ε

$\lim_{x \to \infty}$

e

AP Calculus

$\int$

$\frac{dy}{dx}$

∂

$\oiint$

$f(x)$

Unit 1: Limits and Continuity

Can change occur at an instant?

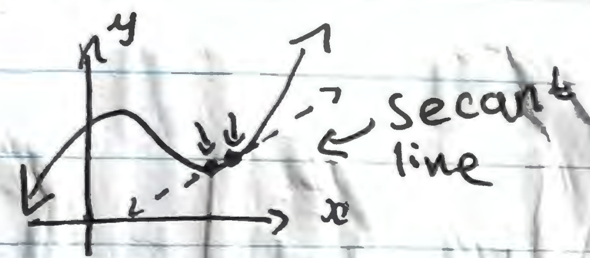
Average Rate of Change : $\frac{\Delta y}{\Delta x}$

How about at an instant?

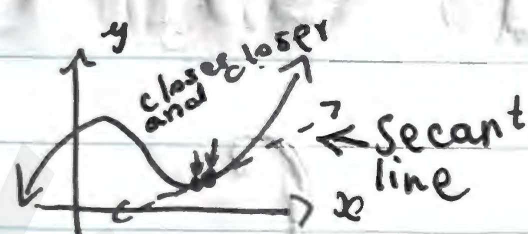
Estimates:

Closer and Closer

$$\frac{f(x)-f(x-0.1)}{x-(x-0.1)}$$



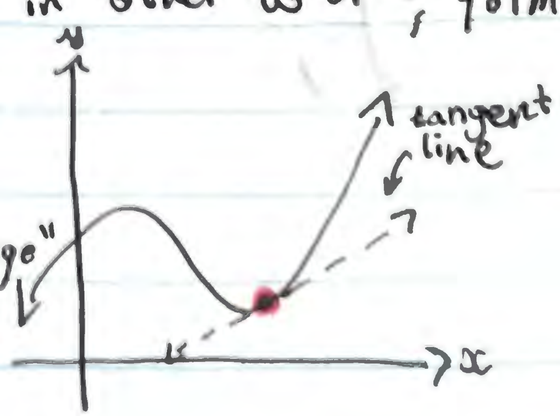
$$\frac{f(x)-f(x-0.001)}{x-(x-0.001)}$$



*NOTE: We cannot do $\frac{f(x)-f(x)}{x-x}$ as that would result in an indeterminate form or $\frac{0}{0}$.

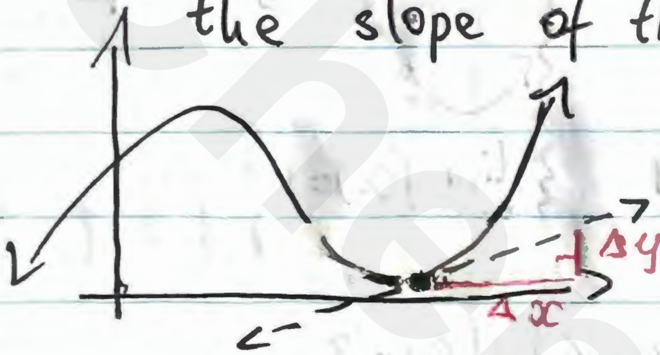
— So... What happens when the two points over lap eachother? Or in other words, form a tangent line?

We can get an "instant rate of change"



By definition... Rate of Change = Slope

Thus... An instantaneous rate of change is the slope of the tangent line:



$\therefore$ Instantaneous Rate of Change $= \dfrac{\Delta y}{\Delta x}$

Limits - Intro

Despite $f(2)$ being different values the limit is the same:

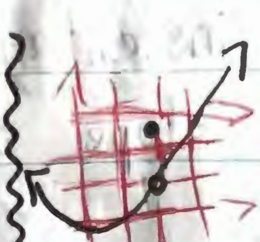
The output you approach with a given input

$\lim_{x \to 2} f(x) = 1$

$f(2) = 1$

$\lim_{x \to 2} f(x) = 1$

$f(2) =$ undefined



$\lim_{x \to 2} f(x) = 1$

$f(2) = 3$

Structure:

$$\lim_{x \to a} f(x) = b$$

a: as x approaches what value?

b: what happens when x approaches that value?

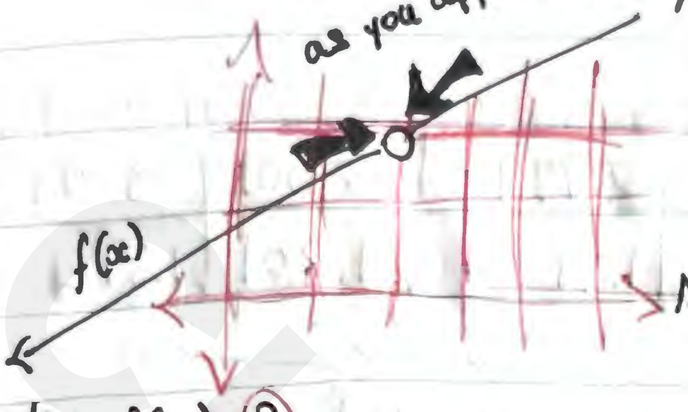
*NOTE: Limits ignore what $f(x)$ EQUALS, instead it refers to what $f(x)$ will equal when you approach it.

as you approach (from both sides)

Even if undefined.

THUS:

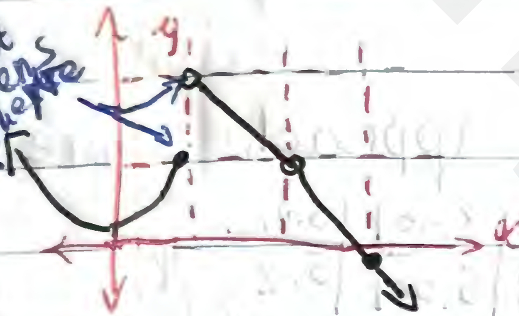
NOT $f(1)=\lim_{x\to 1} f(x)$ nor $f(1)\neq \lim_{x\to 1} f(x)$ in ALL CASES



$\lim_{x\to 2} f(x) = 2$

Special Cases of Limits

what happens when they're different



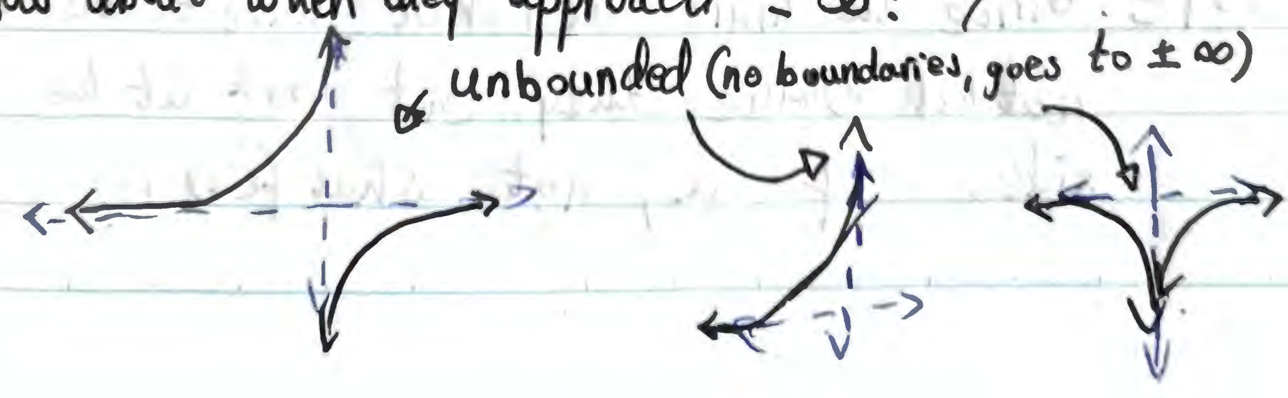
$\lim_{x\to 1} f(x) = \dots$ since the limit does not agree, it DOES NOT EXIST

$\lim_{x\to 1} f(x)$ DOES NOT EXIST

They ALSO Do Not Exist

How about when they approach $\pm\infty$?

unbounded (no boundaries, goes to $\pm\infty$)



Limits from tables

$\lim_{x \to 3} f(x)$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	1.682	1.788	1.7988	1.801	1.812	1.922

wanted value

So you can infer that when $x = 3$, $f(x)$ would approach 1.8.

When 2 sides don't approach same value: E.g.

2.99	2.999	3.001	3.01
1.788	1.7988	5.001	5.2

The limit DOES NOT EXIST

*NOTE: Since the limit does not have to be the actual value, You must look at the sides of x, not what $f(x)$ is.

Limit Properties

Given: $\lim_{x \to c} f(x) = L \quad \lim_{x \to c} g(x) = M$

$$\lim_{x \to c} \left(f(x) + g(x)\right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$$

$$\lim_{x \to c} \left(f(x) - g(x)\right) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$$

$$\lim_{x \to c} \left(f(x) g(x)\right) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = L \cdot M$$

$$\lim_{x \to c} k f(x) = k \lim_{x \to c} = k L$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}$$ (only true when $\lim_{x \to c} g(x) \neq 0$) (to avoid division by 0)

$$\lim_{x \to c} \left(f(x)\right)^{\frac{r}{s}} = \left(\lim_{x \to c} f(x)\right)^{\frac{r}{s}} = L^{\frac{r}{s}}$$

Limits from the left and right:

$\lim_{x \to a^-} f(x)$

FROM the NEGATIVE side, the LEFT

FROM the POSITIVE side, the RIGHT

$\lim_{x \to a^+} f(x)$

when unstated: $\lim_{x \to a} f(x$

it refers to BOTH

Limits of Combined functions

$\lim_{x \to a} (f(x) + g(x)) =$

plug it in

find one by one

When Both seem to be non-existant (as in $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ don't exist), try approaching from the right and then on the left:

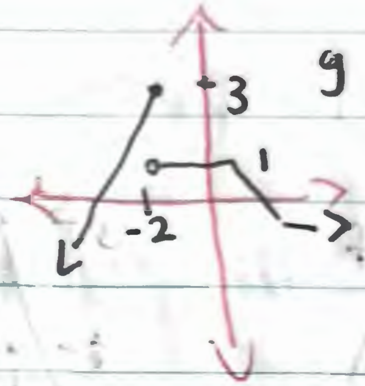
$\lim_{x \to a^-} (f(x) + g(x))$ instead and $\lim_{x \to a^+} (f(x) + g(x))$

If both yield same result, then that's you're answer

ELSE, the limit is non existant

* NOTE: This applies to all limits involving multiple functions.

E.g.



Although $\lim_{x \to -2} f(x)$ does not exist and $\lim_{x \to -2} g(x)$ also does not exist, $\lim_{x \to -2^-} f(x)$ does with $\lim_{x \to -2^+} f(x)$ and the same for $g(x)$

$\lim_{x \to -2^-} (f(x)+g(x)) = 1+3 = 4$ $\lim_{x \to -2^+} (f(x)+g(x)) = 3+1 = 4$

same...

$\therefore \lim_{x \to -2} (f(x)+g(x)) = 4$

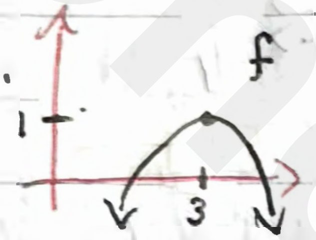
Theorem for limits of composite functions:

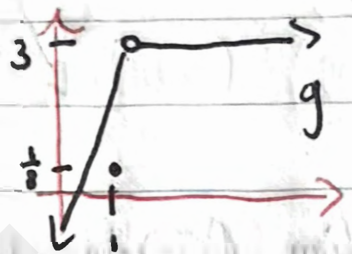
$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$ for $\lim_{x\to a} g(x) = L$ exists

AND

$f(x)$ is CONTINUOUS at L

E.g.:





since function f is continuous at 3 and $\lim_{x\to 1} g(x)$ exists, $\lim_{x\to 1} f(g(x)) = f(\lim_{x\to 1} g(x)) = 1$.

*NOTE: Although some limits don't work with this theorem (i.e. conditions aren't met), they don't necessarily not exist and can be still solved without it.

Limits of Composite functions: when INTERNAL limit does not exist

Let's say for function $g(x)$ and $h(x)$, $\lim_{x\to a} h(x)$ does not exist and you want to solve $\lim_{x\to a} g(h(x))$. So how?

$$\lim_{x\to a} g(h(x)) = C$$

split it into left or right

$$\lim_{x\to a^+} g(h(x)) = \lim_{h(x)\to b^+} g(h(x)) = C_1$$

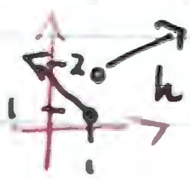
→ solve it. $\lim_{x\to b^+} g(x)$

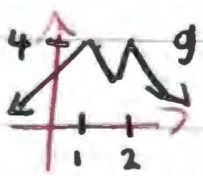
value approaches from positive (up) or negative (down), not to

$$\lim_{x\to a^-} g(h(x)) = \lim_{h(x)\to d^+} g(h(x)) = C_2$$

if both have same answer, $\lim_{x\to a} g(h(x)) = C$

E.g. $\lim_{x\to 1} g(h(x)) = 4$





$$\lim_{x\to 1^+} g(h(x)) = \lim_{h(x)\to 2^+} g(h(x)) = 4$$

$$\lim_{x\to 1^-} g(h(x)) = \lim_{h(x)\to 1^+} g(h(x)) = 4$$

Limits of Composite Functions: EXTERNAL limit does not exist

Similar to internal:

$\lim_{x \to a} g(h(x))$ for $\lim_{x \to a} g(x)$ does not exist

$$\lim_{x \to a^-} g(h(x)) = \lim_{h(x) \to b^-} g(h(x)) = \lim_{x \to b^-} g(x) = c_1$$

$$\lim_{x \to a^+} g(h(x)) = \lim_{h(x) \to d^+} g(h(x)) = \lim_{x \to d^+} g(x) = c_2$$

if same

$\lim_{x \to a} g(h(x)) = c$

ELSE

limit does not exist

Limits by DIRECT SUBSTITUTION

- ONLY if f is continuous at $x = a$ will $\lim_{x \to a} f(x) = f(a)$

E.g. $\lim_{x \to -1} (6x^2 + 5x - 1) = 6(-1)^2 + 5(-1) - 1 = 0$

Undefined limits

E.g. $\lim_{x \to 1} \frac{x}{\ln(x)} = \frac{1}{\ln(1)} = \frac{1}{0}$ } Although undefined, final answer was not in indeterminate form or $0 \div 0$, so limit DOES NOT EXIST

*NOTE: A limit is only non-existant when the reasoning is undefined in a form that is not indeterminate form $\left(\frac{0}{0}\right)$

Limits of Piecewise Functions

When dealing with Piecewise Functions, you must consider both sides.

$\lim_{x \to c}$ to consider $\lim_{x \to c^-}$ OR $f(c-0.001)$ AND $\lim_{x \to c^+}$ OR $f(c+0.001)$. You should not be worried about $f(c)$ as limits refer to the general shape of graph, not considering holes.

If $\lim_{x \to c^-} = \lim_{x \to c^+}$ then they also equal $\lim_{x \to c}$

E.g. $f(x) = \begin{cases} \frac{x+2}{x-1} & \text{for } 0 < x < 4 \\ 5 & \text{for } x = 4 \\ \sqrt{x} & \text{for } x > 4 \end{cases}$

$\lim_{x \to 4^+} f(x) = 2$

$\to \sqrt{4}$

it applies for $x > 4$, although we're dealing with limits so we don't care

$\lim_{x \to 4^-} f(x) = \frac{4+2}{4-1} = \frac{6}{3} = 2$

HOWEVER...

$f(4) = 5$ ← since we don't care about $x = 5$, we neglect it. We only care about when $x = 3.99\bar{9}$ AND $x = 4.0\bar{0}1$

so it doesn't matter

SINCE $\lim_{x \to 4^+} = \lim_{x \to 4^-}$, $\lim_{x \to 4} = 2$

Limits with Absolute Values

An absolute value can be seen as a piece-wise function:

$$|x| = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x & \text{for } x < 0 \end{cases}$$

SO...

a function like $f(x) = \frac{|x-3|}{x-3}$ can also be seen as it

$$f(x) = \begin{cases} \frac{x-3}{x-3} & \text{for } x > 3 \\ \text{undef.} & \text{for } x = 3 \\ \frac{-(x-3)}{x-3} & \text{for } x < 3 \end{cases}$$

since absolute value on top.

SINCE... We're doing limits, we can ignore what happens when $x = C$ in limit $\lim_{x \to c}$

E.g. $\lim_{x \to 3} f(x) = \begin{cases} \frac{x-3}{x-3} & \text{for } x > 3 \\ \frac{-(x-3)}{x-3} & \text{for } x < 3 \end{cases}$

ALSO, when $x = 3$ function is undefined so

Evaluate limits from both side

$\lim_{x \to 3^-} f(x) = \frac{-\cancel{(x-3)}}{\cancel{x-3}} = -1$ $\qquad$ $\lim_{x \to 3^+} f(x) = \frac{\cancel{x-3}}{\cancel{x-3}} = 1$

IN THIS CASE, $-1 \neq 1$ so $\lim_{x \to 3^-} \neq \lim_{x \to 3^+}$ which means $\lim_{x \to 3} f(x)$ D.N.E

Solving Limits (Algebraic Manipulation)

When dealing with limits that cannot be solved through direct substitution (undefined indeterminate form), you can use other methods.

1) Factoring:

You can factor numerator & demominator and then cancel them out.

↳ Since limit is value that it approaches, we don't care about holes.

E.g. $f(x) = \frac{x^2+x-6}{x-2} = \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = x+3, \; x \neq 2$

it refers to a hole, so we won't have to care about it

$\lim_{x \to 2} f(x) = (2)+3 = 5$ ← ANSWER

2) Rationalising (Conjugates)

We can also use the difference of squares.

I.E. $z = a+b \quad \bar{z} = a-b$

$\therefore z \cdot \bar{z} = (a+b)(a-b) = a^2 - b^2$

When working with radicals, this can work similarly.

$$(\sqrt{a}+b)(\sqrt{a}-b) = a-b^2$$

So...

We can use this to rationalise and cancel out expressions.

E.G. $\lim_{x \to -1} \frac{x+1}{\sqrt{x+5}-2} = g(x)$

$$= \frac{x+1}{\sqrt{x+5}-2} \cdot \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} = \frac{(x+1)(\sqrt{x+5}+2)}{x+5-4}$$

$$= \frac{\cancel{(x+1)}(\sqrt{x+5}+2)}{\cancel{(x+1)}} = \sqrt{x+5}+2 \text{ for } \underbrace{x \neq -1}_{\text{limits don't care about holes}}$$

$$\lim_{x \to -1} \frac{x+1}{\sqrt{x+5}-2} = \sqrt{x+5}+2 = 4$$

3) Complex Fractions

We can use least common multiples to simplify an expression.

E.g. $$\lim_{x \to 0} \frac{x}{\frac{1}{x-4}+\frac{1}{4}} = \frac{x}{\frac{\cancel{4}}{4x-16}+\frac{x\cancel{-4}}{4x-16}} = \frac{x}{\frac{x}{4x-16}}$$

$$= x \cdot \frac{4x-16}{\cancel{x}} = 4(x-4) \underbrace{\text{for } x \neq 0}_{\text{we ignore this as usual}}$$

DIRECT SUBSTITUTION

$$4(0-4) = 4 \cdot (-4) = -16$$

SPECIAL TRIG LIMITS

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{OR} \quad \lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \quad \text{OR} \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Similarly, $\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

$$\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \frac{\sin x \cdot \sin x}{x} = 1 \cdot \lim_{x \to 0} \sin x = 0$$

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

Squeeze Theorem (Sandwich Theorem, Pinching Theorem)

states that... when...

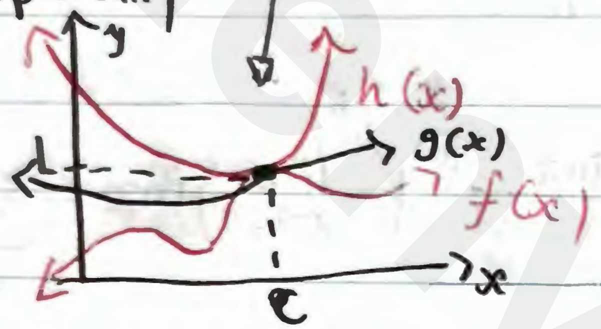
$$f(x) \le g(x) \le h(x)$$

$\lim_{x\to c} f(x) = L$ $\qquad$ $\lim_{x\to c} h(x) = L$

$$\Downarrow$$

$$\lim_{x\to c} g(x) = L$$

Graphically



For Trig Functions:

Since $\sin x$ & $\cos x$ have a range of $[-1, 1]$, we can use squeeze theorem too.

$-1 \le \sin x \le 1$ AND $-1 \le \cos x \le 1$

$\therefore$

$-k \le k\sin x \le k$ AND $-k \le k\cos x \le k$

HOWEVER...

Not all can be determined.

E.g. $1 \leq \lim_{x \to 0} f(x) \leq 2$ cannot be determined

$5 \leq \lim_{x \to 5} f(x) \leq 5$ can be determined

$81 \leq \lim_{x \to 7} f(x) \leq 92$ Cannot be determined

Continuity and discontinuity

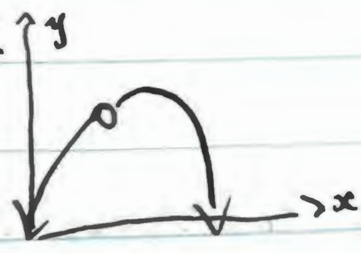
A point is considered continuous if:

1. $f(c)$ is defined, c is in the domain
2. $\lim_{x \to c} f(x)$ exists, $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$
3. $\lim_{x \to c} f(x) = f(c)$

Types of discontinuities

1. Hole (removable)
   - ↳ Can be removed by filling in hole
   - ↳ Single point being undefined
   - ↳ Indeterminate Form

Graph:



$\lim_{x \to c} f(x) \neq f(c)$

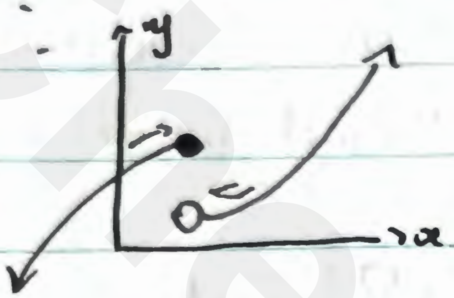
You can "factor" out the hole

2) Jump Discontinuity (non-removable)

↳ Only Piecewise

↳ Where $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

Graph:



$\lim_{x \to c} f(x)$ does not exist

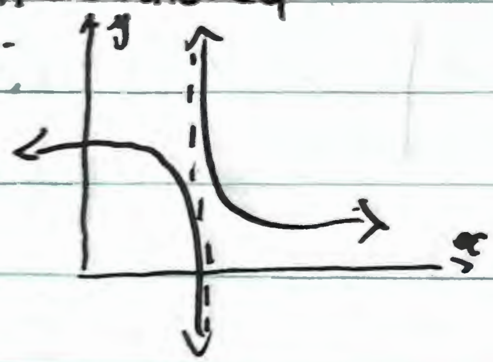
3) Asymptotic Discontinuity

↳ of Vertical Asymptote

↳ Approaches ∞ OR $-\infty$

↳ Unbounded

Graph:



Continuity over an Interval

↳ f is continuous over (a,b) IF and ONLY IF f is continuous over every point in interval.

