing Quadratic Functions (PR1)

**Activity 4.1.10** Consider the following four graphs of quadratic functions:



(a) Which of the graphs above have a maximum?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

(b) Which of the graphs above have a minimum?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

(c) Which of the graphs above have an axis of symmetry at  $x = 2$ ?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

**Graphing Quadratic Functions (PR1)**

(d) Which of the graphs above represents the function  $f(x) = -(x-2)^2 + 4$ ?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

(e) Which of the graphs above represents the function  $f(x) = x^2 - 4x + 1$ ?

A. Graph A

C. Graph C

B. Graph B

D. Graph D

## Graphing Quadratic Functions (PR1)

**Remark 4.1.11** Notice that the maximum or minimum value of the quadratic function is the  $y$ -value of the vertex. How can you determine if the vertex is a maximum or minimum?

### Graphing Quadratic Functions (PR1)

**Activity 4.1.12** A function  $f(x)$  has a maximum value at 7 and its axis of symmetry at  $x = -2$ .

- (a) Sketch a graph of a function that meets the criteria for  $f(x)$ .
- (b) Was your graph the only possible answer? Try to sketch another graph that meets this criteria.



## Graphing Quadratic Functions (PR1)

**Remark 4.1.13** Other points, such as  $x$ - and  $y$ -intercepts, may be helpful in sketching a more accurate graph of a quadratic function.

## Graphing Quadratic Functions (PR1)

**Activity 4.1.14** Consider the following two quadratic functions  $f(x) = x^2 - 4x + 20$  and  $g(x) = 2x^2 - 8x + 24$  and answer the following questions:

- (a) Applying [Definition 4.1.7](#), what is the vertex and axis of symmetry of  $f(x)$ ?
- A. vertex:  $(2, -16)$ ; axis of symmetry:  $x = 2$
  - B. vertex:  $(-2, 16)$ ; axis of symmetry:  $x = -2$
  - C. vertex:  $(-2, -16)$ ; axis of symmetry:  $x = -2$
  - D. vertex:  $(2, 16)$ ; axis of symmetry:  $x = 2$
- (b) Applying [Definition 4.1.7](#), what is the vertex and axis of symmetry of  $g(x)$ ?
- A. vertex:  $(2, -16)$ ; axis of symmetry:  $x = 2$
  - B. vertex:  $(-2, 16)$ ; axis of symmetry:  $x = -2$
  - C. vertex:  $(-2, -16)$ ; axis of symmetry:  $x = -2$
  - D. vertex:  $(2, 16)$ ; axis of symmetry:  $x = 2$
- (c) What do you notice about  $f(x)$  and  $g(x)$ ?
- (d) Now graph both  $f(x)$  and  $g(x)$  and draw a sketch of each graph on one coordinate plane. How are they similar/different?

## 4.2 Quadratic Models and Meanings (PR2)

### Objectives

- Use quadratic models to solve an application problem and establish conclusions.

## Quadratic Models and Meanings (PR2)

**Activity 4.2.1** A water balloon is tossed vertically from a fifth story window. It's height  $h(t)$ , in feet, at a time  $t$ , in seconds, is modeled by the function

$$h(t) = -16t^2 + 40t + 50$$

- (a) Complete the following table. Do all the values have meaning in terms of the model?

**Table 4.2.2**

$t$	$h(t)$
0	
1	
2	
3	
4	
5	

- (b) Compute the slope of the line joining  $t = 0$  and  $t = 1$ . Then, compute the slope of the line joining  $t = 1$  and  $t = 2$ . What do you notice about the slopes?
- (c) What is the meaning of  $h(0) = 50$ ?
- A. the initial height of the water balloon is 50 feet.
  - B. the water balloon reaches a maximum height of 50 feet.
  - C. the water balloon hits the ground after 50 seconds.
  - D. the water balloon travels 50 feet before hitting the ground.
- (d) Find the vertex of the quadratic function.
- A. (0, 50)
  - B. (1, 74)
  - C. (1.25, 75)
  - D. (3.4, 0)
- (e) What is the meaning of the vertex?
- A. The water balloon reaches a maximum height of 50 feet at the start.
  - B. After 1 second, the water balloon reaches a maximum height of 74 feet.

### Quadratic Models and Meanings (PR2)

- C. After 1.25 seconds, the water balloon reaches the maximum height.
- D. After 3.5 seconds, the water balloon hits the ground.

### Quadratic Models and Meanings (PR2)

**Activity 4.2.3** The population of a small city is given by the function  $P(t) = -50t^2 + 1200t + 32000$ , where  $t$  is the number of years after 2015.

(a) When will the population of the city reach a maximum?

A. 2020

C. 2025

B. 2022

D. 2027

(b) Determine when the population of the city is increasing and when it is decreasing.

(c) When will the population of the city reach 36,000 people?

A. 2019

C. 2027

B. 2025

D. 2035

## Quadratic Models and Meanings (PR2)

**Activity 4.2.4** The unit price of an item affects its supply and demand. That is, if the unit price increases, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of \$6. Market research has suggested that if the owners raise the price to \$8, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

- (a) Which of the following represents a linear function which relates the price of the streaming service  $p$  to the number of subscribers  $Q$ ?

A.  $Q(p) = -2p$

C.  $Q(p) = -2p - 4$

B.  $Q(p) = -2p + 84$

D.  $Q(p) = -2p + 96$

- (b) Using the fact that Revenue =  $pQ$ , which of the following represents the Revenue  $R$  in terms of the price  $p$ .

A.  $R(p) = -2p^2$

C.  $R(p) = -2p^2 - 4p$

B.  $R(p) = -2p^2 + 84p$

D.  $R(p) = -2p^2 + 96p$

- (c) What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. \$10

C. \$24

B. \$19.50

D. \$28.25

- (d) How many subscribers would the company have at this price?

A. 39.5 million

C. 57 million

B. 48 million

D. 76 million

- (e) What is the maximum revenue?

A. 760 million

C. 1152 million

B. 1112 million

D. 1116 million

## Quadratic Models and Meanings (PR2)

**Activity 4.2.5** The owner of a ranch decides to enclose a rectangular region with 240 feet of fencing. To help the fencing cover more land, he plans to use one side of his barn as part of the enclosed region. What is the maximum area the rancher can enclose?



## 4.3 Polynomial Long Division (PR3)

### Objectives

- Rewrite a rational function as a polynomial plus a proper rational function.

## Polynomial Long Division (PR3)

### Activity 4.3.1

(a) Use long division to find the quotient and remainder when 346 is divided by 17.

- A. quotient 20, remainder 3
- B. quotient 20, remainder 6
- C. quotient 6, remainder 20
- D. quotient 6, remainder 3

(b) What is the divisor?

- A. 3
- B. 17
- C. 20
- D. 6

(c) Now write the answer as a mixed number.

- A.  $20 + \frac{3}{17}$
- B.  $20 + \frac{6}{17}$
- C.  $6 + \frac{20}{17}$
- D.  $6 + \frac{3}{17}$

(d) How can you check your answer?

### Polynomial Long Division (PR3)

**Activity 4.3.2** Now use long division to find the quotient and remainder.

(a) Simplify the following rational expression using factoring.  $f(x) = \frac{x^2+2x-8}{x+4}$

A.  $x + 4$

B.  $x - 4$

C.  $x - 2$

D.  $x + 2$

(b) What is the quotient?

