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Algorithm Analysis

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1. Use the LCS algorithm developed in class to compute the LCS of "exponential" and "polynomial". Show all of your work including the table.

LCS: PONIAL

			E	X	P	O	N	E	N	T	I	A	L
		0	1	2	3	4	5	6	7	8	9	10	11
	0	0	0	0	0	0	0	0	0	0	0	0	0
P	1	0	0	0	1	1	1	1	1	1	1	1	1
O	2	0	0	0	1	2	2	2	2	2	2	2	2
L	3	0	0	0	1	2	2	2	2	2	2	2	3
Y	4	0	0	0	1	2	2	2	2	2	2	2	3
N	5	0	0	0	1	2	3	3	3	3	3	3	3
O	6	0	0	0	1	2	3	3	3	3	3	3	3
M	7	0	0	0	1	2	3	3	3	3	3	3	3
I	8	0	0	0	1	2	3	3	3	3	4	4	4
A	9	0	0	0	1	2	3	3	3	3	4	5	4
L	10	0	0	0	1	2	3	3	3	3	4	5	6

2. Assume we are computing the LCS of S and T using the algorithm we developed in class (not the bad recursive algorithm).

a) **Find <u>and prove</u> a run-time Big-Oh for the LCS algorithm that computes the length of the LCS.**

```
LCS(s, t) {
   // Fill in row and column 0
   for j = 0 to s c[0][j] = 0; // O(s)
   for i = 0 to t C[i][0] = 0; // O(t)

   for i = 1 to s     // for each row i
      for j = 1 to t  // for each column j of I O(s*t)

            //do constant operation

}
```

The two initial for loops run (S + T) times, the for loop with the nested for loop runs (S * T) times. Thus this algorithm takes (S + T) + (S * T) or $2n+n^2$ which is $O(n^2)$

Proof:

$2n + n^2 \le C_1 n^2$

replace 2n with 2n[2]:

$2n^2 + n^2 \le C_1 n^2 = 3n^2 \le C_1 n^2$

set the constant = 3:

$3n^2 \le 3n^2$

b) **Find <u>and prove</u> a Big-Oh that describes the amount of memory that LCS uses.**

```
LCS(s, t) {
   // Fill in row and column 0
   for j = 0 to s c[0][j] = 0;
   for i = 0 to t C[i][0] = 0;

   for i = 1 to s    // for each row i
      for j = 1 to t  // for each column j of I O(s*t)

           //do constant operation

}
```

In the second pair of for loops, the algorithm fills a two dimensional array of size (S * T). Thus this algorithm takes up (S * T) space in memory or O(ST) space.

Proof:

$\mathrm{ST} \le C_1 \mathrm{ST}$

Set the constant = 1:

$\mathrm{ST} \le \mathrm{ST}$

3. Develop a version of LCS that uses O(n+m) space.

```
// The LCS algorithm only looks at the array position to the top
// of, and directly to the left of the current iteration in order
// to calculate the LCS.  This algorithm utilizes that fact in
// order to implemented an LCS algorithm that uses only the space
// of two arrays in memory - O(n+m).

LCS(m, n) {
   // Fill in the top array with 0s and position 0 of the bottom
   // array with a 0
   for (i = 0 to m.length + 1) { top[i] = 0; }
   bottom[0] = 0;

   for i = 1 to m.length    // for each row i
       for j = 1 to n.length { // for each column j of i

        // if the string chars are equal at the current iteration
        // set the current array position to 1 + the
        // left-diagonal array position
         if m[i] = n[j]
           bottom[i] = 1 + top[j-1];

        // if they are not equal, set the current array position
        // to the max of the left array position, and the
        // the adjacent top array position
         else
           bottom[j] = max(bottom[j-1], top[j])

       }

      // copy the bottom row to the top row and continue
      top = bottom;

   }

   // the algorithm is now complete, return the last position of
   // the bottom array
   return bottom[m.length];

} // end LCS
```

