Some Notes About Topological Band Theory and SPT States

Suyang Lin [a] and Shu Luo [a]
[a] School of Physics , University of Science and Technology of China, Hefei, Anhui 230026, China
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I. INTRODUCTION

In the research of topology in physics, one motivation is important: we can use topology to classify physical model or macroscopic materials. Similar idea dates back to Ancient Greece, when the philosopher used geometric objects to represent the basic elements of the universe, like Platonic solids. In modern physics, we should achieve this goal more cautiously. There are more fundamental questions behind the classification by topology. One of them is the definition of a phase, or how to tell that two physical models are situated in different classes generally. Thus, we should set some criterion to define the phase. In defining the phase, there is a crucial question as to whether we should put the response properties in the definition of a phase. More basically, is the dynamics needed in the definition of a phase? This should be considered carefully, as there are already some attempts to describe all macroscopic observables of a many-body system as its phase [1]. This is related to the criteria of classifying. With clear criteria, we should also clear up the definition of topology in different situations, which means pointing out the mapping relation. In our paper, we roughly divide the topological classification along two lines. One is topological band theory focusing on Berry phase and related response theory. Another focus on the description of many-body quantum state itself, where the classification of symmetry protected topological state is a typical example. We will compare the two philosophies in the two examples.

There has been a lot of researches along the two lines in recent years. First, as the topological band theory is with a clear relation to the structure of energy band which can be detected from materials by ARPES [2] and its imaginable physical picture by quasi-particles method [3], it is used to study kinds of systems. For example, it can be used to design thermal Hall effect which comes from the topological property of phonon [4], which can be achieved by chiral phonon [5]. Also, keeping the picture of band theory in the light-matter interaction system, the topological properties can be analyzed by quantized electromagnetic fields with an effective time-independent Hamiltonian, such as the geometric phase in cavity-QED [6]. Meanwhile, the energy band's picture can be extended to Brillouin zone in the axis of frequency in a Floquet Hamiltonian [7] to engineer the driven material such as vibration or light from environment [8]. In this extended Brillouin zone, there are also many researches about Floquet topological effect [9] aiming to design some topological effect which is hard to attain in static materials, such as Haldane models [10].

In another line of researches focusing on the description of quantum states aiming to solve strong-correlation problem, it requires to understand the many-body phenomenon from the bottom and find the new paradigm in understanding the strong correlation system. There are many researches aiming to classify phases of isolated and open system according to the understanding of quantum many-body states [11, 12]. In recent study, there are indeed some signs implying the importance of entanglement in this retrospect, which could be a candidate to the the new paradigm [13, 14]. It should be mentioned that, the picture in quantum information like entanglement, quantum operation, encoding, and states preparation play an important role in this process of classifying quantum many-body phases.

II. TOPOLOGICAL BAND THEORY: FROM QAHE TO QWZ MODEL

A. QAHE and the Derivation of Topological conductivity

Haldane first proposed a "Quantum Anomalous Hall Effect", in which the quantized conductivity can exist without a vertical magnetic field. In this model, there is a honeycomb lattice and the interaction is between the nearest and next-nearest neighborhood. There are other types of QAHE and among them the most general form of the two-band Hamiltonian is

$$H(\mathbf{k}) = \epsilon(\mathbf{k}) + V d_\alpha(\mathbf{k})\sigma^\alpha \,, \tag{2.1}$$

Now we explain what is the meaning of $d_\alpha(k)$. One interpretation is by spin-orbital coupling of a electron. The picture is that when we consider the magnetic effect to a electron, we should consider the energy contribution of the magnetic filed. When we transform the reference frame to the electron's frame, there will be an effective magnetic field coming from the Lorentz transformation, $B_{eff} \propto \vec{k} \times \vec{E}$, which contributes to a term in energy with $\vec{\sigma} \cdot \vec{B}$. However, this hopping will be linear with momentum k. If we want to construct the term as a nonlinear modification as $\cos(k)$, it should be understood in the tight-binding model. When the hopping term of electrons which represents the intensity of orbital's mixing is coupled to spin, this lattice version coupling will cause a $\cos(k)$ term in $d_\alpha(k)$. in which V is the magnitude of the coupling and $d_\alpha(\mathbf{k})$ can be viewed as a map from the momentum space $\mathbf{T}^2$ to the two-sphere $\mathbf{S}^2$ under periodic boundary condition. The topological information is thus encoded in this mapping pattern.

Haldane's example is given by

$$\epsilon(\mathbf{k}) = 2t_2 \cos\phi \sum_i \cos(\mathbf{k}\cdot\mathbf{b}_i)\,, \tag{2.2}$$

$$d_x(\mathbf{k}) = t_1 \sum_i \cos(\mathbf{k}\cdot\mathbf{a}_i)\,, \qquad d_y(\mathbf{k}) = t_1 \sum_i \sin(\mathbf{k}\cdot\mathbf{a}_i)\,, \tag{2.3}$$

$$d_z(\mathbf{k}) = M - 2t_2 \sin\phi \sum_i \sin(\mathbf{k}\cdot\mathbf{b}_i)\,, \tag{2.4}$$

in which $\mathbf{b}_i$ is the vector between two adjacent lattice a, $\mathbf{a}_i$ is the one between lattice a and b, and $\phi = (2\phi_a + \phi_b)/\phi_0$ is the dimensionless magnetic flux, see Fig.1 for detail. Other extended QAHE models contain the one introduced by Kane and Mele, which has a four-band Hamiltonian and its general form is summarized as

$$H(\mathbf{k}) = \sum_{a=1}^{5} d_a(\mathbf{k})\Gamma^a + \sum_{a<b} d_{ab}(\mathbf{k})\Gamma^{ab}\,, \qquad \Gamma^{ab} = \frac{1}{2i}[\Gamma^a, \Gamma^b]\,. \tag{2.5}$$

In topological band theory, the property is usually reflected by linear response, which is determined by the topology of Hamiltonian/BZ directly through the current expression. However, we should make a distinction between the current in the quasi-particle theory and the field theory. In quasi-particle approximation, $j(k) = ev(\text{group velocity}) = e\frac{\partial E(k)}{\partial k}$. But the current in a gauge theory is usually represented by $\frac{\delta H}{\delta A}$, and that's why we can also attain topological conductivity through the topology of map from other space (like magnetic flux) to Hilbert space. So we should ask if this formulation of current is always effective, what is the condition? Ref:p332 introduction to many-body physics by Piers Coleman [15]. If we simply suppose two approaches are equivalent, we can further attain the expression of the current $j(\mathbf{k})$ in momentum space as

$$\mathbf{j}(\mathbf{k}) = \nabla_{\mathbf{k}} H(\mathbf{k}) = \nabla_{\mathbf{k}}\epsilon + V\nabla_{\mathbf{k}} d_\alpha(\mathbf{k})\sigma^\alpha\,, \tag{2.6}$$

