

| OPERATION | DEFINITION AND NOTATION |
|---|---|
| <i>Union of L and M</i> | $L \cup M = \{s \mid s \text{ is in } L \text{ or } s \text{ is in } M\}$ |
| <i>Concatenation of L and M</i> | $LM = \{st \mid s \text{ is in } L \text{ and } t \text{ is in } M\}$ |
| <i>Kleene closure of L</i> | $L^* = \bigcup_{i=0}^{\infty} L^i$ |
| <i>Positive closure of L</i> | $L^+ = \bigcup_{i=1}^{\infty} L^i$ |

Figure 3.6: Definitions of operations on languages

Example 3.3: Let L be the set of letters $\{A, B, \dots, Z, a, b, \dots, z\}$ and let D be the set of digits $\{0, 1, \dots, 9\}$. We may think of L and D in two, essentially equivalent, ways. One way is that L and D are, respectively, the alphabets of uppercase and lowercase letters and of digits. The second way is that L and D are languages, all of whose strings happen to be of length one. Here are some other languages that can be constructed from languages L and D , using the operators of Fig. 3.6:

1. $L \cup D$ is the set of letters and digits — strictly speaking the language with 62 strings of length one, each of which strings is either one letter or one digit.
2. LD is the set of 520 strings of length two, each consisting of one letter followed by one digit.
3. L^4 is the set of all 4-letter strings.
4. L^* is the set of all strings of letters, including ϵ , the empty string.
5. $L(L \cup D)^*$ is the set of all strings of letters and digits beginning with a letter.
6. D^+ is the set of all strings of one or more digits.

□

3.3.3 Regular Expressions

Suppose we wanted to describe the set of valid C identifiers. It is almost exactly the language described in item (5) above; the only difference is that the underscore is included among the letters.

In Example 3.3, we were able to describe identifiers by giving names to sets of letters and digits and using the language operators union, concatenation, and closure. This process is so useful that a notation called *regular expressions* has come into common use for describing all the languages that can be built from these operators applied to the symbols of some alphabet. In this notation, if *letter* is established to stand for any letter or the underscore, and *digit* is

established to stand for any digit, then we could describe the language of C identifiers by:

$$\textit{letter_} (\textit{letter_} \mid \textit{digit})^*$$

The vertical bar above means union, the parentheses are used to group subexpressions, the star means “zero or more occurrences of,” and the juxtaposition of *letter_* with the remainder of the expression signifies concatenation.

The regular expressions are built recursively out of smaller regular expressions, using the rules described below. Each regular expression r denotes a language $L(r)$, which is also defined recursively from the languages denoted by r ’s subexpressions. Here are the rules that define the regular expressions over some alphabet Σ and the languages that those expressions denote.

BASIS: There are two rules that form the basis:

1. ϵ is a regular expression, and $L(\epsilon)$ is $\{\epsilon\}$, that is, the language whose sole member is the empty string.
2. If a is a symbol in Σ , then **a** is a regular expression, and $L(\mathbf{a}) = \{a\}$, that is, the language with one string, of length one, with a in its one position. Note that by convention, we use italics for symbols, and boldface for their corresponding regular expression.¹

INDUCTION: There are four parts to the induction whereby larger regular expressions are built from smaller ones. Suppose r and s are regular expressions denoting languages $L(r)$ and $L(s)$, respectively.

1. $(r)|(s)$ is a regular expression denoting the language $L(r) \cup L(s)$.
2. $(r)(s)$ is a regular expression denoting the language $L(r)L(s)$.
3. $(r)^*$ is a regular expression denoting $(L(r))^*$.
4. (r) is a regular expression denoting $L(r)$. This last rule says that we can add additional pairs of parentheses around expressions without changing the language they denote.

As defined, regular expressions often contain unnecessary pairs of parentheses. We may drop certain pairs of parentheses if we adopt the conventions that:

- a) The unary operator $*$ has highest precedence and is left associative.
- b) Concatenation has second highest precedence and is left associative.

