A Note on the Aubry–André-Harper Model

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In this note, I will try to present some interesting physical phenomena in incommensurate systems. This mainly concerns the Aubry-André-Harper (AAH) model. I will first give a pedagogical review of the AAH model, and then discuss some recent developments about its energy band, localization and topology.

I. THE ENERGY BAND OF AAH MODEL

The AAH model is a simplified but non-trivial model that describes incommensurate systems [1]. It's a tight-binding model whose eigen equation is:

$$\psi_{n+1} + \psi_{n-1} + \lambda\cos(2\pi\beta n + \phi)\psi_n = E\psi_n$$

where ψ_n is the wave function component at site n, λ represents the relative intensity between the onsite potential and hopping strength between sites, β is an irrational number, ϕ is the additional phase for the onsite potential, and E is the eigen energy. This is the original version published in the 1980 paper. As β is irrational, the problem is that we cannot find a periodicity N that satisfies:

$$\cos(2\pi\beta(n+N) + \phi) = \cos(2\pi\beta n + \phi)$$

This is why we lose the periodicity in this model. The fundamental theorem in solid-state physics is the Bloch theorem, which requires the discrete translational symmetry of the system. Without this symmetry, we lose the ability to describe the degeneracy caused by the periodic potential in a many-body system, meaning it's difficult to find the energy band. However, the energy band is always present, which is a natural phenomenon of many-body interactions. It is simply hard to find a good quantum number.

These difficulties are very common in the research of incommensurate systems, and they make analytic calculations difficult in this region. Therefore, we use numerical methods to investigate this model, and we can uncover many interesting facts.

For the numerical method, we usually use exact diagonalization. We assume a closed boundary condition, which means cutting off the matrix at the ends and connecting the first term with the the final term. The eigen problem will be as follows [2]:

$$\begin{bmatrix} \lambda\cos(2\pi\beta+\phi) & -1 & \cdots & -1 \\ -1 & \lambda\cos(4\pi\beta+\phi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & \lambda\cos(2\Omega\pi\beta+\phi) \end{bmatrix}$$

where Ω is the number of sites. Using exact diagonalization, we can obtain the energy levels. To make the effect of the onsite potential significant, we choose $\lambda = 2$. I compare the energy levels of the commensurate case ($\beta = 2$) and the incommensurate case ($\beta = \frac{\sqrt{5}-1}{2}$), with $\Omega = 100$.

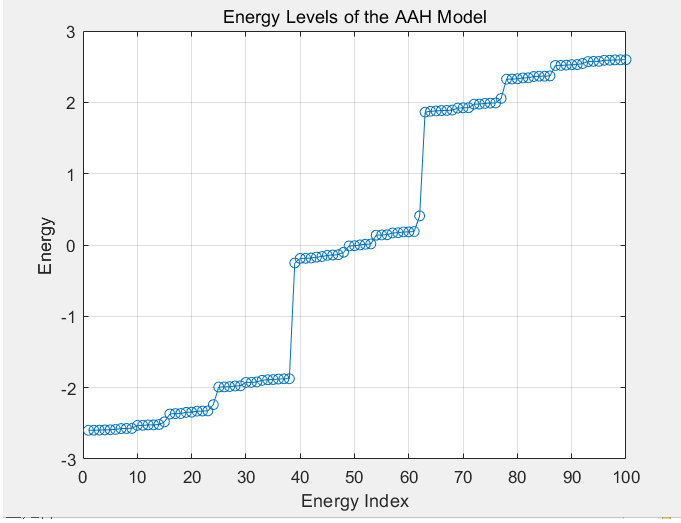

FIG. 1. Energy level of AAH model in incommensurate potential

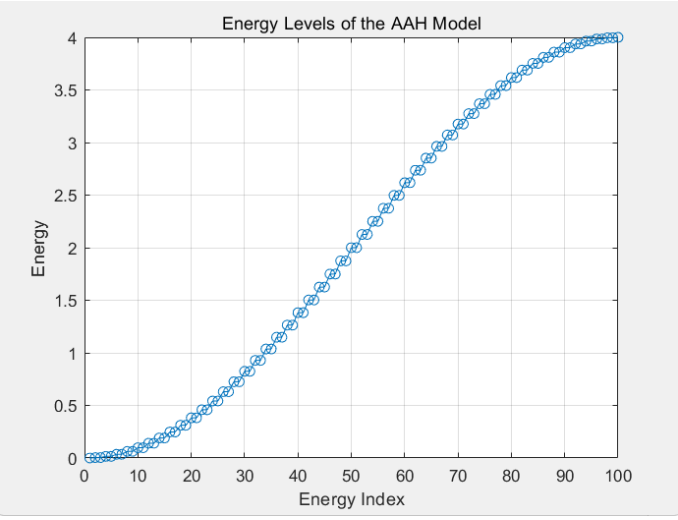

FIG. 2. Energy level in commensurate case.

The effect on the energy levels is similar to the analysis of the Moiré system (twisted bilayer graphene) [3].

This may be misunderstood as being caused by the incommensurate potential. Actually, we can observe the same phenomenon if we set $\beta = 0.7$. The reason will be discussed in detail later.

As analyzed above, it seems possible to obtain an energy band. We can first perform a Fourier transform. For convenience in discussing the dual behavior, the eigen equation is modified as [4]:

$$t(u_{j-1} + u_{j+1}) + V\cos(2\pi\beta j + \delta)u_j = Eu_j$$

where t represents the intensity of hopping explicitly, and V is the intensity of the onsite potential. The Fourier transform is:

$$u_j = \frac{1}{\sqrt{L}}\sum_k \psi_k e^{-ikj}$$

Substituting this into the eigen equation:

$$E\sum_k \psi_k e^{-ikj} = \sum_k \left[(2t\cos k)\psi_k e^{-ikj}\right] + \sum_k \left[V\frac{e^{i2\pi\beta j+i\delta} + e^{-i2\pi\beta j-i\delta}}{2}\psi_k e^{-ikj}\right] \quad (1)$$

We absorb the exponential terms in the final term by relabeling k, and make the coefficient of the plane wave function equal because these plane waves are orthogonal functions:

$$\frac{V}{2}(\psi_{k+2\pi\beta}e^{-i\delta} + \psi_{k-2\pi\beta}e^{i\delta}) + 2t\cos k\psi_k = E\psi_k$$

Usually, we set $\delta = 0$, but we have written it explicitly until now because it is meaningful in the discussion later. For now, we set $\delta = 0$.

