Content

1. Statistics

a. **Handling statistical data**

10%

i. Broad classification of statistical data

- Identify numeric (quantitative) data, such as discrete and continuous (definition and examples) and non-numeric data, (qualitative), such as categorical and ordinal (definitions and examples); and
- State the types of statistical data based on collection process: primary data; definition, methods of collection (such as **mail questionnaire**; interview, observation and telephone); advantages and disadvantages of each method) and secondary data; definition, types (i.e. financial data, health data, oil and gas data, security data, political data); examples and sources of each).

ii. Sampling

- Explain the meaning of sampling, and state the definition of basic sampling terms, such as population, sample, sampling, unit, sampling fraction, sampling frame and sampling survey.
- Explain the purpose of sampling.
- State the method of sampling, such as probability sampling, (simple random, systematic, stratified, cluster and multi-stage); advantages and disadvantages of each method) and non-probability sampling (quota, judgemental, convenience and snowball); advantages and disadvantages of each method).

iii. Data presentation

- Explain tabulation of data including guidelines for constructing frequency tables and cross tabulation.
- State and draw the type of charts such as bar chart (simple, multiple component and percentage), and pie chart.
- State and draw the types of graph, such as histogram, frequency polygon, cumulative frequency curve (ogive),
- Explain the use of statistical application packages, such as Microsoft excel, statistical package for social sciences (SPSS).

b. **Measures of location (grouped and ungrouped data)**

5%

i. State and calculate the measures of central tendency/centre, such as:

- Arithmetic mean;
- Unimodal mode;
- Median; and
- The relationship among the three measures;
- The characteristics/features of each of the above measures;

ii. State and calculate the measures of partitions such as quartiles, deciles and percentiles;

iii. Estimate the mode from histogram; median, quartiles, deciles and percentiles, all from ogive.

c. **Measures of variation/dispersion/spread (grouped and ungrouped data) 5%**

Explain and calculate range, mean deviation, variance, standard deviation, coefficient of variation, quartile deviation and skewness.

d. **Measures of relationship 5%**

i. Correlation

- Explain the meaning of correlation and state its uses.
- Explain the nature of scatter diagrams/plots.
- Draw scatter diagrams/plots.
- State the nature and type of correlation, such as positive, perfect positive, negative, perfect negative and zero/non-correlated.
- Explain the meaning of Correlation coefficient and state its types, such as spearman's rank and Pearson's product moment.
- Calculate any of the above listed correlation coefficient type.

ii. Regression analysis

- Explain the meaning and uses of regression analysis for estimation/prediction/forecasting purposes.
- State the methods of fitting simple linear regression line, such as best line of fit and least squares.
- Determine the simple linear regression line and use it for prediction.
- Interpret regression constant/intercept and regression coefficient/slope/gradient.

e. Times series 5%

i. Explain the meaning of time series and its application to forecasting.

ii. State the basic components of time series, such as secular/trend, seasonal, irregular and cyclical.

iii. State the models of time series, such as additive and multiplicative.

iv. State the methods of constructing trend line, such as eye fitting, semi-averages, moving averages and least squares.

v. State the methods of determining seasonal indices such as moving average and smoothening.

vi. Determine adjusted seasonal variation.

f. **Index numbers**
2%

i. Explain the meaning of index numbers and state their uses.
ii. State the problems associated with the construction of index numbers.
iii. Calculate the simple aggregate index, mean of price relatives as unweighted indices.
iv. Calculate the Laspeyre, Paasche, Fisher and Marshall Edgeworth as weighted indices.

g**.** **Probability**
4%

i. Define probability and its basic terms, such as random experiment, sample space, event and probability of an event.
ii. Explain the types of probability events, such as mutually exclusive, independent and conditional/dependent events.
iii. State the addition and multiplication laws as applied to the above events.
iv. Explain and determine the expected values.

h. **Test of hypothesis**
4%

i. Explain the meaning of hypothesis and state the types of hypothesis such as null and alternative.
ii. State and explain the types of probable decision error, such as types I error and type II error.
iii. Explain the useful concepts in the test of hypothesis such as level of significance, test statistic, critical region, one-tailed/sided and two-tailed/sided tests.
iv. Explain the meaning of test of hypothesis about single population mean for small and large samples and single proportion.
v. Test the hypothesis about single population mean for small and large samples and single proportion at appropriate level of significance.
vi. Explain the meaning of test of hypothesis about the difference between two population means for small and large samples and the difference between two population proportions.
vii. Test the hypothesis about the difference between two population means for small and large samples and the difference between two population proportions at appropriate level of significance.

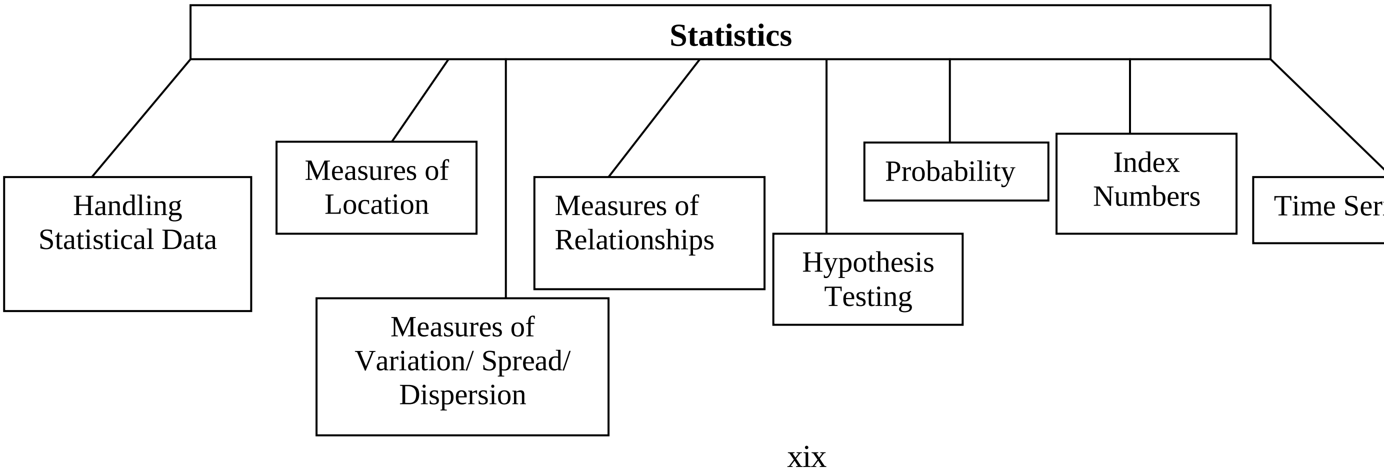


Fig 2: Relational diagram of the main capabilities of the Statistics section of the QA syllabus

2. Business Mathematics

a **Profit and loss based on sales**

2%

i. Explain and calculate selling price, cost price, profit/loss, and profit percent/loss percent.
ii. Explain and calculate discounting.
iii. Explain and calculate marked price.

b**.** **Set theory as applied to business**

2%

i. Explain the relevant terms of set theory such as universal set, element/ member of a set, empty/null set, complement of a set, union of sets, intersection of sets; and
ii. Apply the concept of set theory to business-oriented problems involving at most 3 sets using Euler-Venn diagram.

c**.** **Functional relationships**

3%

i. Define function and state its definition, types (linear, quadratic, polynomial and exponential) and solutions of their equations including graphical treatment.
ii. Apply the concept of functional relationship to cost function, revenue function, profit function, break-even analysis to determining break-even point in quantity and value) and its interpretation.
iii. Explain the simple linear inequalities including graphical approach.

d. **Mathematics of finance**

7%

i. Apply sequences and series (limited to arithmetic and geometric progressions), sum to infinity of a geometric progression; application to business concerns:
ii. Apply simple and compound interests to present value of single amount, present value of series amounts, annuities such as ordinary annuity and annuity due, sum of an ordinary annuity that is sinking fund, present value of an annuity and amortisation to business concerns; and
iii. Determine the net present value (NPV) and internal rate of return (IRR) with their interpretations.

e. **Differentiation**
3%

i. Explain the meaning of slope or gradient or derivative, and state the rules for differentiating polynomials in one variable:
ii. Apply differentiation to finding marginals, minimum and maximum values, elasticity of price and quantity.

f. **Integration**
3%

i. State the rules for integrating polynomial in one variable as a reverse of differentiation.
ii. Apply integration to finding functions such as cost and revenue functions from marginal functions, determination of consumers' and producers' surpluses.

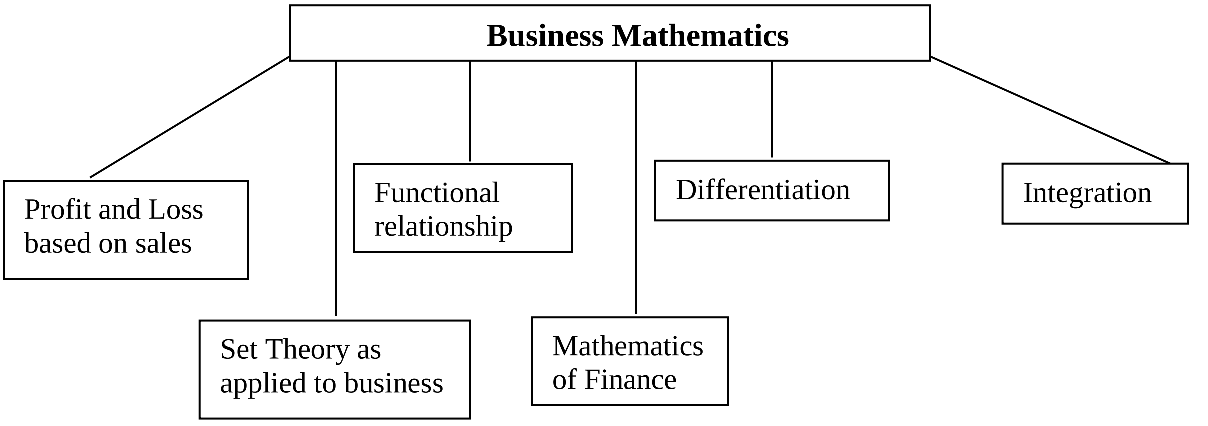


Fig 3: Relational diagram of the main capabilities of the business mathematics section of the QA Syllabus.

