

primitiva ha el mismo fin)

E

$$\lim_{x \rightarrow \infty}$$

e

**AP**

# Calculus

J

$$(100, 0-0) - (0, 0)$$

$$(100, 0-0) - 0$$

$$\frac{dy}{dx}$$

d

\$

f(x)

# Unit 1: Limits and Continuity

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Can change occur at an instant?

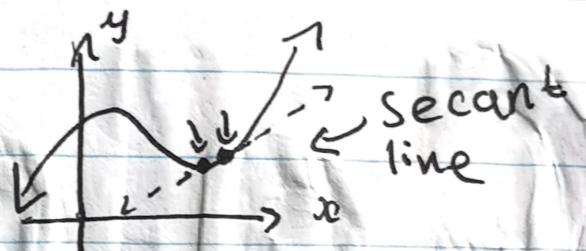
Average Rate of Change:  $\frac{\Delta y}{\Delta x}$

How about at an instant?

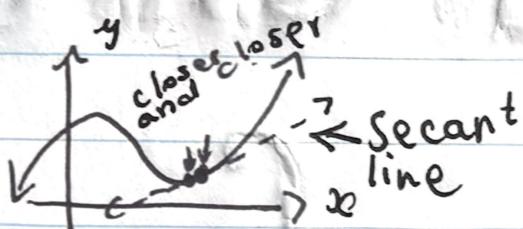
Estimates:

$$\frac{f(x) - f(x-0.1)}{x - (x-0.1)}$$

Closer and Closer



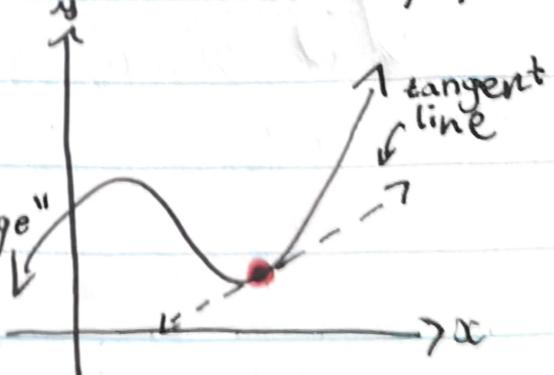
$$\frac{f(x) - f(x-0.001)}{x - (x-0.001)}$$



\*NOTE: We cannot do  $\frac{f(\infty) - f(\infty)}{\infty - \infty}$  as that would result in an indeterminate form or  $\frac{0}{0}$ .

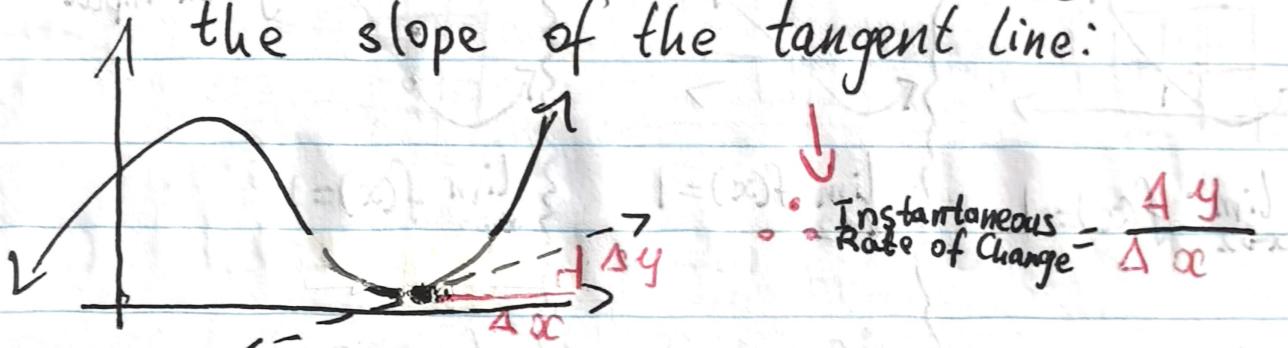
— So... What happens when the two points overlap each other? Or in other words, form a tangent line?

We can get an "instant rate of change"



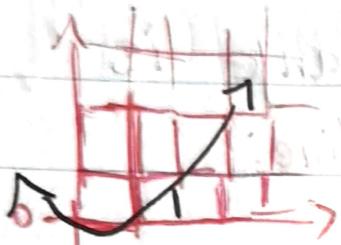
By definition... Rate of Change = Slope

Thus... An instantaneous rate of change is  
the slope of the tangent line:

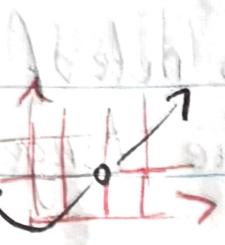


# Limits - Intro

Despite  $f(2)$  being different values, the limit is the same:



$$\lim_{x \rightarrow 2} f(x) = 1$$



$$\lim_{x \rightarrow 2} f(x) = 1$$



$$\lim_{x \rightarrow 2} f(x) = 1$$

The output you approach with a given input

$$f(2) = 1$$

$$f(2) = \text{undefined}$$

$$f(2) = 3$$

Structure:

$$\lim_{x \rightarrow a} f(x) = b$$

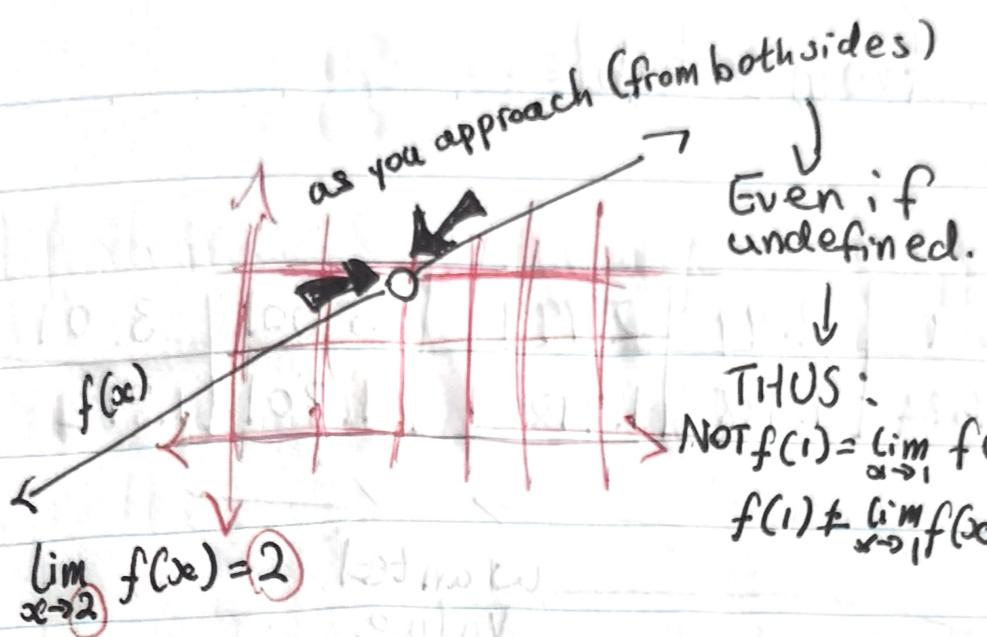
$x \rightarrow a$

$\uparrow$

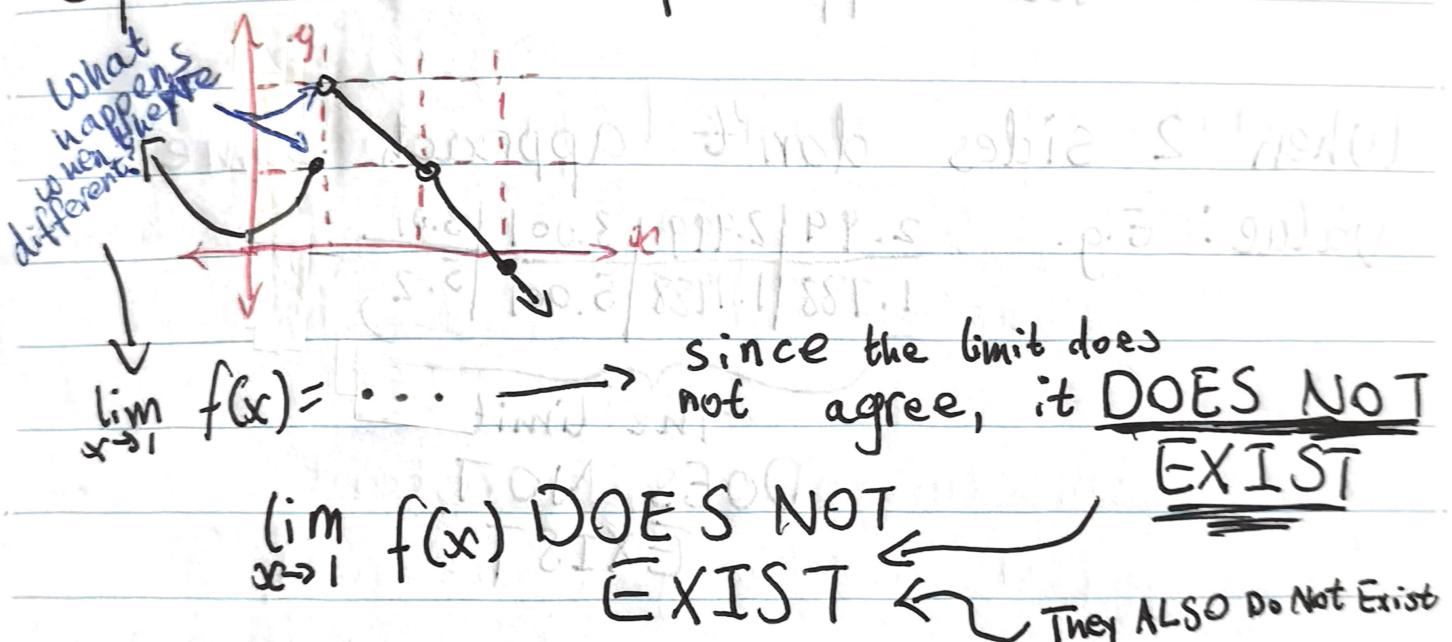
as  $x$  approaches  
what value?

↑  
what happens  
when  $x$  approaches  
that value?

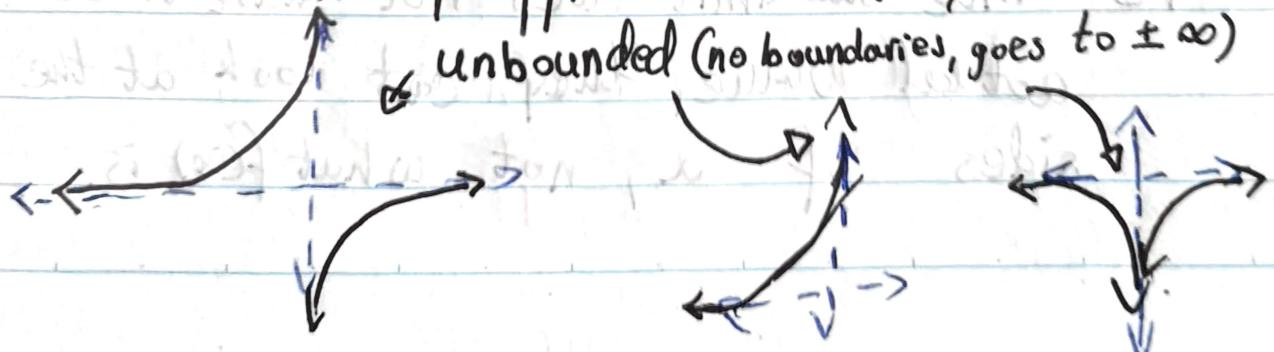
\*NOTE: Limits ignore what  $f(x)$  EQUALS,  
instead it refers to what  $f(x)$  will equal  
when you approach it.



## Special Cases of Limits



How about when they approach  $\pm \infty$ ?



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## Limits from tables

$$\lim_{x \rightarrow 3} f(x)$$

$x$	2. 9	2. 99	2. 999	3. 001	3. 01	3. 1
$f(x)$	1.682	1.788	1.7988	1.801	1.812	1.922



wanted  
value

→ So you can infer that when  $x=3$ ,  $f(x)$  would approach 1.8.

When 2 sides don't approach same

value: E.g.

2. 99	2. 999	3. 001	3. 01
1.788	1.7988	5.001	5.2

The limit  
DOES NOT  
EXIST

\*NOTE: Since the limit does not have to be the actual value, You must look at the sides of  $x$ , not what  $f(x)$  is.

# Limit Properties

Given:  $\lim_{x \rightarrow c} f(x) = L$      $\lim_{x \rightarrow c} g(x) = M$

$$\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = L + M$$

$$\lim_{x \rightarrow c} (f(x) - g(x)) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x) = L - M$$

$$\lim_{x \rightarrow c} (f(x)g(x)) = (\lim_{x \rightarrow c} f(x))(\lim_{x \rightarrow c} g(x)) = L \cdot M$$

$$\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x) = kL$$

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{L}{M}$$
 (Only true when  $\lim_{x \rightarrow c} g(x) \neq 0$ )  
(to avoid division by 0)

$$\lim_{x \rightarrow c} (f(x))^{\frac{r}{s}} = \left( \lim_{x \rightarrow c} f(x) \right)^{\frac{r}{s}} = L^{\frac{r}{s}}$$

# Limits from the left and right.

$$\lim_{x \rightarrow a^-} f(x)$$

FROM the NEGATIVE side, the LEFT

FROM the POSITIVE side, the RIGHT

$$\lim_{x \rightarrow a^+} f(x)$$

when unstated:  $\lim_{x \rightarrow a} f(x)$

it refers to BOTH

## Limits of Combined functions

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \leftarrow$$

plug it in

find one by one

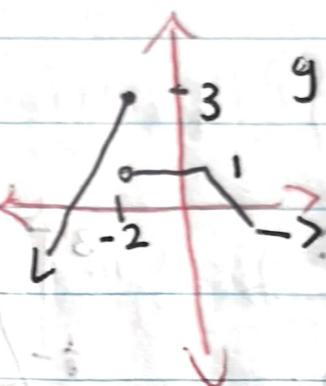
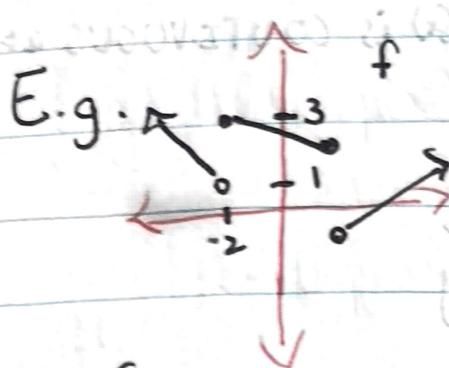
When Both seem to be non-existent  
 (as in  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  don't exist),  
 try approaching from the right and then on  
 the left:

$$\lim_{x \rightarrow a^-} (f(x) + g(x)) \text{ instead and } \lim_{x \rightarrow a^+} (f(x) + g(x))$$

ELSE, the limit  
 is non-existent

If both yield  
 same result,  
 then that's  
 your answer

\* NOTE: This applies to all limits involving multiple functions.



Although  $\lim_{x \rightarrow -2} f(x)$  does not exist and  $\lim_{x \rightarrow -2} g(x)$  also does not exist,  $\lim_{x \rightarrow -2^-} (f(x) + g(x))$  does with  $\lim_{x \rightarrow -2^+} f(x)$  and the same for  $g(x)$

$$\lim_{x \rightarrow -2} (f(x) + g(x)) = 1 + 3 = 4$$

$$\lim_{x \rightarrow -2^+} (f(x) + g(x)) = 3 + 1 = 4$$

Same e..

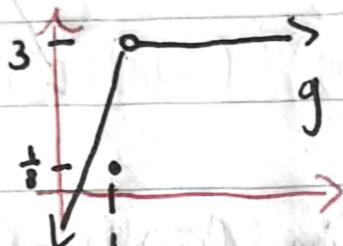
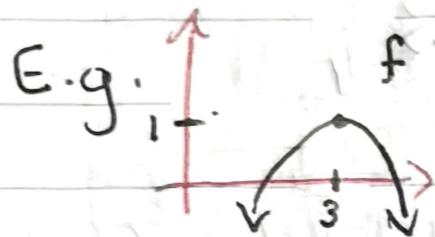
$$\therefore \lim_{x \rightarrow -2} (f(x) + g(x)) = 4 \text{ result}$$

Left hand side has no mention of direction

## Theorem for limits of composite functions:

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) \quad \text{for } \lim_{x \rightarrow a} g(x) = L \quad \text{AND}$$

$f(x)$  is CONTINUOUS at  $L$



since function  $f$  is continuous at 3 and  $\lim_{x \rightarrow 1} g(x)$  exists,  $\lim_{x \rightarrow 1} f(g(x)) = f(\lim_{x \rightarrow 1} g(x)) = 1$

\*NOTE: Although some limits don't work with this theorem (i.e. conditions aren't met), they don't necessarily not exist and can be still solved without it.

Limits of composite functions: When INTERNAL limit does not exist

Let's say for function  $g(x)$  and  $h(x)$ ,  $\lim_{x \rightarrow a} h(x)$  does not exist and you want to solve  $\lim_{x \rightarrow a} g(h(x))$ . So how?

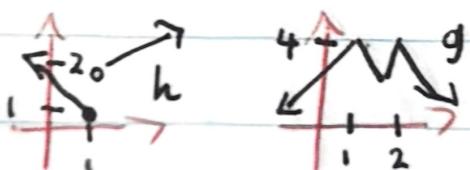
$$\lim_{x \rightarrow a} g(h(x)) = C$$

split it into left or right

$$\begin{aligned} &\hookrightarrow \lim_{x \rightarrow a^+} g(h(x)) = \lim_{h(x) \rightarrow b^+} g(h(x)) = C_1 \\ &\text{value it approaches from positive (up) or negative (down), not to} \\ &\hookrightarrow \lim_{x \rightarrow a^-} g(h(x)) = \lim_{h(x) \rightarrow b^-} g(h(x)) = C_2 \end{aligned}$$

if both have same answer,  $\lim_{x \rightarrow a} g(h(x)) = C$

$$\text{E.g. } \lim_{x \rightarrow 1} g(h(x)) = 4$$



$$\lim_{x \rightarrow 1^+} g(h(x)) = \lim_{h(x) \rightarrow 2^+} g(h(x)) = 4$$

$$\lim_{x \rightarrow 1^-} g(h(x)) = \lim_{h(x) \rightarrow 1^+} g(h(x)) = 4$$

Limits of Composite Functions: EXTERNAL limit does not exist

Similar to internal:

$\lim_{x \rightarrow a^-} g(h(x))$  for  $\lim_{x \rightarrow a} g(x)$  does not exist.

$$\begin{aligned} \lim_{x \rightarrow a^-} g(h(x)) &= \lim_{h(x) \rightarrow b^-} g(h(x)) = \lim_{x \rightarrow b^-} g(x) = c, \\ \text{if } & \\ \lim_{x \rightarrow a^+} g(h(x)) &= \lim_{h(x) \rightarrow b^+} g(h(x)) = \lim_{x \rightarrow b^+} g(x) = c_2, \end{aligned}$$

same

$$\lim_{x \rightarrow a} g(h(x)) = c$$

ELSE

limit does not exist

$$f = ((\omega_d)_P \mid \omega_d)_P$$



$$f = ((\omega_d)_P \mid \omega_d)_P$$

$$g = ((\omega_d)_P \mid \omega_d)_P$$

## Limits by DIRECT SUBSTITUTION

- ONLY if  $f$  is continuous at  $x=a$ , will  
 $\lim_{x \rightarrow a} f(x) = f(a)$

E.g.  $\lim_{x \rightarrow -1} (6x^2 + 5x - 1) = 6(-1)^2 + 5(-1) - 1 = 0$

## Undefined limits

E.g.  $\lim_{x \rightarrow 1} \frac{x}{\ln(x)} = \frac{1}{\ln(1)} = \frac{1}{0}$  } Although undefined,  
final answer was not  
in indeterminate form  
or  $0 \div 0$ , so limit  
DOES NOT EXIST

\*NOTE: A limit is only non-existent when the reasoning is undefined in a form that is not indeterminate form ( $\frac{0}{0}$ )

# Limits of Piecewise Functions

When dealing with piecewise functions, you must consider both sides.

$\lim_{x \rightarrow c}$  to consider  $\lim_{x \rightarrow c^-}$  OR  $f(c-0.001)$  AND  $\lim_{x \rightarrow c^+}$

OR  $f(c+0.001)$ . You should not be worried about  $f(c)$  as limits refer to the general shape of graph, not considering holes.

If  $\lim_{x \rightarrow c^-} = \lim_{x \rightarrow c^+}$  then they also equal  $\lim_{x \rightarrow c}$

E.g.  $f(x) = \begin{cases} \frac{x+2}{x-1} & \text{for } 0 < x < 4 \\ 5 & \text{for } x=4 \\ \sqrt{x} & \text{for } x > 4 \end{cases}$

$$\lim_{x \rightarrow 4^+} f(x) = 2$$

$$\lim_{x \rightarrow 4^-} f(x) = \frac{4+2}{4-1} = \frac{6}{3} = 2$$

HOWEVER...

$f(4) = 5$  — since we don't care about  $x (= 5)$ , we neglect it. We only care about when  $x = 3.999$  AND  $x = 4.001$

SINCE  $\lim_{x \rightarrow 4^+} = \lim_{x \rightarrow 4^-}, \lim_{x \rightarrow 4} = 2$

# Limits with Absolute Values

An absolute value can be seen as a piece-wise function:

$$|x| = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x & \text{for } x < 0 \end{cases}$$

SO...

a function like  $f(x) = \frac{|x-3|}{x-3}$  can also be seen as it

$$f(x) = \begin{cases} \frac{x-3}{x-3} & \text{for } x > 3 \\ \text{undef. for } x = 3 \\ \frac{-(x-3)}{x-3} & \text{for } x < 3 \end{cases}$$

since absolute value  
on top.

SINCE... We're doing limits, we can ignore what happens when  $x = c$  in limit  $\lim_{x \rightarrow c}$

$$\text{E.g. } \lim_{x \rightarrow 3} f(x) = \begin{cases} \frac{x-3}{x-3} & \text{for } x > 3 \\ \frac{-(x-3)}{x-3} & \text{for } x < 3 \end{cases}$$

ALSO, when  $x = 3$ ,  
function is undefined  
so

Evaluate limits from both sides

$$\lim_{x \rightarrow 3^-} f(x) = \frac{(x-3)}{x-3} = -1 \quad \lim_{x \rightarrow 3^+} f(x) = \frac{x-3}{x-3} = 1$$

IN THIS CASE,  $-1 \neq 1$  so  $\lim_{x \rightarrow 3^-} \neq \lim_{x \rightarrow 3^+}$  which means  $\lim_{x \rightarrow 3} f(x)$  D.N.E

# Solving Limits (Algebraic Manipulation)

When dealing with limits that cannot be solved through direct substitution (undefined/ineterminate form), you can use other methods.

## 1) Factoring:

You can factor numerator & denominator and then cancel them out.

↳ Since limit is value that it approaches, we don't care about holes.

$$\text{E.g. } f(x) = \frac{x^2 + x - 6}{x - 2} = \frac{(x+3)(x-2)}{(x-2)} = x+3, x \neq 2$$

$$\lim_{x \rightarrow 2} f(x) = (2) + 3 = 5$$

ANSWER

it refers to  
a hole, so we  
won't have  
to care about  
it

## 2) Rationalising (Conjugates)

We can also use the difference of squares.

$$\text{I.E. } z = a+b \quad \bar{z} = a-b$$

$$\therefore z \cdot \bar{z} = (a+b)(a-b) = a^2 - b^2$$

When working with radicals, this can work similarly.

$$(\sqrt{a} + b)(\sqrt{a} - b) = a - b^2$$

So...

We can use this to rationalise and cancel out expressions.

$$\text{E.G. } \lim_{x \rightarrow -1} \frac{x+1}{\sqrt{x+5}-2} = g(x)$$

$$= \frac{x+1}{\sqrt{x+5}-2} \cdot \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} = \frac{(x+1)(\sqrt{x+5}+2)}{x+5-4}$$

$$= \frac{(x+1)(\sqrt{x+5}+2)}{(x+1)} = \sqrt{x+5}+2 \text{ for } x \neq -1$$

*Limits don't care about holes*

$$\lim_{x \rightarrow -1} \frac{x+1}{\sqrt{x+5}-2} = \sqrt{x+5}+2 = 4$$

### 3) Complex Fractions

We can use least common multiples to simplify an expression.

$$\text{E.g. } \lim_{x \rightarrow 0} \frac{\frac{x}{x-4+\frac{1}{4}}}{\frac{4}{4x-16}} = \frac{\frac{x}{x-4+\frac{1}{4}}}{\frac{4}{4x-16}} = \frac{\frac{x}{x-4+\frac{1}{4}}}{\frac{4}{4x-16}}$$

$$= x \cdot \frac{4x-16}{x} = 4(x-4) \text{ for } x \neq 0$$

DIRECT SUBSTITUTION

$$4(0-4) = 4 \cdot (-4) = \boxed{-16}$$

*We ignore this as usual*

# SPECIAL TRIG LIMITS

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{OR} \quad \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \quad \text{OR} \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

→ Similarly,  $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \frac{\sin x \cdot \sin x}{x} = 1 \cdot \lim_{x \rightarrow 0} \sin x = 0$$

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

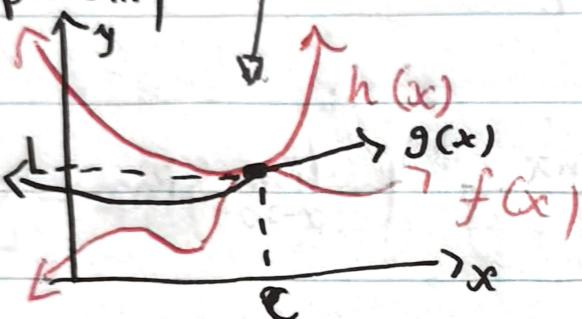
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Squeeze Theorem (Sandwich Theorem, Pinching Theorem)  
states that... when...

$$f(x) \leq g(x) \leq h(x)$$
$$\lim_{x \rightarrow c} f(x) = L \quad \Downarrow \quad \lim_{x \rightarrow c} h(x) = L$$
$$\lim_{x \rightarrow c} g(x) = L$$

Graphically,



For Trig Functions:

Since  $\sin x$  &  $\cos x$  have a range of  $[-1, 1]$ , we can use squeeze theorem too.

$$-1 \leq \sin x \leq 1 \quad \text{AND} \quad -1 \leq \cos x \leq 1$$

∴

$$-k \leq k \sin x \leq k \quad \text{AND} \quad -k \leq k \cos x \leq k$$

HOWEVER...

Not all can be determined.