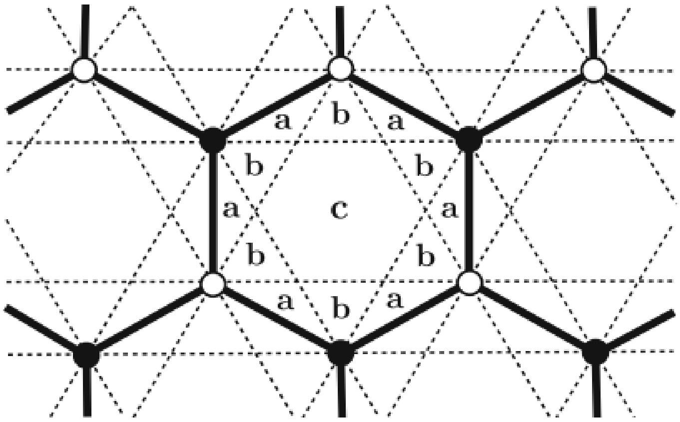

Figure 1: Haldane model

After we obtain the explicit form of the current, we can use the Kubo formula for AC field and take the DC limit to obtain

$$\sigma_{xy} = \lim_{\mathbf{k}\to 0,\omega\to 0} \sigma_{xy}(\mathbf{k},\omega) = \lim_{\mathbf{k}\to 0,\omega\to 0} \frac{i}{\omega} Q_{xy}(\omega + i\delta)\,, \tag{2.7}$$

in which there is a retarded Green function

$$Q_{xy}(\mathbf{k}, \omega + i\delta) = -\frac{i}{\Omega}\int_{-\infty}^{\infty} dt\theta(t)e^{i\omega t}\left\langle 0\right| \left[J_x(\mathbf{k},t), J_y(\mathbf{k},0)\right]\left|0\right\rangle\ , \tag{2.8}$$

and through the analytic extension one can transform it into Matsubara Green function

$$Q_{xy}(0,i\nu_m)=\frac{1}{\Omega\beta}\sum_{\mathbf{k},n}\mathrm{tr}[j_x(\mathbf{k})G^M(\mathbf{k},i(\omega_n+\nu_m))j_y(\mathbf{k})G^M(\mathbf{k},i\omega_n)]\,, \tag{2.9}$$

where $G(\mathbf{k},i\omega_n)$ is the Matsubara Green function

$$G^M(\mathbf{k},i\omega_m)=[i\omega_m-H(\mathbf{k})]^{-1}=\frac{P_+}{i\omega_m-E_+(k)}+\frac{P_-}{i\omega_m-E_-(k)}\,, \tag{2.10}$$

$$P_\pm=\frac{1}{2}[1\pm\hat{d}_\alpha\sigma_\alpha]\,, \tag{2.11}$$

are two projector operators corresponding to the two eigenvalues of $H(\mathbf{k})$, and we have

$$J_i(\mathbf{q},\sigma)=\sum_k c^\dagger_{\mathbf{q}+\frac{\mathbf{k}}{2},\alpha}(\sigma)j_i^{\alpha\beta}(\mathbf{k})c_{\mathbf{q}-\frac{\mathbf{k}}{2},\beta}(\sigma)\,, \tag{2.12}$$

$$G^T_{\alpha\beta}(\mathbf{k},\sigma)=\langle 0|\,T\{c^\dagger_{\mathbf{k},\alpha}(\sigma)c_{\mathbf{k},\beta}(0)\}\,|0\rangle=\sum_n e^{-i\omega_n\sigma}G^M_{\alpha\beta}(\mathbf{k},\sigma)\,, \tag{2.13}$$

so the conductivity can be computed as

$$\sigma_{xy}=\lim_{\omega\to 0}\frac{i}{\omega}Q_{xy}(\omega+i\delta)=-i\sum_{s,t=\pm}\sum_{\mathbf{k}}\frac{\mathrm{tr}[j_x(\mathbf{k})P_s(\mathbf{k})j_y(\mathbf{k})P_t(\mathbf{k})]}{[E_t(\mathbf{k})-E_s(\mathbf{k})]^2}(n_t(\mathbf{k})-n_s(\mathbf{k}))\,, \tag{2.14}$$

where

$$n(\mathbf{k})=\frac{1}{\exp(\beta(E(\mathbf{k})-\mu))+1}\,, \tag{2.15}$$

is the Fermion distribution function. By bringing into the expression of $j_{x/y}$ and $P_{s/t}$ into it we attain exactly the winding number/Chern number expression as

$$\sigma_{xy}=-\frac{1}{2\Omega}\sum_{\mathbf{k}}\left\{\frac{\partial\hat{d}_\alpha}{\partial k_x}\frac{\partial\hat{d}_\beta}{\partial k_y}\hat{d}_\gamma\epsilon^{\alpha\beta\gamma}\right\}[n_-(\mathbf{k})-n_+(\mathbf{k})]\,. \tag{2.16}$$

In the zero-temperature limit, we have $n_+=0, n_-=1$, and then the expression has a geometric meaning

$$\sigma_{xy}=-\frac{1}{8\pi^2}\int dk_x dk_y\hat{\mathbf{d}}\cdot(\partial_x\hat{\mathbf{d}}\times\partial_y\hat{\mathbf{d}})=-\frac{n}{2\pi}\,,\qquad(n=0,1,\cdots) \tag{2.17}$$

the $\partial_{k_x}\hat{d},\partial_{k_y}\hat{d}$ are two tangent vectors, and so $\hat{\mathbf{d}}\cdot(\partial_x\hat{\mathbf{d}}\times\partial_y\hat{\mathbf{d}})$ is the unit area of the image $\hat{d}$ which lives in S^2. As this is a smooth periodic mapping, the area of the image is the integer number times the area of unit sphere 4π.For this reason σ_{xy} here is proportional to the winding number of the map from the FBZ $\mathbf{T}^2$ to a unit sphere $\mathbf{S}^2$.

A tight binding regularized form of the Hamiltonian which gives good insight in the further study of QSHE is given by

$$d_x=\sin k_y\,,\qquad d_y=-\sin k_x\,,\qquad d_z=c(2-\cos k_x-\cos k_y-e_s)\,, \tag{2.18}$$

It can be calculated that this configuration indeed provides a nonzero Hall conductivity with clear condition of parameters in the model. But to discuss this question more generally, is there some methods guiding us to get the construction of $\vec{d}$ with non-zero Hall conductance? A practical method is to construct a repetition of any point in the image space. Taking this construction as an example, we focus on the initial point as $\vec{k}=(0,0)\to\vec{d_i}=(0,0,-ce_s)$. To construct a non-zero topological number, we need the condition of repetition in this mapping and a full covering of the two sphere. The full covering condition is $-e_s<0<4-e_s\to 0<e_s<4$ which is the condition for the nonzero topological number. Also it's meaningful to argue the circumstance when $e_s=2$ in Eq.(2.18), at which point we may encounter a zero-norm d_α and thus the normalization is ill-defined. Turn back to the spectrum analyses we note that this is precisely when the bulk gap between two bands is closed.

Now we offer a intuitive physical explanation of this discontinuity. We note that the whole image surface must be closed and have no edge, and range of d_x and d_y can fill [-1,1]. Thus if d_z is restricted to the north/south semisphere, we expect the mapping originated from north/south pole must have a southmost/northmost latitude and then retreat back to the north/south pole, so the contribution cancels each other. On the other hand, if d_z can cross the equator, normalization always guarantee the image arriving at the opposite pole. This is also precisely guaranteed by the property of trigonometric functions: the extremity of d_z always correspond to the zero point of d_x and d_y, and thus must be a extremity of normalized $d_z = \pm 1$. In this case there is no retreat but the surface is still close. We thus make the following general conjecture:

$$d_x = f'(k_x)\,, \qquad d_y = g'(k_y)\,, \qquad d_z = \alpha + \beta f(k_x) + \gamma g(k_y)\,, \tag{2.19}$$

in which

$$f(k_x + 2\pi) = f(k_x)\,, \qquad g(k_y + 2\pi) = g(k_y)\,, \tag{2.20}$$

$$d_z = 0 \text{ is a closed loop and has at most } 2n_y(n_x) \text{ zero points fixing } k_y(k_x)\,, \tag{2.21}$$

then the topological number will be $\frac{\pm n_x n_y}{2\pi}$.