↳ f is continuous over $[a,b]$ IF and ONLY IF f is continuous over (a,b) and $\lim_{x \to a^+} f(x) = f(a)$ & $\lim_{x \to b^-} f(x) = f(b)$

Scenarios to look out for when looking for domain restrictions

1) Denominators (can't be $=0$) $f(x) = \frac{x-5}{x+1}$, $x+1 \neq 0$

2) Radicals with Even Roots (cannot be <0 inside) $f(x) = \sqrt{7x+3}$, $7x+3 \geq 0$

3) Logarithms (Vertical Asymptote at 0) $f(x) = \ln(2x+1)$, $2x+1 > 0$

If f is not continuous at an end point of an interval $[a,b]$ but $\lim_{x \to a^+} f(x)$ exists and $\lim_{x \to b^-} f(x)$ exists then f is still continuous over that entire interval (closed)

Removing Discontinuities

↳ You can remove removable discontinuities by factoring them out.

$$f(x) = \frac{kx^2 + 2kxb + kb^2}{hx^2 - hb^2} = \frac{k(x+b)\cancel{(x+b)}}{h(x-b)\cancel{(x+b)}}$$

$$= \frac{k(x+b)}{h(x-b)} \text{ for } \underbrace{x \neq -b}_{\text{Restraint on domain}}$$

(hole, removable discontinuity)

To remove it, don't write the restraint

$$f(x) = \frac{k(x+b)}{h(x-b)}$$

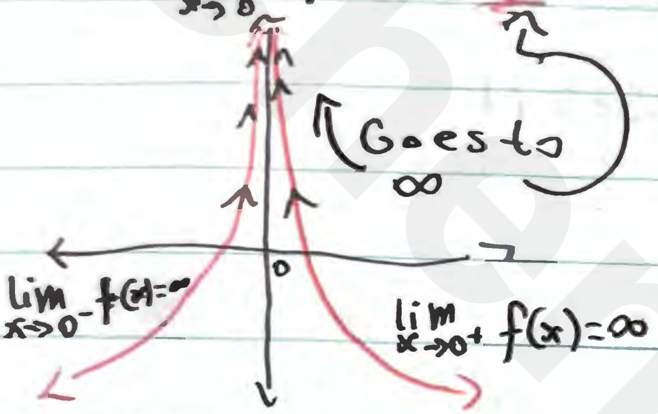
Now, $f(x)$ is defined at $x = -b$

You can also use rationalisation with conjugates, just like finding the limit of it. (see Solving Limits: Algebraic Manipulation)

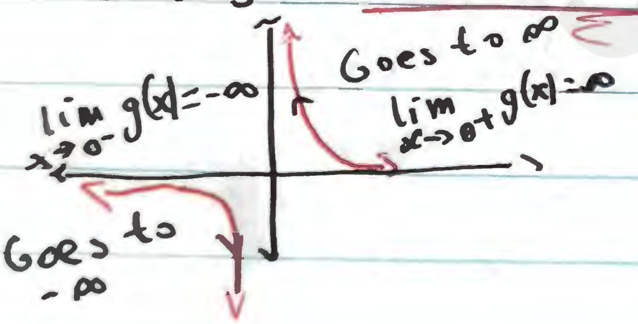
Infinite Limits and Vertical Asymptote

When faced with unbounded limits, they can approach infinity (a concept, not a number though)

E.G $\lim_{x \to 0} f(x) = \infty$



$\lim_{x \to 0} g(x) =$ Does not exist

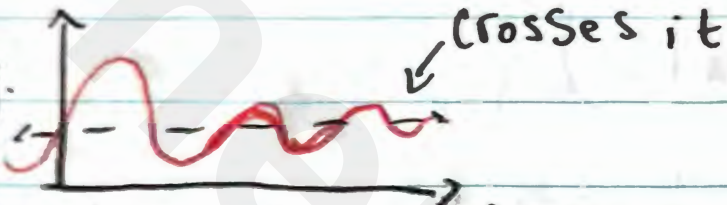


$\lim_{x \to 0^+} g(x) \neq \lim_{x \to 0^-} g(x)$

Key Differences between Horizontal vs Vertical Asymptote.

When dealing with a function, a vertical asymptote cannot be crossed whilst a horizontal asymptote can (function oscillates)

E.g.



Example Function ~~f(x)=~~ $\frac{10\sin x}{x}+3$

Limits at Infinity and Horizontal Asymptote

Horizontal Asymptote: (End Behaviour)
↳ What does the y-value approach as the x→value approaches $-\infty$ AND $+\infty$?
↳ Specific Number or Out Of Bounds?

Basic Rules:

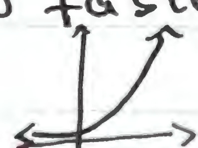
1) If the denominator grows faster...
$\frac{\text{not as big}}{\text{BIG Number}} = 0$ $\left(\frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000} \rightarrow \frac{1}{\infty} = 0\right)$

2) If numerator and denominator grow EQUALLY fast, then you have $\frac{\text{BIG number}}{\text{BIG number}} = 1$
$\left(\frac{10}{10}, \frac{100}{100}, \frac{10000}{10000} \longrightarrow \frac{\infty}{\infty} = 1\right)$

3) If the numerator grows FASTER than the denominator...
$\frac{\text{BIG number}}{\text{not as big}} = \infty$ $\left(\frac{10}{1}, \frac{100}{1}, \frac{10000}{1} \longrightarrow \frac{\infty}{1} = \infty\right)$

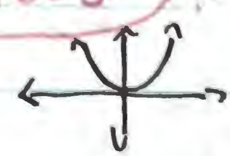
Which Functions grow fastest?

1) exponential: $f(x) = a^x$

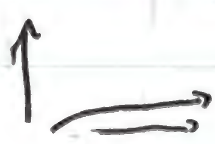


↳ Bigger the a is, faster it grows

2) Polynomials: $f(x) = x^a$



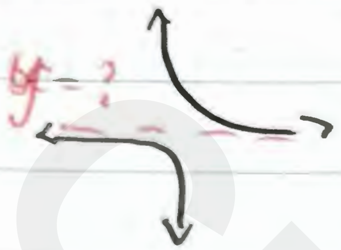
3) Logarithims: $f(x) = \log_b x$



smaller the b, faster it grows

↳ So... $\ln x$ grows faster than $\log x$
↳ $e < 10$

To find Horizontal Asymptotes, evaluate
$\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$



↳ Plug in ∞ / $-\infty$

↳ In rational functions, only care about the highest degree terms

E.g. $\dfrac{5x^4 + 3x + 2}{7x^4 + 2x^3 - x} \approx \dfrac{5x^4}{7x^4} \approx \dfrac{5}{7}$

Change the most

$\dfrac{8x^7 + 34x}{2x^6 - 3} \approx 8x \approx \infty$

→ Replace for ∞

* NOTE: When dealing with square roots and exponents, the signs may change.

E.g.

$\lim_{x \to -\infty} \dfrac{x}{\sqrt{x^2}} = \dfrac{x}{|x|}$ $\quad \dfrac{-(\infty)}{\infty} = -1$

↳ $\lim_{x \to -\infty} \dfrac{x}{|x|} = -1$

Don't forget that after you square a number it becomes positive

Intermediate Value Theorem (IVT)

Used as a proof for when a continuous function will pass a specific number.

E.g.

x	0	3	4	8	9
$f(x)$	1	-5	3	7	-1

0 0 0

Has to pass 0 at least 3 times

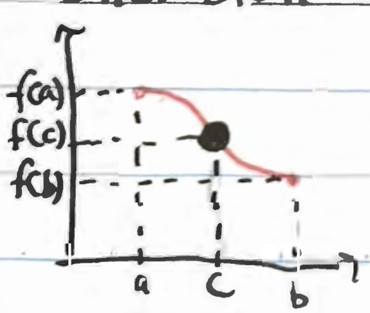
The IVT is used as a justification for this

Justification With the IVT

Conditions (must also be written for a complete proof):

1. The function $f(x)$ is continuous on an interval $[a,b]$.
2. $f(a) < f(b)$ or $f(a) > f(b)$, can't be equal
3. $f(c)$ is between $f(a)$ and $f(b)$

Conclusion: "According to the IVT, there is a value c such that $f(c) =$ *Some Number* and $a < c < b$"



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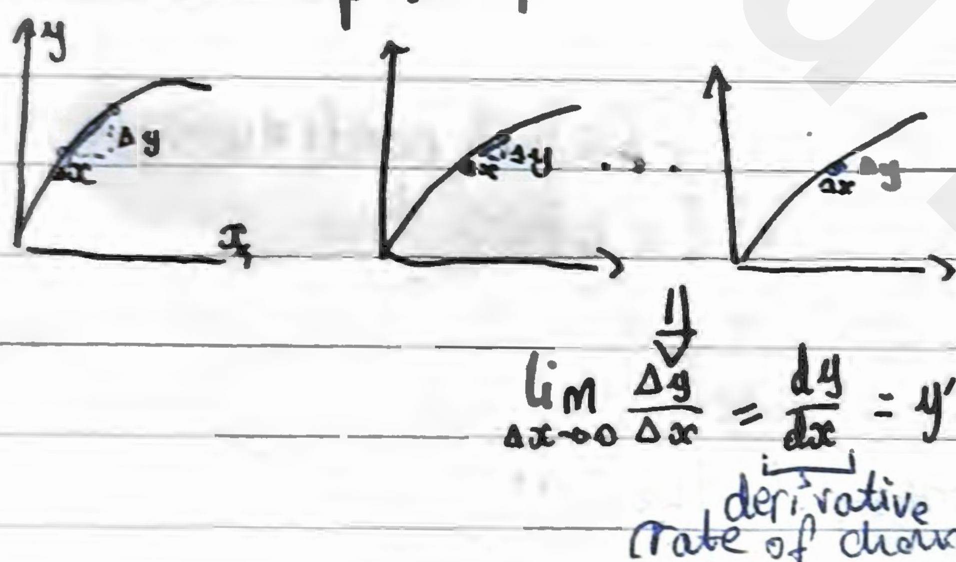
Defining Average and Instantaneous rates of Change at a point:

Differential Calculus
↳ About Instantaneous rate of change

Average Rate of Change = $\frac{\Delta y}{\Delta x}$

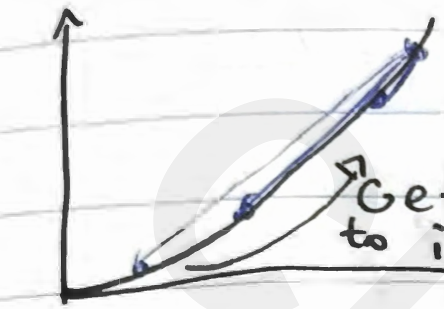
Instantaneous is at an instant (the slope)
↳ So, you get more and more accurate as the Δx gets smaller and smaller (at a point)



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \underbrace{\frac{dy}{dx}}_{\text{derivative}} = y' = f'(x)$$

derivative
(rate of change at a point)

Secant lines and average rate of change



Gets closer and closer to instantaneous rate of change

Lagrange's Notation: $f'(x)$ OR y'

Leibniz's Notation: $\frac{dy}{dx}$

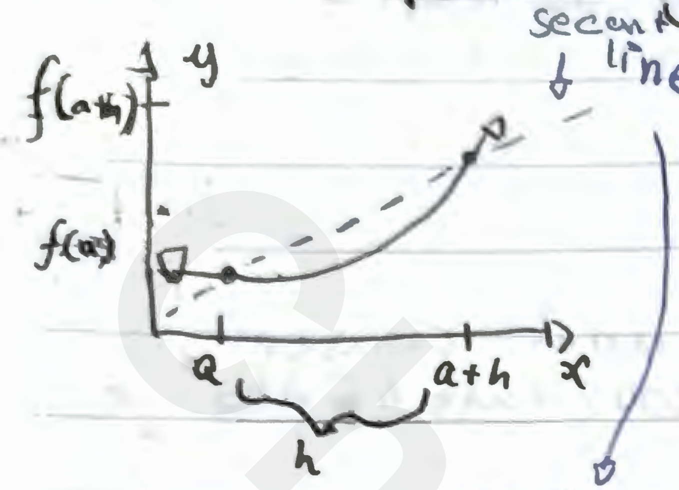
Newton's Notation: $\dot{y}$

Leibniz Notation: $\frac{d}{dx} f(x)$ can be replaced with whatever function

$\frac{dy}{dx}$ only when $y = f(x)$

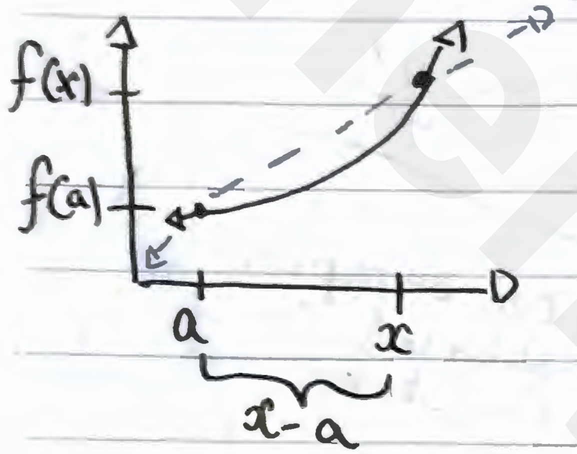
when $y=f(x)$ only, else $f(x)$

Expression for Average rate of change



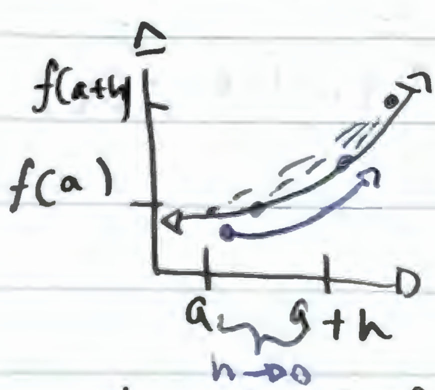
$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

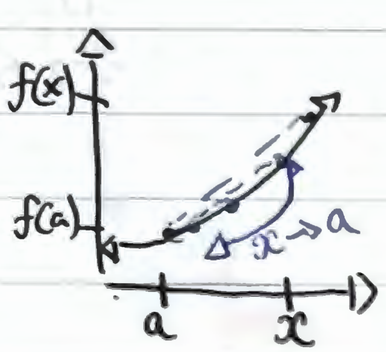
Slope of secant line



$$\frac{f(x)-f(a)}{x-a}$$

Instantaneous Rate of Change is when both points approach one single point





$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{OR} \quad \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

*This is the slope of the tangent line

You can now plug in the function and values for an instantaneous rate of change at $x = a$ (replace a with domain)

↳ when solving, there HAS to be a hole to cancel out in the end

Finding the original function from derivative:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

↳ Identify similarities between $f(a+h)$ & $f(a)$

↓

Usually it always has $a+h$

$x = a$ so x value $= a$

↓

The rest is original function

~~easier~~

↓

function

↓

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$x = ?$

The Derivative

- The Derivative is an expression that calculates the instantaneous rate of change (slope of the tangent line) of a function at any given x-value. In other words, it gives us the slope of the function at a point!

Notation:

Lagrange: $\underbrace{f'(x)}_{\text{when } f(x)}$ y'

when y

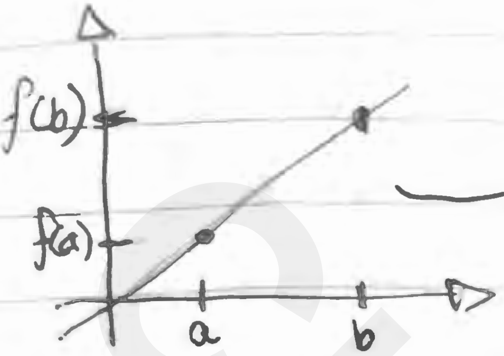
Leibniz: $\frac{dy}{dx}$

To find it, solve for when $x \to x$ OR $x = x$ instead of $x \to a$
(Replace "a" not x with x)

y

$$f'(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

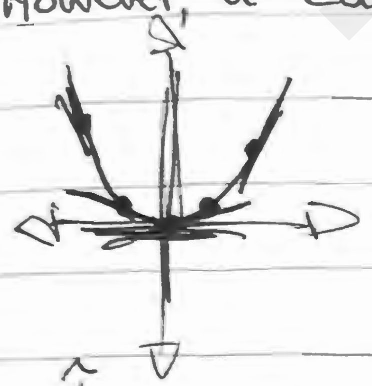
The Derivative as a limit

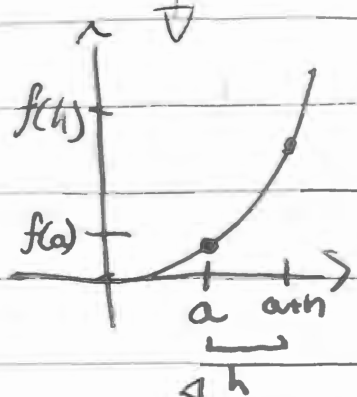


the slope is the average rate of change

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

However, a curves slope changes





$$\text{slope} = \frac{f(a+h)-f(a)}{\cancel{(a}+h)-\cancel{a}}$$

$$= \frac{f(a+h)-f(a)}{h}$$

Slope of secant line

tangent line would be when ~~two points~~ approach one point

As $h \to 0$

Derivative of f

$$f'(x) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Tangent line slope

The Derivative at any point.

→ We can find the derivative at a single given point, so if we plugged in x, we would be given a derivative at any point.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

In this one, you substitute a for x

alternate form does not work.

E.g. $f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2x\cancel{h}+\cancel{h^2}}{\cancel{h}} = \lim_{h\to 0} 2x+h$$

$$= 2x$$

The derivative at any point for $f(x) = x^2$

Plug in x to know instantaneous rate of change at x

Equation of Tangent line

To find equation of a tangent line we need slope and point.

Point is where instantaneous rate of change is at.

Slope is derivative at that point.

E.g. $f'(2)=5$ & $f(2)=7$

Point $= (2, 7)$

Slope $= 5$

Estimating derivatives.

A derivative refers to the rate of change at a single point

- So an estimate would be close to that single point
  - ↳ In other words, average rate of change around that point
  - (make sure graph is differentiable

E.g.

x	0	3	5	6	9
$f(x)$	72	95	112	72	54

$\to f'(4) \approx \frac{17}{2} = 8.5$

(4 is between 3 and 5)

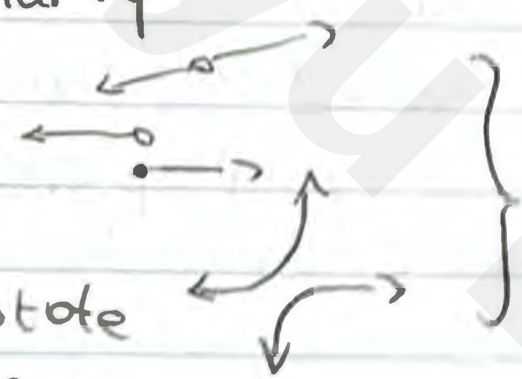
Differentiability and Continuity

A graph that is differentiable must be continuous, but one that is continuous may not be differentiable.

Differentiability is where the derivative exists for each point in the domain. Graph must be a smooth line.

↳ It must not contain:

1. Discontinuity
   - Hole
   - Jump
   - Vertical Asymptote

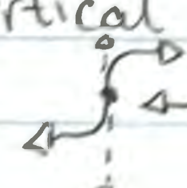


Not differentiable

2. Corner/Cusp

Not differentiable

3. Vertical Tangent



← slope here is $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{0}$

Differentiability at a point

Test if derivative exists at that point

To find derivative in a Piecewise function, do it from BOTH SIDES

E.g.

$$g(x) = \begin{cases} x-1, & x<1 \\ (x-1)^2, & x \geq 1 \end{cases}$$

USE ALTERNATE FORM

$$\lim_{x \to 1} \frac{g(x) - g(1)}{x-1}$$

$$\lim_{x \to 1^-} \frac{g(x)}{x-1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$$

similar to limits, if both sides not equal, then it doesn't exist.

$$\lim_{x \to 1^+} \frac{g(x)}{x-1} = \lim_{x \to 1^+} \frac{(x-1)^2}{x-1} = \lim_{x \to 1^+} x-1 = 0$$

Estimating derivatives with CALCULATOR

1. Go to Calculator Menu (not Graphing nor any other) $\boxed{\begin{matrix}\times & \div \\ + & -\end{matrix}}$
2. Menu → Calculus → Derivative / Derivative at a point
   ↳ If at a point, put in the x value of the point
3. Fill in function

$$\frac{d}{dx}(\square)\underbrace{\Big|\, x = ?}_{\text{only for derivative at a point}}$$

input function here

4. Press ENTER for EXACT VALUE, or ≈
   Ctrl + ENTER

The POWER RULE

Function		Derivative
$f(x) = x$	$\longrightarrow$	$f'(x) = 1$
$f(x) = x^2$	$\longrightarrow$	$f'(x) = 2x$
$f(x) = x^3$	$\longrightarrow$	$f'(x) = 3x^2$
$f(x) = x^4$	$\longrightarrow$	$f'(x) = 4x^3$
$f(x) = x^5$	$\longrightarrow$	$f'(x) = 5x^4$
$\cdots$		$\cdots$
$f(x) = x^n$		$f'(x) = nx^{n-1}$

E.g. $f(x) = x^{37}$ $\quad f'(x) = 37x^{36}$

$f(x) = \frac{1}{x^4}$ $\quad f'(x) = -\frac{4}{x^5}$

$f(x) = \sqrt[7]{x^3}$ $\quad f'(x) = \frac{3}{7\sqrt[7]{x^4}}$

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Basic Derivative Rules:

1. Constant: $\frac{d}{dx} c = 0$

   ↳ $cx^0 = c \cdot 1 = c$

   ↳ $\cancel{0 \cdot c \cdot x^{0-1}} = 0$

2. Constant Multiple: $\frac{d}{dx} cu = c\frac{du}{dx}$

   ↳ $y = 3x^2 \to y' = 3 \cdot (2 \cdot x^{2-1}) = 6x$

3. Sum / Difference: $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

$\therefore$ $f(x) = ax^n \qquad f'(x) = anx^{n-1}$

$f(x) = a \qquad f'(x) = 0$

$f(x) = ax^n + bx^p \qquad f'(x) = anx^{n-1} + bpx^{p-1}$

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) = f'(x) + g'(x)$$

4. Product Rule: $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

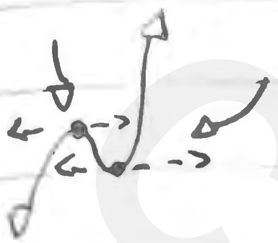
OR

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

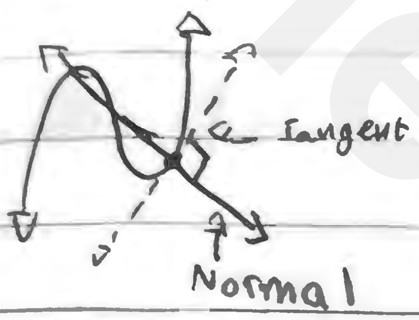
Horizontal Tangent Lines

↳ When slope = 0

↳ So... when derivative = 0



Normal Lines: Lines that go through the same point the tangent does, but is PERPENDICULAR to the TANGENT line.