3. Operations research

a. **Introduction** **2%**

State the main stages of operations research (OR) and explain its relevance in business.

b. **Linear programming (LP)**
10%

i. Explain the concept, meaning and state its underlying basic assumptions.
ii. State the processes of problems formulation in LP.
iii. State and explain the methods of solving a typical LP problem such as graphical method (for 2 decision variables only); and Simplex method (in three decision variables only).
iv. Use the above stated two methods to maximise profit or minimise cost.
v. Explain the meaning of shadow price and dual/shadow cost and their calculations.

c. **Inventory and production control**
5%

i. Explain the meaning and functions of an inventory.
ii. Explain and calculate inventory cost such as holding cost, ordering cost, shortage cost and cost of materials.
iii. State and explain the general inventory models such as deterministic and stochastic models.
iv. Explain the periodic review system and re-order level system (limited to one channel).
v. Explain and calculate the basic economic order quantity (EOQ) and state its assumptions.

d. Network analysis
6%

i. Explain the critical path analysis (CPA) and Programme Evaluation and Review Technique (PERT).
ii. Draw the network diagram based on activity-on-arrow (AOA) concept including the introduction of dummy (where applicable) in the network diagram.
iii. Explain and determine the critical path and the associated duration.
iv. Explain and calculate the floats such as free, independent and total.

e. Replacement analysis
4%

i Explain the concept of replacement of items that wear gradually.
ii. Explain the concept of replacement of items that fail suddenly.
iii. Calculate the cost of group replacement.
iv. Determine the best interval period among group replacements and
v. Determine when and how best equipment can be replaced in order to minimise the total cost of maintaining them.

f. **Transportation and assignment models**
8%

i. Explain the nature of transportation and assignment models.
ii. Explain the concept of balanced transportation problems and unbalanced transportation problems involving dummy (vertical or horizontal).
iii State and explain the methods for calculating initial basic feasible transportation cost, such as North-west corner method (NWCM), least cost method (LCM) and vogel's approximation method (VAM).
iv. Calculate the minimum transportation cost using the above stated methods.
v. Explain the use of Hungarian method for solving assignment problems.
vi Calculate the best allocation of an assignment problem, using the above stated methods.

g. **Simulation**
5%

i. Explain the meaning of simulation and state the applications of simulation technique to business – oriented situations that is, the imitation of the operation of a real-world process of system over time.
ii. Explain the Monte Carlo method of solution involving the use of probabilities to assign random number ranges in construction and running of simple simulation.
iii. Use probabilities to assign a random number range.
iv. Use Monte Carlo method to construct and run simple simulations.

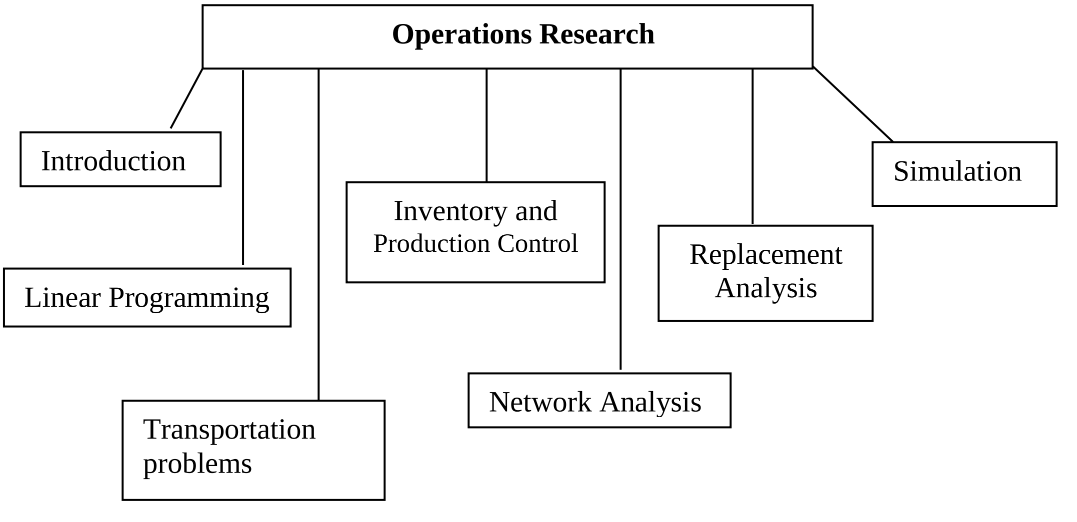


Fig 4: Relational Diagram of Operations Research Section main capabilities of the QA syllabus.

Recommended Texts

1. ATSWA **Study Pack on Quantitative Analysis**
2. *Adamu, S .O. and Johnson, T.L. (1983),* **STATISTICS FOR BEGINNERS** Lagos: Evans, Nigeria.
3. Funk J. (1980). **BUSINESS MATHEMATICS. BOSSON ALLYN** and Bacon Inc.
4. *Lapin, L. (1994)*: **QUANTITATIVE METHODS FOR BUSINESS DECISIONS** (6th EDITION). New York : The Dryden Press.
5. *Lucey, T (2002)*: **QUANTITATIVE TECHNIQUES,** London: ELST/Continuum.

SECTION A

STATISTICS

CHAPTER ONE

HANDLING STATISTICAL DATA

Chapter contents

(a) Introduction.

(b) Broad classification of statistical data.

(c) Types of statistical data based on collection process.

(d) Data presentation.

(e) The use of statistical application packages for data presentation.

(f) Sampling.

Objectives

At the end of the chapter, readers should be able to understand the

a) meaning of statistical data;

b) types of data, how to collect and classify them;

c) concept of data presentation in form of charts and graphs;

d) use of statistical packages for data presentation; and

e) various methods of sampling.

1.1 Introduction

Information is the key needed for a smooth running of an organization or a country. A piece of information or raw facts in either numerical or non-numerical form that are collected, analysed and summarised for presentation and interpretation purposes, are known as data or better still statistical data. Handling statistical data can sometimes be called statistics because it involves the study of the theory and methods in collection, analysis, interpretation and utilization of the results for a set of data. Data are the basic raw facts needed for statistical investigations. These investigations may be needed for planning, policy implementation, and for other purposes.

Statistics is a scientific method that concerns data collection, presentation, analysis, interpretation or inference about data when issues of uncertainties are involved. Definitely, statistics is useful and needed in any human area of endeavour where decision making is of vital importance. Hence, it is useful in Accountancy, Engineering, Education, Business, social Sciences, Law, Agriculture, to mention a few.

Statistics can be broadly classified into two: (i) Descriptive and (ii) Inferential.

Descriptive statistics deals with data collection, summarizing and comparing numerical data, i.e. descriptive statistics is concerned with summarizing a data set rather than using the data to learn about the population, while inferential statistics deals with techniques and tools for collection of data from a population. These data are then studied by taking a sample from the population. By doing this, knowledge of the population characteristics is gained and vital decisions are made about the sampled population.

1.2 Broad classification of statistical data

In view of the above introduction, statistical data can broadly be classified into two namely: numeric and non-numerical data

(a) Numerical data

These are data whose values can be quantified or data that assumes numerical values e.g. number of bank staff, number of candidates that registered for a particular examination diet, ages of individual, height of a person etc. numeric data are sometimes known as quantitative data and can be further classified into two namely: discrete and continuous

(i) Discrete data

These are numeric data that consist of numbers whose values are integers i.e. whole numbers which can be negative, zero or positive. What this implies is that decimal or fractional value cannot be found in a discrete data e.g. number of students in a class, number of typists in an organization etc.

(ii) Continuous data

These are numeric data that consist of numbers whose values can be integer, fraction or decimal which can be negative, zero or positive e.g. wages of workers in a firm, prices of goods and services, weights of employees in a company, marks of candidates in an examination etc.

(b) Non – numeric data

These are data whose values cannot be quantified or data that cannot assume numerical values e.g. gender, marital status, state of origin, boxer's weight, income group, and social – economic status etc. Non – numeric data are sometimes known as quantitative data and can be classified into two as well namely: ordinal or categorical

(i) Ordinal data

These are non – numeric data that can be put on an ordinal scale i.e. data that uses a particular yardstick or condition for its group ranking e.g. Age group, rank of a soldier, social-economic status of individual etc.

(ii) Categorical data

These are non-numeric data that cannot be put on an ordinal scale. The data set is in categorical form where the facts collected are based on classification by group or category and, also known as Nominal data e.g. Gender, Nationality, State of origin, Types of religion, in political affiliation etc.

1.3 Types of statistical data based on collection process

Statistical data can be classified into two, based on collection process, namely: primary and secondary data.

(a) Primary data

These are data that are obtained by direct collection from respondents/informants or through one – on-one interaction. Primary data have different means of collecting them. Also, primary data are collected from a planned experiment that have relevant objectives to the statistical investigation through direct observation or through the conduct of sample survey. Primary data are always collected specifically for the purpose for which the investigation is carried out.

Advantages

i. It allows for detailed and accurate information to be collected.

ii. The methods used and the level of accuracy are known.

iii. It is reliable.

Disadvantages

i. It is time consuming.

ii. It is expensive.

Methods of collection

The major methods of data collection are: The Interview, Mail Questionnaire, Observation, Documents/Reports and Experimental methods.

Interview Method

This is a method in which the medium of data collection is through conversation by face-to-face contact or through a medium such as telephone. In general, the interview method can be accomplished by: (a) Schedule; (b) Telephone; and (c) Group discussion.

(a) **Interview schedule:** This is the use of schedule by the interviewer to obtain necessary facts/data. A schedule is a form or document which consists of a set of questions to be completed by interviewer as he/she asks the respondent questions.

Advantages

i. Pieces of information provided are reliable and accurate;

ii. Non-response problem is drastically reduced;

iii. Questions not understood by the respondents may be reframed; **and**

iv. Difficult respondent can be persuaded.

Disadvantages

i. It is time consuming.

ii. It is expensive.

(b) **Telephone interview:** Here**,** the questions are asked through the use of telephone in order to get the needed data. The respondents and interviewers must have access to telephone.

Advantages

iv. It is fast;

v. Uncooperative respondents can be persuaded;

vi. Call-backs are faster; and

vii. Problem of non-response is reduced.

Disadvantages

i. Data collected is biased towards those who own telephone;

ii. Respondents without telephone cannot be reached; and

iii. Data collected may not be reliable.

(c) **Interview by Group discussion:** The interview is conducted with more than one person with a focus on a particular event.

Mail Questionnaire Method

A questionnaire is a document that consists of a set of questions which are logically arranged and are to be filled by the respondent himself. The interviewers send the copies of the questionnaire by post to be filled by respondents.

The types of questions in a questionnaire can be classified into two; namely:

a. Close- ended or coded questions are the types of questions in which alternative answers are given for the respondent to pick one; and
b. Open-ended or uncoded questions are the types of questions in which the respondent is free to give his own answers and is not restricted to some particular answers.

