*Recap refers to review of concept already shown befo

IB Maths: AA HL, Quadratics

Vieta's Formulas:

$\boxed{\text{Given}}$ $ax^2+bx+c=0$ where α and β are roots

$\boxed{\text{THEN...}}$

$$ax^2+bx+c = a(x-\alpha)(x-\beta) = 0$$

$$\Rightarrow \frac{ax^2}{a} + \frac{b}{a}x + \frac{c}{a} = \frac{a}{a}(x-\alpha)(x-\beta) = 0$$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - \alpha x - \beta x + \alpha\beta$$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha+\beta)x + \alpha\beta$$

$$\therefore -(\alpha+\beta) = \frac{b}{a} \Rightarrow \boxed{\alpha+\beta = -\frac{b}{a}} \text{ SUM}$$

$$\therefore \boxed{\alpha\beta = \frac{c}{a}} \text{ PRODUCT}$$

$\boxed{\text{SINCE...}}$ axis of symmetry is midpoint (mean) of two roots

$$\text{axis}_{\text{symmetry}} = \frac{\text{sum}}{2} = \frac{\alpha+\beta}{2} = -\frac{b}{2a}$$ ← important for optimization problems, min/max is at $x = -\frac{b}{2a}$

Complex numbers must also be symmetrical

comes in conjugates

Roots $\Rightarrow$ $a+bi$ must also mean $a-bi$

Difference of squares $a^2-b^2 = (a-b)(a+b)$

$\therefore$ Sum of squares $a^2+b^2 = a^2 - i^2b^2 = (a-ib)(a+ib)$

Approaches:

Factoring - Use master product $(a\cdot c)$ and think of two numbers that add to b but multiply to get $a\cdot c$

Completing the square - Divide equation so $a=1$, move constant to one side, add $\left(\frac{b}{2}\right)^2$ to both sides and factor, square root.

Writing with a purpose.

E.g. $T(t)$ models average temperature (°C) for year t
where $T(2024)=17$, $T'(t) \approx 0.1$

Pt.-Slope $\Rightarrow$ $T-17=0.1(t-2024)$

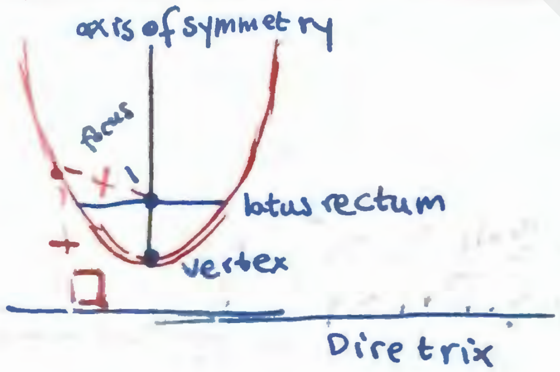
makes sense because year 2024, average temp. is 17°C

HOWEVER...

slope-Intercept $\Rightarrow$ $y=0.1x-185.4$

although true, doesn't make sense that, in year 0, the temperature was -185.4°C

Geometric Definition of a parabola



a^x ← exponentiation

not to be confused with

Alternative ways to convert to vertex form:
h = x-of vertex = $-\frac{b}{2a}$
k = plug h into original

Power Equations: $x^n = k$, $n \neq 0$

- Standard Form (quadratic): $ax^2+bx+c=0$ ← y-intercept
- Vertex Form: $a(x-h)+k=0$ (h,k) is vertex
- Factored Form: $a(x-x_0)(x-x_1)=0$ where $x = x_0, x_1$ are roots

- Standard Form (linear): $Ax+By=C$ ← intercepts (plug in 0s for x, y)
- Slope-intercept: $y=mx+b$ ← m = slope; b is y-intercept
- Point-Slope: $y-y_1=m(x-x_1)$ ← m = slope; (x_1, y_1)

Formative Assessment on Quadratics

Practice

1\. a) $x^2+4x-2 = x^2+4x+4-4-2 = (x+2)^2-6$ ✓

b) $f(x)=x+2 \quad g(f(x)) = x^2+4x-2 = g(x+2) = (x+2)^2-6$ ✓

$\therefore g(x) = x^2-6$

2\. a) $f(x) = ax^2-4x-c$

i) $\because$ L intersects f at $x=-1, 3$; parabola maintains symmetry; L is horiz.

$x=p$ for $p=1 \quad \left(\frac{-1+3}{2}\right)=1$ ✓

ii) $-\frac{b}{2a}=p \Rightarrow 1=-\frac{-4}{2a}=\frac{4}{2a} \Rightarrow \frac{2}{a}=1 \Rightarrow 2=a$ ✓

b) $L \Rightarrow y=5$

$f(x)=2x^2-4x-c \Rightarrow 5=2x^2-4x-c \Rightarrow 2x^2-4x-5-c=0$

$(x+1)(x-3)=0 \Leftarrow x^2-2x-\frac{5+c}{2}=0$

$x^2-2x-3=x^2-2x-\frac{5+c}{2}$

$-3=-\frac{5+c}{2} \Rightarrow 6=5+c \Rightarrow c=1$ ✓

3\. a) $\because f$ is quadratic, 2 x-values (positive, negative) can have same y-value

$\therefore$ 1 y-value matches 2 x-value, $f^{-1}(x)$ not function

$\therefore$ function f has no inverse ✓

b) a = axis of symmetry

$-\frac{b}{2a} = -\frac{4}{-4}=1 \quad \therefore \quad a=1, \quad x \geq 1$ ✓

c) $x=-2y^2+4y+30$

$\frac{x}{-2}=y^2-2y-15 \Rightarrow \frac{x}{-2}+15=y^2-2y \Rightarrow \frac{x}{-2}+16=(y-1)^2$

$\Rightarrow \pm\sqrt{\frac{x}{-2}+16}+1=y$

$\therefore y=\pm\sqrt{\frac{x}{-2}+16}+1$ ✓

$x \geq 1$

$\therefore f^{-1}(x) \geq 1$

$\therefore \sqrt{\frac{x}{-2}+16}+1 \geq 1$

4\. a) $f(x) = -\frac{1}{2}(x+2)^2+5$ ✓

b) $f(x) = -\frac{1}{2}(x^2+4x+4-\frac{5}{2})$

$= -\frac{1}{2}(x^2+4x+\frac{3}{2})$

$= -\frac{1}{2}(x-2-\sqrt{10})(x-2+\sqrt{10})$

5\. $f(x) = x^2+bx+c = (x-3-\sqrt{5}i)(x-3+\sqrt{5}i) = x^2-(3-\sqrt{5}i)x-(3+\sqrt{5}i)x+9+5$

$= x^2-6x+14$

$g(x) = 2f(x+4) = 2(x+4)^2-12(x+4)+28 = 2x^2+16x+32-12x-48+28$

$= 2x^2+4x+12$ ✓ overcomplicated

6\. $10-x=\frac{4}{x-k} \Rightarrow (10-x)(x-k)=4 \Rightarrow -x^2+10x-10k+kx=4$

$\Rightarrow -x^2+(10+k)x-10k-4=0 \Rightarrow \Delta=(10+k)^2-40k-16 = 100+20k+k^2-40k-16 = k^2-20k+84$

$\Rightarrow \Delta=(k-6)(k-14) \quad \frac{d^2\Delta}{dk^2}=1>0$, Δ is ccu

$\therefore \Delta<0$ for $k \in (6, 14)$ ✓

and y does not intersect with g(x)

7. (a) i. $\alpha+\beta = -\frac{b}{a} = \frac{8}{2} = 4$ ✓ ii. $\alpha\beta = \frac{c}{a} = \frac{1}{2}$ ✓

(b) $\frac{2}{\alpha}+\frac{2}{\beta} = -\frac{p}{1} = -p$; $p = -\frac{2}{\alpha}-\frac{2}{\beta}$; $q = \frac{2}{\alpha}\cdot\frac{2}{\beta} = \frac{4}{\alpha\cdot\beta}$

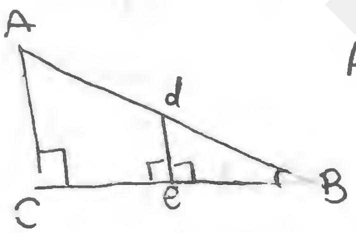
EXACT VALUE $p = -2\left(\frac{\alpha+\beta}{\alpha\beta}\right) = -2\left(\frac{4}{\frac{1}{2}}\right) = -16$, $q = \frac{4}{\frac{1}{2}} = 8$ ✓

8. $f(x) = x^2+4x-1 = (x+2)^2-5$

$g(x) = 3x^2+9x+2 = 3\left(x^2+3x+\frac{2}{3}\right) = 3\left(x+\frac{3}{2}\right)^2-\frac{19}{4}$

1. Vertical stretch by factor of 3 $f_1(x) = 3(x+2)^2-15$ ✓
2. Shift x by $\frac{1}{2}$ units to the right
3. Shift y by $\frac{41}{4}$ units upwards. } ✓

9.



$ABC \sim dBe$, AA similarity

$\therefore \frac{x}{5} = \frac{x+2}{x+8} \Rightarrow x^2+8x = 5x+10$

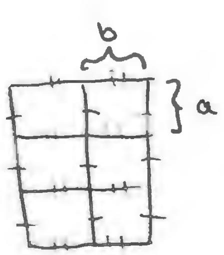
$\Rightarrow x^2+3x-10 = 0$

$\Rightarrow (x+5)(x-2) = 0$

$\because x$ is measurement, $x > 0$

$\therefore x = 2$ ✓

10.



$9a+8b = 200\text{m}$ $\rightsquigarrow b = \frac{200\text{m}-9a}{8}$

$A = ab$, maximize

$= a\left(\frac{200\text{m}-9a}{8}\right) = a\left(25-\frac{9}{8}a\right)$

$= 25a - \frac{9}{8}a^2$

$\Rightarrow$ Axis of symmetry = Max. width $= -\frac{25}{-\frac{9}{4}} = 25\cdot\frac{4}{9} = \frac{100}{9}$

If $a = \frac{100}{9}\text{m}$, $b = \frac{200\text{m}-9a}{8} = 25-\frac{100}{8} = 25-\frac{25}{2} = \frac{25}{2}\text{m}$

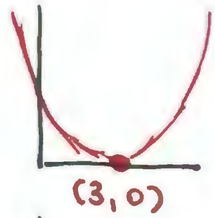
$\therefore$ Length $= \frac{25}{2}\text{m} = 12.5\text{m}$ ✓

Width $= \frac{100}{9}\text{m} = 11.\overline{1}\text{m}$ ✓

Complex Numbers

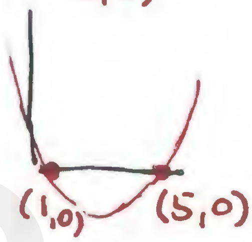
There's an imaginary plane perpendicular to the real plane

E.g. $y = x^2 - 6x + 9$



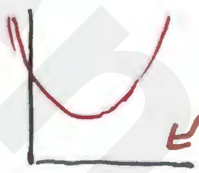
$x = 3$

$y = x^2 - 6x + 5$



$x = 3 \pm 2$ ⇒ when you add 2 roots together ⇓ $= 6$ (both!)

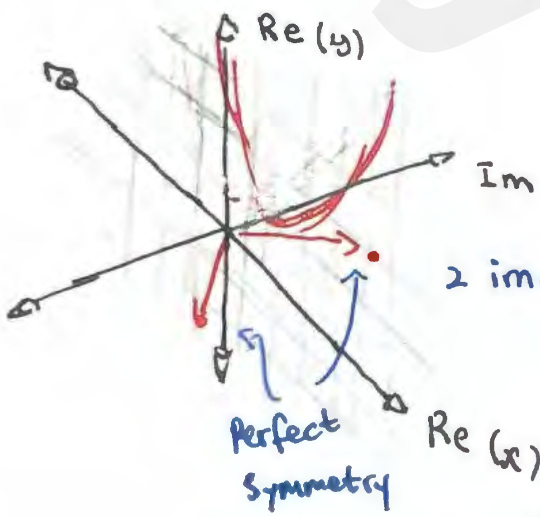
$y = x^2 - 6x + 13$



no roots?
no real roots

$x = 3 \pm 2i$

$\boxed{\text{Actually,}}$ there exists an imaginary plane



2 imaginary roots

i represents a rotation of 90°

$i = \sqrt{-1}$

$a + bi$ has a conjugate of $a - bi$

Inverse Functions

A function only has an inverse when its range (y) can map onto its domain (x) with one-to-one correspondence

$\boxed{\text{input: } 3}$ ⇒ $\boxed{x^2}$ ⇒ $\boxed{\text{output: } 9}$

$\boxed{\text{output: } -3}$ ⇒ $\boxed{x^2}$ ⇒ $\boxed{\text{output: } 9}$

} 2 range maps to one domain (no inverse unless domain of x^2 restricted)

Problem solving methods and approaches.

Approaches

Diagram and label (meaningful letters)

- Write down what you known
- Write down unknown (max / min vertex, zeroes, etc)

Methods

- Quadratic formula, Symmetry (vertex at average of zeroes)
- Factors and roots, discriminant

Equating Coefficients

- Substitution
- Sum / Product of roots
- Ratios (set one value to 1)

Transformations

$-f(x)$ is reflection over x-axis

$f(-x)$ is reflection over y-axis

$f(x) \to af(x)$, scaled vertically by factor of a

$\longrightarrow f(bx)$, scaled horizontally by factor of $\frac{1}{b}$

$\longrightarrow f(x-h)+k$, shifted horizontally to right h units, upwards k units.