Next, we have the freedom to choose the lattice in k-space. We set $k_n = k + 2\pi\beta n$, where $n \in \mathbb{Z}$, then:

$$\frac{V}{2}(\psi_{n+1} + \psi_{n-1}) + 2t\cos(k + 2\pi\beta n)\psi_n = E\psi_n$$

If we perform exact diagonalization in this momentum space, we obtain a relation $E = E(k)$, which has the potential to form an energy band structure.

As discussed in [5], we choose n to range from $-N$ to N considering the parity symmetry in k-space

They chose $\beta = \frac{\sqrt{5}-1}{2}, V = 1.2t = 1.2, N = 6$ to obtain the diagram on the left. The key observation here is the comparison with the diagram generated by translating the cosine potential with the quasi-periodic step. This comparison is an alternative way to understand the gap in the perturbative method. The crossing between near potentials leads to the formation of a gap. You might be interested in where these gaps will open.

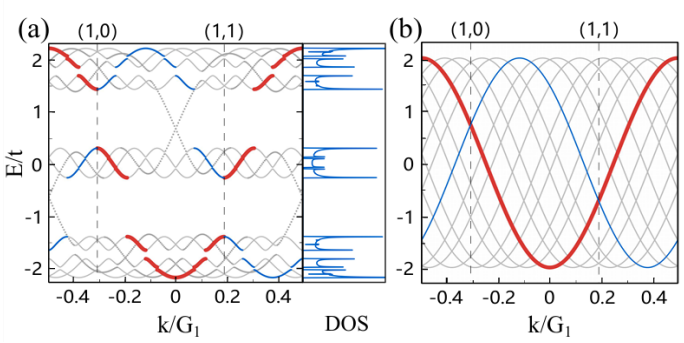

FIG. 3. The energy band of AAH model[6]

In the perturbative method, to observe a significant gap, we need to make the nearest onsite term as close as possible. Additionally, in the case of a large perturbation, we expect the hopping term to be as large as the onsite term, implying a significant modification of the original band structure,but not disturbing the original structure too much.

For the parameters $\beta = \frac{\sqrt{5}-1}{2}, V = t = 1, N = 7$, the spectrum we obtain is as shown below:

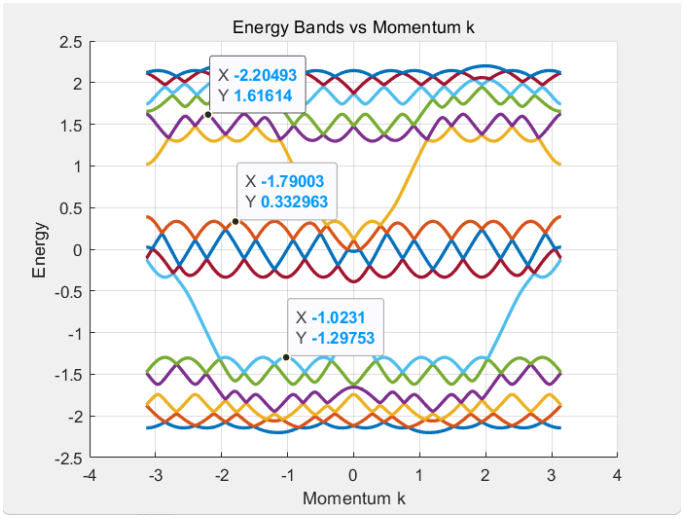

FIG. 4. The energy band of AAH model

The largest gap here is located in the range $[-2.2, -1]$ for $n = 0$. As discussed in the article, if we compare the $\cos(k)$ band with the $n = 1$ band $\cos(k + 2\pi\beta)$ and the $n = -1$ band $\cos(k - 2\pi\beta)$, we find that -1 and -2 are the crossing points of these bands.

If the hopping potential becomes very large, in the extreme limit, we can treat the onsite potential as zero. In this case, the energy band structure is lost, and we are left with discrete energy levels. This can be understood by noting that, when transformed into real space, the system becomes non-interacting.

These results are basic outcomes of the tight-binding model, but they do not fully capture the significance of

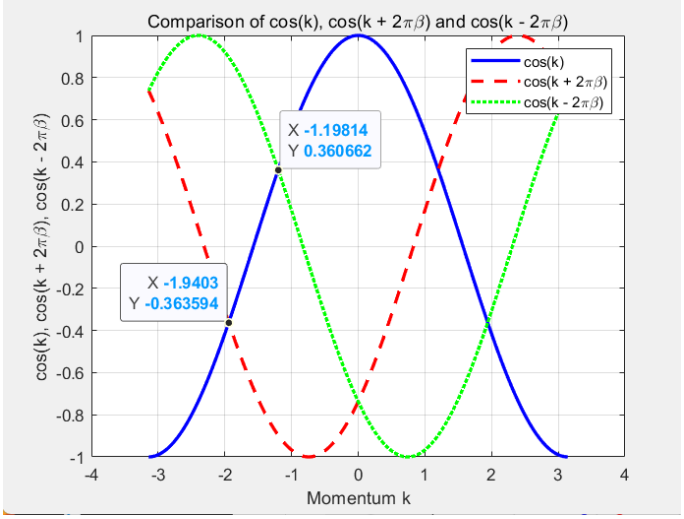

FIG. 5. The crossing points

incommensurate systems. The situation becomes more complex as N increases. In a commensurate model, the structure of the spectrum remains unchanged as N increases, and the dispersion relation remains invariant. In contrast, in an incommensurate system, since n lacks periodicity, the energy bands become covered. This means that for each state labeled by k, the possible energy values will form a bandwidth. The ergodic property is a key feature of incommensurate systems, which can be regarded as a near random property. For now, a nontrivial result we can infer is that the ergodic lines cause the boundaries of each energy band to appear as flat bands.Also, we can observe some lines in the gap which is usually an implication for the topological state. However, these lines change when the cutoff is different. We can't judge whether it's physical phenomenon or numerical error.This will be discussed later.