E.g.  $1 \leq \lim_{x \rightarrow 0} f(x) \leq 2$  cannot be determined

$5 \leq \lim_{x \rightarrow 5} f(x) \leq 5$  can be determined

$81 \leq \lim_{x \rightarrow 7} f(x) \leq 92$  cannot be determined

## Continuity and discontinuity

A point is considered continuous if:

1.  $f(c)$  is defined,  $c$  is in the domain
2.  $\lim_{x \rightarrow c} f(x)$  exists,  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$
3.  $\lim_{x \rightarrow c} f(x) = f(c)$

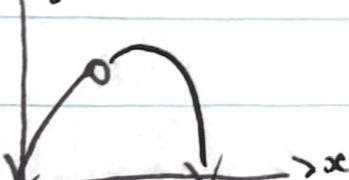
## Types of discontinuities

### 1. Hole (removable)

↳ Can be removed by filling in hole

↳ Single point being undefined

↳ Indeterminate Form

Graph: 

$$\lim_{x \rightarrow c} f(x) \neq f(c)$$

You can "factor" out the hole

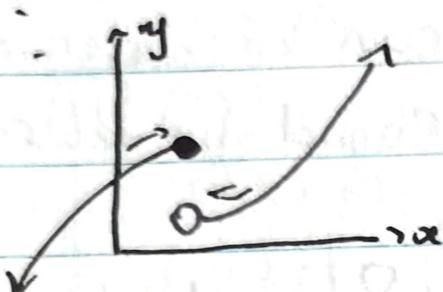
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## 2) Jump Discontinuity (non-removable)

↳ Only Piecewise

↳ Where  $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$

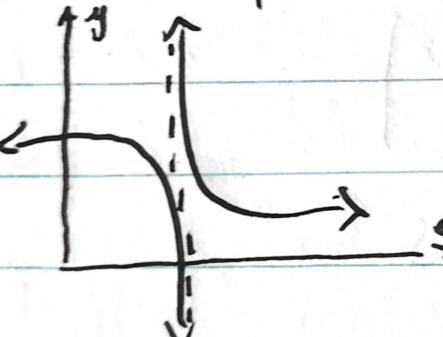
Graph:   $\lim_{x \rightarrow c} f(x)$  does not exist

## 3) Asymptotic Discontinuity

↳ Of Vertical Asymptote

↳ Approaches  $\infty$  OR  $-\infty$

↳ Unbounded

Graph: 

## Continuity over an Interval

- ↳  $f$  is continuous over  $(a, b)$  IF and ONLY IF  $f$  is continuous over every point in interval.
- ↳  $f$  is continuous over  $[a, b]$  IF and ONLY IF  $f$  is continuous over  $(a, b)$  and  $\lim_{x \rightarrow a^+} f(x) = f(a)$  &  $\lim_{x \rightarrow b^-} f(x) = f(b)$

Scenarios to look out for when looking for domain restrictions

1) Denominators  $f(x) = \frac{x-5}{x+1}$ ,  $x+1 \neq 0$   
(can't be = 0)

2) Radicals with Even Roots  $f(x) = \sqrt{7x+3}$ ,  $7x+3 \geq 0$   
(cannot be < 0 inside)

3) Logarithms  $f(x) = \ln(2x+1)$ ,  $2x+1 > 0$   
(Vertical Asymptote at 0)

If  $f$  is not continuous at an end point of an interval  $[a, b]$  but  $\lim_{x \rightarrow a^+} f(x)$  exists and  $\lim_{x \rightarrow b^-} f(x)$  exists then  $f$  is still continuous over that entire interval (closed)

## Removing Discontinuities

↳ You can remove removable discontinuities by factoring them out.

$$f(x) = \frac{kx^2 + 2kb + kb^2}{h(x^2 - b^2)} = \frac{k(x+b)(x+b)}{h(x-b)(x+b)}$$

$$= \frac{k(x+b)}{h(x-b)} \text{ for } x \neq -b$$

Restraint on domain  
(hole, removable discontinuity)

To remove it, don't write the restraint

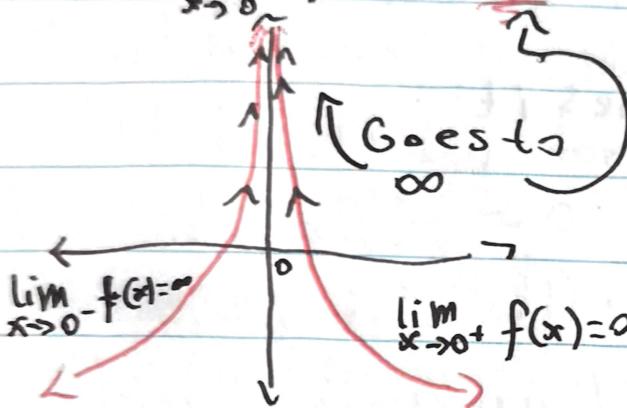
$$f(x) = \frac{k(x+b)}{h(x-b)} \quad \begin{array}{l} \text{Now, } f(x) \text{ is defined} \\ \text{at } x = -b \end{array}$$

You can also use rationalisation with conjugates, just like finding the limit of it. (see Solving limits: Algebraic Manipulation)

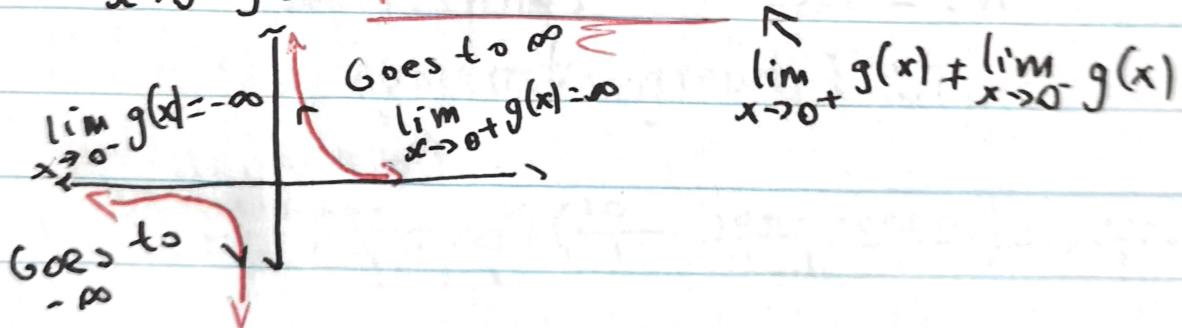
# Infinite Limits and Vertical Asymptote

When faced with unbounded limits, they can approach infinity (a concept, not a number though)

E.G  $\lim_{x \rightarrow 0} f(x) = \infty$



$\lim_{x \rightarrow 0} g(x)$  Does not exist

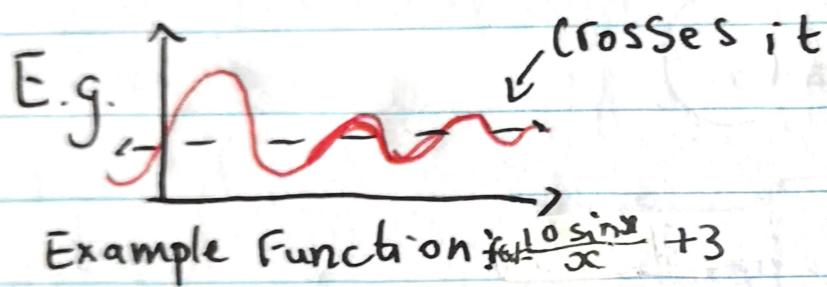


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## key Differences between Horizontal vs Vertical Asymptote.

When dealing with a function, a vertical asymptote cannot be crossed whilst a horizontal asymptote can (function oscillates)



# Limits at Infinity and Horizontal Asymptote

Horizontal Asymptote: (End Behaviour)

↳ What does the  $y$ -value approach as the  $x \rightarrow \text{value}$ , approaches  $-\infty$  AND  $+\infty$ ?

↳ Specific Number or Out Of Bounds?

## Basic Rules:

1) If the denominator grows faster...

$$\frac{\text{not as big}}{\text{BIG Number}} = 0 \quad (\frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000} \rightarrow \frac{0}{\infty} = 0)$$

2) If numerator and denominator grow **EQUALLY** fast, then you have  $\frac{\text{BIG number}}{\text{BIG number}} = 1$

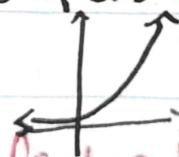
$$(\frac{10}{10}, \frac{100}{100}, \frac{10000}{10000} \rightarrow \frac{\infty}{\infty} = 1)$$

3) If the numerator grows **FASTER** than the denominator...

$$\frac{\text{BIG number}}{\text{not as big}} = \infty \quad (\frac{10}{1}, \frac{100}{1}, \frac{10000}{1} \rightarrow \frac{\infty}{1} = \infty)$$

## Which Functions grow fastest?

1) Exponential:  $f(x) = a^x$



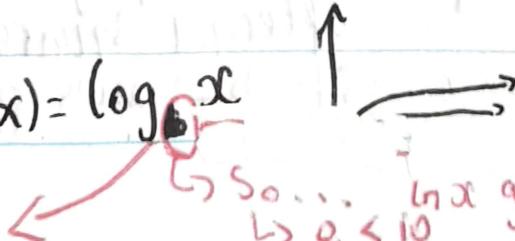
↳ Bigger the  $a$  is, faster it grows

2) Polynomials:  $f(x) = x^a$



3) Logarithms:  $f(x) = \log_b x$

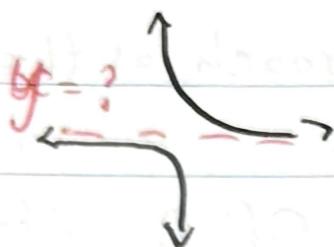
Smaller the  $b$ , faster it grows



↳ So...  $\ln x$  grows faster than  $\log x$   
 $e < 10$

To find Horizontal Asymptotes, evaluate

$$\lim_{x \rightarrow \infty} f(x) \text{ and } \lim_{x \rightarrow -\infty} f(x)$$



↳ Plug in  $\infty / -\infty$

↳ In rational functions, only care about the highest degree terms

E.g.  $\frac{5x^4 + 3x + 2}{7x^4 + 2x^3 - x} \approx \frac{5x^4}{7x^4} \approx \frac{5}{7}$

*Change the most*

$\frac{8x^7 + 3x}{2x^6 - 3} \approx \frac{8x^7}{2x^6} \approx 8x \approx \infty$

*Replace for  $\infty$*

\* NOTE: When dealing with square roots and exponents, the signs may change.

E.g.

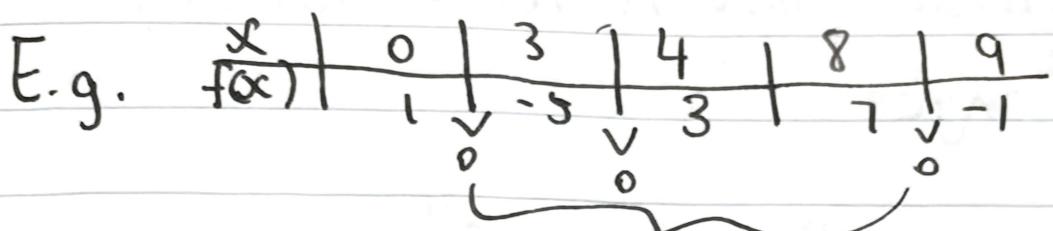
$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2}} = \frac{x}{\cancel{\sqrt{x^2}}} \quad \begin{matrix} -( \infty ) \\ \infty \end{matrix}$$

↳  $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2}} = -1$

*Don't forget that after you square a number, it becomes positive*

# Intermediate Value Theorem (IVT)

Used as a proof for when a continuous function will pass a specific number.



The IVT is used as a justification for this

Has to pass  
at least 3  
times

## Justification With the IVT

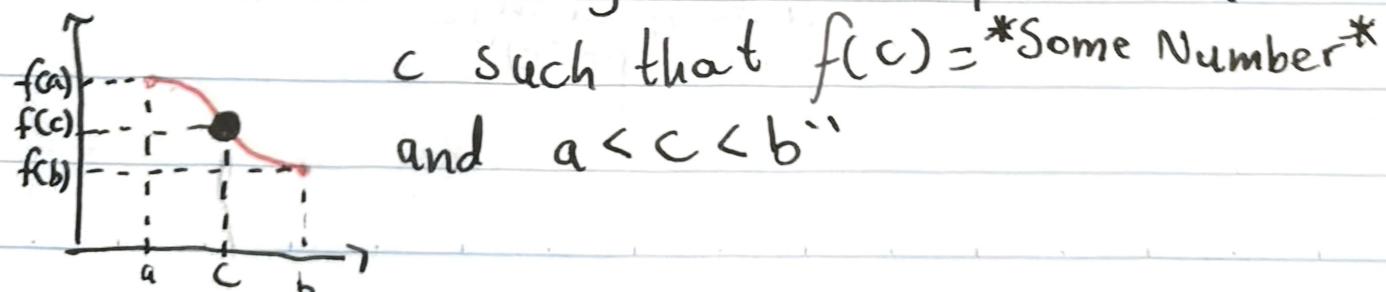
Conditions (must also be written for a complete proof):

1. The function  $f(x)$  is continuous on an interval  $[a, b]$ .

2.  $f(a) < f(b)$  or  $f(a) > f(b)$ , can't be equal

3.  $f(c)$  is between  $f(a)$  and  $f(b)$

Conclusion: "According to the IVT, there is a value



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Unit 2: Differentiation  
(Definition and fundamental properties)

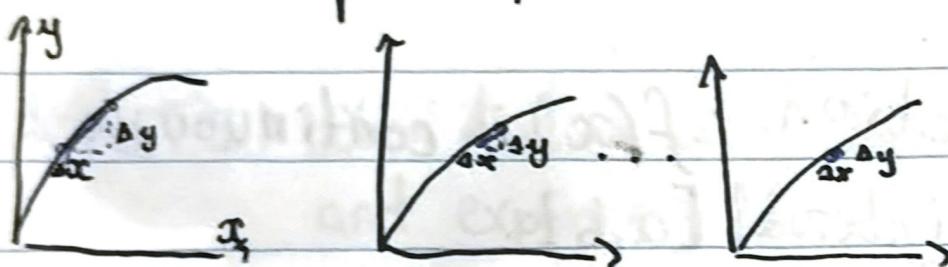
Defining Average and Instantaneous rates of Change at a point.

Differential Calculus

↳ About Instantaneous rate of change

Average Rate of Change =  $\frac{\Delta y}{\Delta x}$

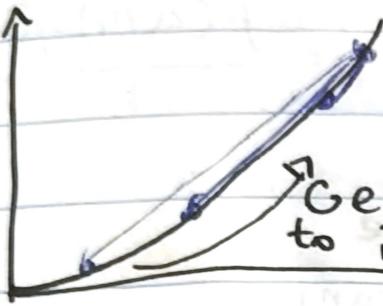
Instantaneous is at an instant (the slope)  
↳ So, you get more and more accurate as the  $\Delta x$  gets smaller and smaller (at a point)



$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y' = f'(x)$$

derivative  
rate of change  
at a point

# Secant lines and average rate of change



Gets closer and closer to instantaneous rate of change

Lagrange's Notation:  $f'(x)$  or  $y'$

Leibniz's Notation:  $\frac{dy}{dx}$

Newton's Notation:  $\dot{y}$

Leibniz Notation:  $\frac{d}{dx} f(x)$  ← can be replaced with whatever function

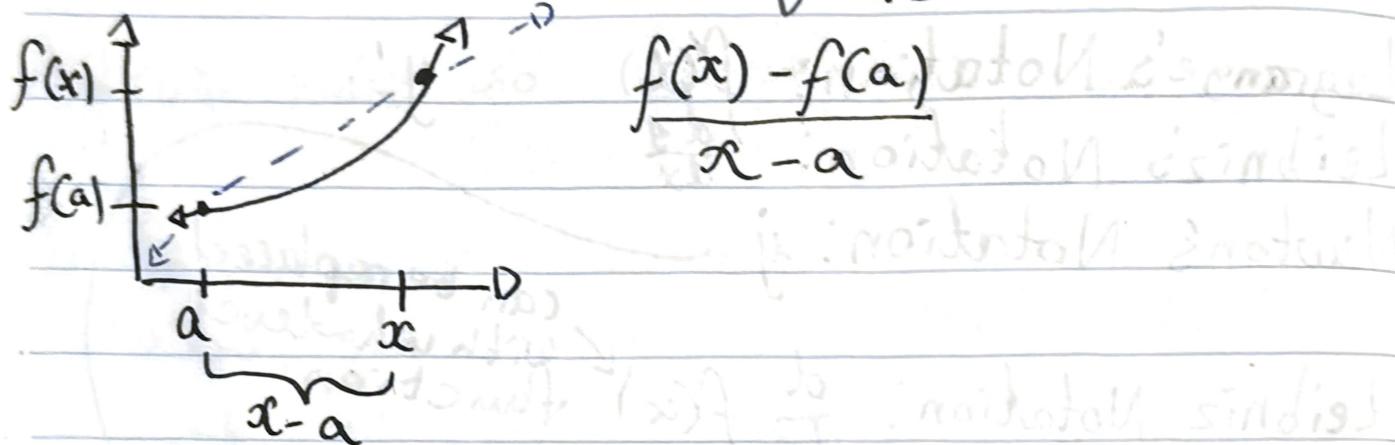
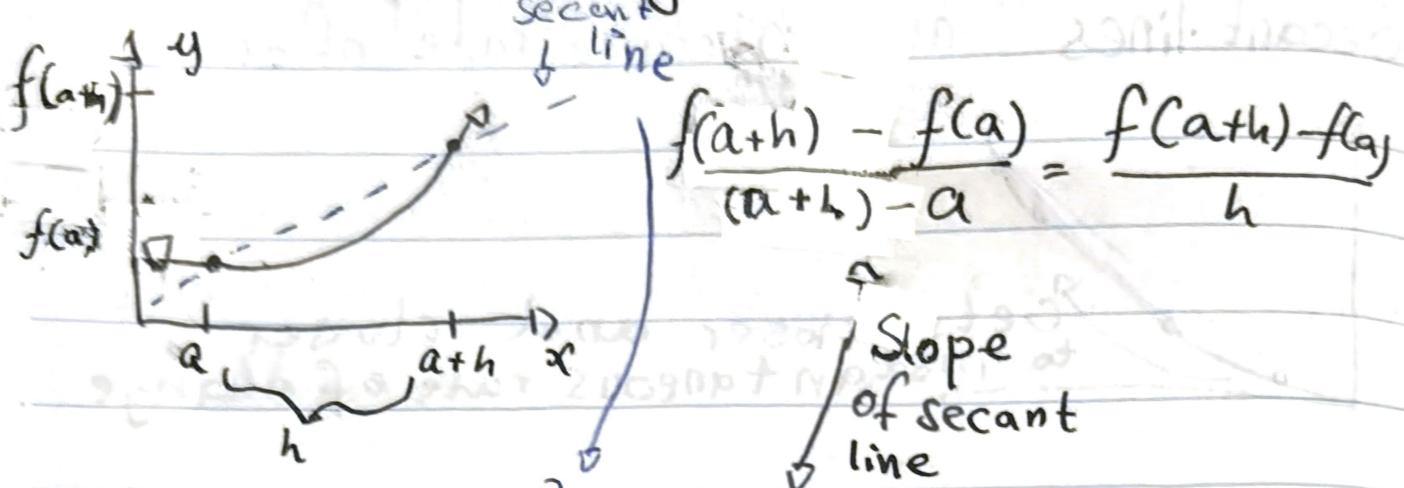
$\frac{dy}{dx}$  ← only when  $y = f(x)$

When  $y = f(x)$  only, else  $f(x)$

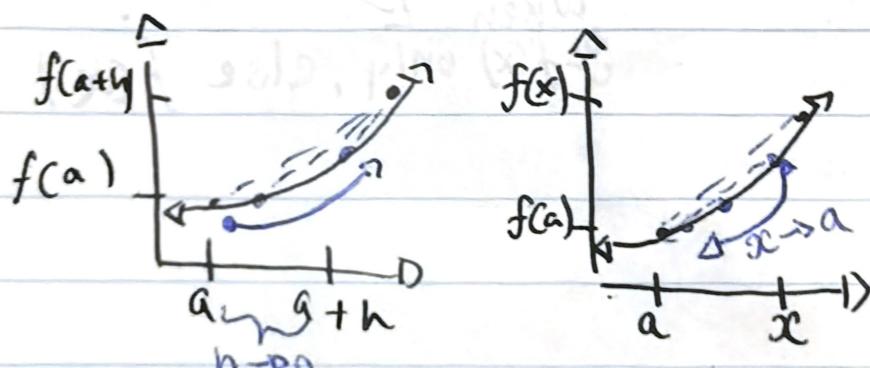
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## Expression for Average rate of change



Instantaneous Rate of Change is  
when both points approach one  
single point



$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{OR} \quad \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

\*This is the slope of the tangent line

You can now plug in the function and values for an instantaneous rate of change at  $x=a$  (replace  $a$  with domain)

L<sub>2</sub> When solving, there HAS to be a hole to cancel out in the end

Finding the original function from derivative:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

→ Identify similarities between  $f(a+h)$  &  $f(a)$

↓  
Usually it always has  $a+h$

$$x=a \text{ so } a \text{ value } = a$$

easier

↓ function

The rest is  
original function

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$x=?$$

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# The Derivative

- The derivative is an expression that calculates the instantaneous rate of change (slope of the tangent line) of a function at any given  $x$ -value. In other words, it gives us the slope of the function at a point!

Notation:

Lagrange:  $f'(x)$   $y'$

$\underbrace{\quad}_{\text{when } f(x)}$   $\uparrow$

Leibniz:  $\frac{dy}{dx}$   $\leftarrow$   $\underbrace{\quad}_{\text{when } y}$

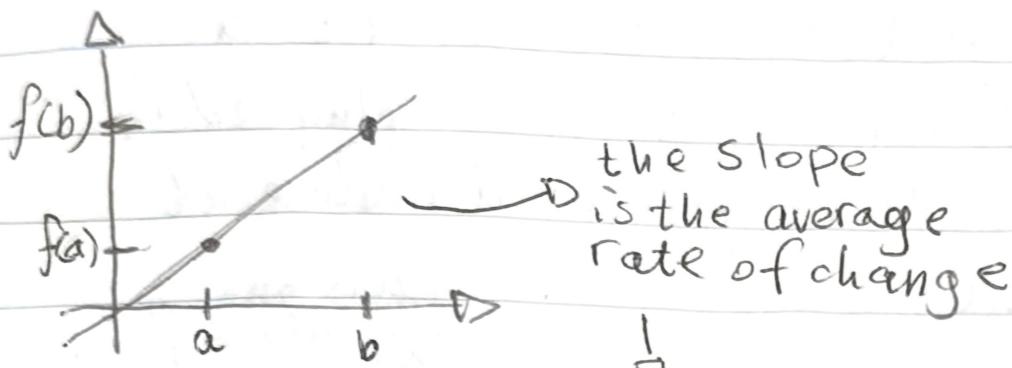
To find it, solve for when  $x \rightarrow x$  or

$x = x$  instead of  $x \rightarrow a$

(Replace "a" not  $x$  with  $x$ )

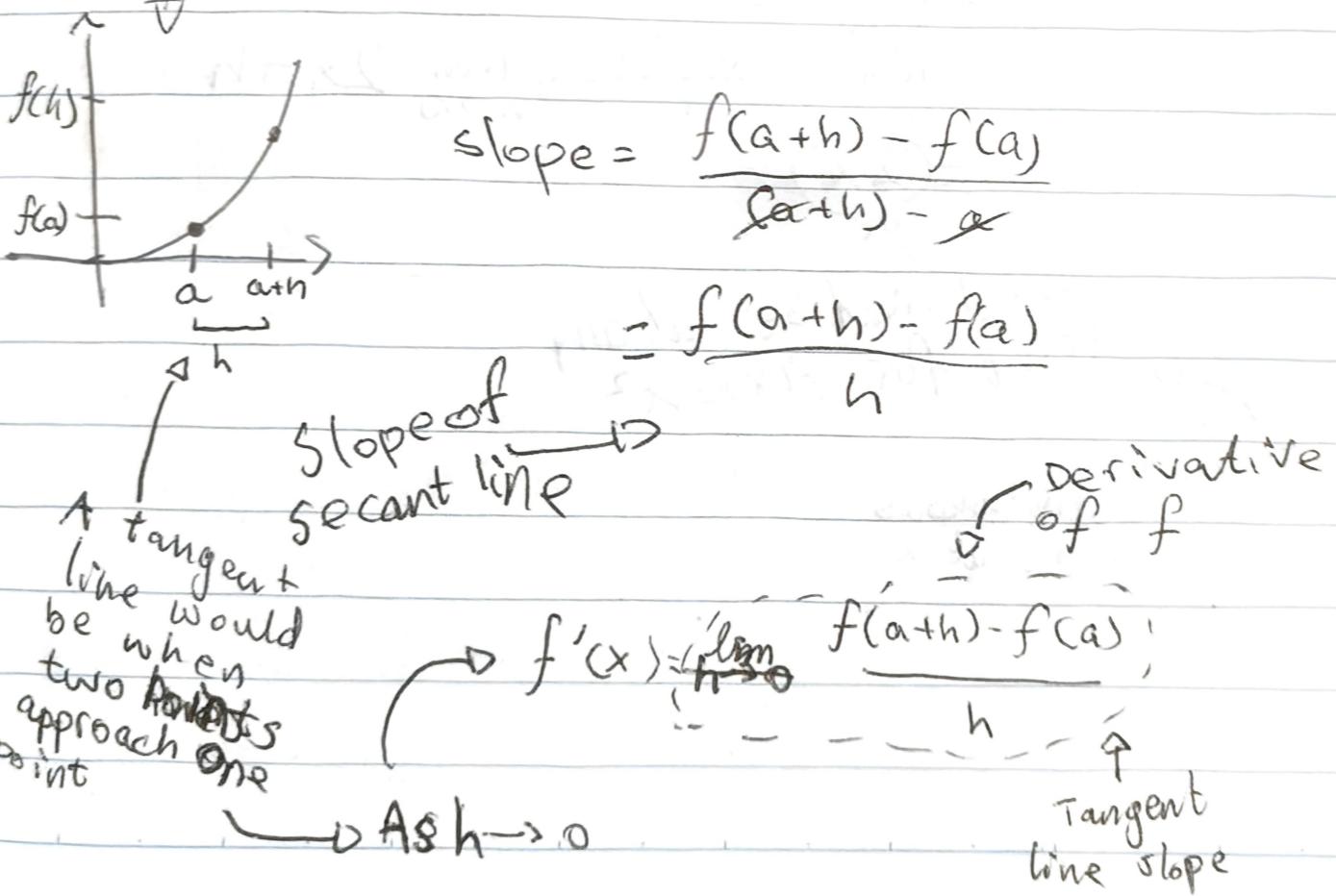
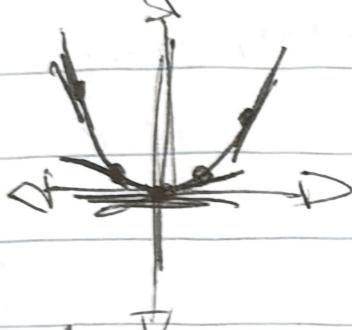
$$f'(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

# The Derivative as a limit



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

However, a curve's slope changes



# The Derivative at any point.

↳ We can find the derivative at a single given point, so if we plugged in  $x$ , we would be given a derivative at any point.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

In this one, you substitute a for  $x$   
alternate does not form work

E.g.  $f(x) = x^2$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} 2x + h$$

$$= 2x$$

The derivative at any point for  $f(x) = x^2$

Plug in  $x$  to know instantaneous rate of change at  $x$

# Equation of Tangent line

To find equation of a tangent line we need slope and point.