In drafting a questionnaire, the following qualities are essentials to note:

i. A questionnaire must be well structured so that major sections are available. The first section usually deals with personal data while other sections consider necessary and relevant questions on the subject matter of investigation;
ii. A questionnaire must be clear in language, not ambiguous and not lengthy;
iii. A question should not be a leading question;
iv. Questions should not require any calculation to be made;
v. Questions should neither be offensive nor resentive; and
vi. Questions should be arranged in a logical order.

Advantages

i. Wide area can be covered;
ii. It is cheap; and
iii. It saves a lot of time.

Disadvantages

i. Postal services are unreliable;
ii. Problem of non-response; and
iii. Information given may be unreliable.

Observation Method

This method enables data to be collected on behaviour, skills etc. of persons, objects that can be observed in their natural ways. Observation method can either be of a controlled or uncontrolled type. It is controlled when issues to be observed are predetermined by rules and procedures; while otherwise, it is termed uncontrolled type.

Experimental Method

Here, experiments are carried out in order to get the necessary data for the desired research. There are occasions when some factors are to be controlled and **some are** not controlled in this method. The method is mostly used in the sciences and engineering.

Documents/Reports

This method allows the existing documents/reports to be checked from an organization.

(a) Secondary data

These are data obtained, collected, or extracted from already existing records or sources. They are derived from existing published or unpublished records of government agencies, trade associations, research bureaus, magazines and individual research work. They are data that can be obtained from data collection agencies which can be for both research and planning purposes.

The following are some of the well - known data collection agencies:

(i) National Bureau of Statistics (NBS);
(ii) Central Bank Of Nigeria (CBN);

(iii) Educational Institutions;

(iv) Ministries; and

(v) Commercial/Private individual companies.

Table showing some types of secondary data with examples and sources

	Types of statistical data	Examples	Sources
a.	Financial data	i. List of commercial banks operating in the country; ii. List of Chartered Accountants in Nigeria; and iii. Exchange rates and interest rates.	i. Central Bank of Nigeria (CBN); ii. ICAN; and iii. Commercial banks.
b.	Health data	i. List of public hospitals; ii. List of medical doctors in the country; and iii. Number of people suffering from COVID-19	i. Federal & State ministries of Health; ii. World health organization (WHO); and iii. National Centre for Disease Control (NCDC).
c.	Oil and Gas data	i. Number of refineries in the country; ii. List of registered petroleum marketers; and iii. List of crude oil exporting countries.	i. Nigeria National Petroleum Corporation (NNPC) ii. Federal of Ministry of Petroleum Resources; and iii. OPEC.
d.	Security data	i. Number of Police Men &Women in the country; ii. Number of Correctional Centers in the country; and iii. List of Military Personnels.	i. Nigeria Police Force; ii. Ministry of Defence; iii. NDA; and iv. Federal Ministry of Interior.
e.	Political data	i. Number of political parties; ii. Number of registered voters; and iii. Number of national law makers.	i. INEC; ii. Federal Government of Nigeria; and iii. National & State Assemblies.
f.	Educational data	i. List of public & private universities; ii. Number of admitted candidates into higher	i. NUC; ii. JAMB; iii. WAEC; and iv. NECO.

		institution in the country; and iii. Number of students writing SSCE annually.	
g.	Economic data	i. The country's Gross Domestic Product; ii. Rate of inflation in the country; and iii. Country's national income & expenditure.	i. Federal of Ministry of Economic Affairs; ii. National Bureau of Statistics (NBS); and iii. Federal Ministry of Budget Planning.

Advantages

i. It is readily available;

ii. It is not expensive: and

iii. It is not time consuming.

Disadvantages

i. It may not be reliable;

ii. It may be misleading; and

iii. It may not be accurate.

Internal and External Sources Data

It will be of great benefit to know that data source can either be internal or external. Internal sources are the data collected within the organization. Such data are used within the organization. Examples are the sales receipts, invoice and work schedule. For the external source, data are collected outside the organization. The data are generated or collected from any other places which are external to the user.

1.4 Data presentation

Statistical data are organised and classified into groups before they are presented for analysis. Four important bases of classification are:

(a) Qualitative – By type or quality of items under consideration;

(b) Quantitative – By range specified in quantities;

(c) Chronological – Time series – Monthly or Yearly: An analysis of time series involving a consideration of trend, cyclical, periodic and irregular movements; and

d) Geographical – By location.

The classified data are then presented in one of the following three methods:

(a) Text presentation; (b) Tabular presentation; and (c) Diagrammatic presentation:

Text Presentation

This is a procedure by which texts and figures are combined. It is usually a report in which much emphasis is placed on the figures being discussed.

For instance, a text presentation can be presented as follows:

The populations of science and management students are 3,000 and 5,000 respectively for year 2006 in a Polytechnic in Ghana.

Tabular presentation

A table is more detailed than the information in the text presentation. It is brief and self-explanatory. A number of tables dealt with in statistical analysis are general reference table, summary table, Time series table, frequency table.

Tables may be simple or complex. A simple table relates a single set of items such as the dependent variable against another single set of items such as the independent variable. A complex table, on the other hand, has a number of items presented and often shows sub-divisions.

Essential features of a table are:

- A title to give adequate information about it;
- Heading for identification of the rows and columns;
- Source i.e. the origin of the figures; and
- Footnote to give some detailed information on some figures in the table.

Example 1.1 (A typical example of simple table)

Classification of two hundred Polytechnic students on departmental basis

Department	**No of Students**
Accountancy	60
Business Administration	50
Marketing	40
Banking / Finance	50
Total	**200**

Example 1.2 **(A Typical Example of Complex Table)**

Departmental classification of 200 University students on the basis of gender

Department	**No of Students**		Total
	Male	**Female**	
Accountancy	40	20	60
Business Administration	36	14	50
Marketing	30	10	40
Banking / Finance	24	26	50
Total	**130**	**70**	**200**

Formulation of Frequency Table

A frequency table is a table showing the number of times a value (figure) or group of value (figure) has occurred in a given set of data.

It can be ungrouped or grouped.

- **Ungrouped frequency table**

  This shows the figure (value) in one column and the number of times (frequency) it has occurred in the given data.

Example 1.3

The marks scored by 20 candidates out of 10 marks in a quiz competition were as follows:

6,	7,	4,	5,	6,	4,	5,	8,	7,	6,
9,	8,	4,	6,	5,	7,	6,	5,	8,	7,

Obtain the frequency distribution for the data solution

Solution

Mark	**No. Of candidates (frequency)**
4	3
5	4
6	5
7	4
8	3
9	1

- **Grouped table:**

Guidelines for constructing a grouped frequency table

viii. The number of class intervals (or classes) should not be too few or too many (say 5 to 8 classes);

ix. The class width should be 5 or multiple of 5 to allow for easy manipulation;

x. The classes should generally be of the same width except where there are extreme values when the opening and closing classes may be wider to take care of the extreme values;

xi. The classes must be such that each observation will have a distinct class. Classes must not be of the types 5 – 10, 10 – 15, etc. These will create a problem of which class 10 belongs;

xii. Class intervals could be of the type 1 – 5, 6 – 10, 11 – 20, 21 – 20, etc.', 10 but less than 20, 20 but less than 30, etc. The size of observations combined with (i) above could determine the width of the classes;

xiii. Open-ended classes (e.g. less than 20, 10 and above) are assumed to have the same width as adjacent classes.

Usually, Tally Method is used to construct a frequency table especially when there are many observations (figures).

A tally is a stroke (|) drawn for each occurrence of an observation. The fifth stroke (tally) is drawn across the first four strokes (~~||||~~); this allows for easy counting.

The great advantage of the tally method is that you go through the data only once. It removes the confusion which may arise when each observation is counted throughout the data.

Example 1.4

The daily sales figure (₦'000) of a supermarket for 40 days were as follows:

12,	33,	23,	48,	56,	18,	22,	55,	57,	35,
45,	28,	36,	44,	17,	39,	58,	25,	31,	48,
26,	32,	45,	24,	35,	47,	56,	33,	27,	31,
34,	19,	21,	35,	41,	32,	45,	37,	29,	49,

You are required to:

Use class intervals of 11 – 20; 21 – 30; etc., to construct a frequency table for the sales.

Solution

Class Intervals	Tally	Frequency
11 – 20	$\cancel{\vert\vert\vert\vert}$	5
21 – 30	$\cancel{\vert\vert\vert\vert}$ $\vert\vert\vert\vert$	9
31 – 40	$\cancel{\vert\vert\vert\vert}$ $\cancel{\vert\vert\vert\vert}$ $\vert\vert\vert\vert$	14
41 – 50	$\cancel{\vert\vert\vert\vert}$ $\vert\vert$	7
51 – 60	$\cancel{\vert\vert\vert\vert}$	5

NOTE:

(a) Class Limits

The first number in a class is called the **lower class limit** while the second number is the **upper class limit**.

e.g. lower class limit of the 3^{rd} class is 31 while its upper class limit is 40

(b) Class Boundaries

i. The lower class boundary of a class is the sum (addition) of the upper class limit of the preceding class and its lower limit divided by 2.

e.g. the lower class boundary of the 2^{nd} class is:

[20 (upper class limit of 1^{st} class) + 21 (its lower class limit)] ÷ 2

$$= \frac{41}{2} = 20.5$$

ii. The upper class boundary of a class is the sum of its upper limit and the lower limit of the succeeding class divided by 2

e.g. the upper class boundary of the 2^{nd} class is:

[30 (its upper limit) + 31 (lower limit of 3^{rd} class)] ÷ 2

$$= \frac{30+31}{2} = 30.5$$

Consequently, the lower class boundary of a class is the upper class boundary of the class preceding it. Class boundaries are used to draw histograms and ogives.

(c) **Class Size (Width)**

The width of a class is the difference between its boundaries.

e.g. The width of the 2^{nd} class is 30.5 – 20.5 = 10

The lower boundary for 1^{st} class and upper boundary for last class are obtained by logic

(d) **Class Mid-point**

The mid-point of a class is the sum of its limits divided by 2.

e.g. the mid-point of the 4^{th} class is $\frac{41+50}{2} = 45.5$

Note that the mid-point of a class is the sum of the mid –point of the preceding class and the class width.

Class marks are used to represent class intervals in the calculation of statistical measures. They are also used in drawing the frequency polygon

Cumulative Frequency Table

A cumulative frequency table is a table showing the sum of frequencies of all classes before a particular class and the frequency of that class.

Example 1.5

Obtain the cumulative frequency table for the frequency table below:

Class Intervals	**Frequency**
11 – 20	5
21 – 30	9
31 – 40	14
41 – 50	7
51 – 60	5

Solution

Class Intervals	**Frequency**	**Cumulative frequency**
11 – 20	5	5
21 – 30	9	5 + 9 = 14
31 – 40	14	14 + 14 = 28
41 – 50	7	28 + 7 = 35
51 – 60	5	35 + 5 = 40

Diagrammatic Presentation:

Diagrams are used to reflect the relationship, trends and comparisons among variables presented on a table. The diagrams are in form of charts and graphs.