To guide us design materials with non-trivial topological number, two conditions matter. One is the covering condition, another is repeating condition (repetition of a D-1 dim sub-manifold). The repeating condition can be argued as the degeneracy of gauged fields. Usually, these degeneracies are protected by symmetry, which behaves like the locally similarity of the Berry curvature on the momentum space. And this will contribute to the degeneracy of the tangent vector. For the covering condition, it can be argued to be the covering of each sectors separated by degenerated points.

B. QWZ Model, or QSHE

We now turn to QWZ model, which proposes a 2D paramagnetic semi-conductors in which there is a intrinsic spin Hall effect [16]. The topological property is exactly reflected by the number of edge states crossing the Fermi level, or equivalently the bulk topological charge number. The conductivity is again carried by the gapless edge state, with response to the adiabatic flux insertion just like in the integer Hall effect, but here the edge states also carries nonzero z-direction spin. The model is another example of “bulk-boundary correspondence”, in which the edge states precisely encodes the topological invariant of the bulk. QSHE can be interpreted as two copies of AHEM E, each of which can break time-reversal symmetry but remain it as a whole. The model here has the form

$$H(\mathbf{k}) = \frac{\gamma_1 + \frac{5}{2}\gamma_2}{2m}\mathbf{k}^2 + \frac{\gamma_2}{m}(\mathbf{S}\cdot\mathbf{k})^2\,, \tag{2.22}$$

and this is in accordance to the Luttinger model, in which $\mathbf{S}$ stand for the 4×4 angular momentum matrix for $j = \frac{3}{2}$ spin electron. By reexpressing them into the form through $\Gamma^a = \xi^a_{ij}\{S^i, S^j\}$, Γ^a satisfies the $SO(5)$ Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$,

$$H(\mathbf{k}) = \epsilon(\mathbf{k}) + V d_a(\mathbf{k})\Gamma^a\,, \tag{2.23}$$

which gives the following energy eigenvalue

$$E_\pm(\mathbf{k}) = \epsilon(\mathbf{k}) \pm V\sqrt{d_a d^a(\mathbf{k})}\,, \tag{2.24}$$

where

$$d_1 = -\sqrt{3}k_y k_z\,, \qquad d_2 = -\sqrt{3}k_x k_z\,, \qquad d_3 = -\sqrt{3}k_x k_y\,, \tag{2.25}$$

$$d_4 = -\frac{\sqrt{3}}{2}(k_x^2 - k_y^2)\,, \qquad d_5 = -\frac{1}{2}(2k_z^2 - k_x^2 - k_y^2)\,. \tag{2.26}$$

Although here the model is 3D, we can imagine applying a well $U(z)$ in z direction and the electron is effectively restricted to 2 dimensions. Then we can replace k_z with $< k_z >$ and then k_z^2 with $< k_z^2 >$. If we suppose $U(z) = U(-z)$, then $< z >= 0$, $< z^2 >= e_s$, then in Hamiltonian we are left with merely $\Gamma^\alpha, (\alpha = 3,4,5)$, which form a $SO(3)$ Clifford subalgebra. Apart from that, we have $[\Gamma^{12}, \Gamma^\alpha] = 0$, which means that Γ^{12} is a conserved sum in 2D effective Hamiltonian, and it can be diagonalized simultaneously

$$\Gamma^{12} = \mathrm{diag}(\mathbf{I}, -\mathbf{I}) \,, \qquad \Gamma^{\alpha} = \mathrm{diag}(\sigma^{\alpha-2}, -\sigma^{\alpha-2}) \,, \tag{2.27}$$

and in this case the 2D Hamiltonian has a block-diagonalized form

$$H_{2D}(\mathbf{k}) = \mathrm{diag}(\epsilon(\mathbf{k}) + V d_{\alpha}(\mathbf{k})\sigma^{\alpha}, \epsilon(\mathbf{k}) - V d_{\alpha}(\mathbf{k})\sigma^{\alpha}) \,, \tag{2.28}$$

while the contribution of two layers to the electric charge cancels each other, we note that these two layers actually carries opposite Γ^{12} charge, so the final expression of the Hall conductivity σ^{Γ}_{xy} is given by

$$\sigma^{\Gamma}_{xy} = 2\sigma_{xy} = \frac{n}{\pi} \,, \qquad (n = \cdots - 1, 0, 1, \cdots) \,, \tag{2.29}$$

when we have winding number n, there are now $2|n|$ edge states which contribute to the whole conductivity, each two of which at the left/right edge with opposite current but the same Γ^{12} current. If we take the tight binding approximation,

$$d_3(\mathbf{k}) = -\sqrt{3} \sin k_x \sin k_y \,, \qquad d_4(\mathbf{k}) = \sqrt{3}(\cos k_x - \cos k_y) \,, \qquad d_5(\mathbf{k}) = 2 - e_s - \cos k_x - \cos k_y \,, \tag{2.30}$$

and when $0 < e_s < 4$ we have $\sigma^{\Gamma}_{xy} = 2/\pi$, otherwise $\sigma^{\Gamma}_{xy} = 0$.

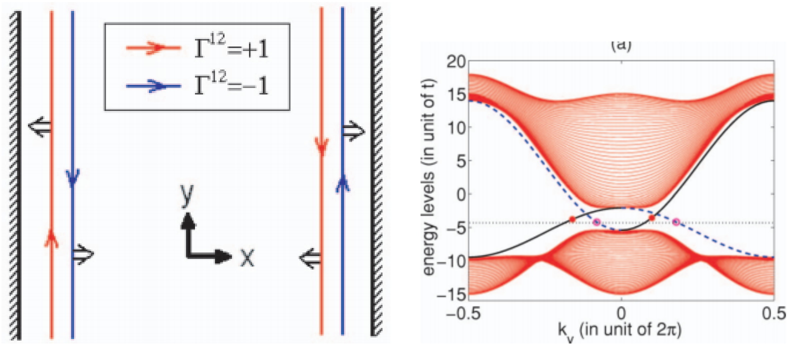

Figure 2: The edge states in QSHE

III. SRE AND SPT STATE

A gapped quantum phase is usually defined as a class of gapped Hamiltonians which can smoothly deform into each other without closing the gap and hence without any singularity in the local properties of the ground state. Two gapped ground states belong to the same phase if and only if they are related by a local unitary (LU) transformation. It should be mentioned that the problem of equivalence between these two description matters. Usually it can be proved that if two phase can be described by a related parent Hamiltonian's smooth deformation without closing gap then the two phases can be related by LU transformation. However, the proof in another direction seems non-trivial. This requires us to think about the different philosophy behind, one keeps the mind of dynamics controlled by Hamiltonian, another focus on the quantum state itself whose dynamics can be described by quantum operation or quantum circuit. We can understand this difference better in the following discussion [17].