Point is where instantaneous rate of change is at.

Slope is derivative at that point.

E.g.  $f'(2)=5$  &  $f(2)=7$

Point =  $(2, 7)$

Slope = 5

## Estimating derivatives.

A derivative refers to the rate of change at a single point

- So an estimate would be close to that single point

↳ In other words, average rate of change around that point

(make sure graph is differentiable)

E.g.  $f(x)$

x	0	3	5	6	9
f(x)	72	95	112	115	114

$\rightarrow f'(4) \approx \frac{17}{2} = 8.5$   
(4 is between 3 and 5)

# Differentiability and Continuity

A graph that is differentiable must be continuous, but one that is continuous may not be differentiable.

Differentiability is where the derivative exists for each point in the domain. Graph must be a smooth line.

→ It must not contain:

## 1. Discontinuity

- Hole

- Jump

- Vertical Asymptote

Not differentiable

## 2. Corners/Cusp

Not differentiable

## 3. Vertical Tangent

slope here is  $\frac{\Delta y}{\Delta x} = \frac{40}{5} = 8$

# Differentiability at a point

Test if derivative exists at that point

To find derivative in a Piecewise function,  
do it from BOTH SIDES

E.g.

$$g(x) = \begin{cases} x-1, & x < 1 \\ (x-1)^2, & x \geq 1 \end{cases}$$

USE ALTERNATE FORM

$$\lim_{x \rightarrow 1^-} \frac{g(x) - g(1)}{x - 1}$$

$$\lim_{x \rightarrow 1^-} \frac{g(x)}{x-1} - \lim_{x \rightarrow 1^+} \frac{x-1}{x-1} = 1$$

similar to limits,  
if both sides  
not equal, then  
it doesn't exist.

$$\lim_{x \rightarrow 1^+} \frac{g(x)}{x-1} = \lim_{x \rightarrow 1^+} \frac{(x-1)^2}{x-1} = \lim_{x \rightarrow 1^+} x-1 = 0$$

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# Estimating derivatives with CALCULATOR

1. Go to calculator Menu (not Graphing nor any other)

2. Menu  $\rightarrow$  Calculus  $\rightarrow$  Derivative/Derivative at a point

$\hookrightarrow$  If at a point, put in the x value of the point

3. Fill in function

$$\frac{d}{dx} (\boxed{\quad}) \quad | \quad x = ?$$

Input  $\nearrow$  function here  $\underbrace{\quad}_{\text{only for derivative at a point}}$

4. Press ENTER for EXACT VALUE, or  $\approx$

Ctrl + ENTER ↵

# The POWER RULE

function

$$f(x) = x$$



Derivative

$$f'(x) = 1$$

$$f(x) = x^2$$



$$f'(x) = 2x$$

$$f(x) = x^3$$



$$f'(x) = 3x^2$$

$$f(x) = x^4$$



$$f'(x) = 4x^3$$

$$f(x) = x^5$$



$$f'(x) = 5x^4$$

• • •

• • •

$$f(x) = x^n$$

$$f'(x) = nx^{n-1}$$

$$\text{E.g. } f(x) = x^{37} \quad f'(x) = 37x^{36}$$

$$f(x) = \frac{1}{x^4} \quad f'(x) = -\frac{4}{x^5}$$

$$f(x) = \sqrt[7]{x^3} \quad f'(x) = \frac{3}{7\sqrt[7]{x^6}}$$

## Basic Derivative Rules:

1. Constant:  $\frac{d}{dx} c = 0$

$$\text{LD } c \cdot x^0 = c \cdot 1 = c$$

$$\text{LD } 0 \cdot c \cdot x^{0-1} = 0$$

2. Constant  $\cdot \frac{d}{dx} cu = c \frac{du}{dx}$   
Multiple:  $\frac{d}{dx} cu = c \frac{du}{dx}$

$$\text{e.g. } y = 3x^2 \rightarrow y' = 3 \cdot 2 \cdot x^{2-1} = 6x$$

3. Sum / Difference:  $\frac{d}{dx} (u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

$$\therefore f(x) = ax^n \quad f'(x) = anx^{n-1}$$

$$f(x) = a \quad f'(x) = 0$$

$$f(x) = ax^n + bx^p \quad f'(x) = anx^{n-1} + bp x^{p-1}$$

$$\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} (f(x)) + \frac{d}{dx} (g(x)) = f'(x) + g'(x)$$

4. Product Rule:  $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

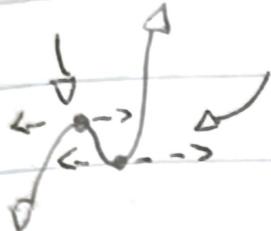
OR

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

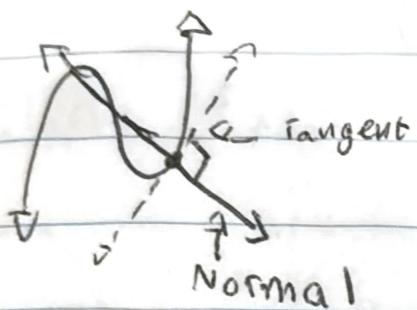
## Horizontal, Tangent Lines

↳ When slope = 0

↳ So... when derivative = 0



Normal Lines: Lines that go through the same point the tangent does, but is PERPENDICULAR to the TANGENT line.



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# Differentiating Polynomials

To differentiate polynomials, separate the terms and find the derivative of each:

$$f(x) = a + b + c$$

$$f'(x) = \frac{d}{dx}(a) + \frac{d}{dx}(b) + \frac{d}{dx}(c)$$

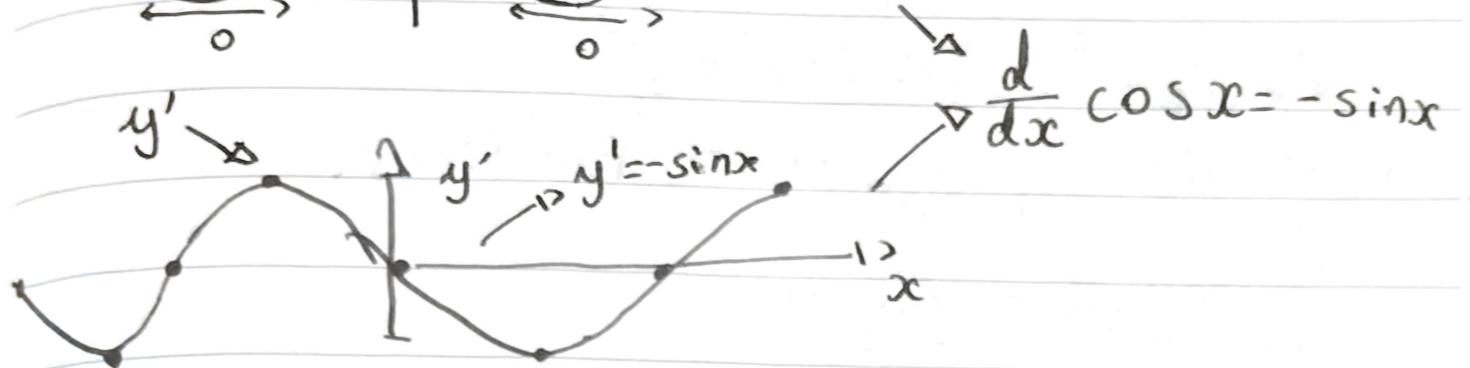
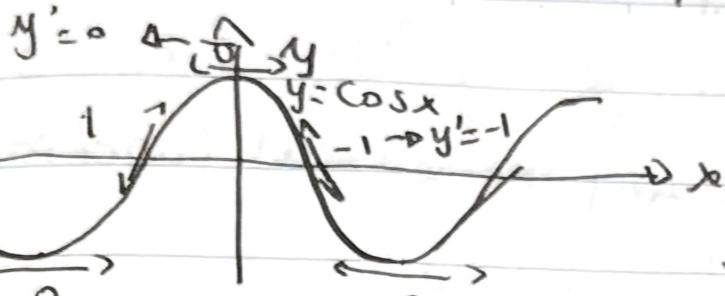
E.g.  $f(x) = x^2 + 2x^3 - x^2$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^5 + 2x^3 - x^2)$$

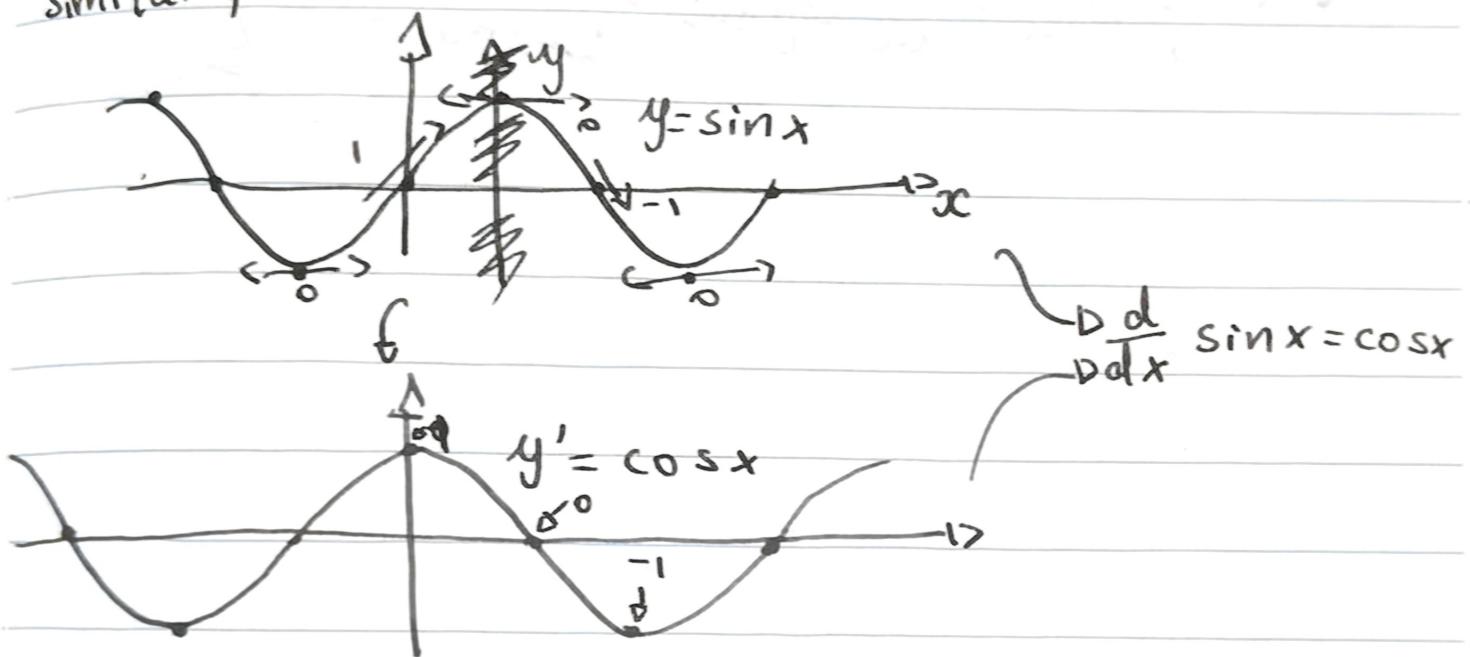
$$f'(x) = \frac{d}{dx}(x^5) + \frac{d}{dx}(2x^3) - \frac{d}{dx}(x^2)$$

$$f'(x) = 5x^4 + 6x^2 - 2x$$

# Derivatives of $\cos x$ , $\sin x$ , $e^x$ and $\ln x$



similarly



Summarised...

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \sin x = \cos x$$

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## Derivatives of Exponential Functions

$$\frac{d}{dx} a^x = \ln(a) \cdot a^x \quad \begin{array}{l} \text{e.g. } 2^x \text{ or } 5^x \\ \text{Any exponential function} \end{array}$$

$$\therefore \frac{d}{dx} e^x = \ln(e) \cdot e^x = \underbrace{e^x}_e$$

## Derivatives of logarithmic Functions

$$\frac{d}{dx} \log_a x = \frac{\ln(a) \cdot x}{\underbrace{e}_{\ln(e)}}$$

$$\therefore \frac{d}{dx} \ln x = \frac{d}{dx} \log_e x = \frac{1}{\ln(e) \cdot x} = \frac{1}{x}$$

# The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$$

In other words...

if  $h(x) = f(x) \cdot g(x)$

then...  $h'(x) = f'(x) \cdot g(x) + f(x)g'(x)$

# The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

You take the derivative of  $f$  multiplied by  $g$ , subtract  $f$  multiplied by the derivative of  $g$ , and divide all that by  $[g(x)]^2$

In other words...

if  $h(x) = \frac{f(x)}{g(x)}$

then...  $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

\*Note: Derivative of top then bottom (order matters as it is subtraction)

# Derivatives of all 6 trig functions

$$f(x) = \sin x \rightarrow f'(x) = \cos x$$

$$g(x) = \cos x \rightarrow g'(x) = -\sin x$$

$$h(x) = \tan x \rightarrow h'(x) = \frac{\cos x \cos x + \sin x + \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$k(x) = \cot x \rightarrow k'(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{csc}^2 x$$

$$m(x) = \sec x \rightarrow m'(x) = \frac{\sin x}{\cos^2 x} = \frac{\tan x}{\cos x} = \sec x \tan x$$

$$p(x) = \csc x \rightarrow p'(x) = -\frac{\cos x}{\sin^2 x} = -\frac{\cot x}{\sin x} = -\csc x \cot x$$

So...

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\operatorname{csc}^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

Notice how  
functions that  
start with  
"C" are always negative

Unit 3: Differentiation  
(composite, implicit, and inverse functions)

Date \_\_\_\_\_

## The Chain Rule

A rule used to find the derivatives of Composite Functions (i.e  $f(g(h(x)))$ )

$$h(x) = f(g(x)) \quad \begin{matrix} \nearrow \text{Inner} \rightarrow \text{Outer} \\ \downarrow \text{Chain rule} \end{matrix}$$

$$h'(x) = \frac{d}{dx}(g(x)) \cdot \frac{d}{d(g(x))}(f(g(x))) = g'(x) \cdot f'(g(x))$$

\* This equation works for as many functions as you wish, e.g.

$$a(x) = b(c(cde(f(\dots x))))$$

$$\frac{d}{dx} a(x) = \frac{d}{dx}(z(x)) \cdot \frac{d}{d(y(x))}(y(z(x))) \dots \cdot \frac{d}{d(e(de\dots x))}(b(\dots x))$$

\* NOTE: You only take the derivative of the outside.

$$\text{i.e. } \frac{d}{dx}(f(g(x))) = \overset{\text{only outermost}}{\underset{\text{is derivative}}{\overbrace{g'(x)}}} \cdot \overset{\text{is not } g'(x),}{\underset{\text{it is } g(x)}{\overbrace{f'(g(x))}}}$$

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## Common Chain Rule Misunderstandings

$$\frac{d}{dx}(\ln(\sin(x)))$$

$f(x) \rightarrow$   $\downarrow$   $g(x)$   
 $f(g(x))$

Requires Chain  
Rule to  
Solve

$$\neq \frac{d}{dx}(\ln(x)\sin(x))$$

$f(x) \rightarrow$   $\downarrow$   $g(x)$   
 $f(g(x))$

Requires Product  
Rule to  
Solve

Is a composite  
function

Not a composite  
function

# Implicit Differentiation

Explicit:

$$\text{e.g. } y = mx + b$$

↑  
Function

VS

Implicit:

$$\text{e.g. } ax^2 + by^2 = 1$$

↑  
Not a function  
(requires both  $x$  and  
 $y$  to solve, or other  
variables)

Usually, when differentiating explicit,

$$\frac{dx}{dx} = 1 \quad \text{so we don't have anything different.}$$

$$\text{i.e. } f(x) \quad f'(x) = \frac{d}{dx} x = 1$$

$$f(x) = x^2 \quad f'(x) = 2 \frac{d}{dx} x = 2x$$

↑  
for every variable "a",  
the derivative is  $\frac{d}{dx}(a)$  or

Since  $\frac{dx}{dx}$   
does not cancel  
be  $\frac{d}{dx}$  for  $y$ , it would  
only be  $\frac{dy}{dx}$

You leave it as is

$$\text{i.e. } x^2 + y^2 = 1$$

$$\text{LD } 2x \cancel{\frac{dx}{dx}} + 2y \frac{dy}{dx} = 0$$

solve if you want to

find  $\frac{dy}{dx}$  (or the instantaneous  
rate of change)

\*NOTE: You will need  
multiple inputs  
for implicit  
differentiation

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# Derivatives of inverse functions

Inverse functions are when the domain and range of a function are swapped.

I.E  $f(x) = mx + b \rightarrow x = f^{-1}(x) + b$   
 $f^{-1}(x) = \frac{x - b}{m}$

The diagram illustrates the relationship between an input  $x$  and its output  $f(x)$ . An arrow labeled  $f$  points from the input circle to the output circle. Another arrow labeled  $f^{-1}$  points from the output circle back to the input circle.

## Derivative Formula:

$$\frac{d}{dx}(f(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

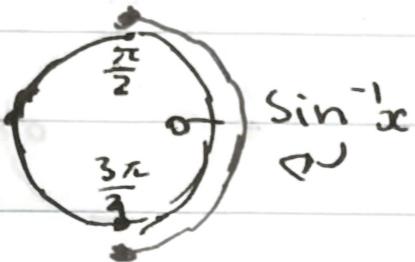
↑  
Derivative  
of regular  
function      ↑  
Inverse  
function

# Derivative of Inverse Sine

$$y = \sin^{-1}(x)$$

Domain:  $-1 \leq x \leq 1$

Range:  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



$$y = \sin^{-1} x \Rightarrow \sin y = x \Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}[x]$$

$$(\cos y) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

Now we write it in terms of sin.

$$\text{Trig identity: } \sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$\leftarrow$  since  $x = \sin y$  (2nd step)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$\therefore \frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$$

No.

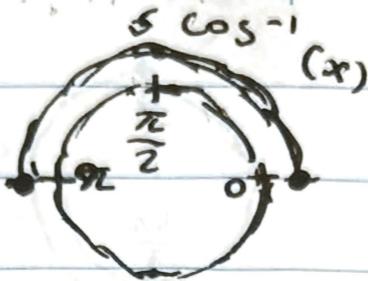
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# Derivative of inverse Cosine

$$y = \cos^{-1}(x)$$

Domain:  $-1 \leq x \leq 1$

Range:  $0 \leq y \leq \pi$



$$y = \cos^{-1} x \Rightarrow x = \cos y \xrightarrow{\text{Take derivative}} \frac{dx}{dy}(x) = \frac{d}{dx}(\cos y)$$

$$1 = (-\sin y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin y = \sqrt{1 - \cos^2 y}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2 y}}$$

$$\therefore \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

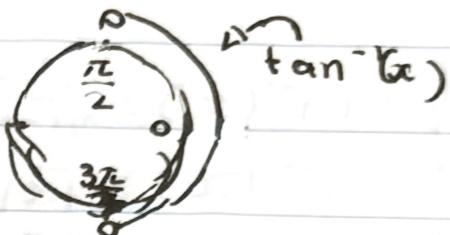
# Derivative of inverse Tangent

$$y = \tan^{-1}(x)$$

Domain:  $-\infty \leq x \leq \infty$

Range:  $-\frac{\pi}{2} < y < \frac{\pi}{2}$

\* Range does not include  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  because  $\tan z = \frac{\sin z}{\cos z}$  and  $\cos x = 0$  when  $x = -\frac{\pi}{2}$  and  $\frac{\pi}{2}$



$$y = \tan^{-1} x \Rightarrow x = \tan y$$

$$\frac{dx}{dx}[x] = \frac{dx}{dx}[\tan y] \Rightarrow 1 = \frac{1}{\cos^2 y} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \cos^2 y \cdot 1 \rightarrow \text{express as tangent of } y$$

$$= \frac{\cos^2 y}{\cos^2 y + \sin^2 y} \cdot \frac{\frac{1}{\cos^2 y}}{\frac{1}{\cos^2 y}} \left\{ \begin{array}{l} = 1 \\ \text{Value did not change} \end{array} \right\} = \frac{1}{1 + \left(\frac{\sin y}{\cos y}\right)^2} = \frac{1}{1 + \tan^2 y}$$

$$* \frac{\sin}{\cos} = \tan$$

Since  $x = \tan y \dots$

$$\therefore \frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}$$

No.

Date

# Derivatives of inverse trig functions

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2+1}$$

$$\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{x^2+1}$$

Just negative

Simplifying  $\sec^{-1}(x)$  derivatives  
(and  $\csc^{-1}(x)$  derivatives)

→ Due to the fact that there exists an absolute value in the derivative, we must always check the sign (plug in negative number to see if we need absolute value or not).

Usually,  $x=-1$  is the easiest

$$\text{E.g. } \frac{9x^2}{|3x|\sqrt{9x^2-1}} \quad \begin{matrix} \leftarrow \text{Plug in negative} \\ \text{number} \end{matrix}$$

$\frac{9(-1)}{|3(-1)|\sqrt{9(-1)^2-1}} \leftarrow \text{Positive, so simplification MUST include absolute value}$

$$\frac{9x^2}{|3x^3|\sqrt{9x^2-1}} = \frac{8x^2 \cdot 3}{3x|x|\sqrt{9x^2-1}} = \frac{3}{|x|\sqrt{9x^2-1}}$$

However, if the original would've been negative, then the absolute value must be removed:

$$\text{E.g. } \frac{4x}{|2x^3|\sqrt{4x^2-1}} = \frac{2}{2x\sqrt{4x^2-1}}$$

# Derivative Rules Recap...

Power Rule:  $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

Sum Rule:  $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference Rule:  $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Product Rule:  $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient Rule:  $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$

Exponential Rule:  $\frac{d}{dx}[a^x] = \ln(a) \cdot a^x$

< Natural Exponential:  $\frac{d}{dx}[e^x] = e^x >$

Logarithm Rule:  $\frac{d}{dx}[\log_a x] = \frac{1}{(ln(a)) \cdot x}$

< Natural Logarithm:  $\frac{d}{dx}[\ln x] = \frac{1}{x} >$

Chain Rule:  $\frac{d}{dx}[f(g(x))] = g'(x) \cdot f'(g(x))$

Inverse Rule:  $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$

Also...

Implicit Differentiation: Differentiate  $y$  as you do

(NOTE:  $\frac{d}{dx}[\cos(x+y)]$  would with  $x$ , just multiply

mean that  $\frac{dy}{dx}$  is derivative with  $\frac{dy}{dx}$  in the end  
not necessary for the outer derivative or  $f'(g(x))$ , only inner) and solve for it.

Trig Derivatives (6 Functions + 6 Inverse functions)

$\frac{d}{dx}[\cos x] = -\sin x$ ,  $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$ ,  $\frac{d}{dx}[\csc x] = -\csc x \cdot \cot x$ ,  
 $\frac{d}{dx}[\tan x] = \sec^2 x$ ,  $\frac{d}{dx}[\cot x] = -\operatorname{csc}^2 x$

$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$ ,  $\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$ ,  $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$ ,  $\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$ ,  
 $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$ ,  $\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{x^2+1}$

# Higher-Order Derivatives

Notation:

$y$	$f(x)$	$y$
1st Derivative: $y'$	$f'(x)$	$\frac{dy}{dx}$
2nd Derivative: $y''$	$f''(x)$	$\frac{d^2y}{dx^2}$
3rd Derivative: $y'''$	$f'''(x)$	$\frac{d^3y}{dx^3}$
$n^{th}$ Derivative: $y^{(n)}$	$f^{(n)}(x)$	$\frac{d^n y}{dx^n}$

The  $n^{th}$  derivative just means that you differentiate it  $n$  times.

i.e. Differentiate function to first derivative, then differentiate the first derivative to the second derivative ... and so on until you reach the  $n^{th}$  derivative.

$n$  times

$$\underbrace{\frac{d}{dx} \left[ \frac{d}{dx} \left[ \frac{d}{dx} \left[ \dots \frac{d}{dx} [f(x)] \dots \right] \right]}_{n \text{ times}}$$

E.g.  $y = x^3 + x^2 + x + 1$

$$\frac{dy}{dx} = 3x^2 + 2x + 1$$

$$\frac{d^2y}{dx^2} = 6x + 2$$

$$\frac{d^3y}{dx^3} = 6$$

$$\frac{d^4y}{dx^4} = 0$$

Due to the way derivatives work, you will eventually reach a point where the derivative is equal to 0.

## Second Derivatives, Implicit Equation

1. Differentiate like you do with first Implicit Derivatives
2. Differentiate it again (same rules)
3. When you have  $\frac{dy}{dx}$  alongside  $\frac{d^2y}{dx^2}$ , plug in the first derivative ( $\frac{dy}{dx} =$ ) instead of  $\frac{dy}{dx}$
4. Simplify.

E.g.  $y^2 - x^2 = 4 \Rightarrow \frac{d}{dx}[y^2 - x^2] = \frac{dy}{dx}[4]$

$$2y \cdot \frac{dy}{dx} - 2x = 0$$

$$\left( \frac{dy}{dx} = \frac{x}{y} \right)$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{x}{y}\right]$$

$$\frac{d^2y}{dx^2} = 1 \cdot y^{-1} + x \cdot (-1) \cdot y^{-2} \cdot \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x^2}{y^3}$$

## Disguised Derivatives

$$\text{E.g. } \lim_{h \rightarrow 0} \frac{5 \log_{10}(2+h) - 5 \log_{10}(2)}{h}$$

Match with Derivative definition

$$f(x) = f(x)$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

So find similarities

Scalar x limit  
so scalar  
can be  
factored.  
But ...

$$\rightarrow f(ath) = \log(2+h)$$

$$\therefore 5 \lim_{h \rightarrow 0} \frac{\log(2+h) - \log(2)}{h} \rightarrow f(a) = \log(2)$$

$$\therefore 5f'(2) = 5 \lim_{h \rightarrow 0} \frac{\log(2+h) - \log(2)}{h} \quad a=2$$

$$\therefore 5 \cdot f'(2) = 5 \cdot \frac{1}{[\ln(10)]2} = \frac{5}{\ln(10) \cdot 2}$$

Unit 4:  
Contextual Applications of  
Differentiation

Interpreting the meaning of the derivative in context

1. Interpret the meaning of function,  
i.e  $f(x)$

- Check to see if it is the rate ALREADY
  - e.g. Velocity ( $v_{\text{cal}}$ ) is a rate but position is not.