(a) Charts

i. Bar charts

A bar chart is a chart where rectangular bars represent the information. The bars must be of equal width with heights or lengths proportional to the values which they represent. The bars can be plotted vertically (usually or horizontally and can take various forms as discussed below.

- Simple bar chart;
- component bar chart;
- Percentage component bar charts; and
- Multiple bar charts**.**

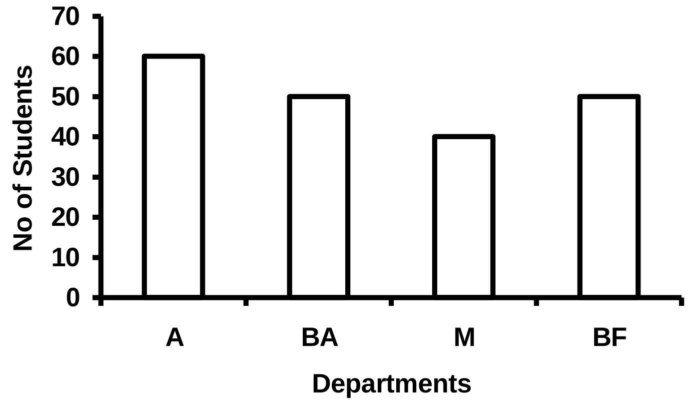
- **Simple Bar Chart**

This consists of a series of bars with the same width while the height of each bar indicates the size of the value it represents.

Example 1.6

Draw simple bar chart for the table below:

Department	No of Students
Accountancy	60
Business Administration	50
Marketing	40
Banking / Finance	50
Total	**200**



Departments	**Key**
Accountancy	A
Business Admin	BA
Marketing	M
Banking/Finance	B/F

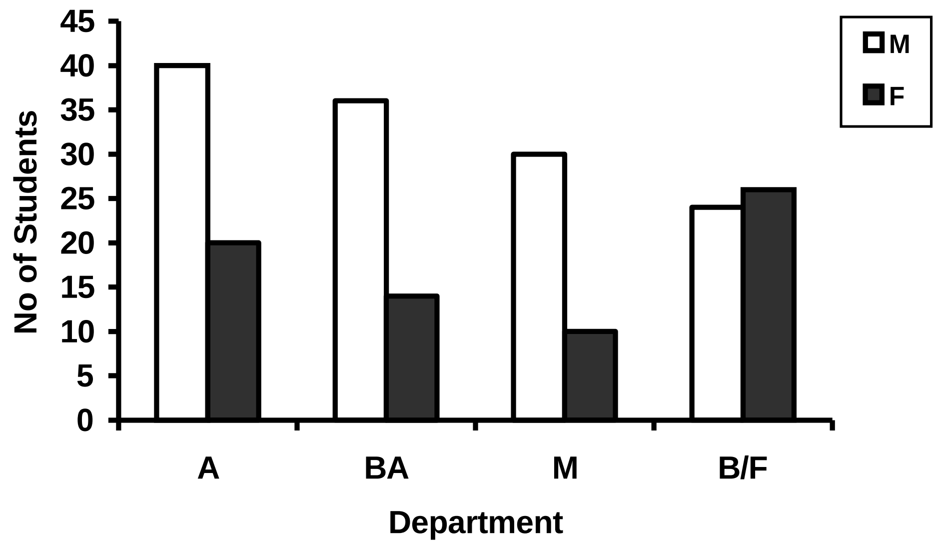
- **Multiple Bar Chart**

  This shows each group as separate bars beside each other.

Example 1.7

Draw Multiple bar chart for the following table:

Department	No of Students	
	Male	**Female**
Accountancy	40	20
Business Administration	36	14
Marketing	30	10
Banking / Finance	24	26

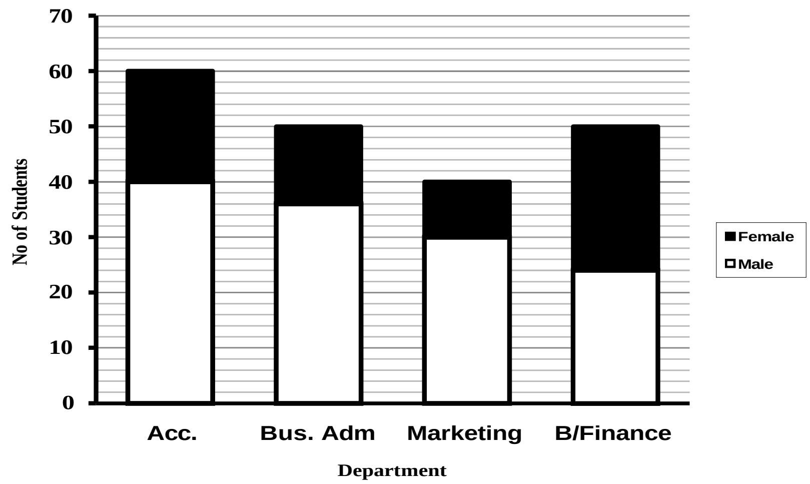


- **Component Bar Chart**

  This is similar to simple bar chart. The heights of each bar are divided into component parts.

Example 1.8
Draw component bar chart for the following table

Department	No of Students		TOTAL
	Male	Female	
Accountancy	40	20	60
Business Administration	36	14	50
Marketing	30	10	40
Banking / Finance	24	26	50



- **Percentage Component Bar Chart**

  Here, the bars are of the same height (100%). Each bar is divided into percentages which each component represents within a group.

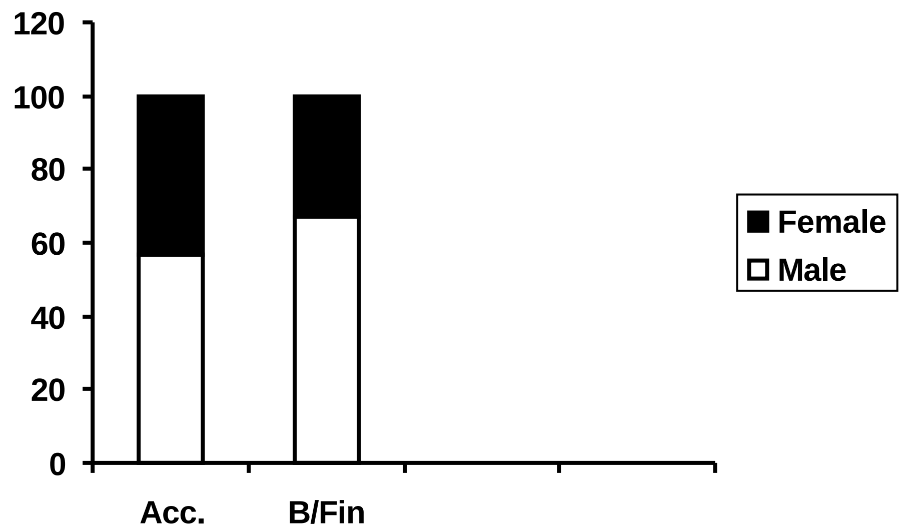
Example 1.9

Draw the percentage component bar chart for the table below:

	Accountancy	**Banking/Finance**
Male	40	20
Female	30	10

Solution

	Accountancy	%	Banking/ Finance	%	Total
Male	40	57%	20	67%	60
Female	30	43%	10	33%	40
	70	100%	30	100%	100



ii. **Pie Chart**

A pie chart is a circular chart which is divided into sectors. Each sectorial angle represents the parts in degrees.

Example 1.10

Draw Pie chart for the table. Classification of two hundred polytechnic students on departmental basis

Department	**No of Students**
Accountancy	60
Business Administration	50
Marketing	40
Banking / Finance	50

Solution

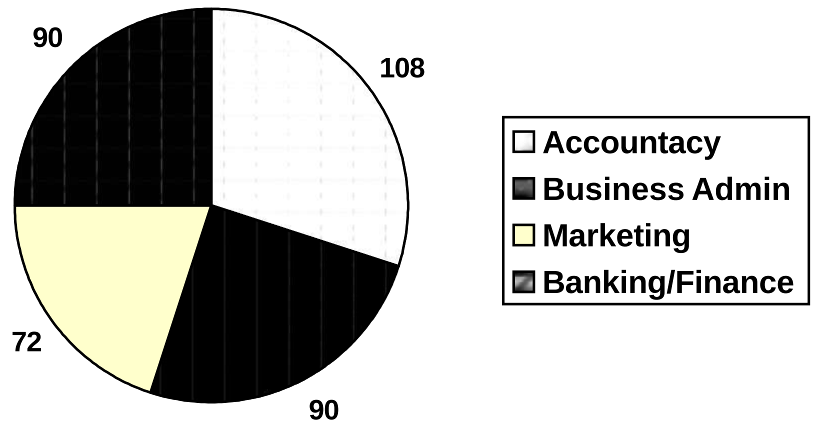
Calculation of angles

Total no of students is 200 and the sum of angles in a circle is $360°$

-

$\therefore$ For Accountancy, the corresponding angle	$=\frac{60}{200}\times\frac{360^\circ}{1}=108^\circ$
For Business Administration, the corresponding angle	$=\frac{50}{200}\times\frac{360^\circ}{1}=90^\circ$
For Marketing, the corresponding angle	$=\frac{40}{200}\times\frac{360^\circ}{1}=72^\circ$
For Banking/Finance, the corresponding angle	$=\frac{50}{200}\times\frac{360^\circ}{1}=90^\circ$
	Check: $108^\circ+90^\circ+72^\circ+90^\circ=360^\circ$

Pie chart



(b) Graphs

A graph shows relationship between variables concerned by means of a curve or a straight line. A graph will, for example, show the relationship between output and cost, or the amount of sales to the time the sales were made.

Typical graphs used in business are the histogram, frequency polygon and cumulative frequency curve (ogive).