4. Develop a modified version of LCS that computes the *minimal* number of insert and delete editsneeded to convert a string S into a string T (rather than the length of the LCS of S and T).

```
//
// This modified version of LCS computes the minimal number of
// edits (inserts and deletes) required to change the input
// string s into the input string t
//
// First it fills the top row and left-most column with
// 0..s.length or 0..t.length
// Second it calculates the edit distance by going through each
// cell in the 2-dimensional array.
//
// It does this in two ways:
// If the chars of the two strings s&t are equal at the current
// iteration, set the current cell block to the key in the
// left-diagonal cell.
//
// If they are not equal, calculate the min of the position to
// the left of the current iteration + 1 and to the top of the
// current iteration + 1 and set the current cell block to the
// min.
//
// Then proceed to fill the rest of the table cells until it is full
//
editDistance(s, t) {

   // Fill in row and column 0 with 0..n

   // fill row 0 with 0..s.length
   for j = 0 to s.length C[0][j] = j;

   // fill column 0 with 0..t.length
   for i = 0 to t.length C[i][0] = i;

   for i = 1 to s.length  {  // for each row i
      for j = 1 to t.length { // for each column j of i

                // if the two strings are equal at this iteration
                if (s[i] == t[j])
                        C[i][j] = C[i-1][j-1];
                // if they are not equal, calculate the min of the
                // the top position + 1 or the left position + 1
                // then place the min in the current table cell
                else
                        C[i][j] = min(
                                        C[i][j-1]+1,
                                        C[i-1][j]+1
                                        );
        }
    }
   // the algorithm is complete, return the final position in
   // the table
   return C[s.length][t.length];
}
```

5. Modify the algorithm you developed in 4 to compute the *minimal* number of edits including the change operation needed to convert S into T. Trace your algorithm on “exponential” and “polynomial”.

```
//
// This modified version of the LCS based editDistance algorithm
// also accounts for the “change operation” required to change a
// string s into the string t.
//
// It is nearly exactly the same as the original algorithm with
// one key difference:
//
// If they are not equal, calculate the min of the position to
// the left of the current iteration + 1, to the top of the
// current iteration + 1, and to the diagonal-left of the current
// position + 1.  Then set the current cell block to the min.
//
editDistance(s, t) {

   // Fill in row and column 0 with 0..n

   // fill row 0 with 0..s.length
   for j = 0 to s.length C[0][j] = j;

   // fill column 0 with 0..t.length
   for i = 0 to t.length C[i][0] = i;

   for i = 1 to s.length  {  // for each row i
      for j = 1 to t.length { // for each column j of i

               // if the two strings are equal at this iteration
               if (s[i] == t[j])
                       C[i][j] = C[i-1][j-1];

               // Here is the key difference:
               // if they are not equal, calculate the min of the
               // the top position + 1, the left position + 1,
               // and the diagonal-left position + 1.  Then place the
               // min in the current table cell.
               else
                       C[i][j] = min(
                                       C[i][j-1]+1,
                                       C[i-1][j]+1,
                                       C[i-1][j-1]+1
                                       );
        }
    }
   // the algorithm is complete, return the final position in
   // the table
   return C[s.length][t.length];
}
```

After the first two for loops run:

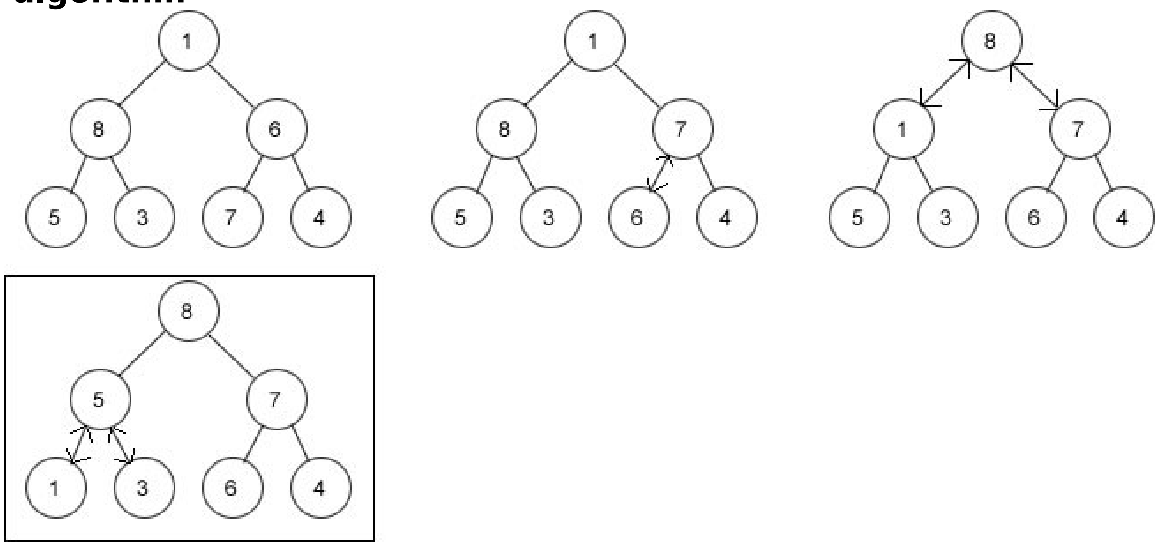
		e	**x**	**p**	**o**	**n**	**e**	**n**	**t**	**i**	**a**	**l**
	0	1	2	3	4	5	6	7	8	9	10	11
p	1											
o	2											
l	3											
y	4											
n	5											
o	6											
m	7											
i	8											
a	9											
l	10											

When the algorithm is complete:

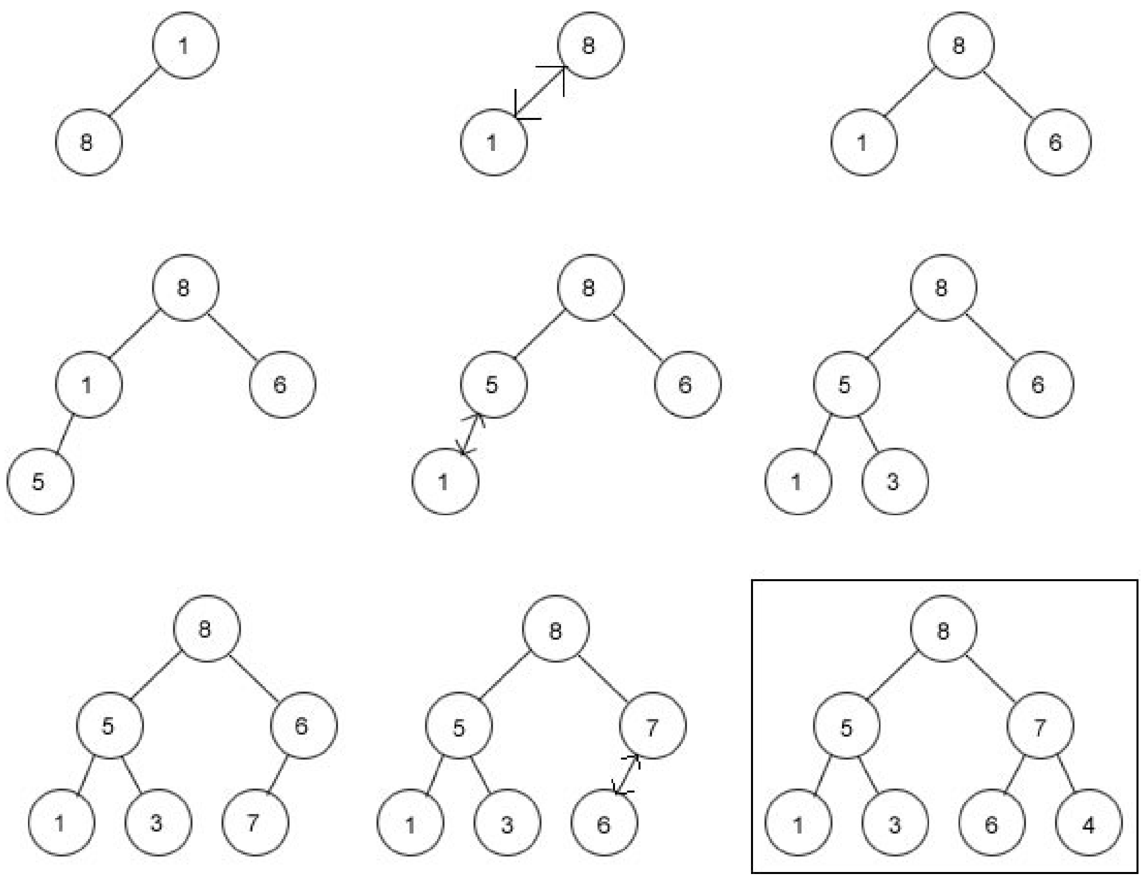
		e	**x**	**p**	**o**	**n**	**e**	**n**	**t**	**i**	**a**	**l**
	0	1	2	3	4	5	6	7	8	9	10	11
p	1	1	2	2	3	4	5	6	7	8	9	10
o	2	2	2	3	2	3	4	5	6	7	8	9
l	3	3	3	3	3	3	4	5	6	7	8	8
y	4	4	4	4	4	4	4	5	6	7	8	9
n	5	5	5	5	5	4	5	4	5	6	7	8
o	6	6	6	6	5	5	5	5	5	6	7	8
m	7	7	7	7	6	6	6	6	6	6	7	8
i	8	8	8	8	7	7	7	7	7	6	7	8
a	9	9	9	9	8	8	8	8	8	7	6	7
l	10	10	10	10	9	9	9	9	9	8	7	**6**