II. LOCALIZATION PHASE TRANSITION IN AAH MODEL

The discussion above mainly concerns the spectrum of the Aubry-André-Harper (AAH) model. These studies are based on the exact diagonalization method. However, in practice, there are several other approaches that provide useful insights.

1. Transfer Matrix Method. This method is actually the one discussed in the original paper [1]. The key idea is that the equation can be written as:

$$\begin{pmatrix} E - t\cos(2\pi\beta n) & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix}$$

Here, we change λ in the original equation to t, as using λ could be misleading when handling the eigenvalue problem. This method can be used to study the energy levels as well. However, we will focus on the localization behavior of the eigenfunctions. In the non-Hermitian case, we can calculate the Lyapunov exponent for this model (the recurrence relation takes the form $E\phi_n = \Omega_n + V\phi_{n-1}$, making the calculation of the Lyapunov exponent easier) [7], which can be generalized to discuss other non-Hermitian models. But we just check the eigenvalues here. The eigenvalue equation for this problem is:

$$\lambda_n^2 + (t\cos(2\pi\beta n) - E)\lambda_n + 1 = 0$$

Our approach here is that when the potential is incommensurate, at long scales, we lose the information about the momentum. To examine the long-range behavior between ψ_n and ψ_1 (for large n), we set $\cos(2\pi\beta n) = \pm 1$. For simplicity, we assume $E = 0$. This simplifies the eigenvalue equation to:

$$\lambda^2 \pm t\lambda + 1 = 0$$

When $|t| < 2$, there are no real roots for this equation and the modulus of the root is 1, meaning the eigenfunction is extended. However, if $|t| \geq 2$, real roots appear, and their modulus is greater than 1. This implies that the eigenfunction will exhibit a singular relationship, with the site-to-site relation given by $(\lambda)^n$. This is one of the most famous results in the AAH model. The value $t = 2$ marks the localization transition point. It is worth noting that the choice of $E = 0$ here is made for symmetry reasons; without this choice, the analysis would be less effective. You can consider this a qualitative analysis. In the next method, I will provide a different explanation.

2. Green Function. Thouless used this method to discuss the localization behavior in the tight-binding model[8], and the original paper of the AAH model cites this result to discuss the phase transition at $t = 2$.

In Thouless's paper , the tight-binding model is given by:

$$\epsilon_i a_i^\alpha - V_{i,i+1}a_{i+1}^\alpha - V_{i-1,i}a_{i-1}^\alpha = E_\alpha a_i^\alpha$$

Here, the index is not summed, and he considers the Green function of the wavefunction. He obtains:

$$(E-\epsilon_i)G_{ij}(E) + V_{i,i+1}G_{i+1,j}(E) + V_{i-1,i}G_{i-1,j}(E) = \delta_{ij}$$

The G_{ij} elements form a matrix, which is the inverse of the tridiagonal matrix $EI - H$, where I is the identity matrix and H is the Hamiltonian of the system. The analytic expression for the elements of the inverse matrix can be computed using the adjoint matrix or the cofactor. Thus, we have:

$$G_{1N}(E) = \prod_{i=1}^{N-1} V_{i,i+1} / \det(EI - H) = \prod_{i=1}^{N-1} V_{i,i+1} / \prod_{\alpha=1}^{N} (E - E_\alpha)$$

The Green function G_{1N} has a pole at $E = E_\beta$ with residue $a_1^\beta a_N^\beta$, which gives:

$$\ln|a_1^\beta a_N^\beta| = \sum_{i=1}^{N-1} \ln|V_{i,i+1}| - \sum_{\alpha\neq\beta} \ln|E_\beta - E_\alpha|$$

This result connects the behavior of the correlation between sites with the spectrum. To more concretely judge the localization transition, Thouless assumed that the wavefunction is localized and peaks at site i. For the exponential decay of the localized state, a_1 and a_N should behave as $e^{-\lambda_\beta(i-1)}$ and $e^{-\lambda_\beta(N-i)}$, respectively. This allows us to determine the decay exponent:

$$\lambda_\beta = \lim_{N\to\infty} \left\{ -(N-1)^{-1} \sum_{i=1}^{N-1} \ln|V_{i,i+1}| + (N-1)^{-1} \sum_{\alpha\neq\beta} \ln|E_\beta - E_\alpha| \right\} \quad (2)$$

Finally, by converting the summation over eigenstates into an integral with the density of states, we obtain:

$$\lambda_\beta = \int \rho(x) \ln|E_\beta - x|\, dx - \ln|V|$$

If $\lambda_\beta > 0$, the corresponding eigenstate is localized.

This method is highly instructive, and I have copied it from the original paper for reference.

Now, we apply this to the AAH model, where $|V| = 1$. The final form of the exponent is then:

$$\gamma = \int_{-\infty}^{\infty} \ln|E - E'|\, dN(E')$$

Typically, this quantity cannot be calculated explicitly. However, in the AAH model, there is another well-known feature: the AAH model is self-dual, which can be used to determine the transition point. As previously discussed, the form of the AAH model is similar in real space and momentum space, but the coefficients differ. Therefore, we use these coefficients to label the solutions of the corresponding model. In real space, the number of states is $N_1 = N_{t,V}(E)$, while in momentum space, the number of states is $N_2 = N_{t,4t^2/V}(2tE/V)$. According to the first Helly theorem, we have $N_1 = N_2$. By substituting this result into the Thouless formula, we obtain:

$$\gamma(E) = \gamma'\left(\frac{2tE}{V}\right) + \ln\left(\frac{V}{2t}\right)$$

This follows from:

$$\int \ln(E'-E)\, dN(E) = \int \ln\left(E' - \frac{2tE}{V}\right) dN\left(\frac{2tE}{V}\right) \approx \int \ln(E) - \ln\left(\frac{2t}{V}\right) dN(E) \quad (3)$$

The approximation assumes $E' = 0$. Thus, we expect $\gamma > 0$, and one necessary condition is $V > 2t$. However, this result is not rigorous.