Specifically, if $H(g)$ is a series of Hamiltonian which are all gapped and thus the system have no phase transition. The bound state of a certain $H(g)$ is given by $|\Phi(g)\rangle$, and if there exist a LU transformation

$$U(s) = \mathcal{T} \exp(\int_0^s -i\bar{H}(g)dg) \,, \tag{3.1}$$

in which $\bar{H}(g)$ is a local Hermitian operator, s.t.,

$$|\Phi(s)\rangle = U(s) \, |\Phi(0)\rangle \,, \tag{3.2}$$

then we find an adiabatic path connecting these two systems and we divide them into the same quantum phase. Under this adiabatic evolution we have

$$H(s) = U(s)H(0)U^\dagger(s) = \sum_i O_i(s)\,, \tag{3.3}$$

is still a local operator. For any Hamiltonian, thus, we can classify it into a certain phase, and one special class is those bound states that can be transformed into a direct product state. These are called Short Ranged Entangled States (SRE), or they belong to Long Ranged Entangled States (LRE), which have nontrivial topological order, and different equivalent LRE corresponds to different topological order. Then a natural question is whether SRE states can further be divided into several classifications.

First focus on the 1D case, when there is no topological order and all states can be transformed into direct product states. But when there is a symmetry to preserve things are different. First review the concept of projective representation. In quantum mechanics, we have an ambiguity coming from the equivalent relation among states up to a certain phase. So

$$u(g_1)u(g_2) = \omega(g_1, g_2)u(g_1, g_2)\,, \tag{3.4}$$

in which $\omega(g_1, g_2)$ is a $U(1)$ number and u forms a projective representation of group G. In order to satisfy associativity, there is

$$\omega(g_2, g_3)\omega(g_1, g_2g_3) = \omega(g_1, g_2)\omega(g_1g_2, g_3)\,, \tag{3.5}$$

on the other hand, there is an equivalent relation

$$\omega(g_1, g_2) \sim \frac{\beta(g_1, g_2)}{\beta(g_1)\beta(g_2)}\omega(g_1, g_2)\,, \tag{3.6}$$

and all such $\omega(g_1, g_2)$ form the cohomology group $H^2(G, U(1))$. Now a natural question arises that why this cohomology group can classify a certain SPT.

To build the relation between cohomology group and the classification of SPT in 1D, we should understand the structure of trivial phase in 1D first, where the classification can be embedded in the structure of triviality. In 1D system, by Mermin-Wegner theorem, there is no stable long-range order with finite temperature. Also, there is usually a long-range order in critical points. A natural general description of the 1D system is short-range entangled state (SRE). To describe the structure of SRE, an inspiring perspective is by quantum circuits [18]. The philosophy is that any quantum state can be prepared by quantum operation from a simple state. The structure of a state can be encoded in the process of quantum preparation. To get a parent process description of the many-body SRE, we should introduce the matrix product state (MPS) here, whose preparation by quantum circuits is already refined to Log-depth recently. We will give two equivalent description of MPS here. The first one is easier to see the method of classification, the latter is easier to design quantum operation.

The MPS can be defined by encoding its information in the parent entangled pairs. For a system with Hilbert space $dim(H) = (\mathbb{C}^d)^{\otimes N}$, where d is the dimension of the onsite operator and N is the size of system, the maximally entangled pair for one site is defined as

$$|\omega_D\rangle = \sum_{i=1}^{D} |i, i\rangle\,. \tag{3.7}$$

Then, the MPS can be defined by a mapping $P : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$ of each site that

$$|\mu[P]\rangle = P^{\otimes N}\,|\omega_D\rangle^{\otimes N}\,. \tag{3.8}$$

By introducing a matrix form of the projection operator as

$$P = \sum_{i,\alpha,\beta} A^i_{\alpha,\beta}\,|i\rangle\,\langle\alpha, \beta|\,, \tag{3.9}$$

another definition is thus behaving as

$$|\mu[P]\rangle = \sum_{i_1,\ldots,i_N} \mathbf{Tr}[A^{i_1}\cdots A^{i_N}]\,|i_1, ..., i_N\rangle\,. \tag{3.10}$$

Next, we will introduce the method of classifying MPS with symmetry. The basic philosophy of classifying MPS by symmetry is still by representation theory. Consider a U_g-invariant MPS, we want to know how the g acts on the parent pairs. As the maximal entangled pairs are invariant under a gauge transformation $V \otimes (V^{-1})^T$,thus a natural representation in the space of parent pairs is $V_g \otimes \bar{V}_g$. However, there is still a gauge redundancy in this representation that $V_g \to e^{i\chi_g} V_g$. Thus V_g is actually a projective representation that $V_g V_h = e^{i\omega(g,h)} V_{gh}$ with a equivalent condition that

$$\omega(g,h) \sim \omega(g,h) + \chi_{gh} - \chi_g - \chi_h \bmod 2\pi. \tag{3.11}$$

The equivalent classes induced by this relation is isomorphic to the second cohomology group $H^2(G, U(1))$. The next work is to pick a typical state in a class. The important characters of this mapping is projection, which can be decomposed as $P = QW$, where W is an isometry with $WW^\dagger = I$ and Q is the non-invertible part. Actually, the non-invertible part of projection can be regarded as a renormalization in the lattice space as the projection in well-defined after blocking, which means a renormalized projection $P' : \mathbb{C}^D \otimes \mathbb{C}^D \to (\mathbb{C}^d)^{\otimes N'}$. After renormalization $Q_\gamma = \gamma Q + (1-\gamma)1, \gamma : 0 \to 1$, we have that $|\mu[P]\rangle$ and $|\mu[W]\rangle$ are in the same phase and $|\mu[W]\rangle$ is the isometric fixed point of different class.

A renormalization process can map all $(A^{[k]}_{i_k})^{(0)}$ on each single site to a certain $(A^{[K]}_{i_K})^{(1)}$ on a certain block containing sites $1 \sim n$. This is done specifically by invariant under LU as [19]

$$\mathrm{E}^{[k]}_{\alpha\beta,\gamma\delta} = \sum_i A^{[k]}_{i,\alpha\gamma} (A^{[k]}_{i,\beta\delta})^* \,, \tag{3.12}$$

$$\mathrm{E} = \mathrm{E}^{[1]} \cdots \mathrm{E}^{[n]} \,, \tag{3.13}$$

$$\mathrm{E}_{\alpha\beta,\gamma\delta} = \sum_i A_{i,\alpha\gamma} (A_{i,\beta\delta})^* \,, \tag{3.14}$$

and here we regard indices α, β as row indices and γ, δ as column indices, and we can suppose its eigenvalues to be $\lambda_i (i = 1, 2 \cdots D^2)$, and thus can be decomposed into the form

$$\mathrm{E}_{\alpha\beta,\gamma\delta} = \sum_i \lambda_i V_{i,\alpha\beta} V^*_{i,\gamma\delta} \,, \tag{3.15}$$

in which V_i are normalized eigenvectors and $A_i = \sqrt{\lambda_i} V_i$. In the end the RG flow will approach a fixed point value $(A^{[k]}_{i,\alpha\beta})^{(\infty)}$. Rather than explicitly attain the form of this, we can define another "double spin representation"

$$(A^{[k]}_{i^l i^r,\alpha\beta})^{(\infty)} = \sqrt{(\mathrm{E}^{[k]}_{i^l i^r,\alpha\beta})^{(\infty)}} = \sqrt{\lambda^{[k]}_{i^l}} \delta_{i^l,\alpha} \sqrt{\lambda^{[k+1]}_{i^r}} \delta_{i^r,\beta} \tag{3.16}$$