Differentiating Polynomials

To differentiate polynomia's, separate the terms and find the derivative of each:

$$f(x) = a + b + c$$

$$f'(x) = \frac{d}{dx}(a) + \frac{d}{dx}(b) + \frac{d}{dx}(c)$$

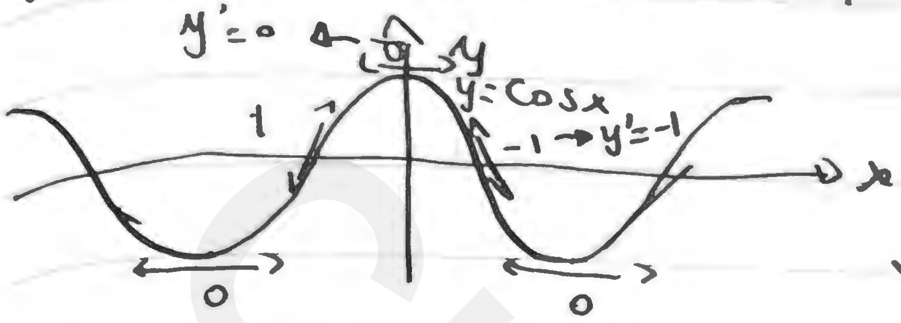
E.g. $f(x) = x^2 + 2x^3 - x^2$

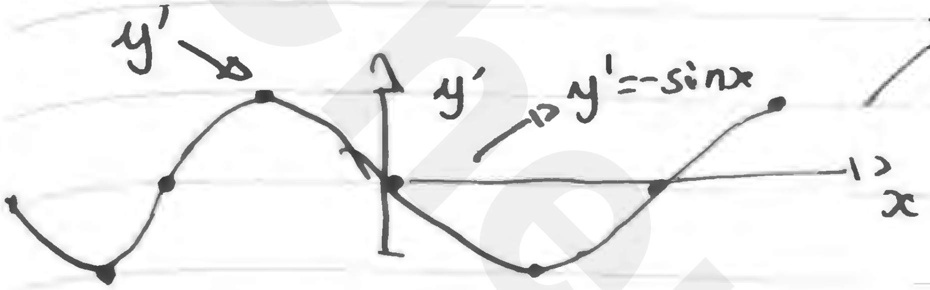
$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^5 + 2x^3 - x^2)$$

$$f'(x) = \frac{d}{dx}(x^5) + \frac{d}{dx}(2x^3) - \frac{d}{dx}(x^2)$$

$$f'(x) = 5x^4 + 6x^2 - 2x$$

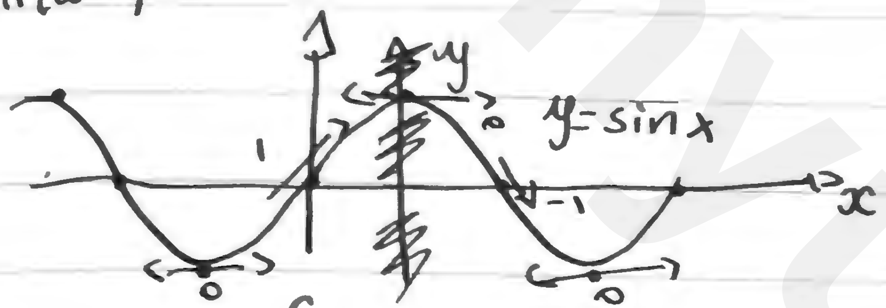
Derivatives of $\cos x$, $\sin x$, e^x and $\ln x$

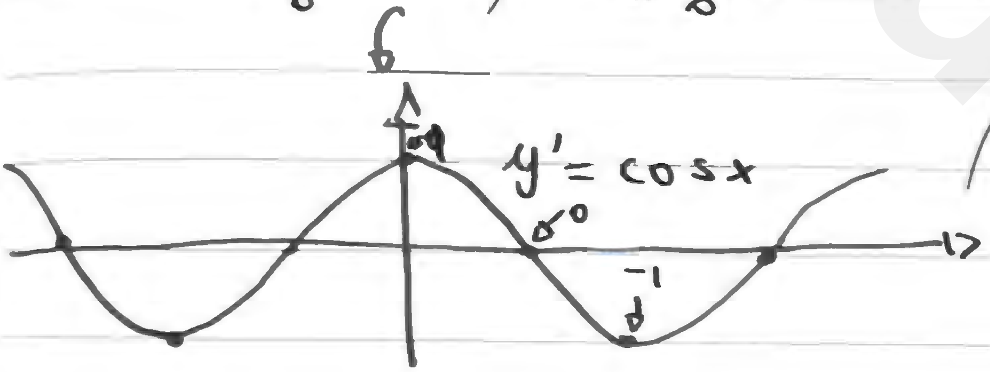




$$\frac{d}{dx}\cos x = -\sin x$$

similarly





$$\frac{d}{dx}\sin x = \cos x$$

Summarised...

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sin x = \cos x$$

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Derivatives of Exponential Functions

$$\frac{d}{dx} a^x = \underline{\ln(a) \cdot a^x}$$

e.g. 2^x OR 5^a

Any exponential function

$$\therefore \frac{d}{dx} e^x = \cancel{\ln(e)} \cdot e^x = \underline{e^x}$$

Derivatives of logarithmic Functions

$$\frac{d}{dx} \log_a x = \underline{\frac{1}{\ln(a) \cdot x}}$$

$$\therefore \frac{d}{dx} \ln x = \frac{d}{dx} \log_e x = \frac{1}{\cancel{\ln(e)} \cdot x} = \underline{\frac{1}{x}}$$

The Product Rule

$$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}[f(x)]\cdot g(x) + f(x)\cdot \frac{d}{dx}[g(x)]$$

In other words...

if $h(x) = f(x)\cdot g(x)$

then...

$$h'(x) = f'(x)\cdot g(x) + f(x)g'(x)$$

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - f(x)\cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

You take the derivative of f multiplied by g, subtract f multiplied by the derivative of g, and divide all that by $[g(x)]^2$

In other words...

if $h(x) = \frac{f(x)}{g(x)}$

then... $h'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

*Note: Derivative of top then bottom (order matters as it is subtraction)

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Derivatives of all 6 trig functions

$f(x) = \sin x \longrightarrow f'(x) = \underline{\cos x}$

$g(x) = \cos x \longrightarrow g'(x) = \underline{-\sin x}$

$h(x) = \tan x \longrightarrow h'(x) = \frac{\cos x \cos x + \sin x + \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \underline{\sec^2 x}$

$k(x) = \cot x \longrightarrow k'(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} = \underline{-\csc^2 x}$

$m(x) = \sec x \longrightarrow m'(x) = \frac{\sin x}{\cos^2 x} = \frac{\tan x}{\cos x} = \underline{\sec x \tan x}$

$p(x) = \csc x \longrightarrow p'(x) = -\frac{\cos x}{\sin^2 x} = -\frac{\cot x}{\sin x} = \underline{-\csc x \cot x}$

So...

$\frac{d}{dx}\sin x = \cos x$ $\qquad$ $\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$ $\qquad$ $\frac{d}{dx}\csc x = -\csc x \cot x$

Notice how functions that start with "C" are always negative

Unit 3: Differentiation
(Composite, implicit, and inverse functions)

The Chain Rule

A rule used to find the derivatives of Composite Functions (i.e $f(g(h(x)))$

Inner → outer

$$h(x) = f(g(x))$$

⇓ Chain rule

$$h'(x) = \frac{d}{dx}(g(x)) \cdot \frac{d}{d(g(x))}(f(g(x))) = g'(x) \cdot f'(g(x))$$

* This equation works for as many functions as you wish, e.g.

$$a(x) = b(c(d(e(f(\cdots z(x))))))$$

⇓

$$a'(x) = \frac{d}{dx}(z(x)) \cdot \frac{d}{d(y(x))}(y(z(x))) \cdots \cdot \frac{d}{d(c(d(e\cdots(x))))}(b(c\cdots(x)))$$

*NOTE: You only take the derivative of the outside.

i.e. $\frac{d}{dx}(f(g(x))) = g'(x) \cdot f'(g(x))$

only outermost is derivative

is not $g'(x)$, it is $g(x)$

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Common Chain Rule Misunderstandings

$\frac{d}{dx}(\ln(\sin(x)))$ $\neq$ $\frac{d}{dx}(\ln(x)\sin(x))$

$f(x) \rightarrow f(g(x)) \leftarrow g(x)$

Requires Chain Rule to Solve

Is a composite function

$f(x) \rightarrow f(x)g(x) \leftarrow g(x)$

Requires Product Rule to Solve

Not a composite function

Implicit Differentiation

Explicit:

e.g $y = mx + b$

Function

vs

Implicit:

e.g. $ax^2 + by^2 = 1$

Not a function (requires both x and y to solve, or other variables)

Usually, when differentiating explicit, $\frac{dx}{dx} = 1$ so we don't have anything different.

i.e. $f(x) = x$ $\quad f'(x) = \frac{dx}{dx} = 1$

$f(x) = x^2$ $\quad f'(x) = 2\frac{dx}{dx}x = 2x$

for every variable "a", the derivative is $\frac{d}{dx}(a)$ or $\frac{da}{dx}$

so for y, it would be $\frac{dy}{dx}$

Since $\frac{dy}{dx}$ does not cancel

you leave it as is

i.e. $x^2 + y^2 = 1$

$\hookrightarrow 2x\cancel{\frac{dx}{dx}} + 2y\frac{dy}{dx} = 0$

solve if you want to find $\frac{dy}{dx}$ (or the instantaneous rate of change)

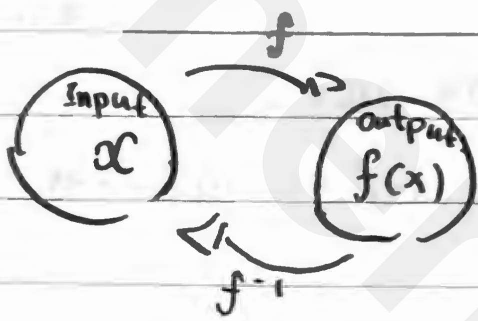
*NOTE: You will need multiple inputs for implicit differentiation

Derivatives of inverse functions

Inverse functions are when the domain and range of a function are swapped.

I.E $f(x) = mx + b \rightarrow x = mf^{-1}(x) + b$

$$f^{-1}(x) = \frac{x-b}{m}$$



Derivative Formula:

$$\frac{d}{dx}(f(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Derivative of regular function

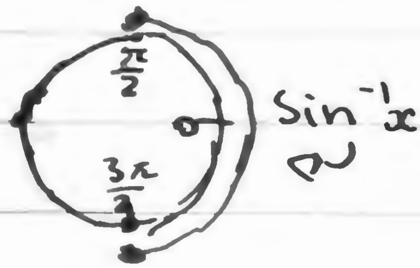
Inverse function

Derivative of Inverse Sine

$y = \sin^{-1}(x)$

Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



$y = \sin^{-1} x \Rightarrow \sin y = x \Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}[x]$

$(\cos y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

Now we write it in terms of sin.

Trig identity: $\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1 - \sin^2 y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$ $\longleftarrow$ since $x = \sin y$ (2nd step)

$\therefore \frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$

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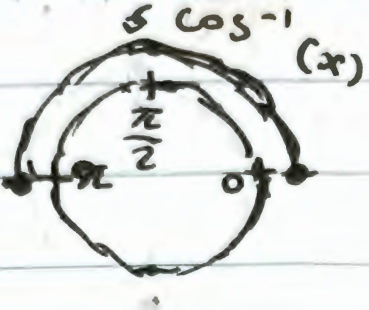
Derivative of inverse Cosine

$y=\cos^{-1}(x)$

Domain: $-1 \le x \le 1$

Range: $0 \le y \le \pi$



$$y=\cos^{-1}x \Rightarrow x=\cos y \xrightarrow{\text{Take derivative}} \frac{d}{dx}(x)=\frac{d}{dx}(\cos y)$$

$$1=(-\sin y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=-\frac{1}{\sin y}$$

$$\sin^2 y+\cos^2 y=1$$

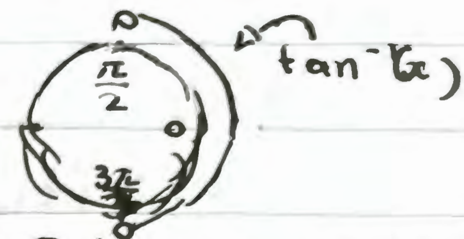
$$\sin y=\sqrt{1-\cos^2 y}$$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-\cos^2 y}}$$

$$\therefore \frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$$

Derivative of inverse Tangent

$y = \tan^{-1}(x)$



Domain: $-\infty \le x \le \infty$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

* Range does not include $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ because $\tan x = \frac{\sin x}{\cos x}$ and $\cos x = 0$ when $x = -\frac{\pi}{2}$ and $\frac{\pi}{2}$

$$y = \tan^{-1} x \Rightarrow x = \tan y$$

$$\frac{d}{dx}[x] = \frac{d}{dx}[\tan y] \Rightarrow 1 = \frac{1}{\cos^2 y} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \cos^2 y \cdot 1 \rightarrow \text{express as tangent of } y$$

$$= \frac{\cancel{\cos^2 y}}{\underbrace{\cos^2 y + \sin^2 y}_{=1}} \cdot \frac{\cancel{\frac{1}{\cos^2 y}}}{\frac{1}{\cos^2 y}} \Bigg\} = 1 = \frac{1}{1 + \left(\frac{\sin y}{\cos y}\right)^2} = \frac{1}{1 + \tan^2 y}$$

value did not change

* $\frac{\sin}{\cos} = \tan$

Since $x = \tan y$ ...

$$\therefore \frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$$

No.

Date

Derivatives of inverse trig functions

$$\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\sec^{-1}(x)=\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x)=-\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}(x)=\frac{1}{x^2+1} \qquad \frac{d}{dx}\cot^{-1}(x)=-\frac{1}{x^2+1}$$

The same but just negative

Simplifying $\sec^{-1}(x)$ derivatives (and $\csc^{-1}(x)$ derivatives)

- Due to the fact that there exists an absolute value in the derivative, we must always check the sign (plug in negative number to see if we need absolute value or not).

Usually, $x=-1$ is the easiest

E.g. $\frac{9x^2}{|3x^3|\sqrt{9x^6-1}}$ ← Plug in negative number

$\frac{9\cdot 1}{|-3|\sqrt{9-1}}$ ← Positive, so simplification MUST include absolute value

$$\frac{9x^2}{|3x^3|\sqrt{9x^6-1}}=\frac{3x^2\cdot 3}{3x^2|x|\sqrt{9x^6-1}}=\frac{3}{|x|\sqrt{9x^6-1}}$$

However, if the original would've been negative, then the absolute value must be removed:

E.g. $\frac{4x}{|2x^2|\sqrt{4x^2-1}}=\frac{2}{x\sqrt{4x^2-1}}$

Derivative Rules Recap...

Power Rule: $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

Exponential Rule: $\frac{d}{dx}[a^x] = \ln(a) \cdot a^x$

< Natural Exponential: $\frac{d}{dx}[e^x] = e^x$ >

Logarithm Rule: $\frac{d}{dx}[\log_a x] = \frac{1}{\ln(a) \cdot x}$

< Natural Logarithm: $\frac{d}{dx}[\ln x] = \frac{1}{x}$ >

Chain Rule: $\frac{d}{dx}[f(g(x))] = g'(x) \cdot f'(g(x))$

Inverse Rule: $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$

Also...

Implicit Differentiation: Differentiate y as you do with x, just multiply derivative with $\frac{dy}{dx}$ in the end and solve for it.

(NOTE: $\frac{d}{dx}[\cos(x+y)]$ would mean that $\frac{dy}{dx}$ is not necessary for the outer derivative or $f'(g(x))$, only inner)

Trig Derivatives (6 Functions + 6 Inverse functions)

$\frac{d}{dx}[\sin x] = \cos x$, $\frac{d}{dx}[\cos x] = -\sin x$, $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$, $\frac{d}{dx}[\csc x] = -\csc x \cdot \cot x$,
$\frac{d}{dx}[\tan x] = \sec^2 x$, $\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$, $\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$
$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{x^2+1}$, $\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{x^2+1}$

Higher-Order Derivatives

Notation:	y	$f(x)$	y
1st Derivative:	y'	$f'(x)$	$\frac{dy}{dx}$
2nd Derivative:	y''	$f''(x)$	$\frac{d^2y}{dx^2}$
3rd Derivative:	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$
n^{th} Derivative:	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$

The n^{th} derivative just means that you differentiate it n times.

i.e. Differentiate function to first derivative, then differentiate the first derivative to the second derivative ... and so on until you reach the n^{th} derivative.

$$\underbrace{\frac{d}{dx}\left[\frac{d}{dx}\left[\frac{d}{dx}\left[\cdots \frac{d}{dx}[f(x)]\cdots\right]\right]\right]}_{n \text{ times}}$$

E.g. $y = x^3 + x^2 + x + 1$

$\frac{dy}{dx} = 3x^2 + 2x + 1$

$\frac{d^2y}{dx^2} = 6x + 2$

$\frac{d^3y}{dx^3} = 6$

$\frac{d^4y}{dx^4} = 0$

Due to the way derivatives work, you will eventually reach a point where the derivative is equal to 0.

Second Derivatives, Implicit Equation

1. Differentiate like you do with First Implicit Derivative
2. Differentiate it again (same rules)
3. When you have $\frac{dy}{dx}$ alongside $\frac{d^2y}{dx^2}$, plug in the first derivative ($\frac{dy}{dx} =$) instead of $\frac{dy}{dx}$
4. Simplify.

E.g. $y^2 - x^2 = 4 \Rightarrow \frac{d}{dx}[y^2 - x^2] = \frac{d}{dx}[4]$

$$2y \cdot \frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{x}{y}\right]$$

$$\frac{d^2y}{dx^2} = 1 \cdot y^{-1} + x \cdot (-1) \cdot y^{-2} \cdot \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x^2}{y^3}$$

Disguised Derivatives

E.g. $\lim_{h \to 0} \frac{5\log_{10}(2+h) - 5\log_{10}(2)}{h}$

Match with Derivative definition

$f(x) = f(x)$

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

So find similarities

Scalar × limit so scalar can be factored out

$\therefore 5 \lim_{h \to 0} \frac{\log(2+h) - \log(2)}{h}$

$\to f(a+h) = \log(2+h)$

$\to f(a) = \log(\underbrace{2}_{a=2})$

$\therefore 5f'(2) = 5 \lim_{h \to 0} \frac{\log(2+h) - \log(2)}{h}$

$\therefore 5 \cdot f'(2) = 5 \cdot \frac{1}{[\ln(10)]2} = \frac{5}{\ln(10) \cdot 2}$

Unit 4:
Contextual Applications of Differentiation

Interpreting the meaning of the derivative in context

1. Interpret the meaning of function, i.e $f(x)$
   - Check to see if it is the rate ALREADY
     - e.g. Velocity ($v(x)$) is a rate. but position is not.
2. Check what is domain and range
   e.g. x, input — e.g. y, output
   - In some scenarios it'll say:
     - The function ___function___ models the ___range___ as a function of the ___domain___.

       OR
     - ___Scenario___ modeled by the function, ___function___ where ___range___ and ___domain___

       OR
     - The function ___function___ ___range___ ___domain___

*Sometimes they'll explicitly say what each variable is.

*Although time is usually domain, it could also be the range (output)

No.

Date

3. Interpret the derivative
   - The range is changing/increasing/decreasing <u>at a rate of</u>

     It is important to state that it is a rate

   ✱ <u>If</u> it was already a rate, then say so in the beginning as well

   $\boxed{\text{I.e}}$ The rate is changing at a rate of ....

4. Units
   - Units for the derivative are based on the slope $\left(\frac{\Delta y}{\Delta x}\right)$ so units are $\frac{\text{range}}{\text{domain}}$.
   - For the second derivative it would be

$$\frac{\text{range of 1st derivative}}{\text{domain}} = \frac{\frac{\text{range}}{\text{domain}}}{\text{domain}} = \frac{\text{range}}{\text{domain}^2}$$

Depending on the context, you can use this format:

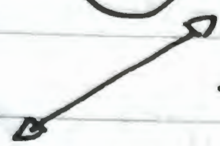
(rate or not rate)

"At $\underbrace{\text{domain}}$ $\underbrace{\text{domain context}}$, the $\underbrace{\text{Range context}}$ changes/increases/decreases at a rate of $\underbrace{\text{derivative}}$ $\underbrace{\text{units}}$."

Depending on context it may be after word

One-Dimensional Motion

Position: e.g. 
in terms of x^2
slope in terms of x

Velocity: e.g. 
in terms of x
slope in terms of constant

Acceleration: e.g. 
in terms of constant
slope $= 0$

Position function: $x(t)$
Velocity function: $v(t) = x'(t)$
Acceleration function: $a(t) = v'(t) = x''(t)$
Jerk Function: $x'''(t)$
Snap (Jounce) Function: $x''''(t)$
Crackle Function: $x^{(5)}(t)$
Pop Function: $x^{(6)}(t)$

slopes (derivatives of previous)

$\therefore$ Velocity = Rate of Change of position

negative — check which graph (pos-time or vel-time) it is

Positive

$v(t) < 0$ means the particle is moving left (x-axis) or down (y-axis)

$v(t) > 0$ means the particle is moving right (x-axis) or up (y-axis)

$v(t) = 0$ means the particle is at rest

Since velocity is just the slope, average velocity is just the average rate of change:

$$\text{Average Velocity} = \frac{s(b) - s(a)}{b - a} \text{ on the interval } [a, b]$$

You can calculate displacement using the times at which it changes direction (x-ints of velocity and plugging it into posit

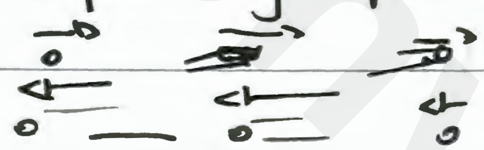
Velocity is a vector, it has direction
Speed is a scalar, it does not have direction

$$\text{Speed} = |\text{Velocity}|$$

Speeding up or slowing down.