¹However, when talking about specific characters from the ASCII character set, we shall generally use teletype font for both the character and its regular expression.

c) $|$ has lowest precedence and is left associative.

Under these conventions, for example, we may replace the regular expression $(\mathbf{a})|((\mathbf{b})^*(\mathbf{c}))$ by $\mathbf{a|b^*c}$. Both expressions denote the set of strings that are either a single a or are zero or more b 's followed by one c .

Example 3.4: Let $\Sigma = \{a, b\}$.

1. The regular expression $\mathbf{a|b}$ denotes the language $\{a, b\}$.
2. $\mathbf{(a|b)(a|b)}$ denotes $\{aa, ab, ba, bb\}$, the language of all strings of length two over the alphabet Σ . Another regular expression for the same language is $\mathbf{aa|ab|ba|bb}$.
3. $\mathbf{a^*}$ denotes the language consisting of all strings of zero or more a 's, that is, $\{\epsilon, a, aa, aaa, \dots\}$.
4. $\mathbf{(a|b)^*}$ denotes the set of all strings consisting of zero or more instances of a or b , that is, all strings of a 's and b 's: $\{\epsilon, a, b, aa, ab, ba, bb, aaa, \dots\}$. Another regular expression for the same language is $\mathbf{(a^*b^*)^*}$.
5. $\mathbf{a|a^*b}$ denotes the language $\{a, b, ab, aab, aaab, \dots\}$, that is, the string a and all strings consisting of zero or more a 's and ending in b .

□

A language that can be defined by a regular expression is called a *regular set*. If two regular expressions r and s denote the same regular set, we say they are *equivalent* and write $r = s$. For instance, $\mathbf{(a|b) = (b|a)}$. There are a number of algebraic laws for regular expressions; each law asserts that expressions of two different forms are equivalent. Figure 3.7 shows some of the algebraic laws that hold for arbitrary regular expressions r , s , and t .

| LAW | DESCRIPTION |
|----------------------------------|--|
| $r s = s r$ | $ $ is commutative |
| $r (s t) = (r s) t$ | $ $ is associative |
| $r(st) = (rs)t$ | Concatenation is associative |
| $r(s t) = rs rt; (s t)r = sr tr$ | Concatenation distributes over $ $ |
| $\epsilon r = r\epsilon = r$ | ϵ is the identity for concatenation |
| $r^* = (r \epsilon)^*$ | ϵ is guaranteed in a closure |
| $r^{**} = r^*$ | $*$ is idempotent |

Figure 3.7: Algebraic laws for regular expressions

3.3.4 Regular Definitions

For notational convenience, we may wish to give names to certain regular expressions and use those names in subsequent expressions, as if the names were themselves symbols. If Σ is an alphabet of basic symbols, then a *regular definition* is a sequence of definitions of the form:

$$\begin{array}{lll} d_1 & \rightarrow & r_1 \\ d_2 & \rightarrow & r_2 \\ & \dots & \\ d_n & \rightarrow & r_n \end{array}$$

where:

1. Each d_i is a new symbol, not in Σ and not the same as any other of the d 's, and
2. Each r_i is a regular expression over the alphabet $\Sigma \cup \{d_1, d_2, \dots, d_{i-1}\}$.

By restricting r_i to Σ and the previously defined d 's, we avoid recursive definitions, and we can construct a regular expression over Σ alone, for each r_i . We do so by first replacing uses of d_1 in r_2 (which cannot use any of the d 's except for d_1), then replacing uses of d_1 and d_2 in r_3 by r_1 and (the substituted) r_2 , and so on. Finally, in r_n we replace each d_i , for $i = 1, 2, \dots, n-1$, by the substituted version of r_i , each of which has only symbols of Σ .

