Using a soft circle to represent the location of the pendulum mass

Carlos E. Álvarez

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1 Soft circle function

The soft circle function centered at c is defined as the product of two sigmoid functions:

$$\begin{aligned} f(x) &= \left(\frac{1}{1+e^{-A((x-c)+R)}}\right)\left(\frac{1}{1+e^{A((x-c)-R)}}\right) \\ &= \frac{1}{1+e^{-2AR}+2e^{-AR}\cosh(A(x-c))}, \end{aligned} \tag{1}$$

where A controls the decay of the rim of the circle, and R the radius of the circle.

2 Soft circle as pendulum mass

In 2D, to represent the pendulum mass as a soft circle we realize that the center of the pendulum is located at

$$\vec{r} = (r_x, r_y) = (p_x + l\sin\theta, p_y + l\cos\theta), \tag{2}$$

where (p_x, p_y) is the location of the pivot of the pendulum, l is the length of the pendulum, and θ is the angle the pendulum makes with the vertical. Which means we have to compute function (1) as a function of the radial distance d from $\vec{r}$, instead of c. That is

$$D = \sqrt{(x - p_x - l\sin\theta)^2 + (y - p_y - l\cos\theta)^2}. \tag{3}$$

Then

$$f(D) = \frac{1}{a + b\cosh(AD)}, \tag{4}$$

where

$$a = 1 + e^{-2AR} \tag{5}$$
$$b = 2e^{-AR}. \tag{6}$$

The angle θ is a function of time, and we want to compute the first and second time derivatives of f with respect to time.

2.1 1st derivative

$$\frac{df}{dt} = \frac{df}{dD}\frac{dD}{d\theta}\dot{\theta}. \tag{7}$$

$$\frac{df}{dD} = -\frac{bA\sinh(AD)}{[a + b\cosh(AD)]^2} = -bA\sinh(AD)f(D)^2. \tag{8}$$

$$\frac{dD}{d\theta} = -\frac{l}{D}[(y - p_y)\sin\theta - (x - p_x)\cos\theta]. \tag{9}$$

Then

$$\dot{f}(D) = \frac{df}{dt} = \frac{lbA}{D}f(D)^2\sinh(AD)[(x - p_x)\cos\theta - (y - p_y)\sin\theta]\dot{\theta} \tag{10}$$

2.2 2nd derivative

$$
\begin{aligned}
\frac{d^2 f}{dt^2} =& lbA\frac{d}{dt}\left[\frac{1}{D}\right] f(D)^2 \sinh(AD)[(y-p_y)\sin\theta-(x-p_x)\cos\theta)]\dot\theta \\
&+ lbA\frac{1}{D}\frac{d}{dt}\left[f(D)^2\right]\sinh(AD)[(y-p_y)\sin\theta-(x-p_x)\cos\theta)]\dot\theta \\
&+ lbA\frac{1}{D}f(D)^2\frac{d}{dt}\left[\sinh(AD)\right][(y-p_y)\sin\theta-(x-p_x)\cos\theta)]\dot\theta \\
&+ lbA\frac{1}{D}f(D)^2\sinh(AD)\frac{d}{dt}\left[(y-p_y)\sin\theta-(x-p_x)\cos\theta)\right]\dot\theta \\
&+ lbA\frac{1}{D}f(D)^2\sinh(AD)\left[(y-p_y)\sin\theta-(x-p_x)\cos\theta)\right]\ddot\theta \\
=& \frac{l^2bA}{D^3}f(D)^2\sinh(AD)[(y-p_y)\sin\theta-(x-p_x)\cos\theta]^2\dot\theta^2 \\
&+ 2\frac{lbA}{D}f(D)\dot f(D)\sinh(AD)[(y-p_y)\sin\theta-(x-p_x)\cos\theta]\dot\theta \\
&- \frac{lbA^2}{D}f(D)^2\cosh(AD)[(y-p_y)\cos\theta+(x-p_x)\sin\theta][(y-p_y)\sin\theta-(x-p_x)\cos\theta]\dot\theta^2 \\
&- \frac{lbA}{D}f(D)^2\sinh(AD)\left[(y-p_y)\cos\theta+(x-p_x)\sin\theta\right]\dot\theta^2 \\
&+ \frac{lbA}{D}f(D)^2\sinh(AD)\left[(y-p_y)\sin\theta-(x-p_x)\cos\theta\right]\ddot\theta \\
=& \dot f(D)\left[\frac{l}{D^2}\dot\theta+2\frac{\dot f(D)}{f(D)}-\frac{\sin\theta}{\dot\theta}\right. \\
&\left.-\left(\frac{A}{\tanh(AD)}+\frac{1}{[(y-p_y)\sin\theta-(x-p_x)\cos\theta)]}\right)\left[(y-p_y)\cos\theta+(x-p_x)\sin\theta\right]\dot\theta\right],
\end{aligned}
\tag{11}
$$

where we have used that, for a pendulum with $l = g$

$$\ddot\theta = -\sin\theta. \tag{12}$$

Numerically, it is better to avoid dividing by $\dot{\theta}$ or $f(D)$, so

$$\begin{aligned}
\ddot{f} =& \frac{lbA}{D} f(D)\sinh(AD)[(x-p_x)\cos\theta - (y-p_y)\sin\theta] \\
&\times \left[\frac{l}{D^2} f(D)\dot{\theta}^2 + 2\dot{f}(D) - f(D)\sin\theta \right. \\
&\left. - \left(\frac{A}{\tanh(AD)} + \frac{1}{[(y-p_y)\sin\theta - (x-p_x)\cos\theta)]}\right)[(y-p_y)\cos\theta + (x-p_x)\sin\theta]\, f(D)\dot{\theta}^2 \right]
\end{aligned} \tag{13}$$