2. Check what is domain and range

e.g.  $x$ , input      e.g.  $y$ , output

- In some scenarios it'll say:

• The function function models the range as a function of the domain.

OR

• Scenario modeled by the function function where range and domain

OR

• The function function range domain

\* Sometimes they'll explicitly say what each variable is.

\* Although time is usually domain, it could also be the range (output)

### 3. Interpret the derivative

- The range is changing/increasing/decreasing at a rate of

*It is important to state that it is a rate*

- \* If it was already a rate, then say so in the beginning as well  
I.e. The rate is changing at a rate of ...

### 4. Units

- Units for the derivative are based on the slope ( $\frac{\Delta y}{\Delta x}$ ) so units are range/domain.

- For the second derivative it would be range of 1st derivative/domain = range/domain = range/domain<sup>2</sup>

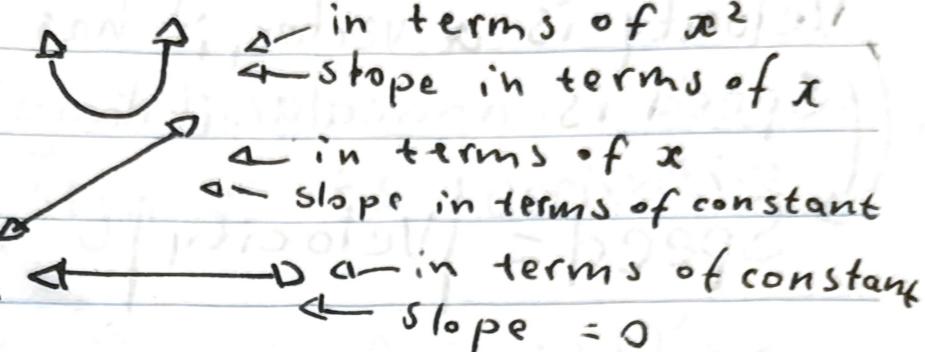
Depending on the context, you can use this format: At domain context, the Range context (rate or not rate)

At domain context, the Range context

changes/increases/decreases at a rate of derivative units."

# One-Dimensional Motion

Position: e.g.



Velocity: e.g.

Acceleration: e.g.

Position function:  $x(t)$

Velocity function:  $v(t) = x'(t)$

Acceleration function:  $a(t) = v'(t) = x''(t)$

Jerk Function:  $x'''(t)$

Snap (Jounce) Function:  $x^{(4)}(t)$

Crackle Function:  $x^{(5)}(t)$

Pop Function:  $x^{(6)}(t)$

slopes (derivatives of previous)

$\therefore$  Velocity = Rate of Change of position

$v(t) < 0$  means the particle is moving left (x-axis) or down (y-axis)  
if it is negative check which graph (pos-time or ve(-time))

$v(t) > 0$  means the particle is moving right (x-axis) or up (y-axis)

$v(t) = 0$  means the particle is at rest

Since velocity is just the slope, average velocity is just the average rate of change:

Average Velocity =  $\frac{s(b) - s(a)}{b - a}$  on the interval  $[a, b]$

You can calculate displacement using the times at which it changes direction (x-ints of velocity) and plugging it into position

Newton's Law of motion -

Velocity is a vector, it has direction  
 (Speed is a scalar, it does not have direction)

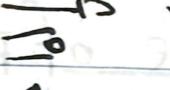
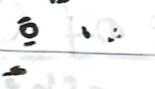
$$\rightarrow \text{Speed} = |\text{Velocity}|$$

Speeding Up or slowing down.

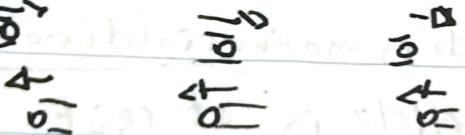
$\rightarrow$  When  $a(t)$  and  $v(t)$  have same signs,  
 it is speeding up.

i.e.  speed up  
 speed up

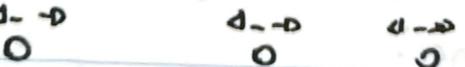
$\rightarrow$  When  $a(t)$  and  $v(t)$  have different signs,  
 it is slowing down.

i.e.  slow down  
 slow down

$\rightarrow$  When  $a(t) = 0$ , it has a constant velocity

i.e. 

$\rightarrow$  When  $v(t) = 0$ , it is at rest

i.e. 

\* since  $a(t) = v'(t)$ , you look at the slope  
 at a point in a velocity-time graph  
 (be careful of non-differentiable points)

# Rates of Change other than Motion

To know if something is increasing or decreasing, check the sign of its derivative.

E.g.

Height is increasing if [height]' > 0

Velocity is decreasing if [velocity]' < 0

$\theta$  is increasing if  $\theta' > 0$

$k$  is decreasing if  $k' < 0$

Is the function already a rate of change?

Not a rate of change:

"If  $f(x)$  is the bunny population after  $x$  years, then what is  $f'(x)$ ?"

↳ The rate of change of the bunny population per year

Is a rate of change:

"If  $f(x)$  is the rate at which a bunny population increases (bunnies per year), then what is  $f'(x)$ ?"

↳ The rate of change of the rate at which a bunny population increases, changes in bunnies per year per year.

# Mathematical Symbol Use:

Union: " $\cup$ " meaning "or"

$\hookrightarrow f(x)$  is increasing for  $t \cup r$

\*Not "and" because either one increases

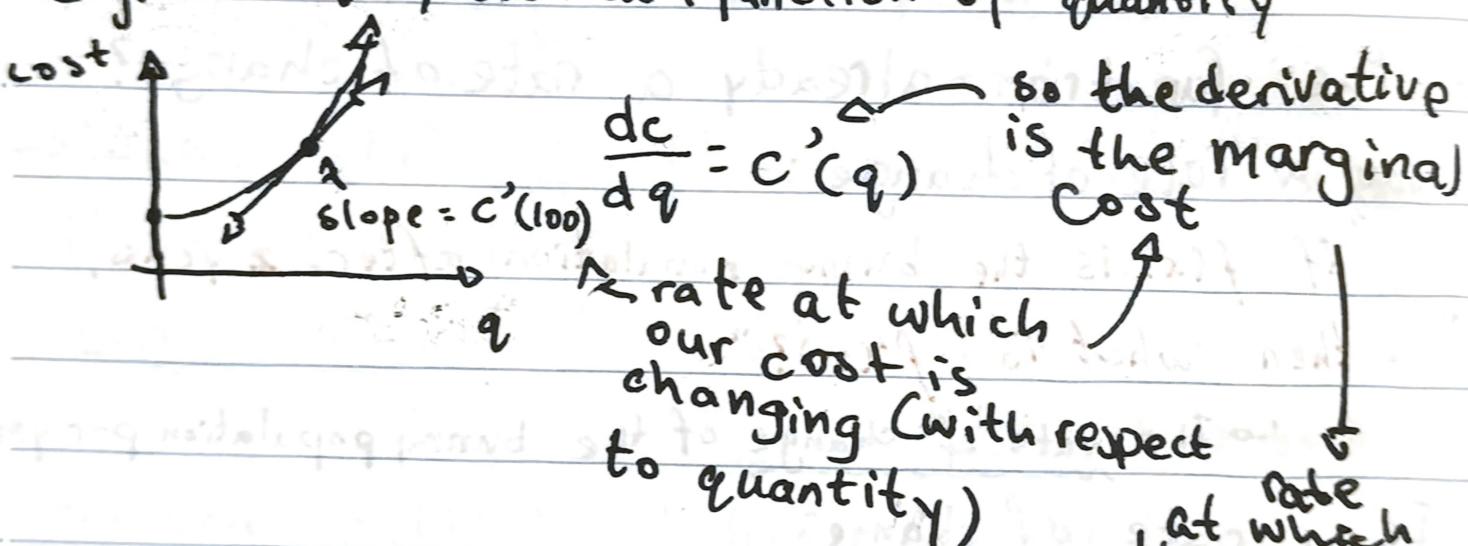
Intersection: " $\cap$ " meaning "and"

$\hookrightarrow f$  exists in  $A \cap B$

\*Both criterias must be met at the same time

## Marginal Cost

E.g.  $c(q)$ , cost as a function of quantity



# Related Rates - Intro

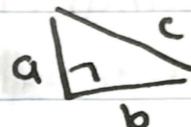
Many variables have a related relationship. Therefore their rates (derivatives) are related.

→ We use implicit differentiation to solve for their relationships.

E.g. Pythagorean Theorem

$$a^2 + b^2 = c^2$$

$$\Rightarrow \frac{d}{dt}[a^2] + \frac{d}{dt}[b^2] = \frac{d}{dt}[c^2]$$



$$\Rightarrow 2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}$$

\* In this case, we chose  $t$  as the domain,  
i.e.  $\frac{da}{dx}$  but we can choose  
anything else (depending on  
domain given scenario)

Differentiate your relationship with respect to one specific variable

(could be time like in our previous example)

→ Some dimensions are constants (i.e. they do not change, derivatives are 0)

↳ in that case, plug in 0 for that specific derivative (to simplify relationship)

↳ E.g. if  $b$  is constant, then  $\frac{db}{dt} = 0$ , so

$$2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}$$

$$\Rightarrow 2a \frac{da}{dt} = 2c \frac{dc}{dt}$$

No.

Date

## Related Rates - Intro (continued...)

To find a relationship between given rates of change:

1. Set up a relationship using the given variables for the situation

2. Differentiate with respect to the given domain (which is found in  $\frac{d}{dt}$ )

- It is usually time ( $t$ ) but not always

3. Simplify (if possible, depending on situation)  
• Look for constant variables  
(derivative = 0)

# Solving Related Rates

1. Draw a picture
  2. Make a list of all known and unknown rates and quantities
  3. Relate the variables in an equation
  4. Differentiate with respect to the given domain (usually time)
  5. Substitute the known quantities/rates and solve.
- \* Substituting a non-constant quantity before differentiating is not allowed.

(Some times you may have to solve for an unknown quantity, in original equation, to get the final answer)

$$\rho d = \frac{N}{P}(d)$$

$$f(d) = \frac{N}{P}(d)$$

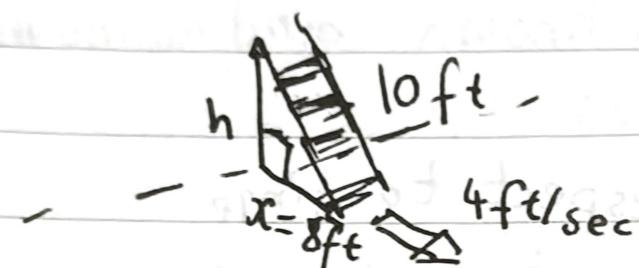
$$f'(d) = \frac{N}{P}'(d)$$

$$f'(d) = \frac{N'}{P}(d)$$

$$f'(d) = \frac{N}{P'}(d)$$

# Solving Related Rates - Example

Falling ladder:



How fast is  $h$  changing  
(at what rate)?

Since a ladder's length physically cannot change,

$$\frac{d[\text{length}]}{dt} = 0$$

Also, the given scenario resembles a right triangle, which we can create a relationship using the pythagorean theorem.

$$h^2 + x^2 = [\text{length}]^2$$

$$\therefore \frac{d}{dt}[h^2 + x^2] = \frac{d}{dt}[\text{length}^2] \text{ constant, so } = 0$$

$$\Rightarrow 2h \frac{dh}{dt} + 2x \frac{dx}{dt} = \frac{d[\text{length}^2]}{dt} 2 [\text{length}]$$

$$\text{Plug in: } 2h \frac{dh}{dt} + 16 - 4 = 0$$

$$2h \frac{dh}{dt} + 64 = 0$$

$\uparrow$   
we need  $h$ , so...

$$h^2 + 64 = 100$$

$$h^2 = 36$$

$$h = 6$$

$$2(6) \frac{dh}{dt} = -64$$

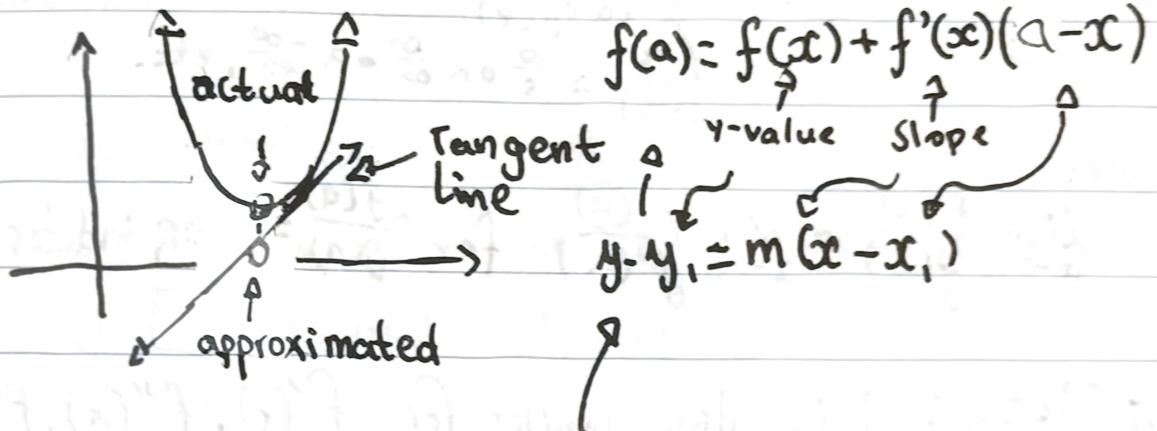
$$\frac{dh}{dt} = -\frac{64}{12}$$

$$= -\frac{16}{3} \text{ ft/s}$$

## Local Linearity

A method to obtain an estimated output using the tangent line of a nearby value.

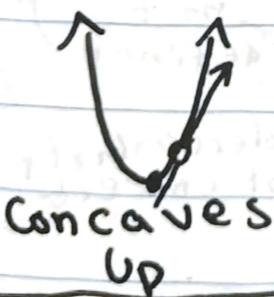
E.g.



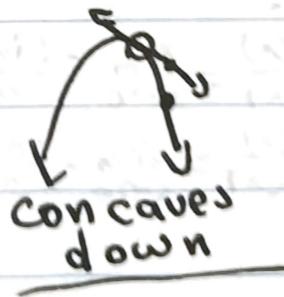
\* You find tangent line equation for a nearby value.

You plug in the domain to obtain an estimate

## Concavity



Underestimates Value



Overestimates Value

\* Since Local Linearity relies on derivatives, the function must be differentiable (around the selected value).

# L'Hôpital's rule introduction

- Using derivatives to find limits

↳ limits that are indeterminate

↳ i.e.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ , with direct substitution  
 $\frac{f(a)}{g(a)} = \frac{0}{0}$  or  $\frac{\infty}{\infty}$  or  $\frac{-\infty}{-\infty}$ , etc.

Rule:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \text{ for } \frac{f(a)}{g(a)} = \frac{0}{0} \text{ or indeterminate form}$$

\* Note: This also works for  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$   
(with 1 level below/high)

However, it only works until it is defined (not indeterminate)

E.g.  $\lim_{x \rightarrow 0} \frac{\sin x}{x} \Rightarrow \frac{\sin(0)}{0} = \frac{0}{0}$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1 \quad \begin{cases} \text{limit is} \\ \text{defined} \end{cases}$$

However...

$$\lim_{x \rightarrow 0} \frac{f''(x)}{g''(x)} = \lim_{x \rightarrow 0} -\frac{\sin x}{0} \Rightarrow -\frac{\sin(0)}{0} = \frac{0}{0} \quad \begin{cases} \text{no t} \\ \text{defined} \end{cases}$$

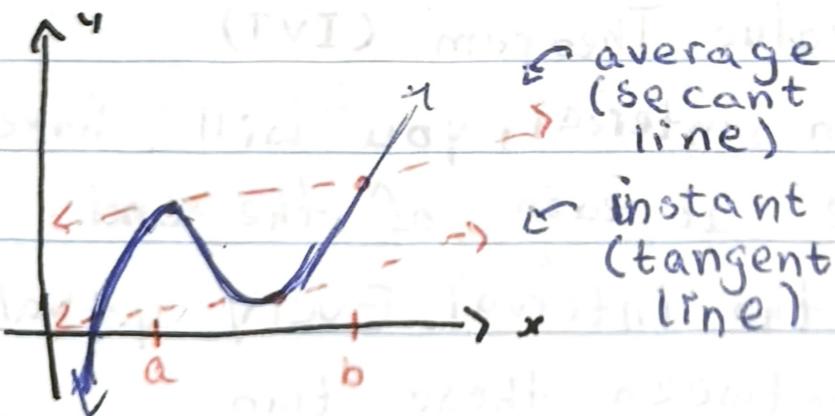
$$\lim_{x \rightarrow 0} \frac{f'''(x)}{g'''(x)} = \lim_{x \rightarrow 0} -\frac{\cos x}{0} = -\frac{1}{0} \quad \begin{cases} \text{Not indeterminate} \\ \text{but not correct} \end{cases}$$

# The Mean Value Theorem

Justification for conclusions about a function over an interval.

Mean Value Theorem:

"If a function  $f$  is continuous over the interval  $[a, b]$  and differentiable over the interval  $(a, b)$ , then there exists a point  $c$  within that open interval where the instantaneous rate of change equals the average rate of change over the interval."



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Conditions:  $f$  must be CONTINUOUS over  $[a, b]$  and DIFFERENTIABLE over  $(a, b)$

2 tip  
continuous graph  
continuous system

## MVT vs IVT

### Mean Value Theorem (MVT)

→ The derivative (instantaneous rate of change) must equal the average rate of change somewhere in the interval given the function is both continuous and differentiable over that interval.

→ E.g. Speed

→ Given Avg Speed is 82 mph  
then at some point speed MUST equal 82 mph exactly.

### Intermediate value Theorem (IVT)

→ On a given interval, you will have a y-value at each of the end points of the interval. Every y-value exists between these two y-values at least once in the interval given that the function is continuous over that interval.

→ E.g. Speed

→ If Speed at one point was 25 mph and at another was 50 mph then the speed between those two points must equal 30 mph at least once.

## Mean Value Theorem Uses

Since the Mean Value Theorem justifies how there always exists a derivative equal to the average rate of change over a continuous interval, we can use it to find at which point specifically

$$\text{E.g. } f(x) = x^2 - 6x + 8 \text{ over } [2, 5]$$

$$f(5) = 25 - 30 + 8 = 3$$

$$f(2) = 0$$

$$\therefore \frac{f(5) - f(2)}{5 - 2} = \frac{3}{3} = 1$$

$$f'(x) = 2x - 6$$

$$\Rightarrow 1 = 2x - 6$$

$$x = \frac{7}{2}$$

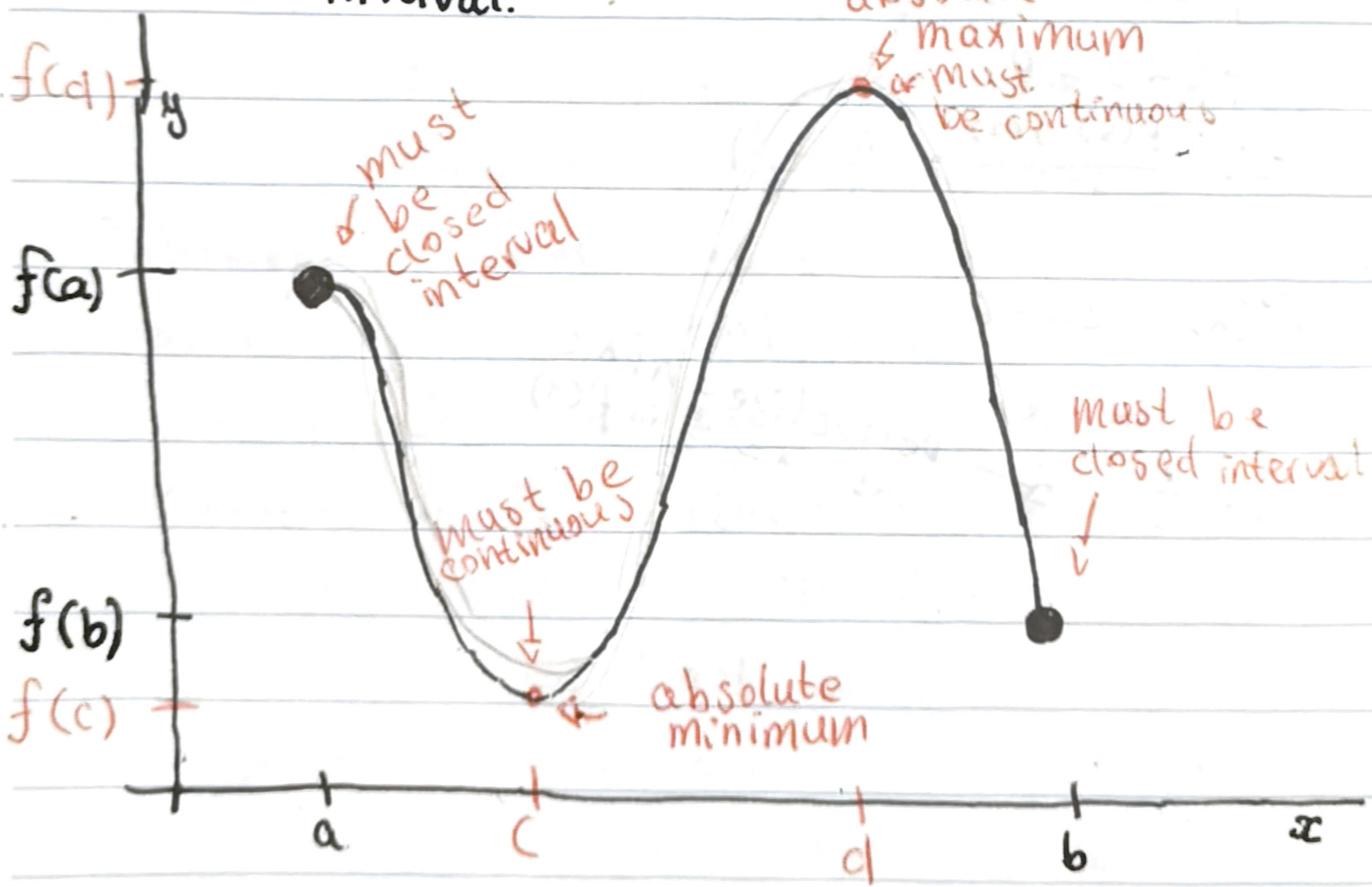
Derivative = A.R.O.C  
at  $x = \frac{7}{2}$  in  $f(x)$

# The Extreme Value Theorem

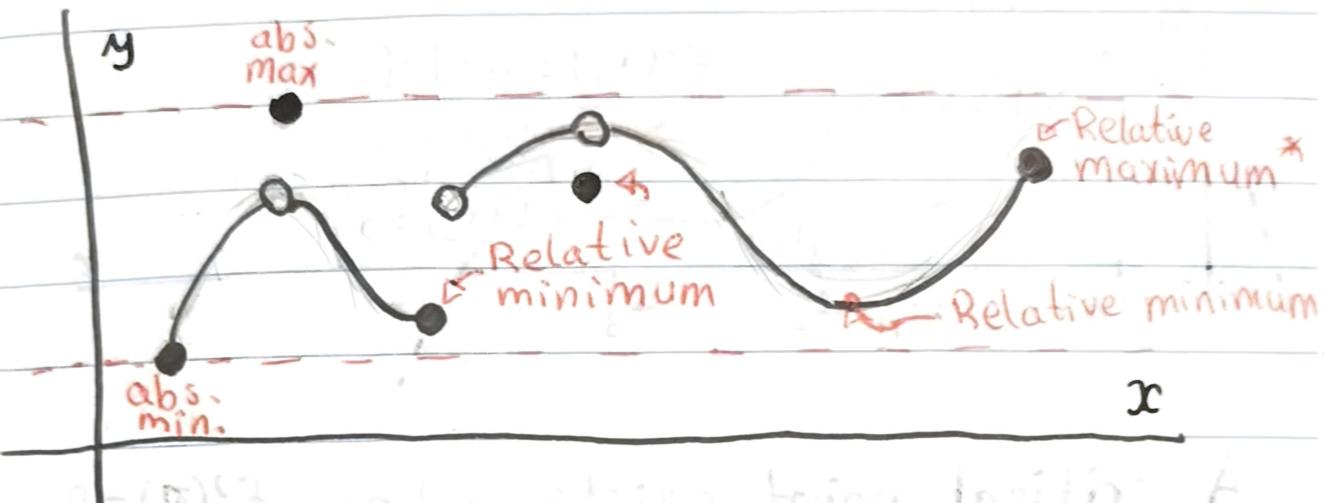
"If a function is continuous over the interval  $[a, b]$ , then  $f$  has at least one minimum value and at least one maximum value on  $[a, b]$ ."

Condition:  $f$  is continuous over  $[a, b]$

Theorem: There exists at least one absolute maximum and one absolute minimum over that interval.



# Extremas



\*Whether an endpoint (an endpoint) can be classified as a relative/local extrema (i.e. rel. max and rel. min) is disputed amongst Mathematicians.

- A relative minimum/maximum can still exist even if it's not continuous (jump discontinuity), it only has to be relatively minimum/maximum compared to the points around it.

Global/Absolute Extrema: True for entire function or interval

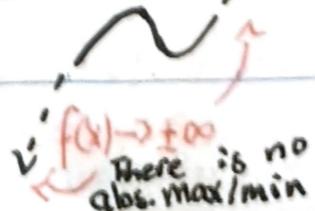
Local/Relative Extrema: True for points around a point or it is the limit of

Horizontal Line:



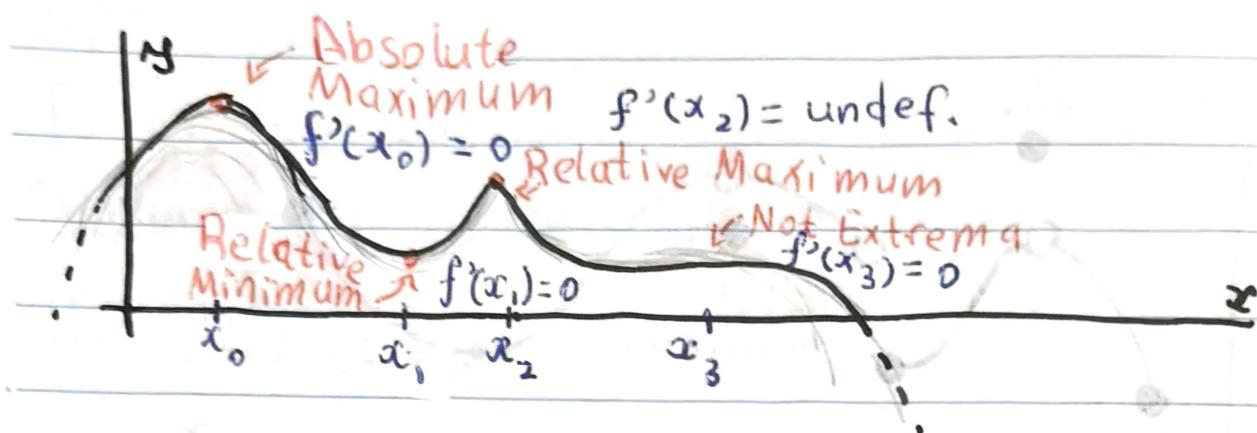
every point is an absolute/relative max./min.