When $f(x) = ax^2 + bx + c$

$\alpha f(x)$: roots don't change, $\therefore$ sum of roots $-\frac{b}{a}$

$\therefore$ product of roots $\frac{c}{a}$

$f(\beta x)$: roots affected by b, $\therefore$ sum of roots $-\frac{\beta b}{\beta^2 a} = -\frac{b}{\beta a}$

$\therefore$ product of roots $\frac{c}{\beta^2 a}$

$f(x-h)$. roots shifted right by h, $\therefore$ sum of roots $-\frac{b}{a} + 2h$

$\therefore$ product of roots $\frac{c}{a} - h(-\frac{b}{a}) + h^2$

It is important to realize order of transformations

$f(ax-h) \Rightarrow$ shift right by h units, stretch horiz. by $\frac{1}{a}$

$f(x-h) \Rightarrow f(ax-h)$

$f(a(x-h)) \Rightarrow$ stretch horiz. by $\frac{1}{a}$, shift right by h units

$f(ax) \Rightarrow f(a(x-h))$

Polynomials

The horizontal asymptote or slant asymptote of a rational function is given by the quotient

E.g. $y = \frac{5x+3}{x-4} \Rightarrow x-4 \overline{)5x+3}$ with quotient (5)

$$\begin{array}{r} 5x+3 \\ 5x-20 \\ \hline 23 \end{array}$$

$\lim_{x \to \infty} y(x) = 5$

Remainders

A cubic divided by a quadratic can have a linear remainder

↳ etc. for higher degree polynomials

Remainder Theorem

When function $P(x)$ is divided by $x-k$, the remainder will be $P(k)$

E.g. $P(x) = x^2 + 3x + 2$ divided by $x-3$, the remainder is $P(3) = 9+9+2 = 20$

Quotient and Remainder

When function $P(x)$ is divided by $f(x)$, the following equation will be true

$$P(x) = Q(x) \cdot f(x) + R(x)$$

Polynomial — Quotient — Factor — Remainder

Alternatively, the following is also true...

$$\frac{P(x)}{f(x)} = Q(x) + \frac{R(x)}{f(x)}$$

*with the remainder theorem... if divide by $x-k$...

$P(x) = Q(x) \cdot (x-k) + R(x)$

$P(k) = \cancel{Q(k) \cdot (k-k)} + \underbrace{R(k)}_{\text{only remainder left}}$

useful when asked to find remainder when divided by a different polynomial (but $x-k$ still factor)

Sum and product

in a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

Product of roots: $(-1)^n \frac{a_0}{a_n}$

Sum: $\frac{-a_{n-1}}{a_n}$

Note that repeated roots are counted the number of times they are repeated

E.g. $x^2 + 2x + 1$ ↳ $x = -1$

$\Sigma = -2 = (-1) + (-1)$ counted twice?

$\Pi = 1 = (-1)(-1)$

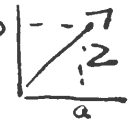
Fundamental Theorem of Algebra

- Every real polynomial of degree n can be factored into n complex linear factors, some of which may be repeated
- Every real polynomial can be expressed as a product of real linear and real irreducible quadratic factors (where $\Delta < 0$)
- Every real polynomial of degree n has exactly n zeros, some of which may be repeated
- If $p+qi$ $(q \neq 0)$ is a zero of a real polynomial then its complex conjugate $p-qi$ is also a zero.
- Every real polynomial of odd degree has at least one real zero

Complex Roots

$Z = a+bi$, $Z^* = a-bi$, $|Z|$ = Modulus / Magnitude of Z

$|Z|^2 = a^2+b^2 = (a-bi)(a-bi)$ $\therefore |Z|^2 = Z \cdot Z^*$



Long Division

$f(x) \div g(x) = ?$

$$\hookrightarrow \quad f(x) \,\overline{)\, g(x)}^{\;a}$$
$$-(a f(x))$$
$$\cdots$$
$r \leftarrow$ remainder

E.g. $(3x^2+2x+1) \div (x+1)$

$$\begin{array}{r} 3x-1 \\ x+1 \,\overline{)\, 3x^2+2x+1} \\ -\,(3x^2+3x) \\ \hline -x+1 \\ -\,(-x-1) \\ \hline 2 \end{array}$$

match leading coefficients and power

can be multiplied by negative number

$\therefore (3x^2+2x+1) \div (x+1) = \underbrace{3x-1}_{\text{quotient}} + \frac{2}{x+1}$ } remainder

Synthetic Division

Linear Divisor:

$$(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0) \div (x - b)$$

when leading coefficient $\neq 1$, divide both by coefficient (e.g. $(5x^2+3x+1)\div(5x+1)$ $\therefore (x^2+\frac{3}{5}x+\frac{1}{5})\div(x+\frac{1}{5})$)

② carry down 1st term

* If $a_x = 0$, then still leave space (0)

$$\begin{array}{c|cccc} b & a_n & a_{n-1} & a_{n-2} \cdots & a_0 \\ & \downarrow & b\cdot a_n & & \\ \hline & a_n & a_{n-1}+b\cdot a_n & \cdots & r \end{array}$$

① opposite from constant in divisor

③ multiply previous result by dividing value (b)

$r \leftarrow$ remainder

④ Add coefficient to product

$$\therefore \underbrace{a_n x^{n-1}}_{} + (a_{n-1}+b\cdot a_n)x^{n-2} \cdots$$

x^{n-1} because $x^n \div x = x^{n-1}$

E.g. $(3x^2+2x+1)\div(x+1)$

$$\begin{array}{c|ccc} -1 & 3 & 2 & 1 \\ & \downarrow & -3 & 1 \\ \hline & 3 & -1 & 2 \end{array}$$

$$\underbrace{3x - 1}_{\text{Quotient}} + \frac{2}{x+1} \text{ remainder}$$

Quadratic Divisor

$$(a_n x^n + a_{n-1}x^{n-1} + \dots + a_0) \div (x^2 + \alpha x + \beta)$$

② carry down 2nd term partially

*same setup as linear

$$\begin{array}{c|cccc} -\alpha \; -\beta & a_n & a_{n-1} & a_{n-2} \cdots & a_0 \\ - & & - & -\beta\cdot a_n & \\ & & a_{n-1} & -\beta\cdot a_n + a_{n-2} & \\ & & -\alpha\cdot a_n & & \\ \hline & a_n & -\alpha\cdot a_n + a_{n-1} & \cdots & r \end{array}$$

④ multiply pre-previous result by dividing constant value

⑤ constant term, after first sum, carry down

③ multiply previous result by dividing value (of power 1)

① carry down 1st term completely

coefficients

⑥ continue...

$r \leftarrow$ remainder

E.g. $(5x^4+2x^3+3x^2+1)\div(x^2+7x+5)$

$$\begin{array}{c|ccccc} -7 \; -5 & 5 & 2 & 3 & 0 & 1 \\ & & \downarrow & -25 & 165 & -1045 \\ & - & 2 & -22 & 165 & 1044 \\ & \downarrow & -35 & 231 & -1463 & \downarrow \\ \hline & 5 & -33 & 209 & -1298 & 1044 \end{array}$$

$$\therefore \underbrace{5x^2 - 33x + 209}_{\text{Quotient}} + \underbrace{(-1298x - 1044)}_{\text{remainder}}/(x^2+7x+5)$$

Symmetric Polynomial

"A polynomial is symmetric if its value doesn't change when you permute its variables."

$x^2+y^2+z^2 = y^2+x^2+z^2$ symmetric

$x-y \neq y-x$ not symmetric

Elementary Symmetric Polynomials

1st: $e_1(x_1, x_2, \dots, x_n) := x_1 + x_2 + x_3 + \cdots + x_n$

2nd: $e_2(x_1, \dots, x_n) := x_1x_2 + x_1x_3 + \cdots = \sum x_i x_j$

3rd: $e_3(x_1, \dots, x_n) := x_1x_2x_3 + x_n x_2 x_{n-1} + \cdots = \sum x_i x_j x_k$

$\vdots$

nth: $e_n(x_1, \dots, x_n) := x_1 x_2 x_3 \cdots x_n$

Newton's Theorem

"Any power-sum polynomial in n variables can be expressed using elementary symmetric polynomials."

E.g.

$$x^3+y^3+z^3 = (x+y+z)^3 - 3(x+y+z)(xy+xz+yz) + 3(xyz)$$
$$= e_1^3 - 3e_1e_2 + 3e_3$$

Stick and Stone Counting

Theorem 1

n items into k groups (where each group must have at least 1 element)

each group is distinct (1 item in group A is different than 1 item in B)

$\#$ of possibilities $= \binom{n-1}{k-1}$

$nC_r = \frac{n!}{(n-r)!r!}$

$\underbrace{\bullet \bullet \bullet \bullet \bullet \bullet}_{n \text{ items}}$

into k groups so $k-1$ dividers $\Rightarrow \cdots | \cdot | \cdots$

Since $k-1$ dividers must be between 2 items, there are $n-1$ spots

$n-1$ choose $k-1$ ($n-1$ spots to choose from for $k-1$ items)

Theorem 2

n items into k groups (where a group may have no elements)

$\#$ of possibilities $= \binom{n+k-1}{k-1}$

[Theorem 1 Proof]

Stick and stone counting

[Theorem 2 proof]

n elements divided into k groups

↳ k groups so $k-1$ dividers

Theorem 2 for when group can have 0 elements

$\bullet\bullet\bullet|\bullet\bullet\bullet|\bullet$ OR $||\bullet\bullet\bullet\bullet\bullet\bullet\bullet$

group 1 group 2 group 3 1 2 group 3

There are $n+k-1$ symbols, you must arrange $k-1$ dividers with n items for $n+k-1$ spots in total

$\therefore$ $n+k-1$ choose $k-1$

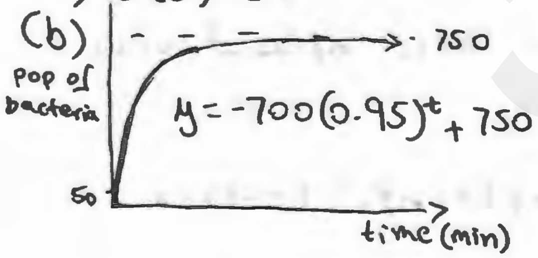
Total spots to fit dividers

Graphing Display Calculator Practice

1) $G(t) = -700(0.95)^t + 750$

(a) $G(0) = 50$

(b)



(c) $G(20) \approx 499$ bacteria

(d) $\lim_{x \to \infty} G(t) = 750$ bacteria

(e) $G_2(t) = -700(0.95)^{\frac{t}{2}} + 750$

(f) $G_3(t) = -700(0.95)^{2t} + 750$

(g) $G_3^{-1}(600) \approx 15.016$

2) $f(x) = e^x \Rightarrow \begin{pmatrix} -p \\ q \end{pmatrix}$ $g(x) = e^{x+p} + q = e^x e^p + q$

$g(0) = 1.5 = e^p + q$

$g(\ln(2)) = 2 = 2e^p + q$

$e^p = 0.5 \Rightarrow p \approx -0.693$

$q = 1$

3) Emily cycles 20km/hr for 15mins, t hrs for 15 km/hr

a) average velocity $= \frac{5+15t}{\frac{1}{4}+t} = \frac{20+60t}{1+4t}$

3) b) $s'(t) = \frac{-t+70}{4t-60}$ c) $s^{-1}(1[illegible]) = 1$

4) a) $P(x) = (2+\pi)x + 2$

b) $x \in (0, 1m]$; $P(x) \in (2, \pi+4\ m]$ c) $P^{-1}(x) = \frac{1}{\pi+2}x - \frac{2}{\pi+2}$

d) 7 m

[illegible] $r(24) = 4\sqrt{[illegible]} \approx 17.36[illegible]$

c) The area, $C^{-1}(2\pi)$, is π when the circumference is 2π

6. a) $s(t)$ is position, so $s(t)$ is cont.

continuity: $\lim_{t \to 6.5} s(t) = s(6.5) \approx 8.4[illegible]95$

($\lim_{t \to 6.5} s(t)$ exists) $\lim_{t \to 6.5} s(t) = a(6.5)^2$

$= s(6.5)$

$a \cdot 6.5^2 = s(6.5)$

$a = 8.429 \div (6.5)^2$

$= 0.1995$

b) same logic as a), cont. if $\lim_{t\to 10} s(t)$ exists and $= s(10)$

$\lim_{t \to 10^+} s(t) = s(10) = 10m - 62$

$\lim_{t \to 10^-} s(t) = 23$

$\therefore 23 = 10m - 62$ ($\lim_{t\to 10^+} s(t) = s(10) = \lim_{t \to 10^-} s(t)$)

$m = \frac{[illegible]}{[illegible]} = 8.5$

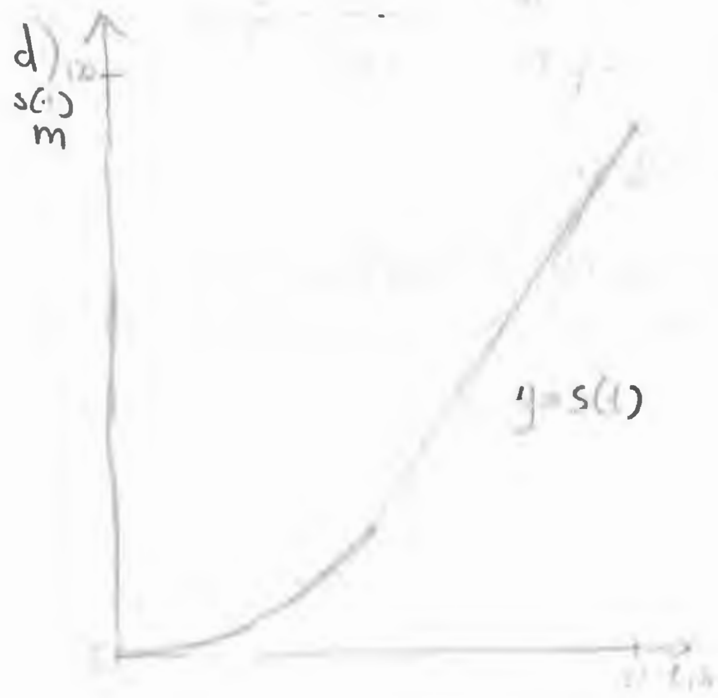
c)

$$s(t) = \begin{cases} 0.1995t^2, & 0 \le t < 6.5 \\ 0.1[illegible]t^2 + 2.4[illegible]t - 12, & 6.5 \le t < 10, \\ 8.5t - 62, & t \ge 10 \end{cases}$$

$s^{-1}(15m) \approx 8.153\ s$

$s^{-1}(200m) \approx 30.824\ s$

d)



e) $s'(t)$ greatest when

$t > 10$

f) $s'(11) = 8.5\ ms^{-1}$

$= 30.6\ kmh^{-1}$

7. a) $10 = \frac{\pi}{3}r^2h$

$= \frac{\pi}{3}r^2 \times \pi r h$

$\therefore h = \frac{180}{\pi r^2}$

$k = rh r^2 = 180$

b) $r = 3.855$ mm

8. a) i) $a + b + c = 94.3\text{m}$ ii) $a = -2.6$
$4a + 2b + c = 93.9\text{m}$ $b = 17.4$
$64a + 8b + c = 42.3\text{m}$ $c = 69.5$

b) 69.5m c) $t \approx 9.59$

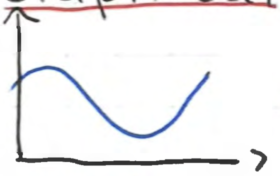
9. a) i) $A = (5, 3) + 3 = (17, 0)$
b) $AB = \sqrt{3^2 + 12^2} \approx 12.4$
c) i) $\frac{1}{4}$ ii) $y - 5 = 4(x - 9)$
d) $D = (8, 1)$
e) $CD = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12$ f) $2\sqrt{17} \cdot \sqrt{17} \cdot \frac{1}{2} = 25.5$

10. a) $1.4 \times 10^6 \text{ cm}^2$ b) 5.26%

$$P(k) = \begin{cases} 8k & \text{for } 0 < k \le 25 \\ 6k + 50 & \text{for } k > 25 \end{cases} \quad \text{where } k \in \mathbb{Z}$$

Multiple ways to interpret functions.

