Flow formalism, Floquet AAH model and emergent symmetry

Big project for computational physics

Shu Luo [a], Suyang Lin [a]
[a] *School of Physics , University of Science and Technology of China, Hefei, Anhui 230026, China*
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In this paper we mainly reproduce the analytical and numerical results in arXiv:2412.10498, in which different aspects of the symmetry law and thermalization of Floquet system are investigated thoroughly through "flow formalism". We introduce the background and mathematical basis of this method and then apply it to three different well-known models in condensed matter physics: harmonic oscillator, spin-chain model and driven AAH model. We observe many interesting phenomenons, including diverse dynamical features in different fRG flow stages, emergent symmetry, reliance of system's evolution on the initial value, and the transition between adjoint unstable attractors accompanied by instanton peaks of off-diagonal Hamiltonian.

I. INTRODUCTION

The time-dependent periodic system, also known as the Floquet system, has been widely discussed in recent years. By applying external periodic driving, one can observe intriguing phenomena that are difficult to achieve in static systems. For example, consider the momentum-energy Brillouin zone (MEBZ) of a space-time crystal, the additional dimension introduces new symmetries [1]. And naturally, the topological effects on the MEBZ are also partly discussed in [2, 3]. From a more basic point of view, the direct consequence of periodic driving is the breaking of time-translational symmetry, which leads to the absence of conventional energy conservation. As a result, the system is expected to thermalize to an "infinite temperature" (i.e. featureless) state [4, 5]. However, this thermalization process may take a long time to fully develop.

The standard approach to analyzing such periodic systems is Floquet theory, which employs an effective Hamiltonian to describe the system's evolution over discrete time periods. To examine dynamics on different timescales, a commonly used method is the Floquet-Magnus expansion, which expands the effective Hamiltonian in powers of the driving frequency [6]. A major challenge associated with this approach is the divergence of the series, which may be linked to the quantum ergodicity of non-integrable systems [7]. The divergence issue has been largely addressed in [7], which demonstrates that at an appropriate order of expansion, the effective Hamiltonian remains a valid description of the system over a finite time range. Before full thermalization sets in, the system can exhibit prethermalization, where an effective Hamiltonian governs its dynamics for an effective period [7]. When considering the dynamical structure of the effective Floquet Hamiltonian, the symmetry behavior plays a crucial role. In some cases, an approximately conserved quantity may emerge, leading to a breakdown of ergodicity [8, 9]. This phenomenon, known as dynamical freezing, prevents the system from thermalizing to featureless states.

To investigate the physics governing prethermalization or the absence of thermalization in Floquet-driven systems, the Floquet flow renormalization method offers an alternative approach [10, 11], utilizing the advantages of the flow equation framework. This serves as the main method of this study. Flow equation methods have a long history in the study of time-independent models, where the diagonalization process is interpreted as a form of renormalization [12]. In this formalism, one obtains a sequence of Hamiltonians related by a series of infinitesimal unitary transformations, all sharing the same spectrum while exhibiting an evolving coupling parameters. This property is particularly useful for identifying emergent phenomena in complex systems. The flow equation method can also be extended to time-dependent periodic systems, a framework known as Floquet flow renormalization [12, 13].

We reproduce some results from [11], using examples such as the harmonic oscillator and the spin chain. In particular, we propose a modified AAH model of our own. The Aubry-Andr-Harper AAH model is a simple yet non-trivial system [14], which can be regarded as a one-dimensional reduction of the electron system with the integer quantum Hall effect [15], or as a simulation of a one-dimensional quasicrystal with an incommensurate potential. The effects of incommensurability, which breaks the space translational symmetry, have been widely discussed. One famous result is the Hofstadter butterfly in the spectrum [16]. Moreover, when the system is incommensurate, a localization phase transition occurs when the effect of the incommensurate potential is strong. Additionally, research [17] finds that the incommensurate AAH model can exhibit a Chern number in the thermodynamic limit of a two-dimensional integer quantum Hall model. This can also be understood as the incommensurate system being the projection of a commensurate system in a higher dimension [18]. Investigating the AAH model analytically is difficult, though some famous results have been obtained by mathematicians, such as those from Artur Avila's global theory [19]. The key parameter controlling the incommensurability or translational symmetry is the lattice constant (magnetic flux). Here, we drive this parameter with a periodicity expecting to observe some average effect.

The content of this paper is organized as follows. In Sec.II, we introduce the main background and mathematical basis for flow formalism. In Sec.III we apply this formalism to a simple harmonic oscillator, in order to visualize some formal propositions in the last section. In Sec.IV A, we study the famous model, Heisenberg spin chain through flow formalism, distinguishing various stages of fRG time evolution. In Sec.IV B we analyses the emergent symmetry of this model. Then we investigate Floquet AAH model analytically and numerically in Sec.V A and Sec.V B respectively. We give a brief conclusion and discussion in Sec.VI, while presenting the connection of our research to the topic introduced in the computational physics curriculum in Sec.VII.

II. THE SETUP

The proposal of fRG flow is to move out the high energy "off-diagonal" elements of the Hamiltonian. A related treatment is Schrieffer-Wolff(SW) transformation [20, 21], which is a unitary transformation:

$$H' = \mathrm{e}^{S} H \mathrm{e}^{-S} = H + [S, H] + \frac{1}{2}[S, [S, H]] + \cdots , \qquad S = -S^{\dagger} , \tag{2.1}$$

in which S satisfies

$$[S, H_0] = -V , \tag{2.2}$$

with H_0, V being the free (diagonal) part and interaction (off-diagonal) part of the Hamiltonian respectively. If we apply this idea to a series transformations labeled by one parameter λ, we may suppose

$$\partial_\lambda H(\lambda) = [\eta(\lambda), H(\lambda)] , \tag{2.3}$$

The choice of $\eta(\lambda)$ here serves for a crucial target: to cancel the off-diagonal opponents of the Hamiltonian as quickly as possible. For stationary Hamiltonian it can be formally written as

$$\eta(\lambda) = [H_{\mathrm{diag}}(\lambda), H_{\mathrm{off-diag}}(\lambda)] , \tag{2.4}$$

which is commonly referred as Wegner generator. However, for a time-dependent one, we need to set

$$\eta(\lambda, t) = [H_{\mathrm{diag}}(\lambda, t), H_{\mathrm{off-diag}}(\lambda, t)] - i\partial_t H_{\mathrm{off-diag}}(\lambda, t) . \tag{2.5}$$

The main goal of this transformation is to make a certain $H_{\mathrm{diag}}(\lambda_c, t)$ which is not only time-independent ($\partial_t H_{\mathrm{diag}}(\lambda_c, t) = 0$) but also serves as an effective Hamiltonian, with a certain given t_{eff} during which the evolution of the system can be approximated by $H_{\mathrm{diag}}(\lambda_c) = H_{\mathrm{eff}}$. Up to this step the flow formalism is rather close to the idea of Floquet-Magnus expansion. However, at most of the time there is no such flow that can transform a time-dependent Hamiltonian into a stationary one that retains all spectrum properties.