Extremas D.N.E.:

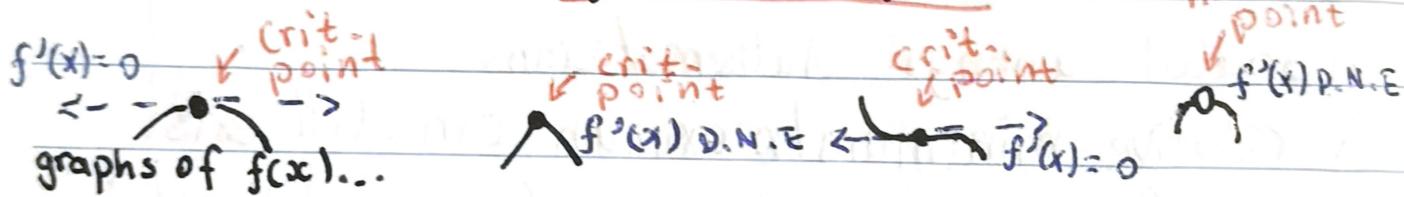


There is no abs. max/min

# Critical Points



A critical point exists when  $f'(a) = 0$  or  $f'(a)$  is undefined. A point may NOT be a critical point if  $f(a)$  is undefined.



A critical point is a point that has a possibility of being an extrema (i.e. max/min.) (the x-value of a critical point is known as a critical number)

- All local extrema that aren't endpoints are crit. Points
- All global extrema that aren't endpoints are crit. Points
- Not all crit. Points are extrema

↳ i.e.  $f'(a) = 0$  but  $x=a$  is not an extrema

Check for zeroes (numerator), undefined (zeroes in denominator, square roots or logarithms)  
 $\langle e^x \text{ will never equal } 0, \text{ H.A. at } y=0 \rangle$

# Increasing and Decreasing Intervals

When the slope of a function is positive, the function is increasing.

When the slope of a function is negative, the function is decreasing.

When the slope of a function is 0, the function is neither increasing nor decreasing.

Critical Points have a possibility of being an extrema or a sign change.

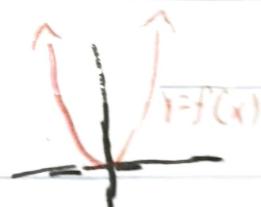
→ We can use Critical Points to build a sign chart.

$$\text{E.g. } f(x) = x^2$$

$$\Rightarrow f'(x) = 2x$$

Crit. Pt(s) at  $x = 0$

So...	$x$	$(-\infty, 0)$	$0$	$(0, \infty)$
	$f'(x)$	-	0	+



Plug in values to  $f''(x)$  to see if result is positive or negative

Justif.

$f$  is increasing for  $x > 0$  because  $f'(x) > 0$

$f$  is decreasing for  $x < 0$  because  $f'(x) < 0$

No.  
Date

Unit 5:  
Applying Derivatives  
to analyse functions  
(continued)

## The First Derivative Test

- Using the first derivative to "test" whether or not a function has a maximum or minimum (only for cont. functions)

### Justification statements

Assume  $c$  and  $d$  are critical numbers of a function  $f$ .

- There is a maximum value at  $x=c$  because  $f'(x)$  changes from positive to negative around  $x=c$ .
- There is a minimum value at  $x=d$  because  $f'(x)$  changes from negative to positive around  $x=d$ .

E.g.  $f(x) = \frac{x^3}{3} - x$   
 $\Rightarrow f'(x) = x^2 - 1 = (x+1)(x-1)$

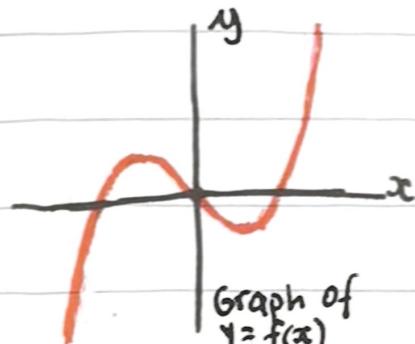
$$\therefore x \begin{array}{|c|c|c|c|c|} \hline & (-\infty, -1) & -1 & (-1, 1) & 1 & (1, \infty) \\ \hline f'(x) & + & 0 & - & 0 & + \\ \hline \end{array}$$

$$f'(-100) = (-99)(101) \quad f'(100) = (101)(99)$$

$$= + \rightsquigarrow \quad = + \rightsquigarrow$$

$$f'(0) = 1 \cdot (-1)$$

$$= -1$$



Graph of  
 $y = f(x)$

$\therefore f'(x)$  changes from + to - around  $x = -1$

$\therefore$  There is a relative maximum at  $x = -1$

$\therefore f'(x)$  changes from - to + around  $x = 1$

$\therefore$  There is a relative minimum at  $x = 1$

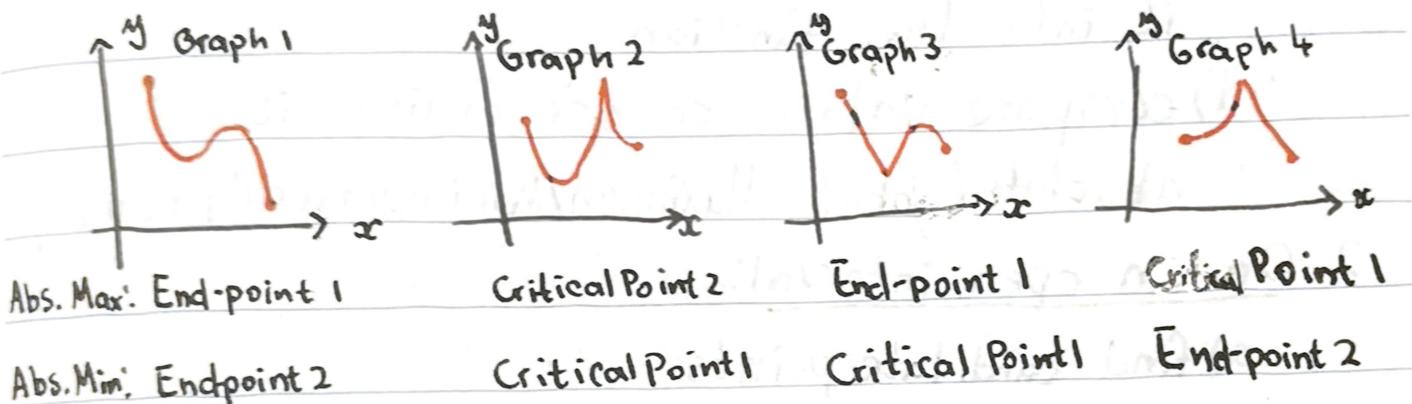
When a question asks for "What is the maximum value?",

it asks for the range (e.g.  $y$ ,  $f(x)$ ,  $g(x)$ )

When a question asks for "Where is the maximum?",  
it asks for the domain (e.g.  $x$ ,  $t$ )

# The Candidates Test

Candidates - The points that have a possibility of being an absolute or global extrema.



End-points (of a closed-interval) and Critical Points both have a possibility of becoming an absolute or global extrema, so they're both candidates.

\*NOTE: Endpoints only classify as a candidate if they're that of a closed interval (in an open interval, there may not be an absolute extrema; in an unbounded interval, the function may approach  $\pm\infty$  meaning there may not be an absolute extrema)

# Determine Absolute / Global Extrema from Candidates

## 1. On a closed interval:

- find critical points
- find end-points
- find the value of said points when you plug it into the function
- compare values to see which is absolute (global) Maximum/Minimum (could be multiple)

## 2. On an open interval:

- find critical points
- find end-points
- find the value of said points when you plug it into the function
- compare values to see which is the absolute or global maximum/minimum

↳ If end-point is, then put D.N.E

↳ If multiple crit. points then put them all

↳ If both end-point and crit. point, then only put the crit. points

test material shift 1st

### 3. On an unbounded interval:

- a) find critical points
- b) evaluate the critical points
- c) use the first derivative test to see which points are minimum or maximum and the end behaviour of function
- d) compare values and take into account the end behaviour of the function  
(if  $\lim_{x \rightarrow \pm\infty} f(x)$  is unbounded then there does not exist an absolute/global maximum/minimum depending on the case).

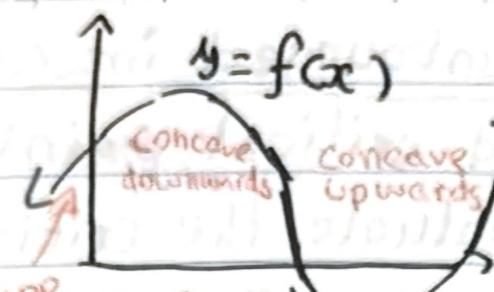
# Determining Concavity

## Concave Up

- Slope or  $f'(x)$  is increasing

- $f''(x) > 0$

- There is a minima

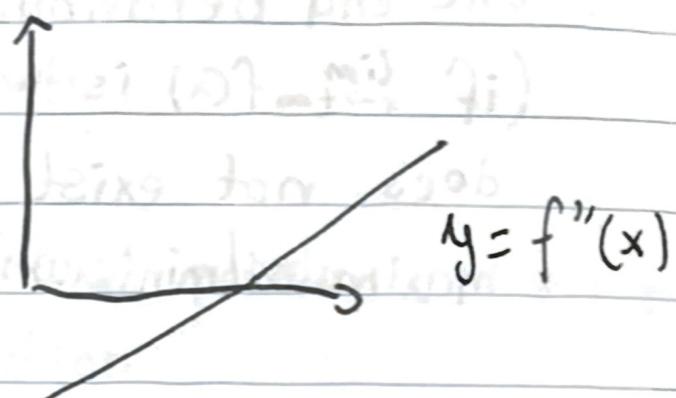
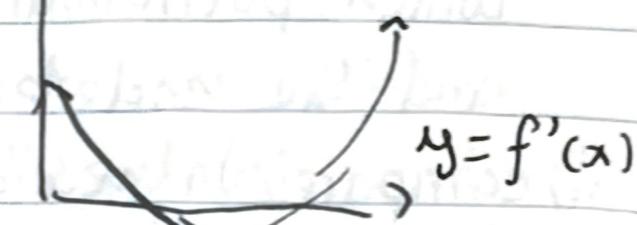


## Concave down

- Slope or  $f'(x)$  is decreasing

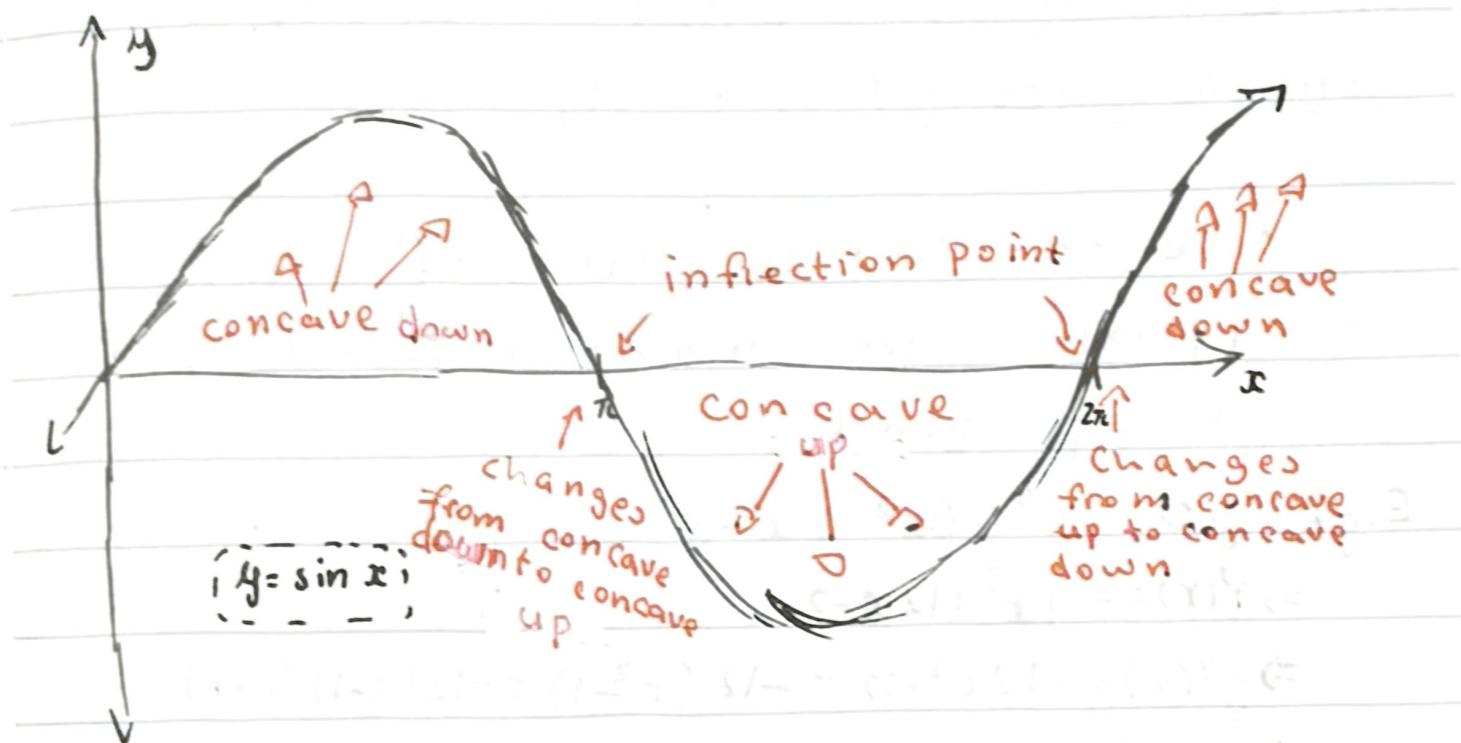
- $f''(x) < 0$

- There is a maxima

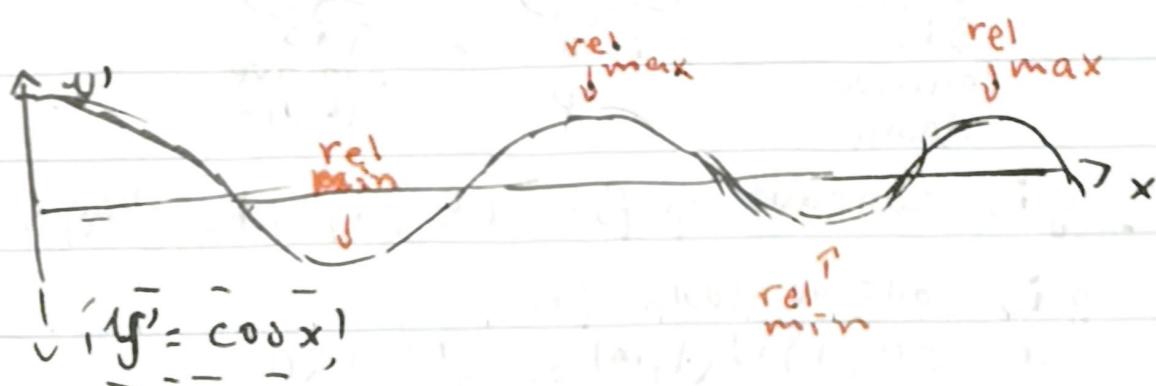


## Inflection Point

- The point of  $f''(x)$  at  $x = c$  if  $f(c)$  is defined and  $f''(x)$  changes signs at  $x = c$
- It's the point where the graph changes concavity



$f'(x)$  changes from increasing to decreasing or vice versa. Thus,  $f'(x)$  has a local or relative extrema at  $x = c$



## Determining Concavity (Algebraic)

Because concavity is determined by the sign of  $f''(x)$ , we can find the intervals where  $f''(x) > 0$  or  $f''(x) < 0$  to find on which intervals  $f(x)$  is concave up or concave down respectively.

- We can find the zeroes of  $f''(x)$  ( $f''(x) = 0$  for which  $x$ -values?) and make a sign chart.

$$\text{E.g.: } g(x) = -x^4 + 6x^2 - 2x - 3$$

$$\Rightarrow g'(x) = -4x^3 + 12x - 2$$

$$\Rightarrow g''(x) = -12x^2 + 12 = -12(x^2 - 1) = -12(x-1)(x+1)$$

$$\therefore x = \pm 1$$

$x$	$(-\infty, -1)$	$-1$	$(-1, 1)$	$1$	$(1, \infty)$
$g''(x)$	-	0	+	0	-

$\underbrace{\phantom{0}}_{g \text{ is concave down}}$        $\underbrace{\phantom{0}}_{g \text{ is concave up}}$        $\underbrace{\phantom{0}}_{g \text{ is concave down}}$

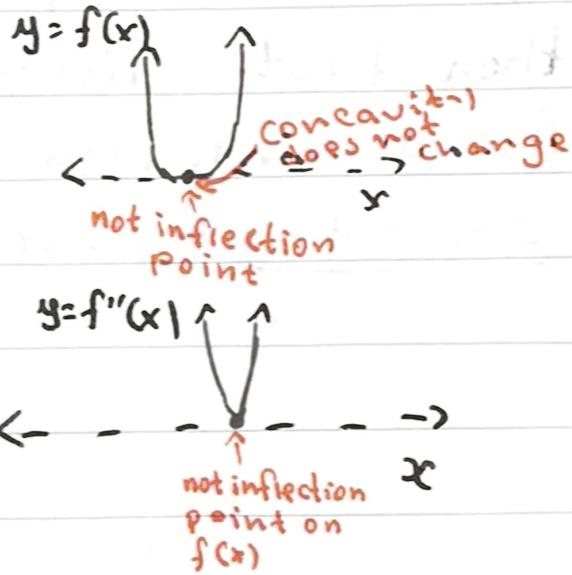
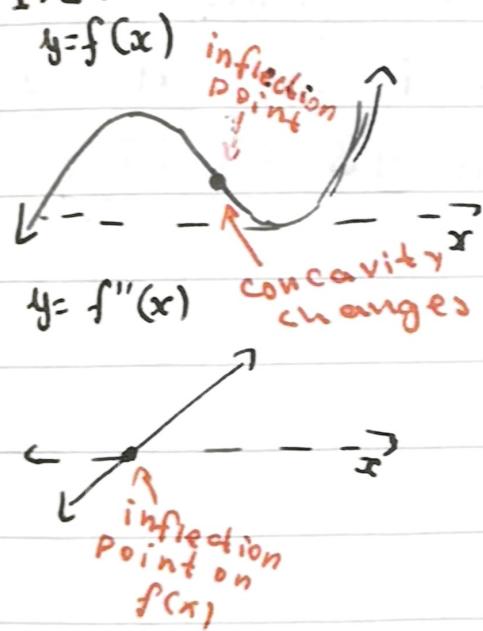
$\therefore g$  is concave up for  $x \in (-1, 1)$  bc  $g''(x) > 0$

$g$  is concave down for  
 $x \in (-\infty, -1) \cup (1, \infty)$  bc  $g''(x) < 0$

# Finding Inflection Points (Algebraic)

To find an inflection point,  $f''(x)$  must equal 0 or is undefined and has to change signs from either positive to negative or negative to positive  
 ↳ It cannot change from one sign to 0 then back to the same sign.

I.E.



$$y = f''(x)$$

not inflection point on  $f(x)$

## Common Mistakes:

- $f''(x)$  can also be undefined
- Not checking if  $f''(x)$  changes signs (crosses x-axis)
- $f(x)$  MUST be defined

## Second Derivative Test

Suppose  $f'(c) = 0 \dots$

•  $f''(c) < 0$ :  $f$  has a relative max value at  $x=c$

•  $f''(c) = 0$ : inconclusive

•  $f''(c) > 0$ :  $f$  has a relative min value at  $x=c$

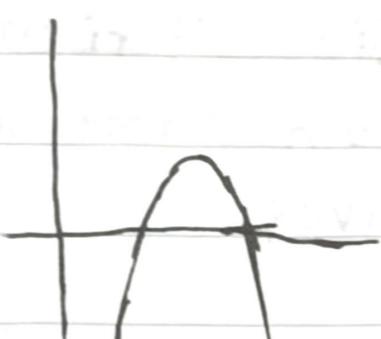
If there is only one critical point, and that critical point is an extremum (max or min), then it is an ABSOLUTE extremum (max or min).

# Relating Derivatives ( $f(x)$ , $f'(x)$ , $f''(x)$ )

$$y = f(x)$$



$$y = f'(x)$$



$$y = f''(x)$$



$$\underline{f(x)}$$

$$\underline{f'(x)}$$

$$\underline{f''(x)}$$

$f(x)$  is increasing

$$f'(x) > 0$$

[Inconclusive]

$f(x)$  is decreasing

$$f'(x) < 0$$

[Inconclusive]

$f(x)$  has a critical point

$$f'(x) = 0 \text{ or is undefined}$$

[Inconclusive]

$f(x)$  has a relative maximum

$$f'(x) \text{ changes from pos. to neg.}$$

[Inconclusive]

$f(x)$  has a relative minimum

$$f'(x) \text{ changes from neg. to pos.}$$

[Inconclusive]

$f(x)$  is concave up

$$f'(x) \text{ is increasing}$$

$$f''(x) > 0$$

$f(x)$  is concave down

$$f'(x) \text{ is decreasing}$$

$$f''(x) < 0$$

$f(x)$  has a point of inflection

$$f'(x) \text{ has a relative extrema}$$

$$f''(x) \text{ changes signs}$$

# Optimization

- Process of selecting the best element given some specific criterion (usually max/min) and condition from some set of available alternatives

- To do so, you must develop equations with the given and simplify accordingly.  
(Then use calculus to solve for the wanted value)

E.g. "What is the smallest possible sum of squares of two numbers, if their product is -16?"

Equation:

$$\text{Minimise } S = x^2 + y^2 \quad \text{where } xy = -16 \Rightarrow y = -\frac{16}{x}$$

$$S(x) = x^2 + \left(-\frac{16}{x}\right)^2$$

$$\text{empic approach} \Rightarrow x^2 + 256x^{-2}$$

Find min:

$$s'(x) = 2x - 512x^{-3}$$

$$s''(x) = 2 + 1536x^{-4}$$

$$x = 4$$

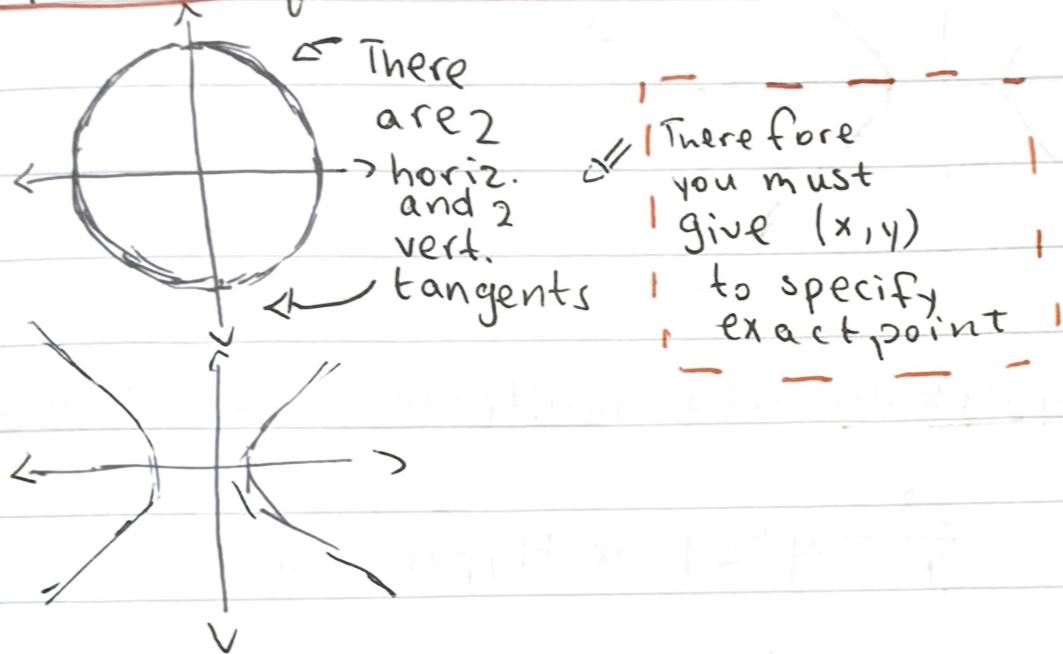
Plug in:

$$S = 16 + 16 = 32$$

$$y = -\frac{46}{4} = -4$$

# Horizontal/Vertical Tangent lines (Implicit)

## Implicit Equations:



E.g.  $x^2 + y^4 + 6x = 7$ , find horiz. tangent above x-axis

$$\frac{dy}{dx} = \frac{-2(x+3)}{4y^3} \Rightarrow \text{Horiz. Tangent: } x = -3$$

$$\therefore (-3)^2 + y^4 + 6(-3) = 7$$

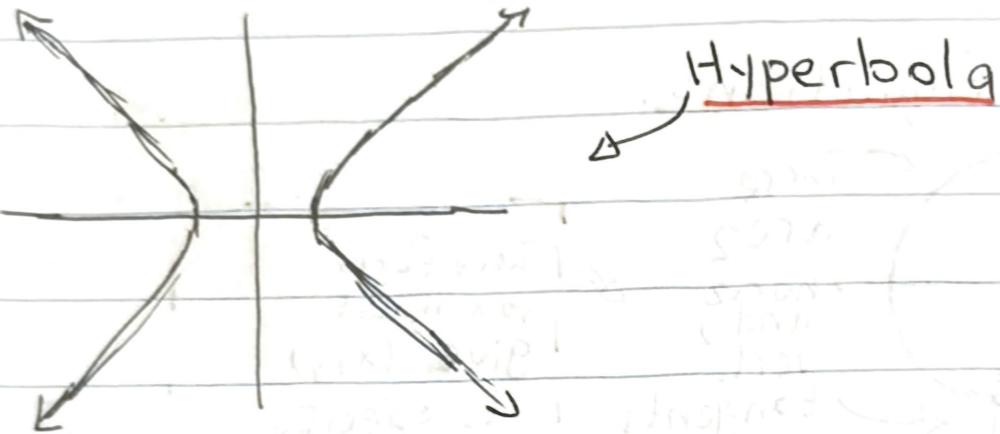
$$y = \pm 2 \quad \text{Therefore, there's two points}$$

$$\{(-3, 2), (-3, -2)\}$$

The one above tangent would be

$$(-3, 2)$$

# Concavity of Implicit Equations



A hyperbola has multiple concentric cavities.