1. Graphical



graphs

interchangeable

2. Numerical

x	$f(x)$
0	2
	4
1	
2	6

input/output table

3. Analytical

$f(x) = x^2$

functions

4. Verbal

...models the amount of people...

word problem/explanation

Exponential Functions

$$f(x) = a \cdot b^{dx-h} + c$$

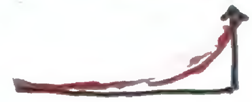
- Initial Value / y-intercept $= a + c = f(0)$

b = growth/decay factor, i.e. for each increase in x, y is changed by a factor of b

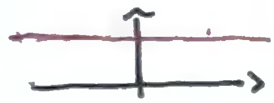
$$f_n = f_{n-1} \cdot b$$
$$f_0 = a + c$$

- $b < 1$, exponential decay

- $b > 1$, exponential growth



- $b = 1$, horizontal line



horizontal asymptotes: $\lim_{x\to\infty} f(x)$ for $b<1$, always

$\lim_{x\to-\infty} f(x)$ for $b>1$, always

$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$ for $b=1$, always

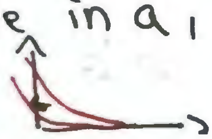
horizontal asymptote $= \lim_{x\to\pm\infty} f(x) = c$

Δb = change in b = growth/decay factor change

Δa = change in a = vertical stretch

decrease in a, "flatter graph", vice versa

change sign in a, reflection by x-axis

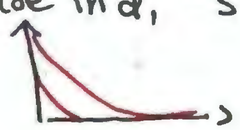


y-intercept changes

Δd = change in d = horizontal stretch

change sign in d, reflection by Y-axis

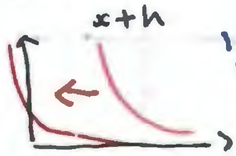
increase in d, "slimmer graph", vice versa



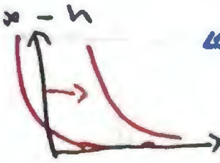
y-intercept does not change

Δh = change in h = horizontal shift

horizontal asymptote does not change

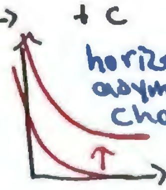


like parabolas

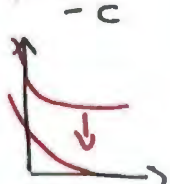


Y-intercept changes

Δc = change in c = vertical shift

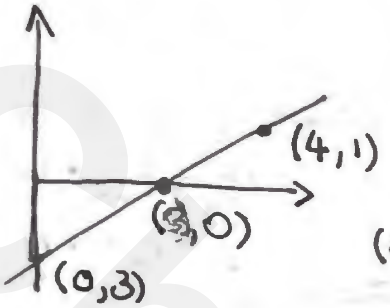


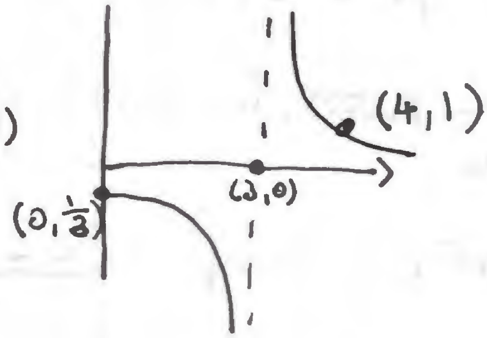
horizontal asymptote changes



Finding reciprocal of functions

E.g. $f(x) = x - 3 \quad \therefore \quad \frac{1}{f(x)} = \frac{1}{x-3}$





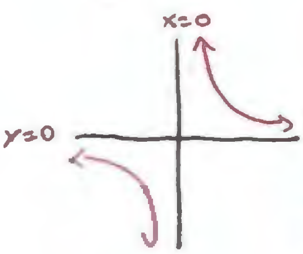
(4,1) is an example of an <u>invariant point</u>

any point that stays the same after a set of mathematical transformations or operations

- A local <u>min</u>. of $f(x)$ is a local <u>max</u>. in $\frac{1}{f(x)}$
- A local <u>max</u>. of $f(x)$ is a local <u>min</u>. in $\frac{1}{f(x)}$

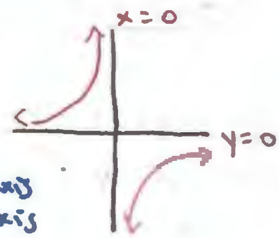
The x-intercept (i.e. (3,0)) of the original function is the vertical asymptote of the reciprocal (i.e. $x=3$) and vice versa if applicable

$y = \frac{1}{x}$

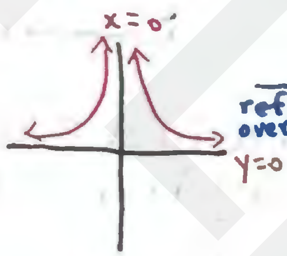


reflect over x-axis or y-axis

$y = -\frac{1}{x}$

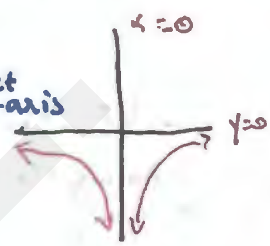


$y = \frac{1}{x^2}$

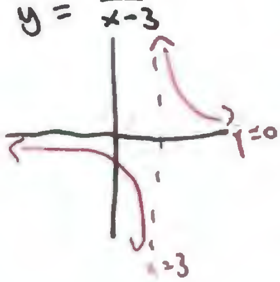


reflect over x-axis

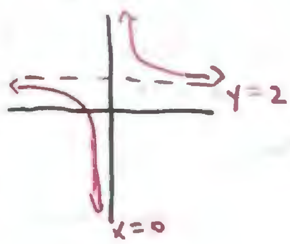
$y = -\frac{1}{x^2}$



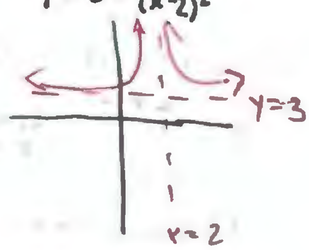
$y = \frac{1}{x-3}$



$y = \frac{1}{x} + 2$



$y = 3 + \frac{1}{(x-2)^2}$



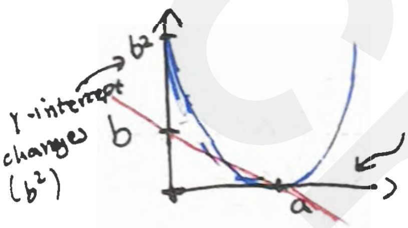
Squaring a function

E.g. $y = f(x) \Rightarrow [f(x)]^2$

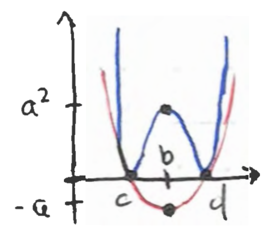
if $f(x)$ is linear, then $[f(x)]^2$ is quadratic...

if $f(x)$ is quadratic, then $[f(x)]^2$ is quartic...

etc...



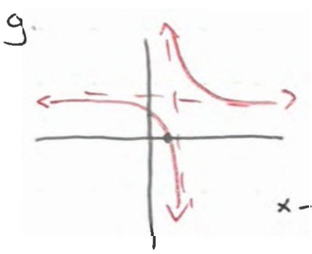
x-intercepts stay the same



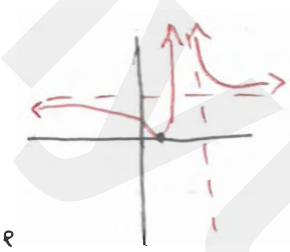
- Always ≥ 0
- instead of crossing x-axis, function touches
- y intercept squared

Sometimes the new function may "cross" the asymptote

E.g.

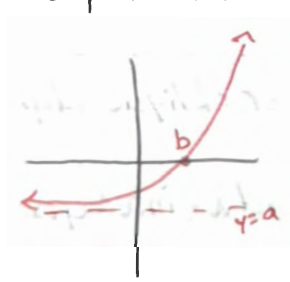


$\Rightarrow$

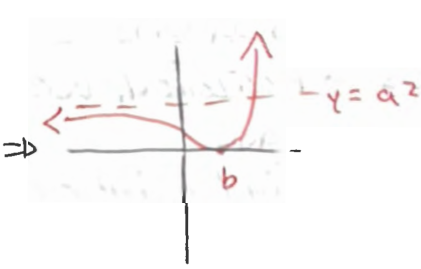


x-int stay the same

or exponential functions



$\Rightarrow$



Graphing Linear / Quadratic functions

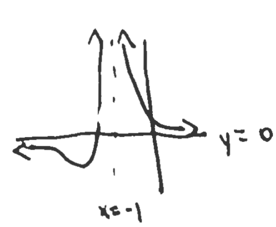
remember to include asymptotes (vertical) and x-intercepts

- Make a sign chart!!
- intercepts
- Asymptotes

E.g. $\dfrac{3x+6}{x^2+2x+1}$

	-2		-1	
−		+		+

$\therefore$

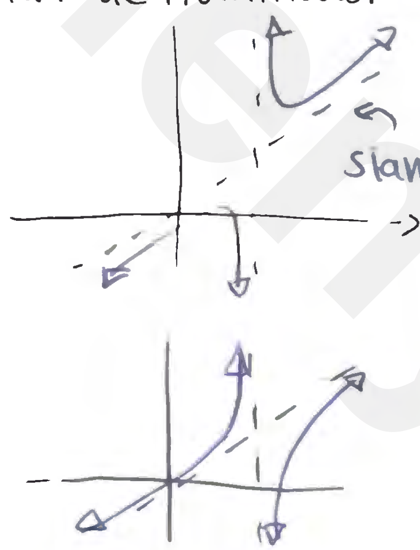


Asymptotes:

- Horizontal Asymptote = $\lim_{x \to \pm\infty} f(x)$
  - $\Rightarrow$ Compare coefficients (when top power $\leq$ bottom power)

- Vertical Asymptote, zeroes of denominator

- Slant asymptote, divide top by bottom
  - Oblique asymptote"
  - is $px+q$. when rational written as $px+q+\frac{r}{x-a}$
  - Exists when numerator is one degree larger than denominator



Slant asymptote

Two types of oblique asymptote

Graphing Rational Functions

1. Find asymptotes - Horizontal, vertical / or oblique depending on the case
2. Find axes intercepts - Find y-intercept and x-intercepts by substituting 0 for x and y
3. Draw a sign chart - Sign chart describes behaviour of graph
   - will approach $\pm\infty$ $\Rightarrow$ i. must have vertical asymptotes (when $f(x)$ is undef.)
   - either crosses or touches x-axis $\Rightarrow$ ii. must have x-intercepts (when $f(x)$ is 0)
4. Plot points and asymptotes; sketch rational function

Partial Fractions

Just like fractions involving numbers, fractions involving algebra can be added by writing them with a common denominator

E.g. $\frac{3}{x-2} + \frac{4}{x+3} = \frac{3}{x-2} \times \frac{x+3}{x+3} + \frac{4}{x+3} \times \frac{x-2}{x-2}$

$$= \frac{3(x+3)+4(x-2)}{(x+3)(x-2)} = \frac{3x+9+4x-8}{x^2+3x-2x-6} = \boxed{\frac{7x+1}{x^2+x-6}}$$ rational function

We can also reverse this process turning a rational function to a sum of partial fractions

E.g. $\frac{7x+1}{x^2+x-6} = \frac{7x+1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

$\therefore 7x+1 = A(x+3) + B(x-2)$

substitute in zeroes $x = 2, -3$

$\hookrightarrow x = 2 \Rightarrow 7(2)+1 = A(2+3)$

$A = 3$

$\hookrightarrow x = -3 \Rightarrow 7(-3)+1 = B(-3-2)$

$B = 4$

$\therefore \frac{7x+1}{x^2+x-6} = \frac{3}{x-2} + \frac{4}{x+3}$

⇐ for the rational function to be able to be converted into partial fractions. A and B must be numbers

This process can be repeated for higher degrees (of the polynomial in the denominator)

E.g. $\frac{33-12x}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

Properties of Rational Functions

$\frac{f(x)}{g(x)}$

1. Vertical Asymptote, roots of $g(x)$
2. Horizontal Asymptote
   i. if degree of $f(x) > g(x)$, then no H.A.
   ii. if degree of $f(x) < g(x)$, then H.A. is $y=0$
   iii. if degree of $f(x) = g(x)$, then H.A is $y = \frac{\text{leading coeff. of } f(x)}{\text{leading coeff. of } g(x)}$
3. Oblique /Slant Asymptote
   Exists when numerator exactly 1 degree larger than denominator
   when $r(x)$ is written as $px+q+\frac{r}{x-a}$, $px+q$ is asymptote

4. x-intercepts = roots of $f(x)$

5. y-intercept = $\frac{\text{y-int of } f(x)}{\text{y-int of } g(x)}$

Sequence

- A sequence is a discrete function such that the domain is purely integers (either positive or natural usually)
- Instead of writing "$f(3)$", we write "a_3" or "u_3", etc.
  - Using subscript instead of parenthesis
  - "a_3" is an index, "a_3, a_4, a_5" are indices
- We use n instead of x to denote natural/integer numbers

Recursive definition of a sequence

$a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$

so... $a_3 = a_1 + a_2 = 1 + 1 = 2$, etc.