One counterexample is the period-driven Hamiltonian. If we have a time-dependent Hamiltonian of the form

$$H = H_0 + \sum_M \mathrm{e}^{-iM\Omega t} H_M , \tag{2.6}$$

then we have a Floquet systems, which are described by Hamiltonian periodic in time, satisfying $H(t + T) = H(t)$ where T is the period. According to Floquet theorem, we can suppose the wave function to be the product of a free plane wave function and a periodic wave function. In other words, we expect

$$|\psi_\alpha(t)\rangle = \sum_n \mathrm{e}^{in\Omega t} |\psi_\alpha^n(t)\rangle , \qquad |\psi_\alpha^n(t + T)\rangle = |\psi_\alpha^n(t)\rangle , \tag{2.7}$$

and if we make the Hilbert space expansion, i.e., to regard $|\psi_\alpha(t)\rangle$ as $\sum_n |\psi_\alpha^n\rangle \otimes |n\rangle$, and to get the Sambe representation [22] ($H = H_F \otimes H_T$), the Floquet quasienergy operator can be written as

$$H = \sum_m (m\Omega \otimes |m\rangle \langle m| + \sum_M H_M \otimes |m + M\rangle \langle m|) , \tag{2.8}$$

so the fRG flow generator $\eta(\lambda)$ can be written as

$$\eta(\lambda) = \sum_m \sum_{M>0} [H_M \otimes |m + M\rangle \langle m| - \mathrm{h.c.}] , \tag{2.9}$$

and if we have $H_M = 0 (M > 1)$, in this representation we can rewrite Eq.(2.3) into the form

$$\partial_\lambda H_0(\lambda) = 2[H_1(\lambda), H_1^{\dagger}(\lambda)] , \tag{2.10}$$

$$\partial_\lambda H_1(\lambda) = -\Omega H_1(\lambda) - [H_0(\lambda), H_1(\lambda)] . \tag{2.11}$$

The general flow equation is

$$\partial_\lambda H_M(\lambda) = -M\Omega H_M(\lambda) + \sum_{M'=0}^{M-1} [H_{M-M'}(\lambda), H_{M'}(\lambda)] + 2\sum_{M'>M} [H_{M'}(\lambda), H^\dagger_{M'-M}(\lambda)]\,. \tag{2.12}$$

For systems with a relatively small Hilbert space we can numerically solve Eq.(2.10) and Eq.(2.11) by exact diagonalization and forth-order Runge-Kutta method (RK4). However, it is difficult to study the global evolution analytically. Numerical results prove that following three stages exist commonly:

1. For $0 < \lambda < 1/\Omega$, the flow is in the ramp-up regime, when $H_1(\lambda)$ decreases rapidly and H_0 is renormalized.

2. For $0 < 1/\Omega < \lambda_{min}$, the flow is in prethermal regime, when $H_0(\lambda)$ is basically independent of λ while $H_1(\lambda)$ decreases as an exponential function $H_1(\lambda) \sim \exp(-\Omega\lambda)$. This is the case until λ_{min}, at which time the norm of $H_1(\lambda)$ arrives at the minimum and H_0 is supposed to be at the edge of an instable fixing point beyond which the system will be caught again into dynamical evolution. The existence of λ_{min} highly relied on the integrability of the system: when the system is integrable (like Heisenberg spin chain model XYY in which there is no interaction except for adjoint spins), $H_1(\lambda)$ goes on to decrease until $\lambda \to \infty$, and H_0 will eventually flow into an exact effective Hamiltonian. Quite opposite from that, when it's non-integrable, we expect a thermalization process.

   Among $1/\Omega$ and λ_{min} there is a certain λ_{ext} at which we suppose the deviation between the flow formalism and Floquet-Magnus expansion is at its minimum. At that point the quantitative deviation is expected to behave like $1/\Omega^2$ as Ω increases. Another crucial point is λ_c, at which the emergent symmetry is the strongest when we set the parameter A/Ω near some freezing points. In other word, we expect a certain $P(\lambda) \sim [H_0(\lambda), Q]$ to reach its minimum, where Q is the conserved charge related to this symmetry.

3. For $\lambda > \lambda_{min}$ (if there exists), the system will experience a series of instanton events, which are phase transitions between adjoint fixing points, meaning all these fixing points are instable ones. The final eigenvalues are expected to be pushed into an interval of width $T \sim 1/\Omega$, which is in accordance to a Floquet Hamiltonian. Still, generally $H_1(\lambda)$ will decay at the rate $H_1(\lambda) \sim 1/\lambda^2$.

The approximate expression of the re-normalized Hamiltonian in the ramp-up stage is accessible. Through Eq.(2.11) we are able to attain

$$H_1(\lambda) = \mathrm{e}^{-\Omega\lambda} U(\lambda) H_1(0) U^\dagger(\lambda)\,, \qquad U(\lambda) = \mathcal{T}\exp\left(-\int_0^\lambda \mathrm{d}\bar{\lambda} H_0(\bar{\lambda})\right), \tag{2.13}$$

and if we set $\bar{H}_1(\lambda) = \mathrm{e}^{\Omega\lambda} H_1$, we have

$$\partial_\lambda^2 H_0(\lambda) + 2\Omega\partial_\lambda H_0(\lambda) + \mathrm{e}^{-2\Omega\lambda}\left\{\mathrm{Ad}_{H_1(\lambda)}, \mathrm{Ad}_{H_1^\dagger(\lambda)}\right\} H_0(\lambda) = 0\,, \tag{2.14}$$

and if we expand the whole solution by series of $1/\Omega$, we can find the zeroth order of H_0 to be

$$H_0^{(0)}(\lambda) = \frac{\pi}{2}\mathrm{e}^{-\Omega\lambda} z[J_1(z\mathrm{e}^{-\Omega\lambda})Y_0(z) - Y_1(z\mathrm{e}^{-\Omega\lambda})J_0(z)]H_0(0)\,, \tag{2.15}$$

in which $z = \frac{2\mathrm{Ad}_{H_1(0)}}{\Omega}$, and in $\lambda \to \infty$ limit we have $H_0^{(0)}(\lambda) \to J_0\left(\frac{2\mathrm{Ad}_{H_1(0)}}{\Omega}\right) H_0(0)$. Yet what is really useful is not the asymptotic behavior of the flow Hamiltonian, but one at λ_{min} instead. Previous works [10] have confirmed that the effective Hamiltonian induced by a certain $H_0(\lambda)$ has a effective time bound of the scale

$$t_{\text{eff}} \sim ||H_1(\lambda)||^{-1}||\hat{1}||\,, \tag{2.16}$$

which means if we take $H_0(\lambda_{min})$ as the effective Hamiltonian, we will have a particularly long effective time of the scale $\exp(\Omega\lambda_{min})$.