E.g.  $\frac{x^2}{4} - y^2 = 1$   $\leftarrow$  Hyperbola

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4}{16y^3} \quad \text{Ans}$$

The concavity changes depending on the sign of  $y$ .

$\Rightarrow$  when  $\gamma > 0$ , it's concave down; when  $\gamma < 0$ , it's concave up.

Concavity must

be marked by

quadrants or range

rather than only

1 domain

Unit 6:  
Integration and accumulation  
of change

No.

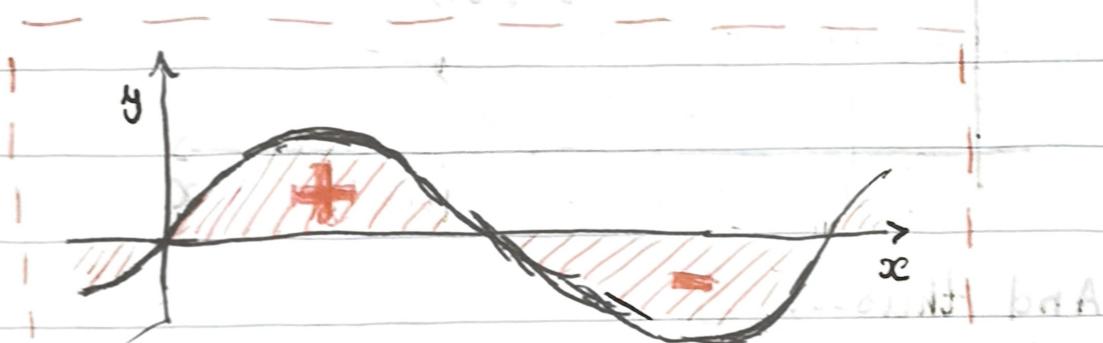
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## Accumulation of Change

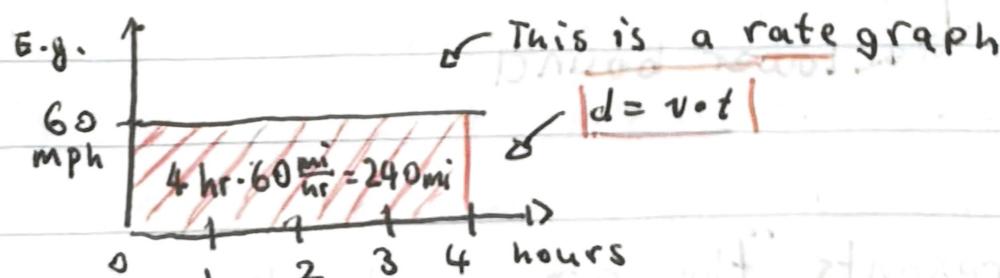
### -Area Under the Curve:

The region between a function and the  $x$ -axis is called the area under the curve:  
(“under” does not necessarily mean below.)

↳ It means the area between the  $x$ -axis and the function.



In an applied example, notice how the area under the curve can give us the accumulation of change.

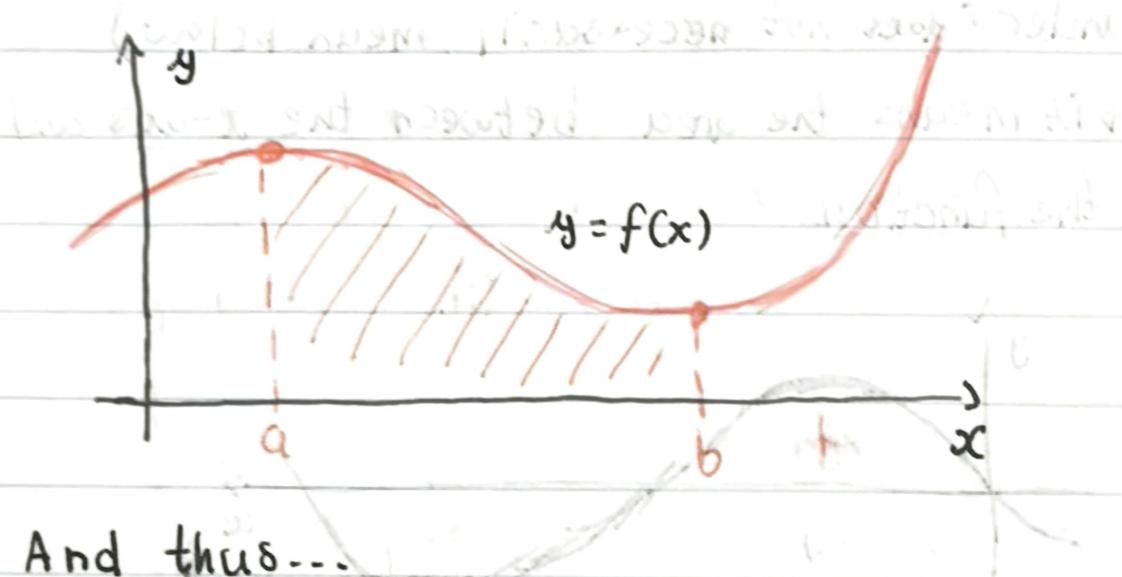


The graph “undid” the derivative. This is the “integral” or the “anti-derivative.”

↳ When it’s given on an interval, it’s the “definite integral”

# Definite Integrals

- An accumulation of change over a definite interval refers to definite integrals



And thus...

this accumulation of change can be written as an integral expression:

$$\int_a^b f(x) dx$$

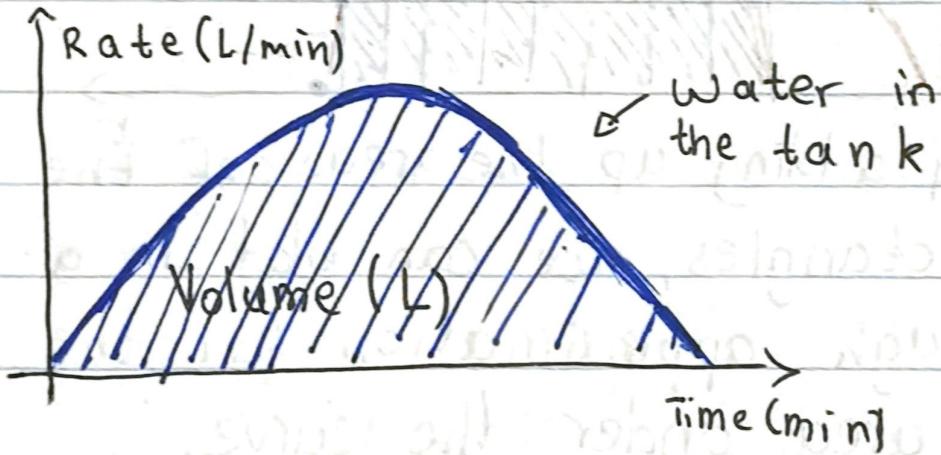
↑  
upper bound  
Function  
↓  
lower bound

- Represents "the area under the curve of the function  $f(x)$  between  $x=a$  and  $x=b$ ."

## Definite Integrals

- The definite integral always gives us the net change in a quantity, not the actual value of that quantity
- You need to add an initial condition to the definite integral to find the actual value of that quantity.

E.g.



We weren't told whether the tank (function) started off with water already or not.

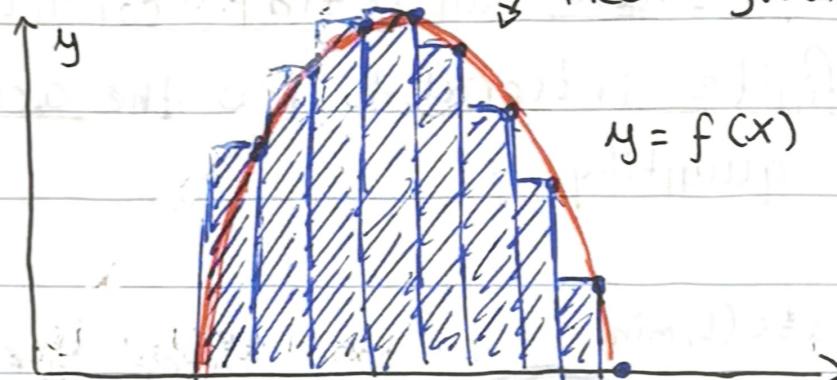
↳ Therefore we also need to know what is the initial value of the quantity.

# Riemann Introduction

How can we find the area under the curve of more complex functions?

(i.e. parabolas, cubic functions, logarithmic functions, etc.)

E.g.

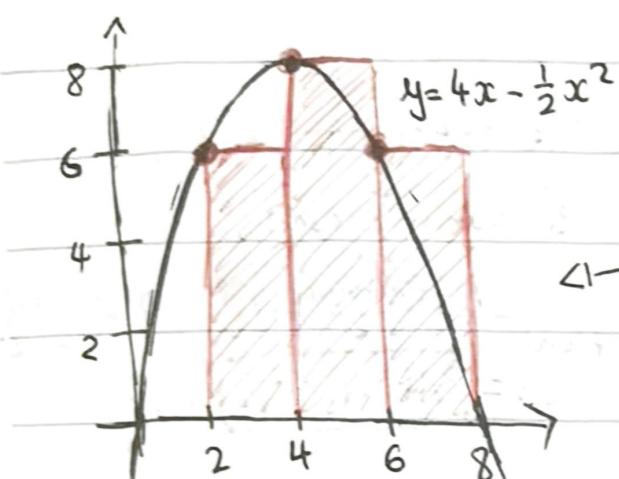


By adding up the areas of the rectangles, we can obtain a rough approximation for the area under the curve.

This approximation method is called a Riemann Sum. It was named after a German mathematician named Benhard Riemann.

# Riemann Sums

## Left Riemann Sum

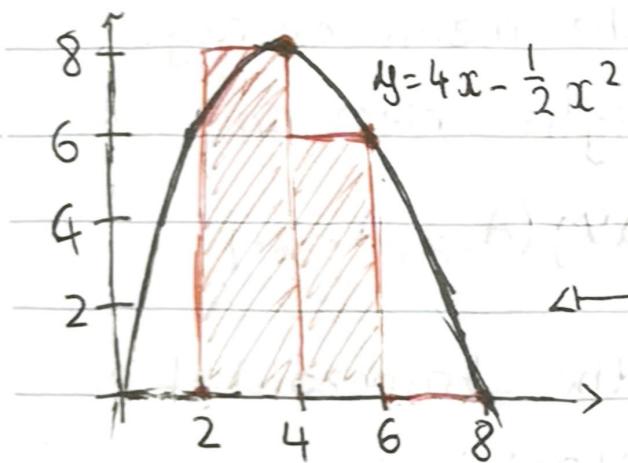


on the interval  $[2, 8]$ ,

use 3 subintervals

$$\begin{aligned} A_{\text{left}} &= 2 \cdot f(2) + 2 \cdot f(4) + 2 \cdot f(6) \\ &= [40 \text{ units}^2] \end{aligned}$$

## Right Riemann Sum

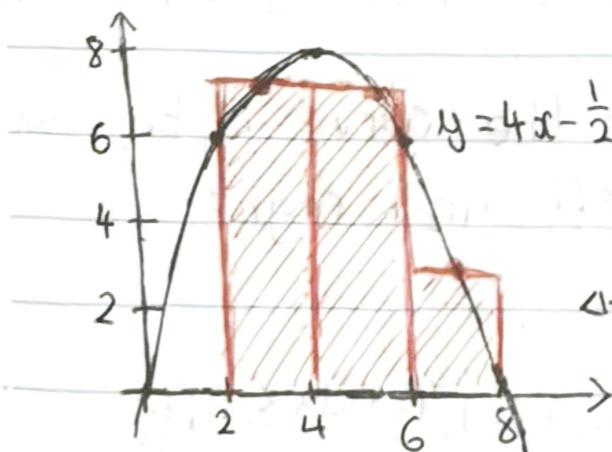


on the interval  $[2, 8]$ ,

use 3 intervals

$$\begin{aligned} A_{\text{right}} &= 2 \cdot f(4) + 2 \cdot f(6) + 2 \cdot f(8) \\ &= [28 \text{ units}^2] \end{aligned}$$

## Midpoint Riemann Sum



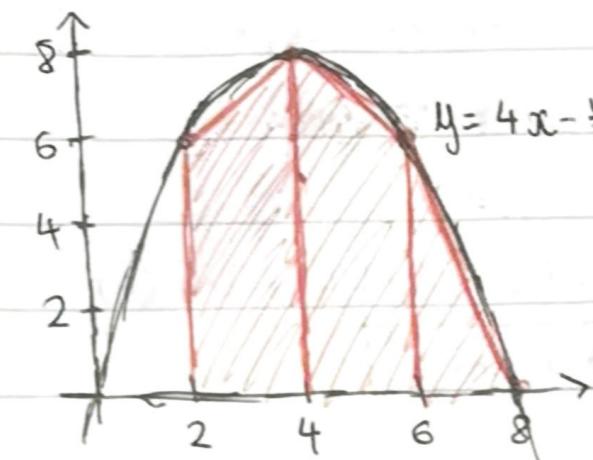
on the interval  $[2, 8]$ ,

use 3 intervals

$$\begin{aligned} A_{\text{mid}} &= 2 \cdot f(3) + 2 \cdot f(5) + 2 \cdot f(7) \\ &= [37 \text{ units}^2] \end{aligned}$$

## Riemann Sums continued...

### Trapezoidal Sum



$y = 4x - \frac{1}{2}x^2$  on the interval  $[2, 8]$ ,

use 3 subintervals

$$\begin{aligned} A_{\text{trapezoid}} &= 2 \cdot \left[ \frac{f(2) + f(4)}{2} \right] + 2 \cdot \left[ \frac{f(4) + f(6)}{2} \right] + 2 \cdot \left[ \frac{f(6) + f(8)}{2} \right] \\ &= 34 \text{ units}^2 \end{aligned}$$

### Special Tricks

→ The trapezoidal sum is the mean average between the left and right riemann sums.

$$A_{\text{trapezoid}} = \frac{A_{\text{left}} + A_{\text{right}}}{2} = \text{avg}(A_{\text{left}}, A_{\text{right}})$$

→ For a linear function, the trapezoidal sum is equal to the midpoint sum.

$$A_{\text{trapezoid}} = A_{\text{mid}} \text{ for } f(x) \text{ is a first degree function.}$$

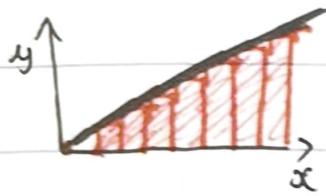
→ The actual area under the curve is between (or equal to) the left and right riemann sums.

$$A_{\text{left}} \leq \int_a^b f(x) dx \leq A_{\text{right}} \text{ OR } A_{\text{right}} \leq \int_a^b f(x) dx \leq A_{\text{left}}$$

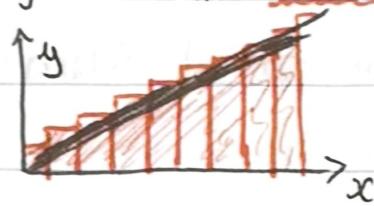
# Over-estimation and Underestimation of Riemann Sums

## Increasing Function

Left Riemann = Underestimate

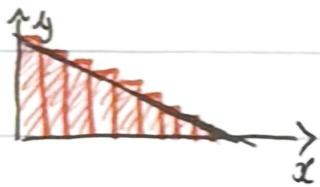


Right Riemann = Overestimate



## Decreasing Function

Left Riemann = Overestimate



Right Riemann = Underestimate



## Trapezoidal Estimates

Concave up = Overestimate



Concave Down = Underestimate



Note that some Riemann Sums can be done with UNEQUAL intervals if equal ones cannot be formed (especially on a table).

# Summation Notation

Riemann Sums are the sums of areas.

↳ We can use the summation notation to find the total sum (i.e.  $\Sigma$ )

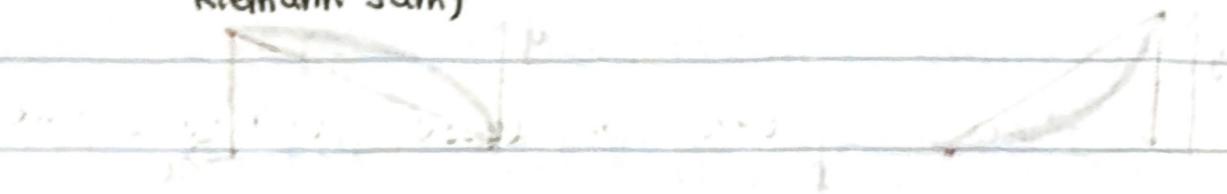
$$A = f(0) \cdot 2 + f(2) \cdot 2 + f(4) \cdot 2 + \dots + f(n) \cdot 2$$

↳ This is the sum of the areas of each Riemann Rectangle

General Formula:  $n$

$$A = \sum_{i=1}^n f(x_{i-1}) \cdot \Delta x$$

height at change in  $x$   
Point  $x_{i-1}$   
(input may  
vary depending  
on type of  
Riemann Sum)

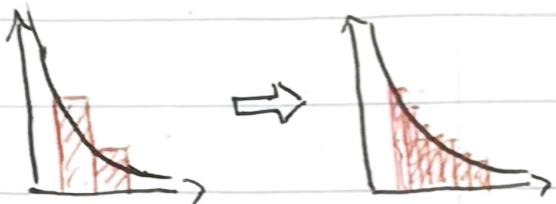


under and over? under account still  
one large triangle and 6 small thin  
(left from all edges) bound at same

# Definite Integrals as the limit of a Riemann Sum

Recalling the sum of rectangles...

$$\sum_{i=1}^n f(x_{i-1}) \Delta x$$



We notice that the sum gets more and more accurate the more rectangles we have...

Therefore...

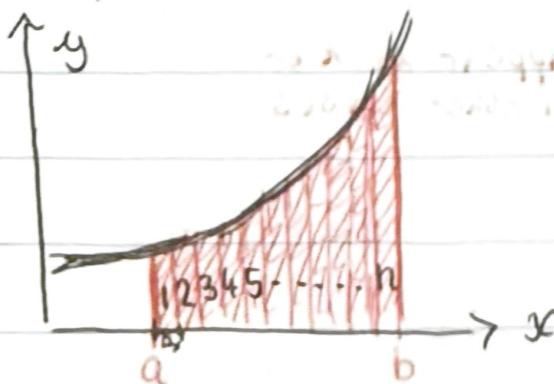
We want  $n$  [i.e. #of rectangles]  $\rightarrow \infty$

and thus:  $\Delta x$  [i.e. width of each rectangle]  $\rightarrow 0$

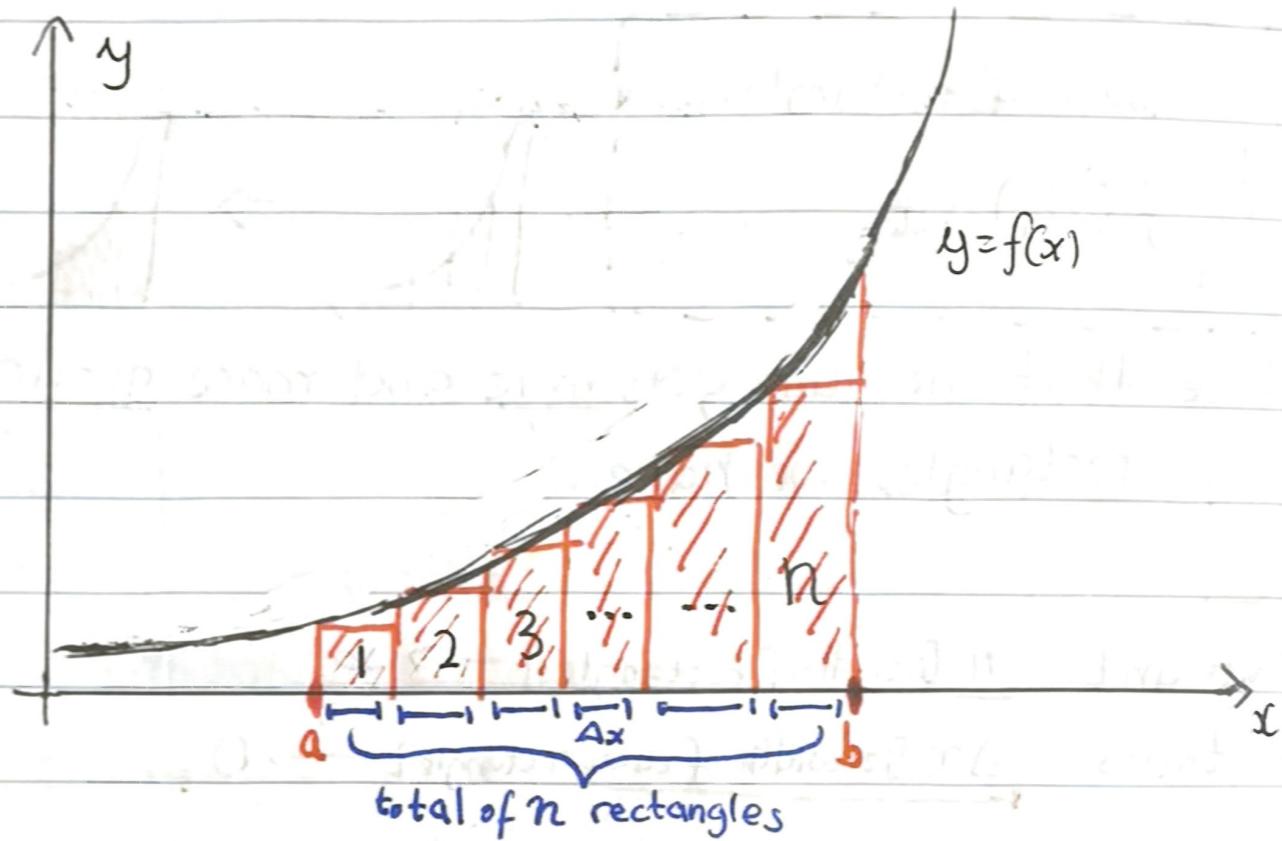
Notice that  $n$  can be expressed as a relation with  $\Delta x$ ...

$$\Delta x = \frac{b-a}{n} \quad \text{OR} \quad n = \frac{b-a}{\Delta x}$$

where 'a' and 'b' are the lower and upper bounds respectively.



# Definite Integral as the limit of a Riemann Sum



For an infinitely small change in  $x$  ( $\Delta x$ )

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1}) \cdot \Delta x$$

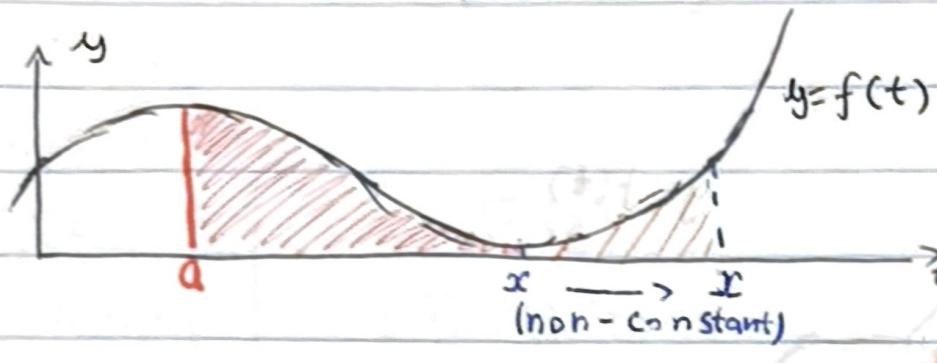
$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + \frac{b-a}{n} \cdot i\right) \cdot \left(\frac{b-a}{n}\right)$$

$i$  can also appear as  $k$  or other variables in other cases

# The fundamental theorem of Calculus

If  $a$  is a constant and  $f$  is a continuous function, then...

$$\frac{d}{dx} \left[ \int_a^x f(t) dt \right] = f(x)$$



This connects derivatives ( $\frac{dy}{dx}$ ) and integrals ( $\int f(x) dx$ )

→ Integrals are the inverses of derivatives or anti-derivatives.

Variations of the theorem...

$$\frac{d}{dx} \left[ \int_{h(x)}^{g(x)} f(t) dt \right] = [f(g(x)) \cdot g'(x)] - [f(h(x)) \cdot h'(x)]$$

where  $g(x)$  and  $h(x)$  are functions (e.g.  $x^2, \sin(x)$ , etc.)

This is really similar to the Chain Rule for Derivatives ( $f'(g(x)) \cdot g'(x)$ )

↗ In this case, since  $\frac{d}{dx} [\int f(x) dx] = f(x)$ , there isn't  $f'(x)$

No.

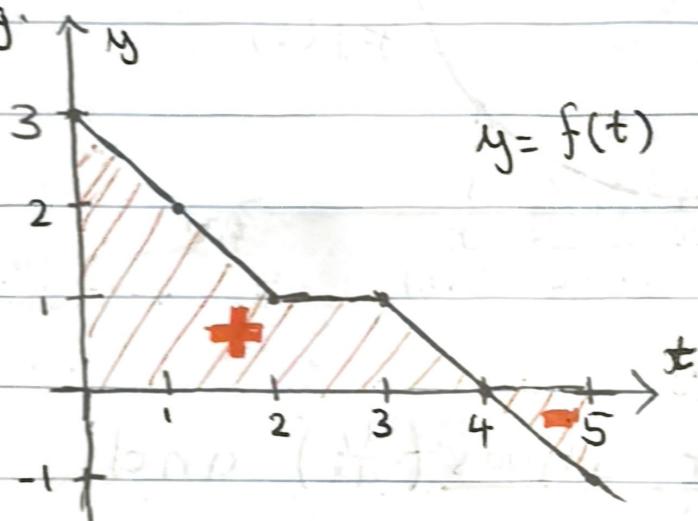
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## Accumulation Functions

$$F(x) = \int_0^x f(t) dt$$

we can use Capital letters to represent integrals.

E.g.



x	0	1	2	3	4	5
F(x)	0	2.5	4	5	5.5	5

This,  $F(x)$  is called an Accumulation Function.

