# AP Calculus BC

## 2024-2025 Syllabus

### 1. Limits & Continuity

a. Limits mean "approach", not "equal"
   i. When a limit doesn't seem to exist (when plugged in directly), evaluate from the left and the right.
   ii. The limit is equal to the left and the right limit (if they're both equal)
   iii.

b. Unbounded limit (approaches infinity) when $n/0$ for $n \neq 0$

c. Limit is indeterminate (use methods to solve for) when $0/0$, $\infty/\infty$
   i. Algebraic manipulation (factoring so "hole" cancels out)
   ii. Rationalization (rationalizing denominator so "hole" cancels out), use conjugates
   iii. Common denominator (fraction/fraction, "hole" cancels out)
   iv. L'Hôpital's Rule: the limit of $f(x)/g(x)$ = limit of $f'(x)/g'(x)$ if $f(x)/g(x)$ is indeterminate
   v. Graph Composite Functions: In composite piecewise functions, the limit might not be apparent. E.g. $f(g(x))$, evaluate $x \to a$ from left and right side and if $g(x)$ approaches that value from up, then evaluate $g(x) \to b^+$. If from below then $g(x) \to b^-$, etc. Then evaluate $f(x)$ accordingly.

d. Squeeze Theorem - Used when the value of a function or its limit is not apparent but you know the function's range
   i. Only conclusive when lower bound = upper bound for the value you want to find
   ii. E.g. $f(x) \le g(x) \le h(x)$ then Squeeze Theorem works when $f(x) = h(x)$, thus $g(x) = f(x) = h(x)$

e. Continuity at a point occurs when...

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

f. Continuity over an interval occurs for polynomial, some trigonometric, exponential, etc. functions and evaluate continuity at a point on uncertain points
g. You can remove a discontinuity with algebraic manipulations (factorization, rationalization, etc.)
h. Limits toward infinity is the end behavior of a function (horizontal or slant asymptote)
i. Discontinuities: Jump, Point (Removable), Asymptotic
j. Intermediate Value Theorem - On a continuous function there must exist at least one point (c, f(c)) if f(a) < f(c) < f(b) or f(b) < f(c) < f(a) and a < c < b.
    i. Justification requires continuity, interval (and the inequality), and stating the IVT

## 2. Differentiation: Basics

a. Definition

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \text{ or } \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

b. Power Rule

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$$

c. Product Rule

$$\frac{d}{dx}[u \cdot v] = u' \cdot v + u \cdot v'$$

d. Quotient Rule

$$\frac{d}{dx}[u \div v] = \frac{u' \cdot v - u \cdot v'}{v^2}$$

e. Trig Functions

| FUNCTION | DERIVATIVE | FUNCTION | DERIVATIVE |
|---|---|---|---|
| sin(x) | cos(x) | sec(x) | sec(x)tan(x) |
| cos(x) | -sin(x) | csc(x) | -csc(x)cot(x) |
| tan(x) | $sec^2(x)$ | cot(x) | $-csc^2(x)$ |

## 3. Differentiation: Advanced

a. Chain Rule

$$\frac{d}{dx}[u(v)] = v' \cdot u'(v)$$

b. Implicit Differentiation

$$\frac{d}{dx}[y] = \frac{dy}{dx}$$

When differentiating a variable you're not explicitly differentiating it to (e.g. t, time), make sure to multiply by the derivative of that variable with respect to the independent variable.

c. Inverse Differentiation

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

d. Higher Derivatives - Differentiate the derivative as normal and substitute lower derivatives if needed for simplification

e. Inverse Trig Functions

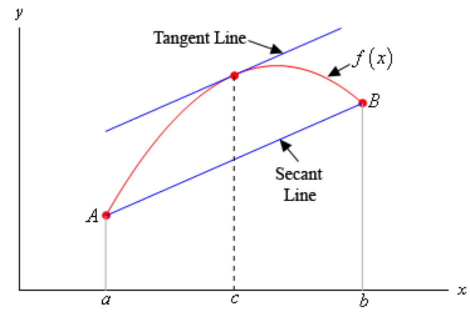
| FUNCTION | DERIVATIVE | FUNCTION | DERIVATIVE |
|---|---|---|---|
| arcsin(x) | $\frac{1}{\sqrt{1-x^2}}$ | arcsec(x) | $\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ |
| arccos(x) | $-\frac{1}{\sqrt{1-x^2}}$ | arccsc(x) | $-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ |
| arctan(x) | $\frac{1}{x^2+1}$ | arccot(x) | $-\frac{1}{x^2+1}$ |

## 4. Differentiation: Contextual application

a. Related Rates - Using implicit differentiation, you can relate the rates of different variables in a multivariable equation (e.g. Pythagorean Theorem).
   i. Be attentive to constant variables (derivative = 0) and substitute in the original equation if needed.

b. Local linearity - You can approximate nearby points with a tangent line to the curve. The value is underestimated when concave up, overestimated when concave down.