Apart from the flow formalism, another well-known and widely-appiled method is Floquet-Magnus expansion, focusing on extracting the long-time averaging effect of a time-dependent Hamiltonian. A simple case is when the Hamiltonian is of the form

$$H = H_0 + \sum_M \mathrm{e}^{-iM\Omega t} H_M\,, \tag{2.17}$$

in which $H_M = H^\dagger_{-M}$, and when we have H_1 being Hermitian while $H_M = 0, M > 1$, we first need to apply a unitary

transformation to the co-moving Hamiltonian:

$$H_{\rm mov}(t) = W^\dagger(t) H W(t), \qquad W(t) = \exp(-\frac{2iH_1}{\Omega}\sin\Omega t)\,, \tag{2.18}$$

$$H_{\rm mov}(t) = \exp\Big(\frac{2i\sin(\Omega t){\rm Ad}_{H_1}}{\Omega}\Big)H_0 = \sum_m h_m {\rm e}^{im\Omega t}\,, \qquad h_m = J_m\Big(\frac{2{\rm Ad}_{H_1}}{\Omega}\Big)H_0\,, \tag{2.19}$$

This transformation is to weaken the reliance of time evolution operator on the high-frequency terms and increase the congruence efficiency, making the transformed Hamiltonian closer to a quasi-static one. Then we apply the Floquet-Magnus expansion

$$H_F^{(0)} = \frac{1}{T}\int_0^T {\rm d}t H_{\rm mov}(t)\,, \tag{2.20}$$

$$H_F^{(1)} = \frac{1}{2iT}\int_0^T {\rm d}t_1 \int_0^{t_1} {\rm d}t_2 [H_{\rm mov}(t_1), H_{\rm mov}(t_2)]\,, \tag{2.21}$$

and by inserting the Fourier expansion in Eq.(2.19) into it we have the following results

$$H_F^{(0)} = h_0, \quad H_F^{(1)} = 2\sum_{m\neq 0}\frac{-1}{m\Omega}[h_m, h_0]\,. \tag{2.22}$$

Now that we have both results from two methods, we get the crucial relation about the equivalence between them:

$$H_F^{(0)} = \lim_{\lambda\to\infty} H_0^{(0)}(\lambda)\,, \tag{2.23}$$

in the following sections we will apply these methods to a specific system (spin-chain model) and study the absolute deviation from the two methods. We will get the expected result. Still, there are some details different from theoretical prediction, which reveals the disadvantage of the approximate calculation above. For instance, as the flow formalism has its best effect when we take $H_0(\lambda_{min})$ as the effective Hamiltonian but leave and $H_0(\lambda)$ else we expect the deviation from these two methods to be the tiniest near λ_{min}, instead of the infinity. The final results are in accordance to this naive prediction. We find the minimum is reached when $1/\Omega <\sim \lambda_{ext} <\sim \lambda_{min}$, which can also be regarded as the beginning point of the prethermal stage.

III. THE EXAMPLE OF HARMONIC OSCILLATOR

As a simple example, the practice of the fRG flow equation for this problem does not need any numerical treatment in its Hilbert space. The Hamiltonian is written as

$$H = \omega_0 a^\dagger a + \omega_1(a + a^\dagger) + A a^\dagger a\cos(\Omega t)\,, \tag{3.1}$$

and if we further suppose

$$H_0(\lambda) = A_0(\lambda)a^\dagger a + B_0(\lambda)a^\dagger + B_0^*(\lambda)a\,, \tag{3.2}$$

$$H_1(\lambda) = A_1(\lambda)a^\dagger a + B_1(\lambda)a^\dagger + C_1^*(\lambda)a\,, \tag{3.3}$$

in this case the fRG flow equation will have the form

$$\partial_\lambda A_0 = 0\,, \tag{3.4}$$

$$\partial_\lambda A_1 = -\Omega A_1\,, \tag{3.5}$$

$$\partial_\lambda B_0 = 2A_1(C_1 - B_1)\,, \tag{3.6}$$

$$\partial_\lambda B_1 = -\Omega B_1 - A_0 B_1 + A_1 B_0\,, \tag{3.7}$$

$$\partial_\lambda C_1 = -\Omega C_1 + A_0 C_1 - A_1 B_0 \,, \tag{3.8}$$

and the initial value is set to be $A_0 = \omega_0, A_1 = A/2, B_0 = \omega_1, B_1 = C_1 = 0$. What's interesting is the possible emergent $U(1)$ symmetry reflected by the conservation law of the charge $a^\dagger a$. It is clear that this can be satisfied when we have $B_0(\lambda_c)$ for a certain proper λ_c. Here, because the system is integrable, λ_c can actually approach infinity, with no thermal stage in the flow formalism.

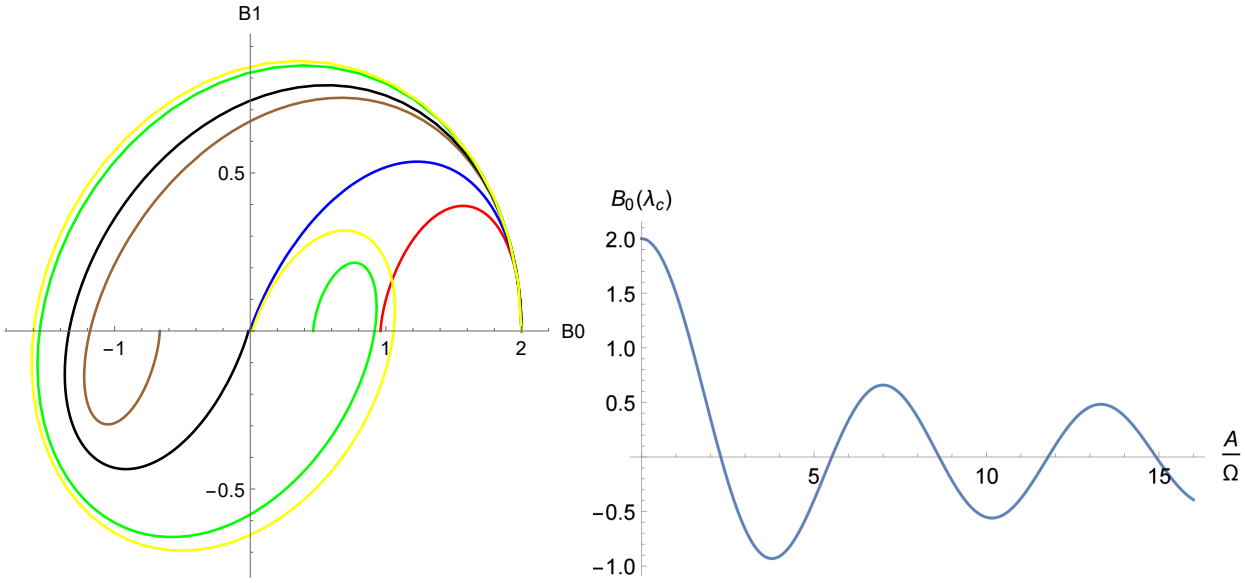

Figure 1: The left panel is the (B_0, B_1) plane of the harmonic oscillator, and different colors correspond to different parameters (A/Ω), either being "FP" (with $B_0(\lambda_c) = 0$) or "NFP" (with $B_0(\lambda) \neq 0$). The right panel is the quantitative relation between $B_0(\lambda_c)$ and A/Ω.

IV. THE EXAMPLE OF SPIN CHAIN MODEL

A. Different stages of fRG time flow

For the spin-chain model we first need to set a Hamiltonian of the form

$$H = -J \sum_i S_i^z S_{i+1}^z - J_2 \sum_i S_i^z S_{i+2}^z + 2A \cos \Omega t \sum_i S_i^x \,, \tag{4.1}$$

and when $J_2 = 0$ this will be a typical integrable quantum model. Now that $J_2 \neq 0$, the fRG flow of the system will eventually evolve into a pure thermal state, i.e., all the eigenvalues of the Hamiltonian $H(\lambda)$ will be pushed into a narrow band width of which is close to $T = 2\pi/\Omega$. According to the convention introduced in Sec.II, we have

$$H_0 = -J \sum_i S_i^z S_{i+1}^z - J_2 \sum_i S_i^z S_{i+2}^z \,, \tag{4.2}$$

and

$$H_1 = A \sum_i S_i^x \,, \tag{4.3}$$

so we can apply the universal fRG flow equation to this system easily. The specific numerical calculation needs to figure out the whole Hilbert space and include them in an index table. Then we can write out the matrix form of both Hamiltonian and use RK4 to discritize the evolution in the time domain.