(i) Histogram

A histogram consists of rectangles drawn to represent a group frequency distribution. This is similar to a bar chart but

- the rectangles must touch each other (continuous); and
- The frequency of a class is represented by the area of the corresponding rectangle (not its height as in the bar chart**).**

If the class sizes are not equal, the frequency of a class with different class size must be adjusted as follows:

- Choose the size common to most of the classes

  Adjust the other frequency as follows:
- Common size divided by the size multiplied by the frequency.
  e.g. if the size is double that of the common size, its frequency is divided by 2

Example 1.11

Draw the histogram for the frequency table of sales figures given below:

Class interval (Sales)	Frequency (No. of days)
11 – 20	5
21 – 30	9
31 – 40	14
41 – 50	7
51 – 60	5

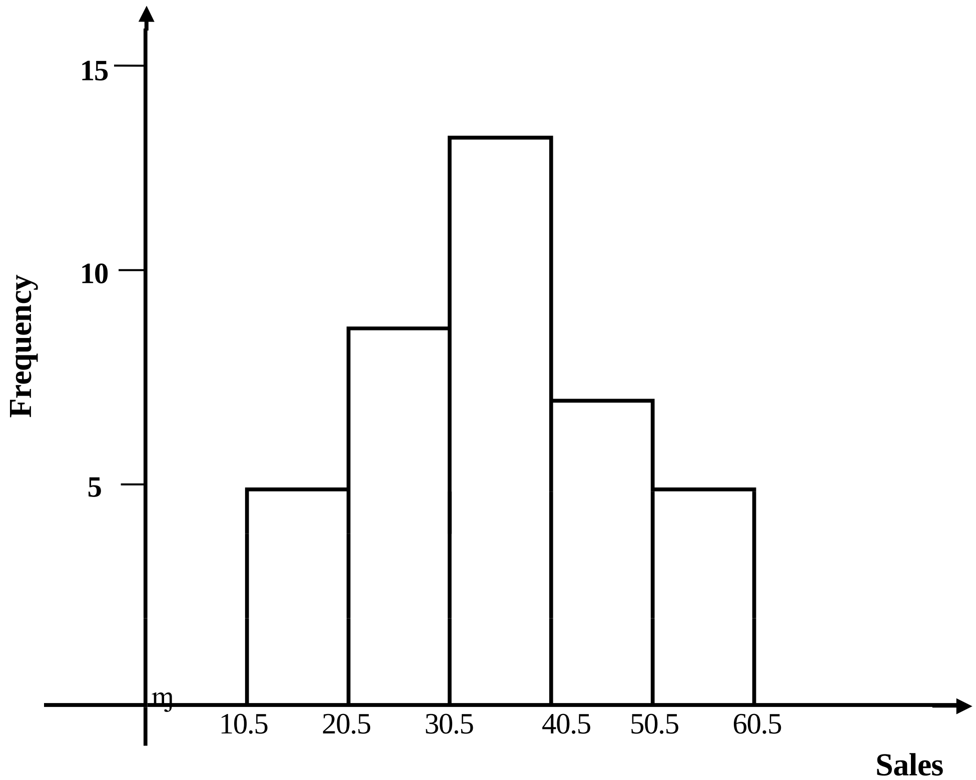
Solution

Since the rectangles in a histogram must be continuous (touch each other), the class intervals are written using the class boundaries thus:

Class interval (Sales)	Frequency (No. of days)
10.5 – 20.5	5
20.5 – 30.5	9
30.5 – 40.5	14
40.5 – 50.5	7
50.5 – 60.5	5

The histogram can be used to estimate the modal value.

Histogram



ʓ indicates that the horizontal scale does not start from 0 (zero)

(ii) Frequency Polygon

A frequency polygon is the graph of frequencies against class marks.

If the histogram to a frequency table has been drawn, the frequency polygon is obtained by joining the mid-points of the top of the rectangles.

The polygon is closed up by joining to the class mark of the class before the first and after the last class intervals with zero frequencies.

Example 1.12

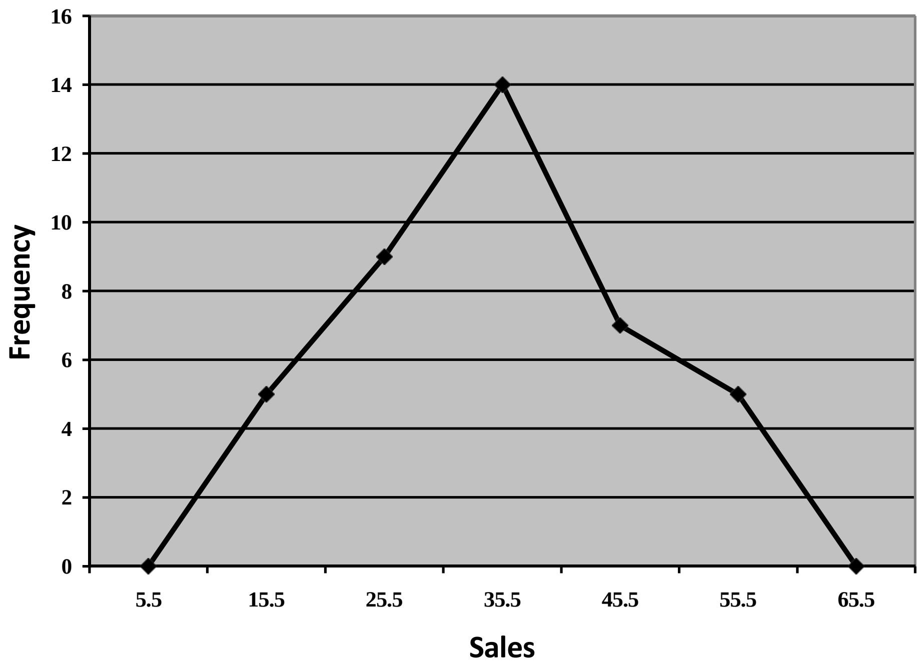
Draw the frequency polygon for the table below:

Class interval (Sales)	Frequency (No. of days)
11 – 20	5
21 – 30	9
31 – 40	14
41 – 50	7
51 – 60	5

Solution

Class interval (Sales)	Class Mark	Frequency (No. of days)
1 – 10	5.5	0
11 – 20	15.5	5
21 – 30	25.5	9
31 – 40	35.5	14
41 – 50	45.5	7
51 – 60	55.5	5
61 – 70	65.5	0

Frequency Polygon



The points are joined with a straight edge but when it is smoothened out it becomes a frequency curve which shows the shape of the frequency distribution.

(iii) **Ogive**

An ogive is a graph of cumulative frequencies against class boundaries. It is also referred to as the cumulative frequency curve. It could be 'less than' type or 'greater than' type.

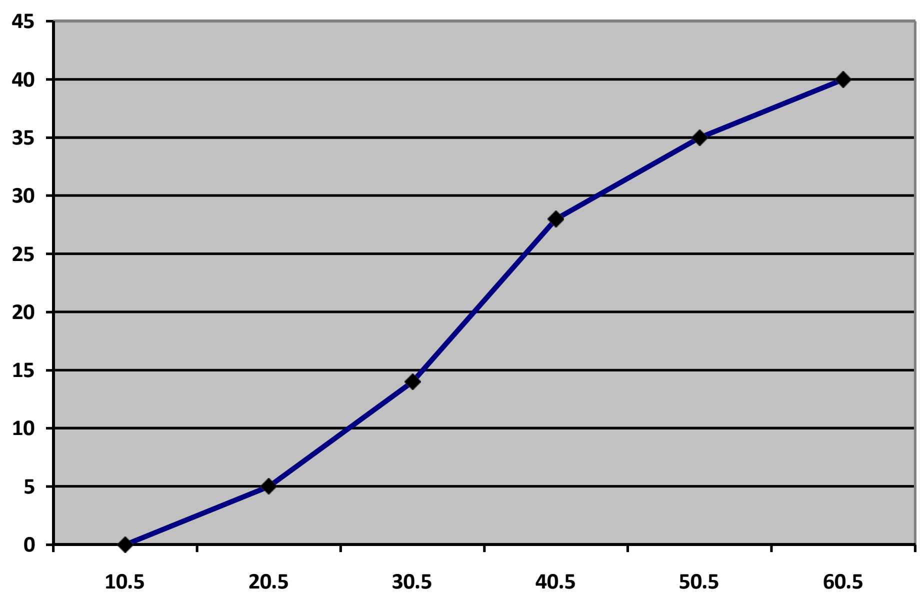
Example 1.13

Draw the ogive for the frequency table below

class interval	Frequency
11 – 20	5
21 – 30	9
31 – 40	14
41 – 50	7
51 – 60	5

Solution

Less than (or equal to)	Cumulative frequency
10.5	0
20.5	5
30.5	14
40.5	28
50.5	35
60.5	40



The points are to be jointed with free hand

NOTE:

- ‘Or equal to’ is understood and usually hidden so only ‘less than’ is used.

- In the frequency table, no value is less than 10.5 (the upper class boundary of the class before the first class), hence, the cumulative frequency of zero (0) and no value is greater than 60.5 (the upper class boundary of the last class), hence, the cumulative frequency of 40 which is the total frequency.

  The ogive can be used to estimate median, quartiles, deciles and the percentiles.

REMARK

- In the construction of ogive (cumulative frequency curve) in example 1.13, the approach used in drawing the ogive is the 'less than' type. However, there is another approach tagged 'more than'. The major differences between the two approaches are as follows:
  (i) Cumulating the frequencies in 'less than' and 'more than' starts respectively from top and bottom frequencies; and
  (ii) The cumulative figures are attached to upper class boundaries for 'less than' while they are attached to lower class boundaries for 'more than'.

  Let us consider the following example for the 'more than' case:

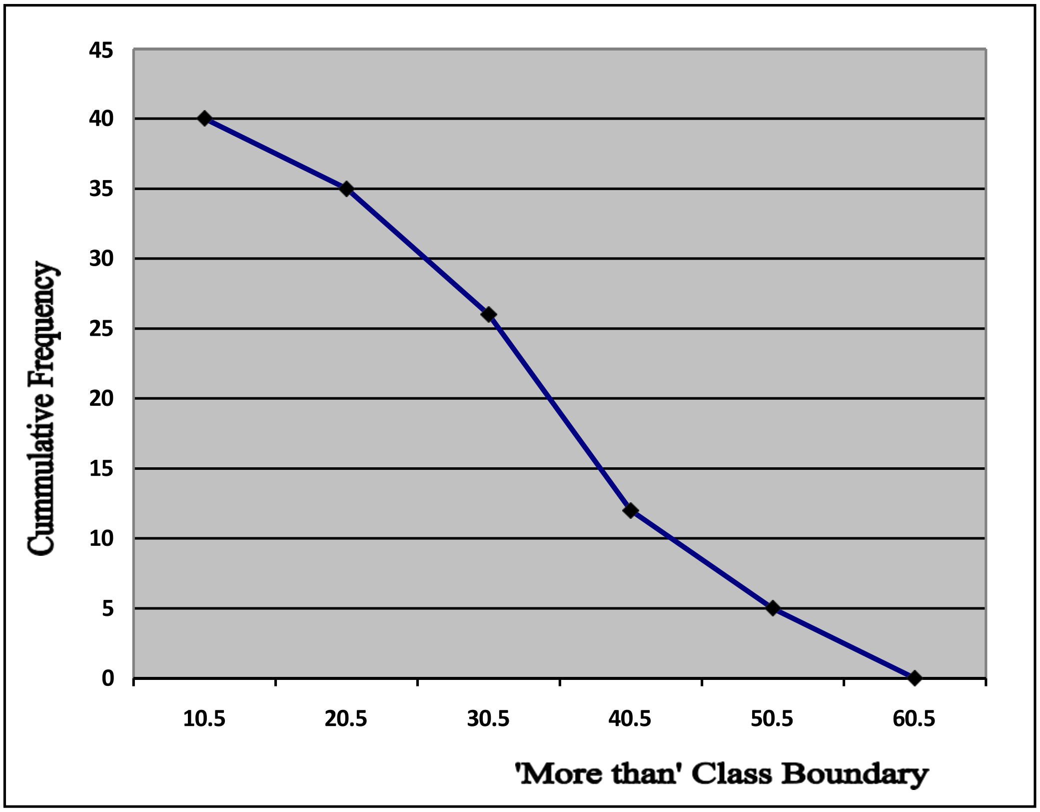
Example 1.14

By using example 1.13 data, construct a 'More than' ogive

The cumulative table for 'more than' is as follows:

class boundary	**frequency**	**cumulative frequency**
More than 10.5	5	40
More than 20.5	9	35
More than 30.5	14	26
More than 40.5	7	12
More than 50.5	<u>5</u>	05
	40	

