AP Calculus BC exclusive

Unit 9

1. Parametric functions

when $x(t)$, $y(t)$ given t (time)

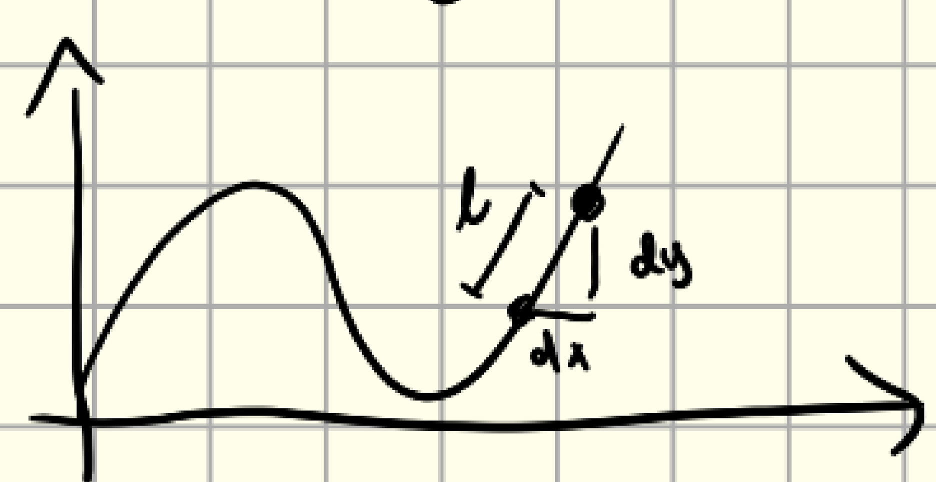
- $\frac{dx}{dt}$ = how fast x is changing per unit of time
- $\frac{dy}{dt}$ = how fast y is changing per unit of time

thus $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ according to chain rule

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

arrows indicate direction as time increases

Arc length

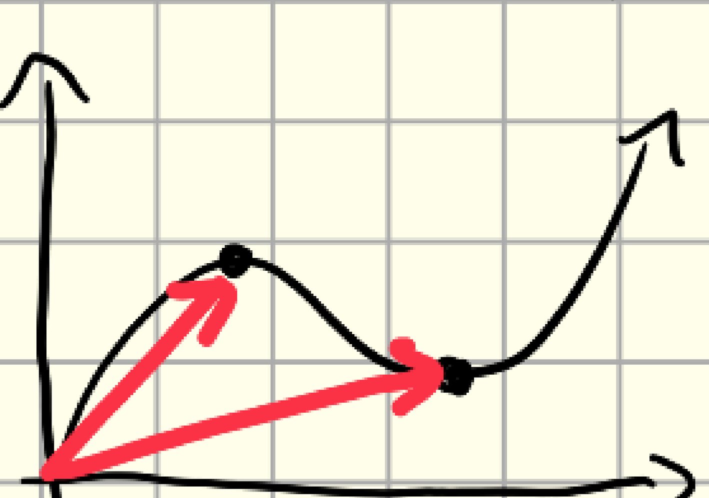


$l = \sqrt{dy^2 + dx^2}$

$\therefore$ infinite sum

arclength of curve $= \int \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

2. Vector-valued functions



● = displacement at time t (position)

position (x, y) can be represented by a vector

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$
$$= (x(t), y(t))$$

the velocity vector (how fast an object is travelling given x, y trajectory) is the derivative

same with acceleration

$$\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$
$$= (x'(t), y'(t))$$

2. Vector Value Functions

New position is given by old + displacement

Displacement can be obtained from $\int_a^b x'(t)\,dt$, $\int_a^b y'(t)\,dt$

The magnitude of the displacement follows the pythagorean theorem where $s = \sqrt{x^2 + y^2}$