In this section we always restrict the system studied to the case that there are $L = 10$ particles, which lead to a Hilbert space with dimension $2^{10} = 1024$, and is appropriate for exact diagonalization. The first step lies on the RG flow itself, the evolution of both the diagonal part H_0 and off-diagonal part H_1. The results are shown in Fig.2 and Fig.3.

We then compare the results attained by fRG flow and Magnus expansion. By the general formula introduced in Sec.II, we

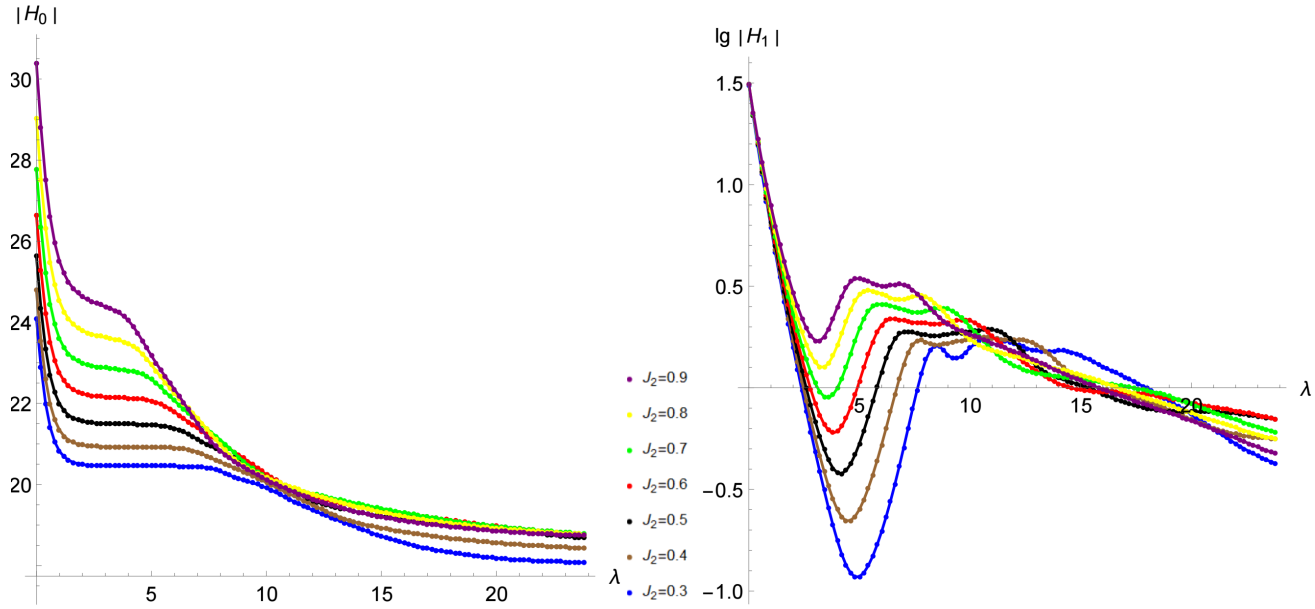

Figure 2: When the system is NFP ($A/\Omega = 0.3$), fixing $J = 1$ and varying J_2, the evolution in the fRG time domain of both the diagonal (left) and off-diagonal (right) part of the Hamiltonian.

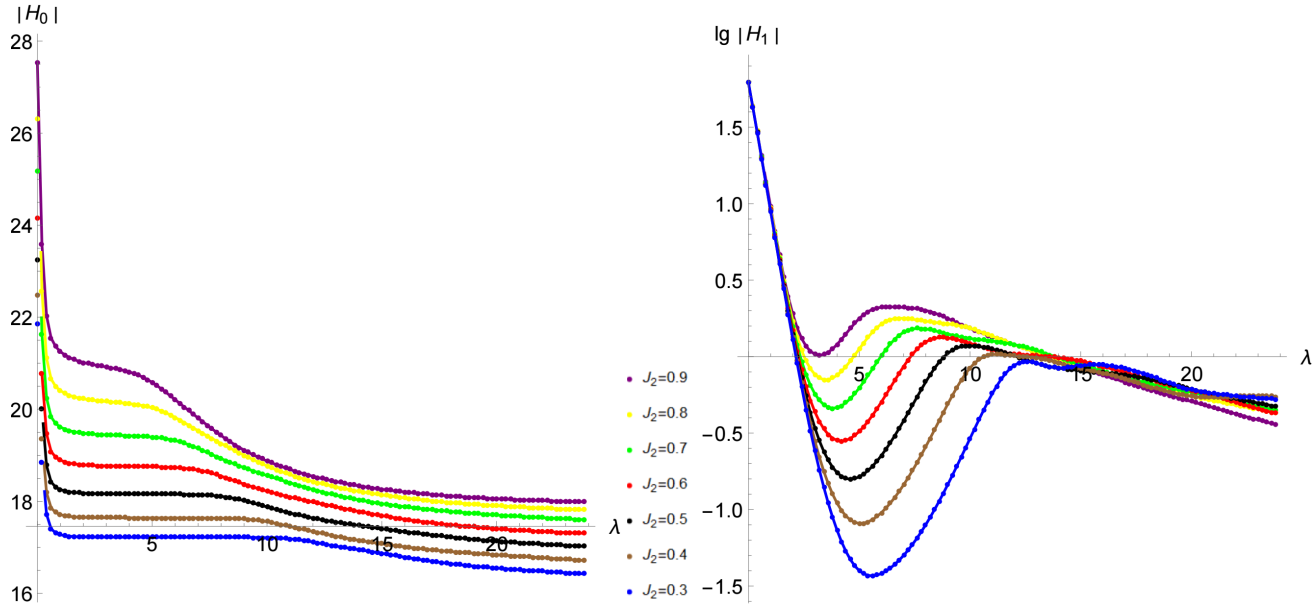

Figure 3: When the system is FP ($A/\Omega = 0.601$) ,fixing $J = 1$ and varying J_2, the evolution in the fRG time domain of both the diagonal (left) and off-diagonal (right) part of the Hamiltonian.

have the co-moving Hamiltonian written as ($\alpha = 4A/\Omega$)

$$H_{\rm mov}(t) \sim \sum_{d=1,2} \frac{J^{(d)}}{2} \sum_i \{S_i^y S_{i+d}^y [1-\cos(\alpha \sin(\Omega t))] + S_i^z S_{i+d}^z [1+\cos(\alpha \sin(\Omega t))] + [S_i^y S_{i+d}^z + S_i^z S_{i+d}^y] \sin(\alpha \sin(\Omega t))\} \tag{4.4}$$

where $J^{(1)} = J$ and $J^{(2)} = J_2$. The Fourier components are

$$\begin{aligned} h_m = -\sum_{d=1,2} \sum_i \Big\{ & S_i^y S_{i+d}^y \Big[\delta_{m,0} - \frac{1+(-1)^m}{2} J_m(\alpha)\Big] + S_i^z S_{i+d}^z \Big[\delta_{m,0} + \frac{1+(-1)^m}{2} J_m(\alpha)\Big] \\ & + [S_i^d S_{i+d}^z + S_i^z S_{i+d}^y] \frac{1-(-1)^m}{2i} J_m(\alpha) \Big\}, \end{aligned} \tag{4.5}$$

and we take h_0 as the Floquet Hamiltonian.The results are shown in the left panel of Fig.4. The typical fRG time scale we study is focused on $0 < \lambda <\approx 1/J$, and there is clearly a certain λ_{min} such that the deviation between these two methods reaches its minimum. If we consider the possible omission of the true minimum when taking the discontinuous λ value (this is the case for $\Omega = 9$), then all the other results in the left panel of Fig.4 satisfy the expected law $1/\Omega^2$, shown in the right panel of Fig.4.