Explicit definition

If we have a recursive definition of a sequence we should be able to convert it into an explicit function

E.g. $a_1 = 2$, $a_{n+1} = a_n + 7$

then

$a_n = 2 + (n-1)\underbrace{7}$ where n is a positive $\mathbb{Z}$

7 is common difference (linear)

Main Patterns

1) Arithmetic

E.g. 5, 8, 11, 14 . . .

The common difference must always be the same

$k = 8-5 = 14-11 = 11-8 = \ldots$

2) Geometric

E.g. 2, 10, 50, 250 . . .

The common ratio must always be the same $k = \frac{10}{2} = \frac{50}{10} = \frac{250}{50}$

Sequences and Series

Sequence is a LIST whereas a series = Sum

Arithmetic Series

$a_n = a_1 + (n-1)d$

S_n = Sum of terms from 1 to n = mean × n
$= \left(\frac{a_1 + a_n}{2}\right) n$

Sigma notation

$$\sum_{i=a}^{n} (f(i))$$

$= f(a) + f(a+1) + \cdots + f(n)$

$$= \underbrace{\frac{f(a) + f(n)}{2}}_{\text{average}} \cdot \underbrace{(n - a + 1)}_{\text{\# of terms}}$$

for arithmetic only

Geometric Sequences

$a_n = a_1 (r)^{n-1}$

conversion is necessary

* The S_n formula is only applicable for $a_1 r^{n-1}$, $a_0 r^n$ does not work

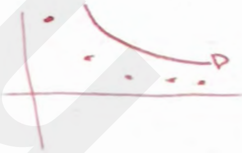
S_n = sum of terms from 1 to n = $a_1 \times \left(\frac{1 - r^n}{1 - r}\right)$

such that a_1 = first term, r = common ratio, n = # of terms

if $|r| < 1$, the infinite geometric series

$$\sum_{n=1}^{\infty} a_1 (r)^{n-1} = \frac{a_1}{1-r}$$

converges



Arithmetic mean $= \frac{a+b}{2}$

∴ in arithmetic sequence a, x, b; $x = \frac{a+b}{2}$

Geometric mean $= \sqrt{ab}$

∴ in geometric sequence a, x, b; $x = \sqrt{ab}$

* Proof that an infinite arithmetic of distinct primes may not exist:

$a_n = a_1 + nd$

a_1 is prime ⇒ but $a_{a_1} = a_1 + a_1 \cdot d$
$= \underbrace{a_1}_{}(1 + d)$

no longer prime b/c factor

Financial Sequence

When money is lent or borrowed, the lender usually charges a fee called interest to the borrower ("the cost of using the other person's money")

There are two main types of interest

Simple Interest

- The interest charge on borrowing calculated solely with an original principle amount and an interest rate that never changes
- Modeled by a linear equation and an arithmetic sequence
- Does not involve compounding

$$A = P(1 + rt)$$

where A is final amount, P is initial principle balance, r is interest rate per interval, t is time intervals

Compound Interest

- Interest generated in one period will itself earn more interest in the next period
- Modeled by an exponential equation and a geometric sequence

$$u_n = u_0(1 + i)^n$$

where u_n is final amount, u_0 is initial investment, i is interest rate per compounding period, n is time periods

*Nominal Values are also a result of compounding

$$\text{Nominal} = \text{Real value} \times (1 + \text{inflation per year})^{\#\text{ of years}}$$

given depreciation rate

Depreciation refers to loss of value over time: $a_n = a_0(1 - d)^n$

Appreciation/inflation refers to gain in nominal value over time: $a_n = a_0(1 + i)^n$

inflation rate

Series

A series is the sum of the terms of a sequence

- For a finite sequence with n terms
  $S_n = u_1 + u_3 + u_3 + \cdots + u_n$ and will always be a finite
- For an infinite sequence
  The sum sometimes may not be calculated. However, other times it may converge to a finite number

Sigma Notation

$u_1 + u_2 + u_3 + \cdots + u_n$ can be written more compactly using sigma or summation notation ...

"$\sum$" is called sigma, equivalent of capital S in Greek alphabet.

- We write $u_1 + u_2 + u_3 + \cdots u_n$ as $\sum_{k=1}^{n} u_k$
  $\sum_{k=1}^{n} u_k$ reads "the sum of all numbers of the form u_k where $k = 1, 2, 3, \cdots n$"

E.g.

$S_n = 1 + 4 + 9 + 16 + 25 + \cdots$, $S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$
$= \sum_{k=1}^{n} k^2$

Properties of Sigma Notation

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

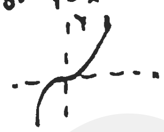
If c is a constant ...

$$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k \quad \text{and} \quad \sum_{k=1}^{n} c = cn$$

Further Functions

Odd Functions - Functions that have rotational symmetry about the origin
- $f(x) = -f(-x)$

E.g. $y = x^3$



Even Functions - Functions that are symmetrical about the y-axis
- $f(x) = f(-x)$

E.g. $y = x^2$

Absolute Value

$|x|$ is a piecewise function

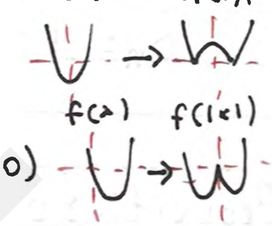
$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

it can also be expressed as $|x| = \sqrt{x^2}$
since $x^2 \geq 0$ *and* $\sqrt{x^2} = x$ *when* $x \geq 0$

Transformations

$y = |f(x)|$ is just $f(x)$ but ALWAYS positive

$y = f(|x|)$ is just $f(x)$ but the right side $(x>0)$ is MIRRORED to the left



Solving Absolute Value equations/inequalities
- Similar to solving other inequalities, set two sides equal
  E.g. $|5x+1| < 4 \Rightarrow |5x+1| = 4$

EQUATION
- Since absolute value is a piecewise function where anything neg. becomes pos
  E.g. $5x+1 = 4$ OR $5x+1 = -4$
- Solve as normal: $x = \frac{3}{5}$, $x = -1$
- Set up a sign chart and solve (try in values)

Inverse Functions

An inverse of a function is a function that undoes the operation the original function

The inverse of f exists if and only if f is bijective (one to one correspondence)

- If not then the domain of a function must be restricted
  E.g. $f(x) = x^2 \Rightarrow f^{-1}(x) = \sqrt{x}$ where domain $f(x)$ is positive

To solve for inverse:

1) Swap x and y
2) Distribute if necessary
3) Isolate all "y" to one side and "x" to another with constants
4) Combine like terms, factor out "y" and divide if necessary

For quadratics...

4) Combine like terms and convert into vertex form to square root both sides if possible

OR...

4) Combine like terms and use quadratic formula to solve for "y" (it's okay to have x as part of the coefficients

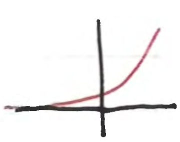
To find the range of the function just find the domain of the inverse!

i.e. in quadratics, check discriminant

- Look for even roots, denominators, logarithms

Inverses are just original functions graphed with x and y switched!

E.g. $y = e^x$



$y = \ln x$

Self inverse functions are functions that satisfy the condition

$$f(x) = f^{-1}(x)$$

Composite Functions

$$(f \circ g)(x) = f(g(x)) = f \text{ of } g \text{ of } x$$

$$f(g(x)) = x \xrightarrow{\text{in}} \boxed{g(x)} \xrightarrow{\text{out} \quad \text{in}} \boxed{f(x)} \xrightarrow{\text{out}}$$

INNER When given $f(x) =$ __, $f(g(x)) =$ __ and you wish to solve for $g(x)$

① just substitute x for $g(x)$ and equate it to the given of $f(g(x))$

E.g. $f(x) = \frac{3}{x}$, $f(g(x)) = 3x$

$\therefore f(g(x)) = 3x = \frac{3}{g(x)}$,

solve for $g(x)$, $g(x) = \frac{1}{x}$

OR...

② Solve for $f^{-1}(f(g(x)))$

E.g. $f(g(x)) = \frac{2x}{x+1}$, $f(x) = \frac{2x-5}{x-2}$

$g(x) = f^{-1}\left(\frac{2x}{x+1}\right) = \frac{x+5}{2}$

OUTER When given $g(x) =$ __, $f(g(x)) =$ __ and you wish to solve for $f(x)$

$g(g^{-1}(x)) = x$; $(f \circ g \circ g^{-1})(x) = f(x)$

Find $g^{-1}(x)$ and plug in $f(g(g^{-1}(x)))$ to obtain $f(x)$

E.g. $f(g(x)) = 2x+3$, $g(x) = x^{\frac{1}{3}}$

$\therefore g^{-1}(x) = x^3$; $f(x) = f(g(g^{-1}(x))) = 2x^3+3$

Inequalities

To solve inequalities, first set the two sides equal:

$x^2-2x-4<-1 \Rightarrow x^2-2x-4=-1$

Solve the equation as normal ← in the case of rationals, look out for vert. asymp.

$x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0$

Set up a sign chart by putting zeroes and asymptotes (undef.) and plugging in values from the intervals between them

$x\in$ $(-\infty,-1)$	-1	$(-1,3)$	3	$(3,\infty)$
false	0	inequality true	0	false

$\therefore$ we know $x^2-2x-4<-1$ is only true for $x\in(-1,3)$

Exponent Rules

- $x^a \cdot x^b = x^{a+b}$
- $x^a \div x^b = \frac{x^a}{x^b} = x^{a-b}$
- $x^a \cdot y^a = (x\cdot y)^a$
- $x^a \div y^a = \frac{x^a}{y^a} = (\frac{x}{y})^a = (x\div y)^a$
- $\frac{1}{a^x} = a^{-x}$
- $\sqrt[x]{a} = a^{\frac{1}{x}}$
- $(a^b)^c = a^{b\cdot c}$ not to be confused with $(a)^{(b^c)}$

Exponentiation

Multiplication is not repeated addition

E.g. 4 mangoes $\times$ 2 mangoes $\neq$ 4 mangoes + 4 mangoes

$=$ 8 mangoes^2

Similarly with exponents

if $a^x = a\cdot a\cdot a\cdot a \cdots a$

then what is $a^{0.1}$, a^{π} ?

Solving for x when x is an exponent

Using exponent rules, get both sides to same base

(i.e $a^{f(x)} = a^b$)

Equate Powers

E.g. $9^{x-1}\cdot \frac{1}{27^x} = \sqrt[2]{3}$

$3^{2x-2}\cdot 3^{-3x} = 3^{\frac{1}{2}} = 3^{2x-2-3x} \Rightarrow -x-2 = \frac{1}{2} \Rightarrow x = -\frac{5}{2}$

Finding Euler's constant

We use euler's constant $e \approx 2.718$ a lot, in e^x or $\ln x = \log_e x$

- This is because...

  e^{bx} has the property such that the rate at which e^{bx} grows is $b \cdot e^{bx}$

  (grows b times)

- In other words...

$$\frac{d}{dx}\left[e^{bx}\right] = b \cdot e^{bx} \text{ for } b \text{ is a constant}$$

This will not work for any other base

E.g. $f'(x) = 2^{0.3x}$ $\quad f'(x) = e^{0.3x}$

$f'(0) \approx 0.2079$ ✗ $\quad f'(0) = 0.3$ ✓

- e is defined to be...

$$\boxed{e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n \text{ and } e^x = \lim_{n \to \infty}\left(1 + \frac{x}{n}\right)^n}$$

Exponential Function Forms

Form 1:

$$P(t) = P_0 \cdot a^t \Rightarrow P(t)\ P_1\ a^{t-1}$$

a = growth rate per unit of time (Discrete)
t = time
P_0 = initial value

Form 2:

$$P(t) = P_0 \cdot b^{\frac{t}{n}} \Rightarrow P(t) = P_0 \cdot 2^{\frac{t}{n}},\ n \text{ is doubling time}$$
$$P(t) = P_0 \cdot \left(\tfrac{1}{2}\right)^{\frac{t}{n}},\ n \text{ is half life}$$

b = growth rate per n units of time
t = time
n = time it takes for P to grow/decay by factor of a
P_0 = initial value

Form 3:

$$P(t) = P_0 \cdot e^{kt} \Rightarrow k = \ln(a)$$

k = continuous growth rate = ln (growth rate per unit of time)
t = time
P_0 = initial time

Conversion between forms

$$P(t) = \begin{cases} P_0 \cdot a^t \\ P_0 \cdot b^{\frac{t}{n}} = P_0 \cdot (b^{\frac{1}{n}})^t \\ P_0 \cdot e^{kt} = P_0 = (e^k)^t \end{cases}$$

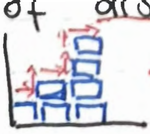
$a = b^{\frac{1}{n}}$; $a = e^k$; $b^{\frac{1}{n}} = e^k$

$n = \frac{1}{\log_b a}$; $k = \ln a$; $n = \frac{1}{\log_b e^k}$

$b = a^n$; $k = \ln(b^{\frac{1}{n}})$

Discrete Growth

- Change happening after a specific period of time
- Thought of discretely

E.g. $y = 2^x$  Doubles after set period of time

Continuous Growth

- Change happening constantly (Compounded infinitely) E.g. $e^{\ln(2)\cdot x}$
- Always in motion, thought of continuously

 ln(2) infinitely compounded

Logarithms

$$\log_a x = m, \quad a>0,\ x>0,\ a\neq 1$$

or

a to the m power is x

or

$$a^m = x$$

Logarithm rules

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^m = m \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

PROOF that $\log_a x = \dfrac{\log_b x}{\log_b a}$

$\log_a x = k$

$\therefore a^k = x$

$\log_b a^k = \log_b x$

$k \cdot \log_b a = \log_b x$

$\therefore \log_a x = \dfrac{\log_b x}{\log_b a}$

Linearization

You can convert any exponential into a linear function by taking the corresponding logarithm to all y values of the original function.