☆ Magnitude of velocity vector $\neq \frac{dy}{dx}$

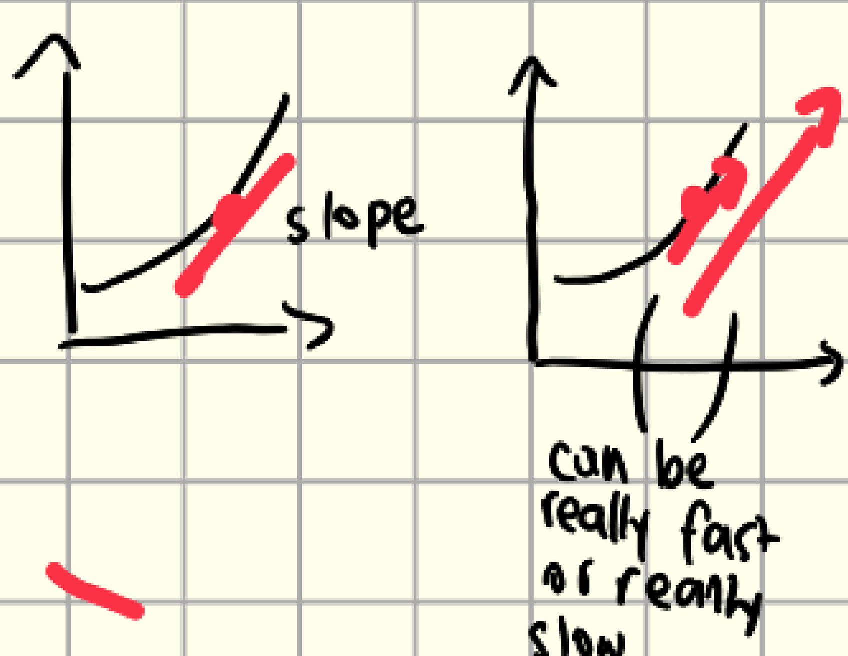
↳ $\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$ how fast the arclength is changing

↳ $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\frac{dy}{dx}$ is the slope (doesn't tell us anything about time)

the direction of velocity matches up with $\frac{dy}{dx}$ but different magnitudes

E.g.



3. Polar Functions

Polar functions are defined as $r(\theta)$ (radius and angle)

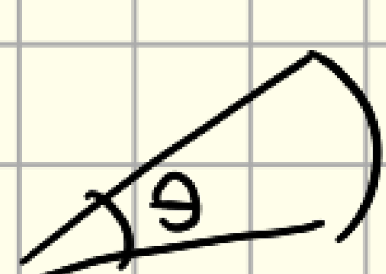
The x, y components is as follows

$$x = r \cdot \cos\theta \qquad y = r \cdot \sin\theta$$

So finding $\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$ and you can differentiate x, y individually to find individual rates of change