and here $A_{i^l i^r}$ are a series of matrices with element indices denoted by α, β. Then we can define the accurate meaning of symmetry invariant as

$$\sum_{j^l,j^r} u^{[k]}_{i^l i^r, j^i j^r}(g) A^{[k]}_{j^i j^r} = \alpha^{(R)}_{[k]}(g) N^{-1}_{[k]}(g) A^{[k]}_{i^l,i^r} M_{[k]}(g) \,, \tag{3.17}$$

where $M_{[k-1]}(g) = N_{[k]}(g)$ and for a certain site k, both $M(g)$ and $N(g)$ form projective representations, i.e.,

$$M(g_1)M(g_2) = \omega_M(g_1,g_2) M(g_1 g_2) \,, \qquad N(g_1)N(g_2) = \omega_N(g_1,g_2) N(g_1,g_2) \,, \tag{3.18}$$

and we have the total projective factor to be $\omega = \omega_M / \omega_N$. In this way we illustrate that the in order to preserve a certain symmetry described by group G, the original class can be further divided into several classifications each of which corresponding to an element in the cohomology group $H^2(G, U(1))$.

A. 1D Example: Haldane Chain

A vivid example is the Haldane phase of the 1D spin chain [20]

$$H_0 = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + B_x \sum_j S^x_j + U_{zz} \sum_j (S^z_i)^2 \,, \tag{3.19}$$

and this model has the following symmetry

$$\text{spatial inversion}: \quad \mathbf{S}_j \to \mathbf{S}_{-j} \,, \tag{3.20}$$

$$\text{time reversal}: \quad \mathbf{S}_j \to -\mathbf{S}_j \,, \tag{3.21}$$

$$\text{rotation of } \pi \text{ around x axis}: \quad S_j^{y,z} \to -S_j^{y,z} \,, \tag{3.22}$$

$$\text{rotation of } \pi \text{ around y axis and time reversal}: \quad S_j^{y} \to -S_j^{y} \,, \tag{3.23}$$

and we can restrict further the action of symmetry on the states to be $N^{[k]} = M^{[k]}$ thus the transformation is uniform for all sites. Then we expect a certain relation such as

$$\Sigma_I A_m^{[k]} = e^{i\theta_I} U_I^\dagger A_m^{[k]} U_I \,, \tag{3.24}$$

where Σ_I refers to a certain symmetry transformation and U_I is the projective representation of the symmetry group. First focus on the spatial inversion, when

$$\Sigma_I A_m = A_m^T = e^{i\theta_I} U_I A_m U_I^\dagger \,, \tag{3.25}$$

which lead to

$$A_m = (A_m^T)^T = e^{2i\theta_I} (U_I U_I^*)^\dagger A_m U_I U_I^* \,, \tag{3.26}$$

As the MPS representation is in its canonical from, i.e., there is a “quasi unitary” condition for A_m

$$\sum_m A_m (A_m)^\dagger = \mathbf{1} \,, \tag{3.27}$$

which is equivalent to say that transfer matrix

$$T_{\alpha\alpha',\beta\beta'} = \sum_m A_{m,\alpha\beta} (A_{m,\alpha'\beta'})^* \,, \tag{3.28}$$

has a eigenvalue 1 with respect to eigenvector $\delta_{\alpha\alpha'}$ or $\delta_{\beta\beta'}$. We further require that this is the unique eigenvalue λ satisfying $|\lambda| \geq 1$. This requirement is equivalent to say that $|\Phi\rangle$ is a pure state. Then bring Eq.(3.26) into Eq.(3.27) we find

$$\sum_m A_m^\dagger U_I U_I^* A_m = e^{2i\theta_I} U_I U_I^* \,, \tag{3.29}$$

thus we must have $U_I U_I^* = e^{i\phi_I}\mathbf{1}$, together with the condition $e^{2i\theta_I} = 1$. This further restrict $\{\theta_I, \phi_I\} = \{0, \pi\}$. Haldane phase of the spin chain, in AKLT example, is that $\theta_I = \phi_I = \pi$, and here $U_I = \sigma_y$ satisfies $U_I^T = -\sigma_y U_I \sigma_y = -U_I$. Similar argument also make sense for time inversion and rotation of π around a certain axis. As the value of ϕ_I can only take discrete value, we conclude that it is impossible to adiabatically evolve a state in a certain SPT phase into another without breaking these two symmetries. At the same time, the classification of SPT state is in accordance to cohomology description $H^2(G, U(1)) = \mathbf{Z}^2$.

B. General Construction Through Cohomology

The classification of SPT state in higher dimension is not proved vigorously, but one can construct examples through the cohomology technique [21]. For a system in general dimension with merely on-site symmetry, we flow the RG current to a fixed point at which the system behaves like a auto-similar one. That means the group action on the SRE states would be a linear representation and is invariant under repetitive tensor product. Then protecting the symmetry of group G means we can find a SRE state $|\Psi_p\rangle$at any site scale such that

$$\otimes^i U^i \left|\Psi_p\right\rangle = \left|\psi_p\right\rangle \,, \tag{3.30}$$

so a doublet $(U^i, |\Psi_p\rangle)$ is enough to classify one kind of SPT.

In general dimension, one can define the group cohomology through the concept of a group module (representation), which is an Abelian group the group element can act on and whose action is compatible to the module multiplication. We now further restrict M on the unit circle $S^1 = U(1)$. Then we can define d-cochain as a map from G^{d+1} to $M = U_T(1)$, where elements in $U_T(1)$ is also $U(1)$ but the group action on the module is nontrivial: if $g \in G$ contains no time reversal, then simply $g \cdot a = a$, $a \in U_T(1)$, other wise $g \cdot a = a^{-1}$:

$$|\nu_d(g_0, \cdots g_d)| = 1\,, \qquad g \cdot \nu_d(g_0, \cdots g_d) = \nu_d(gg_0, \cdots gg_d) = \nu_d^{s(d)}(g_0, \cdots g_d)\,, \tag{3.31}$$

in which $s(d) = 1$ if there is no time reversal in G or $s(d) = -1$. A special class in cochain satisfies the condition $\mathrm{d}\nu_d = 1$ is called d-cocycle, in which d is a map from d-cochain to $d+1$-cochian

$$\mathrm{d}\nu_d(g_0, \cdots g_{d+1}) = \prod_{i=0}^{d} \nu_d^{(-1)^i}(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots g_d)\,, \tag{3.32}$$

and such cocycle resembles the "closed form" we meet in de-Rham cohomology theory. Another concept, d-coboundary, is naturally defined as $\nu_d = \mathrm{d}\nu_{d-1}$, which corresponds to "exact form". $d-$ cohomology is thus the equivalent class of cocycle with respect to coboundary. One can prove that every $\nu_2(g_0, g_1, g_2)$ in 2-cohomology defined above can be one-to-one mapped to the projective phase $\omega(g_1, g_2)$ defined to satisfy Eq.(3.5) and Eq.(3.6). The geometric interpretation is similar to the d-complex graph plotted for d-homology. The relation between the cohomology defined here and the de-Rham cohomology may be important in transferring the discrete lattice model to continuous topological quantum field theory. In the spacetime manifold, we can set several elements as dots and then link them through an edge. An edge can further compose a surface and so on, and the spacetime is now descretized into a complex. Then on the d-dimensional subspace we construct now there can be a d-form Ω_d defined on it, such that

$$\nu_d(g_0, \cdots, g_d) = \exp(i \int_{(g_0, \cdots, g_d)} \Omega_d)\,, \tag{3.33}$$

The exponential map here has a complex unit i here in order to keep the 2π periodicity of θ in topological field theory. Now the definition of Eq.(3.32) can be naturally translated into the form language, for instance,

$$\int_{(g_0, g_1, g_2)} \mathrm{d}\Omega_1 = (\int_{(g_0, g_1)} - \int_{(g_0, g_2)} + \int_{(g_1, g_2)})\Omega_1\,, \tag{3.34}$$

in Sec.III C we will discuss about this translation into continuous model in detail.