The edit distance between exponential and polynomial is 6.

6a. Construct a heap for the list1, 8, 6, 5, 3, 7, 4 by the bottom-up algorithm



6b. Construct a heap for the list 1, 8, 6, 5, 3, 7, 4 by successive key insertions (top-down algorithm)



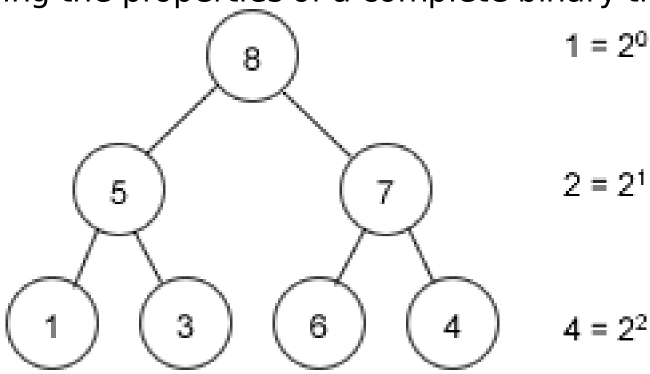
7. Outline an algorithm for checking whether an array H[1…n] is a heap and determine its time efficiency.

```
   // initially call isHeap with the array h and 1
   // precondition: the array has no value set, or an
   // abstract value set, in position 0

   isHeap(array h, index i) {

         // base case, if we've reached the end of the parents
      // without returning false, it is a heap, return true

         if(i > h.length/2) return true;

         // verify that i is >= its children

         // we must first verify that the right child of i exists
         if(h[i*2+1] != NULL) {

            // right child of i exists
            if(h[i] >= h[i*2] && h[i] >= h[i*2+1])
                 isHeap(h, i+1);
            else
                 return false; // not a heap
      }

      else {

            // right child of i does not exist
            if(h[i] >= h[i*2])
                 isHeap(h, i+1);
            else
                 return false; // not a heap
      }
} // end isHeap
```

The isHeap algorithm will recur, at most, n/2 times where n is the length of the array. Each recurrence is C time, n/2 is O(n). Thus this is a O(n) algorithm.