III. PHYSICAL BACKGROUND OF AAH MODEL

Here, we provide a possible explanation based on previous work [2, 9]. Consider a one-dimensional bichromatic potential obtained by superimposing two optical lattices. The Hamiltonian for this system is given by:

$$H = \int dx\, \psi^\dagger(x) \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{opt}}(x)\right) \psi(x)$$

where the optical potential is:

$$V_{\text{opt}}(x) = s_1 E_{k_1} \sin^2(k_1 x) + s_2 E_{k_2} \sin^2(k_2 x + \phi)$$

Here, $E_{k_i} = \frac{(\hbar k_i)^2}{2m}$, and s_1, s_2 are dimensionless coefficients. This provides a physically reasonable model, and we aim to express it in a form analogous to the Aubry-André-Harper (AAH) model.

To do this, we apply a tight-binding approximation:

$$\psi(x) = \sum_i b_i w(x - x_i)$$

where $w(x - x_i)$ is the Wannier function centered at x_i. We choose the Wannier basis to ensure the orthogonality of the basis functions. We also set $k_1 = \frac{2\pi}{a}$, where a is the lattice constant, representing the crystal momentum of the lattice. This implies that the left potential represents the periodic potential of the lattice. The incommensurate relation is defined as $k_2 = \beta k_1$, where β is an irrational number.

For our purposes, we assume that $s_2\beta^2 \ll s_1$, so that the contribution from the right potential is perturbative and consists only of an onsite term. The left potential can include a nearest-neighbor hopping term. By applying these relations, we obtain the following form for the Hamiltonian:

$$H = -\sum_{i,j} J_{ij} b_i^\dagger b_j + \sum_{i,j} \Delta_{ij} b_i^\dagger b_j + \text{h.c.}$$

where:

$$J_{ij} = -\int dx\, w^*(x - x_i) \left(-\frac{\hbar^2}{2m}\nabla^2 + s_1 E_{k_1} \sin^2(k_1 x)\right) w(x - x_j)$$

and

$$\Delta_{ij} = s_2 E_{k_1} \beta^2 \int dx\, w^*(x - x_i) \sin^2(\beta k_1 x + \phi) w(x - x_j)$$

Note that J_{ii} is independent of i, so the onsite potential contribution can be neglected. We will only consider the nearest-neighbor hopping term.

Next, we analyze Δ_{ij}. We will demonstrate that, when considering only the onsite potential contribution, this model reduces to the AAH model. Ignoring a constant term, we obtain:

$$\Delta_{ii} = -\frac{s_2 E_{k_1} \beta^2}{2} \int dy\, |w(y)|^2 \cos(2\beta k_1 y + 2\beta k_1 x_i + 2\phi)$$

where $y = x - x_i$ is a change of variable. Since $|w(y)|^2$ is an even function, when we expand the cosine potential, only the cosine term remains. Thus, we have:

$$\Delta_{ii} = -\frac{s_2 E_{k_1} \beta^2}{2} \cos(2\pi\beta i + \phi) \int dy \, |w(y)|^2 \cos(2\beta k_1 y)$$

This simplifies to:

$$\Delta_{ii} = \cos(2\pi\beta i + \phi)\Delta$$

Finally, the Hamiltonian takes the form of the AAH model in second quantization:

$$H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j + \text{h.c.} + \Delta \sum_i \cos(2\pi\beta i + \phi) n_i$$

where n_i is the particle number operator.

Next, we will discuss more recent developments in this model. In the field of mathematics, there has been some research on the numerical calculation of this model[6]. However, the problem lies in the incommensurability, which causes ambiguity in the accuracy of the results . One instructive method involves treating an incommensurate system as a projection from a commensurate model in higher dimensions. A well-known example of this approach is the 1D Fibonacci quasicrystal [10]

IV. TOPOLOGY IN AAH MODEL

When considering a higher-dimensional version of the Aubry-André-Harper (AAH) model, some interesting phenomena arise.One of the most interesting property is about the topology. We will first introduce the topological aspects. The main focus of this section is to review key results concerning the topology of the 1D AAH model defined in a higer dimension as discussed in [11].

To begin with a fundamental concept in the topology of quantum mechanics, the geometric phase of adiabatic evolution is particularly instructive [12]. When considering a Hamiltonian with extreme time dependence, one of the extreme cases is the adiabatic approximation.

In this case, we imagine a Hamiltonian that depends on a set of parameters. The parameter defining a curve in the parameter space can regarded as time t. The interesting phenomenon occurs when the evolution of the system is considered along a closed loop in parameter space. In such a scenario, the wave function accumulates an additional phase, which is related to the topology of the parameter space. The detailed discussion follows:

In general, at each time t, we can expand the wave function in terms of the instantaneous eigenbasis:

$$|\psi(t)\rangle = \sum_m c_m(t) \, |\psi_m(t)\rangle$$

For convenience, we can rewrite this as:

$$|\psi(t)\rangle = \sum_m c_m(t) e^{-\frac{i}{\hbar}\int_0^t \epsilon_m(\tau)\, d\tau} \, |\psi_m(t)\rangle$$

Substituting this into the Schrödinger equation, we obtain an equation for the evolution of the coefficients $c_m(t)$:

$$\dot{c}_n(t) e^{-\frac{i}{\hbar}\int_0^t \epsilon_n(\tau)\, d\tau} = -\sum_m c_m(t) e^{-\frac{i}{\hbar}\int_0^t \epsilon_m(\tau)\, d\tau} \, \langle \psi_n(t)| \, |\dot{\psi_m}(t)\rangle$$

The challenge is to understand how the instantaneous eigenbasis evolves over time. A subtle issue arises because the instantaneous eigenbasis does not, in general, satisfy the Schrödinger equation. If one incorrectly assumes this, one may find that the instantaneous eigenbasis is also an eigenfunction of $\frac{d\hat{H}(t)}{dt}$, but this has counterexamples.