A.  $x + 4$

B.  $x - 4$

C.  $x - 2$

D.  $x + 2$

(c) What is the remainder?

A.  $x + 4$

B.  $x - 4$

C.  $x - 2$

D. 0

### Polynomial Long Division (PR3)

**Activity 4.3.3** Repeat this process, using long division, given  $f(x) = \frac{6x^2+5x-10}{2x+3}$ .

(a) What is the result when  $6x^2$  is divided by  $2x$ ? Place this number at the top left when doing long division.

- A. 3
- B.  $3x$
- C.  $3x^2$

(b) Next, multiply this result by  $2x + 3$  and write this below  $6x^2 + 5x - 10$ . Be sure to line up like terms in the same column. What is the result?

- A.  $6x + 9$
- B.  $6x^2 + 9x$
- C.  $6x^3 + 9x^2$

(c) Now subtract the like terms by placing them in the corresponding columns.

- A.  $6x^2 - x - 19$
- B.  $-4x - 10$
- C.  $-6x^3 + 5x - 10$
- D.  $14x - 10$

(d) Next, divide the first term in the previous result by  $2x$ . Write the answer at the top beside the first answer. What is the expression at the top now?

- A.  $3 - 3x^2$
- B.  $3x - 2$
- C.  $3x^2 - 3x^2$
- D.  $3x + 7$

(e) Multiply this resulting term by  $2x + 3$  and write that answer in the proper columns of like terms. What is this result?

- A.  $-6x^3 - 9x^2$

### Polynomial Long Division (PR3)

B.  $-4x - 6$

C.  $14x + 21$

(f) Subtract the like terms. This is the remainder. What is the remainder?

A.  $-6x^3 - 9x^2 - 4x - 10$

B.  $-4$

C.  $-18x - 31$

(g) What is the quotient?

A.  $3 - 3x^2$

B.  $3x - 2$

C.  $0$

D.  $3x + 7$

(h) How can you check your answer? (Hint: Think of regular long division with positive integers.)

### Polynomial Long Division (PR3)

**Activity 4.3.4** Using long division, find the quotient and remainder for the given rational function. Rewrite the function as a polynomial plus a proper rational function, given  $f(x) = \frac{3x^5 - 5x^2 + 2}{x^2 + x - 1}$ .

- (a) What is the quotient?
- (b) What is the remainder?
- (c) What is the divisor?
- (d) Write the rational function as a polynomial plus a proper rational function.
- (e) How can you check your answer? (Hint: Think of regular long division with positive integers.)

### Polynomial Long Division (PR3)

**Activity 4.3.5** Using long division, find the quotient and remainder for the given rational function,  $f(x) = \frac{3x^4 - 5x^2 + 2}{x - 1}$ .

- (a) What is the quotient?
- (b) What is the remainder?
- (c) How can you check your answer? (Hint: Think of regular long division with positive integers.)

### Polynomial Long Division (PR3)

**Activity 4.3.6** Using synthetic division, find the quotient and remainder for the given rational function.  $f(x) = \frac{x^2+2x-8}{x+4}$

- (a) Write only the coefficients of the numerator in a row.
  - (b) Set the denominator equal to zero and solve for x. Place this number out in front, in the far left corner.
  - (c) Make 3 rows and drop the 1st coefficient down from the top row to the bottom row.
  - (d) Now multiply the number in the top left corner by the first coefficient and place this number under the 2nd coefficient.
  - (e) Add the numbers in the 2nd column and write the result in the same column, third row.
  - (f) Repeat the process with the resulting number. Multiply this number by the number in the upper left corner. Place it in the 2nd row, third column. Add the numbers in the third column and place this in the third row and column. What number did you get?
- A. 1
  - B. -2
  - C. 0
  - D. -4

How do the numbers in the third row relate to Activity 4.3.2?



### Polynomial Long Division (PR3)

**Activity 4.3.7** Now use synthetic division, to find the quotient and remainder for the given rational function.  $f(x) = \frac{x^3+2x^2-3x+4}{x-2}$

(a) What is the quotient?

A.  $x^2 - 3$

B.  $x^3 + 2x - 3x + 4$

C.  $x^2 + 4x + 5$

D. 5

(b) What is the remainder?

A. 14

B. -10

C. 5

D. 6

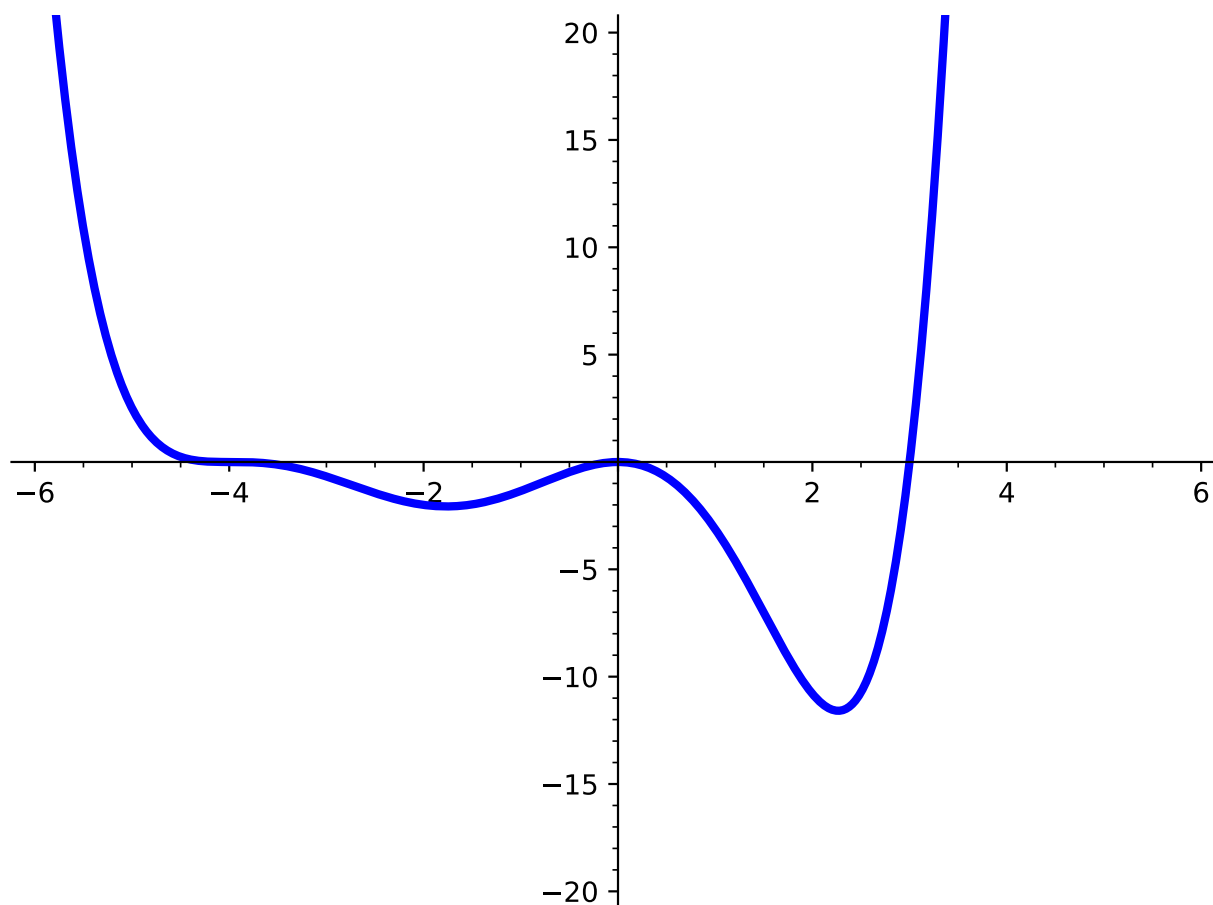
## 4.4 Zeroes of Polynomial Functions (PR4)

### Objectives

- Determine the zeros of a real polynomial function, write a polynomial function given information about its zeros and their multiplicities, and apply the Factor Theorem and the Fundamental Theorem of Algebra.