Solution



Example 1.15

In a class of accounting students, the students were tested on 'Quantitative Analysis'. The following table depicts the scores of these students in a tabular from:

Marks in interval	Number of Students
0 – 10	5
10 – 20	10
20 – 30	15
30 – 40	20
40 – 50	25
50 – 60	10
60 – 70	5
70 – 80	4
80 – 90	3
90 – 100	3

Draw the ogive for the table.

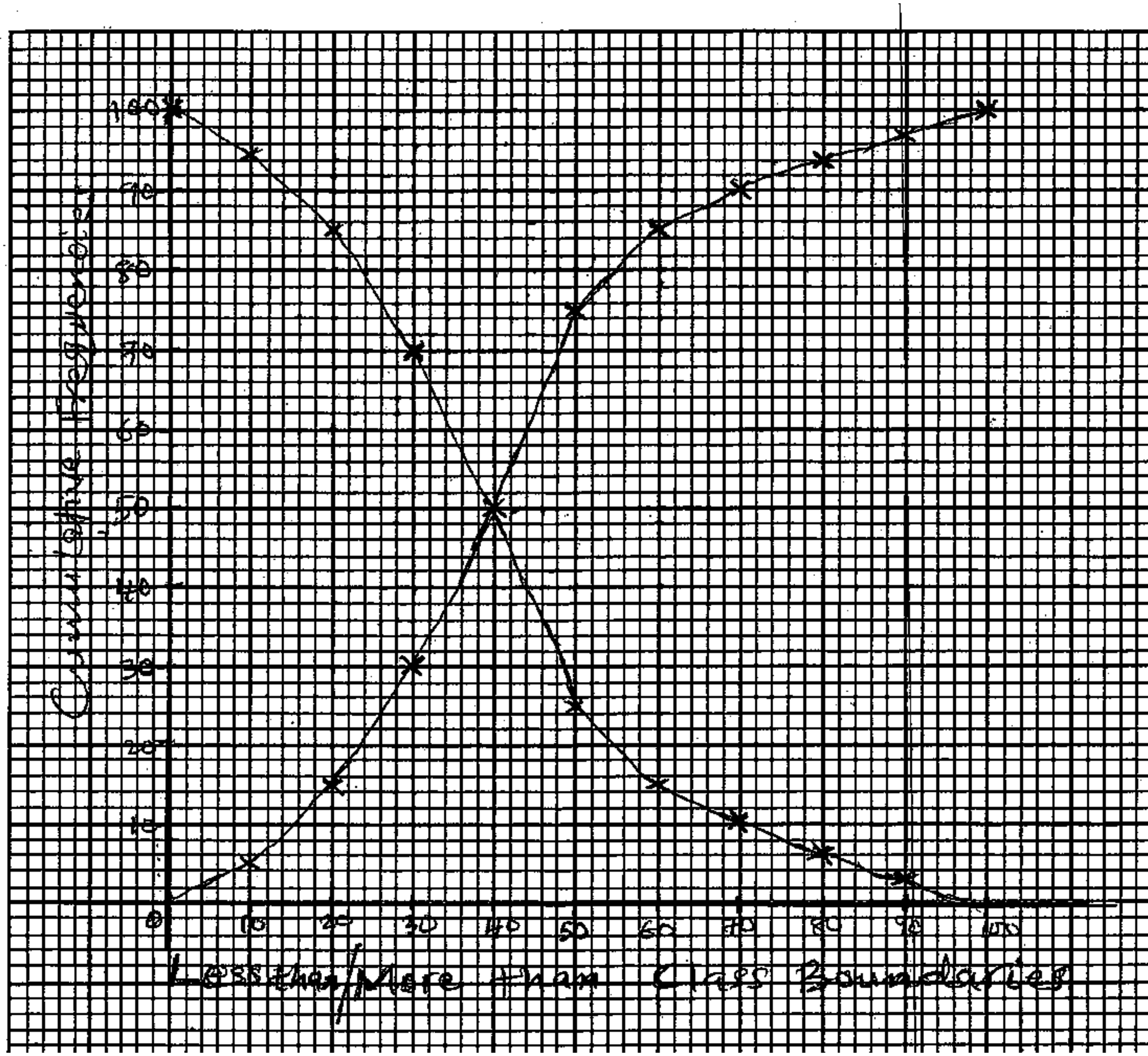
Solution

More than

Class boundary	Frequency	Cumulative frequency
More than 0	5	100
More than 10	10	95
More than 20	15	85
More than 30	20	70
More than 40	25	50
More than 50	10	25
More than 60	5	15
More than 70	4	10
More than 80	3	06
More than 90	3	03
	100	

Less than

Class boundary	Frequency	Cumulative frequency
Less than 10	5	05
Less than 20	10	15
Less than 30	15	30
Less than 40	20	50
Less than 50	25	75
Less than 60	10	85
Less than 70	5	90
Less than 80	4	94
Less than 90	3	97
Less than 100	3	100
	100	



Comment on the Ogives

The above graphs are the ogives of ‘less than ‘and ‘more than ‘on the same axes.

From the drawn ogives, it could be seen that the two ogives intercept at the class boundary of 40 at cumulative frequency 50. The intersection implies that the median value is 40

1.5 The use of statistical application packages for data presentation.

In the advent of computer, there are lots of computer applications available to make operations like accounting process, statistical process etc. easy to comprehend. Statistical Application Package is one of such computer applications. There are many statistical application packages that have been developed for different purposes such as data presentation and analysis. But the two commonly used one are Microsoft Excel and SPSS.

However, a good knowledge of statistics and computer know-how are required to interpret and understand these packages.

The statistical packages make use of the speed, efficiency and accuracy of the computer to analysis data effectively.

We will now explain the use of the two packages mentioned above for data presentation.

(a) MICROSOFT EXCEL

This is a spreadsheet developed by Microsoft for windows android, MAC OS and IOS, which can sometimes be called; “EXCEL” (short form). It has a lot of features which include graphic tools, pivot tables, calculations and visual basic (programming language) for applications.

Excel forms part of installed Microsoft office in your personal computer or laptop. It has a set of in-built functions that are used to answer appropriate engineering, statistical and financial questions. It is used to perform variety of computations which include a data analysis tool pack and collection of statistical functions which can display data in different types of table, charts and graphic.

The steps below explain how excel application can be used for data presentation:

(i) Clock on all program options from the start menu;

(ii) Search for Microsoft office (which always comes with year i.e. for example Microsoft office 2020) from the sub menu and then click on it;

(iii) From the Microsoft office sub menu, search for Microsoft excel (which also comes with a particular year i.e. Microsoft excel 2020) from the sub menu and then click on it;

(iv) The Microsoft excel application chosen would be launched and a new spreadsheet (which contains a lot of cells i.e. intersection between rows & Colum);

(v) Data to be analyzed are then entered into the different cells along the columns;

(vi) Highlight the part of data entered that are to be worked on;

(vii) Go to the menu, select the chart wizard icon and click on it or click on insert from the menu and then click on chart and select and click on the desired type of chart (i.e. pie chart or bar chart). Also, for further analysis, select data from the menu bar and locate the data analysis and search for the desired statistical tools; and

(viii) The result/output is displayed and it is then saved for use.

NOTE:

In the Microsoft excel menu, the help or tutorial menu can be used as a guide to carry out the appropriate data presentation correctly.

(b) SPSS

The acronym SPSS which stands for "Statistical Package for the Social Sciences" is a set of programs for presentation, manipulation and analysis of different data sets. The SPSS was originally developed by SPSS INC. which has now been acquired by international business machine (IBM) and it is now known as IBM SPSS statistics. The SPSS was originally developed for researchers in the field of social sciences like & psychology, sociology etc., but since the acquisition by IBM, it has been expanded into different fields. The SPSS software has been developed in a customised manner to allow one to enter the needed exact data like numbers (quantitative) and variables (quantitative). It has a user interface like that of Microsoft excel i.e. spreadsheet set up. Unlike the Microsoft excel that comes with the Microsoft office, the SPSS software has to be installed on your personal computer or laptop (the free trial version can be downloaded from the IBM SPSS site or it can be from appropriate software site). SPSS software has four different windows, namely: data editor window, output window, syntax window and script window but data editor and output window are of major concern in the data analysis environment. The data editor window consists of two view, namely: the data view and the variable view, while the data view displays the actual data entered and variables created, the variable view contains the definition label of each variable in the data set. On the other hand, the output window displays result of the analysis for interpretation.

The steps below explain how SPSS can be used for data:

(i) Click on all programs option from the start menu;

(ii) Search for the SPSS software installed i.e. SPSS 22.0 and click on it:

(iii) SPSS data editor window will be launched;

(iv) A dialog box will appear asking if one wants to" open an existing data file" or to "enter a new data set" where one has to select one, most of the time, "enter a new data set" is selected;

(v) The data editor window will be launched where one has to click on the variable view to define and label the data set and then click on the data view to input/enter the data based on the definition and labelling under the variable view;

(vi) From the SPSS menu at the top, search for analyze and click on it and select the desire statistical tool to be performed i.e. frequency distribution, chart and graphs for the set of data selected as well and click ok; and

(vii) The output window will be launched showing the result output of data the analysis presentation and it can then be saved for the SPSS menu.

NOTE:

In the SPSS menu, the help or tutorial menu can be used as a guide to carry out the appropriate data presentation correctly.