Area bounded by polar curve

↳



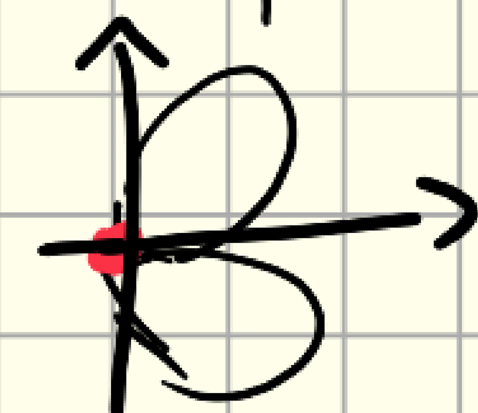
$$A = \frac{r^2\theta}{2}$$

$$\Sigma A = \frac{1}{2}\int r^2\,d\theta$$

3 Polar functions

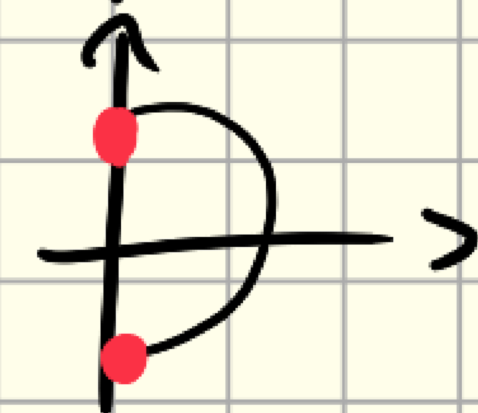
Finding bounds is hard part

Look out for $r=0$



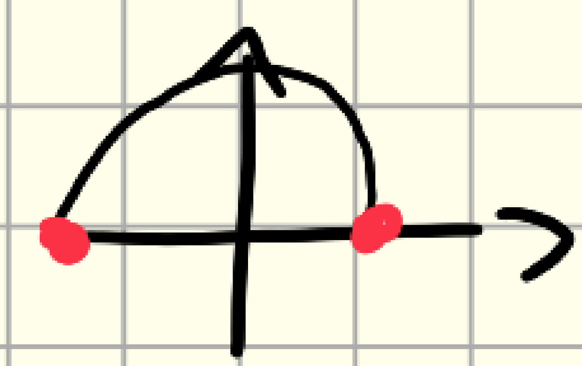
$x=0$

$x=r\cos\theta$



$y=0$

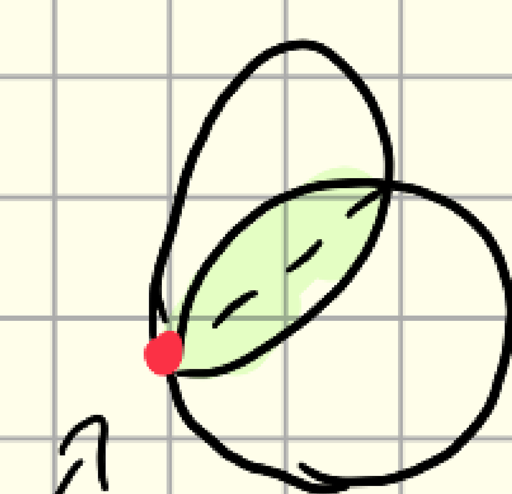
$y=r\sin\theta$



sometimes bounds might be given

When looking at areas bounded by two polar graphs...

ADD

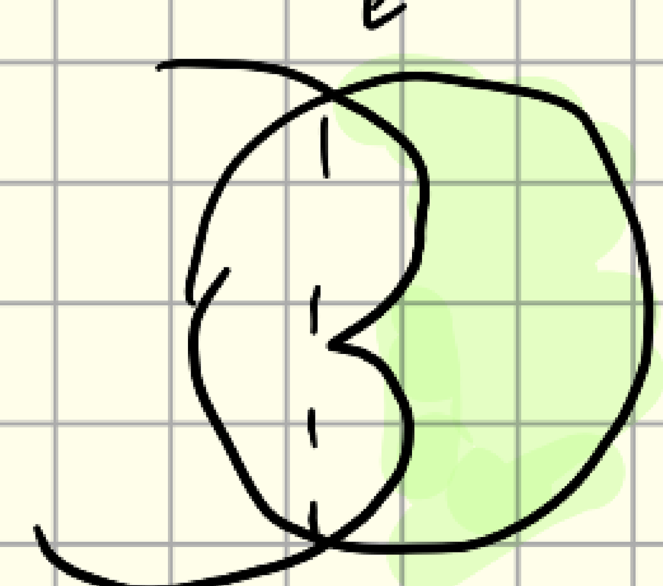


$$A=\frac{1}{2}\int_a^b r_1^2+r_2^2\,d\theta$$

area is composed of two parts (add them up!)
↳ each corresponding to a function

SUBTRACT

area is difference between two parts



$$A=\frac{1}{2}\int_a^b r_2^2-r_1^2\,d\theta$$

Unit 10

Convergence: $\lim_{n\to\infty} a_n = 0$

a_1, a_2, a_3 etc. is a convergent SEQUENCE when $\lim_{n\to\infty} a_n = 0$

$a_1 + a_2 + a_3$ etc. is a convergent SERIES when $\sum_{n=1}^{\infty} a_n = k$

for k is real

Divergence: when $\lim_{n\to\infty} a_n \neq 0$

(does not reach a particular value)