- When $a(t)$ and $v(t)$ have same signs, it is speeding up.

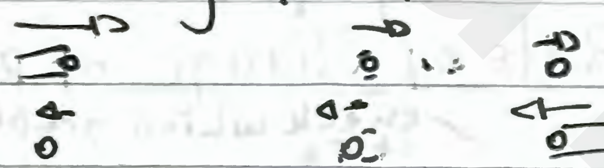
I.e



speed up
speed up

- When $a(t)$ and $v(t)$ have different signs, it is slowing down.

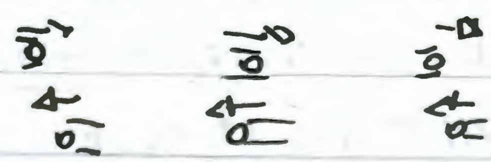
I.e



slow down
slow down

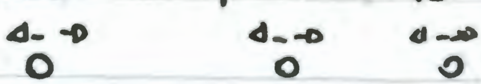
- when $a(t) = 0$, it has a constant velocity

I.e.



- when $v(t) = 0$, it is at rest

I.e.



* since $a(t) = v'(t)$, you look at the slope at a point in a velocity-time graph (be careful of non-differentiable points)

Rates of Change other than Motion

To know if something is increasing or decreasing, check the sign of its derivative.

E.g.

Height is *increasing* if $[\text{height}]' > 0$

Velocity is *decreasing* if $[\text{velocity}]' < 0$

☺ is *increasing* if ☺$' > 0$

k is *decreasing* if $k' < 0$

Is the function already a rate of change?

Not a rate of change:

"If $f(x)$ is the bunny population after x years, then what is $f'(x)$?".

↳ The rate of change of the bunny population per year

Is a rate of change:

"If $f(x)$ is the rate at which a bunny population increases (bunnies per year), then what is $f'(x)$?"

↳ The rate of change of the rate at which a bunny population increases, changes in bunnies per year per year.

Mathematical Symbol Use:

Union: "$\cup$" meaning "or"

↳ $f(x)$ is increasing for $t \cup v$

*Not "and" because either one increases

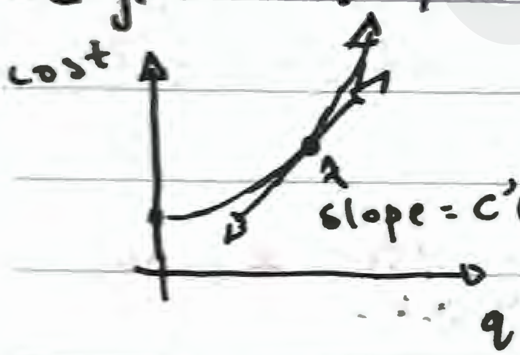
Intersection: "$\cap$" meaning "and"

↳ f exists in $A \cap B$

*Both criterias must be met at the same time

Marginal Cost

E.g. $c(q)$, cost as a function of quantity



$$\frac{dc}{dq} = c'(q)$$

so the derivative is the marginal cost

rate at which our cost is changing (with respect to quantity)

rate at which the cost is increasing for that one incremental unit.

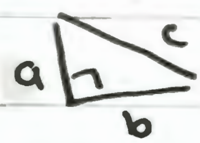
Related Rates - Intro

Many variables have a related relationship. Therefore their rates (derivatives) are related.

→ We use implicit differentiation to solve for their relationships.

E.g. Pythagorean Theorem

$$a^2 + b^2 = c^2$$



$$\Rightarrow \frac{d}{dt}[a^2] + \frac{d}{dt}[b^2] = \frac{d}{dt}[c^2]$$

$$\Rightarrow 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

* In this case, we chose t as the domain, but we can choose anything else (depending on given scenario)

i.e $\frac{da}{dx}$ ← domain

Differentiate your relationship with respect to <u>one specific variable</u> (could be time like in our previous example)

→ Some dimensions are constants (i.e they do not change, derivatives are 0)

↳ in that case, plug in 0 for that specific derivative (to simplify relationship)

↳ E.g. if b is constant, then $\frac{db}{dt} = 0$, so

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$\Rightarrow 2a\frac{da}{dt} = 2c\frac{dc}{dt}$$

Related Rates - Intro (Continued...)

To find a relationship between given rates of change:

1. Set up a relationship using the given variables for the situation
2. Differentiate with respect to the given domain (which is found in $\frac{d}{d\square}$)
   - It is usually time (t) but not always
3. Simplify (if possible, depending on situation)
   - Look for constant variables (derivative = 0)

Solving Related Rates

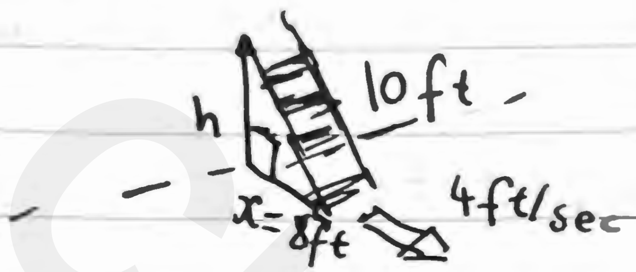
1. Draw a picture
2. Make a list of all known and unknown rates and quantities
3. Relate the variables in an equation
4. Differentiate with respect to the given domain (usually time)
5. substitute the known quantities/rates and solve.

* Substituting a non-constant quantity before differentiating is not allowed.

(Some times you may have to solve for an unknown quantity, in original equation, to get the final answer)

Solving Related Rates - Example

Falling ladder:-



How fast is h changing (at what rate)?

Since a ladder's length physically cannot change, $\frac{d[\text{length}]}{dt} = 0$

Also, the given scenario resembles a right triangle, which we can create a relationship using the pythagorean theorem.

$$h^2 + x^2 = [\text{length}]^2$$

$$\therefore \frac{d}{dt}[h^2 + x^2] = \frac{d}{dt}[\text{length}^2]$$ constant, so $=0$

$$\Rightarrow 2h\frac{dh}{dt} + 2x\frac{dx}{dt} = \frac{d[\text{length}]}{dt} 2[\text{length}]$$

Plugin: $2h\frac{dh}{dt} + 16 \cdot 4 = 0$

$$2h\frac{dh}{dt} + 64 = 0$$

we need h, so...

$$h^2 + 64 = 100$$
$$h^2 = 36$$
$$h = 6$$

$$2(6)\frac{dh}{dt} = -64$$

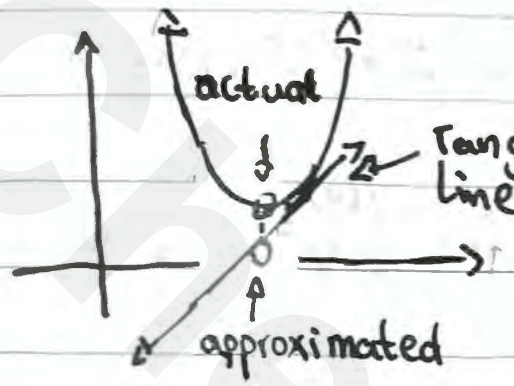
$$\frac{dh}{dt} = -\frac{64}{12}$$

$$= -\frac{16}{3} \text{ ft/s}$$

Local Linearity

A method to obtain an estimated output using the tangent line of a nearby value.

E.g.



$$f(a) = f(x) + f'(x)(a-x)$$

y-value Slope

$$y - y_1 = m(x - x_1)$$

* You find tangent line equation for a nearby value.

You plug in the domain to obtain an estimate

Concavity

Concaves up

Underestimates Value

concaves down

overestimates value

*Since Local Linearity relies on derivatives, the function must be differentiable (around the selected value)

L'Hôpital's rule introduction

- Using derivatives to find limits
  - ↳ limits that are indeterminate
    - ↳ i.e $\lim_{x \to a} \frac{f(x)}{g(x)}$ with direct substitution
      $\frac{f(a)}{g(a)} = \frac{0}{0}$ OR $\frac{\infty}{\infty}$ OR $\frac{-\infty}{\infty}$, etc.

Rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{for } \frac{f(a)}{g(a)} = \frac{0}{0} \text{ OR indeterminate form}$$

*Note: This also works for $f'(x), f''(x), f'''(x)$ (with 1 level below/high)
However, it only works until, it is defined (not indeterminate)

E.g. $\lim_{x \to 0} \frac{\sin x}{x} \Rightarrow \frac{\sin(0)}{(0)} = \frac{0}{0}$ ($f(x)$, $g(x)$)

$\therefore \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$ ← limit is defined

} You can only use it up to a certain point

However...

$\lim_{x \to 0} \frac{f''(x)}{g''(x)} = \lim_{x \to 0} -\frac{\sin x}{0} \Rightarrow -\frac{\sin(0)}{0} = -\frac{0}{0}$ ← not defined

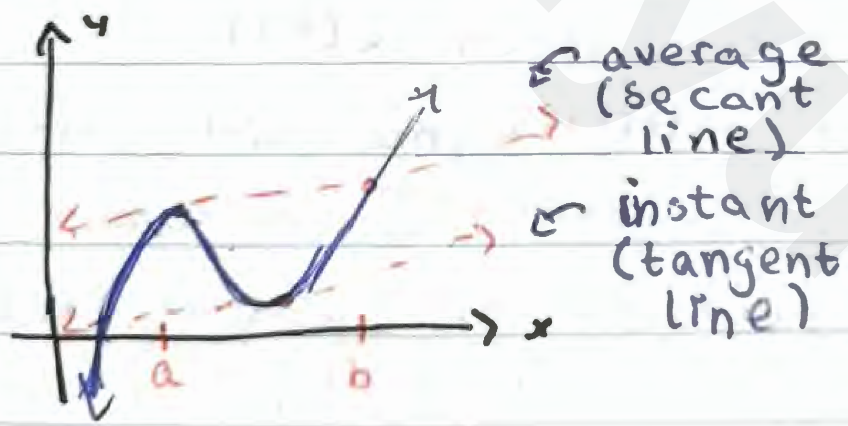
$\lim_{x \to 0} \frac{f'''(x)}{g'''(x)} = \lim_{x \to 0} -\frac{\cos x}{0} = -\frac{1}{0}$ ← Not indeterminate but not correct

The Mean Value Theorem

Justification for conclusions about a function over an interval.

Mean Value Theorem:

"If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b), then there exists a point c within that open interval where the instantaneous rate of change equals the average rate of change over the interval."



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Conditions: f must be CONTINUOUS over $[a, b]$
and DIFFERENTIABLE over (a, b)

MVT vs IVT

- **Mean Value Theorem (MVT)**
  - → The derivative (instantaneous rate of change) must equal the average rate of change somewhere in the interval given the function is both continuous and differentiable over that interval.
  - → E.g. Speed
    - ↳ Given Avg Speed is 82 mph then at some point speed MUST equal 82 mph exactly.
- Intermediate value Theorem (IVT)
  - → On a given interval, you will have a y value at each of the end points of the interval. Every y-value exists between these two y-values at least once in the interval given that the function is continuous over that interval
  - → E.g. Speed
    - ↳ If Speed at one point was 25 mph and at another was 50 mph then the speed between those two points must equal 30 mph at least once.

Mean Value Theorem Uses

Since the Mean Value Theorem justifies how there always exists a derivative equal to the average rate of change over a continuous interval, we can use it to find at which point specifically.

E.g. $f(x) = x^2 - 6x + 8$ over $[2,5]$

$f(5) = 25 - 30 + 8 = 3$

$f(2) = 0$

$\therefore \dfrac{f(5) - f(2)}{5 - 2} = \dfrac{3}{3} = 1$

$f'(x) = 2x - 6$

$\Rightarrow 1 = 2x - 6$

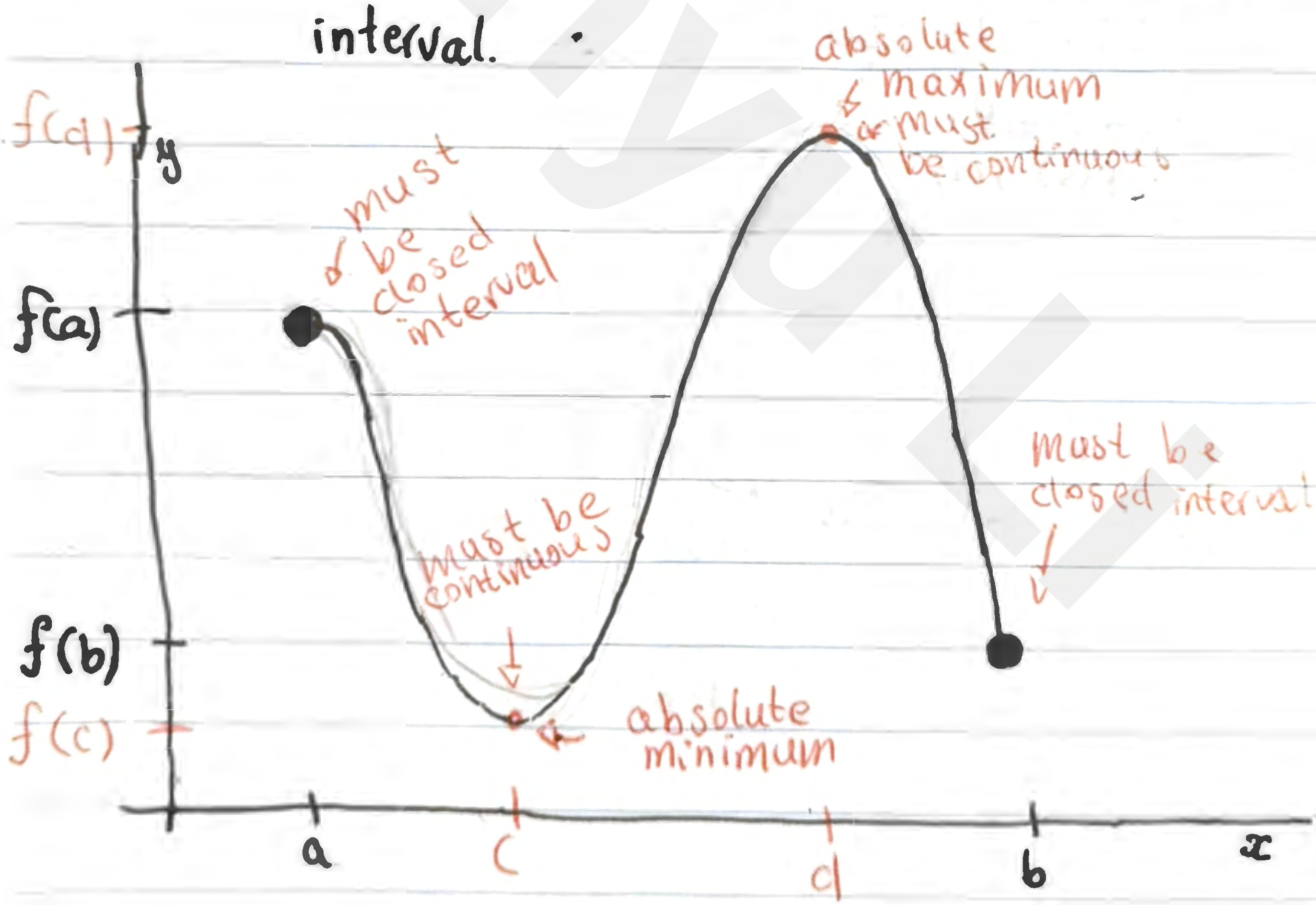
$x = \dfrac{7}{2}$ ← Derivative = ARoC at $x = \frac{7}{2}$ in $f(x)$

The Extreme Value Theorem

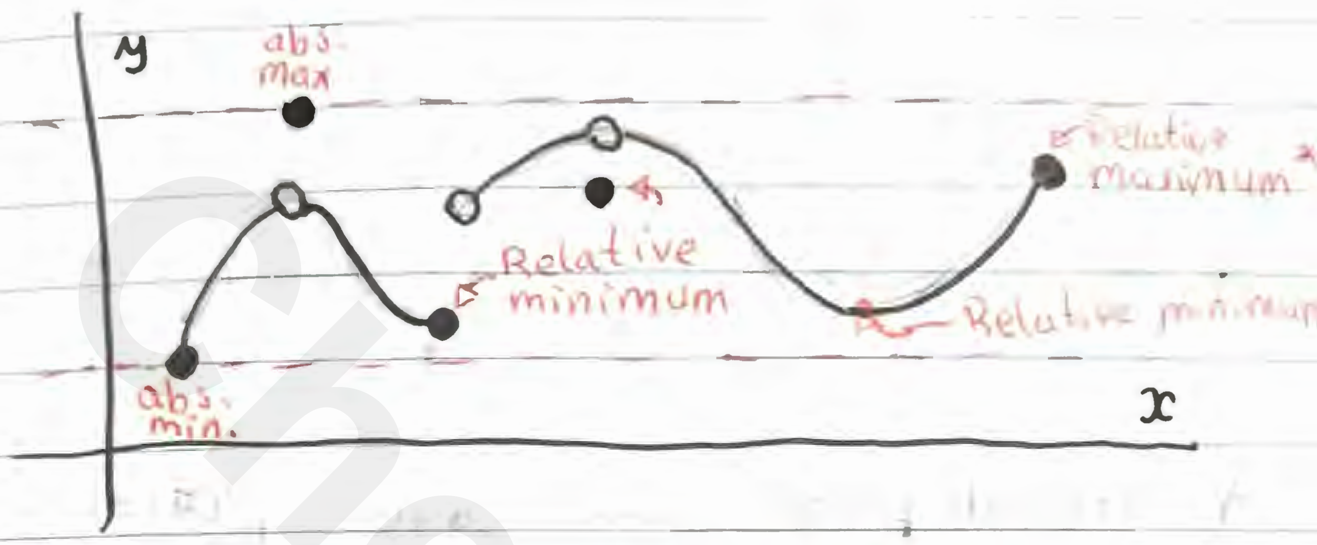
"If a function is continuous over the interval [a, b], then f has at least one minimum value and at least one maximum value on [a, b]."

Condition: f is continuous over [a, b]

Theorem: There exists at least one absolute maximum and one absolute minimum over that interval.



Extremas



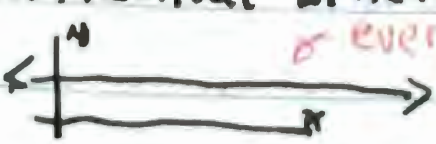
*whether an endpoint can be classified as a relative/local extrema (i.e. rel. max and rel. min) is disputed amongst Mathematicians

- A relative minimum/maximum can still exist even if it's not continuous (jump discont.), it only has to be relatively minimum/maximum compared to the points around it.

Global/Absolute Extrema: True for entire function or interval

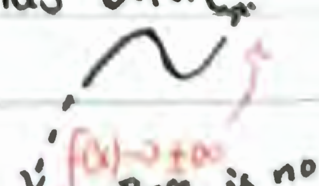
Local/Relative Extrema: True for points around it

Horizontal Line:

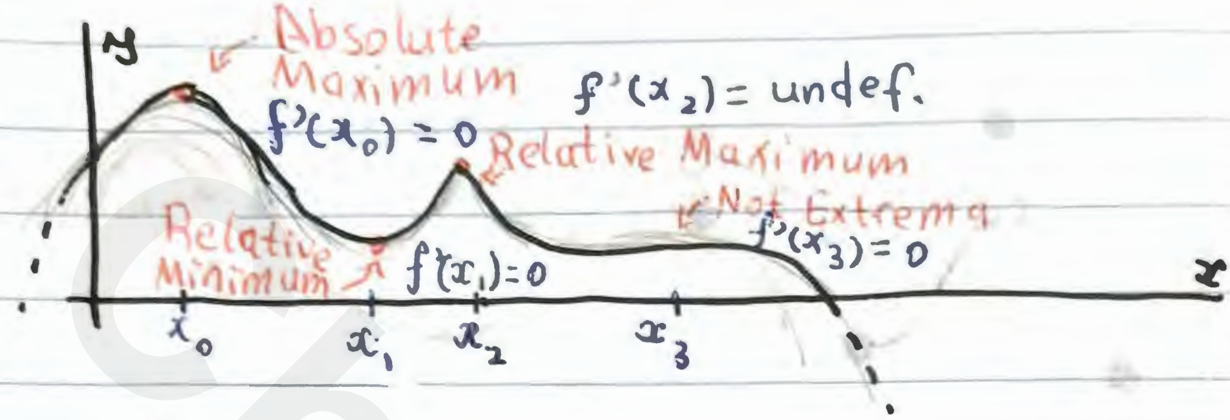


every point is an absolute/relative max./min

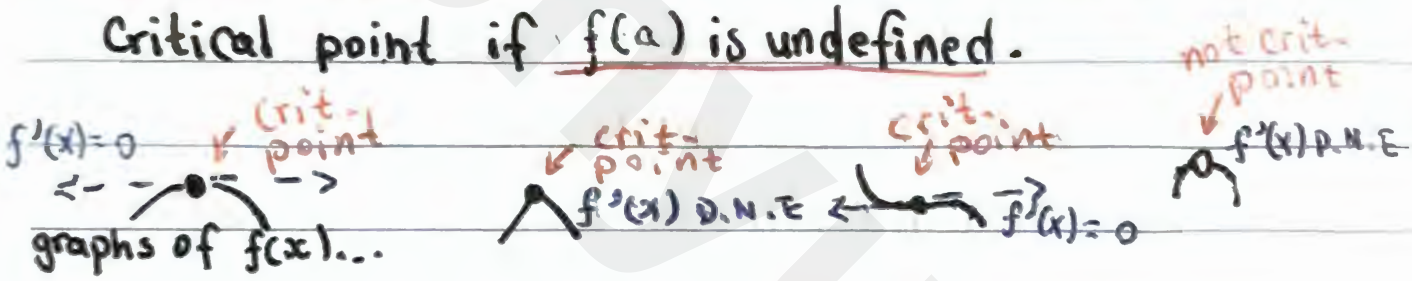
Extremas D.N.E.:



Critical Points



A critical point exists when $f'(a)=0$ OR $f'(a)$ is undefined. A point may NOT be a critical point if $f(a)$ is undefined.



A critical point is a point that has a posibility of being an extrema (i.e. max./min.)
(the x-value of a critical point is known as a critical number)

- → All local extremas that aren't endpoints are Crit. Points
- → All global extremas that aren't endpoints are Crit. Points
- → Not all Crit. Points are extremas
  - ↳ i.e. $f'(a)=0$ but $x=a$ is not an extrema.