8b) Prove that the height of a heap with n nodes is equal to $\lfloor \log_2(n) \rfloor$

For i = the current depth in the heap
The max nodes per level $=2^i$
(using the properties of a complete binary tree)



By summation formula #5, Page 470 (Levitin), the max amount of nodes n in a tree of height h is equal to:

$$\sum_{i=0}^{h} 2^i = 2^{h+1} - 1 = n$$

Add one to the right side to make an inequality:

$$n < 2^{h+1}$$

2^h is less than or equal to n since 2^{h+1} is strictly greater than n

$$2^h \le n < 2^{h+1}$$

Take the log of all three:

$$\log(2^h) \le \log(n) < \log(2^{h+1})$$

By log law #3, Page 469 (Levitin):

$$\log(2^h) = h$$
$$\log(2^{h+1}) = h + 1$$

On a number line, h is less than or equal to log(n) and h+1 is strictly greater than log(n):

$$h \le \log(n) < h + 1$$

Taking the floor of log(n) makes log(n) = h

Thus: $h = \lfloor \log_2(n) \rfloor$

9. Prove the following equality:

$$\sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n+1)) \quad \text{where} \quad n = 2^{h+1} - 1$$

Express the right-hand-side as a function of h:

$$\sum_{i=0}^{h-1} 2(h-i)2^i = 2(2^{h+1} - 1 - \log_2(2^{h+1}))$$

Simplify the summation through distribution:

$$\sum_{i=0}^{h-1} 2(h-i)2^i = \sum_{i=0}^{h-1} (2h-2i)2^i = \sum_{i=0}^{h-1} 2h2^i - 2i2^i$$

By Sum Manipulation Rule #2, Page 470 (Levitin)

$$\sum_{i=0}^{h-1} 2h2^i - 2i2^i = \sum_{i=0}^{h-1} 2h2^i - \sum_{i=0}^{h-1} 2i2^i$$

By Sum Manipulation Rule #1, Page 470 (Levitin)

$$\sum_{i=0}^{h-1} 2h2^i = 2h\sum_{i=0}^{h-1} 2^i$$

By Summation Formula #6, Page 470 (Levitin)

$$\sum_{i=0}^{h-1} 2i2^i = 2\left[((h-2)2^h + 2)\right]$$

By Summation Formula #5, Page 470 (Levitin)

$$2h\sum_{i=0}^{h-1} 2^i = 2h(2^h - 1)$$

Thus:

$$\sum_{i=0}^{h-1} 2h2^i - \sum_{i=0}^{h-1} 2i2^i =$$

$$2h(2^h - 1) - 2\left[((h-2)2^h + 2)\right]$$

Simplify:

$$2h(2^h - 1) - 2\left[((h-2)2^h + 2)\right]$$

Factor out 2:

$$2\left[h(2^h - 1) - ((h-2)2^h + 2)\right]$$

Distribute:
$2(h2^h - h - h2^h + 2^{h+1} - 2)$

Cancel like terms:
$2(-h + 2^{h+1} - 2)$

We are done simplifying, put the simplified left term = the un-simplified right term
$2(-h + 2^{h+1} - 2) = 2(2^{h+1} - 1 - \log_2(2^{h+1}))$

By log law #3, Page 469 (Levitin):
$\log_2(2^{h+1}) = (h+1)\log(2)$

By log law #2, Page 469 (Levitin):
$\log_2(2) = 1$

Thus:
$\log_2(2^{h+1}) = h + 1$

Put h+1 back into the equation for $\log_2(2^{h+1})$:
$2(-h + 2^{h+1} - 2) = 2(2^{h+1} - 1 - (h+1))$

Distribute the negative on the right side:
$2(-h + 2^{h+1} - 2) = 2(2^{h+1} - 1 - h - 1)$

Combine like terms and reorder:
$2(2^{h+1} - h - 2) = 2(2^{h+1} - h - 2)$

The two sides are equal, thus $\sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n+1))$ is true.