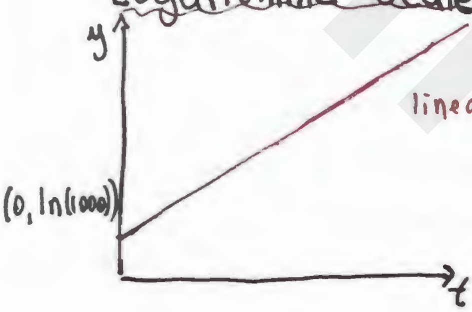
E.g.

$$f(x) = 1000 \cdot e^{0.035t} \quad \text{exponential}$$

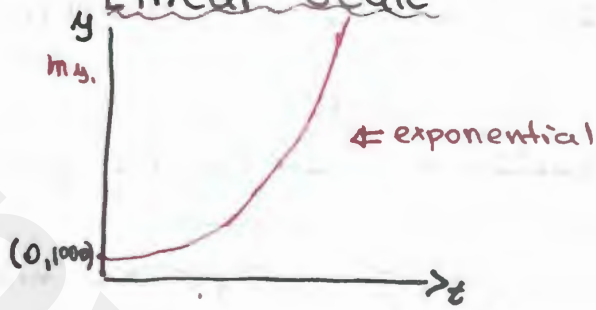
$$g(x) = \ln(f(x)) = \ln(1000 \cdot e^{0.035t}) \quad \text{linear}$$

$$= \ln 1000 + 0.035t$$

Logarithmic Scale (y)



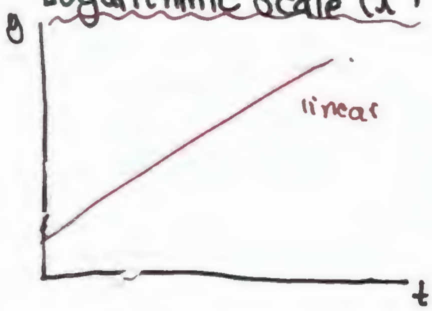
Linear Scale



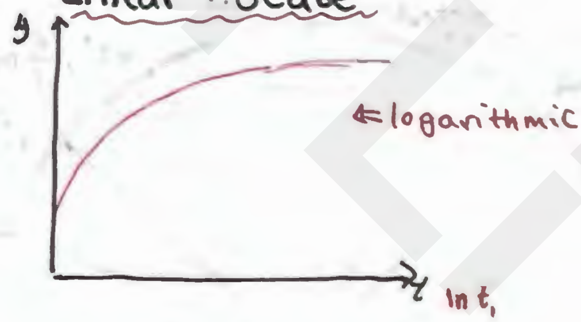
- Exponential Increments
  E.g. e, e^2, e^3, etc.
  1, 10, 100, etc.

- Linear Increments
  E.g. 1, 2, 3, 4, 5, etc.
  e, $2e$, $3e$, $4e$, etc.

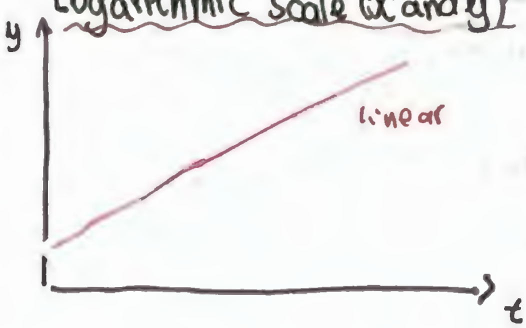
Logarithmic Scale (x)



Linear Scale



Logarithmic scale (x and y)



Linear Scale

Trigonometry

Radians

- One radians is defined to be the ratio between arclength and radius

$\therefore\ \theta = \dfrac{\text{arclength}}{\text{radius}}$

There are multiple ways to describe angles

degrees (°) gradians (^{g}) radians

Reference angle

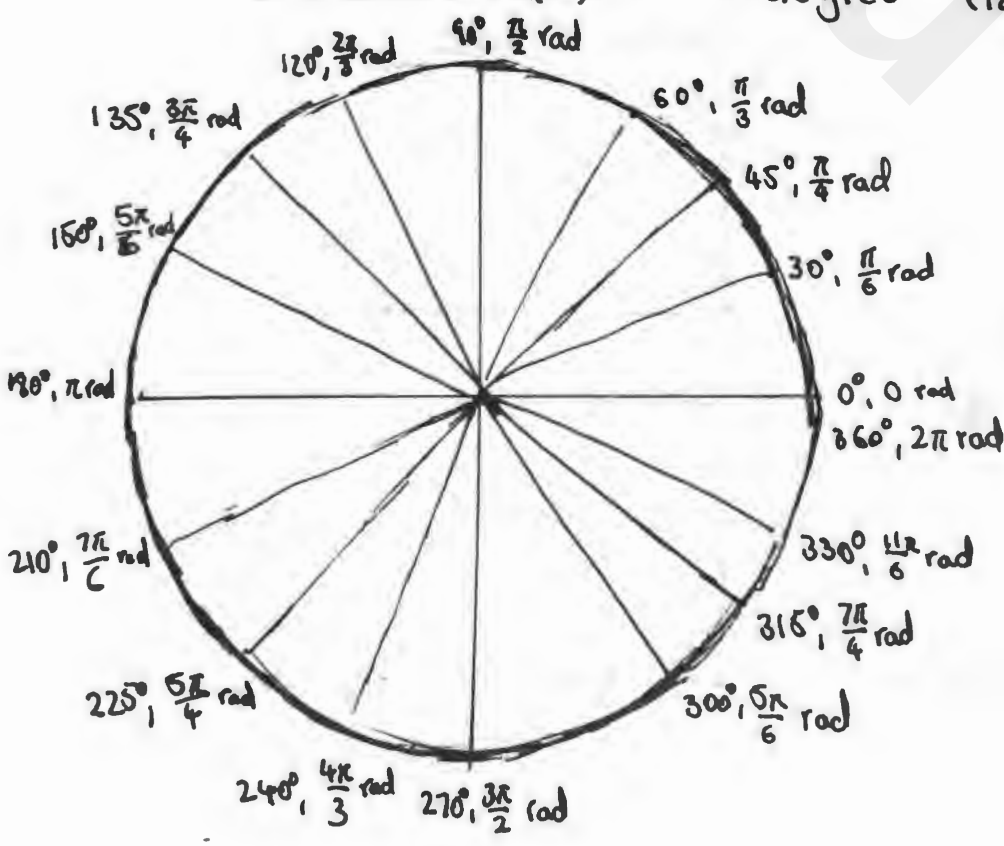
- Equivalent angle in first quadrant

E.g.

α=reference angle

Conversion between radians and degrees

$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ}$ $\quad$ $1 \text{ degree} = \left(\frac{\pi}{180}\right) \text{radians}$



Angles that represent the same "angle" in a circle but represent different rotations are "coterminal" (e.g. 0 rad, 2π rad)

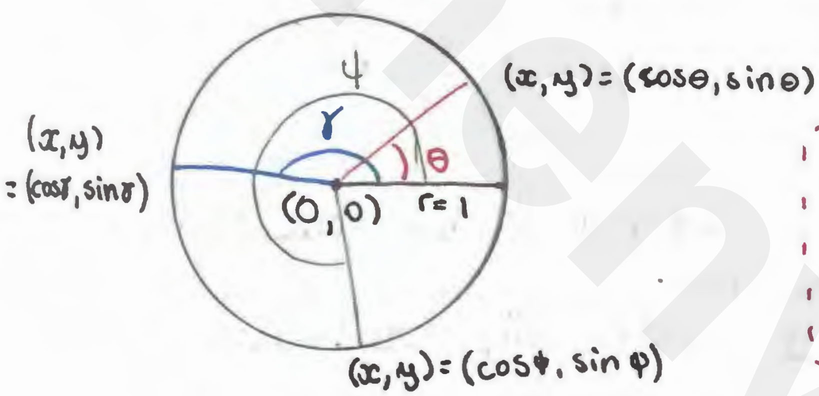
Trigonometric functions

$\sin\theta$ and $\cos\theta$ simply represent how far up or right a given point is (at angle θ to center) in a circle, with respect to the circle's radius.

In other words...

$\sin\theta$ is the y-value of a point on a unit circle (centered $(0,0)$)

$\cos\theta$ is the x-value of that same point



As such, the (x,y) values for any circle, with radius r, can be expressed as:
$(x,y) = (r\cos\theta,\ r\cdot\sin\theta)$

Deriving trig. identities

By the equation of a circle of radius r...

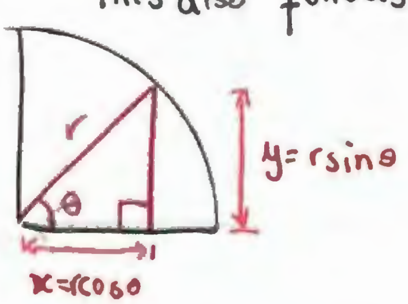
$$x^2 + y^2 = r^2$$

...we can rewrite as

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2 \quad \text{...or...} \quad \cos^2\theta + \sin^2\theta = 1$$

Alternatively

This also follows from the pythagorean theorem



$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ \cancel{r^2} &= \cancel{r^2}\cos^2\theta + \cancel{r^2}\sin^2\theta \\ 1 &= \sin^2\theta + \cos^2\theta \end{aligned}$$

Trigonometric formulas

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

$$\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}}$$

$$\boxed{\begin{aligned}\sec\theta &= \frac{1}{\cos\theta}\\ \csc\theta &= \frac{1}{\sin\theta}\end{aligned}}$$

Following these properties

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \text{AND} \quad \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

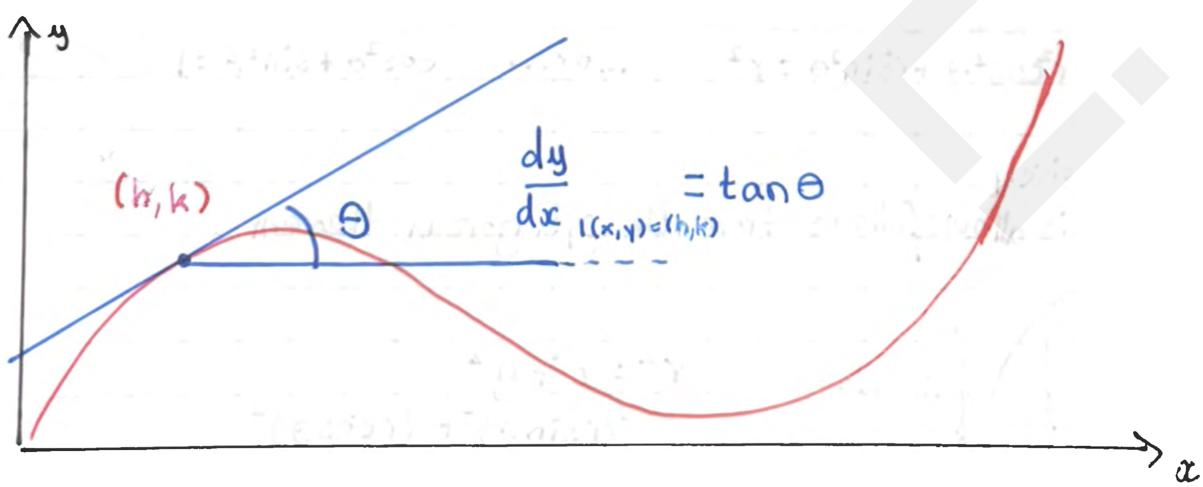
$$\boxed{1 + \cot^2\theta = \csc^2\theta} \qquad \boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Tangent

The tangent function is defined to be the ratio between sine and cosine.
As such, the following relationship can be modeled.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\Delta y}{\Delta x}$$

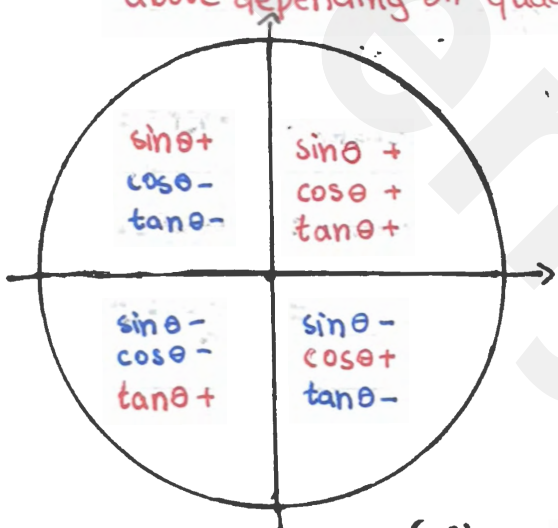
In this case, $\tan\theta$ can be modeled as the slope of a curve at a given interval or point



Important Trig Values

$\cos 0° = \cos(0) = 1$
$\cos 15° = \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$
$\cos 30° = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
$\cos 45° = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
$\cos 60° = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\cos 75° = \cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$
$\cos 90° = \cos\left(\frac{\pi}{2}\right) = 0$

... The rest of circle are just positive/negative of values above depending on quadrants



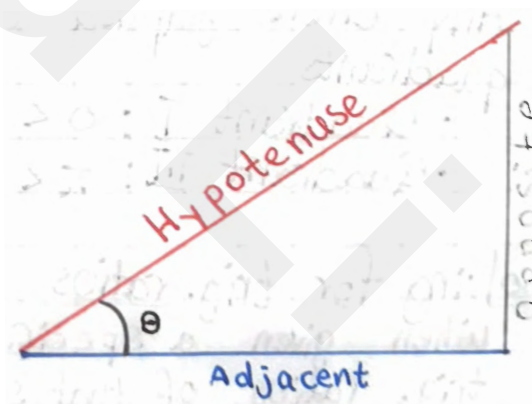
$\sin 0° = \sin(0) = 0$
$\sin 15° = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$
$\sin 30° = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
$\sin 45° = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
$\sin 60° = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$\sin 75° = \sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$
$\sin 90° = \sin\left(\frac{\pi}{2}\right) = 1$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, the following trig values can also be derived...