when $\sum_{1}^{\infty} a_n$ is unbounded

Partial Sum: denoted by S

equation usually given

S_n means $\sum_{k=1}^{n} a_k$

S means $\sum_{k=1}^{\infty} a_k$

$S = \lim_{n\to\infty} S_n$

The a_n (nth term) is given by $S_n - S_{n-1}$

Geometric Sequence

Partial sum given by: $S_n = a_0 \dfrac{(1-r^n)}{1-r}$ ⇐ converges for $|r|<$

for $a_n = a_0 r^n$

Geometric sequence

Infinite partial sum given by: $S = \frac{a_0}{1-r}$

Divergence Test

n-th test

if a sequence does not converge, then its series musn't converge either

↳ if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ will <u>diverge</u>

* if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ will not necessarily converge

Integral Test

if $f(x)$ is positive, continuous, decreasing for $[k, \infty)$

1. if $\int_k^\infty f(x)\,dx$ converges, so will $\sum_{n=k}^{\infty} f(n)$

2. if $\int_k^\infty f(x)\,dx$ diverges, so will $\sum_{n=k}^{\infty} f(n)$

Harmonic Series

refers to $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$

the generalized series for...

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \ldots$$

is the P-series

harmonic series is where $p=1$

P-series

The p-series will converge when $p > 1$

if $p \leq 1$, then p-series will diverge

Comparison test

talking about $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$

if $a_n \leq b_n$ (i.e. $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$)

then if $\sum_{n=1}^{\infty} b_n$ converges, so must $\sum_{n=1}^{\infty} a_n$

if $\sum_{n=1}^{\infty} a_n$ diverges, so must $\sum_{n=1}^{\infty} b_n$

Limit comparison test

if $\sum_{n=k}^{\infty} a_n$, $\sum_{n=k}^{\infty} b_n$ for $a_n \geq 0$, $b_n \geq 0$

then

if $\lim_{n \to \infty} \frac{a_n}{b_n}$ is positive and finite

either both will converge or both will diverge

(if you're asked to find a a_n or b_n, choose a function growing at a similar rate, e.g. $\frac{2^n}{3^n - 1}$ and $\frac{2^n}{3^n}$)

Alternating Series Test

if $\lim_{n\to\infty} b_n = 0$ where b_n is a decreasing sequence

then if $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$

$(b_n \geq 0)$

then... $\sum_{n=k}^{\infty} a_n$ converges

(if this doesn't work then it doesn't mean it'll diverge)

Ratio test for convergence

Based on how a geometric series converges when $|r| < 1$

for $\sum_{n=k}^{\infty} a_n$ ...

- if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, series must converge
- if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, series must diverge

if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the test is inconclusive

<u>Absolute convergence</u> - when a series AND its absolute value converges

<u>Conditional convergence</u> - when a series converges but its absolute value does not

* A series that diverges may never have its absolute value converge

Absolute convergence implies convergence

Alternating Series Error bound

It is possible to estimate the value an alternating series converges to and the error

- Partial sum...

remainder

$$S = S_k + R_k$$

$$|R_k| \leq |a_k|$$

E.g. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16}$ etc.

$$S = S_2 + R_2$$

$$R_2 \leq \frac{1}{9}$$

when $R_k > 0$, S_k is underestimate for S;
when $R_k < 0$, S_k is overestimate for S

Useful for when you need to keep error below a certain value and you want to know how many terms you should evaluate

↳ $|R_k| \leq$ wanted error bound

$\Downarrow$

$|a_{k+1}| \leq$ error bound

careful!

Maclaurin Series

- Used to approximate a function

since
$p(0)=f(0)$
$p'(0)=f(0)$
$p''(0)=f''(0)$
etc.

$$P(x) = f(0) + f'(0)x + \frac{1}{2}f'(x)x^2 + \cdots f^{(n)}(0)\cdot\frac{x^n}{n!}$$

a more generalized form for not just $x=0$ is...