### Zeroes of Polynomial Functions (PR4)

**Activity 4.4.1** Write the polynomial function in factored form using information from the graph below.



**Figure 4.4.2**

- (a) Using the given graph, what are the real zeros of this function? Select all that apply.
- A. 0
  - B. 1
  - C. -3
  - D. 3
  - E. 4
  - F. -4
- (b) What are the least possible multiplicities for each zero?

**Zeroes of Polynomial Functions (PR4)**

- (c) What is the least degree of the function?
- A. 3
  - B. 4
  - C. 5
  - D. 6
- (d) Describe the end behavior of the graph.
- A. As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$
  - B. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$
  - C. As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$
  - D. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$
- (e) Combining the information from the end behavior with the degree of the function, will the leading coefficient be positive or negative?
- A. positive
  - B. negative
- (f) Given the point  $(2, \frac{-54}{5})$  is on the curve, and using the information in parts (a) through (e), write the function for the graph above in factored form.

### Zeroes of Polynomial Functions (PR4)

**Activity 4.4.3** Given the function  $f(x) = x^6 - 3x^4 - 2x^3$ ,

- (a) Find all the zeros and their corresponding multiplicities.
- (b) Write the function  $f(x)$  in factored form using linear factors.

### Zeroes of Polynomial Functions (PR4)

**Activity 4.4.4** Given the function  $f(x) = 2x^4 - 4x^3 + 10x^2 - 16x + 8$ ,

- (a) Find all the zeros and their corresponding multiplicities.
- (b) Write the function  $f(x)$  in factored form using linear factors.

### Zeroes of Polynomial Functions (PR4)

**Activity 4.4.5** The zeros of a function are  $x=2$ , with multiplicity 1,  $x=-1$ , with multiplicity 2 and  $x=i$ .

- (a) Given the information above, find a polynomial function with real coefficients of least degree.

## 4.5 Graphs of Polynomial Functions (PR5)

### Objectives

- Find the intercepts, estimated locations of maxima and minima, and end behavior of a polynomial function, and use this information to sketch the graph.



## Graphs of Polynomial Functions (PR5)

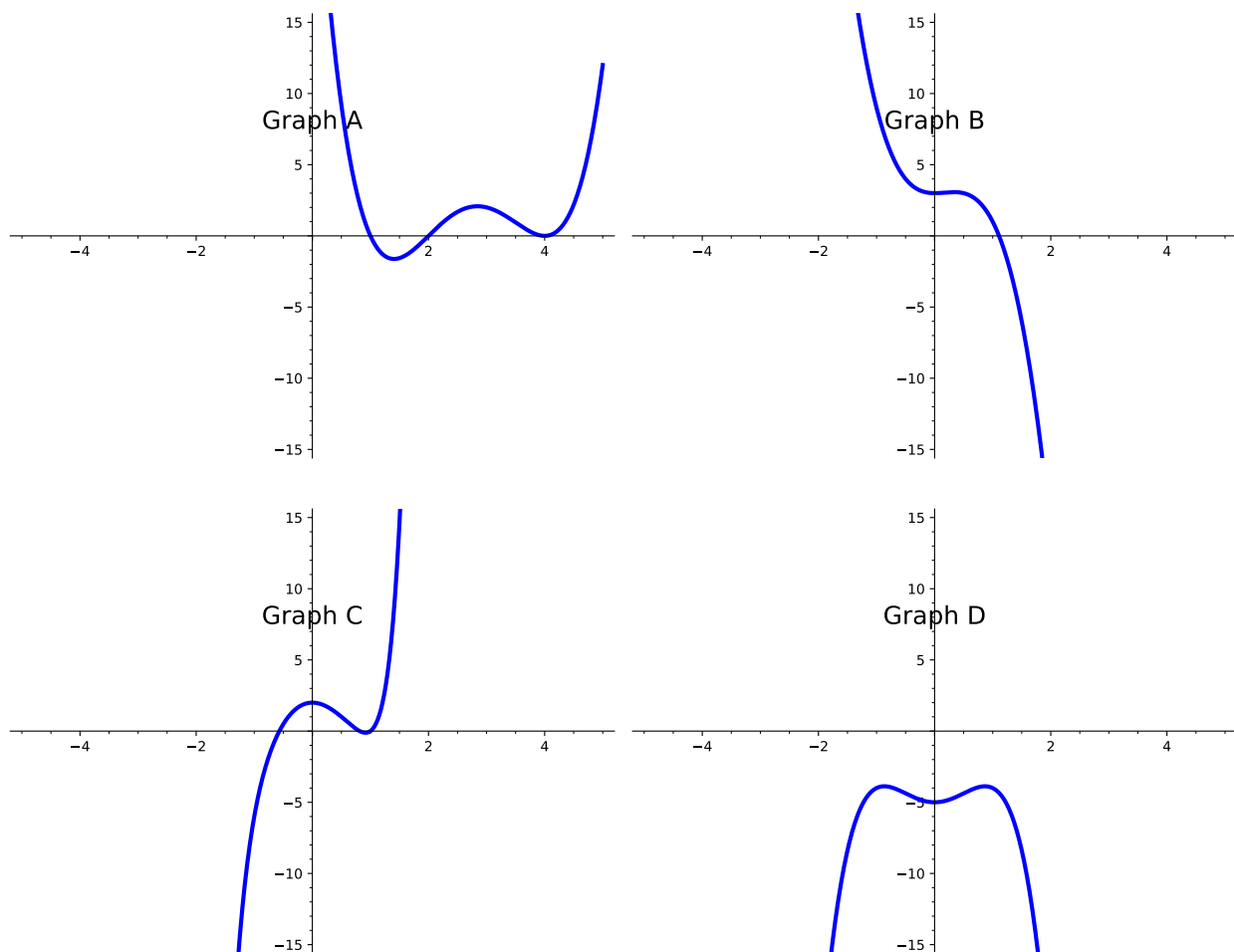
**Remark 4.5.1** Just like with quadratic functions, we should be able to determine key characteristics that will help guide us in creating a sketch of any polynomial function. We can start by finding both  $x$  and  $y$ -intercepts and then explore other characteristics polynomial functions can have. Recall that the **zeros** of a function are the  **$x$ -intercepts** - i.e., the values of  $x$  that cross or touch the  $x$ -axis. Just like with quadratic functions, we can find the zeros of a function by setting the function equal to 0 and solving for  $x$ .