## Behaviour of Accumulation Functions

Accumulation Functions are of the form

$$F(x) = \int_a^x f(t) dt \quad \text{where } a \text{ is constant.}$$

Recognize that  $F'(x) = f(x)$

### Behaviour

<u><math>F(x)</math></u>	<u><math>F'(x)</math></u>	<u><math>f(x)</math></u>
increasing	$F'(x) > 0$	$f(x) > 0$
decreasing	$F'(x) < 0$	$f(x) < 0$
rel. max	$F'(x)$ changes signs (+) to (-)	$f(x)$ pos. to neg.
rel. min	$F'(x)$ changes signs (-) to (+)	$f(x)$ neg. to pos.
Concave up	$F'(x)$ is increasing	$f(x)$ is increasing
Concave down	$F'(x)$ is decreasing	$f(x)$ is decreasing
pt. of inflection	$F'(x)$ has a rel. extrema	$f(x)$ has a rel. extrema

In this case -

→ We can treat the graph of  $f(x)$  as

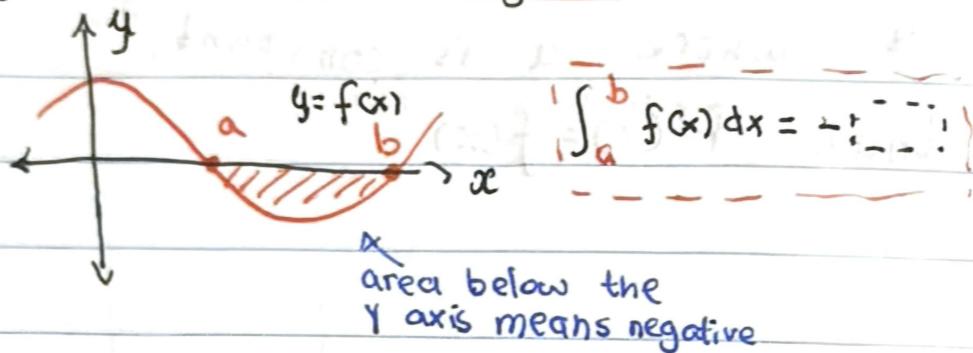
$F'(x)$  since  $F'(x) = f(x)$

No.

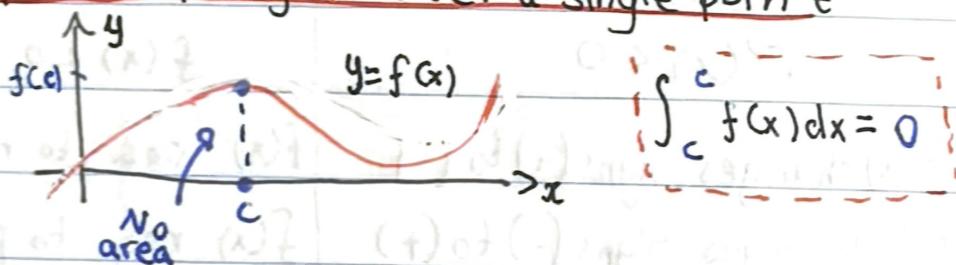
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# Properties of Definite Integrals

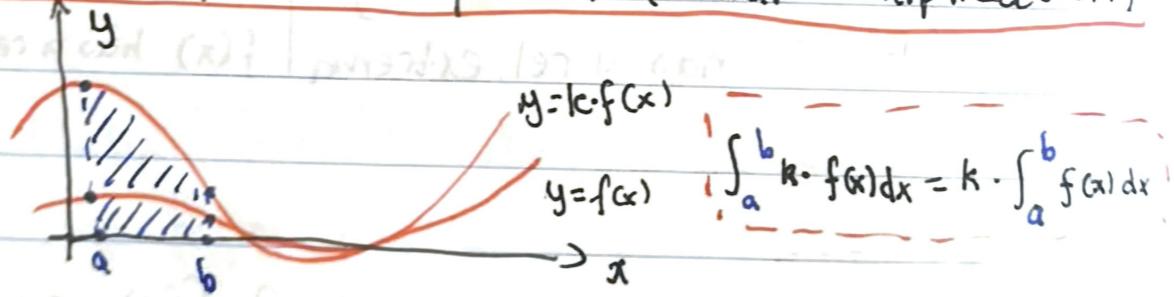
## Negative Definite Integrals



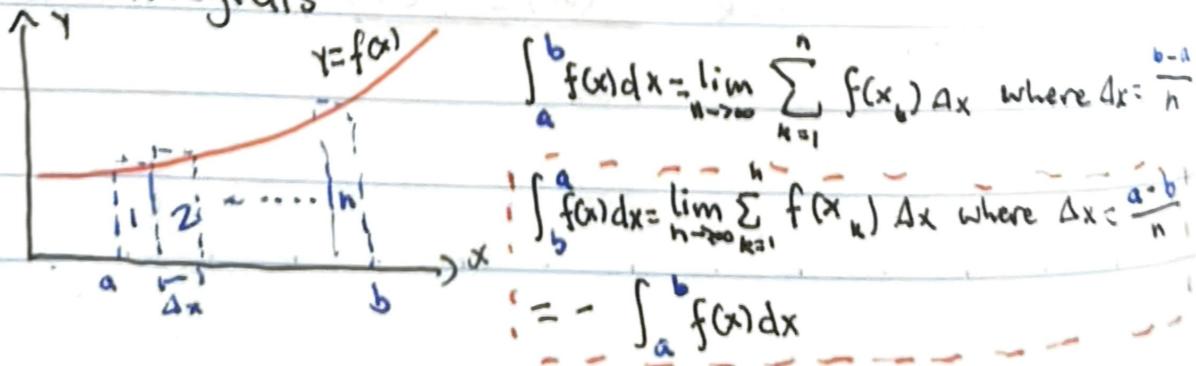
## Definite integral over a single point



## Integration of scaled function (scalar multiplication)

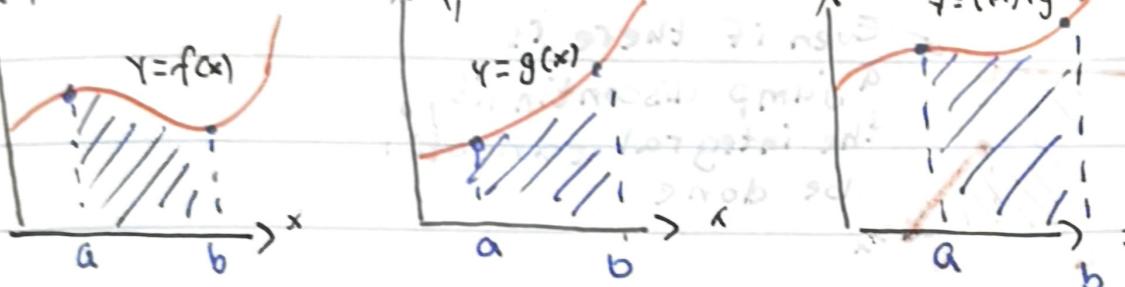


## Reverse Integrals



# Properties of Definite Integrals

## Integrating sum of functions

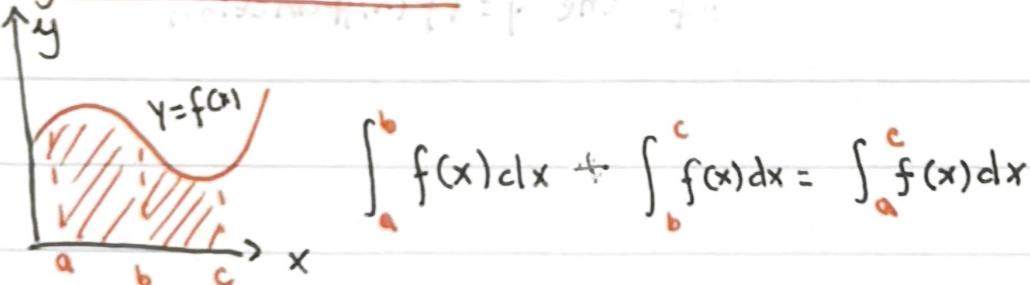


$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

similarly...

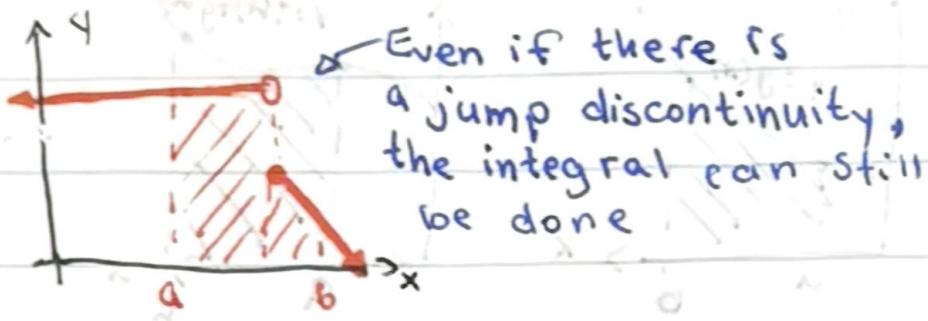
$$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

## Adjacent intervals

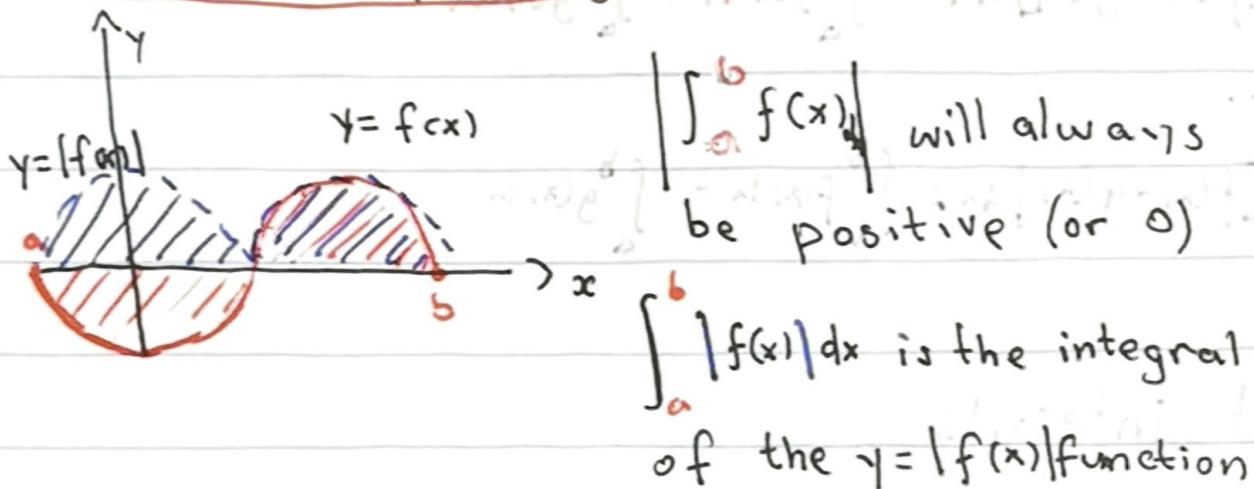


# Properties of definite integrals

## Piecewise-functions



## Absolute Value of integrals



## Anti-Derivative

An antiderivative of a function  $f(x)$  is a function  $F(x)$  whose derivative is  $f(x)$ .

When we take an integral, it is taking the antiderivative of a function. The area under the curve is represented by an antiderivative.

## The fundamental theorem of Calculus

If  $f$  is continuous on the interval  $[a, b]$ , then the area under the curve of  $f$  from  $[a, b]$  can be represented by ...

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F(x)$  is the antiderivative of  $f$ .

when we re-arrange ...

$$F(b) = \int_a^b f(x) dx + F(a)$$

end value      integral  
 ↓      ↓  
 ↓ (change between  
the two values)

↓ initial value  
 ↓ initial value

No.

Date

# Indefinite Integrals

⇒ Integrals without a boundary are called indefinite integrals

E.g.  $f(x) = x^2 + 1$        $f(x) = x^2 - 5$        $f(x) = x^2 + 1000$   
 $f'(x) = 2x$        $f'(x) = 2x$        $f'(x) = 2x$

Antiderivative of  $2x$ ?

we know  $f(x) = x^3 + C$  vise versa, OR  $f(x) = x^2 + 1$

and so we get the antiderivative of  $2x$ .

General Solution and Particular Solution

- ↳ Includes a constant, "C", that represents all solutions, i.e. it represents only one solution.
- ↳ Includes a constant number, e.g. that represents only one solution.

# Reverse Power Rule

$$f(x) = x^n$$

1. Add one to exponent

$$\Rightarrow F(x) = \frac{x^{n+1}}{n+1}$$

2. Divide by the new exponent.

# Exponential

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

## Logarithm as the answer

$$\int \frac{1}{x} dx = \ln(|x|) + C$$

## Trigonometric Integrals

$$\int \cos x dx = \sin x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

## Inverse Trigonometric Integrals

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1}(x) + C$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C$$

# Indefinite Integrals ctd.

## Separating rational functions

*Rewrite  
before  
integrating*

$$\text{E.g. } \int \left( \frac{3x^2+x-2}{x} \right) dx = \int (3x+1-2x^{-1}) dx$$

$$= \frac{3}{2}x^2 + x - 2 \cdot \ln(|x|) + C$$

## Combining expressions

*Distribute  
before integrating*

$$\text{E.g. } \int 3x(\sqrt{x} + x^2) dx = \int (3x\sqrt{x} + 3x^3) dx$$

$$= \int (3x^{\frac{3}{2}} + 3x^3) dx = \frac{6}{5}x^{\frac{5}{2}} + \frac{3}{4}x^4 + C$$

# Integrating Using Substitution

In order to find the integrals of chain functions (i.e. used with chain rule), we must use a method called u-substitution

E.g.  $\int (3x-4)^5 dx \Rightarrow$  we substitute the inner-function with "u"

$$\begin{aligned} & \int u^5 dx \\ & \text{However, } u = 3x-4 \\ & \text{we need } du \text{ to integrate } u \\ & (\text{solve for } du) \quad \frac{du}{dx} = [3x-4] \\ & = \int \frac{u^5}{3} du \quad \frac{du}{3} = dx \\ & = \frac{1}{3} \cdot \frac{1}{6} u^6 + C \quad \text{integrate} \quad \text{Substituting} \quad \Rightarrow \frac{du}{dx} = 3 \\ & = \frac{1}{18} u^6 + C \quad \Rightarrow \frac{du}{3} = dx \\ & = \boxed{\frac{1}{18} (3x-4)^6 + C} \quad \text{write in terms of } x \end{aligned}$$

When you choose the incorrect "u", you cannot cancel some other term (simplify)

↳ A correct u MUST simplify expression.

$$\begin{aligned} & \text{E.g. } \int \sin x e^{\cos x} dx \\ & u = \sin x \Rightarrow \frac{du}{dx} = \cos x \quad u = \cos x \Rightarrow -\frac{du}{\sin x} = dx \\ & \int u e^{\cos x} \cdot \frac{du}{\cos x} \quad \int \frac{\sin x}{-\sin x} e^u \cdot du \\ & \quad \times \quad = -e^u + C = -e \quad \checkmark \end{aligned}$$

# Integrating Using Substitution

## Trig functions

E.g.  $\int \cot(3x) dx$

$$\begin{aligned} u &= 3x \Rightarrow \frac{du}{3} = dx \\ \int \cot(u) \frac{du}{3} &= \frac{1}{3} \int \frac{\cos u}{\sin u} du \end{aligned}$$

No integral for cotangent

rewrite

$$\begin{aligned} &= \frac{1}{3} \cdot \int \frac{1}{w} \cdot \frac{\cos u}{\cos u} dw = \frac{1}{3} \cdot \ln|w| + C \\ &= \frac{1}{3} \cdot \ln|\sin(3x)| + C \end{aligned}$$

## Double substitution

- ↳ Re do substitution (if further simplification not possible) with another variable (e.g. w)
- ↳ make sure to find "dw"

## Inverse trig can be confused with u-sub.

E.g.  $\int \frac{1}{\sqrt{1-4x^2}} dx$

not u-sub  $\rightarrow$  only  $u=2x$

$$\begin{aligned} &= \int \frac{1}{\sqrt{1-(2x)^2}} dx = \int \frac{1}{\sqrt{1-u^2}} \frac{du}{2} = \frac{1}{2} \sin^{-1}(u) + C \\ &= \frac{1}{2} \sin^{-1}(2x) + C \end{aligned}$$

# Integrating using Substitution

Solving for x (when  $du = dx$ , i.e. leading coefficient is "1" for a linear u-substitute)

$$\begin{aligned}
 \text{E.g. } & \int \frac{x}{\sqrt{x+1}} dx \\
 &= \int \frac{x}{\sqrt{u}} du \quad \text{← still does not cancel} \\
 &= \int \frac{u-1}{\sqrt{u}} du = \int \left( \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} \right) du \quad \text{so instead solve for } x \dots \\
 &= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\
 &= \frac{2}{3} (x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C
 \end{aligned}$$

## Definite Integrals with Substitution

with definite integrals, you can use a simple trick

$$\int_a^b f(x^2+1) dx = \int_a^b f(u) \frac{du}{2x} = \int_{u(a)}^{u(b)} \frac{f(u)}{2x} du$$

$$\text{E.g. } \int_0^2 t^2 \sqrt{t^3+1} dt$$

$$= \frac{1}{3} \int_1^9 \sqrt{u} du$$

$$= \frac{1}{3} \cdot u^{\frac{3}{2}} \cdot \frac{2}{3} \Big|_1^9$$

$$= \frac{2}{9} [57 - 1] = \frac{52}{9}$$

Plugin  
'a' and 'b'  
into 'u'

→ In the end,  
directly substitute  
 $u(b)$  and  $u(a)$  in  
for  $u$  instead of  
 $x$

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# Integration Using long division

When encountering integrals of rational functions, it might be necessary to use long division to simplify.

E.g.

$$\int \frac{x-5}{-2x+2} dx = \int \left( \frac{1}{2} + \frac{\frac{1}{2}}{-2x+2} \right) dx = -\frac{1}{2}x + 2 \ln|x-1| + C$$

$\overline{-2x+2}$   
 $x-5$   
 $\underline{-(-2x+2)}$   
 $-1$

When there exists a remainder, you leave it as a part of the rational equation

# Integration using Completing the Square

Notice how the inverse trig function derivatives involve "x<sup>2</sup>"

$$\frac{d}{dx} [\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}} + C$$

$$\frac{d}{dx} [\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}} + C$$

$$\frac{d}{dx} [\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2-1}} + C$$

$$\frac{d}{dx} [\csc^{-1}(x)] = -\frac{1}{|x|\sqrt{x^2-1}} + C$$

$$\frac{d}{dx} [\tan^{-1}(x)] = \frac{1}{x^2+1} + C$$

$$\frac{d}{dx} [\cot^{-1}(x)] = -\frac{1}{x^2+1} + C$$

Therefore, we can use a combination of u-substitution and complete the square to find the integral

$$\text{E.g. } \int \frac{1}{5x^2-30+65} dx = \frac{1}{5} \int \frac{1}{x^2-6x+13} dx$$

$$= \frac{1}{5} \int \frac{1}{x^2-6x+9-9+13} dx = \frac{1}{5} \int \frac{1}{(x-3)^2+2^2} dx$$

$$= \frac{1}{5} \int \frac{\frac{1}{2^2}}{\frac{1}{2^2}} \cdot \frac{1}{(\frac{x-3}{2})^2+2^2} dx = \frac{1}{20} \int \frac{1}{(\frac{x-3}{2})^2+\frac{2^2}{2^2}} dx$$

$$= \frac{1}{20} \int \frac{1}{u^2+1} du = \frac{1}{10} \tan^{-1}(u) + C$$

$$\begin{aligned} u &= \frac{x-3}{2} \\ 2du &= dx \end{aligned}$$

$$\frac{1}{10} \cdot \tan^{-1}\left(\frac{x-3}{2}\right) + C$$

# Differential Equations

## Differential Equation

$$\begin{aligned} y'' + 2y' &= 3y \\ f''(x) + 2f'(x) &= 3f(x) \\ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} &= 3y \end{aligned} \quad \left. \begin{array}{l} \text{All represent} \\ \text{same thing} \end{array} \right\}$$

↳ Relation between a function and its derivatives

↳ Solution is a function (or multiple)

## Algebraic Equation

$$x^2 + 3x + 2 = 0$$

$$(x+2)(x+1) = 0 \quad \text{Solution is a set of}$$

$$x = -2 \text{ or } x = -1 \quad \text{values}$$

## Writing a differential equation

### - Proportional (directly proportional)

$$\Delta y = k \cdot x \rightarrow \frac{dy}{dx} = k \cdot x \text{ etc.}$$

### - Inversely Proportional (proportional to reciprocal)

$$\rightarrow y = \frac{k}{x} \rightarrow \frac{dy}{dx} = \frac{k}{x^2} \text{ etc.}$$

E.g. speed is inversely proportional to the square of the distance  $S$ , if traveled

$$S = \text{distance} \quad \frac{ds}{dt} = \text{Speed}$$

$$\therefore \frac{ds}{dt} = \frac{k}{S^2}$$

" $k$ " refers to  
some constant

# Verifying Solutions to differential equations

Derivatives can be used to verify that a function is a solution to a given differential equation

E.g.  $\frac{dy}{dx} = \frac{4y}{x}$   $\checkmark$  <sup>Verify</sup>  $y = 4x$   $\Rightarrow$  <sup>Plug function</sup>  $4 = \frac{dy}{dx} = \frac{4(4x)}{x} = 16$

Not a solution

## Finding particular solutions

1. Integrate both sides to get  $f(x)$
2. Plug in point into result and solve for "c"
3. Substitute "c" and simplify if needed

E.g.  $\frac{dy}{dx} = e^{2x} - 2x^2$ ,  $(0, 4)$ :

$$y = \int [e^{2x} - 2x^2] dx$$

$$= \frac{1}{2} \int [e^{4u}] du - 2 \int x^2 dx \quad \text{for } u = 2x$$

$$= \frac{1}{2} e^{2x} - \frac{2x^3}{3} + C$$

Solve for  $C$

$$4 = \frac{1}{2} e^{0 \cdot 2} - \frac{2(0)^3}{3} + C \Rightarrow C = \frac{7}{2}$$

$$\therefore y = \frac{1}{2} e^{2x} - \frac{2}{3} x^3 + \frac{7}{2}$$

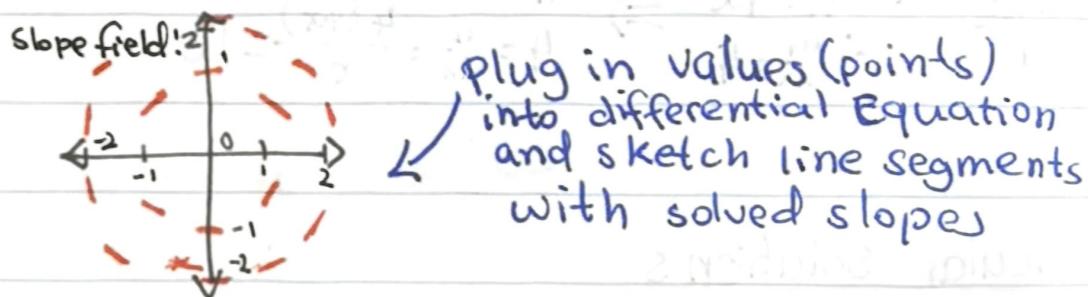
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## Slope Fields

A slope field represents a differential equation on an  $xy$ -plane. It shows the "slope" of all the particular solutions to the differential equation.

E.g.  $\frac{dy}{dx} = -\frac{x}{y}$



### Differential Equation from Slope Field

Look for easy slopes

↳ Compare horizontal line segments,

Vertical line segments

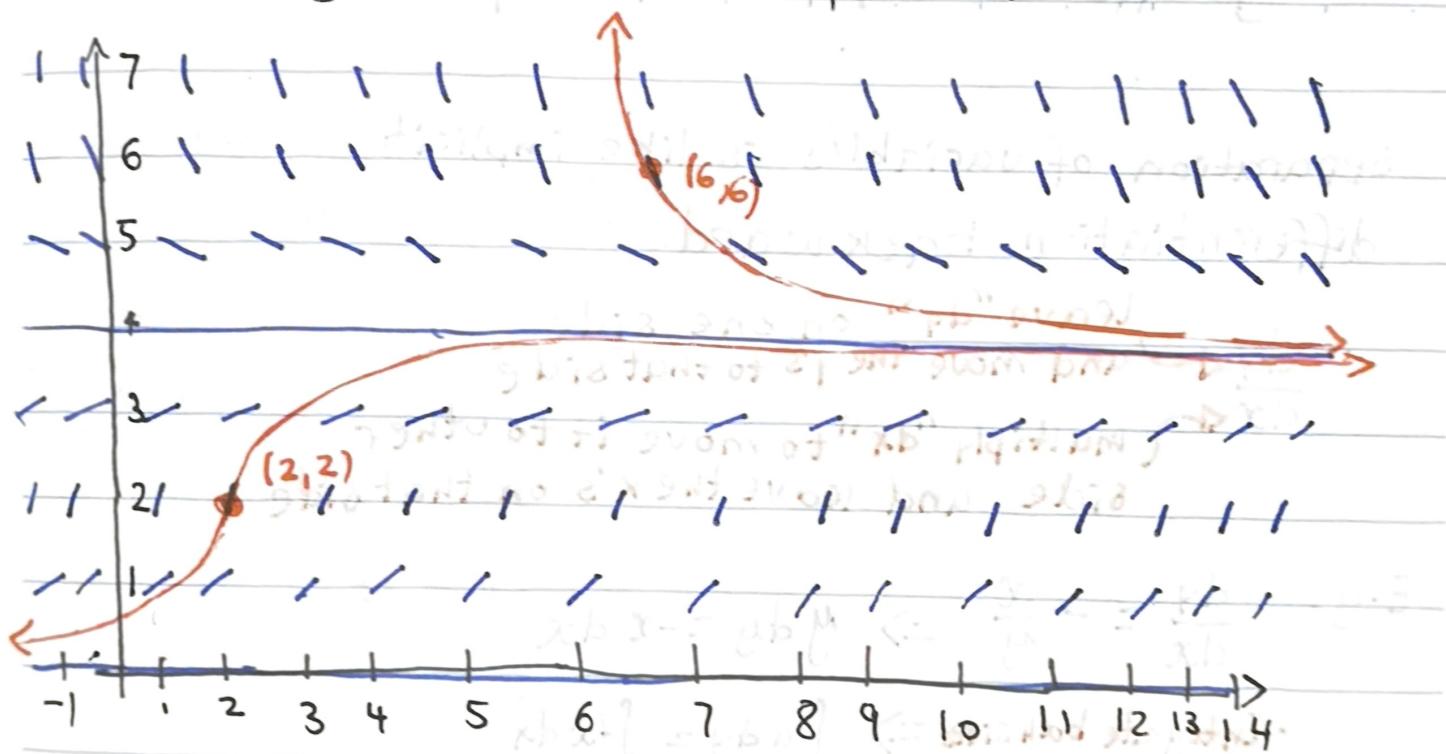
↳ Positive or negative slopes?

↳ Watch the behavior of the slope field

↳ Gets steeper?

⇒ Rule out equations that do not match

## Visualizing Solutions with Slope Field



You can visualize solutions with a slope field.