Taylor Series

where...

again... $p(c)=f(c)$
$p'(c)=f'(c)$
etc.

$$P(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f'(c)(x-c)^2 + \cdots f^{(n)}(c)\cdot\frac{(x-c)^n}{n!}$$

Maclaurin series a special case of taylor series where $c=0$

The series layed out as a polynomial is known as a Taylor polynomial
(can be used to "convert" a function into polynomial)

Error bound

- The error for a taylor series estimate evaluated until the nth derivative

$$|E^{(n+1)}(x)| = |f^{(n+1)}(x)| \leq M$$

max. value (from point of taylor series to x)

$$\Downarrow$$

$$|E(x)| \leq \frac{M(x-a)^{n+1}}{(n+1)!}$$

(< error stated by question)

$n \in \mathbb{Z}$ so you should try values of n in calculator

Power Series

refers to $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$

similar to $\sum_{n=0}^{\infty} a x^n$

geometric

Interval for x for which the series converges

E.g. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^n}$

converges for

$-5 \leq x < 5$

interval of convergence

Radius of convergence

$\pm a$ for which x still converges

(e.g. $-1 < x < 1$, radius of 1)

It is best to use ratio test (since it gives interval) and other tests to check for closed or open

Geometric series and functions

A geometric series ∞ sum $= a \frac{1}{1-r}$ $\Leftarrow$ rational

even when r is not a constant, it can work

$\Downarrow$

as long as

$|r| < 1$

E.g. $\frac{6}{1+x^3} \Rightarrow 6 \cdot \frac{1}{1-(-x^3)}$

$r = -x^3$

$\Downarrow$

$6 - 6x^3 + 6x^6$ etc.

We can also perform the opposite

E.g. $2 - 8x^2 + 32x^4 - 128x^6 \dots$

$r = -4x^2$ $\quad a = 2$

$f(x) = 2 \cdot \frac{1}{1+4x^2}$ for $|x| < \frac{1}{2}$

we can convert polynomial into ∞ sum; however, convergence must be met where $|r| < 1$

Geometric series and functions

using this idea, we can convert more functions into maclaurin series by differentiating and then integrating

E.g.

$$\arctan(2x) = \int 2 \cdot \frac{1}{4x^2+1}\,dx \qquad a=2, \; r=-4x^2$$

$$\therefore \int \frac{2}{4x^2+1}\,dx = \int \sum_{n=0}^{\infty} 2(-4x^2)^n\,dx$$

$$= \int 2 - 8x^2 + 32x^4 - 128x^6 + \text{etc.}\; dx$$

$$= 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \text{etc.} + C$$

$$= \sum_{k=0}^{\infty} \frac{2\cdot(-4)^k \cdot x^{2k+1}}{2k+1} + C$$

$$\arctan(2x) = \sum_{k=0}^{\infty} \frac{2(-4)^k x^{2k+1}}{2k+1} + \cancel{C}^{\,0} \qquad \arctan(0)=0$$

Maclaurin series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Integrating and differentiating Power series

↳ Recall $\int(\Sigma) = \Sigma(\int)$ and $\frac{d}{dx}(\Sigma) = \Sigma(\frac{d}{dx})$

So you can integrate/differentiate what's inside the sigma to get the function
- n is a constant, x is the variable

Alternatively (when finding derivative at $x=0$), you can expand the power series ($a_0 + a_1$ etc.) and differentiate it there (easier)

E.g. $f(x) = \sum_{n=1}^{\infty} \frac{n+1}{4^{n+1}} x^n$ $\quad F(x) = \int_0^1 f(x)\,dx$

$$F(x) = \int_0^1 \sum_{n=1}^{\infty} \frac{n+1}{4^{n+1}} x^n \, dx$$

$$= \sum_{n=1}^{\infty} \frac{n+1}{4^{n+1}} \int_0^1 x^n \, dx = \sum_{n=1}^{\infty} \frac{n+1}{4^{n+1}} \left. \frac{x^{n+1}}{n+1} \right|_0^1$$

$$= \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{4^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n+1}$$

$$= \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots$$

$$= \frac{1}{4^2} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{4^2} \cdot \frac{4}{3} = \frac{1}{12}$$

☆ be careful of Σ bounds (when differentiating, it may start at different initial value due to constant)

This will also work for indefinite integrals/derivatives
- By converting a power series into a rational and then applying calculus