Example 3.5: C identifiers are strings of letters, digits, and underscores. Here is a regular definition for the language of C identifiers. We shall conventionally use italics for the symbols defined in regular definitions.

$$\begin{array}{lll} \textit{letter_} & \rightarrow & A \mid B \mid \dots \mid Z \mid a \mid b \mid \dots \mid z \mid _ \\ \textit{digit} & \rightarrow & 0 \mid 1 \mid \dots \mid 9 \\ \textit{id} & \rightarrow & \textit{letter_} (\textit{letter_} \mid \textit{digit})^* \end{array}$$

□

Example 3.6: Unsigned numbers (integer or floating point) are strings such as 5280, 0.01234, 6.336E4, or 1.89E-4. The regular definition

$$\begin{array}{lll} \textit{digit} & \rightarrow & 0 \mid 1 \mid \dots \mid 9 \\ \textit{digits} & \rightarrow & \textit{digit} \textit{digit}^* \\ \textit{optionalFraction} & \rightarrow & . \textit{digits} \mid \epsilon \\ \textit{optionalExponent} & \rightarrow & (E (+ \mid - \mid \epsilon) \textit{digits}) \mid \epsilon \\ \textit{number} & \rightarrow & \textit{digits} \textit{optionalFraction} \textit{optionalExponent} \end{array}$$

is a precise specification for this set of strings. That is, an *optionalFraction* is either a decimal point (dot) followed by one or more digits, or it is missing (the empty string). An *optionalExponent*, if not missing, is the letter E followed by an optional + or − sign, followed by one or more digits. Note that at least one digit must follow the dot, so *number* does not match 1., but does match 1.0.

□

3.3.5 Extensions of Regular Expressions

Since Kleene introduced regular expressions with the basic operators for union, concatenation, and Kleene closure in the 1950s, many extensions have been added to regular expressions to enhance their ability to specify string patterns. Here we mention a few notational extensions that were first incorporated into Unix utilities such as `Lex` that are particularly useful in the specification lexical analyzers. The references to this chapter contain a discussion of some regular-expression variants in use today.

1. *One or more instances.* The unary, postfix operator $^+$ represents the positive closure of a regular expression and its language. That is, if r is a regular expression, then $(r)^+$ denotes the language $(L(r))^+$. The operator $^+$ has the same precedence and associativity as the operator $*$. Two useful algebraic laws, $r^* = r^+|\epsilon$ and $r^+ = rr^* = r^*r$ relate the Kleene closure and positive closure.
2. *Zero or one instance.* The unary postfix operator $?$ means “zero or one occurrence.” That is, $r?$ is equivalent to $r|\epsilon$, or put another way, $L(r?) = L(r) \cup \{\epsilon\}$. The $?$ operator has the same precedence and associativity as $*$ and $^+$.
3. *Character classes.* A regular expression $a_1|a_2|\dots|a_n$, where the a_i ’s are each symbols of the alphabet, can be replaced by the shorthand $[a_1a_2\dots a_n]$. More importantly, when a_1, a_2, \dots, a_n form a logical sequence, e.g., consecutive uppercase letters, lowercase letters, or digits, we can replace them by $a_1\text{-}a_n$, that is, just the first and last separated by a hyphen. Thus, `[abc]` is shorthand for `a|b|c`, and `[a-z]` is shorthand for `a|b|\dots|z`.

Example 3.7: Using these shorthands, we can rewrite the regular definition of Example 3.5 as:

$$\begin{aligned} \text{letter_} &\rightarrow [\text{A-Za-z}] \\ \text{digit} &\rightarrow [0-9] \\ \text{id} &\rightarrow \text{letter_} (\text{letter} | \text{digit})^* \end{aligned}$$

The regular definition of Example 3.6 can also be simplified:

$$\begin{aligned} \text{digit} &\rightarrow [0-9] \\ \text{digits} &\rightarrow \text{digit}^+ \\ \text{number} &\rightarrow \text{digits} (. \text{digits})? (\text{E} [+-]? \text{digits})? \end{aligned}$$

□

3.3.6 Exercises for Section 3.3

Exercise 3.3.1: Consult the language reference manuals to determine (i) the sets of characters that form the input alphabet (excluding those that may only appear in character strings or comments), (ii) the lexical form of numerical constants, and (iii) the lexical form of identifiers, for each of the following languages: (a) C (b) C++ (c) C# (d) Fortran (e) Java (f) Lisp (g) SQL.

