Description of Locality and Causality: A Note

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In this note, we have reviewed the basic idea of locality in Bell inequalities with simple but inspiring examples. At the same time, as causality is an important intuition related to locality, we discuss the concepts of causality from a fundamental view. This is actually related to the more fundamental question about the origin of space and time. After distinguishing space-time as presupposed or emergent, we discuss different interpretations of causality based on the two ideas. Meanwhile, the possible application of Bell inequalities in many-body physics is also discussed.
Keywords: Bell inequalities, locality, causality

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I. INTRODUCTION

Since the birth of quantum mechanics, there are many debates about the counterintuitive prediction in this framework. One famous problem is the loss of locality in quantum mechanics. To solve this question, Bell organized and clarified some important concepts with regard to locality. At the same time, he put forward Bell inequalities to test whether a system is local. However, these arguments cannot totally clear people's confusion of the intuition about locality. One unclear intuition related to locality is causality. In this paper, we first review the basic ideas and application of Bell inequalities. After that, causality is discussed in detail based on some recent progress.

II. LOCALITY

In this part, we focus on the definition of locality related to the Bell inequality [1].

A. Locality and Bell inequality

The locality here is defined to disambiguate the nonlocal correlation in quantum mechanics. The misleading nonlocal correlation can be understood by a simple example. When there are two local observers located at A and B, we prepare a quantum state on $\mathscr{H}^A \otimes \mathscr{H}^B$ as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle), \tag{1}$$

where $\mathscr{H}^A$ is the Hilbert space at A, $|0\rangle, |1\rangle$ are the spin down and spin up states of a qubit. Then, if the observer at A measures the quantum state and the result is spin up, then she will know that the observer at B will get the spin up state, too. There seems an instantaneous correlation between them, which is regarded impossible in modern physics. However, the obvious disadvantage here is that the shared information of the total quantum state is assumed before the experiment. To eliminate this misunderstanding, the locality is defined when the equality below can be found, which is

$$P(ab|xy) = \int_\Lambda d\lambda Q(\lambda) P(a|x,\lambda) P(b|y,\lambda), \tag{2}$$

where $Q(\lambda)$ is the assumed distribution of the rule and $P(a|x,\lambda)$ is the conditional probability at A when the input is x and the rule λ is assumed. In addition, $P(ab|xy)$ is the joint probability of A and B under the input x, y. This definition can be understood on a game background. The probability distribution $Q(\lambda)$ representing the assumed global information can be regarded as a dealer's rule. When the dealer hand out cards to A and B, A and B share knowledge of the probability of distribution of their cards at hands. The quantum interpretation of the probability is based on the Born rule. The input x, y can be the direction of the measurement to a spin and the output can be up or down along the direction.

However, under this definition of locality, the quantum system can still break locality, which is described by the Bell inequality. We give a simple example breaking the Clauser-Home-Shimony-Holt (CHSH) inequality to show how the Bell inequality works.

We assumed the input $x, y \in \{0, 1\}$ and the output $a, b \in \{-1, 1\}$, which is the measurement of a qubit. Related quantities are defined as follows.

$$\begin{aligned} \langle a_x b_y\rangle &\equiv \sum_{a,b} abP(ab|xy), \\ S &\equiv \langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle. \end{aligned} \tag{3}$$

Then we will use the locality condition to derive an upper bound of $\langle S\rangle$, which is the CHSH inequality. As the locality is satisfied, we have

$$\langle a_x b_y\rangle = \sum_{a,b} \int d\lambda\ abQ(\lambda)P(a|x,\lambda)P(b|y,\lambda) = \int d\lambda\ Q(\lambda)\langle a_x\rangle_\lambda \langle b_y\rangle_\lambda, \tag{4}$$

where $\langle a_x\rangle_\lambda \equiv \sum_a aP(a|x,\lambda) \in [-1, 1]$. With an inequality

$$\begin{aligned} S_\lambda &\equiv [\langle a_0\rangle_\lambda\langle b_0\rangle_\lambda + \langle a_0\rangle_\lambda\langle b_1\rangle_\lambda + \langle a_1\rangle_\lambda\langle b_0\rangle_\lambda - \langle a_1\rangle_\lambda\langle b_1\rangle_\lambda] = [\langle a_0\rangle_\lambda(\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda) + \langle a_1\rangle_\lambda(\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda)] \\ &\leq |\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda| + |\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda| \leq 2\langle b_0\rangle_\lambda \leq 2, \end{aligned} \tag{5}$$

where the triangle equality is used, we can derive that

$$S = \int d\lambda Q(\lambda) S_\lambda \leq 2, \tag{6}$$

as $\int d\lambda Q(\lambda) = 1$. So, the CHSH inequality is

$$S \equiv \langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle \leq 2 \tag{7}$$

Next, we will test two methods of measurement to check this inequality in a quantum system, which partially tells us the reason for the break of locality.