Check for zeroes (numerator), undefined (zeroes in denominator, squareroots or logarithms)

< e^x will never equal 0, H.A. at $y=0$ >

Increasing and Decreasing Intervals

When the slope of a function is positive, the function is increasing.

When the slope of a function is negative, the function is decreasing.

When the slope of a function is 0, the function is neither increasing nor decreasing.

Critical Points have a posibility of being an extrema or a sign change.

→ We can use Critical Points to build a sign chart.

E.g. $f(x) = x^2$

$\Rightarrow f'(x) = 2x$

Crit. Pt(s) at $x = 0$

So...

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	0	$+$

Plug in values to $f'(x)$ to see if result is positive or negative

Justif.

f is increasing for $x > 0$ because $f'(x) > 0$

f is decreasing for $x < 0$ because $f'(x) < 0$

Unit 5:
Applying Derivatives
to analyse functions
(continued)

The First Derivative Test

- Using the first derivative to "test" whether or not a function has a maximum or minimum (only for cont. functions)

Justification statements

Assume c and d are critical numbers of a function f.

1. There is a maximum value at $x = c$ because $f'(x)$ changes from positive to negative around $x = c$.
2. There is a minimum value at $x = d$ because $f'(x)$ changes from negative to positive around $x = d$.

E.g. $f(x) = \frac{x^3}{3} - x$

$\Rightarrow f'(x) = x^2 - 1 = (x+1)(x-1)$

$\Rightarrow x = \pm 1$

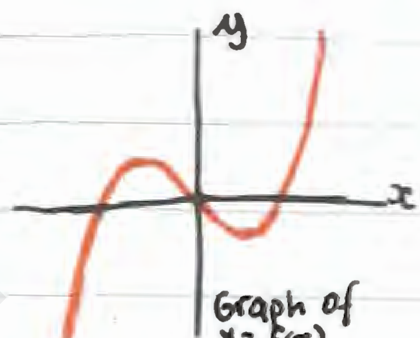
$\therefore$

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$f'(x)$	+	0	-	0	+

$f'(-100) = (-99)(-101) = +$

$f'(0) = 1 \cdot (-1) = -1$

$f'(100) = (101)(99) = +$



Graph of $y = f(x)$

$\because f'(x)$ changes from + to - around $x = -1$

$\therefore$ There is a relative maximum at $x = -1$

$\because f'(x)$ changes from - to + around $x = 1$

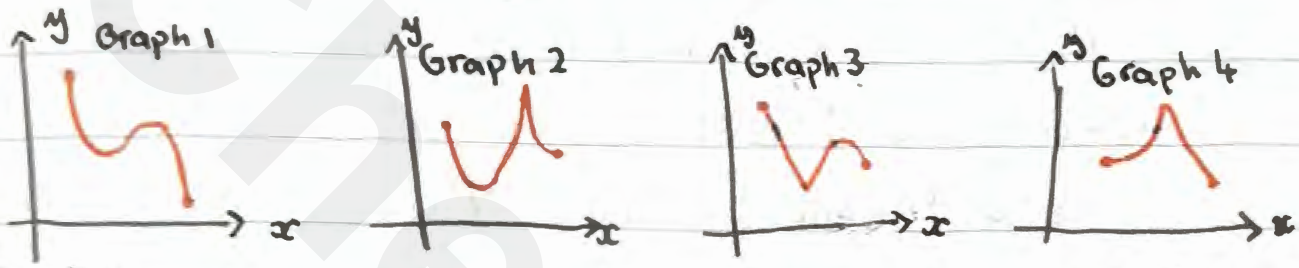
$\therefore$ There is a relative minimum at $x = 1$

When a question asks for "What is the maximum value?", it asks for the range (e.g. y, $f(x)$, $g(x)$)

When a question asks for "Where is the maximum?", it asks for the domain (e.g. x, t)

The Candidates Test

<u>Candidates</u> - The points that have a posibility of being an absolute or global extrema.



	Graph 1	Graph 2	Graph 3	Graph 4
Abs. Max:	End-point 1	Critical Point 2	End-point 1	Critical Point 1
Abs. Min:	Endpoint 2	Critical Point 1	Critical Point 1	Endpoint 2

End-points (of a closed-interval) and Critical Points both have a posibility of becoming an absolute or global extrema, so they're both candidates.

*NOTE: Endpoints only classify as a candidate if they're that of a closed interval
(in an open interval, there may not be an absolute extrema;
in an unbounded interval, the function may approach $\pm\infty$ meaning there may not be an absolute extrema)

Determine Absolute/Global Extrema from Candidates

1. On a closed interval:
   a) find critical points
   b) find end-points
   c) find the value of said points when you plug it into the function
   d) compare values to see which is absolute/global Maximum/Minimum (could be multiple)
2. On an open interval:
   a) find critical points
   b) find end-points
   c) find the value of said points when you plug it into the function
   d) compare values to see which is the absolute or global maximum/minimum
      - If end-point is, then put D.N.E
      - If multiple crit. points then put them all
      - If both end-point and crit. point, then only put the crit. points

3. On an unbounded interval:

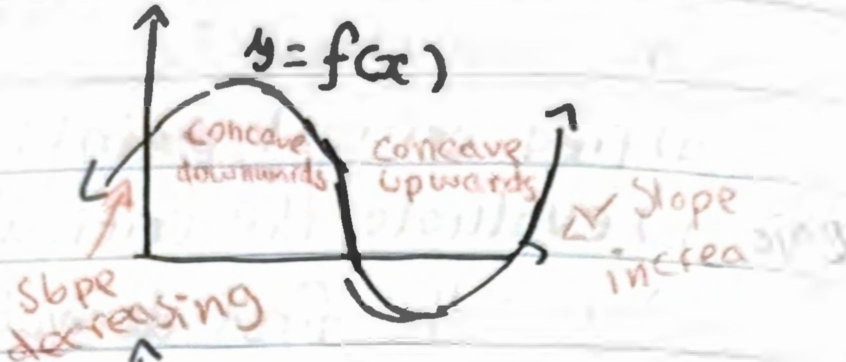
a) find critical points

b) evaluate the critical points

c) use the first derivative test to see which points are minimum or maximum and the end behaviour of function

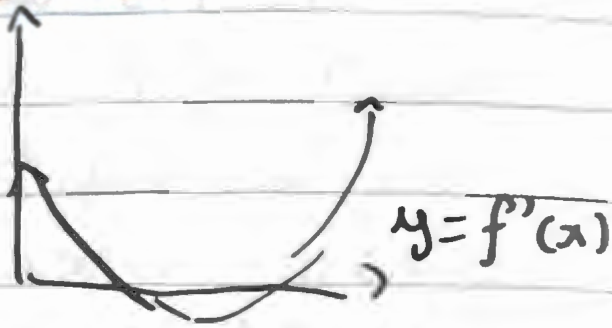
d) compare values and take into account the end behaviour of the function (if $\lim_{x \to \pm\infty} f(x)$ is unbounded then there does not exist an absolute/global maximum/minimum depending on the case).

Determining Concavity



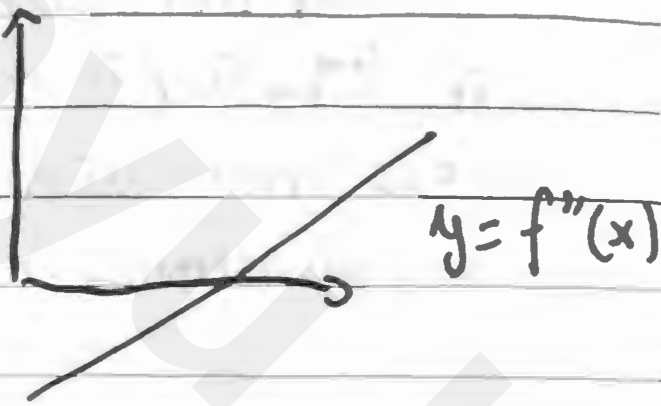
Concave Up

- Slope or $f'(x)$ is increasing
- $f''(x) > 0$
- There is a minima



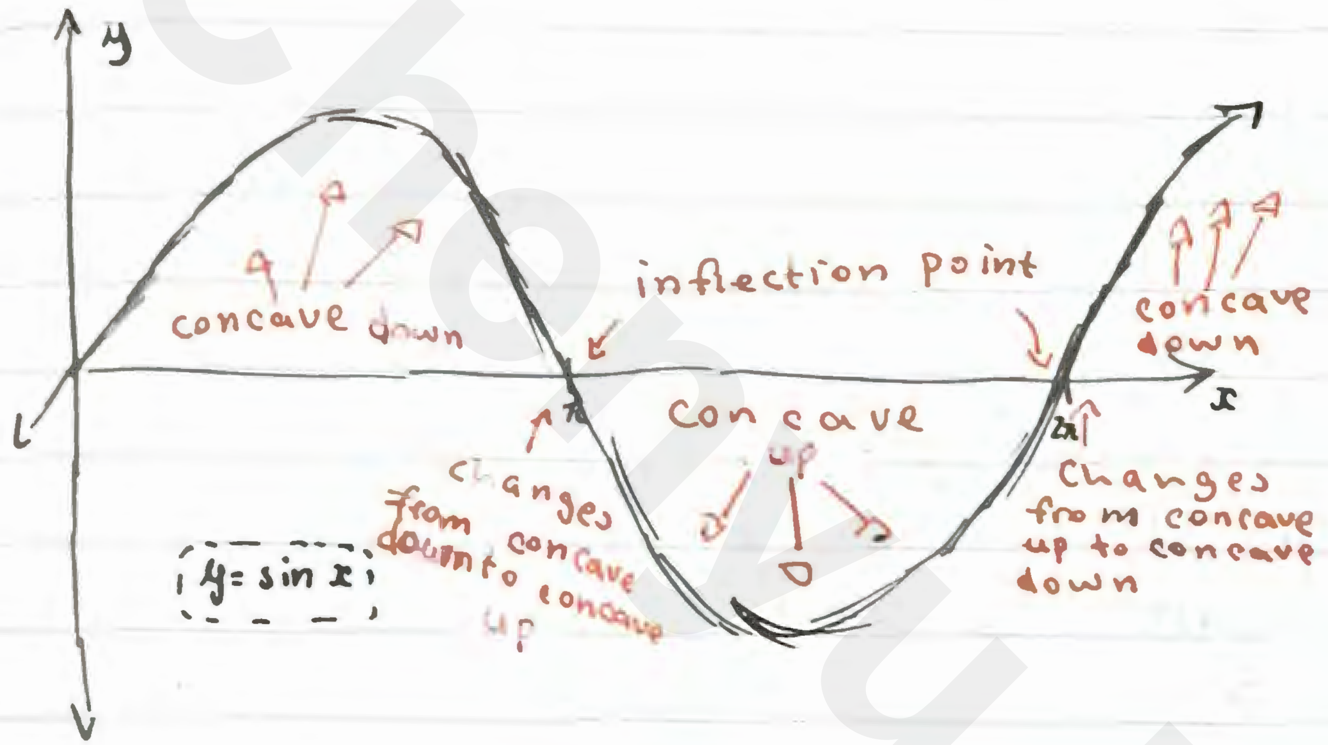
Concave down

- Slope or $f'(x)$ is decreasing
- $f''(x) < 0$
- There is a maxima

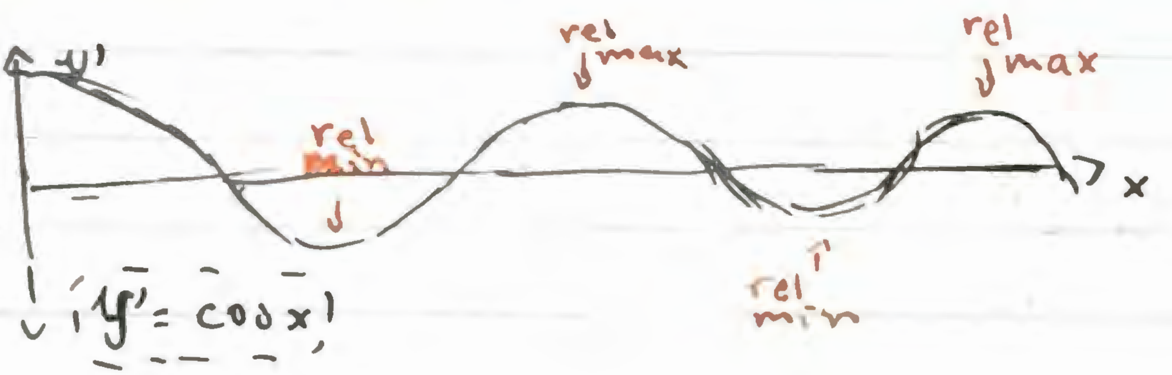


Inflection Point

- The point of $f(x)$ at $x=c$ if $f(c)$ is defined and $f''(x)$ changes signs at $x=c$
- It's the point where the graph changes concavity



$f'(x)$ changes from increasing to decreasing or vice versa. Thus, $f'(x)$ has a local or relative extrema at $x=c$



No.

Date

Determining Concavity (Algebraic)

Because concavity is determined by the sign of $f''(x)$, we can find the intervals where $f''(x)>0$ OR $f''(x)<0$ to find on which intervals $f(x)$ is concave up or concave down respectively.

- We can find the zeroes of $f''(x)$ ($f''(x)=0$ for which x-values?) and make a sign chart.

E.g. $g(x)=-x^4+6x^2-2x-3$

$\Rightarrow g'(x)=-4x^3+12x-2$

$\Rightarrow g''(x)=-12x^2+12=-12(x^2-1)=-12(x-1)(x+1)$

$\therefore x=\pm 1$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$g''(x)$	$-$	0	$+$	0	$-$
	g is concave down		g is concave up		g is concave down

$\therefore$ g is concave up for $x\in(-1,1)$ bc $g''(x)>0$

"$\in$" is in the set of

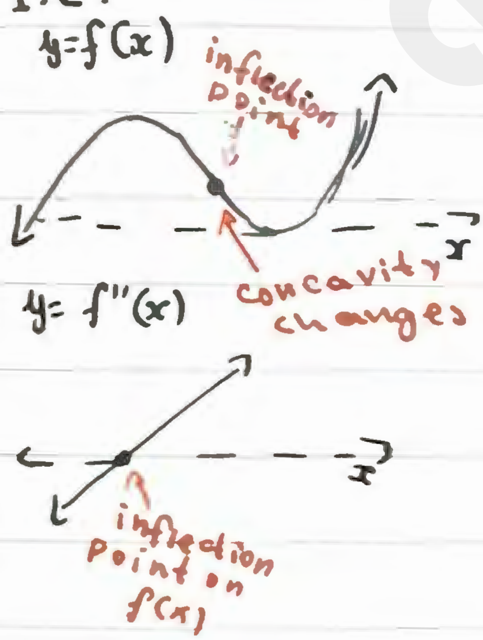
g is concave down for $x\in(-\infty,-1)\cup(1,\infty)$ bc $g''(x)<0$

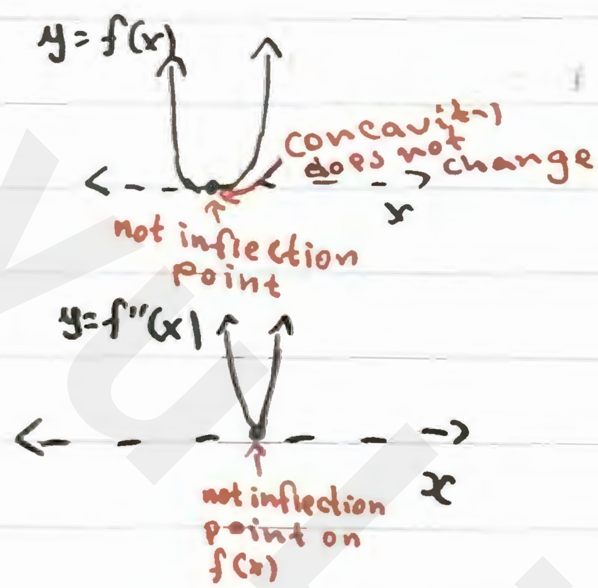
Finding Inflection Points (Algebraic)

To find an inflection point, $f''(x)$ must equal 0 or is undefined And has to change signs from either positive to negative or negative to positive

↳ It cannot change from one sign to 0 then back to the same sign.

I.E.





Common Mistakes:

- $f''(x)$ can also be undefined
- Not checking if $f''(x)$ changes signs (crosses x-axis)
- $f(x)$ MUST be defined

Second Derivative Test

Suppose $f'(c) = 0$ ...

- $f''(c) < 0$: f has a relative max value at $x = c$
- $f''(c) = 0$: inconclusive
- $f''(c) > 0$: f has a relative min value at $x = c$

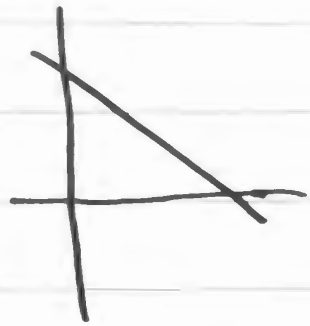
If there is only one critical point, and that critical point is an extremum (max or min), then it is an ABSOLUTE extremum (max or min).

Relating Derivatives ($f(x)$, $f'(x)$, $f''(x)$)

$y = f(x)$

$y = f'(x)$

$y = f''(x)$



$f(x)$	$f'(x)$	$f''(x)$
$f(x)$ is increasing	$f'(x) > 0$	[Inconclusive]
$f(x)$ is decreasing	$f'(x) < 0$	[Inconclusive]
$f(x)$ has a critical point	$f'(x) = 0$ OR is undefined	[Inconclusive]
$f(x)$ has a relative maximum	$f'(x)$ changes from pos. to neg.	[Inconclusive]
$f(x)$ has a relative minimum	$f'(x)$ changes from neg. to pos.	[Inconclusive]
$f(x)$ is concave up	$f'(x)$ is increasing	$f''(x) > 0$
$f(x)$ is concave down	$f'(x)$ is decreasing	$f''(x) < 0$
$f(x)$ has a point of inflection	$f'(x)$ has a relative extrema	$f''(x)$ changes signs

Optimization

- Process of selecting the best element given some specific criterion (usually max/min) and condition from some set of available alternatives

- To do so, you must develop equations with the given and simplify accordingly.
(Then use calculus to solve for the wanted value)

E.g. "What is the smalest possible sum of squares of two numbers, if their product is -16?"

Equation:

Minimise $S = x^2 + y^2$ where $xy = -16 \Rightarrow y = -\frac{16}{x}$

$$S(x) = x^2 + \left(-\frac{16}{x}\right)^2$$
$$= x^2 + 256x^{-2}$$

Find min:

$s'(x) = 2x - 512x^{-3}$ $\qquad$ $s''(x) = 2 + 1536x^{-4}$

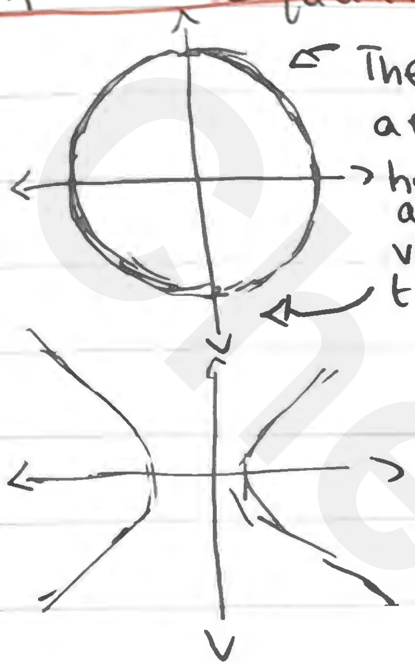
$x = 4$

Plug in:

$S = 16 + 16 = 32$ $\qquad$ $y = \frac{-16}{4} = -4$

Horizontal / Vertical Tangent lines (Implicit)

Implicit Equations:



There are 2 horiz. and 2 vert. tangents

Therefore you must give (x, y) to specify exact point

E.g. $x^2 + y^4 + 6x = 7$, find horiz. tangent above x-axis

$$\frac{dy}{dx} = \frac{-2(x+3)}{4y^3} \Rightarrow \text{Horiz. Tangent: } x = -3$$

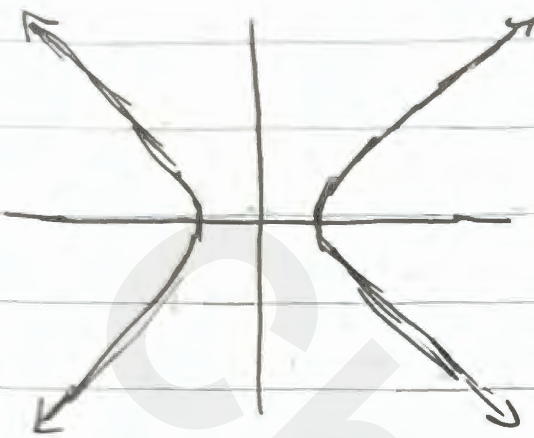
$$\therefore (-3)^2 + y^4 + 6x = 7$$

$$y = \pm 2$$

Therefore, there's two points $\{(-3, 2), (-3, -2)\}$

The one above tangent would be $(-3, 2)$

Concavity of Implicit Equations



Hyperbola

A hyperbola has multiple concavities.

E.g. $\frac{x^2}{4} - y^2 = 1$ ← Hyperbola

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-4}{16y^3}$$

The concavity changes depending on the sign of y.

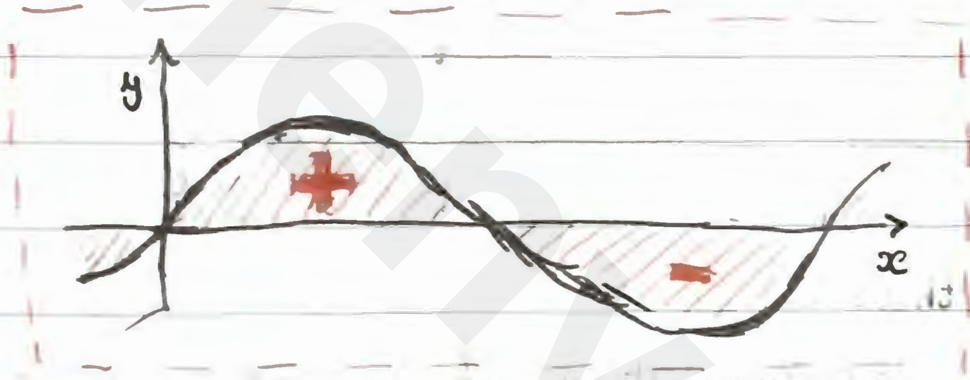
⇒ when $y > 0$, it's concave down; when $y < 0$, it's concave up.