## Graphs of Polynomial Functions (PR5)

**Definition 4.5.2** The **end behavior** of a polynomial function describes the behavior of the graph at the "ends" of the function. In other words, as we move to the right of the graph (as the  $x$  values increase), what happens to the  $y$  values? Similarly, as we move to the left of the graph (as the  $x$  values decrease), what happens to the  $y$  values?  $\diamond$

## Graphs of Polynomial Functions (PR5)

**Activity 4.5.3** Use the graphs of the following polynomial functions to answer the questions below.



- (a) How would you describe the behavior of Graph A as you approach the ends?
- A. Graph A rises on the left and on the right.
  - B. Graph A rises on the left, but falls on the right.
  - C. Graph A rises on the right, but falls on the left.
  - D. Graph A falls on the left and on the right.
- (b) How would you describe the behavior of Graph B as you approach the ends?
- A. Graph B rises on the left and on the right.
  - B. Graph B rises on the left, but falls on the right.

### Graphs of Polynomial Functions (PR5)

- C. Graph B rises on the right, but falls on the left.
  - D. Graph B falls on the left and on the right.
- (c) How would you describe the behavior of Graph C as you approach the ends?
- A. Graph C rises on the left and on the right.
  - B. Graph C rises on the left, but falls on the right.
  - C. Graph C rises on the right, but falls on the left.
  - D. Graph C falls on the left and on the right.
- (d) How would you describe the behavior of Graph D as you approach the ends?
- A. Graph D rises on the left and on the right.
  - B. Graph D rises on the left, but falls on the right.
  - C. Graph D rises on the right, but falls on the left.
  - D. Graph D falls on the left and on the right.

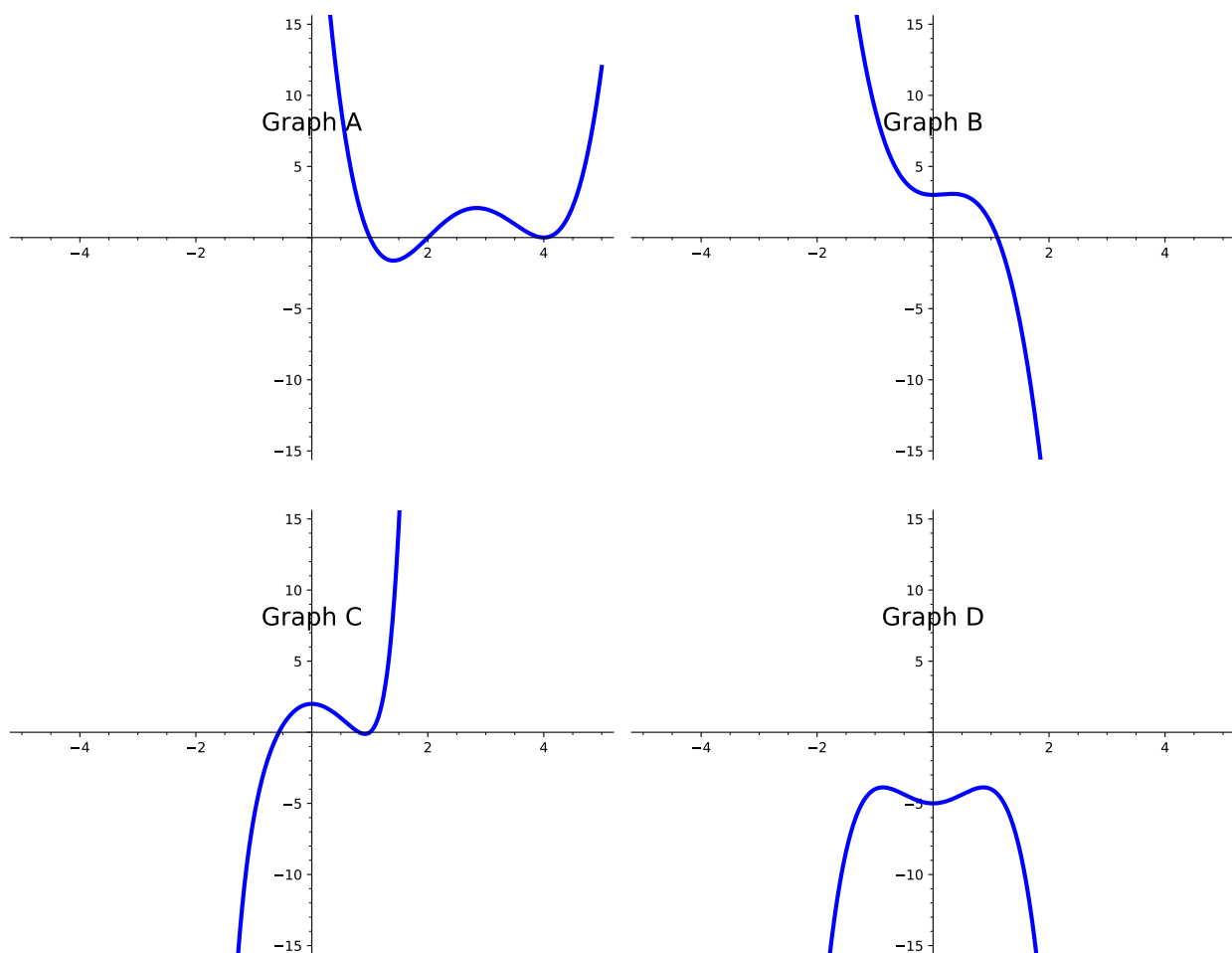
## Graphs of Polynomial Functions (PR5)

**Definition 4.5.4** Typically, when given an equation of a polynomial function, we look at the **degree** and **leading coefficient** to help us determine the behavior of the ends. The **degree** is the highest exponential power in the polynomial. The **leading coefficient** is the number written in front of the variable with the highest exponential power.  $\diamond$

## Graphs of Polynomial Functions (PR5)

**Activity 4.5.5** Let's refer back to the graphs in [Activity 4.5.3](#) and look at the equations of those polynomial functions. Let's apply [Definition 4.5.4](#) to see if we can determine how the degree and leading coefficients of those graphs affect their end behavior.

- Graph A:  $f(x) = -11x^3 + 32 + 42x^2 + x^4 - 64x$
- Graph B:  $g(x) = 2x^2 + 3 - 4x^3$
- Graph C:  $h(x) = x^7 + 2x^3 - 5x^2 + 2$
- Graph D:  $j(x) = 3x^2 - 2x^4 - 5$



(a) What is the degree and leading coefficient of Graph A?

