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Bonus Question

By induction, prove that: $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Base Case:
n == 1

$$\sum_{i=1}^{1} i^3 = \qquad \left(\frac{1(1+1)}{2}\right)^2 =$$

$$1^3 = 1 \qquad \left(\frac{2}{2}\right)^2$$

$$1 \qquad 1^2 = 1$$

$$1$$

Both sides of the formula are equal to one when n is 1, thus the formula is correct, and our base case is correct.

Induction Hypothesis:
Our original formula:
is true

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

To prove:
That our induction hypothesis remains true for k+1

Proof:

$$\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

$$\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

We take out k+1 from the summation to make it equal to our original hypothesis:

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + i^{k+1}$$

By our induction hypothesis:

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Thus:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Simplify the left side of the equation:

$$\left(\frac{k(k+1)}{2}\right)^2 = \left(\frac{k^2+k}{2}\right)^2$$

Square the above fraction:

$$(k^2+k)(k^2+k) = k^4 + 2k^3 + k^2$$
$$2*2 = 4$$
$$\frac{k^4 + 2k^3 + k^2}{4}$$

Simplify:

$$(k+1)^3 =$$
$$(k+1)(k+1)(k+1)$$
$$= k^3 + 3k^2 + 3k + 1$$

Obtain 4 as the common denominator:

$$k^3 + 3k^2 + 3k + 1 \cdot \frac{4}{4}$$
$$\frac{4k^3 + 12k^2 + 12^k + 4}{4}$$

Thus:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

Is equal to:

$$\frac{k^4+2k^3+k^2}{4}+\frac{4k^3+12k^2+12^k+4}{4}$$
$$=\frac{k^4+6k^3+13k^2+12k+4}{4}$$

Simplify the right side of the equation:

$$\left(\frac{(k+1)(k+2)}{2}\right)^2$$
$$(k+1)(k+2)=k^2+3k+2$$
$$=\left(\frac{k^2+3k+2}{2}\right)^2$$
$$=\frac{k^4+6k^3+13k^2+12k+4}{4}$$

Thus:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Is equal to:

$$\frac{k^4+6k^3+13k^2+12k+4}{4}=\frac{k^4+6k^3+13k^2+12k+4}{4}$$

Thus our induction hypothesis is true.