Concavity must be marked by quadrants or range rather than only domain

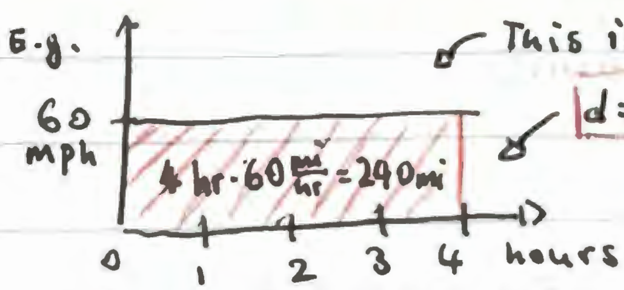
Unit 6: Integration and accumulation of change

Accumulation of Change

- Area Under the curve:
  - The region between a function and the x-axis is called the "area under the curve".
    ("under" does not necessarily mean below)
    ↳ It means the area between the x-axis and the function.



In an applied example, notice how the area under the curve can give us the accumulation of change



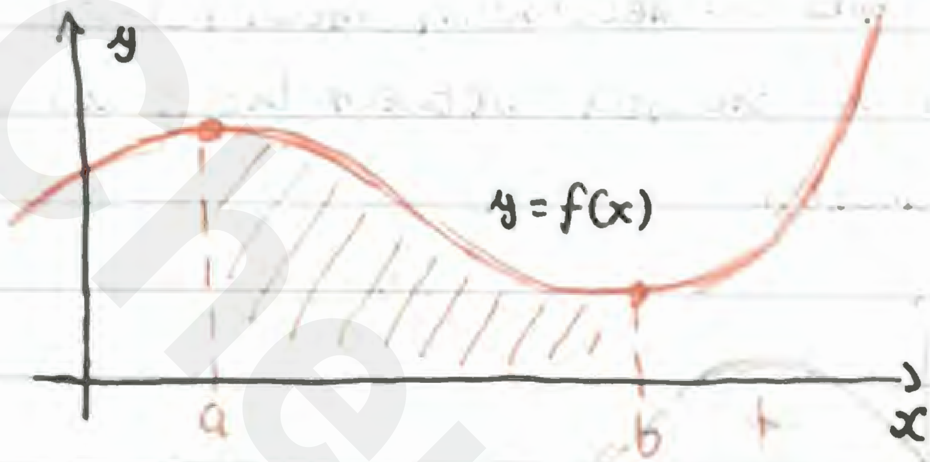
This is a rate graph

$\boxed{d = v \cdot t}$

The graph "undid" the derivative. This is the "integral" or the "anti-derivative."
↳ When it's given on an interval, it's the "definite integral"

Definite Integrals

- An accumulation of change over a definite interval refers to definite integrals



And thus...

this accumulation of change can be written as an integral expression:

upper bound

Function

$$\int_a^b f(x)\,dx$$

lower bound

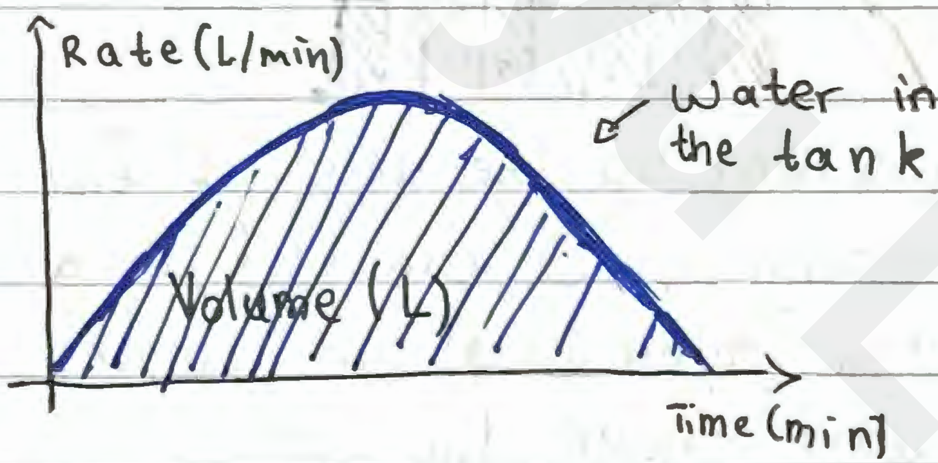
- Represents "the area under the curve of the function $f(x)$ between $x = a$ and $x = b$."

Definite Integrals

- The definite integral always gives us the net change in a quantity, not the actual value of that quantity

- You need to add an initial condition to the definite integral to find the actual value of that quantity.

E.g.

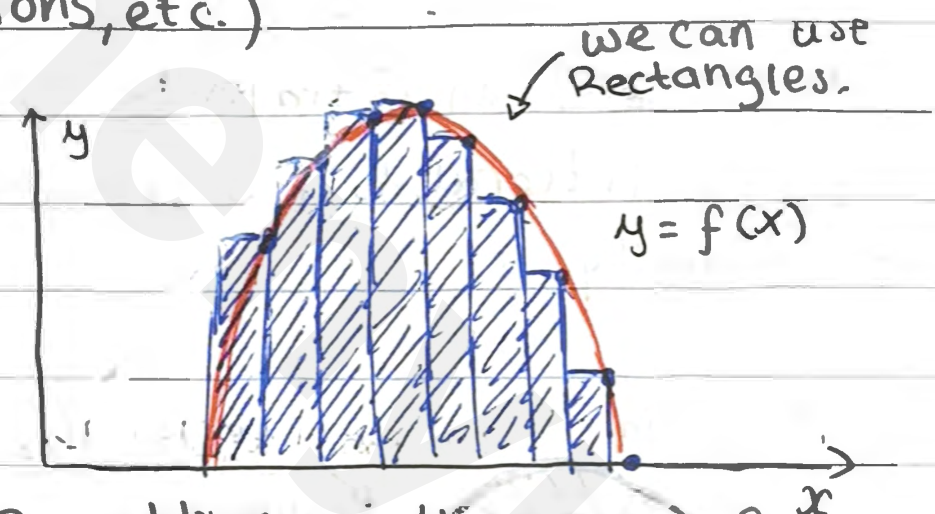


We weren't told whether the tank (function) started off with water already or not.
↳ Therefore we also need to know what is the initial value of the quantity

Riemann Introduction

How can we find the area under the curve of more complex functions? (i.e. parabolas, cubic functions, logarithmic functions, etc.)

E.g.

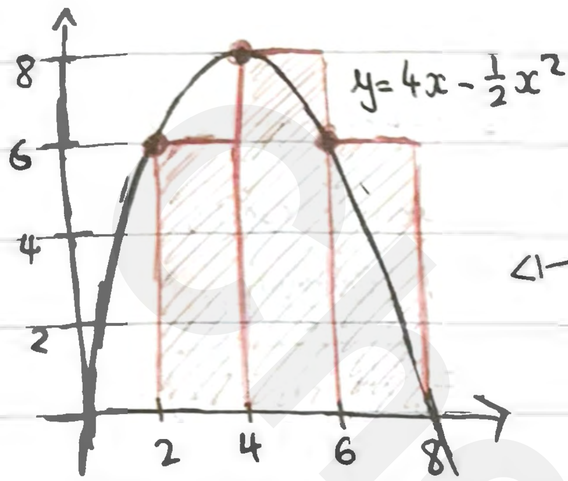


By adding up the areas of the rectangles, we can obtain a rough approximation for the area under the curve.

This approximation method is called a Riemann Sum. It was named after a German mathematician named Benhard Riemann.

Riemann Sums

Left Riemann Sum



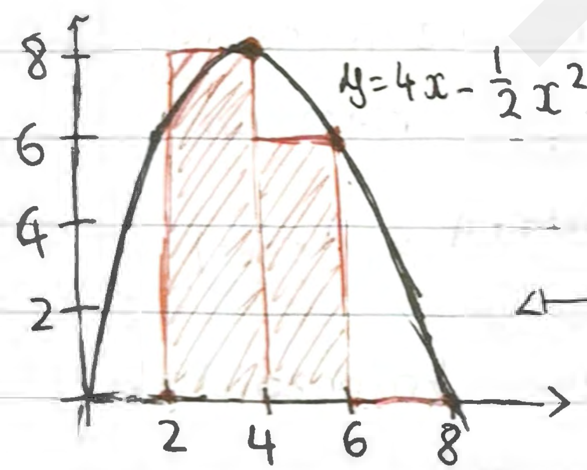
on the interval $[2,8]$,

use 3 subintervals

$A_{left} = 2\cdot f(2) + 2\cdot f(4) + 2\cdot f(6)$

$= \boxed{40 \text{ units}^2}$

Right Riemann Sum



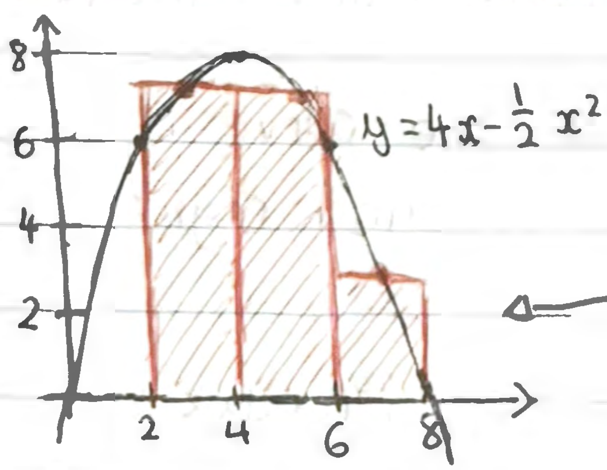
on the interval $[2,8]$,

use 3 intervals

$A_{right} = 2\cdot f(4) + 2\cdot f(6) + 2\cdot f(8)$

$= \boxed{28 \text{ units}^2}$

Midpoint Riemann Sum



on the interval $[2,8]$,

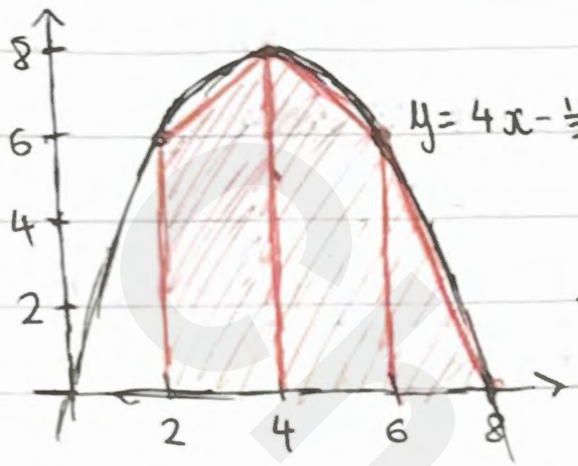
use 3 intervals

$A_{mid} = 2\cdot f(3) + 2\cdot f(5) + 2\cdot f(7)$

$= \boxed{37 \text{ units}^2}$

Riemann Sums continued...

Trapezoidal Sum



$y = 4x - \frac{1}{2}x^2$ on the interval $[2, 8]$, use 3 subintervals

$A_{trapezoid} = 2\cdot\left[\frac{f(2)+f(4)}{2}\right] + 2\left[\frac{f(4)+f(6)}{2}\right] + 2\cdot\left[\frac{f(6)+f(8)}{2}\right]$

$= \boxed{34 \text{ units}^2}$

Special Tricks

- The trapezoidal sum is the mean average between the left and right riemann sums.

$$A_{trapezoid} = \frac{A_{left} + A_{right}}{2} = avg(A_{left}, A_{right})$$

- For a linear function, the trapezoidal sum is equal to the midpoint sum.

$$A_{trapezoid} = A_{mid} \text{ for } f(x) \text{ is a first degree function}$$

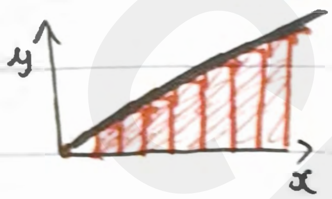
- The actual area under the curve is between (or equal to) the left and right riemann sums.

$$A_{left} \le \int_a^b f(x)\,dx \le A_{right} \text{ OR } A_{right} \le \int_a^b f(x)\,dx \le A_{left}$$

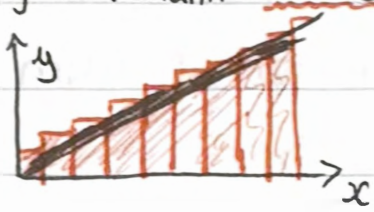
Over-estimation and Underestimation of Riemann Sums

Increasing Function

Left Riemann = Underestimate

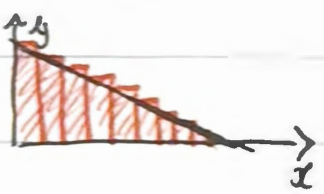


Right Riemann = Overestimate

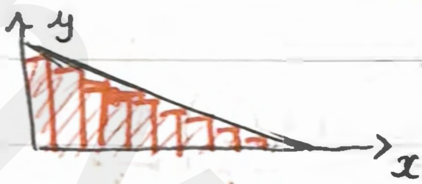


Decreasing Function

Left Riemann = Overestimate



Right Riemann = Underestimate



Trapezoidal Estimates

Concave up = Overestimate

Concave Down = Underestimate

Note that some Riemann Sums can be done with UNEQUAL intervals if equal ones cannot be formed (especially on a table).

Summation Notation

Riemann Sums are the sums of areas.

↳ We can use the summation notation to find the total sum (i.e. Σ)

$$A = f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 + \cdots f(n)\cdot 2$$

↳ This is the sum of the areas of each Riemann Rectangle

General Formula:

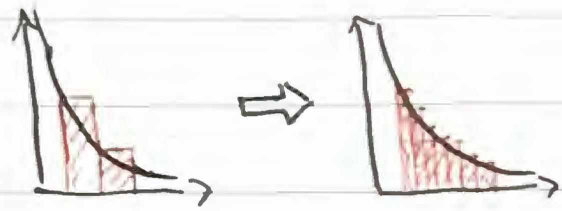
$$A = \sum_{i=1}^{n} \underbrace{f(x_{i-1})}_{\text{height at point } x_{i-1}} \cdot \underbrace{\Delta x}_{\text{change in } x}$$

(input may vary depending on type of Riemann Sum)

Definite Integrals as the limit of a Riemann Sum

Recalling the sum of rectangles...

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x$$



We notice that the sum gets more and more accurate the more rectangles we have...

Therefore...

We want n [i.e. # of rectangles] $\longrightarrow \infty$

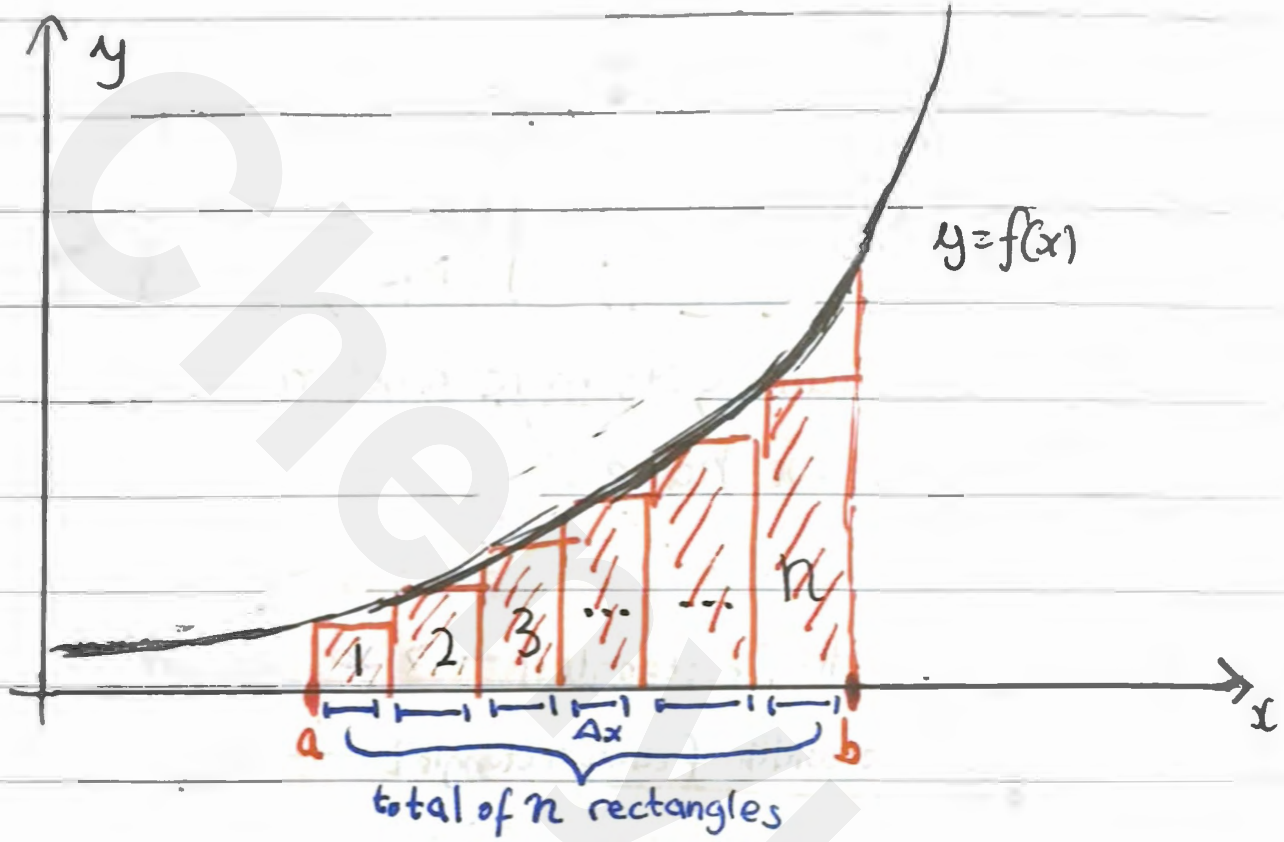
and thus Δx [i.e. width of each rectangle] $\longrightarrow 0$

Notice that n can be expressed as a relation with Δx...

$$\Delta x = \frac{b-a}{n} \quad \text{OR} \quad n = \frac{b-a}{\Delta x}$$

where 'a' and 'b' are the lower and upper bounds respectively.

Definite Integral as the limit of a Riemann Sum



For an infinitely small change in x (Δx)

$$\int_a^b f(x)\,dx$$

$$\Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

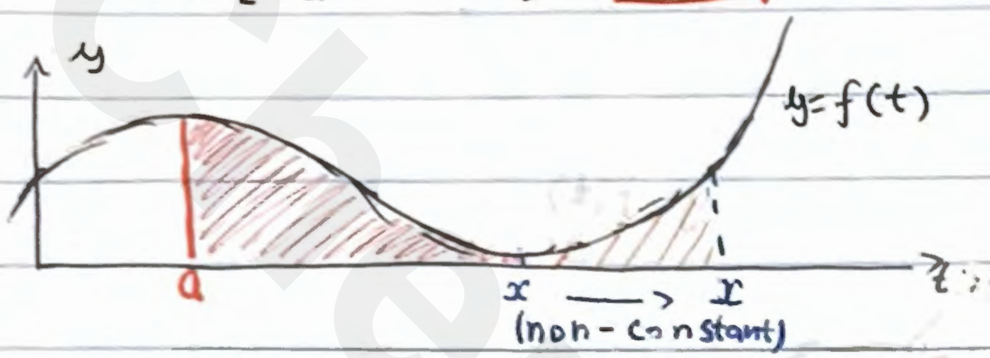
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot i\right) \cdot \left(\frac{b-a}{n}\right)$$

i can also appear as k or other variables in other cases

The fundamental theorem of Calculus

If a is a constant and f is a continuous function, then...

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = \boxed{f(x)}$$



This connects derivatives ($\frac{dy}{dx}$) and integrals ($\int f(x)\,dx$)

→ Integrals are the inverses of derivatives or anti-derivatives.

Variations of the theorem...

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right] = \left[f(g(x))\cdot g'(x)\right] - \left[f(h(x))\cdot h'(x)\right]$$

where $g(x)$ and $h(x)$ are functions (e.g. x^2, $\sin(x)$, etc.)

This is really similar to the Chain Rule for Derivatives ($f'(g(x))\cdot g'(x)$)

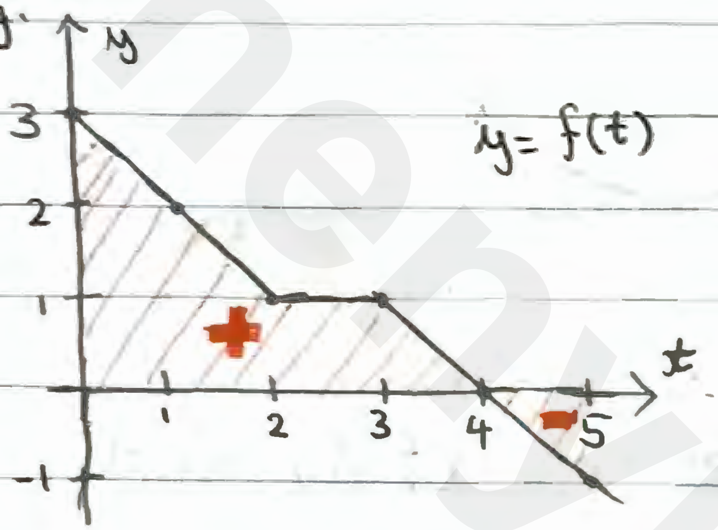
↳ In this case, since $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$, there isn't $f'(x)$

Accumulation Functions

$$\underbrace{F(x)}_{\text{}} = \int_0^x f(t)\,dt$$

we can use capital letters to represent integrals.

E.g.



x	0	1	2	3	4	5
$F(x)$	0	2.5	4	5	5.5	5

This, $F(x)$ is called an Accumulation Function.

Behaviour of Accumulation Functions

Accumulation Functions are of the form
$F(x) = \int_a^x f(t)\,dt$ where a is constant.
Recognize that $F'(x) = f(x)$.

Behaviour

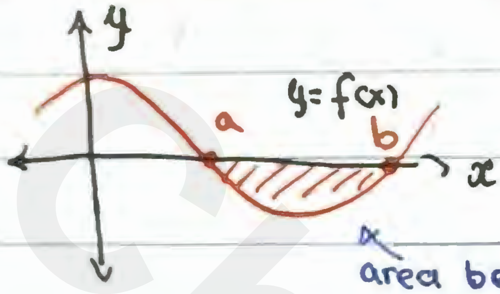
$F(x)$	$F'(x)$	$f(x)$
increasing	$F'(x) > 0$	$f(x) > 0$
decreasing	$F'(x) < 0$	$f(x) < 0$
rel. max	$F'(x)$ changes signs $(+)$ to $(-)$	$f(x)$ pos. to neg.
rel. min	$F'(x)$ changes signs $(-)$ to $(+)$	$f(x)$ neg. to pos.
Concave up	$F'(x)$ is increasing	$f(x)$ is increasing
Concave down	$F'(x)$ is decreasing	$f(x)$ is decreasing
pt. of inflection	$F'(x)$ has a rel. extrema	$f(x)$ has a rel. extrema

In this case...