Now one can construct the SPT state through the cohomology. Explicitly, the classification of SPT of a lattice model with merely on-site symmetry G in 1+d dimension is given by $1+d$-cohomology group $H^{1+d}(G, U_T(1))$. We first turn to a "dimmer" model where SRE states are constructed through the entanglement between adjoint units

$$|\Psi_p\rangle = \otimes(\sum_g |\alpha_2 = g, \beta_1 = g\rangle) \otimes (\sum_g |\beta_2 = g, \gamma_1 = g\rangle) \otimes \cdots = \sum_{\beta_1, \beta_2, \cdots} |\alpha_1, \beta_1, \beta_1, \beta_2, \beta_2, \cdots, \alpha_2\rangle \equiv |\alpha_1, \alpha_2\rangle\,. \tag{3.35}$$

here all $\alpha, \beta \cdots$ refer to group elements and thus can be naturally acted by an element in G. The number of β_i in the second definition can be arbitrary, as all of them will be summed over. The state on every single site is a maximally entangled state. A most general group action on the state may map it into a linear combination of all states in Hilbert space $\{|\alpha, \beta\rangle \,|\alpha, \beta \in G\}$, but as a construction we can restrict it into a simple subspace

$$U^i(g)\,|\alpha_1 \alpha_2\rangle = f_2(\alpha_1, \alpha_2, g, g')\,|g\alpha_1, g\alpha_2\rangle\,, \tag{3.36}$$

in which $f_2(\alpha_1, \alpha_2, g, g')$ is a $U(1)$ phase defined through 2-cocycle

$$f_2(\alpha_1, \alpha_2, g, g') = \frac{\nu_2(\alpha_1, g^{-1}g', g')}{\nu_2(\alpha_2, g^{-1}g', g')}\,, \tag{3.37}$$

where g' is a fixed element in G, and now one can show why U^i forms a linear representation

$$U^i(g_2)U^i(g_1)\,|\alpha_1, \alpha_2\rangle = f_2(g_1\alpha_1, g_1\alpha_2, g_2, g')f_2(\alpha_1, \alpha_2, g_1, g')\,|g_2 g_1 \alpha_1, g_2 g_1 \alpha_2\rangle\,, \tag{3.38}$$

$$f_2(g_1\alpha_1, g_1\alpha_2, g_2, g')f_2(\alpha_1, \alpha_2, g_1, g') = \frac{\nu_2(g_1\alpha_1, g_2^{-1}g', g')}{\nu_2(g_1\alpha_2, g_2^{-1}g', g')}\frac{\nu_2(\alpha_1, g_1^{-1}g', g')}{\nu_2(\alpha_2, g_1^{-1}g', g')}$$
$$= \frac{\nu_2(\alpha_1, g_1^{-1}g_2^{-1}g', g_1^{-1}g')}{\nu_2(\alpha_2, g_1^{-1}g_2^{-1}g', g_1^{-1}g')}\frac{\nu_2(\alpha_1, g_1^{-1}g', g')}{\nu_2(\alpha_2, g_1^{-1}g', g')} = \frac{\nu_2(\alpha_1, g_1^{-1}g_2^{-1}g', g')}{\nu_2(\alpha_2, g_1^{-1}g_2^{-1}g', g')} = f_2(\alpha_1, \alpha_2, g_2g_1, g'), \tag{3.39}$$

here third equality can be easily proved through the complex graph representation of cocycle. Thus

$$U^i(g_2)U^i(g_1)|\alpha_1, \alpha_2\rangle = U^i(g_2g_1)|\alpha_1, \alpha_2\rangle, \tag{3.40}$$

so U^i forms a linear representation. Through the definition in Eq.(3.37), one can easily prove that the group action on all site scale of $|\alpha_1, \alpha_2\rangle$ is universal, so this state is a RG fixed point. From this the construction of SPT state is also easy, as $U^i(g)|\alpha, \alpha\rangle = |g\alpha, g\alpha\rangle$. Because the phase factor here is factorized, this group action on the entangled state can also be divided into the action on $|\alpha_1\rangle$ and $|\alpha_2\rangle$, where

$$U^i(g)|\alpha_1\rangle = \nu_2(\alpha_1, g^{-1}g', g')|\alpha_1\rangle, \tag{3.41}$$

is in its projective representation $M(g)$ while the action on $|\alpha_2\rangle$ is instead in its anti-projective representation, and $U^i(g) = M(g) \otimes [M(g)]^\dagger$.

Now we may ask a question: does different $\nu_2(\alpha_1, g, g')$ indicates different SPT ? With the definition of coboundary in hand, we note that

$$\nu_2'(\alpha_1, g, g') = \nu_2(\alpha_1, g, g')\frac{\mu_1(g, g')\mu_1(\alpha_1, g)}{\mu_1(\alpha_1, g')}, \tag{3.42}$$

is also a cocycle and thus can also lead to a linear representation for dimmer state and permits the existence of a SPT state, yet it can be proven that the state defined by ν_2' and ν_2 can be transformed into each other through a LU transformation and thus is equivalent at the cohomology level. Up to this point we have finished the construction in which we see explicitly how the classification of SPT in 1+1 D is precisely captured by 2-cohomology group $H^2(G, U_T(1))$.

Similar argument and construction can be extended to case with higher dimensions. For 2+1 D case, one can similarly define a plaquette state, which is also a SRE state with entanglement merely existing for adjoint squares:

$$|\Psi_P\rangle = \otimes \sum_g |\alpha_1 = g, \beta_2 = g, \gamma_3 = g, \lambda_4 = g\rangle, \tag{3.43}$$

and the action of a group element on the SRE state will be

$$U^i(g)|\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle = f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4, g, g')|g\alpha_1, g\alpha_2, g\alpha_3, g\alpha_4\rangle, \tag{3.44}$$

where

$$f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4, g, g') = \frac{\nu_3(\alpha_1, \alpha_2, g^{-1}g', g')\nu_3(\alpha_2, \alpha_3, g^{-1}g', g')}{\nu_3(\alpha_4, \alpha_3, g^{-1}g', g')\nu_3(\alpha_1, \alpha_4, g^{-1}g', g')}. \tag{3.45}$$