First, the quantum state is prepared as

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|-1,1\rangle + |1,-1\rangle). \tag{8}$$

The measurement is restricted to be along the original direction, which means that the projective measurement is $|-1\rangle\langle-1|$ or $|1\rangle\langle1|$ corresponding to $x, y \in \{0,1\}$ and this is exactly the same as the problem mentioned in Eq. 1. In this situation, $P(11|xy) = P(-1-1|xy) = 0$ and only when x, y are different, the probability can take the value 1. Thus, the CHSH is calculated to be $S = -2$, which does not break the CHSH inequality. It should be mentioned that satisfying the CHSH inequality is not an equivalent condition of locality, which means that this situation is still not confirmed to be local.

Second, to break the CHSH inequality, we focus on the freedom of quantum mechanics in the measurement, which means that the input can be changed as $x \in \{0,1\} \to \{\hat{P}_{\hat{e}_1}, \hat{P}_{\hat{e}_2}\}, y \in \{0,1\} \to \{\hat{P}_{\hat{e}'_1}, \hat{P}_{\hat{e}'_2}\}$. These bases are chosen as

$$\hat{e}_1 \perp \hat{e}_2, \hat{e}'_1 = \frac{-(\hat{e}_1 + \hat{e}_2)}{\sqrt{2}}, \hat{e}'_2 = \frac{-\hat{e}_1 + \hat{e}_2}{\sqrt{2}}. \tag{9}$$

In this setting, the conditional probability is calculated as

$$P(ab|00) = |\langle ab|\hat{P}_{\hat{e}_1} \otimes \hat{P}_{\hat{e}'_1}|\psi^-\rangle|^2, \tag{10}$$

which is an example of the condition $(x, y) = (0, 0)$. Then, it can be calculated that $S = 2\sqrt{2} > 2$, which breaks the CHSH inequality.

In this comparison, the possible reasons for the nonlocality of the quantum system lie in that a single state can be decomposed to a linear superposition of other states which is impossible in the classical system. In other words, there is extra direction or dimension of the inputs and outputs in the quantum system.

This actually gives some hint for the relation between the quantum system and the classical system. Another well-known related result is in the picture of partition function, which states that via the transfer matrix method, the 2D classical Ising model's partition function is equivalent to the 1D quantum Ising chain with a transverse field.

B. Construction of Bell inequality

The CHSH inequality is one of the Bell inequalities. The general definition is a linear inequality for the probabilities $P(ab|xy)$ that is necessarily verified by any model satisfying the locality condition but which can be violated by suitable measurements on a pair of quantum particles in an entangled state.

To construct other specific Bell inequality, we can imagine based on the analogy of the card games before, which is called the "nonlocal" game as shown in Fig. 1. In this game, Alice and Bob can decide a strategy to maximize the

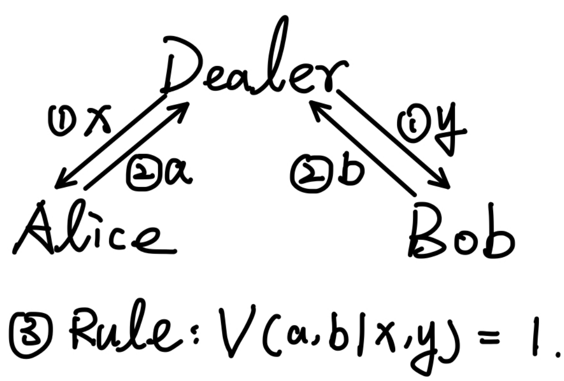

FIG. 1. Illustration of nonlocal game. The game consists of three processes: 1. The dealer send the input x, y to Alice and Bob. 2. Alice and Bob send the output to the dealer when they know the distribution of x, y and the rule of winning $V(a, b|x, y) = 1$. 3. The dealer check the data (a, b, x, y) to decide if the dealer wins.

probability of winning. The strategy is marked as $a = a(x), b = b(y)$. More generally, the strategy can be considered as $P(a, b|x, y)$. Thus, the probability of winning is

$$P_{win} = \sum_{x,y} \pi(x, y) \sum_{a,b} V(a, b|x, y)P(a, b|x, y), \tag{11}$$

where $\pi(x, y)$ is the distribution of the dealer's cards, which can also be understood as a shared randomness between Alice and Bob. Maximizing P_{win} means finding the best strategy $P_{best}(a, b|x, y)$ according to the known rules $\pi(x, y)$ and $V(a, b|x, y)$. At the same time, the maximal quantity also provides a linear inequality of the general $P(a, b|x, y)$, which is a Bell inequality. To make the structure more clear, we denote the inequality as

$$\vec{s} \cdot \vec{p} \le P_{max} = \vec{s} \cdot \vec{p}_{max}, \tag{12}$$

where the components of the vectors are $s^{ab}_{xy} = \pi(x, y)V(a, b|x, y), p^{ab}_{xy} = P(a, b|x, y)$. A famous construction is the XOR game discussed in [1].