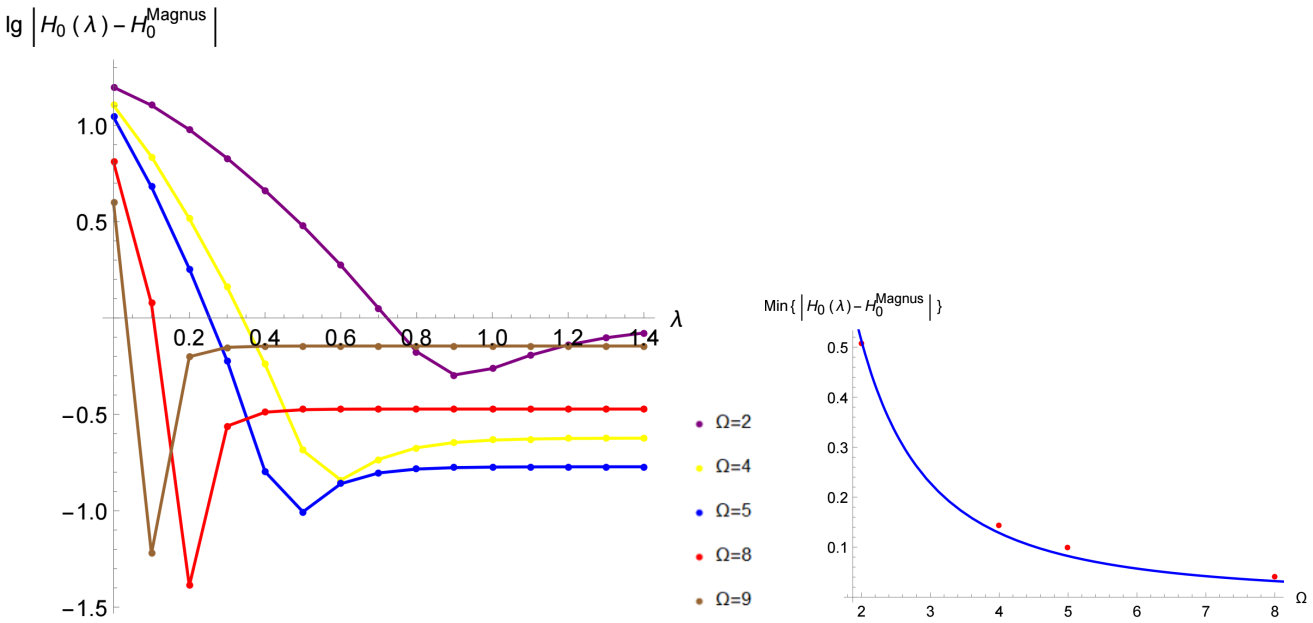

Figure 4: Left panel: Fixing $J = 1, J_2 = 0.2$, and $\frac{A}{\Omega} = 0.601$, while varying Ω, the fRG evolution of the deviation between $H_0(\lambda)$ and H_0^{Magnus}. Right panel: The fitting of the relation between the minimum value $P(\lambda_c)$ and Ω, which satisfies inverse square relation.

B. Emergent symmetry

About the emergent symmetry, just as the discussion in Sec.III, we consider the possible existing $SU(2)$ symmetry reflected by the following commutation relation:

$$[H_0(\lambda), H_1(0)] = 0\,, \tag{4.6}$$

and in order to quantify the degree of symmetry breaking we introduce the following sum

$$P(\lambda) = \frac{||[H_0(\lambda), O]||}{||H_0(\lambda)||}\,, \tag{4.7}$$

in which $O = \sum_i S_i^x$, and the results are shown in Fig.5, from which we find $P(\lambda)$ first to show a rapid decrease before the prethermal stage, and when the system transfers into the prethermal stage, $P(\lambda)$ basically arrives at a plateau. This is of course rational considering that $H_0(\lambda)$ itself is independent of λ in this stage. Thus we can suppose a typical fRG time $\lambda_c \sim< \lambda_{min}$ at which the approximate symmetry degree arrives at its best possibility. In fact, this approximate symmetry may even be broken completely when λ flows on to the scale of λ_{th} at which the system arrives at its thermal stage. The reason is relatively clear. In thermal state $||H_1(\lambda)||$ continues to decrease as $1/\lambda^2$, which means t_{eff} simultaneously increases to infinity. However, we expect no global quasi-conservation law or symmetry, and emergent symmetry can only exist for the time scale $\exp(\Omega\lambda_c)$. With this interpretation we conclude that emergent symmetry must be broken when λ flows to infinity. By numerical calculations, one can confirm that there is approximately $\lambda_c \sim 1/J$, in which J is the energy scale of the system.

Still, there is a problem why the emergent symmetry is merely an approximate one, as $P(\lambda)$ never exactly vanishes. In fact, with the fRG flow, a variety of operators can be generated, which cannot be suppressed simply by modifying A/Ω. Still, these asymmetry can be suppressed as $\Omega \to \infty$, with the inverse square law $P(\lambda_c) \sim 1/\Omega^2$. Apart from the observation in the λ domain, we also need to study the general influence on this emergent symmetry of the dynamical parameter of the system A/Ω, and the results are shown in the right panel of Fig.5.

After attaining the quantitative symmetry breaking degree as well as the effective time for the conservation law to make sense, we may naively look forward to a certain simple quasi-conservation behavior, i.e., the expectation value of a certain symmetry charge remains the same value approximately in a short time but then either decays to 0 or continues to oscillate in a certain

range. Yet the true phenomenon is quite different from that illusion, and related results are shown in Fig.6. Basically there are three phases determined by the initial value of the wave function. The first one is the standard quasi-conservation phase, in which we see a temporary plateau and then it decays to 0. However, we may also encounter stepwise decay which may last a much longer time than usual. The above two circumstance all happen when we set the initial value of the wave function to be one eigenstate, but unfortunately when we set it to be away from eigenstate things turn out to be even more complicated (see the right panel of Fig.6).