C. From Discrete to Continuous

How is the SPT state defined above related to the field theory? The well-known path integral formalism of field theory provide the definition of partition function

$$Z = \int Dn\, e^{-\int d\mathbf{x}d\tau\mathcal{L}[n(\mathbf{x},\tau)]}, \tag{3.46}$$

as well as correlation function, propagator or Green function

$$U(n_1, n_2, \tau_1, \tau_2) = \int_{n(\mathbf{x},\tau_1)=n_1}^{n(\mathbf{x},\tau_2)=n_2} Dn\, e^{-\int d\mathbf{x}\int_{\tau_1}^{\tau_2} d\tau\mathcal{L}[n(\mathbf{x},\tau)]}, \tag{3.47}$$

and a theory with no anomaly means that a true symmetry (thus, according to Noether's theorem, conserved current) will be kept even at the quantum level as long as the action is invariant. In a nonlinear sigma model, the symmetric space is given by the coset of the Lagrangian symmetry G and field redefinition symmetry H as G/H. Explicitly speaking,

$$\mathcal{L}[g\cdot n]=\mathcal{L}[n]\,,\qquad h\cdot n=n\,,\tag{3.48}$$

and a maximally symmetric space correspond to n taking its value in the group G itself. If we then further require that the theory is a low energy effective one and does not rely on the specific form of the metric, i.e., at a RG fixed point, the theory must be topological quantum field theory, such as YM instanton, WZW and CS. All these theories can be expressed by a integration of a closed 1+d form on 1+d-dimensional spacetime, i.e., $S_{topo}=\int_{\mathcal{M}}\omega_{1+d}$. These are thus corresponding to the concept of 1+d-cocycle, and from the definition and the fact that differential form can be defined without providing any metric or measurement on a manifold naturally lead to a topological field theory. A trivial topological Lagrangian is $\mathcal{L}=0$, or we call it a (nontrivial) topological θ term. Among these WZW is a special one as it has similar properties to that of 1+d-cohomology: it can never be realized by a integration of local Lagrangian on a d-dimensional spacetime, so the close form here can never be written as an exact form. Naturally, the classification of WZW term can also be realized as a cohomology group $H^{1+d}(G,U_T(1))$.

Thus a conjecture would be to construct a discrete version of quantized WZW term, as a natural result of the map from group cohomology to de Rham cohomology discussed in the last subsection

$$Z=|G|^{-N}\sum_{\{g_i\}}e^{-S[\{g_i\}]}\tag{3.49}$$

here in order to normalize the partition function we add a factor before the summation, and the summation is over all possible group element configuration on a fixed complex. For topological θ terms we have $Z=1$ and thus correspond to $e^{-S[\{g_i\}]}=1$, so we can define it as

$$e^{-S[\{g_i\}]}=\prod_{\{i_0,\cdots,i_{1+d}\}}\nu_{1+d}^{s_{i_0\cdots i_d}}(g_{i_0},\cdots g_{i_{1+d}})=1\,,\tag{3.50}$$

and here the summation is over the set of all simplex that compose the complex, $s_{i_0\cdots i_d}=\pm1$ depends on the orientation of the simplex. Choosing ν_{1+d} to satisfy Eq.(3.50) can be proven to equal to choose an element in cohomology group $H^{1+d}(G,U_T(1))$ as well, so up to this point we have built up a bridge between the continuous WZW action and discrete topological object. For some examples of SPT in field theory and Higgs mechanism/superconductor, refer to [22, 23].

D. Possible Extensions

Apart from on-site symmetry, SPT argument can also be extended to include spatial translational symmetry. Such results are also obtained in [19]. In (1+1)-D it was shown that with translational symmetry the classification of SPT states is labeled by $H^1(G,U_T(1))\times H^2(G,U_T(1))$. In (2+1)-D it is labeled by $H^1(G,U_T(1))\times H^2(G,U_T(1))\times H^2(G,U_T(1))\times H^3(G,U_T(1))$.

Although lacking a rather vigorous proof of this classification, one can interpret the result in the following way. A translational symmetry restrict the configuration in which one divide the whole spacetime into a complex. In (1+1)-D, the existence of an x-translational symmetry means to divide the whole $x-t$ plane into several $[t]$ line and then multiply them together. Thus we expect another topological θ-term which can be written as

$$e^{-S_{top}}=\prod_{[t]}\prod_{i,j\in[t]}\nu_1^{s_{ij}}(i,j)\,,\tag{3.51}$$

and a more intuitive argument can be found in the construction of the group representation and SPT state in Eq.(3.36), which tightly rely on the fixed start and end point of the state; g and g' there can be regarded as a extension into the t-domain. However this is not possible in translational symmetry, so the only pivot point would be in the $[t]$. So in total the (1+1)-D classification may be labeled by $H^1(G,U_T(1))\times H^2(G,U_T(1))$. Similar argument can make sense for (2+1)-D, as this time we expect two $H^2(G,U_T(1))$ for two slices $[xt]$ plane or $[yt]$, and one $H^1(G,U_T(1))$ for $[t]$, and one for simply on-site symmetry.

Another direction to extend the concept of SPT is to conclude the mixed state and open system [24]. Operator-State correspondence provide us with a systematic method to map a general mixed state to pure state. For a many-body Hilbert space $\mathcal{H}=\otimes\mathcal{H}_i$, we can construct a doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}^*$, and then a maximally entangled state $|\Omega\rangle$ in it. Then for any operator $O\in\mathcal{B}(\mathcal{H})$, we can construct a pure state (Choi state) in the doubled Hilbert space as

$$|O\rangle=(O\otimes I)\,|\Omega\rangle=(I\otimes O^*)\,|\Omega\rangle\ ,\tag{3.52}$$

and this transmission is crucial in mapping any mixed state described by a density matrix ρ to a pure state and continue to use the method we meet before to define and discuss the SRE and SPT state. However, now there are more subtlety, as apart from the exact symmetry, in which satisfies $U_k\rho=\rho U_k=e^{i\theta_k}\rho$, there is also average symmetry, which satisfies $U_g\rho U_g^\dagger=\rho$. Moreover, in the construction of the Choi state, there emerges a natural anti-symmetry, modular conjugation J, which acts on the doubled Hilbert space as $J(|a\rangle |b^*\rangle) = |a^*\rangle |b\rangle$. The whole symmetry group will be

$$\mathcal{G} = (K \times K) \times G \rtimes J \tag{3.53}$$

in which K is the exact symmetry and G is the averaged symmetry. It is proven in [24] that to this sense the classification of mixed state SPT (MSPT) is labeled by

$$\oplus_{p=0}^{d} H^p[G, H^{d+1-p}(K, U(1))]\,, \tag{3.54}$$

and this is the subgroup of $H^{1+d}(K \times G, U(1))$, which is classification of pure SPT state in (1+d)-D under the symmetry group $K \times G$, both being exact symmetry.

In fact, considering we define the SPT of a mixed state of ρ by mapping it to a doubled pure state $|\rho\rangle$ and then using the definition of SPT of pure state, we treat $\mathcal{G}$ as a symmetry of $|\rho\rangle$, thus the classification would be given by $H^{1+d}(\mathcal{G}, U(1))$. Then Kunneth theorem claims the following decomposition

$$H^{1+d}(K_1 \times K_2, U(1)) = \oplus_{p+q=1+d} H^p[K_1, H^q(K_2, U(1))] = H^{1+d}[K_1, U(1)] \oplus H^d[K_1, H^1(K_2, U(1))] \oplus \cdots H^{1+d}[K_2, U(1)]\,. \tag{3.55}$$

It is proven in [24] that if we take both K_1 and K_2 being the exact symmetry of ρ, the nonnegativity of it rules out those protected jointly by K_1 and K_2, so it can only be classified by $H^{1+d}(K, U(1))$. As for the case of K being an exact symmetry and J being an modular conjugation, it can be proven that nonnegativity requires the state to dissatisfy any nontrivial reaction of the modular conjugation, thus is is still classified by $H^{1+d}(K, U(1))$. When we finally combine this with averaged symmetry G, we have the geometric interpretation that G-symmetric state $|\rho\rangle$ is a superposition of domain wall (or in general, symmetry defect) configurations associated with G of all dimensions. On each codimension-p defect we can decorate a $(d-p)$-dimensional SPT of the symmetry $K \times K \rtimes J$, labeled by an element in the group cohomology $H^{1+d-p}(K, U(1))$, and such label pattern is given by $H^p[G, H^{d+1-p}(K, U(1))]$. Thus we obtain the formula in Eq.(3.54).