! Exercise 3.3.2: Describe the languages denoted by the following regular expressions:

- a) $a(a|b)^*a$.
- b) $((\epsilon|a)b^*)^*$.
- c) $(a|b)^*a(a|b)(a|b)$.
- d) $a^*ba^*ba^*ba^*$.
- !! e) $(aa|bb)^*((ab|ba)(aa|bb)^*(ab|ba)(aa|bb)^*)^*$.

Exercise 3.3.3: In a string of length n , how many of the following are there?

- a) Prefixes.
- b) Suffixes.
- c) Proper prefixes.
- ! d) Substrings.
- ! e) Subsequences.

Exercise 3.3.4: Most languages are *case sensitive*, so keywords can be written only one way, and the regular expressions describing their lexeme is very simple. However, some languages, like SQL, are *case insensitive*, so a keyword can be written either in lowercase or in uppercase, or in any mixture of cases. Thus, the SQL keyword **SELECT** can also be written **select**, **Select**, or **sElEcT**, for instance. Show how to write a regular expression for a keyword in a case-insensitive language. Illustrate the idea by writing the expression for “select” in SQL.

! Exercise 3.3.5: Write regular definitions for the following languages:

- a) All strings of lowercase letters that contain the five vowels in order.
- b) All strings of lowercase letters in which the letters are in ascending lexicographic order.
- c) Comments, consisting of a string surrounded by `/*` and `*/`, without an intervening `*/`, unless it is inside double-quotes (“”).

- !! d) All strings of digits with no repeated digits. *Hint:* Try this problem first with a few digits, such as {0, 1, 2}.
- !! e) All strings of digits with at most one repeated digit.
- !! f) All strings of *a*'s and *b*'s with an even number of *a*'s and an odd number of *b*'s.
- g) The set of Chess moves, in the informal notation, such as p-k4 or kbp×qn.
- !! h) All strings of *a*'s and *b*'s that do not contain the substring *abb*.
- i) All strings of *a*'s and *b*'s that do not contain the subsequence *abb*.

Exercise 3.3.6: Write character classes for the following sets of characters:

- a) The first ten letters (up to "j") in either upper or lower case.
- b) The lowercase consonants.
- c) The "digits" in a hexadecimal number (choose either upper or lower case for the "digits" above 9).
- d) The characters that can appear at the end of a legitimate English sentence (e.g., exclamation point).

The following exercises, up to and including Exercise 3.3.10, discuss the extended regular-expression notation from *Lex* (the lexical-analyzer generator that we shall discuss extensively in Section 3.5). The extended notation is listed in Fig. 3.8.

Exercise 3.3.7: Note that these regular expressions give all of the following symbols (*operator characters*) a special meaning:

\ " . ^ \$ [] * + ? { } | /

Their special meaning must be turned off if they are needed to represent themselves in a character string. We can do so by quoting the character within a string of length one or more; e.g., the regular expression "***" matches the string **. We can also get the literal meaning of an operator character by preceding it by a backslash. Thus, the regular expression ** also matches the string **. Write a regular expression that matches the string "\.

Exercise 3.3.8: In *Lex*, a *complemented character class* represents any character except the ones listed in the character class. We denote a complemented class by using ^ as the first character; this symbol (caret) is not itself part of the class being complemented, unless it is listed within the class itself. Thus, [^A-Za-z] matches any character that is not an uppercase or lowercase letter, and [^\^] represents any character but the caret (or newline, since newline cannot be in any character class). Show that for every regular expression with complemented character classes, there is an equivalent regular expression without complemented character classes.