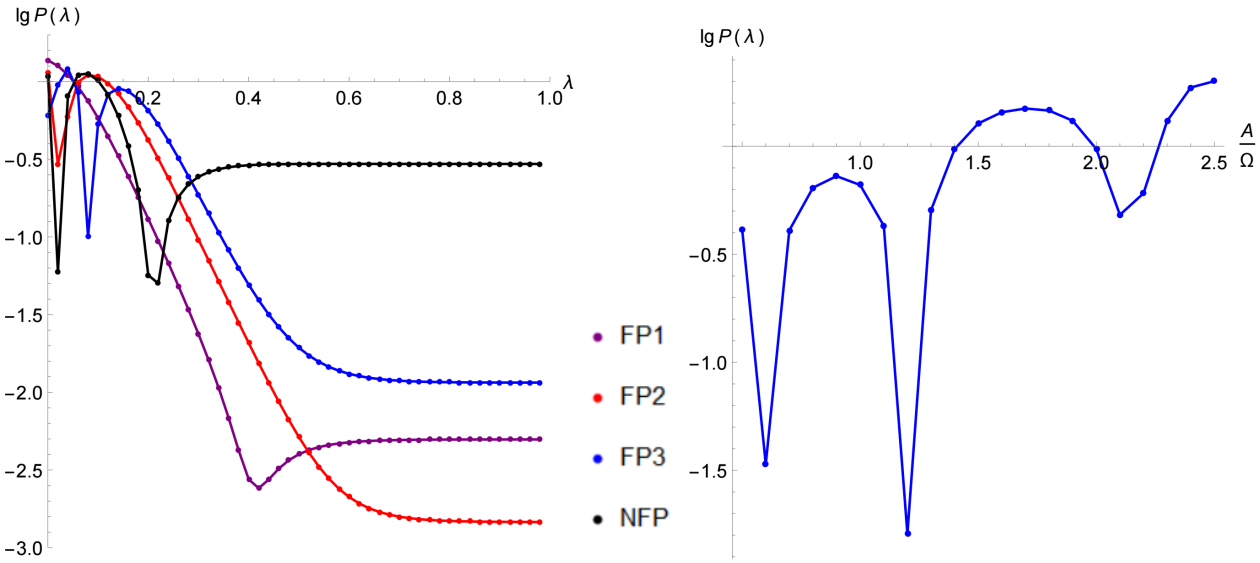

Figure 5: Left panel: Fixing $J = 1, J_2 = 0.2, \Omega = 8$, and varying the value of A/Ω, the fRG evolution of the quantitative symmetry broken degree $P(\lambda)$. Right panel: Varying A/Ω in the range $[0.5, 2.5]$, the quantitative symmetry broken degree at the crucial fRG time $P(\lambda_c)$.

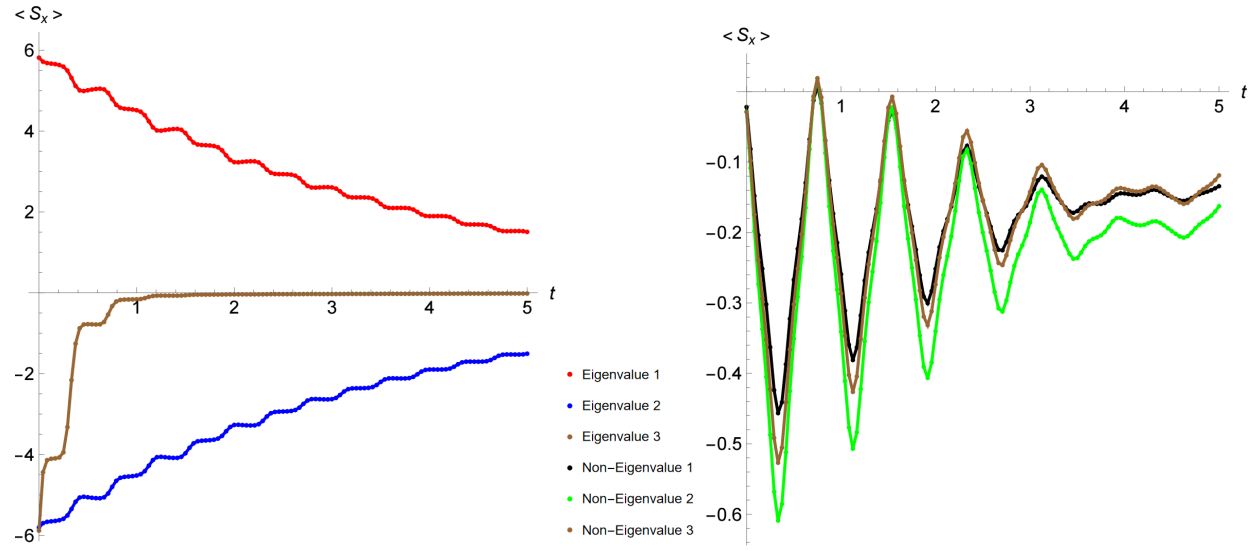

Figure 6: Set the system at the first freezing point $A/\Omega = 0.601$, the evolution of the expectation value of S_x, with different initial value (some being the eigenstates (left panel) while some being the non-eigenstates (right panel)).

V. THE EXAMPLE OF DRIVEN AUBRY-ANDR-HARPER MODEL

A. Analytic Results

We propose a modified AAH model [14, 23] with a varying lattice constant here. In this context, this single-particle model may exhibit discrete translational symmetry at certain points in time. Additionally, the lattice constant may be irrational at some

points. Our goal is to determine whether there exists an averaged translational symmetry and whether there is an averaged irrational constant that leads to the localization.

$$H=\sum_i V\cos(2\pi a\sin(\Omega t)i)c_i^\dagger c_i+t(c_i^\dagger c_{i+1}+c_{i+1}^\dagger c_i)\,, \tag{5.1}$$

and for this Hamiltonian we have

$$H_{2M}(0)=H_{2M}^\dagger(0)=\sum_i Vc_i^\dagger c_i J_{2M}(2\pi ai)+\delta_{M,0}t(c_i^\dagger c_{i+1}+c_{i+1}^\dagger c_i)\,,\qquad H_{2M+1}(0)=0\,, \tag{5.2}$$

from which we find $[H_M(0),H_N(0)]=0$ except for the case that $NM=0$, $M,N\in 2\mathbb{Z}$. The nonzero components are only those with even M. As $M\to\infty$, we have $J_M(x)\to 0$ ($J_M(x)$ is the Mth first kind of Bessel function), so we can naturally make a cutoff at a certain M_c of Eq.(2.12), resulting in a series of consistent equations. From numerical estimation, we can confirm that terms with $2M>3\pi aN$ is negligible. We can also apply analytical calculation perturbatively as follows. Suppose $\tilde{H}_M=\exp(M\Omega\lambda)H_M$, and then through the following expansion

$$\tilde{H}_M=\sum_{k=0}^{\infty}\exp(-4k\Omega\lambda)\tilde{H}_M^{(2k)}\,, \tag{5.3}$$

we are able to re-express Eq.(2.12) in the form

$$\begin{aligned}\partial_\lambda(\sum_{k=0}^{\infty}\exp(-4k\Omega\lambda)\tilde{H}_M^{(2k)})&=\sum_{k=0}^{\infty}\exp(-4k\Omega\lambda)\sum_{k'=0}^{k}[\tilde{H}_M^{(2k')},\tilde{H}_0^{(2(k-k'))}]\\&+2\sum_{k=1}^{\infty}\exp(-4k\Omega\lambda)\sum_{k'\geq 0,k''\geq 0,k'+k''\leq k}[\tilde{H}_{2(k-k'-k'')+M}^{(2k')},\tilde{H}_{2(k-k'-k'')}^{(2k'')\dagger}]\,,\end{aligned} \tag{5.4}$$

and here we only consider large $\Omega\lambda$ limit through which the decay effect of higher order's terms is predominant. Then we can solve order by order to attain ($\lambda>\lambda_c$)

$$\tilde{H}_0^{(0)}(\lambda)=H_0^{(0)}(\lambda)=H_0\,,\qquad \tilde{H}_M^{(0)}(\lambda)=\exp(-(\lambda-\lambda_c)\mathrm{Ad}_{H_0})\tilde{H}_M^{(0)}(\lambda_c)\,, \tag{5.5}$$

$$\tilde{H}_0^{(2)}(\lambda)\exp(-4\Omega\lambda)=\exp(-(\lambda-\lambda_c)\mathrm{Ad}_{H_0})C_0^{(2)}+2\int_{\lambda_c}^{\lambda}\exp(-4\Omega\lambda')[\tilde{H}_2^{(0)}(\lambda'),\tilde{H}_2^{(0)\dagger}(\lambda')]\mathrm{d}\lambda'\,, \tag{5.6}$$

$$\begin{aligned}\tilde{H}_M^{(2)}(\lambda)\exp(-4\Omega\lambda)&=\exp(-(\lambda-\lambda_c)\mathrm{Ad}_{H_0})C_M^{(2)}\\+2\int_{\lambda_c}^{\lambda}\exp(-4\Omega\lambda')\exp(-(\lambda-\lambda')\mathrm{Ad}_{H_0})\{[\tilde{H}_M^{(0)}(\lambda'),H_0^{(2)}(\lambda')]&+2[\tilde{H}_{2+M}^{(0)}(\lambda'),H_2^{(0)\dagger}(\lambda')]\}\mathrm{d}\lambda'\,,\end{aligned} \tag{5.7}$$