$\tan 0° = \tan(0) = 0$
$\tan 15° = \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$
$\tan 30° = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$
$\tan 45° = \tan\left(\frac{\pi}{4}\right) = 1$
$\tan 60° = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$
$\tan 75° = \tan\left(\frac{5\pi}{12}\right) = 2 + \sqrt{3}$
$\tan 90° = \tan\left(\frac{\pi}{2}\right) \rightarrow \infty$

Trig. in a triangle:

$$\begin{cases} \sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \\ \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \\ \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} \\ \sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}} \\ \csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}} \\ \cot\theta = \frac{\text{Adjacent}}{\text{Opposite}} \end{cases}$$



Unit Circle Identities

$\sin(-\theta) = -\sin(\theta)$
$\cos(\theta) = \cos(-\theta)$
$\tan(-\theta) = -\tan(\theta)$

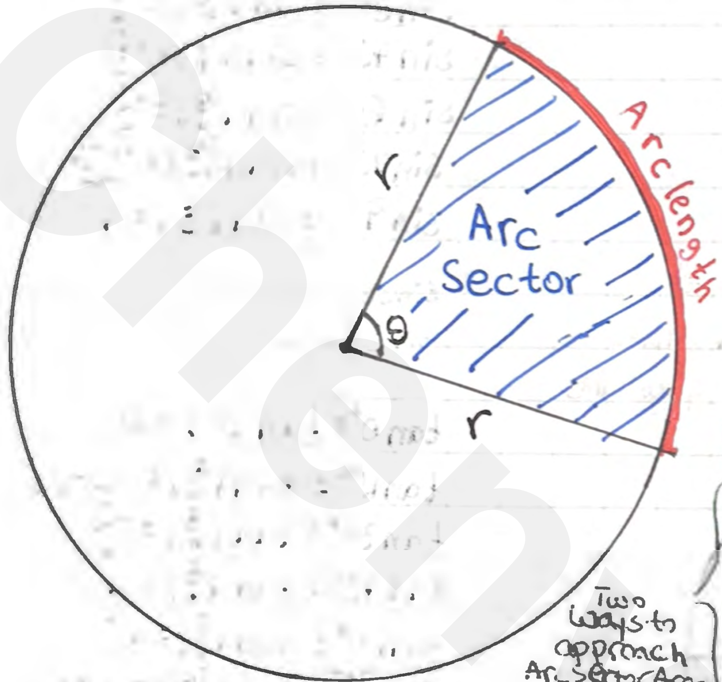
$\sin\theta = \sin(180° - \theta) = \sin(\pi - \theta)$
$\cos\theta = -\cos(180° - \theta) = -\cos(\pi - \theta)$
$\tan\theta = -\tan(180° - \theta) = -\tan(\pi - \theta)$

$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$

$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$

Arc sector

An arc sector is the shape bounded by two radii of a circle and the arc length of that circle



The following ratio can be set up:

$$\frac{A_{\text{Arc sector}}}{A_{\text{circle}}} = \frac{\theta}{2\pi} = \frac{\text{Arc Length}}{\text{Circumference}}$$

$\boxed{\text{THEREFORE}}$

Two ways to approach Arc Sector Area

$$A_{\text{Arc Sector}} = \frac{\theta}{2\pi} \cdot \pi r^2 = r^2\theta$$

$$A_{\text{Arc sector}} = \frac{\text{Arc length}}{2\pi r} \cdot \pi r^2$$

$$= \frac{\text{Arc length}}{2} r$$

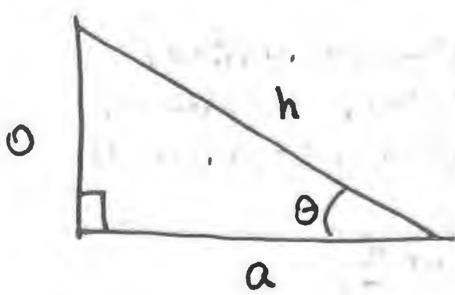
Quadrants

Any circle graphed at $(x,y) = (0,0)$ will have 4 quadrants

- Quadrant I: $0 < \theta < \frac{\pi}{2}$
- Quadrant II: $\frac{\pi}{2} < \theta < \pi$
- Quadrant III: $\pi < \theta < \frac{3\pi}{2}$
- Quadrant IV: $\frac{3\pi}{2} < \theta < 2\pi$

Solving for trig. ratios

When given a specific trig ratio, you can solve for other trig. ratios of that same angle by drawing a right triangle



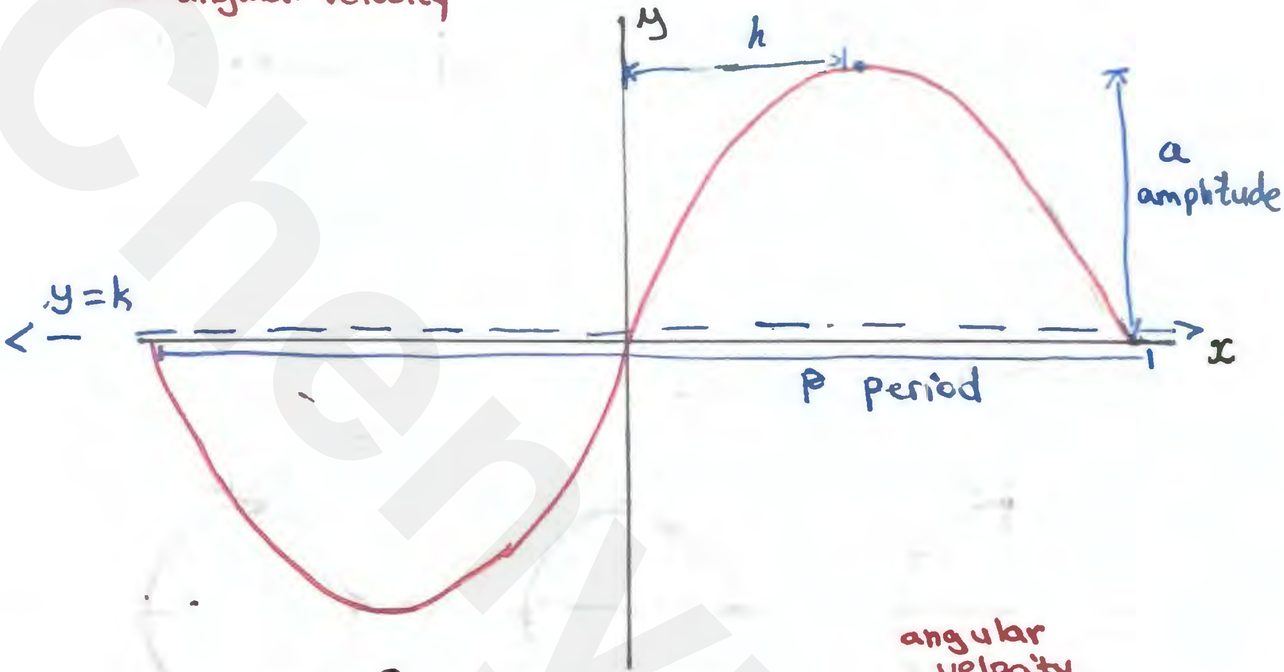
$\sin\theta = \frac{o}{h}$ so $\cos\theta = \frac{a}{h} = \frac{\sqrt{h^2 - o^2}}{h}$ } using pythagorean theorem, you can solve for other sides

make sure to check pos/neg for quadrants ⇒

* Make sure to set up triangle correctly
You should use integer values if possible (easier to calculate with

Shifting trig. functions

$$f(x) = \underbrace{a}_{|a| = \text{amplitude/radius}} \cdot \begin{matrix}\sin \\ \text{or} \\ \cos\end{matrix} (\underbrace{\omega}_{\omega = \text{angular velocity}}(x + \underbrace{h}_{h = \text{horizontal shift}})) + \underbrace{k}_{k = \text{Principal axis}}$$



Recalling distance formula

$$d = r \cdot t$$

dist = rate × time

we can apply it to trig as well...

$$\underbrace{2\pi}_{\text{total angle}} = \overbrace{\omega}^{\text{angular velocity}} \cdot \underbrace{P}_{\text{period}}$$

$$\therefore \text{Period} = \frac{2\pi}{\omega}$$

$$y = a \sin(b(x - c)) + d$$

$y = \sin x$ ⇒ vertical stretch by factor a
⇒ horizontal stetch by factor $\frac{1}{b}$
⇒ horizontal translation of c units
⇒ vertical translation of d units

Inverse Trig Function

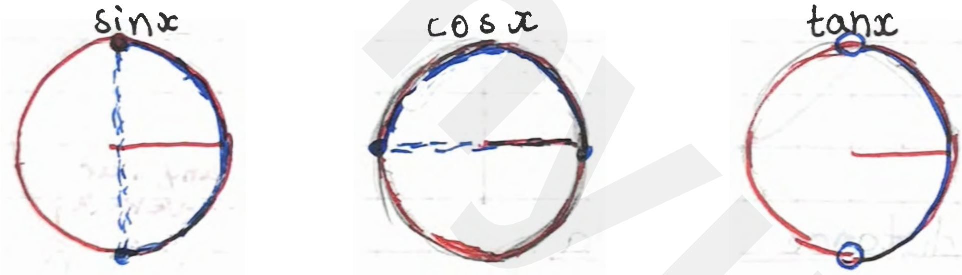
Inverse trig functions are the inverses of trig functions restricted on a domain

INPUT → $f(x)$ → OUTPUT → $f^{-1}(x)$ → INPUT

θ, angle → $\sin(x)$, $\cos(x)$, $\tan(x)$ $\xrightarrow{\text{ratio}}$ $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$ → θ, angle

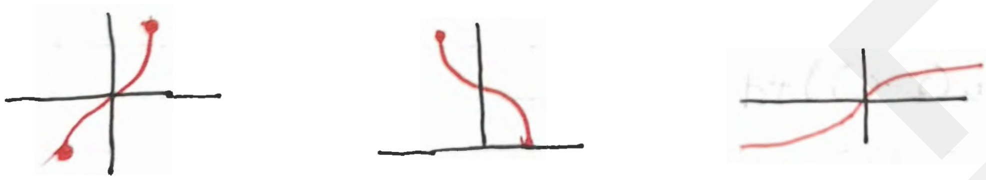
Restriction of domains

sin x | cos x | tan x



$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ | $0 \leq x \leq \pi$ | $-\frac{\pi}{2} < x < \frac{\pi}{2}$

arcsin x | arccos x | arctan x



"Show that"

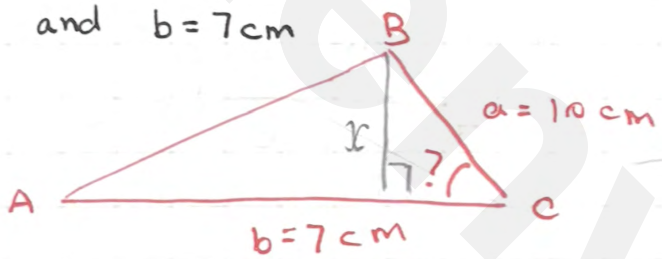
When encountering "Show that" questions, the purpose is trying to demonstrate that the initial equation can be manipulated into the equation.

You can use the latin phrase QED (Quod Erat Demostratum) once you've proved it.

You must end with what you're trying to prove.

Trig Review

E.g. Triangle ΔABC such that $Area_{\Delta abc} = 25cm^2$, $a = 10cm$ and $b = 7cm$



$x = 10cm \cdot \sin(m\angle C)$

$Area = \frac{1}{2} x \cdot 7 = 25cm^2$

$\therefore 50cm^2 = 7x$

$x = \frac{50}{7} cm$

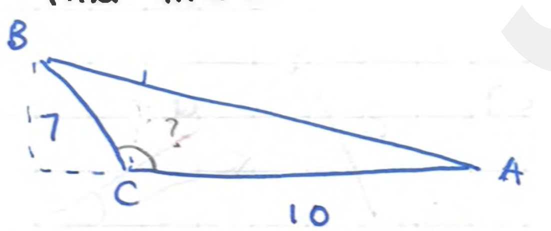
$10cm \cdot \sin(m\angle C) = \frac{50}{7} cm$

$\sin(m\angle C) = \frac{5}{7} cm$

$m\angle C = \arcsin\left(\frac{5}{7} cm\right)$

$\approx 45.5847°$

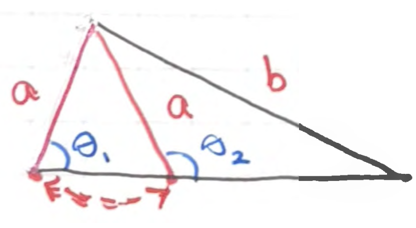
Find $m\angle C$



$m\angle C = 180° - \arcsin\left(\frac{5}{7}\right)$

$\approx 134°$

When solving questions like these, be careful of the different solutions

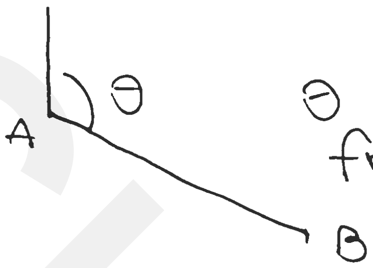


θ has two solutions (side a can pivot and touch at two different points given b is greater than a and other side unknown)

*only when given 2 sides

Bearing

Bearing - The angle from North



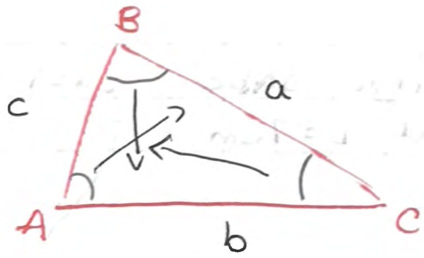
θ is the bearing from A to B

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

OR

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

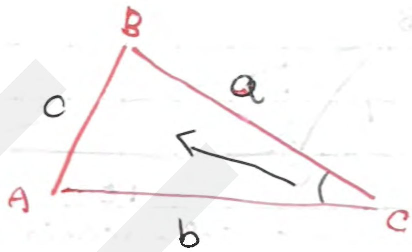


Law of cosines

$$C^2 = a^2 + b^2 - 2ab\cos(c)$$

OR

$$C = \arccos\left(\frac{c^2 - a^2 - b^2}{-2ab}\right)$$



*Useful in relating trig/triangles with quadratics (you can use discriminant to determine how many possibilities of sides, etc.)

Calculus

Limit Notation

$\lim_{x \to a} f(x)$ As x approaches a
Find the value of $f(x)$ as $x \to a$, not necessarily $f(a)$

$\lim_{x \to a^-}$ From the left side

$\lim_{x \to a^+}$ From the right side

when $\lim_{x \to a^-} = \lim_{x \to a^+}$, then $\lim_{x \to a}$ exists (and equal)

Composite Function

$\lim_{x \to a} f(g(x))$, evaluate from both sides

depending on how $\lim_{x \to a^{\pm}} g(x)$ approaches

- If from top then evaluate $\lim_{x \to g(x)^+} f(x)$
  If from bottom then evaluate $\lim_{x \to g(x)^-} f(x)$

Continuity

Continuity is when the curve is defined at all points (on an interval), and no discontinuities

- In other words...

$$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$$

Derivative

Instantaneous rate of change = tangent line

$\frac{dy}{dx}$ leibnitz

$y'(x)$ lagrange

Derivative

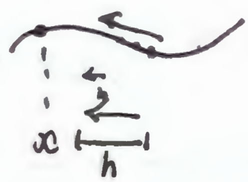
The instantaneous rate of change just refers to the average rate on a very small scale

Two ways...