- A. Degree:  $-64$ ; Leading Coefficient:  $4$
- B. Degree:  $4$ ; Leading Coefficient:  $0$
- C. Degree:  $1$ ; Leading Coefficient:  $-64$

### Graphs of Polynomial Functions (PR5)

- D. Degree: 4; Leading Coefficient: 1
- (b) What is the degree and leading coefficient of Graph B?
- A. Degree: 3; Leading Coefficient:  $-4$
  - B. Degree:  $-4$ ; Leading Coefficient: 3
  - C. Degree: 2; Leading Coefficient: 3
  - D. Degree: 3; Leading Coefficient: 4
- (c) What is the degree and leading coefficient of Graph C?
- A. Degree:  $-5$ ; Leading Coefficient: 2
  - B. Degree: 0; Leading Coefficient: 7
  - C. Degree:  $-5$ ; Leading Coefficient: 3
  - D. Degree: 7; Leading Coefficient: 1
- (d) What is the degree and leading coefficient of Graph D?
- A. Degree:  $-2$ ; Leading Coefficient: 4
  - B. Degree: 3; Leading Coefficient: 2
  - C. Degree:  $-2$ ; Leading Coefficient: 4
  - D. Degree:  $-5$ ; Leading Coefficient: 4
- (e) Notice that Graph A and Graph D have their ends going in the same direction. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?
- (f) Notice that Graph B and Graph C have their ends going in opposite directions. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

## Graphs of Polynomial Functions (PR5)

**Remark 4.5.6** From [Activity 4.5.5](#), we saw that the degree and leading coefficient of a polynomial function can give us more clues about the behavior of the function. In summary, we know:

- If the degree is even, the ends of the polynomial function will be going in the same direction. If the leading coefficient is positive, both ends will be pointing up. If the leading coefficient is negative, both ends will be pointing down.
- If the degree is odd, the ends of the polynomial function will be going in opposite directions. If the leading coefficient is positive, the left end will fall and the right end will rise. If the leading coefficient is negative, the left end will rise and the right end will fall.



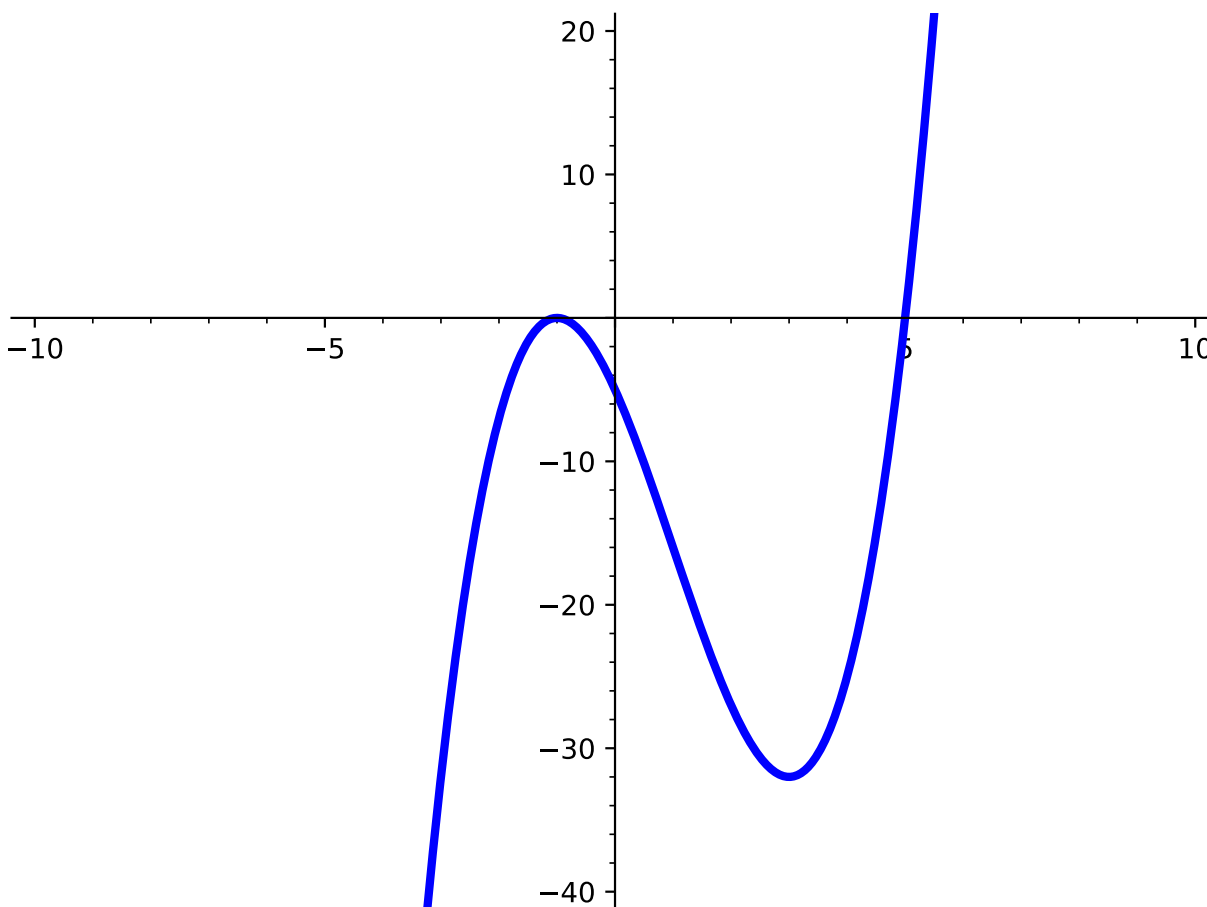
## Graphs of Polynomial Functions (PR5)

**Definition 4.5.7** When describing end behavior, mathematicians typically use **arrow notation**. Just as the name suggests, arrows are used to indicate the behavior of certain values on a graph.

For end behavior, students are often asked to determine the behavior of  $y$ -values as  $x$ -values either increase or decrease. The statement "As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$ " can be translated to "As  $x$  approaches infinity (or as  $x$  increases),  $f(x)$  (or the  $y$ -values) go to negative infinity (i.e., it decreases)."  $\diamond$

## Graphs of Polynomial Functions (PR5)

**Activity 4.5.8** Use the graph of  $f(x)$  to answer the questions below.



**Figure 4.5.9**

- (a) How would you describe the end behavior of  $f(x)$ ?
- A.  $f(x)$  rises on the left and on the right.
  - B.  $f(x)$  rises on the left, but falls on the right.
  - C.  $f(x)$  rises on the right, but falls on the left.
  - D.  $f(x)$  falls on the left and on the right.
- (b) How would you describe the end behavior of  $f(x)$  using arrow notation?
- A. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$   
As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$
  - B. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$   
As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$

### Graphs of Polynomial Functions (PR5)

C. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$

D. As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$

## Graphs of Polynomial Functions (PR5)

**Definition 4.5.10** When graphing polynomial functions, you may notice that these functions have some "hills" and "valleys." These characteristics of the graph are known as the **local maxima** and **local minima** of the graph - similar to what we've already seen with quadratic functions. Unlike quadratic functions, however, a polynomial graph can have many local maxima/minima (quadratic functions only have one).  $\diamond$

### Graphs of Polynomial Functions (PR5)

**Activity 4.5.11** Sketch the function,  $f(x) = (x - 2)(x + 1)(x - 3)^2$ , by first finding the given characteristics.

- Find the zeros of  $f(x)$ .
- Find the multiplicities at each zero.
- Find the  $y$ -intercept of  $f(x)$ .
- Describe the end behavior of  $f(x)$ .
- Estimate where any local maximums and minimums may occur.