C. Locality and entanglement

In this part, we continue the discussion of nonlocality in the quantum system focusing on the entanglement.

To describe the quantum system, the density matrix formulation is a must and the reasons are as follows. First, to describe a general quantum system with a possibility prepared in different quantum states $|\psi_i\rangle$, the density matrix is a natural formulation that $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. Second, in an isolated quantum system containing all freedoms, the quantum state is indeed pure $\rho = |\psi\rangle \langle\psi|$, but when considering open quantum systems, which describe the quantum subsystem surrounded by an environment, the state of the subsystem should be described as $\rho_A = \mathrm{Tr}_B[|\psi_{AB}\rangle \langle\psi_{AB}|]$, which could be a mixed state. More generally, the density matrix provides a convenient tool for investigating the entanglement between different freedoms. Finally, the formulation of density matrix provides a method to discuss the probability structure of a quantum system in the language of quantum ensembles. At the same time, when discussing the locality in a quantum system, we usually regard the total quantum state as a pure state like Eq. 1. Thus, using density matrix is a natural way to investigate the probability structure between different subsystems, which is the structure of locality.

A central concept in quantum system is entanglement, which represents a special correlation between different subsystems. The definition of entanglement is based on the separability of the total density matrix.

$$\text{no entanglement} \Leftrightarrow \rho_{AB} = \sum_\lambda P_\lambda \rho^\lambda_A \otimes \rho^\lambda_B \tag{13}$$

It should be mentioned that ρ_{AB} here does not have to be a pure state. Under this definition, we can derive that if there is no entanglement between A and B, the quantum system is local, whose derivation is

$$p(ab|xy) = \mathrm{Tr}[\sum_\lambda P_\lambda(\rho^\lambda_A \otimes \rho^\lambda_B)M_{a|x} \otimes M_{b|y}] = \sum_\lambda P_\lambda \mathrm{Tr}[\rho^\lambda_A M_{a|x}]\mathrm{Tr}[\rho^\lambda_B M_{b|y}] = \sum_\lambda P_\lambda P(a|x, \lambda)P(b|y, \lambda). \tag{14}$$

Here, $M_{a|x}$ is the measurement operator. However, if there exists an entanglement between A and B, the quantum system can still be local. The study in this direction is much more complex. We just show some results in the simple construction of ρ_{AB}.

First, all pure entangled states are nonlocal, and the only pure state not violating the Bell inequality is $|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$. Then, there exists an equivalent condition of breaking CHSH for any 2-qubit state as discussed in [1], which shows that not all entangled 2-qubit mixed state will violate CHSH. However, as mentioned before, one Bell inequality cannot represent the locality, and thus the relation in this direction is still not clear even in this simple setup.

To better understand the separability of a mixed state which partially represents the condition of locality, we discuss the Werner state in detail as a typical example, which goes through a "phase transition" between entanglement and no entanglement.

First, we show the basic properties of Werner state to have a good picture of it. The Werner state is defined as

$$\rho_W = p\,|\psi^-\rangle \langle\psi^-| + \frac{1-p}{4}\hat{I}, \tag{15}$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the single state in the permutation symmetry and thus for any $U \in SU(2)$, $(U \otimes U)\rho_W(U \otimes U)^\dagger = \rho_W$, which shows that the Werner state is locally unitary invariant. To understand the properties of the Werner state, an important method is the SWAP representation. We define a SWAP operator as changing the location of two sites:

$$Swap\,|\psi^{+}\rangle = |\psi^{+}\rangle\,, Swap\,|\psi^{-}\rangle = -\,|\psi^{-}\rangle\,, \tag{16}$$

where $|\psi^{+}\rangle$ is the triple state in the permutation representation. As $\langle\psi^{-}|\,\psi^{+}\rangle = 0$, we have

$$|\psi^{-}\rangle\,\langle\psi^{-}| = \frac{1}{2}(\hat{I} - Swap), \rho_W = \frac{1+p}{4}\hat{I} - \frac{p}{2}Swap. \tag{17}$$

Thus, the Werner state has a three-fold degenerate eigenvalue of $\frac{1-p}{4}$ and a nondegenerate eigenvalue of $\frac{1+3p}{4}$, whose sum is 1. After these discussions, we can interpret the probability p in the definition as a deviation from the uniform distribution of $|\psi^{-}\rangle$ and $|\psi_i^{+}\rangle\,, i = 1, 2, 3$. When p is large, the quantum system tends to be a single state. If $p = 1$, the system is an entangled Bell state. If $p = 0$, the system is a mixed state with uniform distribution. We can expect that there should be a transition of entanglement at some specific $p^{\star}$.