1.6 Sampling

Sampling is an important statistical method that entails the use of fractional part of a population to study and make decisions about the characteristics of the given population. As an introduction to the sampling, the following **concepts** shall be discussed:

Definition of Some Important Concepts on Sampling

(a) Population: A population is a collection or group of items satisfying a definite characteristic. Items in the definition can be referred to

(i) human beings (as population census**);**

(ii) animate objects as cattle, sheep, goats etc.;

(iii) Inanimate objects such as chairs, tables etc.; and

(iv) a given population like candidates of ATS.

There are two types of population, namely: finite and infinite population:

By finite population, we mean the population which has countable number of items; while the items in infinite population are uncountable. Population of students in a college is an example of finite population while sand particles on the beach are that of infinite population.

(b) Sample: A sample is the fractional part of a population selected in order to observe or study the population for the purpose of making scientific statement or decision about the said population.

(c) Census: This is a complete enumeration of all units in the entire concerned population with the aim of collecting vital information on each unit. Year 2006 head count in Nigeria is a good example of census.

(d) Sample Survey: A sample survey is a statistical method of collecting information by the use of fractional part of population as a representative sample.

(e) Sampling Frame: This is a list containing all the items or units in the target population and it is normally used as basis for the selection of sample. Examples are list of church members, telephone directory, etc.

(f) Sampling unit: This is any individual member of a population

Some sampling notations and terms:

(i) "N" represent the whole population i.e. population size, then the sample size is denoted by "n";

(i) $f = \frac{n}{N}$, where f is called the sampling fraction, N and n are as defined in (i);and

(ii) The expansion or raising factor is given as $g = \frac{N}{n}$, where g is the expansion factor, N and n are as defined in (i).

Methods of Sampling

Sampling methods can be broadly classified into two types, namely:

(a) Probability sampling methods; and

(b) Non-probability sampling methods.

(a) The Probability Sampling Method

The probability sampling method/technique is a method involving random selection. It is done in such a way that each unit of the population is given a definite chance or probability of being included in the sample. Among these sampling techniques are simple random sampling, systematic sampling, stratified sampling, multi-stage sampling and cluster sampling.

(i) Simple Random Sampling (SRS): This is a method of selecting a sample from the population with every member of the population having equal chance of being selected. The sampling frame is required in this technique.

The method can be achieved by the use of

- table of random numbers; and
- lottery or raffle draw approach.

We are to note that SRS can be with replacement or without replacement. When it is with replacement, a sample can be repeatedly taken or selected; while in the SRS without replacement, a sample cannot be repeatedly taken or selected.

Advantages of SRS

- SRS gives equal chance to each unit in the population which can be termed a fair deal;
- SRS method is simple and straight forward; and
- SRS estimates are unbiased.

Disadvantages of SRS

- The method is not good for a survey if sampling frame is not available; and
- There are occasions when heavy drawings are made from one part of the population, the idea of fairness is defeated.

(ii) Systematic Sampling (SS): This is a method that demands for the availability of the sampling frame with each unit numbered serially. Then from the frame, selection of the samples is done on regular interval k i.e. (sampling interval) after the first sample is selected randomly within the first given integer number k; where

$k = \frac{N}{n}$, and must be a whole number, and any decimal part of it should be cut off or cancelled. N and n are as earlier defined. That is, after the first selection within the k interval, other units or samples of selection will be successfully
equidistant from the first sample (with an interval of k).

For instance, in a population N = 120 and 15 samples are needed, the SS method will give $k = \frac{120}{15} = 8$

It means that the first sample number will be between 1 and 8. Suppose serial number 5 is picked, then the subsequent numbers will be 13, 21, 29, 37,… (Note that the numbers are obtained as follows: 5 + 8 = 13,13 + 8 =

21, 21 + 8 = 29, 29 + 8 = 37 etc.)

Advantages of SS

- It is an easy method to handle; and
- The method gives a good representative if the sampling frame is available.

Disadvantage of SS

- If there is no sampling frame, the method will not be appropriate.

(iii) Stratified Sampling (STS): This is a method that is commonly used in a situation where the population is heterogeneous. The principle of breaking the heterogeneous population into a number of homogeneous groups is called stratification. Each group of unique characteristics is known as stratum and there must not be any overlapping in the groups or strata. Some of the common factors that are used to determine the division or breaking into strata are income levels, employment status, etc.

The next step in stratification is the selection of samples from each group (stratum) by simple random sample. An example of where to apply STS can be seen in the survey concerning standard of living or housing pattern of a town.

Advantages of STS

- By pooling the samples from the strata, a more and better representative of the population is considered for the survey or investigation; and
- Breaking the population into homogeneous, goes a long way to have more precise and accurate results.

Disadvantages of STS

- There are difficulties in deciding the basis for stratification into homogeneous group; and
- STS suffers from the problems of assigning weights to different groups (strata) when the condition of the population demands it.

(iv) **Multi–Stage Sampling:** This is a sampling method involving two or more stages. The first stage consists of breaking down the population into first set of distinct groups and then select some groups randomly.

The list of selected groups is termed the primary sampling units. Next, each group selected is further broken down into smaller units from which samples are taken to form a frame of the second-stage sampling units. If we stop at this stage, we have a two-stage sampling.

Further stages may be added and the number of stages involved is used to indicate the name of the sampling. For instance, five-stage sampling indicates that five stages are involved.

An example where the sampling method [MSS] can be applied is the survey of students' activities in higher institutions of a country. Here, first get the list of all higher institutions in the country and then select some randomly. The next stage is to break the selected institutions into faculties and select some. One can go further to break the selected faculties into departments and select some departments to have the third-stage sampling.

Advantages of MSS

- The method is simple if the sampling frame is available at all stages; and
- It involves little cost of implementation because of the ready-made availability of sampling frame.

Disadvantages of MSS

- If it is difficult to obtain the sampling frame, the method may be tedious; and
- Estimation of variance and other statistical parameters may be very complicated.

(v) **Cluster Sampling [CS]:** Some populations are characterized by having their units existing in natural clusters. Examples of these can be seen in farmers' settlement (farm settlements are clusters); students in schools (schools are clusters). Also, there could be artificial clusters when higher institutions are divided into faculties.

The cluster method involves random selection of A clusters from M clusters in the population which represents the samples.

(b) Non-Probability Sampling

The sampling techniques which are not probabilistic are

(i) Purposive Sampling (PS).

In this sampling, no particular probability for each element is included in sample. It is, at times, called the judgment/judgmental sampling.

In purposive sampling, selection of the sample depends on the discretion or judgment of the investigator. For instance, if an investigation is to be carried out on students' expenditures in the university hostels, the investigator may select those students who are neither miserly nor extravagant in order to have a good representation of the targeted population.

Advantages of PS

- It is very cheap to handle; and
- No need for sampling frame.

Disadvantages of PS

- The method is affected by the prejudices of the investigator; and
- Estimates of sampling errors cannot be possible by this method.

(ii) Quota Sampling (QS)

This is a sampling technique in which the investigator aims at obtaining some balance among the different categories of units in the population as selected samples. A good example of this can be seen in the quota selection of students to Federal Colleges in Nigeria.

Advantages of QS

- It has a fair representation of various categories without probability Basis; and
- No need for sampling frame.

Disadvantage of QS

- Sampling error cannot be estimated.

1.7 Chapter Summary

Data are defined as raw facts in numerical form. Its classification by types, method of collection and the forms of presentation are discussed. Some sampling terms and techniques with their advantages and disadvantages are presented.

1.8 MULTIPLE-CHOICE AND SHORT-ANSWER QUESTIONS

1. Which of the following is a non-numeric ordinal data?
   A. Income
   B. Price of commodity
   C. Occupation
   D. Rating in beauty contest
   E. Students number in a class

2. A schedule in statistics refers to______
   A. Examination time table
   B. Set of questions used to gather pieces of information which is filled by an informant or respondent.
   C. A set of questions used to gather information and filled by the investigator him/herself.
   D. A set of past examination questions.
   E. Paper used by bankers to carry out investigation.

3. Which of the following sampling methods does not need sampling frame?
   A. Simple random sampling.
   B. Purposive sampling
   C. Systematic sampling
   D. Cluster sampling
   E. Stratified sampling.

4. The following are qualities of a good questionnaire except
   A. That each question in the questionnaire must be precise and unambiguous.
   B. Avoidance of leading question in the questionnaire.
   C. That a questionnaire must be lengthy in order to accommodate many questions.
   D. That a questionnaire must be well structured into sections such that questions in

each section are related.

E. Avoidance of double barrelled questions in the questionnaire

5. In sampling, a list consisting of all units in a target population is known as_

6. A small or fractional part of a population selected to meet some set objective is known as________

7. A Spreadsheet which has a set of in-built functions that are used to answer appropriate engineering, statistical and financial questions is called__________________________

8. A procedure in the form of report involving a combination of text and figures is known as________

9. A histogram is similar to bar chart except that its bars __________ each other

10. Age of an employee is an example of___________ type of data

Answers

1. D
2. C
3. B
4. C
5. Sampling frame
6. Sample
7. Microsoft Excel
8. Text presentation
9. Touch
10. Continuous numeric

CHAPTER TW0

MEASURES OF LOCATION

Chapter contents

(a) Introduction.

(b) Mean.

(c) Mode and Median.

(d) Measures of Partition.

Objectives

At the end of the chapter, readers should be able to understand

a) the meaning of "Measures of Central Tendency";

b) and solve problems on mean (arithmetic mean);

c) and handle problems on the mode and median;

d) the concept of measures of partition; and

e) and handle problems on quartiles, deciles and percentiles.

2.1Introduction

Measure of location is a summary statistic which is concerned with a figure which represents a series of values. Measures of location can be further classified into

(a) Measures of Average or Measures of Central Tendency; and

(b) Measures of Partition.

In average or measures of central tendency, there is an average value which is a representative of all the values in a group of data. These typical values of averages tend to lie centrally within the set of data arranged in an array, hence they are called measures of central tendency.

The common and usable types of averages or measures of central tendency are

i. Mean (i.e. Arithmetic Mean) ;

ii. Mode; and

iii. Median;

Each of the above averages will be discussed in the subsequent sections.

2.1 Mean (Arithmetic Mean)

This is the sum of all the values $(x_1, x_2, x_3, \Lambda\ , x_n)$ in the data group divided by the total number of the values. In symbolic form, for discrete or ungrouped values without frequency distribution, the mean can be expressed as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \Lambda\ + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad 2.1$$

where $\bar{x}$ represents the arithmetic mean;

$\sum$ is the summation symbol usually called "sigma" and n is the number of values.

In the case of discrete or ungrouped values with frequency distribution, the above formula can be modified as

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \text{or simply} \quad \bar{x} = \frac{\sum fx}{\sum f} \qquad 2.2$$

where, $\sum f = n$ (as defined as 2.1) and f represents frequency.