The general expression for kth order solution is given by

$$\begin{aligned}\tilde{H}_0^{(2k)}(\lambda)\exp(-4k\Omega\lambda)&=\exp(-(\lambda-\lambda_c)\mathrm{Ad}_{H_0})C_0^{(2k)}\\+2\int_{\lambda_c}^{\lambda}\exp(-4\Omega\lambda')&\sum_{k'\geq 0,k''\geq 0,k'+k''\leq k-1}[\tilde{H}_{2(k-k'-k'')}^{(2k')}(\lambda'),\tilde{H}_{2(k-k'-k'')}^{(2k'')\dagger}(\lambda')]\mathrm{d}\lambda'\,,\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\tilde{H}_M^{(2k)}(\lambda)\exp(-4k\Omega\lambda)&=\exp(-(\lambda-\lambda_c)\mathrm{Ad}_{H_0})C_M^{(2k)}\\+2\int_{\lambda_c}^{\lambda}\exp(-4k\Omega\lambda')\exp(-(\lambda-\lambda')\mathrm{Ad}_{H_0})\{\sum_{k'=0}^{k-1}&[\tilde{H}_M^{(2k')}(\lambda'),H_0^{(2(k-k'))}(\lambda')]+\\2\sum_{k'\geq 0,k''\geq 0,k'+k''\leq k-1}&[\tilde{H}_{2(k-k'-k'')+M}^{(2k')}(\lambda'),H_{2(k-k'-k'')}^{(2k'')\dagger}(\lambda')]\}\mathrm{d}\lambda'\,,\end{aligned} \tag{5.9}$$

Here we need to notice two points: first, this solution is actually a iteration formula and the specific solution of $H_0^{(2k)}$ is crucial for attaining $H_M^{(2k)}(M \geq 1)$ and has to be solved first. Second, here $C_M^{(2k)}$ is set to guarantee that $H_M^{(2k)} \exp(-4k\Omega\lambda) \to 0(k \geq 0)$ as $\lambda \to \infty$ in order to keep the effectiveness of the perturbation series. In order to achieve this we have to suppose $[H_0, C_M^{(2k)}] = 0$, so $\exp(-\lambda \mathrm{Ad}_{H_0}) C_M^{(2k)} \equiv C_M^{(2k)}$. This enforces the following consistency condition:

$$[\tilde{H}_M^{(2k)*}(\infty), H_0] = 0, \tag{5.10}$$

where $H_M^{(2k)*}(\lambda)$ is the homogeneous part of the solution. Also, we need $C_M^{(2k)} \ll C_M^{(2(k-1))}$ to keep the effectiveness of the perturbation theory at $\lambda = \lambda_c$ side.

B. Numerical Results

To guarantee the mirror symmetry of the system, the diagonal part of the Hamiltonian is altered to $V \cos[2\pi a \sin(\omega t)(i - L/2)]$, where L is the lattice size. The off-diagonal part represents the hopping term t between neighboring sites. We use exact diagonalization first. This method has advantages in simulating the evolution of wave function in large lattices as the Hamiltonian is tridiagonal. We investigate the evolution of an eigenstate of the translational operator T,which is given by:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{5.11}$$

The eigenstate we choose is a plane wave with momentum $k = \pi$.Under appropriate parameters, we observe oscillations between quasi-conserved platforms, as shown in Fig.7. This behavior can be interpreted as a deterministic simulation of stochastic resonance [24]. Since the Hamiltonian is constructed to keep parity symmetry, the potential at site $i = L/2$ is time-independent,

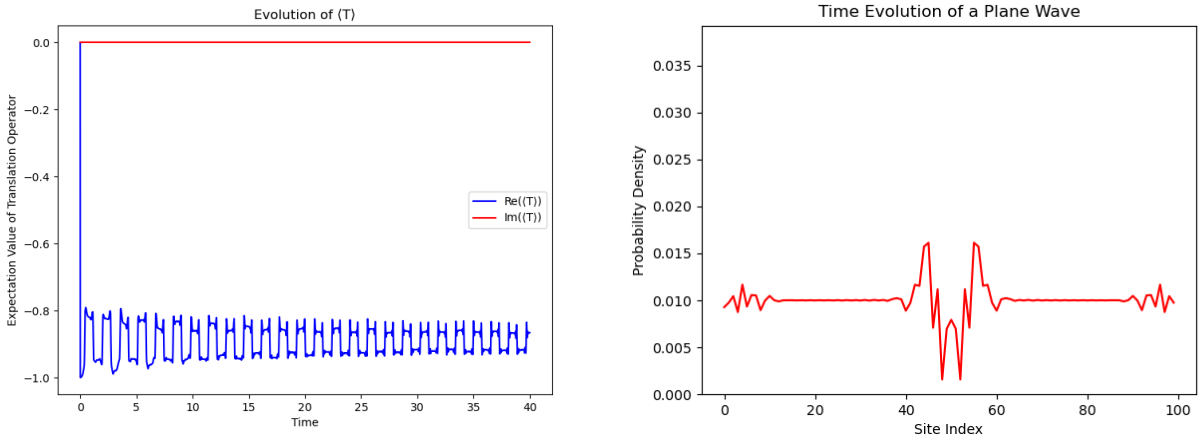

Figure 7: Fixing the lattice size $L = 100, V = 4.0, a = 0.5, \omega = 4.0$, and $t = 1.0$,with a time step of 0.01.Oscillations between quasi-conserved eigenstates and weak localization near the walls are observed.

effectively acting as a fixed boundary. When the potential becomes incommensurate at a given time, the periodic boundary condition no longer holds, and both ends of the lattice should be treated as boundaries. A weak localization phenomenon is observed near the boundary in Fig.7, which can be interpreted as the accumulation of the wave function near the walls. Additionally, from the perspective of the Floquet lattice, it exhibits quasi-conservation behavior. However, not all initial wave functions will exhibit this behavior, suggesting its reliance on the initial conditions. This reliance implies that the whole system does not exhibit the symmetry associated with the operator$T$.