# 5. Differentiation: Analyzing functions




a. Mean Value Theorem - Differentiable function over interval must have an instantaneous slope somewhere on the interval equal to the average rate of change of the interval.
   i. Justify with differentiability, f'(c) = (f(b)-f(a))/(b-a) for a < c < b, and MVT
b. Critical Point - Points where the derivative = 0 or undefined (vertical tangents included). Has a chance of being a relative extremUM.
c. Extreme Value Theorem - If a function is continuous over a closed interval, then there must exist absolute extrema value over that interval
d. Candidates Test - To find absolute extrema. Find critical points and evaluate end point values and critical point values. Determine the absolute extrema.
e. First Derivative Test - A relative minimum occurs when the function's derivative changes from negative to positive. A relative maximum occurs when the function's derivative changes from positive to negative.
f. Second Derivative Test - A relative minimum occurs when the function's derivative is 0 or undefined and the function's second derivative is positive. A relative maximum occurs when the function's derivative is 0 or undefined and the function's second derivative is negative.
g. Inflection point is a point at which a function changes concavity (second derivative changes signs)
h. Optimization - To maximize or minimize a value in a related rates problem (through first/second derivative tests or candidate tests). Analyze given variables carefully.

## 6. Integration and accumulation of change

a. Integration is anti-differentiation (accumulating change), area under the curve
b. Riemann sum can approximate area under the curve

| **APPROXIMATION** | **INCREASING** | **DECREASING** | **CONCAVE UP** | **CONCAVE DOWN** |
|---|---|---|---|---|
| Left Riemann Sum | Underestimate | Overestimate | No effect | No effect |
| Right Riemann Sum | Overestimate | Underestimate | No effect | No effect |
| Midpoint Riemann Sum | No effect | No effect | Overestimate | Underestimate |
| Trapezoidal Riemann Sum | No effect | No effect | Underestimate | Overestimate |

c. Riemann approximation with infinite summation

Left Riemann

$$\int_a^b f(x)dx = \lim_{n \to \infty} \left( \sum_{k=0}^{n-1} \left[ \left( \frac{b-a}{n} \right) \cdot f\left(a + \frac{b-a}{n} k\right) \right] \right)$$

Right Riemann

$$\int_a^b f(x)dx = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \left[ \left( \frac{b-a}{n} \right) \cdot f\left(a + \frac{b-a}{n} k\right) \right] \right)$$

d. Definite integrals - Accumulation with definite bounds
e. Indefinite integrals - General formula with no definite bounds (has a +c in the end to denote vertical shifts)

f. Integral Properties

| |
|---|
| $\int_{a}^{a} f(x)dx = 0$ |
| $\int_{a}^{b} f(x)dx =- \int_{b}^{a} f(x)dx$ |
| $\int_{a}^{c} f(x)dx = \int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx$ |

g. Fundamental Theorem of Calculus

| |
|---|
| $\frac{d}{dx}[\int_{a}^{x} f(t)dt] = f(x)$ |
| $\frac{d}{dx}[\int_{h(x)}^{g(x)} f(t)dt] = [f(g(x)) \cdot g'(x)] - [f(h(x)) \cdot h'(x)]$ |
| $F(b) = F(a) + \int_{a}^{b} f(t)dt$ |

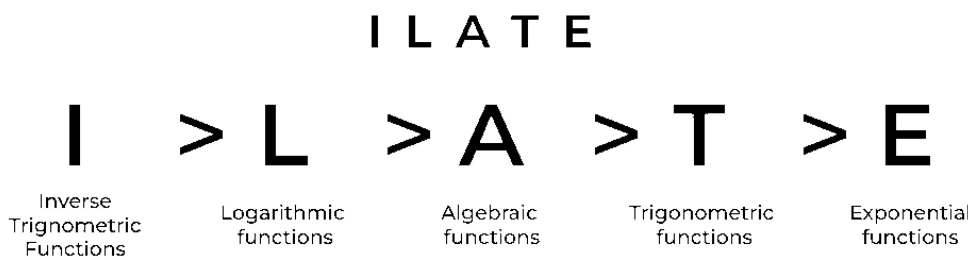
h. U-Substitution - Method to integrate composite functions (opposite of chain rule). Substitute u for the internal function and then solve for du in terms of dx (by differentiating u). This allows you to differentiate in u instead of x. Be aware of bounds in the case of definite integrals (x= is not the same as u=)

i. Integration by Parts - Method to integrate product of functions (opposite of product rule).

$$\int f(x) \cdot g(x)dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)dx$$

$$\int uv'dx = uv - \int v\,du$$

Select u or f(x) based on the following priorities:




j. Integration using linear partial fractions - Method to integrate by splitting a fraction into multiple.

$$\int \frac{ax+b}{f(x) \cdot g(x)}dx = \int \frac{A}{f(x)} + \frac{B}{g(x)}dx$$

Solve for A and B by equating A*g(x) + B*f(x) with ax + b

k. Improper integrals - Some integrals are asymptotic or involve an infinite bound. You can evaluate this by setting a limit (this does not guarantee that the integral will exist).

$$\int_a^\infty f(x)dx = \lim_{n \to \infty} \int_a^n f(x)dx = \lim_{n \to \infty} F(n) - F(a)$$

## 7. Differential Equations

a. Euler's method - Method to approximate a differential equation solution. Start at an initial value, calculate dy/dx at that x-value. Move onto the next x-value (+step size), to calculate the new y, increment the old y by dy/dx at the previous point (divided by step size). E.g:

| X-VALUE (step = 0.5) | Y-VALUE | DY/DX (dy/dx=y) |
|---|---|---|
| 0 | 1 (given) | 1 |
| 0.5 | 1.5 | 1.5 |
| 1 | 2.25 | 2.25 |
| 1.5 | 3.375 | 3.375 |