Second, to derive the specific transition point, we introduce the positive partial transposition criterion (PPT criterion), which is the equivalent condition of separability in low-dimensional $(2\times 2, 2\times 3)$ system. The idea comes from a necessary condition that if a quantum state is separable, after partially transposing on one subsystem (B), the total system is still a well defined quantum state. In the case of the Werner state,

$$\rho_W = \frac{1}{4}\begin{bmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & -2p & 0 \\ 0 & -2p & 1+p & 0 \\ 0 & 0 & 0 & 1-p \end{bmatrix}, \rho_W^{pT} = \frac{1}{4}\begin{bmatrix} 1-p & 0 & 0 & -2p \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+p & 0 \\ -2p & 0 & 0 & 1-p \end{bmatrix}, \tag{18}$$

and the minimal eigenvalue of ρ_W^{pT} is $\frac{1-3p}{4}$. Thus, the transition point is $p^{\star} = \frac{1}{3}$. This actually shows that another difficulty in the study of locality is the condition of entanglement or separability. The PPT criterion is rather restricted, but a general criterion is lack now.

As a conclusion of this part, to understand the locality in the quantum system, a possible way is to solve two important problems, one is the general criterion of separability or entanglement, and the other is the relation between entanglement and locality, especially in a mixed entangled state.

D. Bell inequality and quantum phase transition

Why have we paid much attention to the locality? A direct and fundamental reason is to answer questions about the characters of a quantum system and whether these characters go against common intuition. As a result in the study of locality, we can answer that the nonlocal behavior is an important character of a quantum system which violates our common intuition. Thus, a natural question is asked: Can we use this new property, which is absent in the classical world, to achieve more applications?

To answer this question, although there are many discussions about application of the nonlocal effect in quantum communication, we focus on the application of the Bell inequality in quantum many-body systems, as there are many recent inspiring progress in topological order collecting the community of quantum information and condensed matter physics [2]. For now, one direct application of the Bell inequality is to signal the quantum phase transition (QPT).

To discuss the application in QPT, we first give a brief explanation of what is different from the classical phase transition in QPT [3]. A fundamental observation in QPT is that the finite temperature partition function of a D+1-dimensional classical field theory can usually be regarded as a zero-temperature imaginary-time path integral of a D-dimensional quantum field theory, which comes from the similarity between $\rho \propto e^{-\beta H}$ and $U \propto e^{-itH}$. Thus, the zero-temperature D-dimensional quantum field theory will give us a correlated time scale by analyzing its dynamics by calculating the time-order correlation function, and this correlated lifetime can tell the boundary of effectiveness of a semi-classical description of this quantum field theory. This lifetime is defined as $C(0,t) = \langle \mathscr{T}\hat{n}_i(t)\hat{n}_i(0)\rangle \propto e^{-\frac{t}{\tau_f}}$. We can imagine a process preparing a Guassian wave packet located at i initially and τ_f tells us how long the packet will remain localized, since the definition of C should be irrelevant with the position. If the system keeps localized, then we claim that the semi-classical description is effective. Thus, the semi-classical description with a long τ_f in the quantum field theory corresponds to a low-temperature situation in the D+1 dimensional classical field theory. At the same time, high-temperature behavior of a classical field theory corresponds to quantum criticality, which means a lack of local particles in the corresponding quantum systems. The phase diagram of a classical statistical field theory is shown in Fig. 2. Although our explanation depends on the time-order correlation of the quantum field theory, the truly important phenomenon is quantum criticality, which is the special phase transition in the quantum many-body

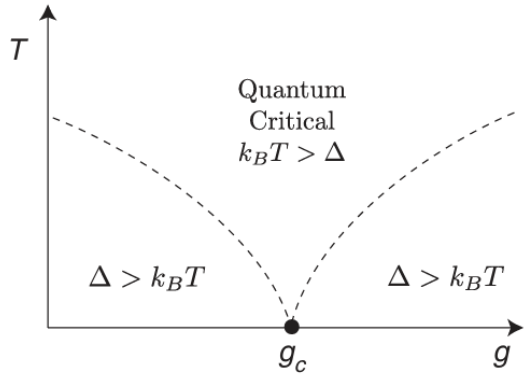

FIG. 2. Illustration of the phase diagram in classical system [3].

system. The loss of low-energy excitation's effective quasi-particle description is related to quantum criticality in the ground state and thermal state. (not equivalent!) Our introduction just provides one aspect of understanding quantum phase transitions. From this correspondence, it can be seen that the finite-temperature quantum field theory without classical counterpart will have some new phenomenon in phase transition.

After a brief introduction of QPT, it can be found that there is actually a relation between QPT and locality. Following this intuition, there is some work that attempts to make the Bell inequality a better quantity to signal QPT which is hard to indicate from energy like Kosterlitz-Thouless QPT [4, 5]. In their study of spin-$\frac{1}{2}$ XXZ model, they find (i) The Bell inequality is not only able to reveal the first-order quantum phase transition but also the Kosterlitz-Thouless quantum phase transition. This cannot be indicated by the energy of the system nor by the bipartite entanglement given in terms of the concurrence. (ii) The two spins, despite being entangled, never violate the Bell inequalities. Thus their correlations can then be described by a realistic local theory as shown in Fig. 3, where the quantity measuring the Bell inequality is the well known $\mathscr{B}_{CHSH}$ discussed in [6].