→ We can treat the graph of $f(x)$ as $F'(x)$ since $F'(x) = f(x)$

Properties of Definite Integrals

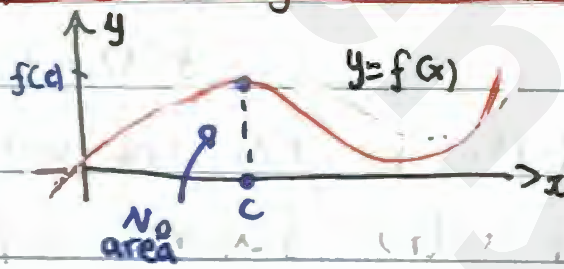
Negative Definite Integrals



area below the
Y axis means negative

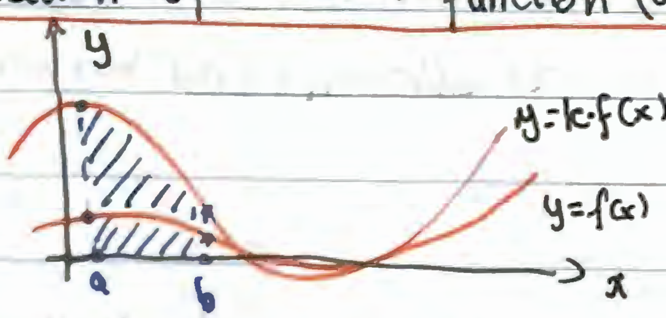
$$\int_a^b f(x)\,dx = -\boxed{}$$

Definite integral over a single point



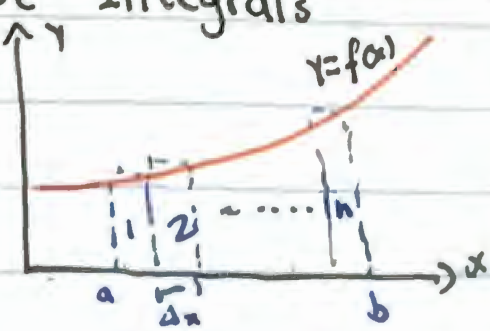
$$\int_c^c f(x)\,dx = 0$$

Integration of scaled function (scalar multiplication)



$$\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$$

Reverse Integrals



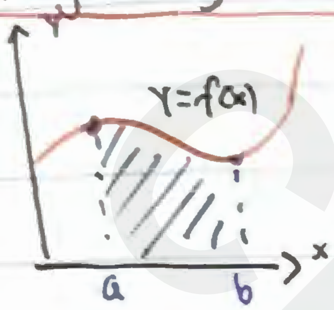
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}$$

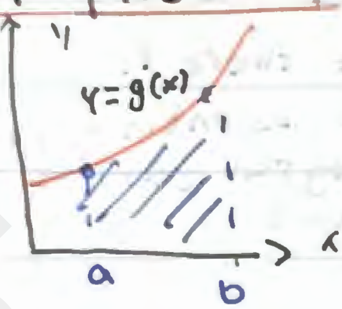
$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x \quad \text{where } \Delta x = \frac{a-b}{n}$$

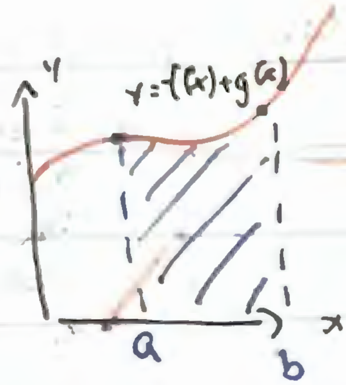
$$= -\int_a^b f(x)\,dx$$

Properties of Definite Integrals

Integrating sum of functions





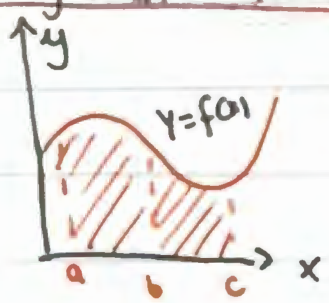


$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

similarly...

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

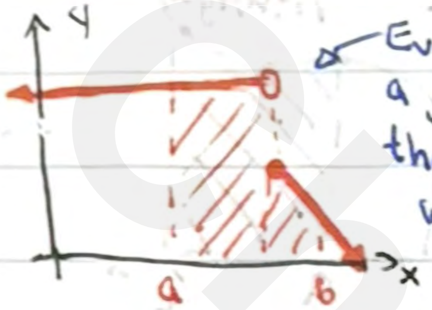
Adjacent intervals



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

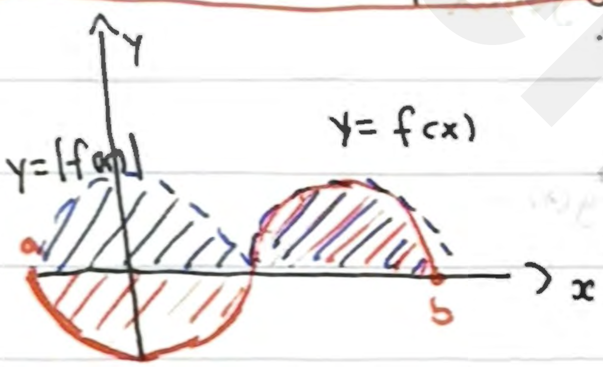
Properties of definite integrals

Piecewise-functions



Even if there is a jump discontinuity, the integral can still be done

Absolute Value of integrals



$\left|\int_a^b f(x)\right|$ will always be positive (or 0)

$\int_a^b |f(x)|dx$ is the integral of the $y=|f(x)|$ function

Anti-Derivative

An antiderivative of a function $f(x)$ is a function $F(x)$ whose derivative is $f(x)$.

When we take an integral, it is taking the antiderivative of a function. The area under the curve is represented by an antiderivative.

The fundamental theorem of Calculus

If f is continuous on the interval $[a,b]$, then the area under the curve of f from $[a,b]$ can be represented by...

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of f.

when we re-arrange...

end value

integral (change between the two values)

$$F(b) = \int_a^b f(x)dx + F(a)$$

initial value

No.

Date

Indefinite Integrals

⇒ Integrals without a boundary are called indefinite integrals

E.g.: $f(x) = x^2+1$ $\quad f(x) = x^2-5$ $\quad f(x) = x^2+1000$

$f'(x) = 2x$ $\quad f'(x) = 2x$ $\quad f'(x) = 2x$

Antiderivative of $2x$?

$f(x) = x^2 + c$ OR $f(x) = x^2 + 1$

General Solution

↳ Includes a constant, "C", that represents all solutions

Particular Solution

↳ Includes a constant number, e.g. 1, that represents only one solution.

Reverse Power Rule

$$f(x) = x^n$$

$$\Rightarrow F(x) = \frac{x^{n+1}}{n+1}$$

1. Add one to exponent
2. Divide by the new exponent.

Exponential

$$\int e^x dx = e^x + c \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

Logarithm as the answer

$$\int \frac{1}{x}\,dx = \ln(|x|) + c$$

Trigonometric Integrals

$$\int \cos x\,dx = \sin x + c$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \csc^2 x\,dx = -\cot x + c$$

Inverse Trigonometric Integrals

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + c$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + c$$

Indefinite Integrals ctd.

Separating rational functions

Rewrite before integrating

E.g. $\int \left(\frac{3x^2+x-2}{x} \right) dx = \int (3x + 1 - 2x^{-1})\, dx$

$= \frac{3}{2}x^2 + x - 2 \cdot \ln(|x|) + c$

Combining expressions

Distribute before integrating

E.g. $\int 3x(\sqrt{x} + x^2)\, dx = \int (3x\sqrt{x} + 3x^3)\, dx$

$= \int (3x^{\frac{3}{2}} + 3x^3)\, dx = \frac{6}{5}x^{\frac{5}{2}} + \frac{3}{4}x^4 + c$

Integrating Using Substitution

In order to find the integrals of chain functions (i.e. used with chain rule), we must use a method called u-substitution

E.g. $\int (3x-4)^5\,dx$ ⇒ we substitute the inner-function with "u"

$u = 3x-4$

$\int u^5\,dx$ → However, we need "du" to integrate "u" (solve for du)

$\frac{du}{dx} = \frac{d}{dx}[3x-4]$

$\Rightarrow \frac{du}{dx} = 3$

$\Rightarrow \frac{du}{3} = dx$

substitute

$= \int \frac{u^5}{3}\,du$

integrate

$= \frac{1}{3}\cdot\frac{1}{6}u^6 + C$

$= \frac{1}{18}u^6 + C$

write in terms of x

$= \boxed{\frac{1}{18}(3x-4)^6 + C}$

When you choose the incorrect "u", you cannot cancel some other term (simplify)

↳ A correct u MUST simplify expression.

E.g. $\int \sin x\, e^{\cos x}\,dx$

$u = \sin x \Rightarrow \frac{du}{\cos x} = dx$

$\int u e^{\cos x}\cdot\frac{du}{\cos x}$ ✗

$u = \cos x \Rightarrow -\frac{du}{\sin x} = dx$

$\int \frac{\sin x}{-\sin x} e^u \cdot du$

$= -e^u + C = -e$ ✓

Integrating Using Substitution

Trig functions...

E.g. $\int \cot(3x)\,dx = \int \cot(u)\,\frac{du}{3} = \frac{1}{3}\int \frac{\cos u}{\sin u}\,du$ $\quad u = 3x \Rightarrow \frac{du}{3} = dx$

No integral for cotangent → rewrite

$w = \sin(u) \qquad \frac{dw}{\cos(u)} = du$

$= \frac{1}{3}\cdot\int \frac{1}{w}\cdot\frac{\cos u}{\cos u}\,dw = \frac{1}{3}\cdot\ln|w| + c$

$= \boxed{\frac{1}{3}\cdot\ln|\sin(3x)| + c}$

Double substitution

- Re do substitution (if further simplification not possible) with another variable (e.g. w)
- make sure to find "dw"

Inverse trig can be confused with u-sub

E.g. $\int \frac{1}{\sqrt{1-4x^2}}\,dx$ not u-sub → only $u=2x$

$= \int \frac{1}{\sqrt{1-(2x)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,\frac{du}{2} = \frac{1}{2}\sin^{-1}(u) + c$

$= \frac{1}{2}\sin^{-1}(2x) + c$

Integrating using Substitution

Solving for x (when $du = dx$, i.e. leading coefficient is "1" for a linear u-substitute)

E.g. $\int \frac{x}{\sqrt{x+1}}\,dx$

$u = x+1 \Rightarrow \boxed{du = dx}$

$= \int \frac{x}{\sqrt{u}}\,du$ ← still does not cancel so instead solve for x ...

$x = u-1$

$= \int \frac{u-1}{\sqrt{u}}\,du = \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$

$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$

$= \boxed{\frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C}$

Definite Integrals with Substitution

With definite integrals, you can use a simple trick

$u = x^2+1 \Rightarrow du \cdot \frac{1}{2x} = dx$

$$\int_a^b f(x^2+1)\,dx = \int_a^b f(u)\,\frac{du}{2x} = \int_{u(a)}^{u(b)} \frac{f(u)}{2x}\,du$$

Plug in 'a' and 'b' into 'u' ⇒ In the end, directly substitute $u(b)$ and $u(a)$ in for u instead of x

E.g. $\int_0^2 t^2\sqrt{t^3+1}\,dt$

$= \frac{1}{3}\int_1^9 \sqrt{u}\,du$

$= \frac{1}{3} \cdot u^{\frac{3}{2}} \cdot \frac{2}{3} \Big|_1^9$

$= \frac{2}{9}[27-1] = \frac{52}{9}$

Integration using long division

When encountering integrals of rational functions, it might be necesoary to use long division to simplify.

E.g.: $\int \frac{x-5}{-2x+2}\,dx = \int\left(-\frac{1}{2} + \frac{\cancel{4}^{\,2}}{\cancel{2}x - \cancel{2}}\right)dx = -\frac{1}{2}x + 2\ln|x-1| + C$

$$\begin{array}{r} -\frac{1}{2} \\ -2x+2 \,\big)\, \overline{x-5} \\ -(x-1) \\ \hline -4 \end{array}$$

$\leftarrow$ when there exists a remainder, you leave it as a part of the rational equation

Integration using Completing the Square

Notice how the inverse trig function derivatives involve "x^2"

$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}} + c$

$\frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}} + c$

$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2-1}} + c$

$\frac{d}{dx}[\csc^{-1}(x)] = -\frac{1}{|x|\sqrt{x^2-1}} + c$

$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{x^2+1} + c$

$\frac{d}{dx}[\cot^{-1}(x)] = -\frac{1}{x^2+1} + c$

Therefore, we can use a combination of u-substitution and complete the square to find the integral

E.g.

$$\int \frac{1}{5x^2-30+65}\,dx = \frac{1}{5}\int \frac{1}{x^2-6x+13}\,dx$$

$$= \frac{1}{5}\int \frac{1}{x^2-6x+9-9+13}\,dx = \frac{1}{5}\int \frac{1}{(x-3)^2+2^2}\,dx$$

$$= \frac{1}{5}\int \frac{\frac{1}{2^2}}{\frac{1}{2^2}} \cdot \frac{1}{(x-3)^2+2^2}\,dx = \frac{1}{20}\int \frac{1}{\left(\frac{x-3}{2}\right)^2+\frac{2^2}{2^2}}\,dx$$

$$= \frac{1}{20}\int \frac{1}{u^2+1}\,dx = \frac{1}{10}\tan^{-1}(u) + c$$

$u = \frac{x-3}{2}$

$2du = dx$

$$= \frac{1}{10} \cdot \tan^{-1}\left(\frac{x-3}{2}\right) + c$$

No

Date

Differential Equations

Differential Equation

$y'' + 2y' = 3y$

$f''(x) + 2f'(x) = 3f(x)$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3y$

} All represent same thing

↳ Relation between a function and its derivatives

↳ Solution is a function (or multiple)

Algebraic Equation

$x^2 + 3x + 2 = 0$

$(x+2)(x+1) = 0$ → Solution is a set of values

$x = -2$ or $x = -1$

Writing a differential equation

- Proportional (directly proportional)

↳ $y = k \cdot x \rightsquigarrow \frac{dy}{dx} = k \cdot y$ etc.

- Inversely Proportional (proportional to reciprocal)

↳ $y = \frac{k}{x} \rightarrow \frac{dy}{dx} = \frac{k}{y}$ etc.

"k" refers to some constant

E.g. speed is inversely proportional to the square of the distance S, it traveled

S = distance $\frac{ds}{dt}$ = Speed

$\therefore \frac{ds}{dt} = \frac{k}{S^2}$

Verifying solutions to differential equations

Derivatives can be used to verify that a function is a solution to a given differential equation

E.g. $\frac{dy}{dx} = \frac{4y}{x}$ verify $y = 4x$ $\therefore \frac{dy}{dx} = 4$ $\Rightarrow$ Plug in function $4 = \frac{dy}{dx} = \frac{4(4x)}{x} = 16$

Not a solution

Finding particular solutions

1. Integrate both sides to get $f(x)$
2. Plug in point into result and solve for "c"
3. Substitute "c" and simplify if needed

E.g. $\frac{dy}{dx} = e^{2x} - 2x^2$, $(0, 4)$:

$$y = \int [e^{2x} - 2x^2] dx$$

$$= \frac{1}{2} \int [e^{u}] du - 2 \int x^2 dx \quad \text{for } u = 2x$$

$$= \frac{1}{2} e^{2x} - \frac{2x^3}{3} + c$$

solve for c

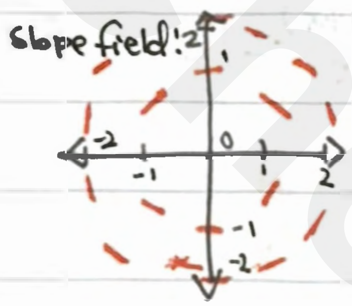
$$4 = \frac{1}{2} e^{0 \cdot 2} - \frac{2(0)^3}{3} + c \Rightarrow c = \frac{7}{2}$$

$$\therefore y = \frac{1}{2} e^{2x} - \frac{2}{3} x^3 + \frac{7}{2}$$

Slope Fields

A slope field represents a differential equation on an xy-plane. It shows the "slope" of all the particular solutions to the differential equation.

E.g. $\frac{dy}{dx} = -\frac{x}{y}$



plug in values (points) into differential Equation and sketch line segments with solved slopes

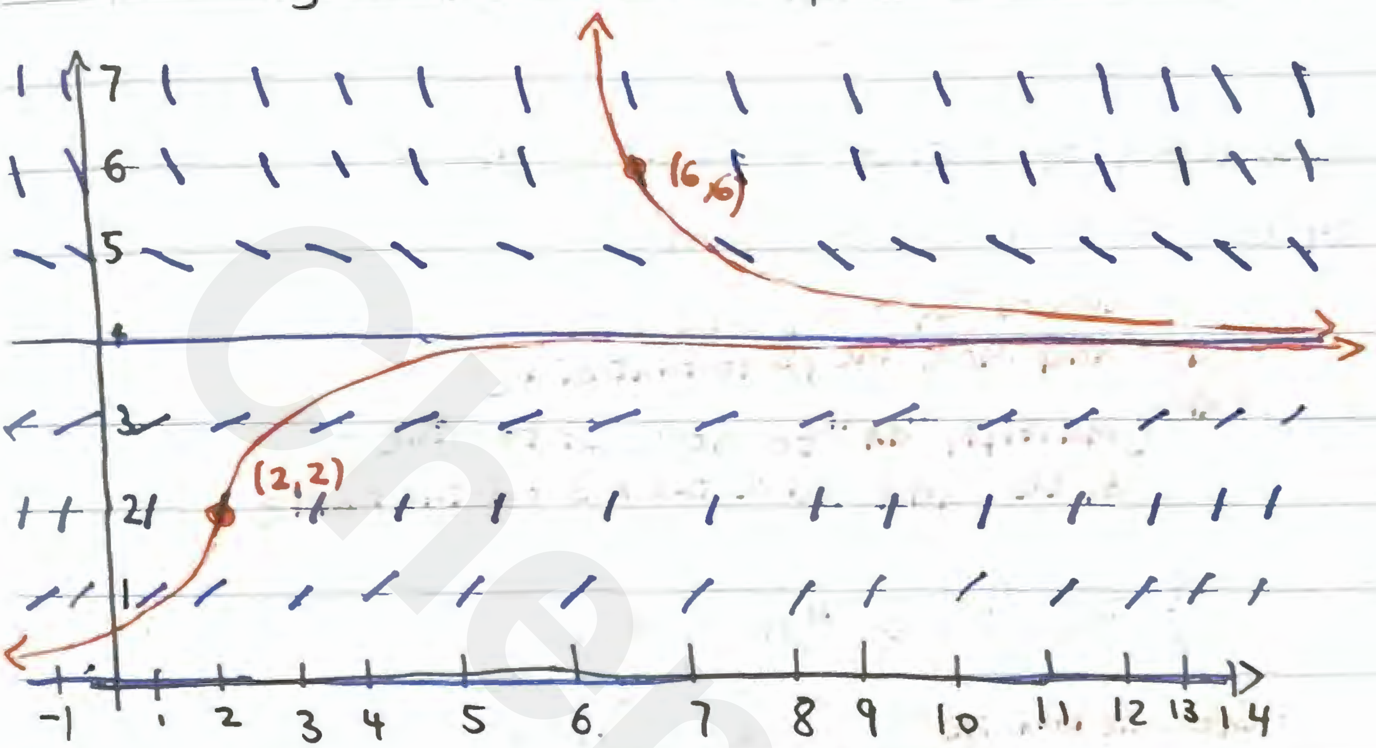
Differential Equation from Slope Field

Look for easy slopes

- Compare horizontal line segments, vertical line segments
- Positive or negative slopes?
- Watch the behavior of the slope field
  - Gets steeper?

⇒ Rule out equations that do not match

Visualizing Solutions with Slope Field



You can visualize solutions with a slope field.

To visualize a particular solution, follow the slopes that link to the selected point (As shown above)

Range of a solution
↳ Watch for domain restrictions (end points) and horizontal asymptotes
↳ End points should be closed (i.e. "[" or "]")
↳ Horizontal asymptotes should be open (i.e. "(" or ")")

Solving Differential Equations with Separation

Separation of variables is like implicit differentiation backwards

$\frac{dy}{dx}$

leave "dy" on one side and move the y's to that side

multiply "dx" to move it to other side and leave the x's on that side.

E.g. $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx$

· Integrate both sides $\Rightarrow \int y\,dy = \int -x\,dx$

$\Rightarrow \frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$

· Simplify ... $\Rightarrow y = \pm\sqrt{-x^2 + C}$

Factoring

→ Sometimes you need to factor before separating

e.g. $\frac{dy}{dx} = 2xy + 4x = 2x(y+2)$

$\frac{1}{y+2}\,dy = 2x\,dx$

$y = \pm Ce^{x^2} - 2$

Separation of Variables

Addition / Subtraction

→ Sometimes you need to move terms before separating

e.g. $\frac{dy}{dx} + y = 10$

$\frac{dy}{dx} = 10 - y$

$\frac{1}{10-y} dy = dx$

$y = \pm ce^{-x} + 10$

Notes:

→ You can simplify constants in an exponential with exponential properties

e.g. $e^{x+C_1} = e^{C_1} \cdot e^{x} = C_1 \cdot e^{x}$

e^{C_1} is also a constant value

→ Integrating "dx" alone

⇒ $\int dx = \int 1\, dx = x + C$

You can only integrate when they're multiplying by the variables e.g. $\int y\, dy$ but not $dy + y$

→ "dy" and "dx" can be treated algebraically

$\underbrace{dy}$ and $\underbrace{dx}$: differentials

⇒ for an infinitely small change in [variable]

⇒ "d" and variable should be treated together (not separated)

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Particular Solutions with Separation

1. Separate

→ separate the equation so that the two variables lie on their respective sides.

e.g. $\frac{dy}{dx} = xy^3 \Rightarrow \frac{dy}{y^3} = x \cdot dx$

2. Integrate

→ Integrate both sides accordingly

e.g. $\int \frac{1}{y^3}\, dy = \int x\, dx \Rightarrow \frac{y^{-2}}{-2} = \frac{x^2}{2} + C$

3. Solve for C

→ Plug in the x-values and y values given and solve for the constant. Then substitute the value for the constant in the equation.

e.g. $\frac{(-2)^{-2}}{-2} = \frac{(1)^2}{2} + C \Rightarrow C = -\frac{5}{8}$

4. Isolate (simplify)

→ Isolate the "y" or function "f(x)" on one side and simplify accordingly.

e.g. $\frac{y^{-2}}{-2} = \frac{x^2}{2} - \frac{5}{8} \Rightarrow y = \pm\sqrt{\frac{-4}{4x^2-5}}$

5. Select (if applicable)

→ Select the correct sign ($\pm$) by plugging in the x-value and y-value again.