To visualize a particular solution, follow the slopes that link to the selected point (As shown above)

### Range of a solution

- ↳ Watch for domain restrictions (end points) and horizontal asymptotes
- ↳ End points should be closed (i.e. "[" or "]")
- ↳ Horizontal asymptotes should be open (i.e. "(" or ")" )

# Solving Differential Equations with Separation

separation of variables is like implicit differentiation backwards

$\frac{dy}{dx}$  leave "dy" on one side  
 and move the y's to that side  
 $\frac{dx}{dx}$  multiply "dx" to move it to other  
 side and leave the x's on that side

E.g.  $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y dy = -x dx$

• Integrate both sides  $\Rightarrow \int y dy = \int -x dx$   
 $\Rightarrow \frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$

• Simplify ...  $\Rightarrow y = \pm \sqrt{-x^2 + C}$

## Factoring

→ Sometimes you need to factor before separating

e.g.  $\frac{dy}{dx} = 2xy + 4x = 2x(y+2)$

$\frac{1}{y+2} dy = 2x dx$

$y = \pm C e^{\frac{x^2}{2}} - 2$

# Separation of Variables

## Addition/Subtraction

→ Sometimes you need to move terms before separating

$$\text{e.g. } \frac{dy}{dx} + y = 10$$

$$\frac{dy}{dx} = 10 - y$$

$$\frac{1}{10-y} dy = dx$$

$$y = \pm ce^{-x} + 10$$

## Notes:

→ You can simplify constants in an exponential with exponential properties

$$\text{e.g. } e^{x+C_1} = e^x \cdot e^{C_1} = (C_1 \cdot e^x)$$

$e^x$  is also a constant value

You can only integrate when they're multiplying by the variables  
e.g.  $\int y dy$  but not  $\int dy$

→ Integrating "dx" alone

$$\Rightarrow \int dx = \int 1 dx = x + C$$

→ "dy" and "dx" can be treated algebraically differentials

⇒ for an infinitely small change in [variable]

⇒ "d" and variable should be treated together (not separated)

# Particular Solutions with Separation

## 1. Separate

→ separate the equation so that the two variables lie on their respective sides.

$$\text{e.g. } \frac{dy}{dx} = xy^3 \Rightarrow \frac{dy}{y^3} = x \cdot dx$$

## 2. Integrate

→ Integrate both sides accordingly.

$$\text{e.g. } \int \frac{1}{y^3} dy = \int x dx \Rightarrow \frac{y^{-2}}{-2} = \frac{x^2}{2} + C$$

## 3. Solve for C

→ Plug in the x-values and y values given

and solve for the constant. Then substitute

the value for the constant in the equation.

$$\text{e.g. } \frac{(-2)^{-2}}{-2} = \frac{(1)^2}{2} + C \Rightarrow C = -\frac{5}{8}$$

## 4. Isolate (simplify)

→ Isolate the "y" or function "f(x)" on one

side and simplify accordingly.

$$\text{e.g. } \frac{y^{-2}}{-2} = \frac{x^2}{2} - \frac{5}{8} \Rightarrow y = \pm \sqrt{\frac{-4}{4x^2 - 5}}$$

## 5. Select (if applicable)

→ Select the correct sign ( $\pm$ ) by plugging in the x-value and y-value again.

→ Applicable for even exponents and absolute values

$$y = \pm \sqrt{\frac{-4}{4x^2 - 5}} = \pm 2 \Rightarrow y = -\sqrt{\frac{-4}{4x^2 - 5}}$$

## Particular Solutions with Separation ctd.

### 4/5. For implicit solutions

- Sometimes you cannot explicitly express the solution and must leave it as implicit
- When point given is  $(x_0, 0)$  then you cannot really determine sign (+/-)

## Exponential Models with Differential Equations

⇒ A function is exponential when its differential equation is proportional to that function

$$f'(x) = k \cdot f(x) \quad \frac{dy}{dx} = k \cdot y$$

Proof:

$$\begin{aligned} \frac{dy}{dx} = k \cdot y &\Rightarrow \int \frac{1}{y} dy = \int k dx \Rightarrow \ln|y| = x + C, \\ &\Rightarrow y = \pm e^{kx+C} = \pm C \cdot e^{kx} \end{aligned}$$

The solution to  $\frac{dy}{dt} = ky$  is  $y = Ce^{kt}$ , where  $C$  represents the initial value of the model

# Exponential Models

A growth model will have a positive exponent. e.g.  $y = 5e^{kt}$

A decay model will have a negative exponent. e.g.  $y = 7e^{-kt}$

## Using initial value...

Sometimes, you may be given  $\frac{dy}{dt}$  or a point to find the particular solution...

e.g.  $\frac{dy}{dt} = ky$ . If  $y$  doubles every 7 years, what is  $k$ ?

$$y = C \cdot e^{kt}$$

initial value

$$2 \cdot C = C \cdot e^{k \cdot 7} \Rightarrow 2 = e^{7k} \Rightarrow k = \frac{\ln(2)}{7}$$

doubles every 7 yrs

Notice how  $e^{ky} = e^{\frac{dy}{dt}t}$ ...  $k = \frac{\ln(\text{factor})}{\text{given } x} = \frac{dy}{dt} \div t$

## General Formula

$$y = y(0) \cdot e^{\frac{\ln(\text{factor})}{\text{given } x} \cdot t} = y(0) \cdot e^{\frac{\ln(\frac{\text{given } y}{y(0)})}{\text{given } x} \cdot t}$$

$$= y(0) \cdot \left(\frac{\text{given } y}{y(0)}\right)^{\frac{t}{\text{given } x}} = y(0) \cdot \text{factor}^{\frac{t}{\text{given } x}}$$

$y$  is [factor] times for every [given  $x$ ] years  
 OR  
 $y$  has point (given  $x$ , given  $y$ )

# Average Value of a function on an interval

Recall... Average Rate of change

$$\text{f'(x)}_{\text{avg}} = \frac{\text{f}(b) - \text{f}(a)}{b - a}$$

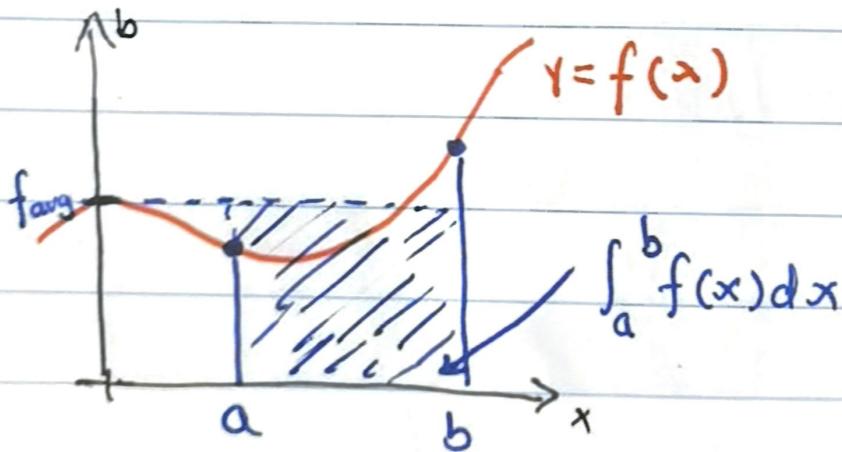
$$\text{f(x)}_{\text{avg}} = \frac{\text{F}(b) - \text{F}(a)}{b - a}$$

} On closed interval  
[a, b]

How do we express f(x)<sub>avg</sub> with f(x)?

Remember...

$$\int_a^b \text{f}(x) dx = \text{F}(b) - \text{F}(a)$$



ALTERNATIVELY...

Area = width × height

$$\int_a^b \text{f}(x) dx = (b-a) \times \text{f}_{\text{avg}}$$

$$\text{f}_{\text{avg}} = \frac{1}{b-a} \int_a^b \text{f}(x) dx$$

$$\text{f(x)}_{\text{avg}} = \frac{1}{b-a} \int_a^b \text{f}(x) dx$$

# Mean Value Theorem for Integrals

## Normal MVT

If  $f(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists some number such that  $a < c < b$  and  $f'(c) = \frac{f(b) - f(a)}{b - a}$

## Similarly ... MVT with integrals

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

$$\Rightarrow f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

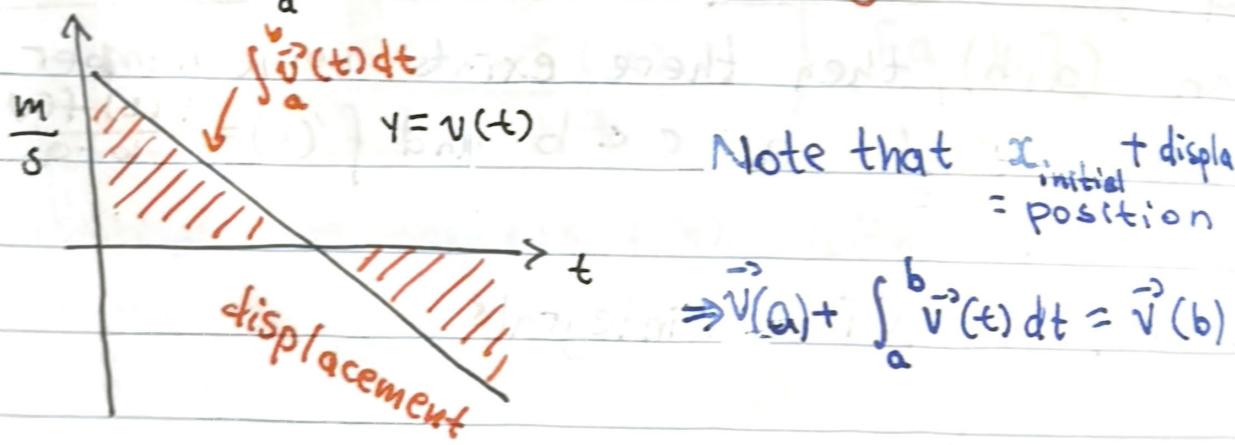
If  $f(x)$  is continuous on  $[a, b]$ , then there exists a  $c$  such that  $f(c) = f_{\text{avg}}$

# Position, Velocity, Acceleration with Integrals

Displacement: Change in position

$$\vec{d} = \Delta \vec{x}$$

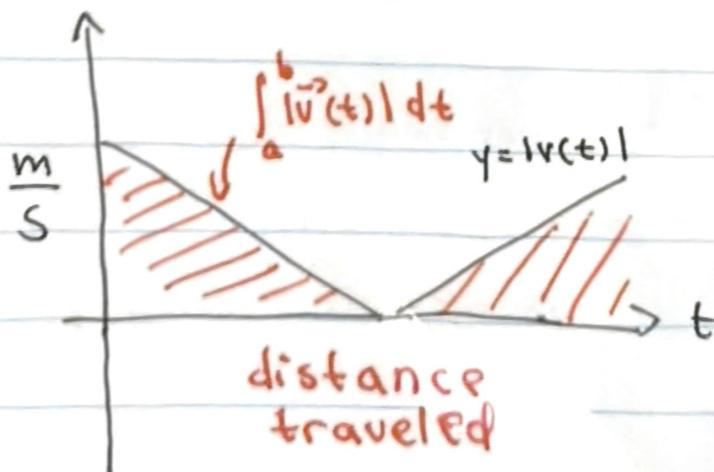
$$= \int_a^b \vec{v}(t) dt \quad \text{velocity integral}$$



Distance: Total length of path

$$d = \int |\vec{v}(t)| dt$$

$|\text{velocity}| = \text{speed}$



To manually solve:  
You can use integral sum property:

$$\int_a^b |\vec{v}(t)| dt \quad \begin{array}{l} \text{use appropriate} \\ \text{sign for } v(t) > 0 \end{array}$$

$$= \int_a^b |\vec{v}(t)| dt + \int_b^c |\vec{v}(t)| dt$$

# Applying Accumulation and Integrals

When you integrate a rate, you get net change  
→ Area under rate function

$$\begin{cases} \text{rate of change} = \text{net change} \\ \Rightarrow \int_a^b f'(x) dx = \Delta f(x) = f(b) - f(a) \end{cases}$$

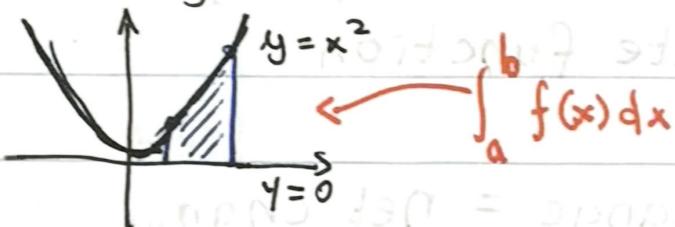
In other words...

$f(x)$  increased by  $\int_a^b f'(x) dx$  over  $[a, b]$  interval  
or decreased

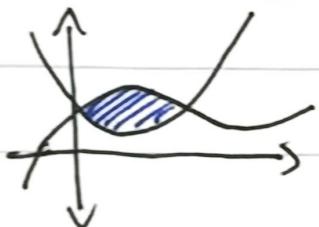
Final Value = Initial Value + NET CHANGE

# Area Between Curves ( $x$ -axis)

Recall: Finding area under the curve



How about area between two curves?



$$A = \int_a^b [f(x) - g(x)] dx$$

$$f(x) \geq g(x) \text{ for all } x \in [a, b]$$

Note that we're dealing with geometric area, which **MUST** be positive. Therefore  $f(x) \geq g(x)$ .

## 1. Given Endpoints

when given endpoints  $x=a, x=b\dots$

$$\boxed{A = \int_a^b [f(x) - g(x)] dx}$$

E.g. Area bounded by the curves  $y = x^2 + 2$ ,  $y = -x$ ,  $x = 0$  and  $x = 1$

$$\int_0^1 [x^2 + 2 - (-x)] dx$$

$$= \int_0^1 [x^2 + x + 2] dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_0^1 = \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}$$

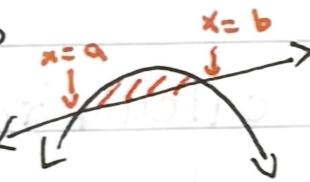
## 2. Not Given Endpoints

You must solve for the point of intersection

and set those points as endpoint boundaries.

$$f(x) = g(x) \Rightarrow x=a, x=b$$

$$A = \int_a^b [f(x) - g(x)] dx$$



E.g. Area bounded by  $y=2-x^2$  and  $y=x$

$$x=2-x^2$$

$$x=1, x=-2$$

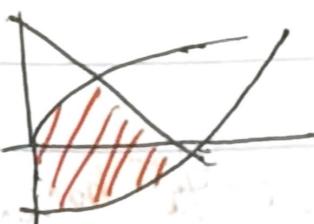
$$\int_{-2}^1 (2-x^2-x) dx = \frac{9}{2}$$

## 3. Composite Area (Multiple Curves)

Find intersections for all curves

$$f(x)=g(x), g(x)=h(x), f(x)=h(x), f(x)=i(x), \text{etc.}$$

$$A = \int_a^b [f(x)-g(x)] dx + \int_b^c [g(x)-h(x)] dx + \dots \text{etc.}$$



E.g. Area bounded by the curves  $y=2-x$ ,  
 $y=\sqrt{x}$ ,  $y=\frac{x^2}{4}-1$ ,  $x=0$

$$\sqrt{x}=2-x \Rightarrow x=1 \quad \frac{x^2}{4}-1=2-x \Rightarrow x=2$$

$$\begin{aligned} & \int_0^1 (\sqrt{x} - (\frac{x^2}{4} - 1)) dx + \int_1^2 (2-x - (\frac{x^2}{4} - 1)) dx \\ &= \left( \frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{12} + x \right) \Big|_0^1 + \left( 3x - \frac{x^2}{2} - \frac{x^3}{12} \right) \Big|_1^2 \\ &= \frac{5}{2} \end{aligned}$$

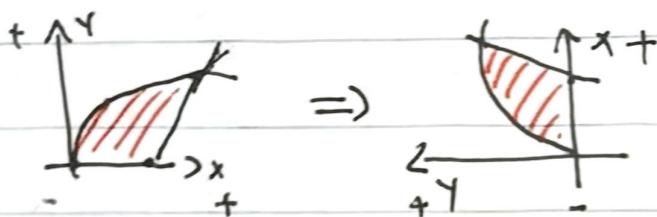
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## Area Between Curves (with respect to $y$ )

An area can be represented by an integral with respect  $x$  or an integral with respect to  $y$ .

When calculating an integral with respect to  $y$ , make sure to solve for " $x =$ " and view the graph rotated...



Integrate with respect to "y"!

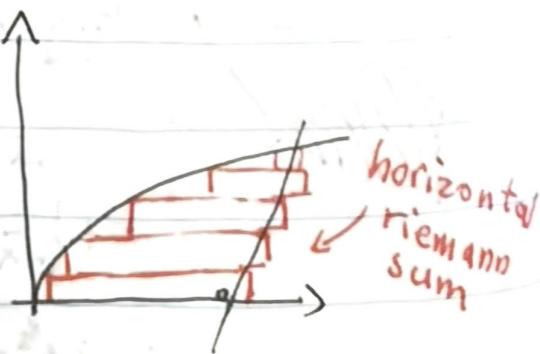
E.g. Set up an integral that allows you to find the area in the first quadrant that is bounded above by  $y=\sqrt{x}$  and below by  $y=x-6$ .

$$y = \sqrt{x} \Rightarrow x = y^2 \quad y = x - 6 \Rightarrow x = y + 6$$

$$y+6 = y^2 \Rightarrow y = 3$$

$$\int_0^3 (y+6 - y^2) dy$$

$x = y+6$  is above  $x = y^2$  relative to  $y$ . The given talks about relative to  $x$ .



## Area - More than Two Intersections

Some graphs require multiple integrals for multiple intersections

→ Find the  $x$ -values of the points of intersections and order from smallest to largest.

E.g. Set up the integrals that represent the total area of the regions bounded by the functions  $f(x) = 3x^3 - x^2 - 10x$  and  $g(x) = -x^2 + 2x$ .

$$3x^3 - x^2 - 10x = x^2 + 2x$$

$$3x^3 - 12x = 0$$

$$3x(x+2)(x-2) = 0$$

$$x = -2, x = 0, x = 2$$

$$\int_{-2}^0 [3x^3 - x^2 - 10x - (-x^2 + 2x)] dx + \int_0^2 [x^2 + 2x - (3x^3 - x^2 - 10x)] dx$$

ALTERNATIVELY...

You can also use the absolute value to get the same answer!

$$\int_A^C |f(x) - g(x)| dx$$

# Volumes with Cross Sections

Volume of a solid with known cross-sections

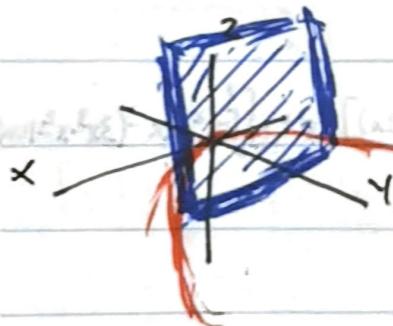
$$V = \int_a^b A(x) dx$$

| A(x) is the area of the cross section  
perpendicular to the x-axis

$$V = \int_a^b A(y) dy$$

| ..... perpendicular to the y-axis

## 1. Square



$$V = \int_a^b s^2 dx$$

where  $s = f(x) - g(x)$

E.g. A region is bounded by  $y = x^2$  and  $y = \sqrt{x}$ , and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume of the solid?

$$V = \int_a^b s^2 dx = \int_0^1 (\sqrt{x} - x^2)^2 dx$$

$$\sqrt{x} = x^2 \Rightarrow x = 0$$

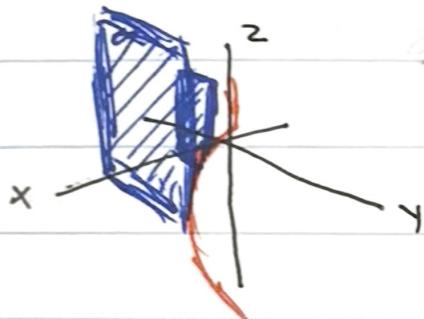
$$1 = x^3$$

$$1 = x$$

Square cross sections are taken perpendicular to the y-axis

$$V = \int_a^b s^2 dy = \int_0^1 (\sqrt{y} - y^2)^2 dy$$

## 2. Rectangle



$$V = \int_a^b (\text{width} \cdot \text{height}) dx$$

where width =  $f(x) - g(x)$ ,  
height = given in the problem

E.g. The base of a solid is bounded by  $y = x^3$ ,

$y = 0$ , and  $x = 2$ .

- a) Find the volume if the cross sections, taken perpendicular to the x-axis, form a rectangle whose height is 2 times its width

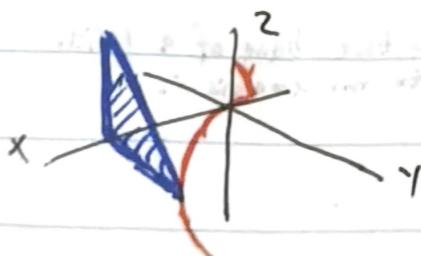
$$V = 2 \int_0^2 (x^3 - 0)^2 dx = 2 \int_0^2 x^6 dx$$

- b) Find the volume if the cross sections, taken perpendicular to the y-axis, form a rectangle whose height is 6.

$$V = 6 \int_0^{3\sqrt[3]{2}} (2 \cdot 3\sqrt[3]{y}) dy$$

height is constant  
so no need to square

## 3. Equilateral Triangle



$$V = \int_a^b \frac{\sqrt{3}}{4} s^2 dx$$

where  $s = f(x) - g(x)$



$$h = \sin(60^\circ) b$$

$$\begin{aligned} A &= \frac{1}{2} b \cdot h \\ &= \frac{1}{2} \sin(60^\circ) b^2 \\ &= \frac{\sqrt{3}}{4} b^2 \end{aligned}$$

E.g. A region is bounded by  $y = x^2$  and  $y = \sqrt{3}x$ , and forms the base of a solid.

For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. What is the volume of the solid?

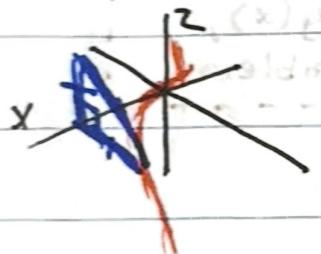
$$V = \int_a^b \frac{\sqrt{3}}{4} s^2 dx = \int_0^1 \frac{\sqrt{3}}{4} (\sqrt{3}x - x^2)^2 dx$$

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## Volumes with cross Sections

### 4. Isosceles Right Triangle



$$V = \int_a^b \frac{1}{2} s^2 dx$$

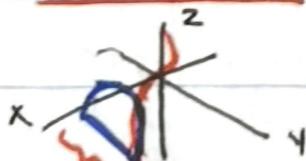
where  $s = f(x) - g(x)$

E.g. A region is bounded by  $y = x^2$  and  $y = \sqrt{x}$ , and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle

$$V = \int_a^b \frac{1}{2} s^2 dx = \int_0^1 \frac{1}{2} (\sqrt{x} - x^2)^2 dx$$

$$\pi b^2 x \left[ \frac{1}{3} x^2 + \frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^1$$

### 5. Semicircle



$$V = \int_a^b \frac{1}{2} \pi r^2 dx$$

THE DISTANCE HERE IS THE DIAMETER

$\therefore$  we need to  $\div 2$  to find radius instead

where  $r = \frac{f(x) + g(x)}{2}$

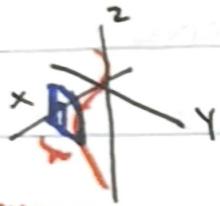
E.g. A region is bounded by  $y = x^2$  and  $y = \sqrt{x}$ , and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle.

$$V = \int_0^1 \frac{1}{2} \pi r^2 dx = \int_0^1 \frac{\pi}{2} \cdot \left( \frac{\sqrt{x} - x^2}{2} \right)^2 dx$$

SOMETIMES  $\pi \cdot 2^2$   
IS BROUGHT OUT...

$\frac{\pi}{8}$  instead of  $\frac{\pi}{2} \cdot \frac{1}{2}$

## 6. Quarter Circle



THE DISTANCE  
HERE IS THE  
RADIUS

→ Don't get confused  
with semicircle

$$V = \int_a^b \frac{1}{4} \pi r^2 dx$$

where  $r = f(x) - g(x)$

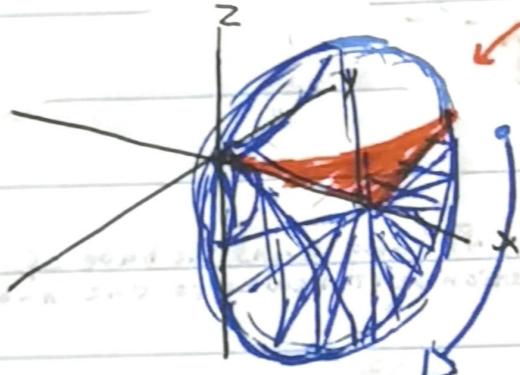
E.g. A region is bounded by  $y=x^2$  and  $y=\sqrt{x}$ , and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is a quarter circle.

$$V = \int_a^b \frac{1}{4} \pi r^2 dx = \int_0^1 \frac{\pi}{4} \cdot (\sqrt{x} - x^2)^2 dx$$

Quarter circles  
are ALMOST ALWAYS  
→ 4. Don't get confused  
with semicircle.

## Disc Method : Revolve around axes

When areas bounded by a curve and an axis is rotated around that axis, a circular cross section is formed.



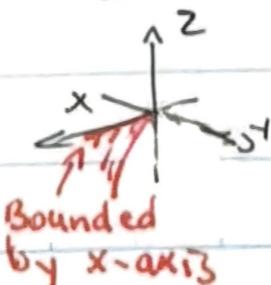
EACH CROSS SECTION IS A CIRCLE  
 → Unlike semicircles, the distance between the curve and the axis is the radius.

## Volume of a Solid of Revolution

$$V = \int_a^b \pi [R(x)]^2 dx$$

where  $R(x)$  is the "distance" between the axis of revolution and the outside of the solid.

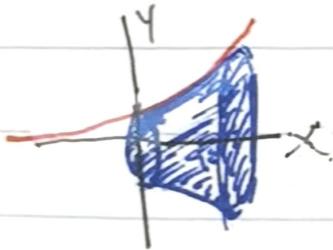
THE AXIS OF REVOLUTION MUST ALSO BE A BOUNDARY OF THE SOLID



## 1. x-axis

Must be integrated with respect to x: "dx"

E.g. Take the region bounded by  $y = e^x$ ,  $y = 0$ ,  $x = 0$ , and  $x = 3$ . Resolve this region about the x-axis. Find the volume of the solid formed.



$$\begin{aligned}
 & \int_0^3 \pi (e^x)^2 x \, dx \\
 &= \frac{\pi}{2} e^{2x} \Big|_0^3 \\
 &= \frac{e^6 \pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}(e^6 - 1)
 \end{aligned}$$

## 2. y-axis

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