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{OR} \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

as change in point approaches 0

as point approaches another point

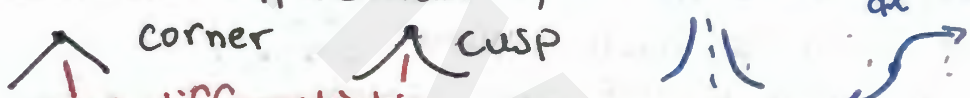


The derivative only exists when (given function is continuous)

$$\lim_{x \to a^-} \text{ or } \lim_{h \to 0^-} = \lim_{x \to a^+} \text{ or } \lim_{h \to 0^+}$$ (two sides equal)

- We call this differentiability

OR when $\frac{dy}{dx} \to \infty$



not differentiable

differentiability implies continuity, not vice versa

Evaluating derivatives

$\frac{d}{dx}[x^n] = n x^{n-1}$ POWER RULE

$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$ ADDITION RULE

$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]\, g(x) + \frac{d}{dx}[g(x)]\, f(x)$ PRODUCT RULE

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\, g(x) - f(x)\frac{d}{dx}[g(x)]}{(g(x))^2}$ QUOTIENT RULE

$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) = \frac{d[f(x)]}{d[g(x)]} \times \frac{d(g(x))}{dx}$ CHAIN RULE

$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$ INVERSE RULE

can be derived from first principles

Be aware of composite functions! (when you see one, use chain!)

$f(g(h(x))) = h' \cdot g'(h) \cdot f'(g(h))$

Tangent line

Finding the equation of the line tangent to the graph at a point just asks you to find the slope and the point.

$$y - y_1 = \frac{dy}{dx}\Big|_{(x,y)=(x_1,y_1)} (x - x_1)$$

Point slope form is very useful

L'Hôpital's Rule

when $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate because $f(a)=0$, $g(a)=0$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

This will work until $\frac{f'(a)}{g'(a)}$ is defined or not indeterminate

This works because the rates at which they change form a ratio

Implicit Differentiation

Sometimes we need to differentiate with respect to other variables or the function takes in multiple variables

E.g. $x^2 + y^2 = 1 \Rightarrow \frac{d}{dt}[x^2+y^2] = \frac{d}{dt}[1]$

the slope changes with respect to x and y

In that case, differentiate normally but multiply $\frac{dx}{dt}$ where x is the variable you're differentiating and t is what you're differentiating with.

E.g. $\frac{d}{dt}[x^2 + y^2] = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

rate of x with respect to t (useful for related rates and parametric equations)

$\frac{dy}{dy} = \frac{dt}{dt} = \frac{dx}{dx} = 1$

(you can simplify)

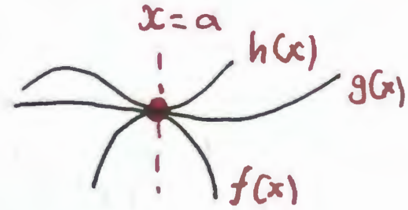
Then, you can isolate $\frac{dy}{dx}$ or any other rate if necessary

Squeeze Theorem

if...

$$f(x) \le g(x) \le h(x)$$

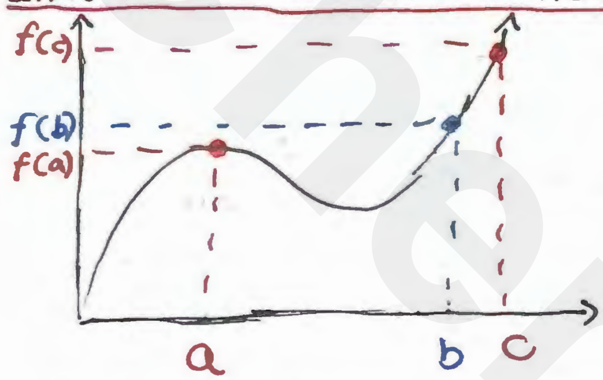
and you want to find $g(a)$



then...

if... $f(a) = h(a)$, $g(a) = f(a) = h(a)$

Intermediate value theorem



If $f(x)$ is continuous over $a \le x \le c$.

Then the IVT guarantees a point $x=b$ such that $a \le b \le c$ where $f(a) \le f(b) \le f(c)$

A function is one-to-one when one x value corresponds to one unique y-value

- In other words, $f(x)$ is one-to-one when $f(x)$ is strictly increasing/decreasing and continuous for the interval on which it is defined

$f(x)$ is cont. for $x \in [a, b]$

and $f'(x) > 0$ for $x \in (a, b)$ (or < 0)

then $f(x)$ is one-to-one for $x \in (a, b)$

Exponential Differentiation

Given $y = a^x$ for $a > 0$, then $\frac{dy}{dx} = a^x \cdot \ln x$

As such, since $\ln(e) = 1$, $\frac{d}{dx}[e^x] = e^x$

- The slope of e^x is the same function

⚠ Be aware of when to use product and when to use chain

In some cases, Product rule does not have to be used

$a^x \cdot a^x = a^{2x}$ ⇒ use chain rule

E.g. $f(x) = x^2 \cdot e^{-x}$

$f'(x) = 2x \cdot e^{-x} + x^2 (-e^{-x})$

This is chain rule! ($e^{-x} = e^{(-x)}$; e^u where $u = -x$)

Logarithmic Differentiation

Given $y = \log_a x$ for $a > 0$, then $\frac{dy}{dx} = \frac{1}{x \cdot \ln(a)}$

As such, since $\ln(e) = 1$, $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

Using logarithm properties, some functions can be differentiated more easily

E.g. $\ln(x \cdot 5x)$ — $\ln(a \cdot b) = \ln(a) + \ln(b)$ (you can use addition rule)

E.g. $\ln(\frac{x}{x+1})$ — $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$ (you can use addition rule)

E.g. $\ln(x^2)$ — $\ln(a^n) = n \cdot \ln(a)$

* $a > 0$, $b > 0$ and $n \in \mathbb{R}$

E.g. $f(x) = (x+5)\ln(5^x)$

$f'(x) = \ln(5^x) + (x+5) \cdot \underbrace{5^x \cdot \ln(5) \cdot \frac{1}{5^x}}_{\text{chain rule}} = \ln(5^x) + (x+5)\ln(5)$

Trigonometric Differentiation

Sinusoidal functions are repeating as they arise from circular motion

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

for this reason, $\sin x$ and $\cos x$ can be differentiated infinitely

- They loop back to the original function every 4 differentiations

We can also find tangent

following quotient rule

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

Trigonometric Differentiation

Eg. $f(x) = 4\tan^2(3x)$ — chain rule (multi-composite function)

$$f'(x) = 4(3\sec^2(3x) \cdot 2\tan(3x))$$
$$= 24\sin(3x)\sec^3(3x)$$

$g(x) = x\sin x$ — Product rule

$$g'(x) = \sin x + x\cos x$$

Trigonometric reciprocal functions can also be differentiated easily

$$\frac{d}{dx}[\sec(x)] = \frac{d}{dx}[(\cos x)^{-1}] = -\sin x(-(\cos x)^{-2}) = \sec x \cdot \tan x$$
$$\frac{d}{dx}[\csc(x)] = \frac{d}{dx}[(\sin x)^{-1}] = \cos x(-(\sin x)^{-2}) = -\csc x \cdot \cot x$$
$$\frac{d}{dx}[\cot(x)] = \frac{d}{dx}[(\tan x)^{-1}] = \sec^2 x(-(\tan x)^{-2}) = -\csc^2 x$$

TIP: any trig function that starts with "c", its derivative will be negative

Inverse trig

$y = \arcsin x$ can be seen as $x = \sin y$ and we differentiate implicitly: $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$ — Pythagorean theorem

THEREFORE

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}[\arctan x] = \frac{1}{x^2+1} \qquad \frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{x^2+1}$$

Higher Derivatives

The second derivative is what you obtain when you differentiate the derivative

- It represents the rate at which the derivative changes

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2} = y''$$

You can keep differentiating a derivative to obtain higher derivatives

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^ny}{dx^n}$$

Higher Derivatives

Implicit Equations

When differentiating implicit equations, you may obtain a function of x and y.

- Differentiating that again may result in $\frac{d^2y}{dx^2}$ be a function of x, y, and $\frac{dy}{dx}$

It is necessary in that case to substitute $\frac{dy}{dx}$ in for the derivative of y to obtain $\frac{d^2y}{dx^2}$ in terms of only x and y

Normal Lines

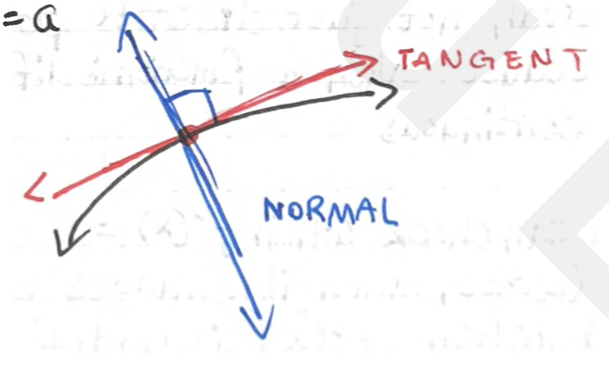
- Tangents are lines that touch a curve

(it may intersect the curve multiple times in other places)

$y - f(a) = f'(a)(x-a)$ represents the tangent line at $x=a$

A normal line is a line perpendicular to the tangent

$y - f(a) = -\frac{1}{f'(a)}(x-a)$ represents the normal line at $x=a$

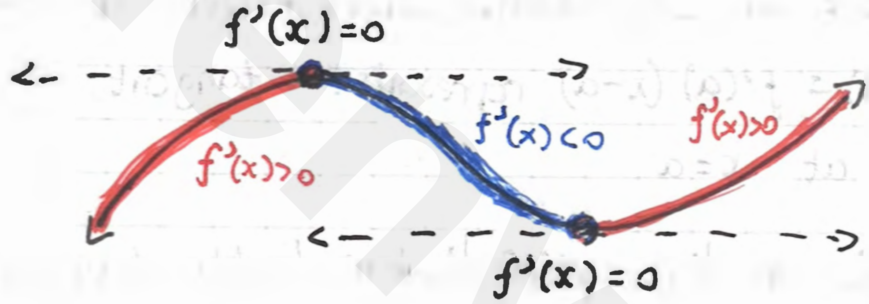


Shape of curves

Using a function's first and second derivative, we can make out a general shape of a curve

The first derivative tells us the slope of the function

when $f'(x)>0$, $f(x)$ is increasing
when $f'(x)<0$, $f(x)$ is decreasing
when $f'(x)=0$, $f(x)$ is stationary



A sign diagram is useful because you can check the slopes over an interval, not just discrete points

- This is because when a function is differentiable (mostly), $f'(x)$ is continuous

Critical points

You can check when $f'(x)=0$
(zeroes, when $f'(x)$ changes signs)
and when $f'(x)$ is undef.
(when $f'(x)$ has a discont.)

By evaluating the signs around these points and inputting them into a sign diagram, it helps make the shape much clearer

E.g. $f(x)=\frac{2x-3}{x^2+2x-3}$, find $f'(x)$ and hence intervals when f is inc. or dec

$$f(x)=\frac{2x-3}{x^2+2x-3}$$

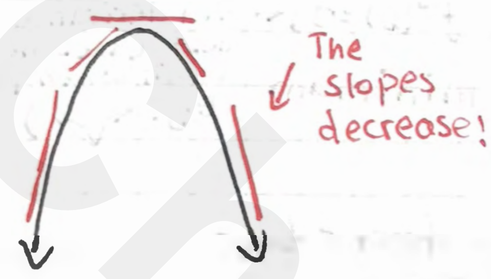
$$f'(x)=\frac{2(x^2+2x-3)-(2x-3)([illegible]x+2)}{(x^2+2x-3)^2}=\frac{-2x(x-3)}{(x-1)^2(x+3)^2}$$

x		-3		0		1		3	
$f'(x)$	$-$		$-$		$+$		$+$		$-$

$\therefore f$ is inc. for $0<x<1$, $1<x<3$ f is dec. for $x<-3$, $-3<x<0$, $x<3$

Concavity

The second derivative of a function tells us the concavity of that function's curve



Noticing that when $f'(x)$ decreases, $f(x)$ is concave down and when $f'(x)$ increases, $f(x)$ is concave up, we can formulate the following relation

when $f''(x) > 0$, $f(x)$ is concave up
when $f''(x) < 0$, $f(x)$ is concave down

Similar to $f'(x)$, we can use sign diagrams to check when $f''(x)$ changes signs

E.g. $y = 2x^3 - 3x^2 + 4x - 6$
$y' = 6x^2 - 6x + 4$
$y'' = 12x - 6$
$\therefore$ y ccu for $x > \frac{1}{2}$,
ccd for $x < \frac{1}{2}$

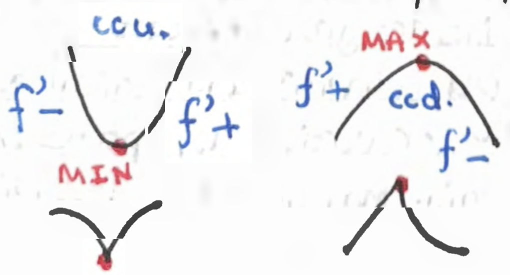
The points for which $f''(x)$ change signs and $f(x)$ change concavity are known as <u>points of inflection</u>

Minima and maxima

LOCAL/RELATIVE

not all critical points are extrema

Some graphs may have local minima or maxima (collectively known as local extrema), they're points lower/greater than surrounding values



Notice how it is at these points in which the function's derivative also changes signs

Minima and Maxima

We can conclude that...

when $f'(x)$ changes signs from $-$ to $+$ ← 1st derivative test

OR... $f''(x) > 0$ and $f'(x) = 0$, ← 2nd derivative test (does not work when f' undef. E.g.)

$f(x)$ has local minimum

when $f'(x)$ changes signs from $+$ to $-$

OR... $f''(x) < 0$ and $f'(x) = 0$,

$f'(x)$ has local maximum

*Points of inflection occur when the first derivative has a local extrema

E.g. $f(x) = x^3 - 3x^2 - 9x + 5$

$f'(x) = 3(x-3)(x+1)$

x		-1		3	
f'	$+$		$-$		$+$

$\therefore$ f has local max. at $x = -1$, local min at $x = 3$

GLOBAL / ABSOLUTE

A relative extrema is not THE max. or min. of a curve.