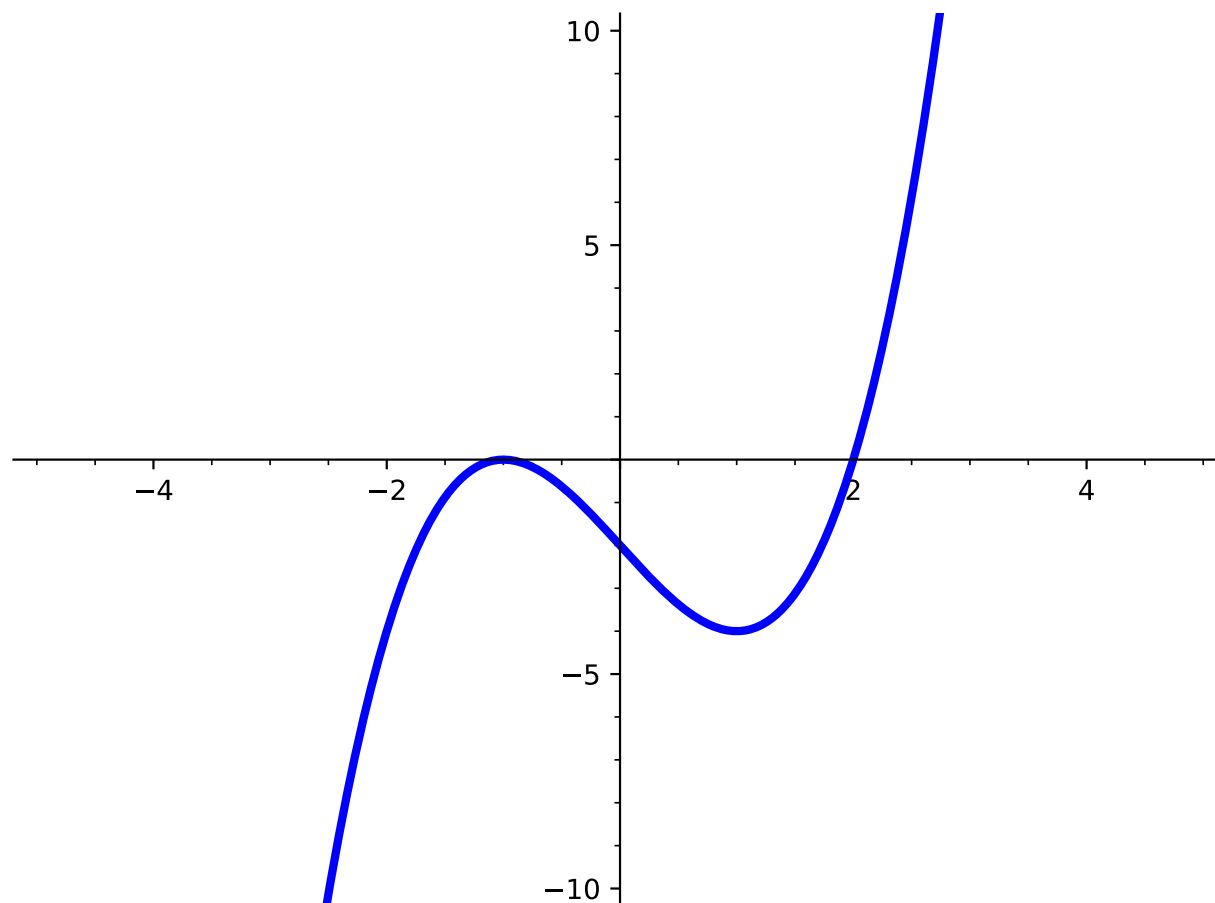
### Graphs of Polynomial Functions (PR5)

**Activity 4.5.12** Sketch the graph of a function  $f(x)$  that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

- (a) The  $x$ -intercepts of  $f(x)$  are 0, 2, and 5.
- (b)  $f(x)$  has one maximum at 0.  $f(x)$  has one minimum at  $-5$  and another at  $-16$ .
- (c) The end behavior of  $f(x)$  is given as:
  - As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$
  - As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$

### Graphs of Polynomial Functions (PR5)

**Activity 4.5.13** Now that we know all the different characteristics of polynomials, we should also be able to identify them from a graph. Use the graph below to find the given characteristics.



**Figure 4.5.14**

- (a) What are the  $x$ -intercept(s) of the polynomial function? Select all that apply.
- |              |              |
|--------------|--------------|
| A. $(1, 0)$  | C. $(2, 0)$  |
| B. $(-1, 0)$ | D. $(0, -2)$ |
- (b) What are the  $y$ -intercept(s) of the polynomial function?
- |              |              |
|--------------|--------------|
| A. $(1, 0)$  | C. $(2, 0)$  |
| B. $(-1, 0)$ | D. $(0, -2)$ |
- (c) How many zeros does this polynomial function have?

**Graphs of Polynomial Functions (PR5)**

A. 0

C. 2

B. 1

D. 3

(d) At what point is the local minimum located?

A.  $(2, -4)$

D.  $(1, -4)$

B.  $(-1, 0)$

E.  $(2, 0)$

C.  $(-2, 0)$

(e) At what point is the local maximum located?

A.  $(2, -4)$

D.  $(1, -4)$

B.  $(-1, 0)$

E.  $(2, 0)$

C.  $(-2, 0)$

(f) How do you describe the behavior of the polynomial function as  $x \rightarrow \infty$ ?

A. the  $y$ -values go to negative infinity

C. the  $y$ -values go to positive infinity

B.  $f(x) \rightarrow \infty$

D.  $f(x) \rightarrow -\infty$

(g) How do you describe the behavior of the polynomial function as  $x \rightarrow -\infty$ ?

A. the  $y$ -values go to negative infinity

C. the  $y$ -values go to positive infinity

B.  $f(x) \rightarrow \infty$

D.  $f(x) \rightarrow -\infty$



### Graphs of Polynomial Functions (PR5)

**Activity 4.5.15** Use the given function,  $f(x) = (x + 1)^2(x - 5)$ , to answer the following questions.

(a) What are the zeros of  $f(x)$ ?

- |             |            |
|-------------|------------|
| A. $-1, -5$ | C. $1, -5$ |
| B. $-1, 5$  | D. $1, 5$  |

(b) What are the multiplicities at each zero?

- A. At  $x = -1$ , the multiplicity is even.  
At  $x = 5$ , the multiplicity is even.
- B. At  $x = -1$ , the multiplicity is even.  
At  $x = 5$ , the multiplicity is odd.
- C. At  $x = -1$ , the multiplicity is odd.  
At  $x = 5$ , the multiplicity is even.
- D. At  $x = -1$ , the multiplicity is odd.  
At  $x = 5$ , the multiplicity is odd.

(c) What is the end behavior of  $f(x)$ ?

- A.  $f(x)$  rises on the left and on the right.
- B.  $f(x)$  rises on the left, but falls on the right.
- C.  $f(x)$  rises on the right, but falls on the left.
- D.  $f(x)$  falls on the left and on the right.

(d) Using what you know about the zeros, multiplicities, and end behavior, where on the graph can we estimate the local maxima and minima to be?

(e) Now look at the graph of  $f(x)$ . At which zero does a local maximum or local minimum occur? Explain how you know.