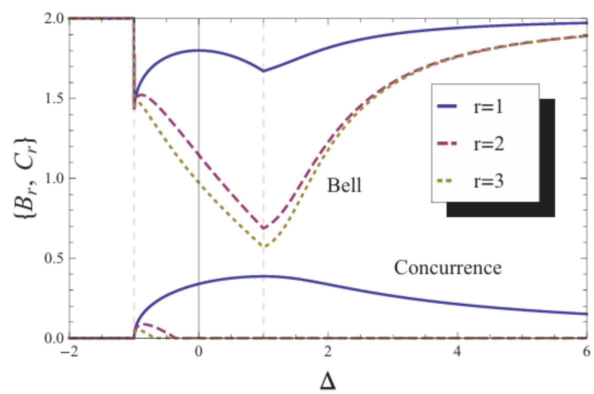

FIG. 3. (Coloronline) Bellmeasurement(three upper curves)and concurrence (three lower curves) for first, second, and third neighbors of the XXZ model as a function of the anisotropy Δ. It can be seen that the Bell inequality is never violated, even when the two spins are entangled. While both the concurrence and the Bell measurement are able to signal the first-order quantum phase transition at $\Delta = -1$, only the Bell measurement reveals the infinite-order QPT at $\Delta = 1$ [4].

As these work show that although the Bell inequality can signal QPT, the locality in these quantum many body models does not break. It is proved in [7] that, if ρ is the ground state of a gapped Hamiltonian of the lattice, then there is $C, \lambda > 0$ such that given X, Y disjoint regions we have that ρ is ϵ-local state for CHSH with respect to these two regions, where $\epsilon = 4|X|Ce^{-\lambda r}$, which is defined as $B^{X,Y}_{CHSH} \leq 2 + \epsilon$. In addition, the situation of the thermal state is also discussed in [7, 8].

With these results, there actually seems to be a relation between Bell inequality and QPT. However, these systems do not break the CHSH locality, which is contradictory to our intuition. The possible correct way to totally understand the relation between QPT and locality requires a fundamental understanding of the relation between the correlation function and probability $P(ab|xy)$, which is not yet clear. To achieve this aim, the definition of time order in probability structure is required, which will be discussed in the next chapter.

E. No-signaling and locality

When we describe the locality, it is usually stated that "Alice cannot send signal to Bob". However, the definition of locality is different to no-signaling condition. In this part, we will discuss the no-signaling constraint and its relation to locality.

The definition of no-signaling is

$$\text{no-signaling} \Leftrightarrow \sum_b P(ab|xy) = \sum_b P(ab|xy'), \sum_a P(ab|xy) = \sum_a P(ab|x'y), \forall a, b, x, x', y, y'. \tag{19}$$

This condition can be interpreted as $P(a|xy) \equiv \sum_b P(ab|xy) = P(a|x)$, which means that the marginal probability of Alice is independent of Bob's measurement setting y. This constraint represents that Bob cannot signal to Alice by his choice of input and vice versa. It can be proven that $\mathscr{L} \subset \mathscr{Q} \subset \mathscr{N}\mathscr{S}$, which means that a local system must be a quantum system and a quantum system must be a no-signaling system. It should be mentioned that the quantum constraint is that the probability satisfies the representation in Eq. 14. A graphical description of their relation is shown in Fig. 4.

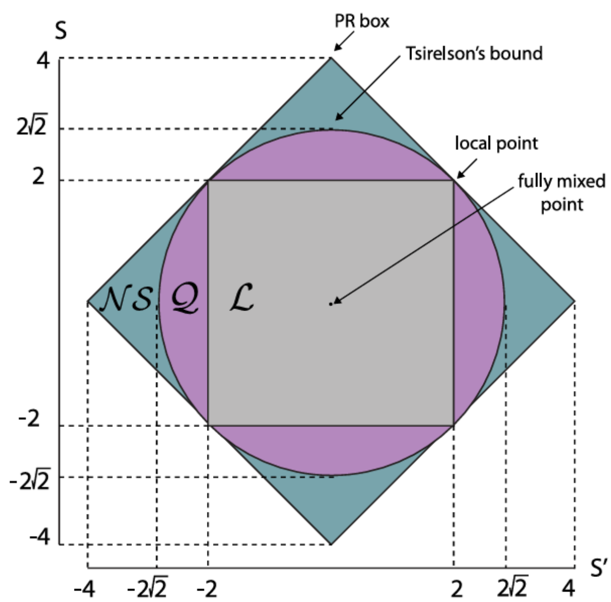

FIG. 4. Illustration of the relation between different constraints [1].