Here we plot the evolution of the system under fRG flow, as shown in Fig.8 and Fig.9. We take the transitional operator T as the symmetry charge to quantify the degree of symmetry breaking of the renormalized Hamiltonian, and we summarize the contribution from all the non-diagonal Hamiltonian in $||H_{\text{off}}|| \equiv \sqrt{\sum_{i=1}^{M_c} ||H_i||^2}$. The result is more than surprising because unlike previous examples, here we cannot even naively suppose $||H_{\text{off}}||$ to decay as λ flows to infinity. Instead, we find the

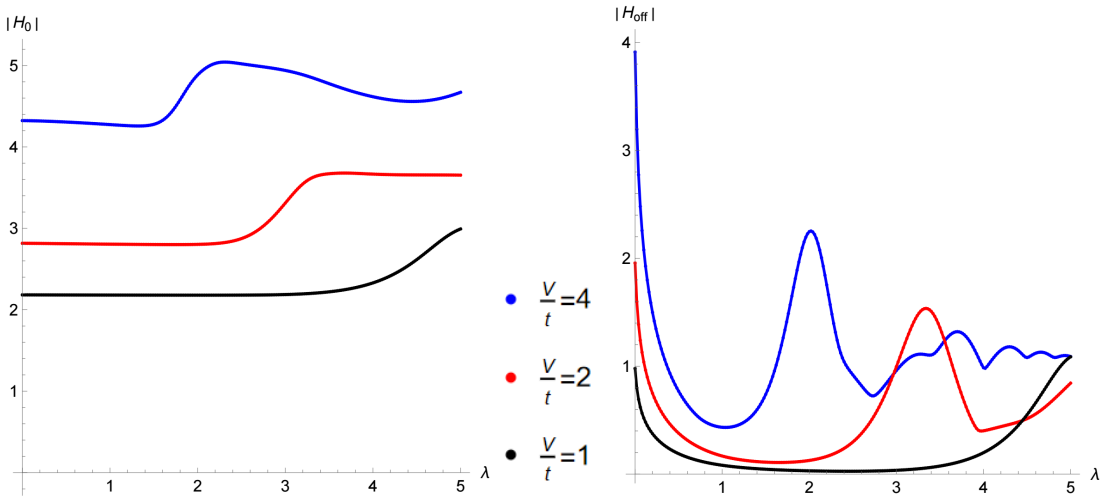

Figure 8: The fRG flow of the AAH model with parameters $L = 100, \Omega = 1, t = 1, a = 0.1$.

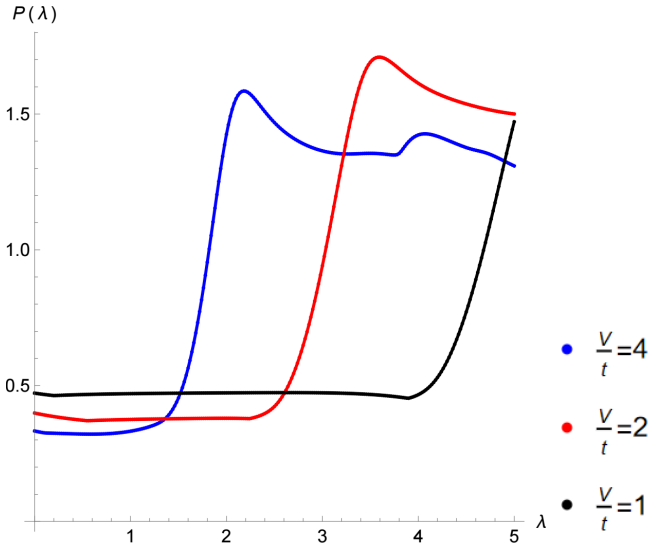

Figure 9: The emergent symmetry of the AAH model with parameters $L = 100, \Omega = 1, t = 1, a = 0.1$.

best fRG time λ_c at which $||H_{\text{off}}||$ reaches its minimum. In this way, we can construct the best effective Hamiltonian $H_0(\lambda)$. Meanwhile, after the prethermal stage, we observe a typical transition to thermal stage, after which the diagonal Hamiltonian H_0 behaves completely "randomly", leading to several phase transitions between adjoint unstable attractors with H_{off} showing an instanton peak. As the model is non-integrable, we expect this behavior to last continuously with λ flowing on. The breaking of symmetry is rather serious compared to the spin-chain model, and there is no obvious λ in which the system explicitly shows transitional symmetry. It seems contradictory to the results of real time evolution (see the left panel of Fig.7), and the reasons remain to be further studied.

VI. CONCLUSION AND DISCUSSION

The flow formalism is developed to provide a continuous version of unitary diagonalization transformation [20]. In this big project we introduce both analytical and numerical realization of the flow formalism for three common models in condensed matter theory. We first present the fundamental proposal and ideas of the flow formalism, and combine it with the universal Floquet driven system, so as to solve and analyze problems with periodic time-dependent Hamiltonian. We compare the results of flow formalism and usual Floquet-Magnus expansion, and show how close these two methods to each other. The flow formalism is found to reveal many new features in Floquet system, first of which turns out to be the hidden symmetry.

One interpretation is that some slightly broken symmetry may lead to quasi-conserved charge Q which has a certain lasting time. Note that this statement is based on the periodic behavior of the system, i.e., to compare $Q(t)$ and $Q(t + nT)$. This

quasi-conservation law is expected to make sense particularly when the parameters are very close to "freezing points", at which leading-order calculation supports an exact conservation law. However, numerical results show that there is no such strict symmetry even at the freezing points, whose cause may come from NL contribution or other symmetry breaking operators. Flow formalism is proved to be particularly appropriate for unveiling these non-perturbative features. The naive estimation of the upper bound of real time for the effective Hamiltonian attained through flow formalism t_{eff} is the same as in Floquet-Magnus expansion. What really differs from usual treatment is the possibility to get the best estimation of the quasi-conservation time of a certain symmetry. Thanks to the freedom given by flow parameter λ, we can confirm the best possible time for the symmetry to exist λ_c. Then by calculating the norm of non-diagonal part of renormalized Hamiltonian $||H_1(\lambda_c)||$, we are able to give the best estimation of the quasi-conservation real time t_{eff}.

Flow formalism also reveals other fancy results, such as the comparison of pre-thermal and thermal behaviors between integrable and non-integrable models. The transition from pre-thermal stage to thermal stage is peculiar for non-integrable models, and the reason for this transition is still an open question. For the behavior of the system in the thermal stage, while some former research support the idea that in this stage the Hamiltonian turns into a real "random matrix", leading to the evolution plateau of the off-diagonal $H_1(\lambda)$ and instanton transition of $H_0(\lambda)$ [11], other investigations support the opposite statements that in this stage the decay of $||H_1(\lambda)||$ is still stable, obeying a $1/\lambda^2$ law [10]. While the numerical results of spin chain model in this study basically accords to the latter argument, the results of AAH model supports the first one instead. We also examine the conservation criterion in real-time evolution, finding that the behavior is sensitive to the initial conditions, meaning the criterion is insufficient for predicting the actual behavior. More details are needed for future investigation in thermalization.

VII. CONNECTION TO COMPUTATIONAL PHYSICS CURRICULUM

In this paper, we use 4th order Runge-Kutta method to solve the flow equation in renormalization and exact diagonalization to simulate the evolution of the wave function. Both of these numerical techniques are contained in the curriculum. Our paper discusses three models: harmonic oscillator, spin chain model, and Aubry-Andr-Harper (AAH) model, all of which have been introduced in the course. Macroscopically, we consider Floquet system, which is also a frontier topic often explored by exact diagonalization and other related numerical method. The form of the translation operator is roughly mentioned in the curriculum.

Regarding physical phenomena, we reproduce behaviors related to fixed points in non-linear differential equations, a topic highlighted in the course. Our development of the new model also draws on knowledge of special functions, specifically Bessel functions. Furthermore, the phenomenon observed in the modified AAH model is related to a deterministic simulation of stochastic resonance, a concept appearing at our homework assignments.

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