## Graphs of Polynomial Functions (PR5)

**Remark 4.5.16** We can estimate where these local maxima and minima occur by looking at other characteristics, such as multiplicities and end behavior.

From [Activity 4.5.15](#), we saw that when the function touches the  $x$ -axis at a zero, then that zero could be either a local maximum or a local minimum of the graph. When the function crosses the  $x$ -axis, however, the local maximum or local minimum occurs between the zeros.

## 4.6 Properties and Graphs of Rational Functions (PR6)

### Objectives

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

## Properties and Graphs of Rational Functions (PR6)

**Definition 4.6.1** A function  $r$  is **rational** provided that it is possible to write  $r$  as the ratio of two polynomials,  $p$  and  $q$ . That is,  $r$  is rational provided that for some polynomial functions  $p$  and  $q$ , we have

$$r(x) = \frac{p(x)}{q(x)}.$$



## Properties and Graphs of Rational Functions (PR6)

**Observation 4.6.2** Rational functions occur in many applications, so our goal in this lesson is to learn about their properties and be able to graph them. In particular we want to investigate the domain, end behavior, and zeros of rational functions.

## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.3** Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

- (a) Find  $r(1)$ ,  $r(2)$ ,  $r(3)$ , and  $r(4)$ .
- (b) Label each of these four points as giving us information about the DOMAIN of  $r(x)$ , information about the ZEROES of  $r(x)$ , or NEITHER.

## Properties and Graphs of Rational Functions (PR6)

**Definition 4.6.4** Let  $p$  and  $q$  be polynomial functions so that  $r(x) = \frac{p(x)}{q(x)}$  is a rational function. The **domain** of  $r$  is the set of all real numbers except those for which  $q(x) = 0$ .  $\diamond$

## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.5** Let's investigate the domain of  $r(x)$  more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

- (a) Rewrite  $r(x)$  by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?
- (b) The function was not defined for  $x = 3$ . Make a table for values of  $r(x)$  near  $x = 3$ .

**Table 4.6.6**

$x$	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

- (c) Which of the following describe the behavior of the graph near  $x = 3$ ?
- A. As  $x \rightarrow 3$ ,  $r(x)$  approaches a finite number
  - B. As  $x \rightarrow 3$  from the left,  $r(x) \rightarrow \infty$
  - C. As  $x \rightarrow 3$  from the left,  $r(x) \rightarrow -\infty$
  - D. As  $x \rightarrow 3$  from the right,  $r(x) \rightarrow \infty$
  - E. As  $x \rightarrow 3$  from the right,  $r(x) \rightarrow -\infty$
- (d) The function was also not defined for  $x = 1$ . Make a table for values of  $r(x)$  near  $x = 1$ .



# Properties and Graphs of Rational Functions (PR6)

**Table 4.6.7**

$x$	$r(x)$
0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

- (e) Which of the following describe the behavior of the graph near  $x = 1$ ?
- A. As  $x \rightarrow 1$ ,  $r(x)$  approaches a finite number
  - B. As  $x \rightarrow 1$  from the left,  $r(x) \rightarrow \infty$
  - C. As  $x \rightarrow 1$  from the left,  $r(x) \rightarrow -\infty$
  - D. As  $x \rightarrow 1$  from the right,  $r(x) \rightarrow \infty$
  - E. As  $x \rightarrow 1$  from the right,  $r(x) \rightarrow -\infty$
- (f) The function is behaving differently near  $x = 1$  than it is near  $x = 3$ . Can you see anything in the factored form of  $r(x)$  that may help you account for the difference?

## Properties and Graphs of Rational Functions (PR6)

**Remark 4.6.8 Features of a rational function.** Let  $r(x) = \frac{p(x)}{q(x)}$  be a rational function.

- If  $p(a) = 0$  and  $q(a) \neq 0$ , then  $r(a) = 0$ , so  $r$  has a **zero** at  $x = a$ .
- If  $q(a) = 0$  and  $p(a) \neq 0$ , then  $r(a)$  is undefined and  $r$  has a **vertical asymptote** at  $x = a$ .
- If  $p(a) = 0$  and  $q(a) = 0$  and we can show that there is a finite number  $L$  such that  $r(x) \rightarrow L$ , then  $r(a)$  is not defined and  $r$  has a **hole** at the point  $(a, L)$ .

## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.9** Another property of rational functions we want to explore is the end behavior. This means we want to explore what happens to a given rational function  $r(x)$  when  $x$  goes toward positive infinity or negative infinity.

(a) Consider the rational function  $r(x) = \frac{1}{x^3}$ . Plug in some very large positive numbers for  $x$  to see what  $r(x)$  is tending toward. Which of the following best describes the behavior of the graph as  $x$  approaches positive infinity?

- A. As  $x \rightarrow \infty$ ,  $r(x) \rightarrow \infty$ .
- B. As  $x \rightarrow \infty$ ,  $r(x) \rightarrow -\infty$ .
- C. As  $x \rightarrow \infty$ ,  $r(x) \rightarrow 0$ .
- D. As  $x \rightarrow \infty$ ,  $r(x) \rightarrow 1$ .

(b) Now let's look at  $r(x)$  as  $x$  tends toward negative infinity. Plug in some very large negative numbers for  $x$  to see what  $r(x)$  is tending toward. Which of the following best describes the behavior of the graph as  $x$  approaches negative infinity?

- A. As  $x \rightarrow -\infty$ ,  $r(x) \rightarrow \infty$ .
- B. As  $x \rightarrow -\infty$ ,  $r(x) \rightarrow -\infty$ .
- C. As  $x \rightarrow -\infty$ ,  $r(x) \rightarrow 0$ .
- D. As  $x \rightarrow -\infty$ ,  $r(x) \rightarrow 1$ .

## Properties and Graphs of Rational Functions (PR6)

**Observation 4.6.10** We can generalize what we have just found to any function of the form  $\frac{1}{x^n}$ , where  $n > 0$ . Since  $x^n$  increases without bound as  $x \rightarrow \infty$ , we find that  $\frac{1}{x^n}$  will tend to 0. In fact, the numerator can be any constant and the function will still tend to 0!

Similarly, as  $x \rightarrow -\infty$ , we find that  $\frac{1}{x^n}$  will tend to 0 too.

## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.11** Consider the rational function  $r(x) = \frac{3x^2-5x+1}{7x^2+2x-11}$ .

Observe that the largest power of  $x$  that's present in  $r(x)$  is  $x^2$ . In addition, because of the dominant terms of  $3x^2$  in the numerator and  $7x^2$  in the denominator, both the numerator and denominator of  $r$  increase without bound as  $x$  increases without bound.

- (a) In order to understand the end behavior of  $r$ , we will start by writing the function in a different algebraic form.

Multiply the numerator and denominator of  $r$  by  $\frac{1}{x^2}$ . Then distribute and simplify as much as possible in both the numerator and denominator to write  $r$  in a different algebraic form. Which of the following is that new form?

A.  $\frac{3x^4 - 5x^3 + x^2}{7x^4 + 2x^3 - 11x^2}$

B.  $\frac{3 - \frac{5}{x} + \frac{1}{x^2}}{7 + \frac{2}{x} - \frac{11}{x^2}}$

C.  $\frac{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}{\frac{7x^2}{x^2} + \frac{2x}{x^2} - \frac{11}{x^2}}$

D. another wrong answer?

- (b) Now determine the end behavior of each piece of the numerator and each piece of the denominator. Hint: Use [Observation 4.6.10](#) to help!
- (c) Simplify your work from the previous step. Which of the following best describes the end behavior of  $r(x)$ ?