However, this no-signaling condition is still not compatible with our intuition of a causal structure. As there is a lack of time in the previous discussion, it seems that our description based on the probability is not compatible with a causal order, which will be discussed in detail next.

III. CAUSALITY

When it comes to causality, one famous construction is in relativistic quantum field theory, which is formulated as

$$[\phi(x), \phi(y)] = 0, \text{if x and y are not causally correlated,} \tag{20}$$

where ϕ is the field operator and x, y are space-time coordinates. A natural question is the relation between this causality and the locality defined before, which is the famous Tsirelson's problem [9]. As recent research finds that the two conditions are not equivalent [10], we still need more investigation of causality in quantum mechanics. For example, there is Everett picture [11], whose spirit can be partially interpreted as: a non-unitary evolution can be unitary seen from higher dimension, whose causality structure is also maintained in high dimension.

In our note, however, we focus on another fundamental question that how the space-time structure appears in quantum mechanics from an operational picture aiming to generalize the meaning of Bell inequality.

A. Causality under a presupposed space-time structure

The usual way to investigate the space-time structure is along the way from Schrödinger equation to Dirac equation when describing the dynamics of quantum states. The causal constraint in quantum field theory mentioned earlier is also along this path. The premise of this path is that we assume the existence of space-time structure before quantum mechanics. The quantum fields are fiber bundles over spacetime manifolds. With this assumption, we give more interpretations of causality from the view of lattice structure and the temporal Bell inequality.

1. Lieb-Robinson bound

We start with a simple lattice model to claim the inconsistency between quantum mechanics and causality [12]. Consider a one-dimensional tight-bounding model with periodic boundary condition, which is

$$H = -h \sum_{r \in \mathbb{Z}} |r\rangle \langle r+1| + |r\rangle \langle r-1| . \tag{21}$$

As it has discrete translation symmetry, its eigen wavefunction is described by momentum state

$$|k\rangle \equiv \sum_{r \in \mathbb{Z}} e^{ikr} |r\rangle , H |k\rangle = E_k |k\rangle , E_k = -2h\cos(k). \tag{22}$$

Thus, the maximal group velocity is $\frac{\partial E(k)}{\partial k} = 2h\sin(k) \leq 2h$. If we regard the group velocity as the speed of information's transition, there is no constraint in quantum mechanics to keep the velocity below the speed of light.

More generally, the Lieb-Robinson (LR) bound gives us a rigorous tool to describe the transition of information. The idea of generalization is to change from the Schrödinger picture to the Heisenberg picture, which is more convenient in a many-body problem [12]. Then, the transition of information can be imagined as a spreading of the initially local onsite operator which is dragged by the local interaction (hopping term in the Hamiltonian). To be more precise, we have

$$A_0(t) = e^{iHt} A_0 e^{-iHt} = A_0 + i[H_{0,1}, A_0]t + i[H_{-1,0}, A_0]t - \sum_{e_2, e_1} \frac{[H_{e_2}, [H_{e_1}, A_0]]t^2}{2!} + \cdots , \tag{23}$$

where A_i is the operator on site i and $H_e, e = (i, j)$ is the hopping term in the Hamiltonian. With these pictures, the LR bound states that in locally interacting systems, we have

$$||[A_X(t), B_Y(0)]|| \leq Ce^{-s_{XY}/\xi}(e^{v|t|/\xi} - 1), \tag{24}$$

where s_{XY} is the distance between two regions X, Y. If our world consists of qubits living on the lattice [13] and the presupposed space-time structure is right, there should be conflict about its consistency with special relativity.

2. Leggett-Garg inequality

The previous discussion reminds us of the possibility of temporal Bell inequalities [6, 14]. A famous discussion of it is the Leggett-Garg inequality (LGI) [15, 16]. We give a simple example to see how it works.

First, we answer what the LGI is. According to the original paper, their assumption is (A1) Macroscopic realism: A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states. (A2) Noninvasive measurability at the macroscopic level: It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics.

To be more precise, we setup the correlation between observables as

$$C_{ij} = \sum_{Q_i, Q_j} Q_i Q_j P_{ij}(Q_i, Q_j) = \langle Q_i Q_j \rangle. \tag{25}$$

Then, (A1) can be interpreted as [16] "Q has a well-defined value at all times, even when left unmeasured". It claims that

$$P_{ij}(Q_i,Q_j)=\sum_{Q_k,k\neq i,j}P_{ij}(Q_k,Q_i,Q_j). \tag{26}$$

At the same time, (A2) can be interpreted as "Macro-valuable Q are left unaltered by the measurements", which claims that

$$P_{ij}(Q_3,Q_2,Q_1)=P(Q_3,Q_2,Q_1). \tag{27}$$

If $Q_i,Q_j\in\{-1,1\}$, with the condition of Eq. 26 and Eq. 27, it can be calculated directly that

$$K_3\equiv C_{21}+C_{32}-C_{31}=1-4[P(1,-1,1)+P(-1,1,-1)]\leq 1, \tag{28}$$

which is the LGI.