The correct method to resolve this is to take the time derivative of both sides of the equation:

$$\hat{H}(t) \, |\psi_n(t)\rangle = \epsilon_n(t) \, |\psi_n(t)\rangle$$

which gives:

$$\langle \psi_n(t)| \, |\dot{\psi_m}(t)\rangle = \frac{\langle \psi_n(t)| \, \dot{H}(t) \, |\psi_m(t)\rangle}{\epsilon_m(t) - \epsilon_n(t)}$$

Now, we can apply the adiabatic approximation to solve the problem analytically. We assume that the change in the Hamiltonian $\dot{H}(t)$ is very slow, and that there is no degeneracy between the states n and m. In this case, we obtain:

$$\dot{c}_n(t) = -c_n(t) \, \langle \psi_n(t)| \, |\dot{\psi_n}(t)\rangle$$

We then define the Berry phase:

$$i\gamma_n(t) = -\int_0^t \langle \psi_n(\tau)| \, |\dot{\psi_n}(\tau)\rangle \, d\tau$$

and the time-dependent coefficient as:

$$c_n(t) = e^{i\gamma_n(t)} c_n(0)$$

This additional phase $\gamma_n(t)$ is known as the Berry phase.

As the time dependence of the Hamiltonian is usually expressed through parameters, we can write the Berry phase as:

$$\gamma_n(t) = i\int_0^t \langle \psi_n(\tau)| \, |\dot{\psi_n}(\tau)\rangle \, d\tau = i\int_{Q(0)}^{Q(t)} \langle \psi_n[Q'(\tau)]| \, |\partial_{Q'}\psi_n[Q'(\tau)]\rangle$$

Here, $\vec{A}_n(Q)$ is defined as:

$$\vec{A}_n(Q) = i \, \langle \psi_n[Q']| \, |\partial_{Q'}\psi_n[Q']\rangle$$

Thus, the Berry phase becomes:

$$\gamma_n(t) = \int_C \vec{A}_n(Q) \cdot dQ$$

where Q is a vector in the parameter space, and this represents the second-kind line integral.

In general, this effect can be eliminated by a gauge transformation:

$$|\psi_n'(Q)\rangle = e^{-i\theta_n(Q)}\,|\psi_n(Q)\rangle$$

However, when the integral curve forms a loop, the phase factor can't be eliminated any more, indicating a real physical effect. In such cases, we can apply Stokes' theorem, which leads to the definition of the Berry curvature:

$$\vec{\Omega}_n(Q) = \nabla \times \vec{A}_n(Q)$$

The Berry phase can then be written as:

$$\gamma_n(t) = \int_C \vec{A}_n(Q)\cdot dQ = \iint_S \vec{\Omega}_n(Q)\cdot d\vec{S}$$

Here, $\vec{A}_n(Q)$ can be interpreted as a vector potential, while $\vec{\Omega}_n(Q)$ is analogous to the magnetic field. The Berry phase is thus the flux through a surface.

More importantly, the Berry curvature is gauge-invariant, which reflects its physical significance. In fact, there has been research on the thermal Hall effect, where, despite the absence of a magnetic field, the Berry curvature of phonons behaves like a magnetic field and leads to a Hall effect for phonons [13].

For a concrete expression, the Berry curvature is given by:

$$\vec{\Omega}_n(Q) = i\,\langle \nabla_Q \psi_n(Q)| \times |\nabla_Q \psi_n(Q)\rangle$$

This form can be viewed as a kind of metric on the Hilbert space with respect to the parameter space. Specifically, when considering the line element:

$$\begin{aligned} ds^2 &= |\psi(Q+dQ) - \psi(Q)|^2 \\ &= \langle \partial_\mu \psi|\,|\partial_\nu \psi\rangle\; dQ^\mu dQ^\nu \\ &= (\gamma_{\mu\nu} + i\sigma_{\mu\nu})\, dQ^\mu dQ^\nu \end{aligned}$$

we find that the real part of this "metric" is not gauge-invariant. To ensure invariance under gauge transformations, we must switch to the Fubini-Study metric. We define:

$$\beta_\mu = i\,\langle \psi_n[Q]|\,|\partial_\mu \psi_n[Q]\rangle$$

and the gauge-invariant metric is given by:

$$g_{\mu\nu}(Q) = \gamma_{\mu\nu}(Q) - \beta_\mu(\lambda)\beta_\nu(\lambda)$$

The final form of the quantum metric is:

$$G_{\mu\nu}(Q) = \langle \partial_\mu \psi(Q)|\,|\partial_\nu \psi(Q)\rangle - \langle \partial_\mu \psi(Q)|\,|\psi(Q)\rangle\,\langle \psi(Q)|\,|\partial_\nu \psi(Q)\rangle$$

The imaginary part is $\sigma_{\mu\nu} = -\frac{1}{2}\Omega_{\mu\nu}$, where $\Omega_{\mu\nu}$ is the Berry curvature. This is a fundamental observation that can be applied in strongly correlated systems, which is highly non-trivial[14–16].

In order to establish a connection with topology, the loop in the parameter space must satisfy the condition of the Chern theorem. This means that the surface integral of the Berry curvature must be taken over a closed manifold. A good example of this is the Brillouin zone (BZ) in two dimensions. The periodic boundary condition in the reduced first Brillouin zone makes the topology of the BZ a torus, which is a closed surface.

When we bring the Berry phase into the context of the crystal lattice, it is called the Zak phase [17]. In the original paper, the system is described by the equation:

$$i\hbar\frac{\partial \psi}{\partial t} = \left(\frac{1}{2m}\left(p - \frac{e}{c}A(t)\right)^2 + V(x)\right)\psi$$

This form is similar to the equation for an electron in an electromagnetic field (from the minimal coupling principle in relativistic quantum mechanics).This can be understood more clearly in the context of path integral quantization(or Dirac non-integrable factor)[18], where the phase of the wavefunction $e^{i\int A\,dl}$ is equivalent to coupling the momentum p with a "momentum" A. But the vector potential $A(t)$ here is independent of x, which preserves the validity of Bloch's theorem. In fact, considering A as a vector potential, we can recover Bloch's theorem by using the magnetic translation group[19]. Here, $V(x) = V(x+a)$ is a periodic potential.

According to Bloch's theorem, the wave function takes the form $\psi_{nkt}(x) = e^{ikx}u_{nkt}(x)$, where the eigenvalue equation becomes:

$$\left[\frac{1}{2m}\left(p + \hbar k - \frac{e}{c}A(t)\right)^2 + V(x)\right] u_{nkt} = \epsilon_n(k,t)u_{nkt}$$

This form can be understood that the e^{ikx} acts as a Dirac factor.