| EXPRESSION | MATCHES | EXAMPLE |
|----------------|---|----------------------|
| c | the one non-operator character c | <code>a</code> |
| $\backslash c$ | character c literally | <code>*</code> |
| $"s"$ | string s literally | <code>"**"</code> |
| $.$ | any character but newline | <code>a.*b</code> |
| \wedge | beginning of a line | <code>^abc</code> |
| $\$$ | end of a line | <code>abc\\$</code> |
| $[s]$ | any one of the characters in string s | <code>[abc]</code> |
| $[\wedge s]$ | any one character not in string s | <code>[^abc]</code> |
| r^* | zero or more strings matching r | <code>a*</code> |
| r^+ | one or more strings matching r | <code>a^+</code> |
| $r^?$ | zero or one r | <code>a?</code> |
| $r\{m,n\}$ | between m and n occurrences of r | <code>a[1,5]</code> |
| $r_1 r_2$ | an r_1 followed by an r_2 | <code>ab</code> |
| $r_1 \mid r_2$ | an r_1 or an r_2 | <code>a b</code> |
| (r) | same as r | <code>(a b)</code> |
| r_1/r_2 | r_1 when followed by r_2 | <code>abc/123</code> |

Figure 3.8: Lex regular expressions

! Exercise 3.3.9: The regular expression $r\{m,n\}$ matches from m to n occurrences of the pattern r . For example, `a[1,5]` matches a string of one to five a 's. Show that for every regular expression containing repetition operators of this form, there is an equivalent regular expression without repetition operators.

! Exercise 3.3.10: The operator \wedge matches the left end of a line, and $\$$ matches the right end of a line. The operator \wedge is also used to introduce complemented character classes, but the context always makes it clear which meaning is intended. For example, `^[^aeiou]*\$` matches any complete line that does not contain a lowercase vowel.

- How do you tell which meaning of \wedge is intended?
- Can you always replace a regular expression using the \wedge and $\$$ operators by an equivalent expression that does not use either of these operators?

! Exercise 3.3.11: The UNIX shell command `sh` uses the operators in Fig. 3.9 in filename expressions to describe sets of file names. For example, the filename expression `*.o` matches all file names ending in `.o`; `sort1.?` matches all file names of the form `sort.c`, where c is any character. Show how `sh` filename

| EXPRESSION | MATCHES | EXAMPLE |
|------------|------------------------------|-------------|
| 's' | string <i>s</i> literally | '\' |
| \c | character <i>c</i> literally | \' |
| * | any string | *.o |
| ? | any character | sort1.? |
| [s] | any character in <i>s</i> | sort1.[cso] |

Figure 3.9: Filename expressions used by the shell command `sh`

expressions can be replaced by equivalent regular expressions using only the basic union, concatenation, and closure operators.

! Exercise 3.3.12: SQL allows a rudimentary form of patterns in which two characters have special meaning: underscore (`_`) stands for any one character and percent-sign (`%`) stands for any string of 0 or more characters. In addition, the programmer may define any character, say *e*, to be the escape character, so *e* preceding an *e* preceding `_`, `%`, or another *e* gives the character that follows its literal meaning. Show how to express any SQL pattern as a regular expression, given that we know which character is the escape character.

3.4 Recognition of Tokens

In the previous section we learned how to express patterns using regular expressions. Now, we must study how to take the patterns for all the needed tokens and build a piece of code that examines the input string and finds a prefix that is a lexeme matching one of the patterns. Our discussion will make use of the following running example.

$$\begin{array}{ll}
 stmt & \rightarrow \text{if } expr \text{ then } stmt \\
 & \quad | \text{if } expr \text{ then } stmt \text{ else } stmt \\
 & \quad | \epsilon \\
 expr & \rightarrow term \text{ relop } term \\
 & \quad | term \\
 term & \rightarrow \text{id} \\
 & \quad | \text{number}
 \end{array}$$

Figure 3.10: A grammar for branching statements

Example 3.8: The grammar fragment of Fig. 3.10 describes a simple form of branching statements and conditional expressions. This syntax is similar to that of the language Pascal, in that **then** appears explicitly after conditions.