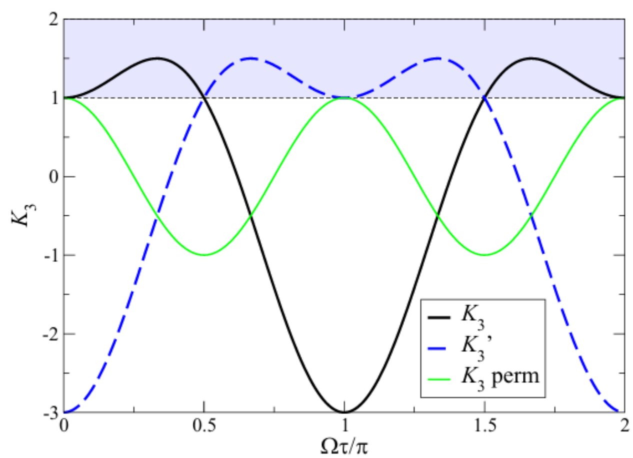

FIG. 5. Illustration of the relation between different K_3 and $\Omega\tau$. When K_3 is larger than 1, there is a violation of LGI [16].

Then, we will show why a quantum system can break the LGI [6]. An inspiring method is discussed in [6], which shows a connection of C_{ij} and anti-commutators. In a single quantum system, Alice and Bob measure with operators $\hat{a},\hat{b}$ respectively satisfying $\hat{a}^2=\hat{b}^2=1$. Then we define $P(r,s)$ as the probability that Alice observes r and Bob get s upon measuring the state of the system after the state collapse due to r. Then the probability can be calculated as

$$\begin{aligned}P(r,s)&=\langle\psi|\,\frac{1+r\hat{a}}{2}(\frac{1+s\hat{b}}{2})\frac{1+r\hat{a}}{2}\,|\psi\rangle\\&=\frac{1}{4}+\frac{1}{4}r\,\langle\psi|\,\hat{a}\,|\psi\rangle+\frac{1}{8}s\,\langle\psi|\,\hat{b}\,|\psi\rangle+\frac{1}{8}rs\,\langle\psi|\,\{\hat{a},\hat{b}\}\,|\psi\rangle+\frac{1}{8}s\,\langle\psi|\,\hat{a}\hat{b}\hat{a}\,|\psi\rangle\;\; r,s\in\{-1,1\}.\end{aligned} \tag{29}$$

Then the result is

$$C=\sum_{r,s}rsP(r,s)=\frac{1}{2}\,\langle\psi|\,\{\hat{a},\hat{b}\}\,|\psi\rangle\,. \tag{30}$$

It should be mentioned that there is an obvious time order between $\hat{a}$ and $\hat{b}$. The next step is to construct a quantum system that breaks the LGI in this background. We choose a spin-$\frac{1}{2}$ system whose Hamiltonian is

$$H=\frac{1}{2}\Omega\hat{\sigma}_x. \tag{31}$$

The observable is chosen to be

$$\hat{a}=\hat{Q}(t_i),\hat{b}=\hat{Q}(t_j),t_i<t_j,\hat{Q}(0)=\hat{\sigma}_z. \tag{32}$$

In the Heisenberg picture, the evolution of operator is

$$\hat{Q}(t) = \cos(\Omega t)\hat{\sigma}_z + \sin(\Omega t)\hat{\sigma}_y. \tag{33}$$

Finally, using the relation $\hat{\sigma}_i^2 = 1, \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{I}$, we can derive that

$$C_{ij} = \frac{1}{2}\langle\psi|\{\hat{Q}(t_i), \hat{Q}(t_j)\}|\psi\rangle = \cos\Omega(t_i - t_j), K_3 = 2\cos\Omega\tau - \cos 2\Omega\tau, \tau = t_j - t_i \tag{34}$$

which is irrelevant to the chosen $|\psi\rangle$. The relation between K_3 and $\Omega\tau$ and the violation of LGI are plotted in Fig. 5.

This result tells us that there should be something wrong in the assumptions (A1) and (A2) for a quantum system. More detailed discussion of this result's implication can be found in [16].

B. Causality with a emergent space-time structure

The origin of space and time is an eternal topic in the study of physics. Most physics are actually based on the presupposed space-time manifolds. In modern physics, with the development of emergent theory, there is some research claiming that the space-time structure can be emergent from the sea of qubits [13], the more precise description of which is that the low-energy excitation of a lattice model can be equivalent to the field-theory description of kinds of particles such as electrons and photons. If we want to understand the origin of spacetime along this idea, two important concepts remain to be distinguished: The structure of space-time and the existence of space-time. The structure of space-time can be defined by dynamics as we test this structure by the evolution of objects. Meanwhile, the existence of space and time is a rather philosophical problem. However, it reminds us to think, when do we realize that there is space and time? A possible answer may be

The difference is the space, the order is the time

which partially defined the existence of space and time. So when we want to ask the origin of space-time, maybe a first step is to answer that if we can get the structure of space-time based on the existence of space and time. Then the next step is to answer: What is the origin of difference (or different "phase" [17]) and order?