A key observation here is that, in the adiabatic evolution of $A(t)$, we can treat the change in $A(t)$ as equivalent to a change in k. Thus, when A changes a periodicity, the wave vector k can be regarded as changing by $\frac{2\pi}{a}$. The Berry phase in this context is given by:

$$\gamma_n = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{2\pi i}{a}\,\langle u_{nk}|\,|\frac{\partial u_{nk}}{\partial k}\rangle\; dk$$

The Berry curvature does not appear only in adiabatic problems. One typical model is the Thouless pump [20]. When considering the adiabatic motion of electrons in a periodic time-dependent Hamiltonian on the Wannier basis, the pumped charge is quantized. The reason for this is as follows : The Wannier basis is given by:

$$|W_n(R;t)\rangle = \frac{V}{(2\pi)^3}\int_{BZ} dk\, e^{-ikR}\,|\psi_{nk}(t)\rangle$$

The expectation value of the electron's position is:

$$\langle r_n\rangle = \langle W_n(0;t)|\,r\,|W_n(0;t)\rangle = i\frac{V}{(2\pi)^3}\int dk\;\langle \psi_{nk}(t)|\,\partial_k\,|\psi_{nk}(t)\rangle$$

This represents the displacement of the electron on site 0 in the Wannier basis, which is a modified version of

the wavefunction on site in the tight-binding model (thus providing a picture of the electron on the site). Now, we can compute the current:

$$\begin{aligned} j_n &= -\frac{e}{V}\frac{d\langle r_n\rangle}{dt} \\ &= -\frac{ei}{(2\pi)^3}\int dk\,[\langle\partial_t\psi_{nk}|\,|\partial_k\psi_{nk}\rangle - \langle\partial_k\psi_{nk}|\,|\partial_t\psi_{nk}\rangle] \end{aligned}$$

which simplifies to:

$$j_n = -\frac{e}{(2\pi)^3}\int dk\,\Omega^n_{tk}$$

The important physics here lies in the negative sign, which comes from the normalization of the eigen state.

For pumping a single band, we can calculate the pumped charge per cycle:

$$Q = \int_0^T dt\, j = -e\int_0^T dt\int_{-\pi}^{\pi}\frac{dk}{2\pi}\Omega_{tk}$$

This integral describes the total charge pumped over one cycle. The periodicity of time and the Brillouin zone cause this surface to be closed, and the integer obtained here is known as the first Chern number. In this spirit of describing the quasi-particle dynamics in a semi-classical view, there has been further research in this area [21].

A more famous result is the TKNN formula [22, 23], which is clearly discussed in this lecture note. For our goal, here we will discuss only the origin of the Harper equation.

The background involves electrons moving in a plane with a perpendicular magnetic field. As mentioned before, we can use the Dirac phase factor to represent the influence of the magnetic field. Using this method, we can easily write the lattice model:

$$H = -t\sum_x\sum_{j=1,2}|\vec{x}\rangle\, e^{-ieaA_j(x)/\hbar}\,\langle\vec{x}+\vec{e}_j| + \text{h.c.}$$

where $\vec{e}_1 = (a,0)$ and $\vec{e}_2 = (0,a)$. We use the Landau gauge here, with $A_1 = 0$ and $A_2 = Bx_1$. It appears that hopping occurs only in the direction of $\vec{e}_2$. Formally, we can directly define the magnetic translation operator, which can also be constructed using the Dirac phase factor. In fact, multiplying the wave function by the Dirac phase factor is a method to transform into an equivalent free wave function, which naturally retains the translational symmetry:

$$T_j = \sum_x |\vec{x}\rangle\, e^{-ieaA_j(x)/\hbar}\,\langle\vec{x}+\vec{e}_j|\,.$$

Typically, the translation operator is:

$$T_j = \sum_x |\vec{x}\rangle\,\langle\vec{x}+\vec{e}_j|\,.$$

Thus, this is a natural construction. With this definition, the Hamiltonian becomes:

$$H = -t\sum_{j=1,2}(T_j + T_j^\dagger).$$

And we have:

$$T_2T_1 = e^{\frac{ie\phi}{\hbar}}T_1T_2,$$

where $\phi = Ba^2$, which can be easily derived in the Landau gauge. This can be understood as an exchange effect. In topological quantum field theory (TQFT), when we substitute the translation operator with the Wilson loop operator, a similar relation holds. This phase factor introduces some complications; $\{T_1, T_2, H\}$ is not a complete set, but we need two good quantum numbers to reconstruct the Brillouin zone (BZ). To solve this problem, modified magnetic translational operators are introduced.

$$\tilde{T}_j = \sum_x |\vec{x}\rangle\, e^{-ieaA_j(x)/\hbar}\,\langle\vec{x}+\vec{e}_j|\,.$$

satisfying $\partial_k\tilde{A}_j = \partial_j A_k$.In Landau gauge, we take $\tilde{A}_1 = Bx_2, \tilde{A}_2 = 0$, so that$[H,\tilde{T}_j] = 0$,but $[\tilde{T}_1,\tilde{T}_2] \neq 0$. We set $\phi = \frac{p}{q}\phi_0$, where $\phi_0 = \frac{2\pi\hbar}{e}$, and p and q are coprime. Then, the commutation relation $[\tilde{T}_1^q,\tilde{T}_2] = 0$ holds. Thus, there exists a common eigenstate $|k\rangle$, with

$$\tilde{T}_1^q\,|k\rangle = e^{iqk_1a}\,|k\rangle\,,\quad \tilde{T}_2\,|k\rangle = e^{ik_2a}\,|k\rangle\,,\quad H\,|k\rangle = \epsilon_k\,|k\rangle\,.$$

The magnetic Brillouin zone is reduced to $k_1 \in \left(-\frac{\pi}{qa},\frac{\pi}{qa}\right]$, and $k_2 \in \left(-\frac{\pi}{a},\frac{\pi}{a}\right]$. Any energy eigenvalue in a given band is q-fold degenerate. This occurs because $[\tilde{T}_1, H] = 0$, so $|k\rangle$ and $\tilde{T}_1\,|k\rangle$ share the same energy. Using the relation

$$\tilde{T}_2\tilde{T}_1\,|k\rangle = e^{i\left(2\pi\frac{p}{q}+k_2a\right)}\tilde{T}_1\,|k\rangle\,,$$

we find that

$$\tilde{T}_1\,|k\rangle = |\left(k_1, k_2+\frac{2\pi p}{qa}\right)\rangle\,.$$

Since p and q are coprime, this operation can generate the q-fold degeneracy. This is similar to the discussion of the degeneracy of the ground state in TQFT.