On an interval, we must consider the end points (candidates test)

- Critical Points (or rel. extrema)
- End Points

after finding these points, solve for $f(a)$ where a is the x-coordinate of that point. Compare their values to check which points are greatest / least

For an entire domain, check the end behaviors and vertical asymptotes

- When end behavior $\to \pm\infty$ or vertical asymptotes exists, then an absolute max. or min. may not exist depending on the curve

Rates of change

Rates of change are present all across the real world
Typically, rates are taken with respect to time but
can also be anything else!

E.g. Velocity, $\frac{ds}{dt}$ where s = displacement, t = time
acceleration, $\frac{dV}{dt} = \frac{d^2s}{dt^2}$

The units will always be y unit per x unit (in the format $\frac{dy}{dx}$)

E.g. $\frac{d\,\text{Cycles}}{dt}$ = cycles per second if t = seconds (Hz)

but could also be

$\frac{dt}{d\,\text{cycle}}$ = seconds per cycle if t = seconds (Period!)

Optimization

Optimization is the process of finding the most optimal solution, typically via a min. or a max.

The question will ask you to maximize or minimize something

You can either...

1. Use technology to solve or graph for extrema
2. Use calculus to solve for extrema
3. Use analysis like vertex of a parabola to deduce

Related Rates

By using implicit differentiation, you can find how fast a value changes with respect to other variables

E.g. $A = w \cdot l \Rightarrow \frac{dA}{dt} = l \cdot \frac{dw}{dt} + w \cdot \frac{dl}{dt}$

Usually, you need to be careful of which variables are constant

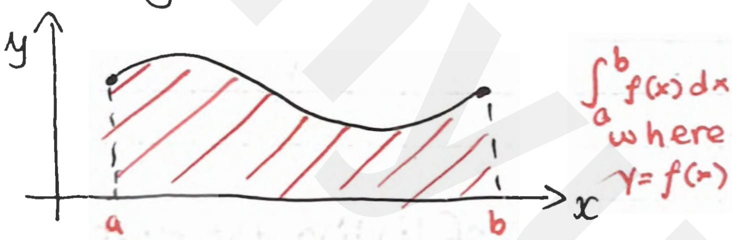
↳ Be attentive to possible relationships you notice

Integration

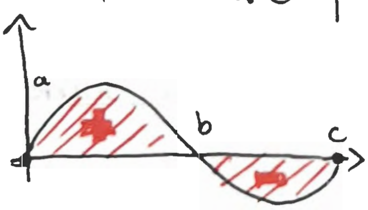
Integrals are the inverse of derivatives

· Known as an "antiderivative"

Represents the accumulation of change (net change)



You can have positive and negative accumulations



if $\int_a^b f(x)\,dx = -\int_b^c f(x)\,dx$

then the areas cancel out

$\int_a^c f(x)\,dx = 0$ (ends up where it started)

Properties of integrals

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Riemann Sums

There are many ways to approximate the area under the curve

Left riemann rectangle

$$\int_0^c f(x)\,dx$$

$$\approx (a-0)f(0) + (b-a)f(a) + (c-b)f(b)$$

overestimate when decreasing
underestimate when increasing

Right riemann rectangle

$$\int_0^c f(x)\,dx$$

$$\approx (a-0)f(a) + (b-a)f(b) + (c-b)f(c)$$

overestimate when increasing
underestimate when decreasing

Midpoint riemann rectangle

$$\int_0^c f(x)\,dx$$

Overestimate when concave down
Underestimate when concave up.

$$\approx (a-0)\cdot f\left(\frac{a+0}{2}\right) + (b-a)\cdot f\left(\frac{b+a}{2}\right) + (c-b)\,f\left(\frac{c+b}{2}\right)$$

Trapezoidal riemann

$$\int_0^c f(x)\,dx$$

$$\approx (a-0)\cdot \frac{f(a)+f(0)}{2} + (b-a)\,\frac{f(a)+f(b)}{2} + (c-b)\cdot \frac{f(c)-f(b)}{2}$$

Overestimate when concave up
Underestimate when concave down

∴ The area under the curve is...

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot \Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + \frac{(b-a)}{n}k\right)\cdot \frac{b-a}{n}$$

Fundamental theorem of calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

net change!

and...

$$f(x) = \frac{d}{dx}\left[\int f(x)\,dx\right]$$

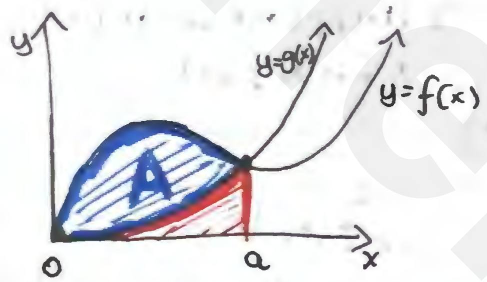
Integrals

Displacement $= \int_a^b f'(x)dx = f(b) - f(a)$

Total distance $= \int_a^b |f'(x)|dx$

Average value $= \dfrac{\int_a^b f'(x)dx}{b-a}$

Since integrals are the area under the curve, we can also find the area between curves by finding the difference between integrals



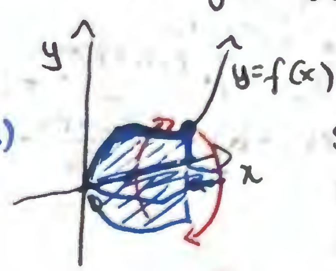
$$A = \int_0^a f(x)\,dx - \int_0^a g(x)dx$$

$$A = \int_0^a (f(x) - g(x))dx$$

for dy if easier

If we were to rotate this area around an axis, we can get volume

(disc method)



Since we're using the function's y-value as the radii, and the cross-sections are circular.

integrate dy if around y-axis

$$V = \pi \int_0^a (f(x))^2 dx$$

If we have a hollowed center:

(washer method)

$$V = \pi \int_0^a (f(x))^2 dx - \pi \int_0^a (g(x))^2 dx$$

$$V = \pi \int_0^a (f(x))^2 - (g(x))^2\,dx$$

dy if around y-axis

Differential equations

Differential equations give the slopes of a curve

e.g. $\frac{dy}{dx} = y$

$y = ce^x$

Integrating the differential equation (with a constant $+C$) gives the general solution.

when given a specific point, you can solve for the constant's value and thus obtain the differential equation's specific solution.

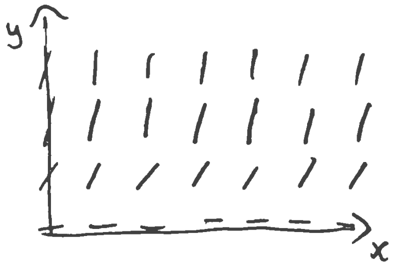
$(x, y) = (0, 1)$

$y(0) = 1 = ce^0$

$C = 1$

$\therefore \ y = e^x$

when you graph all the general solutions at every point, you get a slope field

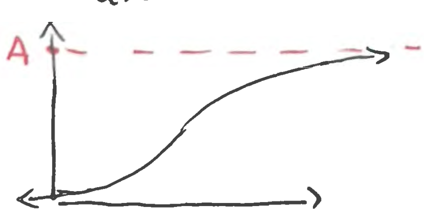


Logistic differential equations

- when modelling a specific scenario, it might not make sense for the quantity to increase infinitely

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{A}\right)$$

where k is the initial slope
A is the maximum value (asymptote)



To integrate, you can use integration through partial fractions

$$\int \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}\,dx = \int \frac{A}{x-a} + \frac{B}{x-b}\,dx$$

Homogeneous Differential Equations

Another way to solve differential equations is through using a substitution such that the sum of the exponents on each term is the same

Typically, these differential equations are unseparable so a substitution is needed to make it separable

A homogeneous differential equation will be of the form

$$\boxed{\frac{dy}{dx} = f(x, y)}$$

where it can be rearranged as

$$\boxed{\frac{dy}{dx} = G\left(\frac{y}{x}\right)}$$

in this case, let $v = \frac{y}{x}$, $y = vx$ and equate $\frac{dy}{dx}$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \quad \frac{dy}{dx} = G(v)$$

$$\therefore \quad x\frac{dv}{dx} = G(v) - v$$

$$\boxed{\int \frac{1}{G(v) - v}\, dv = \int \frac{1}{x}\, dx}$$

which, in turn, becomes separable

E.g. $(xy + y^2 + x^2)dx - x^2 dy = 0$

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \frac{y^2}{x^2} + 1$$

let $v = \frac{y}{x}$, $y = x \cdot v$

$$\int \frac{1}{v^2+1}\, dv = \int \frac{1}{x}\, dx$$

$$\arctan(v) = \ln|x| + c$$

$$\frac{y}{x} = \tan(\ln|x| + c)$$

$$y = x\tan(\ln|x| + c)$$

Integration by parts

Derived from the product rule

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$
$$\Downarrow$$
$$u \cdot v' = \frac{d}{dx}(u \cdot v) - u'v$$
$$\int u v' = uv - \int u'v$$
$$\int u\,dv = uv - \int v\,du$$

or... $\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$

*Typically with cos, sin, e^x, this method may result in a loop. Thus, it may be necessary to move integral all to one side (if they're the same) and divide

$\rightarrow \int a\,dx = uv - \int a\,dx$

$2\int a\,dx = uv$

$\int a\,dx = \frac{uv}{2}$

where u should make the problem simpler...

$u =$ logs — preferred
inverse
algebraic
trig
exponential — least preferred

dv can sometimes be 1 (if nothing is multiplied by u)

Integrating factors method

You can make a differential equation much easier by multiplying an integrating factor.

Works on $y' + \underbrace{P(x)y}$ $= Q(x)$

notice that this is remniscent of chain rule with e^x
$\frac{d}{dx}[\underbrace{e^{f(x)}}_{y}] = f'(x)e^{f(x)} = f'(x)\,y$

Thus, multiplying by $I = e^{\int P(x)\,dx}$ ← no + c in this case
can simplify it

Integrating factors method

once multiplied by I

$$Iy' + IP(x)y = Q(x)\cdot I$$

$I \cdot P(x) = I'$ $\Rightarrow$ matches product rule

$y' = y'$

$$\frac{d}{dx}[u \cdot v] = u'v + v'u$$

as such $\int \underbrace{Iy'}_{u \cdot v'} dx + \int \underbrace{IP(x)}_{u'} \underbrace{y}_{v} dx = \int Q(x) \cdot I \, dx$

$$\therefore \; I \cdot y = \int Q(x) \cdot I \, dx$$

$$\boxed{y = \frac{\int Q(x) \cdot I \, dx}{I}}$$

Integration by substitution

- Used on products or quotients
- By substituting and differentiating with a new variable

$$\boxed{\int f'(g(x))g'(x)\,dx}$$

Resulting from chain rule...

make a substitution (e.g. $u = g(x)$)
solve and integrate with respect to u (or du)

$$du = g'(x)\,dx$$

$$\therefore \int f'(u) \cdot g'(x) \cdot \frac{1}{g'(x)}\,du$$

$$= \int f'(u)\,du$$

* This allows the differential equation to be much easier to integrate

Integration by partial fractions

Recalling partial fractions, where...

$$\frac{\alpha x+\beta}{(ax+b)(cx+d)} = \frac{A(cx+d)}{(ax+b)(cx+d)} + \frac{B(ax+b)}{(ax+b)(cx+d)}$$

$$= \frac{A}{ax+b} + \frac{B}{cx+d}$$

we can use this when u-sub doesn't work
or integrating by other methods is difficult
- given that integrating $\frac{1}{ax+b}$ is much better than one of degree -2

$\Rightarrow$ Integration by parts is used to integrate homogeneous differential equations

Integration through variable separation

Used to integrate differential equations of both x and y of the form...

$$\frac{dy}{dx} = f(x)g(y)$$

such that rearranging the equation separates y's and x's

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$\Rightarrow$ This makes it much easier to integrate

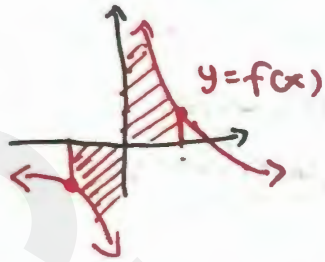
THEREFORE...

method	corresponding rule	notes
Substitution	chain rule	integral involves composite
By parts	Product rule	matches integral to part of product rule
Partial fractions	Logarithmic differentiation	Integrate rationals by breaking into simp
Homogeneous	implicit differentiation	differential equations of y/x
Integrating factor method	Product rule	multiply variables to "create" product ru

Improper integrals

- involves integrals to infinity or ~~discontinuous~~ interval

E.g.



discontinuity:

split the integral at the point of discontinuity and use limits to determine convergence

$$\int_a^c f(x)\,dx = \lim_{n\to b^-} \int_a^n f(x)\,dx + \lim_{n\to b^+} \int_n^c f(x)\,dx$$

infinity:

Since infinity is not a number, use limits to evaluate an integral to ∞

$$\int_a^\infty f(x)\,dx = \lim_{n\to\infty} \int_a^n f(x)\,dx$$

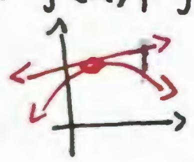
may be undefined (will only yield value if it converges)

Linear approximation

when given a differential equation, you can approximate points on the original curve with a linear equation.

E.g. $f(a),\ f'(a) \Rightarrow\ y - f(a) = f'(a)\,(x-a)$

↳ plugging in x with a domain value close to a gives an approximation



- The approximation is overestimate when...

original function is concave down around 'a' $\Rightarrow$ approximation > curve

- The approximation is underestimate when...

original function is concave up around 'a' $\Rightarrow$ approximation < curve

This only works for nearby values

- Another approximation is Euler's method
- Recursive method to approximate where...

given $y(a)$ and $y'(x)$ $\Rightarrow$ $y(a+\Delta x) \approx y(a) + y'(a)\cdot\Delta x$

use a table

x	y	$\frac{dy}{dx}$
1	3	2
2	5	6
⋮	⋮	⋮

Kinematics – Applying integral and differential calculus to motion of bodies

$a(t) = v'(t) = s''(t)$

accel. vel. displ. position

Total Dist. $= \int_a^b |v(t)|\,dt$

Displacement $= \int_a^b v(t)\,dt$

* Total dist. $\geq$ displ.