Under this guide of philosophy, we introduce some work which tries to complete the first step. It should be mentioned that we are still far from the destination.

1. Process formulation

The method we introduce here is a probability theory that describes operations based on the existence of space and time [18]. The interpretation of probability here tends to be consistent with the Bayesian school [19, 20], which emphasizes subjective probability, and so we should be careful with the conditional probability appearing in this formulation.

We will set our language based on the graph similar to a logic circuit which is shown in Fig. 6. The different boxes represent measurements with inputs and outputs at different places or times. The time order is along the vertical direction. A process without causal order is defined as

$$\bar{W}^{A,B,\dots} \equiv \{P(o^A, o^B, \dots | s^A, s^B, \dots, w^{A,B,\dots})\}, \tag{35}$$

where $w^{A,B,\dots}$ is the condition that these measurements occur. In this diagram, the possible causal relation can be represented by the line with arrow between boxes. More precisely, the causal order between these measurements can be represented by strictly partial order (SPO). With this structure of causal order, the process is changed to be

$$\bar{W}_c^{A,B,\dots} \equiv \{P(\kappa(A, B, \dots), o^A, o^B, \dots | s^A, s^B, \dots, w^{A,B,\dots})\}, \tag{36}$$

where $\kappa(A, B, \dots)$ represents the lines with arrow between boxes. The consistent condition of this causal order is

$$\sum_\kappa P(\kappa(A, B, \dots), o^A, o^B, \dots | s^A, s^B, \dots, w^{A,B,\dots}) = P(o^A, o^B, \dots | s^A, s^B, \dots, w^{A,B,\dots}). \tag{37}$$

These are the basic concepts of the process formulation. Then, we will discuss some detailed structures in this formulation.

First, the constraints of causality can be formulated as

$$P(\kappa(A, \chi), A \geq \chi, o^\chi | s^A, s^\chi) = P(\kappa(A, \chi), A \geq \chi, o^\chi | s^\chi), \tag{38}$$

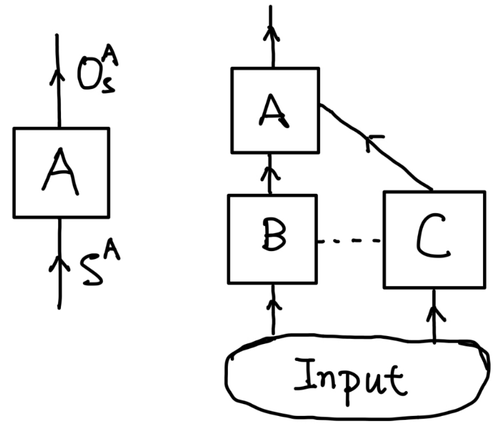

FIG. 6. Illustration of the measurement operation on different places with time order.

which means that the input of the measurement in the future cannot influence the present. Then we can prove that causal constraint ⇒ no-signal condition when the time order $B < A$ is assumed, whose process is

$$\begin{aligned} P(o^B|s^A, s^B) &= P(A < B, o^B|s^A, s^B) + P(B < A, o^B|s^A, s^B) + P(B = A, o^B|s^A, s^B) \\ &= P(B < A, o^B|s^A, s^B) \\ &= P(B < A, o^B|s^B) = P(o^B|s^B). \end{aligned} \tag{39}$$

Second, there is another scenario where the appearance of different causal orders is random. In this scenario,

$$\begin{aligned} P(o^A, o^B|s^A, s^B) = &P(A < B|s^A, s^B)P(o^A, o^B|s^A, s^B, A < B) \\ &+ P(A = B|s^A, s^B)P(o^A, o^B|s^A, s^B, A = B) \\ &+ P(A > B|s^A, s^B)P(o^A, o^B|s^A, s^B, A > B), \end{aligned} \tag{40}$$

which gives a generalization of the first scenario. This equality is called a causal condition similar to the local condition in Eq. 2.

More detailed relations between different constraints in this formulation are discussed in [18].

IV. DISCUSSION

Although much effort has been made to understand mysterious quantum mechanics, we are still far from the truth. Getting control of the quantum world requires a revolution in our viewpoint of the classical world. In our note, we make some fundamental discussions of causality aiming to understand space and time in the quantum world. It can be found that the philosophy of Bayesian school may help us to describe the structure in quantum world by probability in detail. With these detailed structures, we can find the importance of many-body interaction which behaves as a joint probability that is not easily decomposed of two-body interactions [18]. Maybe the description of physics based on probability theory has been underestimated. It should be mentioned that there are many advances in understanding quantum mechanics based on categorical theory that deserve attention [21].

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