A. As  $x \rightarrow \pm\infty$ ,  $r(x)$  goes to 0.

B. As  $x \rightarrow \pm\infty$ ,  $r(x)$  goes to  $\frac{3}{7}$ .

C. As  $x \rightarrow \pm\infty$ ,  $r(x)$  goes to  $\infty$ .

D. As  $x \rightarrow \pm\infty$ ,  $r(x)$  goes to  $-\infty$ .

## Properties and Graphs of Rational Functions (PR6)

**Observation 4.6.12** If the end behavior of a function tends toward a specific value  $a$ , then we say that the function has a **horizontal asymptote** at  $y = a$ .

## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.13** Find the horizontal asymptote (if one exists) of the following rational functions. Follow the same method we used in [Activity 4.6.11](#).

(a)  $f(x) = \frac{4x^3 - 3x^2 + 6}{9x^3 + 7x - 5}$

A.  $y = 0$

B.  $y = \frac{4}{9}$

C.  $y = -\frac{3}{7}$

D.  $y = -\frac{6}{5}$

E. There is no horizontal asymptote.

(b)  $g(x) = \frac{4x^3 - 3x^2 + 6}{9x^5 + 7x - 5}$

A.  $y = 0$

B.  $y = \frac{4}{9}$

C.  $y = -\frac{3}{7}$

D.  $y = -\frac{6}{5}$

E. There is no horizontal asymptote.

(c)  $h(x) = \frac{4x^5 - 3x^2 + 6}{9x^3 + 7x - 5}$

A.  $y = 0$

B.  $y = \frac{4}{9}$

C.  $y = -\frac{3}{7}$

D.  $y = -\frac{6}{5}$

E. There is no horizontal asymptote.

## Properties and Graphs of Rational Functions (PR6)

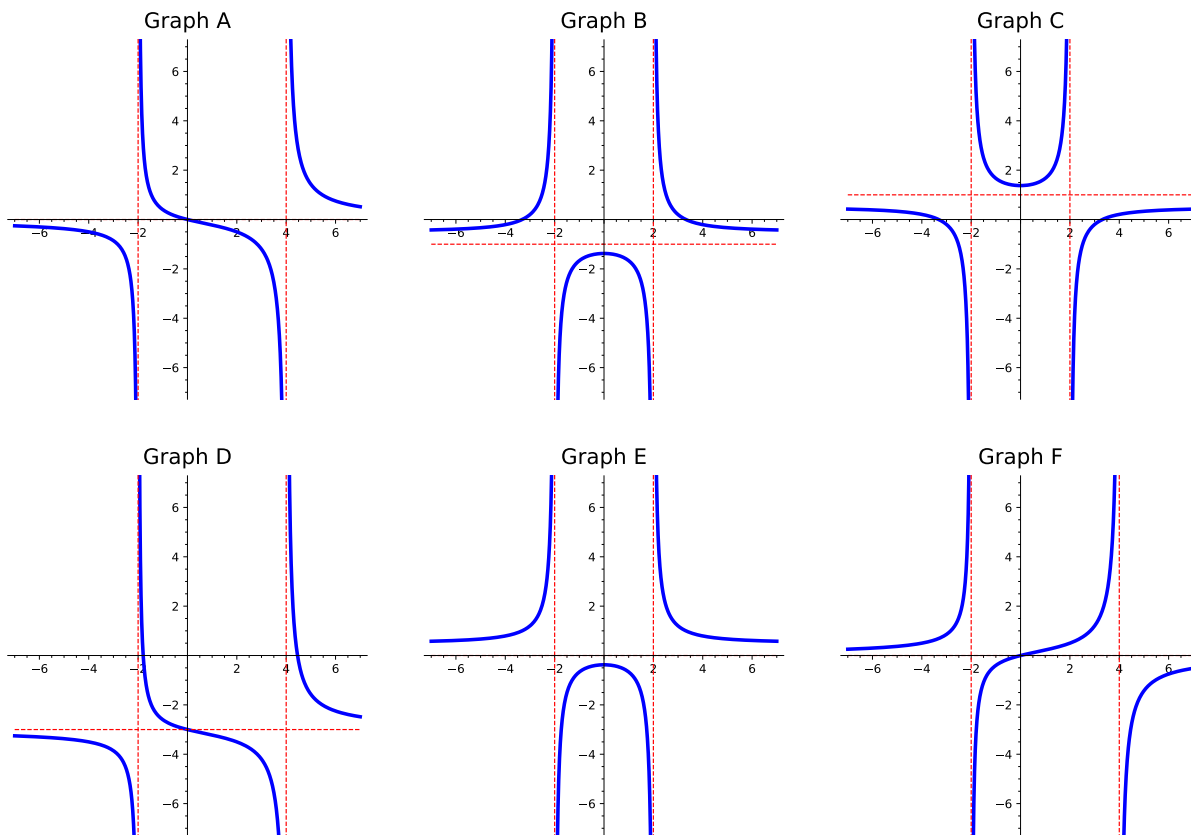
**Activity 4.6.14** Some patterns have emerged from the previous problem. Fill in the rest of the sentences below to describe how to find horizontal asymptotes of rational functions.

- (a) If the degree of the numerator is the same as the degree of the denominator, then...
- (b) If the degree of the numerator is less than the degree of the denominator, then...
- (c) If the degree of the numerator is greater than the degree of the denominator, then...



## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.15** Consider the following six graphs of rational functions:



- (a) Which of the graphs above represents the function  $f(x) = \frac{2x}{x^2 - 2x - 8}$ ?
- (b) Which of the graphs above represents the function  $g(x) = \frac{x^2 + 3}{2x^2 - 8}$ ?

## Properties and Graphs of Rational Functions (PR6)

**Activity 4.6.16** MATCHING ACTIVITY INCLUDING GRAPHS WITH HOLES

(a) QUESTIONS HERE!

## Properties and Graphs of Rational Functions (PR6)

### Activity 4.6.17

(a) Find the roots of the rational function  $f(x) = \frac{-(x-1)(x-4)}{2(x+3)^2(x-1)}$ .

(b) Find the  $y$ -intercept of  $y = \frac{-(x-1)(x-4)}{2(x+3)^2(x-1)}$ .

(c) Find any horizontal asymptotes of  $y = \frac{-(x-1)(x-4)}{2(x+3)^2(x-1)}$ .

(d) Find any vertical asymptotes of  $y = \frac{-(x-1)(x-4)}{2(x+3)^2(x-1)}$ .

(e) Find any holes of  $y = \frac{-(x-1)(x-4)}{2(x+3)^2(x-1)}$ .

(f) Sketch  $y = \frac{-(x-1)(x-4)}{2(x+3)^2(x-1)}$ .

**Properties and Graphs of Rational Functions (PR6)**

**Activity 4.6.18** EXTENSION ACTIVITY ABOUT HOLES

(a) QUESTIONS HERE!

## **Colophon**

This book was authored in PreTeXt.