→ Applicable for even exponents and absolute values

$y = \pm\sqrt{\frac{-4}{4 \cdot 1^2 - 5}} = \pm 2 \Rightarrow y = -\sqrt{\frac{-4}{4x^2-5}}$

Particular Solutions with Separation ctd.

4/5. For implicit solutions

- → Sometimes you cannot explicitly express the solution and must leave it as implicit
- → When point given is $(x, 0)$ then you cannot really determine sign $(+/-)$

Exponential Models with Differential Equations

⇒ A function is exponential when its differential equation is proportional to that function

$$f'(x) = k \cdot f(x) \qquad \frac{dy}{dx} = k \cdot y$$

Proof:

$$\frac{dy}{dx} = ky \Rightarrow \int \frac{1}{y}\,dy = \int dx \Rightarrow \ln|y| = x + c_1$$

$$\Rightarrow y = \pm e^{kx + c_1} = \pm c \cdot e^{kx}$$

The solution to $\frac{dy}{dt} = ky$ is $y = Ce^{kt}$, where C represents the initial value of the model

Exponential Models

A growth model will have a positive exponent. e.g. $y = 5e^{2t}$

A decay model will have a negative exponent. e.g. $y = 7e^{-2t}$

Using initial value...

Sometimes you may be given $\frac{dy}{dt}$ or a point to find the particular solution...

e.g. $\frac{dy}{dt} = ky$. If y doubles every 7 years, what is k?

$$y = \underbrace{C}_{\text{initial value}} \cdot e^{kt}$$

$$\underbrace{2 \cdot C}_{\text{Doubles}} = \underbrace{C}_{\text{every}} \cdot e^{k \cdot \underbrace{7}_{7\text{ yrs}}} \Rightarrow 2 = e^{7k} \Rightarrow k = \frac{\ln(2)}{7}$$

Notice how $e^{ky} = e^{\frac{dy}{dt}}$ ... $k = \frac{\ln(\text{factor})}{\text{given } x} = \frac{dy}{dt} \div t$

General Formula

$$y = \underbrace{y(0)}_{= C} \cdot e^{\frac{\ln(\text{factor})}{\text{given } x} \cdot t} = y(0) \cdot e^{\frac{\ln\left(\frac{\text{given } y}{y(0)}\right)}{\text{given } x} \cdot t}$$

$$= y(0) \cdot \left(\frac{\text{given } y}{y(0)}\right)^{\frac{t}{\text{given } x}} = y(0) \cdot \text{factor}^{\frac{t}{\text{given } x}}$$

Y is [factor] times for every [given x] years

OR

Y has point (given-x, given-y)

Unit 8: Applications of Integration

Average Value of a function on an interval

Recall... Average Rate of change

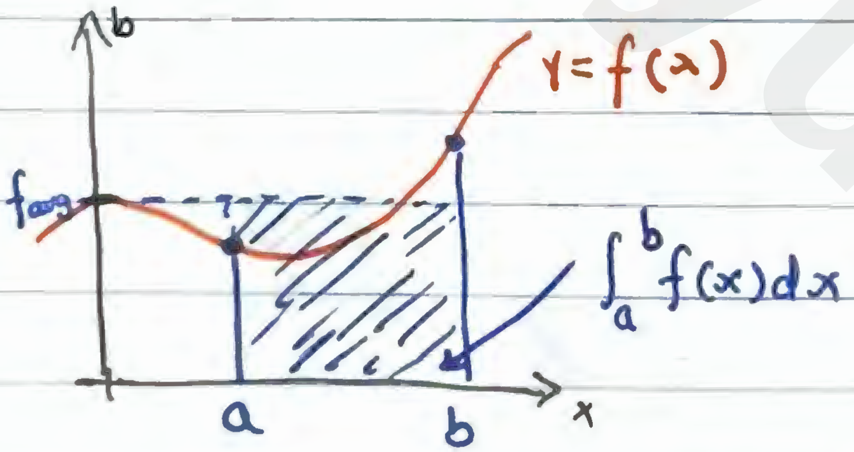
$$\boxed{\begin{aligned} f'(x)_{avg} &= \frac{f(b)-f(a)}{b-a} \\ &\text{OR} \\ f(x)_{avg} &= \frac{F(b)-F(a)}{b-a} \end{aligned}}$$

On closed interval $[a,b]$

How do we express $f(x)_{avg}$ with $f(x)$?

Remember...

$$\int_a^b f(x)\,dx = F(b)-F(a)$$



ALTERNATIVELY...

Area = Width × Height

$$\int_a^b f(x)\,dx = (b-a) \times f_{avg}$$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\boxed{f(x)_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Mean Value Theorem for Integrals

Normal MVT

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists some number such that $a<c<b$ and $f'(c)=\frac{f(b)-f(a)}{b-a}$

Similarly ..... MVT with integrals

$$f_{avg}=\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\Rightarrow f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

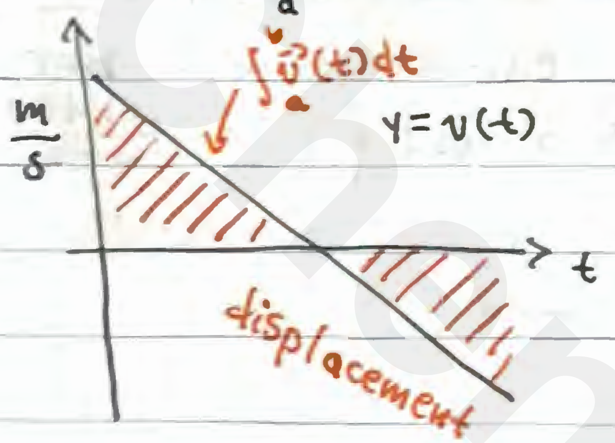
If $f(x)$ is continuous on $[a,b]$, then there exists a c such that $g(c)=g_{avg}$

Position, Velocity, Acceleration with Integral:

Displacement: Change in position

$$\vec{d} = \Delta x$$

$$= \underbrace{\int_a^b \vec{v}(t)\,dt}_{\text{velocity integral}}$$



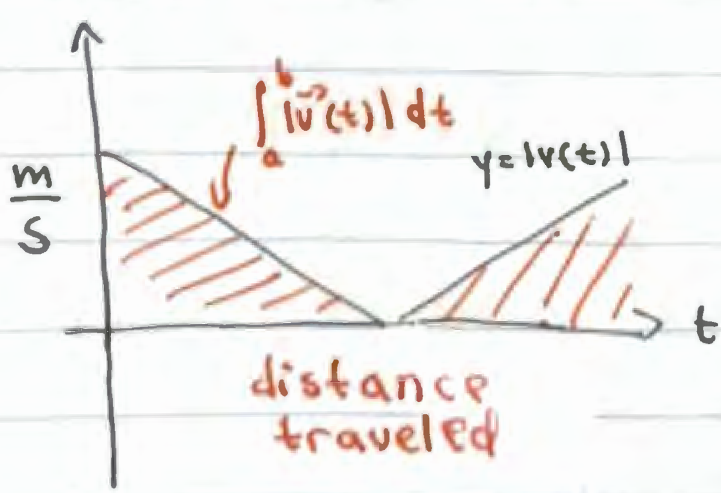
Note that $x_{initial} + \text{displacement} = \text{position}$

$$\Rightarrow \vec{v}(a) + \int_a^b \vec{v}(t)\,dt = \vec{v}(b)$$

Distance: Total length of path

$$d = \int |\vec{v}(t)|\,dt$$

|velocity| = speed



To manually solve:
You can use integral sum property!

$$\int_a^c |v(t)|\,dt$$

use appropriate sign so $v(t) > 0$

$$= \int_a^b |v(t)|\,dt + \int_b^c |v(t)|\,dt$$

Applying Accumulation and Integrals

When you integrate a rate, you get net change
→ Area under rate function

$$\int \text{rate of change} = \text{net change}$$
$$\Rightarrow \int_a^b f'(x)\,dx = \Delta f(x) = f(b) - f(a)$$

In other words...

$f(x)$ increased or decreased by $\int_a^b f'(x)\,dx$ over $[a, b]$ interval

$$\boxed{\text{Final Value} = \text{Initial Value} + \text{NET CHANGE}}$$

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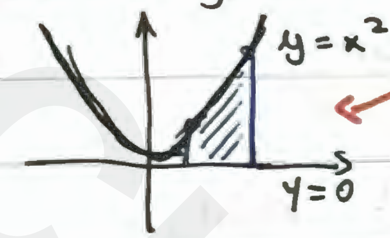
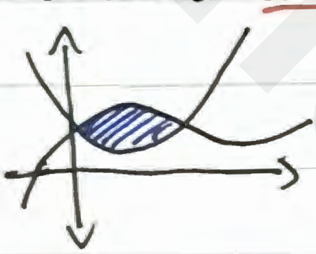
Area Between Curves (x-axis)

Recall: Finding area under the curve



$\int_a^b f(x)\,dx$

How about area between two curves?



$$A = \int_a^b [f(x) - g(x)]\,dx$$

$f(x) \geq g(x)$ for all x in $[a, b]$

Note that we're dealing with geometric area, which MUST be positive. Therefore $f(x) \geq g(x)$.

1. Given Endpoints

when given endpoints $x = a$, $x = b$...

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$$

E.g. Area bounded by the curves $y = x^2 + 2$, $y = -x$, $x = 0$ and $x = 1$

$\int_0^1 [x^2 + 2 - (-x)]\,dx$

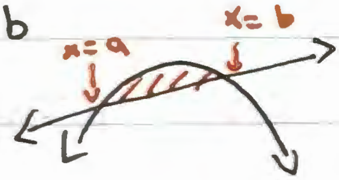
$= \int_0^1 [x^2 + x + 2]\,dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_0^1 = \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}$

2. Not Given Endpoints

You must solve for the point of intersection and set those points as endpoint boundaries.

$$f(x) = g(x) \Rightarrow x = a,\ x = b$$

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$$



E.g. Area bounded by $y = 2 - x^2$ and $y = x$

$x = 2 - x^2$
$x = 1,\ x = -2$

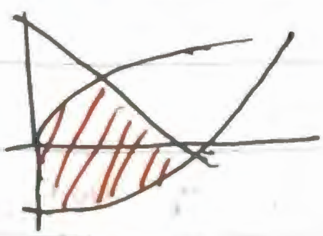
$$\int_{-2}^{1} (2 - x^2 - x)\,dx = \frac{9}{2}$$

3. Composite Area (Multiple Curves)

Find intersections for all curves

$f(x) = g(x),\ g(x) = h(x),\ f(x) = h(x),\ f(x) = i(x)$, etc.

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - h(x)]\,dx + \cdots \text{ etc.}}$$



E.g. Area bounded by the curves $y = 2 - x$, $y = \sqrt{x}$, $y = \frac{x^2}{4} - 1$, $x = 0$

$\sqrt{x} = 2 - x \Rightarrow x = 1$ $\qquad \frac{x^2}{4} - 1 = 2 - x \Rightarrow x = 2$

$$\int_0^1 \left(\sqrt{x} - \left(\frac{x^2}{4} - 1\right)\right) dx + \int_1^2 \left(2 - x - \left(\frac{x^2}{4} - 1\right)\right) dx$$

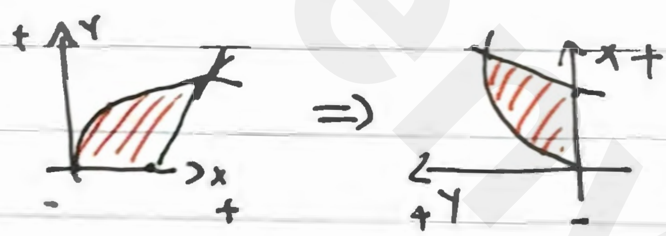
$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{12} + x\right)\Big|_0^1 + \left(3x - \frac{x^2}{2} - \frac{x^3}{12}\right)\Big|_1^2$$

$$= \frac{5}{2}$$

Area Between Curves (with respect to y)

An area can be represented by an integral with respect x or an integral with respect to y.

when calculating an integral with respect to y, make sure to solve for "x =" and view the graph rotated...



Integrate with respect to "y"!

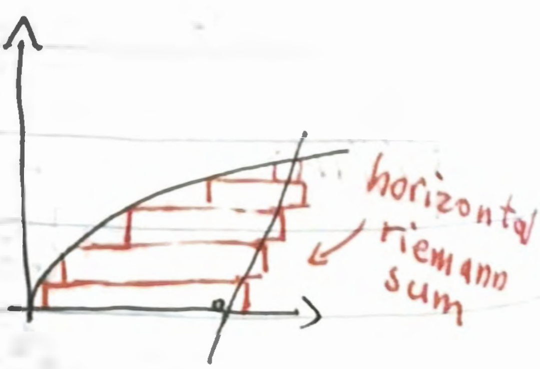
E.g. Set up an integral that allows you to find the area in the first quadrant that is bounded above by $y=\sqrt{x}$ and below by $y=x-6$.

$\cancel{y=\sqrt{x}\Rightarrow}\ x=y^2 \qquad y=x-6 \Rightarrow x=y+6$

$y+6=y^2 \Rightarrow y=3$

$$\int_0^3 (\underbrace{y+6}-y^2)\,dy$$

$x=y+6$ is above $x=y^2$ relative to y. The given talks about relative to x.



Area - More than Two Intersections

some graphs require multiple integrals for multiple intersections

→ Find the x-values of the points of intersections and order from smallest to largest.

E.g. Set up the integrals that represent the total area of the regions bounded by the functions $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

$$3x^3 - x^2 - 10x = x^2 + 2x$$
$$3x^3 - 12x = 0$$
$$3x(x+2)(x-2) = 0$$
$$x = -2,\ x = 0,\ x = 2$$

$$\int_{-2}^{0} [3x^3 - x^2 - 10x - (-x^2 + 2x)]\,dx + \int_{0}^{2} [-x^2 + 2x - (3x^3 - x^2 - 10x)]\,dx$$

ALTERNATIVELY...

You can also use the absolute value to get the same answer!

$$\int_A^C |f(x) - g(x)|\,dx$$

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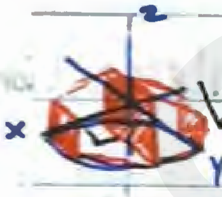
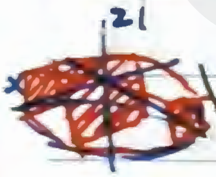
Volumes with Cross Sections

Volume of a solid with known cross-sections

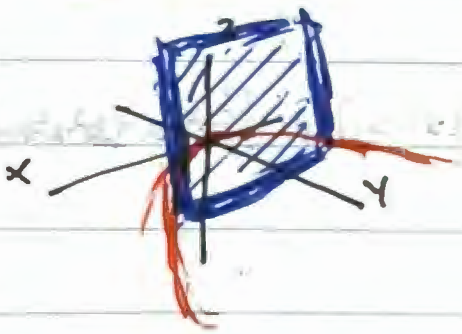


$$V=\int_a^b A(x)\,dx$$ A(x) is the area of the cross section perpendicular to the x-axis



$$V=\int_a^b A(y)\,dy$$ .... perpendicular to the y-axis

1. Square



$$V=\int_a^b s^2\,dx$$
where $s=f(x)-g(x)$

E.g. A region is bounded by $y=x^2$ and $y=\sqrt{x}$, and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume of the solid?

$$V=\int_a^b s^2\,dx=\int_0^1(\sqrt{x}-x^2)^2\,dx$$

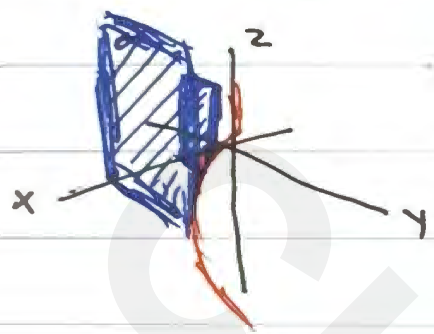
$\sqrt{x}=x^2 \Rightarrow x=0$
$1=x^3$
$1=x$

square cross sections are taken perpendicular to the y-axis

$y=\sqrt{x}$ $\quad y=x^2$
$y^2=x$ $\quad \sqrt{y}=x$

$$V=\int_a^b s^2\,dy=\int_0^1(\sqrt{y}-y^2)^2\,dy$$

2. Rectangle



$$V = \int_a^b (\text{width} \cdot \text{height})\,dx$$

where width $= f(x) - g(x)$,
height = given in the problem

E.g. The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 2$.

a) Find the volume if the cross sections, taken perpendicular to the x-axis, form a rectangle whose height is 2 times its width

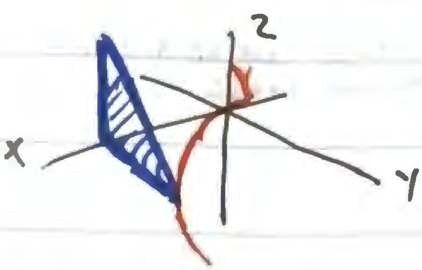
$$V = 2\int_0^2 (x^3 - 0)^2\,dx = 2\int_0^2 x^6\,dx$$

b) Find the volume if the cross sections, taken perpendicular to the y-axis, form a rectangle whose height is 6.

$$V = 6\int_0^{\sqrt[3]{2}} (2 - \sqrt[3]{y})\,dy$$

height is constant so no need to square

3. Equilateral Triangle



$$V = \int_a^b \frac{\sqrt{3}}{4} s^2\,dx$$

where $s = f(x) - g(x)$

$h = \sin(60°)\,b$

$\therefore A = \frac{1}{2} b \cdot h$

$= \frac{1}{2}\sin(60°)\,b^2$

$= \frac{\sqrt{3}}{4} b^2$

E.g. A region is bounded by $y = x^2$ and $y = \sqrt{x}$, and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. What is the volume of the solid?

$$V = \int_a^b \frac{\sqrt{3}}{4} s^2\,dx = \int_0^1 \frac{\sqrt{3}}{4}(\sqrt{x} - x^2)^2\,dx$$

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Volumes with cross sections

4. Isosceles Right Triangle

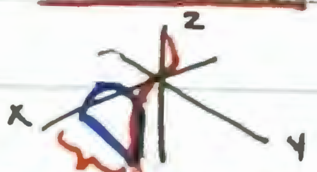
$$V = \int_a^b \frac{1}{2} s^2 \, dx$$

where $s = f(x) - g(x)$

E.g. A region is bounded by $y=x^2$ and $y=\sqrt{x}$, and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is is an isosceles right triangle

$$V = \int_a^b \frac{1}{2} s^2 dx = \int_0^1 \frac{1}{2} (\sqrt{x} - x^2)^2 dx$$

5. Semicircle



THE DISTANCE HERE IS THE DIAMETER

$\therefore$ we need to $\div 2$ to find radius instead

$$V = \int_a^b \frac{1}{2} \pi r^2 dx$$

where $r = \frac{f(x) + g(x)}{2}$

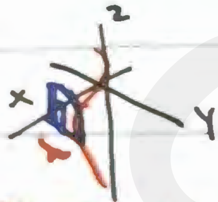
E.g. A region is bounded by $y=x^2$ and $y=\sqrt{x}$, and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle.

$$V = \int_0^1 \frac{1}{2} \pi r^2 dx = \int_0^1 \frac{\pi}{2} \cdot \Big(\underbrace{\frac{\sqrt{x} - x^2}{2}} \Big)^2 dx$$

SOMETIMES THE 2^2 IS BROUGHT OUT...

$\frac{\pi}{8}$ instead of $\frac{\pi}{2} \cdot \frac{\dots}{2^2}$

6. Quarter Circle



THE DISTANCE HERE IS THE RADIUS
→ Don't get confused with semicircle

$$V = \int_a^b \frac{1}{4}\pi r^2 \, dx$$

where $r = f(x) - g(x)$

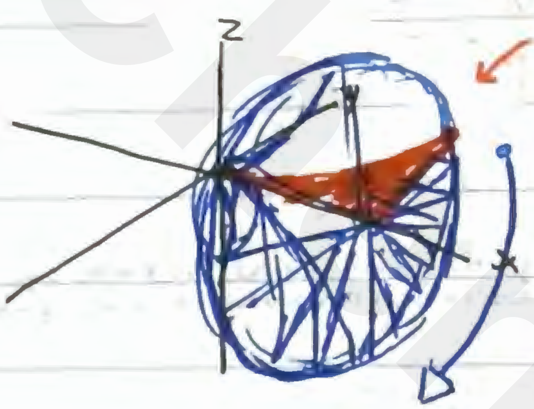
E.g. A region is bounded by $y=x^2$ and $y=\sqrt{x}$, and forms the base of a solid. For this solid, each cross section perpendicular to the x-axis is a quarter circle.

$$V = \int_a^b \frac{1}{4}\pi r^2 dx = \int_0^1 \underbrace{\frac{\pi}{4}}_{} \cdot \left(\sqrt{x} - x^2\right)^2 dx$$

Quarter circles are ALMOST ALWAYS ÷4. Don't get confused with semicircle.

Disc Method : Revolve around axes

When areas bounded by a curve and an axis is rotated around that axis, a circular cross section is formed.



EACH CROSS SECTION IS A CIRCLE

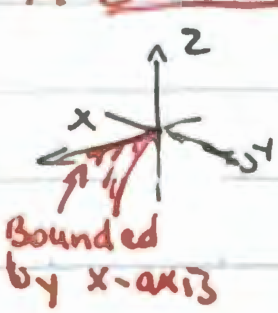
→ Unlike semicircles, the distance between the curve and the axis is the radius.

Volume of a Solid of Revolution

$$\boxed{V = \int_a^b \pi \left[R(x)\right]^2 dx}$$

where $R(x)$ is the "distance" between the axis of revolution and the outside of the solid.

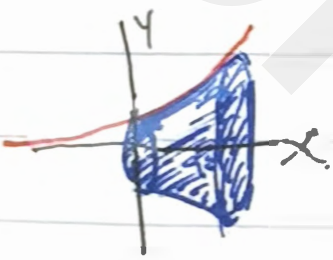
THE AXIS OF REVOLUTION MUST ALSO BE A BOUNDARY OF THE SOLID



1. x-axis

Must be integrated with respect to x: "dx"

E.g. Take the region bounded by $y=e^x$, $y=0$, $x=0$, and $x=3$. Resolve this region about the x-axis. Find the volume of the solid formed.



$$\int_0^3 \pi (e^x)^2 x \, dx$$

$$= \frac{\pi}{2} e^{2x} \Big|_0^3$$

$$= \frac{e^{6x}}{2} - \frac{\pi}{2} = \frac{\pi}{2}(e^6 - 1)$$

2. y-axis

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