Apart from these extensions with other symmetry in different systems, an application of SPT in real physical problem deserves to be discussed here. Recent studies argue that zero-temperature superconductors are topological phases protected by generalized symmetries [25]. More specifically, this study focus on the interpretation of SPT from the perspective of gauge fields. One of the result is the current of Cooper pair originates from the gauge field of magnetic symmetry, whose coupling Lagrangian is similar to quantum spin Hall effect. The magnetic symmetry comes from the absence of monopole in superconductor which is also a result in experiments. The symmetry in $2+1$D behaves

$$\frac{1}{2\pi}\int B d^2x = \text{constant of motion} \tag{3.56}$$

where B is the magnetic field. So, the corresponding generator is $U = e^{i\theta \int B/2\pi d^2x}$. When situated in $3+1$D, the symmetry will be a 1-form symmetry. With this symmetry, the corresponding gauge field A_M will be coupled to the original Lagrangian of zero-temperature superconductor (spin is considered also as gauge field coupled in the covariant derivative). The transformation is

$$L \to L + \frac{1}{2\pi} A_M dA \tag{3.57}$$

where A is the gauge field of electromagnetic field. Solve the ground state of this new Lagrangian when the order parameter of superconductor is $\langle\phi\rangle \neq 0$, which means a condensation. We will get $A = -q^S A_S$, where A_S is the gauge field of spin. Thus, the final coupling term of condensation phase is $\frac{1}{2\pi} A_M dA_S$ which is similar to quantum spin Hall effect.

IV. THE RELATION BETWEEN SPT AND TOPOLOGICAL BAND THEORY

After these introductions, we can see that the mappings with topological information in the two theories are different. The topological band theory focus on the map from Brillouin zone with PBC to a Berry connection living on the momentum space. The philosophy behind is the mapping from the parameter space of states, such as momentum and flux to some gauge fields with imaginable and definite physical meaning as the curvature of motion to a quasi-particle. The SPT classification focus on the map from the group parameters of the symmetric renormalized onsite internal states to the gauge phase of the representation. As there are constraints requiring the linear representation, the algebra of the phase factors can be classified by the cohomology theory. When the dimensional is higher, the quantity of renormalized onsite internal states will be more, which corresponds to a high-dimensional cohomology algebra. To tell the difference in the philosophy of the two theories, the key is in the image space of mapping. The first one is actually based on more physical conjecture, like quasi-particle approximation and perturbative response theory like Kubo formula to describe a phase. The second one actually focus on the complete description of many-body states, with some possible observables like entanglement spectrum and Chern numbers coming from three-parties entanglement.

Another possible relation between the two theories may be the boundary behavior. We note that the boundary excitations of the SPT phases characterized by (1+d)-cocycle ν_{1+d} are described by an effective boundary non-linear σ-model that contains a None-Local Lagrangian (NLL) term characterized by the same (1 + d)- cocycle ν_{1+d}. A non-linear σ-model with a nontrivial NLL term cannot describe a SPT state. Thus the boundary state must be gapless, or break the symmetry, or have degeneracy due to non-trivial topological order. However, the SPT state is a direct product state. The degrees of freedom on the boundary also form a product state. Therefore the boundary state must be gapless, or break the symmetry. Thus a non-trivial SPT state described by a non-trivial (1 + d)-cocycle must have gapless excitations or degenerate ground states at the boundary.

A simple comparison is between SSH model and Haldane chain. The edge modes of SSH model are protected by chiral symmetry, which contributes a symmetric distribution in the energy band with a non-trivial topological number. Haldane chain's edge modes are protected by the time-reversal symmetry and $SO(3)$ symmetry. So the phenomenological similarity of the two theories is the idea of symmetry protection, which means that the topological property will not be changed under symmetric perturbation without closing gap.

[1] L. Kong, X.-G. Wen, and H. Zheng, Journal of High Energy Physics **2022**, 10.1007/jhep03(2022)022 (2022).
[2] H. Yang, A. Liang, C. Chen, C. Zhang, N. B. M. Schroeter, and Y. Chen, Nature Reviews Materials **3**, 341 (2018).
[3] N. Qian, G. Yang, and X. Cong, PHYSICS **53**, 460 (2024).
[4] T. Qin, J. Zhou, and J. Shi, Phys. Rev. B **86**, 104305 (2012).
[5] L. Zhang and Q. Niu, Physical Review Letters **115**, 10.1103/physrevlett.115.115502 (2015).
[6] K. Masuki and Y. Ashida, Phys. Rev. B **107**, 195104 (2023).
[7] S. Xu and C. Wu, Phys. Rev. Lett. **120**, 096401 (2018).
[8] T. Oka and S. Kitamura, Annual Review of Condensed Matter Physics **10**, 387 (2019).
[9] S. Yao, Z. Yan, and Z. Wang, Physical Review B **96**, 10.1103/physrevb.96.195303 (2017).
[10] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, Nature **515**, 237240 (2014).
[11] T. D. Ellison and M. Cheng, PRX Quantum **6**, 10.1103/prxquantum.6.010315 (2025).
[12] L. Kong, T. Lan, X.-G. Wen, Z.-H. Zhang, and H. Zheng, Journal of High Energy Physics **2020**, 10.1007/jhep09(2020)093 (2020).
[13] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[14] B. Shi and I. H. Kim, Physical Review B **103**, 10.1103/physrevb.103.115150 (2021).
[15] P. Coleman, Introduction to Many-Body Physics (Cambridge University Press, 2015).
[16] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B **74**, 085308 (2006).
[17] M. B. Hastings and X.-G. Wen, Phys. Rev. B **72**, 045141 (2005).
[18] N. Schuch, D. Prez-Garca, and I. Cirac, Physical Review B **84**, 10.1103/physrevb.84.165139 (2011).
[19] X. Chen, Z.-C. Gu, and X.-G. Wen, Physical Review B **83**, 10.1103/physrevb.83.035107 (2011).
[20] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).
[21] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Physical Review B **87**, 10.1103/physrevb.87.155114 (2013).
[22] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[23] R. H. Wilke, H. Schlmer, S. M. Linsel, A. Bohrdt, and F. Grusdt, Topology meets symmetry breaking: Hidden order, intrinsically gapless topological states and finite-temperature topological transitions (2025), arXiv:2506.03146 [cond-mat.str-el] .
[24] R. Ma and A. Turzillo, PRX Quantum **6**, 010348 (2025).
[25] R. Thorngren, T. Rakovszky, R. Verresen, and A. Vishwanath, Higgs condensates are symmetry-protected topological phases: Ii. $u(1)$ gauge theory and superconductors (2023), arXiv:2303.08136 [cond-mat.str-el] .