Finally, when we consider the eigenwave function of this Hamiltonian, we obtain the Harper equation. Expanding $|\psi\rangle = \sum_x\psi(x)\,|x\rangle$ and considering $H\,|\psi\rangle = E\,|\psi\rangle$, we perform a Fourier transform:

$$\tilde{\psi}_r(k) = \sum_x e^{-i\left(k_1+\frac{2\pi pr}{qa},k_2\right)\cdot\vec{x}}\psi_x.$$

The method of constructing the lattice in the k-space is similar to the Aubry-André-Harper (AAH) model, and the final result is:

$$2\cos\left(k_1a+\frac{2\pi pr}{q}\right)\tilde{\psi}_r(k)+e^{ik_2a}\tilde{\psi}_{r+1}(k)+e^{-ik_2a}\tilde{\psi}_{r-1}(k) = -\frac{\epsilon(k)}{t}\tilde{\psi}_r$$

which is the famous Harper equation. Some interesting recent research includes [24, 25]. One of the most

important ideas in the quantum Hall effect is the magnetic translation group. The natural question arises: Is there an electric translation group? In the context of the covariant form of Maxwell's equations, the effect of an electric translation group may be involved considering the time dimension. Consequently, there has been some research on this effect in the context of space-time crystals [25]. The comparison between these effects is very clear, as illustrated in the graph6.

	Magnetic translation	Electric translation
Operators	$T_{\Delta x} = e^{-\Delta x \partial_x}$ $T_{\Delta y} = e^{-\Delta y \partial_y} e^{-ieB\Delta y x/\hbar}$ $[T_{\Delta x}, H] = [T_{\Delta y}, H] = 0$	$T_{\Delta t} = e^{-\Delta t \partial_t}$ $T_{\Delta x} = e^{-\Delta x \partial_x} e^{ieE\Delta x t/\hbar}$ $[T_{\Delta x}, i\hbar\partial_t - H] = [T_{\Delta t}, i\hbar\partial_t - H] = 0$
Quantization	$eB\Delta x\Delta y = 2\pi\hbar$	$eE\Delta x\Delta t = 2\pi\hbar$
Gauge	$\vec{A} = (-By, 0, 0), \phi = 0$	$\vec{A} = 0, \phi = Ex$

FIG. 6. This table summarizes the properties of magnetic/electric translation operators[25]

Now, we return to the relationship between the Harper model and the 1D Aubry-André-Harper (AAH) model in topology [11].

Why is the topology of the 1D quasicrystal a problem? As mentioned earlier, the appearance of the Chern number requires periodic boundary conditions, but in a quasi-crystal, we cannot define a proper periodic boundary condition. To verify this, a natural approach is to introduce parameters in the AAH model, which will be eliminated when considering the thermodynamic limit. This also approximates the incommensurate potential by a commensurate potential with large L, which becomes incommensurate in the thermodynamic limit as well. Our goal is to show that the topological properties are preserved in the thermodynamic limit and that the 1D AAH model has topological properties analogous to those of the quantum Hall model.

The construction is given by:

$$H_b(\theta,\phi)\psi_n = te^{\frac{i\theta}{L}}\psi_{n+1} + te^{\frac{-i\theta}{L}}\psi_{n-1} + \lambda\cos(2\pi\bar{b}_L n + \phi)\psi_n$$

where $\bar{b}_L = \frac{[b\cdot L]}{L}$ is a rational approximation to b. As $L \to \infty$, this model approaches the AAH model.

First, it is claimed that the structure of the spectrum is independent of ϕ in the thermodynamic limit. This is because a change in ϕ, specifically $\phi \to \phi + \epsilon$, where $\epsilon = \frac{2\pi l}{L}$ and l is an integer, can be equivalent to a change in n such that $n \to n + n_\epsilon$. The proof here is not entirely rigorous and the condition is:

$$n_\epsilon[b \cdot L] = l + mL$$

where m is an integer. We should confirm that for any large L and l in the lower order of the linear order of L ,we can find an integrt m so that n_ϵ exists. This is not immediately obvious. However, if we relax this condition, we can assert that in the thermodynamic limit, this always holds. Considering translational invariance, we conclude that the spectrum of the AAH model does not depend on ϕ.

Next, we make the most important statement. We calculate the Chern number in terms of the projection operator. Consider the topological states in the gap:

$$P(\phi,\theta) = \sum_{E_n < E_{\text{gap}}} |n\rangle\langle n|$$

The topological number corresponding to the periodic adiabatic evolution can be calculated as:

$$\nu = \frac{1}{2\pi i}\iint_S d\theta d\phi\, C(\theta,\phi)$$

where

$$C(\theta,\phi) = \text{Tr}\left(P\left[\frac{\partial P}{\partial\phi}, \frac{\partial P}{\partial\theta}\right]\right).$$

It can be shown that:

$$C(\theta,\phi) = \sum_n \Omega^n_{\theta\phi}$$

This is the familiar form used in the calculation of the Berry phase previously.

In the thermodynamic limit, the contribution of θ will vanish, and the integral will reduce to an integration of ϕ from 0 to 2π, which is the usual Berry phase for the pump in the AAH model. The important claim is that the Chern number we calculate can be naturally extended to this limit.

