A simple article

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1.1 Iterative methods for linear system

Let:

(1.1.1)
$$\begin{aligned} AX &= B \Leftrightarrow \\ (L+D+U)\,X &= B \Leftrightarrow \end{aligned}$$

(1.1.2)
$$X = A_1 X + B_1$$

where:

$$\begin{aligned} l_{ij} &= \begin{cases} 0, i \leq j \\ a_{ij},, i > j \end{cases} \\ d_{ij} &= \begin{cases} 0, i \neq j \\ a_{ij},, i = j \end{cases} \\ u_{ij} &= \begin{cases} 0, i \geq j \\ a_{ij},, i < j \end{cases} \end{aligned}$$

1.1 Jacobi’s method

$$X^{(k+1)} = -D^{-1}\,(L+U)\,X^{(k)} + D^{-1}B$$

or:

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}}, i = \overline{1, n}$$

1.2 Gauss Seidel method

$$X^{(k+1)} = -\left(D+L\right)^{-1} U X^{(k)} + \left(D+L\right)^{-1} B$$

or:

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}}, i = \overline{1, n}$$

Teorema 1.1 *If matrix A is diagonal-dominant, e.g.:*

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

then Jacobi and Gauss-Seidel methods are convergent to the solution of the system (1.1.1) for every starting matrix $X^{(0)}$.

Teorema 1.2 *If the spectral radius of the matrix A_1 from (1.1.2) is less than 1, then the sequence: $X^{(k+1)} = A_1 X^{(k)} + B_1$ is convergent to the solution of the system (1.1.1) for every starting matrix $X^{(0)}$.*

Exemplul 1.1 *Compare the two methods for the linear system with:*

$$a_{ij} = \begin{cases} 2, i = j \\ -1, |i-j| = 1 \quad i, j = \overline{1, 20} \\ 0 \text{ otherwise} \end{cases}$$

and

$$b_i = \sum_{j=1}^{20} a_{ij}, i = \overline{1, 20},$$

starting vector being the null vector.

1.2 Iterative methods for equation

Let $f : [a, b] \to \mathbb{R}$.

2.1 Bisection method

Teorema 2.1 *If f is continuous and $f(a) f(b) < 0$ then f has a zero on (a, b) and the* *following sequences:*

$$\begin{aligned} a_0 &= a; b_0 = b \\ for\ n &> 0 \\ if \qquad f(a_{n-1}) f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) &< 0 \\ then\ a_n &= a_{n-1}, b_n = \frac{a_{n-1}+b_{n-1}}{2} \\ else\ b_n &= b_{n-1}, a_n = \frac{a_{n-1}+b_{n-1}}{2}. \end{aligned}$$

converge to a zero of function f.

2.2 Newton Raphson Kantorovici method

Teorema 2.2 *If is twice differentiabily on $[a, b]$ and f', f" have constant sign, and $f(a) f(b) < 0$ then f has an unique zero x^* on (a, b) and the sequence:*

$$\begin{aligned} x_0 &= a,\ \text{if}\ f(a) f''(a) > 0 \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

converges to x^*, Moreover:

$$|x_{n+1} - x^*| \leq K |x_n - x^*|^2$$

where K is a constant wich depend on f, f', f''.

2.3 The secant method

Teorema 2.3 *If* f *is twice differentiabily on* $[a,b]$ *and* $f', f"$ *have constant sign, and* $f(a)f(b) < 0$ *then* f *has an unique zero* x^* *on* (a,b) *and the sequence:*

$$
\begin{aligned}
x_0 &= a, x_1 = b \\
x_{n+1} &= x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}}
\end{aligned}
$$

converges to x^**, Moreover:*

$$|x_{n+1} - x^*| \leq K|x_n - x^*|^{\frac{1+\sqrt{5}}{2}}$$

where K *is a constant wich depend on* f, f', f''*.*

2.4 Fix point theorem

Teorema 2.4 *If* f*:*$[a,b] \to [a,b]$ *satisfies:*

$$|f(x) - f(y)| \leq \alpha|x - y|$$

for all $x, y \in [a,b]$ *and* $\alpha \in (0,1)$ *then* f *has an umique zero on* $[a,b]$ *and for every* $x_0 \in [a,b]$ *the following sequence*

$$x_{n+1} = f(x_n)$$

converge to x^*. *Moreover*

$$\left|x_{n+1}-x^*\right| \leq \frac{\alpha^n}{1-\alpha}\left|x_1-x_0\right|.$$

Exemplul 2.1 *Aply the above methods for solving equation:*

$$x^5-7x+1=0$$

on $[0,1]$.