The proof is as follows: First, we can write the symmetry behavior of the Hamiltonian formally as:

$$H_b(\phi+\epsilon) = T_{n_\epsilon} H_b(\phi) T^{-1}_{n_\epsilon}$$

and since the projector differs from the Hamiltonian only in its eigenvalues, we have:

$$P(\phi+\epsilon) = T_{n_\epsilon} P(\phi) T^{-1}_{n_\epsilon}$$

$$\partial_\phi P(\phi+\epsilon) = T_{n_\epsilon} \partial_\phi P(\phi) T^{-1}_{n_\epsilon}$$

This can be proven by the definition of the derivative. Using these relations and considering the property of the trace, $\text{Tr}(ABA^{-1}) = \text{Tr}(B)$, it follows that:

$$C(\phi+\epsilon) = C(\phi)$$

Note that here we are considering the rational approximation when L does not go to the limit. When considering the periodicity of $C(\phi)$, a similar issue arises. For $\epsilon = \frac{2\pi l}{L}$, not all l will correspond to a valid n_ϵ. Therefore, we segment by specific ϵ with a definite l, and we have:

$$\begin{aligned}\int_0^{2\pi l} d\phi\, C(\phi) &= l\int_0^{2\pi} d\phi\, C(\phi) \\ &= lL\int_0^{\frac{2\pi}{L}} d\phi\, C(\phi) \\ &= l2\pi C(\phi=0) + O\left(\frac{1}{L}\right)\end{aligned}$$

Thus, in the thermodynamic limit, we obtain:

$$\int_0^{2\pi} d\phi\, C(\phi) = 2\pi C(\phi = 0)$$

This implies that the Chern number can be defined and is independent of ϕ in the thermodynamic limit.

In comparison to the Harper equation, the parameters θ and ϕ correspond to k_1 and k_2.

V. AAH MODEL IN HIGER DIMENSION

Next, we will consider some generalizations to higher dimensions. There is also a natural generalization of the AAH model to higher dimensions while maintaining self-duality [26]. The form of the Hamiltonian is:

$$H = \sum_{\vec{r}} \sum_{i=1}^{d} \left(c^\dagger_{\vec{r}+\hat{u}_i} c_{\vec{r}} + \text{h.c.} \right) + \sum_{\vec{r}} V(\vec{r}) c^\dagger_{\vec{r}} c_{\vec{r}}$$

where $V(\vec{r}) = 2v \sum_{i=1}^{d} \cos\left(2\pi \vec{b}_i \cdot \vec{r} + \phi_i\right)$, and the set $\{\vec{b}_i\}$ forms a $d \times d$ matrix B. Some restrictions are required to satisfy both the periodic condition and the self-dual condition.

When considering localization, a famous example is Anderson localization caused by random potentials. In the theory of random matrix theory, we focus on a statistical quantity:

$$r_n = \frac{\min(\delta_n, \delta_{n+1})}{\max(\delta_n, \delta_{n+1})}$$

where $\delta_n = E_n - E_{n-1}$. If the distribution of δ_n follows a Poisson distribution, the picture is that there is no repulsion between energy levels, indicating weak interactions and the hopping term tending to zero, resulting in localized eigenstates. If the distribution matches the GOE (without time-reversal symmetry, TRS) or the GUE (with TRS), the eigenstates tend to be extended. The quantity r_n is sampled over eigenstates within energy intervals ΔE and over different choices of ϕ. There are some numerical results in this paper.

VI. SUMMARY

In this review, we focus on the localization and topological properties. These developments have great potential. The research of a quasiperiodic system is closely related to the irrational number. In numerical methods, we create pseudorandom numbers using irrational numbers to mimic the random phenomenon in the natural world which has the profound philosophy about deterministic chaos. This simple model has many metaphors to understand this philosophy. In addition, the topological property may indicate some intrinsic invariants in the chaos system.

[1] S. Aubry and G. André, Proceedings, VIII International Colloquium on Group-Theoretical Methods in Physics **3** (1980).
[2] G. A. Domínguez-Castro and R. Paredes, European Journal of Physics **40**, 045403 (2019).
[3] R. Bistritzer and A. H. MacDonald, Proceedings of the National Academy of Sciences **108**, 12233 (2011), https://www.pnas.org/doi/pdf/10.1073/pnas.1108174108.
[4] Y. Liu, X.-P. Jiang, J. Cao, and S. Chen, Phys. Rev. B **101**, 174205 (2020).
[5] X.-Y. Guo, J.-R. Chen, C. Zhao, M. Liang, Y.-H. Wu, J.-H. Gao, and X. C. Xie, Energy bands of incommensurate systems (2024), arXiv:2410.09793 [cond-mat.mes-hall].
[6] K. Jiang, S. Li, and P. Zhang, On the approximation of quasiperiodic functions with diophantine frequencies by periodic functions (2024), arXiv:2304.04334 [math.NT].
[7] T. Liu, H. Guo, Y. Pu, and S. Longhi, Phys. Rev. B **102**, 024205 (2020).
[8] D. J. Thouless, Journal of Physics C: Solid State Physics **5**, 77 (1972).
[9] M. Modugno, New Journal of Physics **11**, 033023 (2009).
[10] A. Jagannathan, Rev. Mod. Phys. **93**, 045001 (2021).
[11] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Phys. Rev. Lett. **109**, 106402 (2012).
[12] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 3rd ed. (Cambridge University Press, 2020).
[13] T. Qin, Q. Niu, and J. Shi, Phys. Rev. Lett. **107**, 236601 (2011).
[14] A. Avdoshkin, J. Mitscherling, and J. E. Moore, The multi-state geometry of shift current and polarization (2024), arXiv:2409.16358 [cond-mat.str-el].
[15] R. Cheng, Quantum geometric tensor (fubini-study metric) in simple quantum system: A pedagogical introduction (2013), arXiv:1012.1337 [quant-ph].
[16] I. Komissarov, T. Holder, and R. Queiroz, Nature Communications **15**, 10.1038/s41467-024-48808-x (2024).
[17] J. Zak, Phys. Rev. Lett. **62**, 2747 (1989).
[18] T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).
[19] J. Zak, Phys. Rev. **134**, A1602 (1964).
[20] D. J. Thouless, Phys. Rev. B **27**, 6083 (1983).
[21] M.-C. Chang and Q. Niu, Phys. Rev. B **53**, 7010 (1996).
[22] D. Tong, Lectures on the quantum hall effect (2016), arXiv:1606.06687 [hep-th].
[23] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[24] J. Wang, J. J. He, and Q. Niu, Fractional stark ladders and novel quantum dynamics of space-time ssh lattices (2024), arXiv:2409.13260 [cond-mat.other].
[25] C. Ke, K. Yang, and C. Wu, Electric space-time translation and floquet-bloch wavefunction (2024), arXiv:2409.15851 [cond-mat.mes-hall].
[26] T. Devakul and D. A. Huse, Phys. Rev. B **96**, 214201 (2017).