Also for grouped frequency distribution table, formula 2.2 will still be equally applied. However, the x_i/x values in the formula are the class marks.

Example 2.1

From the under listed data generated on the number of sales of cement (in bags):

12, 7, 2, 6, 13, 17, 14, 5, 9, 4, determine the mean

Solution

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \Lambda\ + x_{10}}{10} = \frac{\sum_{i=1}^{10} x_i}{10}$$

$$\bar{x} = \frac{12+7+2+6+13+17+14+5+9+4}{10} = \frac{89}{10}$$

$$\bar{x} = 8.9 bags \approx 9 bags$$

Example 2.2

From the under listed data generated on the volume (litres) of water: 2.7, 6.3, 4.1, 6.4, 3.5, 4.7, 13.8, 7.9, 12.1, 10.4, determine the mean

Solution

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \Lambda + x_{10}}{10} = \frac{\sum_{i=1}^{10} x_i}{10}$$

$$\bar{x} = \frac{2.7 + 6.3 + 4.1 + 6.4 + 3.5 + 4.7 + 13.8 + 7.9 + 12.1 + 10.4}{10} = \frac{71.9}{10}$$

$\bar{x} = 71.9$ litres

Note: The unit of the mean is the same as that of the data

Example 2.3

The table below contains the data collected on the number of students that register for QA at some examination centres, obtain the mean.

x	1	2	3	4	5
f	8	7	10	6	2

Solution

x	f	fx
1	8	8
2	7	14
3	10	30
4	6	24
5	2	10
	$\sum f = 33$	$\sum fx = 86$

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{86}{33} = 2.6061$

Note: This is an unrealistic figure since 2.6061 students do not exist. This is one of the shortcomings of the mean

Example 2.4

These are the data generated on the weights(kilograms) of soap produced in the production section of a company. Obtain the mean

Weight (*x*)	1.0	2.0	3.0	4.0	5.0
Frequency (*f*)	4	3	7	5	6

Solution

x	f	fx
1.0	4	4
2.0	3	6
3.0	7	21
4.0	5	20
5.0	6	30
	$\sum f = 25$	$\sum fx = 81$

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{81}{25} = 3.24$ kg

Example 2.5

The following are data collected on the ages (in years) of the people living in Opomulero village.

Class interval	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Frequency	12	8	7	7	6

Obtain the mean age of the people.

Solution

Class interval	***x***	***f***	***fx***
0 – 10	5	12	60
10 – 20	15	8	120
20 – 30	25	7	175
30 – 40	35	7	245
40 – 50	45	6	270
		$\sum f = 40$	$\sum fx = 870$

$$\text{Mean, } \bar{x} = \frac{\sum fx}{\sum f} = \frac{870}{40} = 21.75 \text{ years}$$

Example 2.6

From the following data generated on the ages (in years)of people living in an Estate.

Class interval	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	6	17	7	12	8

Obtain the mean age of the data.

Solution

Class interval	***x***	***f***	***Fx***
1 – 10	5.5	6	33.0
11 – 20	15.5	17	263.5
21 – 30	25.5	7	178.5
31 – 40	35.5	12	426.0
41 – 50	45.5	6	273
		$\sum f = 50$	$\sum fx = 1174$

$$\text{Mean, } \bar{x} = \frac{\sum fx}{\sum f} = \frac{1174}{50} = 23.48 \text{ years}$$

Assumed Mean

It is possible to reduce the volume of figures involved in the computation of the mean by the use of "assumed mean" method. In this method, one of the observed values of x_i is chosen, (preferably the middle one) as the assumed mean. If A denotes the assumed mean, a new variable d is introduced by the expression

$d_i = x_i - A$, then the actual mean is obtained by

$$\bar{x} = A + \frac{\sum_{i=1}^{n} d_i}{n} \qquad 2.3$$

$$\text{Or } \bar{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i} \qquad 2.4$$

Example 2.7

Use the data in Example 2.1, to obtain the arithmetic mean by using 9 as the assumed mean.

Solution

x	$d_i = x_i - 9$
12	**3**
7	**-2**
2	**-7**
6	**-3**
13	**4**
17	**8**
14	**5**
5	**-4**
9	**0**
4	**-5**
	$\sum d_i = -1$

$$\bar{x} = A + \frac{\sum d_i}{n} = 9 + \frac{(-1)}{10} = 9 - 0.1 = 8.9 \approx 9$$

Note that it gives the same answer as Example 2.1

Example 2.8

Use the data in Example 2.5 to obtain the mean age using 25 years as the assumed mean.

Solution

Class interval	***x***	***f***	**If *A* = 25** ***d*i = *x* – 25**	***fd*i**
0 – 10	5	12	-20	-240
10 – 20	15	8	-10	-80
20 – 30	25	7	0	0
30 – 40	35	7	10	70
40 – 50	45	6	20	120
–		$\sum f = 40$		$\sum fd_i = -130$

$$\bar{x} = A + \frac{\sum fd_i}{\sum f} = 25 + \frac{(-130)}{40} = 25 - 3.25 = 21.75 \text{ years}$$

which is the same as the answer obtained in Example 2.5

Assumed mean and common factor approach can equally be used to obtain the mean. Here, it is only applicable to group data with equal class intervals.

By this approach, equation 2.4 becomes:

$$\bar{x} = A + \left(\frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i} \right) c \qquad 2.5$$

where c is the common factor or the class size.

Example 2.9

Use the data in Example 2.6 to obtain the mean, using the assumed mean of 25 years and the appropriate common factor method.

Solution

Class interval	***x***	***f***	**If *A* = 25** $d_i = x - 25$	$d_i' = \frac{d_i}{10}$	fd_i'
0 – 10	5	12	-20	-2	-24
10 – 20	15	8	-10	-1	-8
20 – 30	25	7	0	0	0
30 – 40	35	7	10	1	7
40 – 50	45	6	20	2	12
		$\sum f = 40$			$\sum fd_i' = -13$

$$\bar{x} = A + \left(\frac{\sum_{i=1}^{n} f_i d_i'}{\sum_{i=1}^{n} f_i} \right) c \qquad (where\ c = 10 - 0 = 20 - 10 = 10)$$

$$\bar{x} = 25 + \left(\frac{-13}{40} \right) 10$$

$$\bar{x} = 25 - 3.25 = 21.75\, years$$

2.3Mode and Median

(i) Mode

The mode is the value which occurs most frequently in a set of data. For a data set in which no measured values are repeated, there is no mode.

For a grouped data with frequency distribution, the mode is determined by either graphical method through the use of Histogram or the use of formula.

(a) In the graphical method, the following steps are to be followed:

i. Draw the histogram of the given distribution.;

ii. Identify the highest bar (the modal class);

iii. Identify the two linking bars to the modal class, i.e. the bar before and after the modal class;

iv. Use the two flanking bars to draw diagonals on the modal class; and

v. Locate the point of intersection of the diagonals drawn in step (iv) above, and vertically, draw a line from the point of intersection to the horizontal axis which gives the desired mode.

(b) By formula, the mode is given by

$$Mode = L_{mo} + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right) c$$

Where

L_{mo}	=	Lower class boundary of the modal class;
Δ_1	=	Modal class frequency – frequency of the class before the modal class;
Δ_2	=	Modal class frequency – frequency of the class after the modal class; and
c	=	Modal class size.

(ii) Median

The median of a data set is the value of the middle item of the data when all the items in the data set are arranged in an ordered array form (either ascending or descending order).

For an ungrouped data, the position (Median) is located by $\frac{N+1}{2}$th position, where N is the total number of items in the data set.

In a grouped data, the position of the Median is located by $\frac{N}{2}$ th position. Here, the specific value of the Median is determined by either graphical method through the use of Ogive (Cumulative frequency curve) or the use of formula

(a) In the graphical method, the following procedure is used:

i) Draw the Ogive of the given distribution;

ii) Locate the point $\frac{N}{2}$ th on the cumulative frequency axis of the Ogive; and

iii) Draw a parallel line through the value of $\frac{N}{2}$ th position to the curve and then draw a perpendicular (or vertical) line from the curve intercept to the horizontal axis in order to get the Median value.

(b) The Median is obtained by the use of the following formula:

$$Median = L_{me} + \left(\frac{\frac{N}{2} - \sum f_{me}}{f_{me}} \right) c$$

Where $L_{me} =$ Lower class boundary of median class;

$N =$ Total number of items in the data set;

$\sum f_{me} =$ Summation of all frequencies before the median class;

$f_{me} =$ Frequency of the median class; and

$c =$ Median class size or width.

Note: The units of the Mode and Median are the same as that of the data

Example 2.10 (n is odd and discrete variable is involved)

The following are the data generated from the sales of oranges within eleven days: 4, 7, 12, 3, 5, 3, 6, 2, 3, 1, 3, calculate both the mode and median.

Solution

Mode is the most occurring number

$\therefore$ Mode = 3

Put the data in an array i.e. 1, 2, 3, 3, 3, 3, 4, 5, 6, 7, 12

Median position = $\frac{n+1}{2}$th (since n is odd i.e. n = 11)

Median position = $\frac{11+1}{2}$th = $\frac{12}{2}$th = 6^{th} position

$\therefore$ The median = 3

Example 2.11 (n is odd and continuous variable is involved)

The following are the data generated from the measurement of the lengths (in meters) of iron-rods; 2.5, 2.0, 2.1, 3.5, 2.5, 2.5, 1.0, 2.5, 2.2, calculate both the mode and median.

Solution

Mode is the most occurring number

$\therefore$ Mode = 2.5m.

Put the data in an array i.e. 1.0, 2.0, 2.1,2.2, 2.5, 2.5, 2.5, 2.5, 3.5

Median position = $\frac{n+1}{2}$th (since n is odd i.e. n = 9)

Median position = $\frac{9+1}{2}$th = $\frac{10}{2}$th = 5^{th} position

$\therefore$ The median = 2.5m,

Example 2.12 (*n* is even and discrete variable is involved)

The following are the data generated from the sales of orange within 12 days 4, 7, 12, 3, 5, 3, 6, 2, 3, 1, 3, 4, calculate the mode and median.

Solution

Mode is the most occurring number

$\therefore$ Mode = 3

Put the data in an array i.e. 1, 2, 3, 3, 3, 3, 4, 4, 5, 6, 7, 12

Median position = $\frac{n+1}{2}$th (since n is even i.e. n = 12)

Median position = $\frac{12+1}{2}$th = $\frac{13}{2}$th = 6.5^{th} position

Since the median position is 6.5^{th}, the median lies between the 6^{th} value and the 7^{th} value, then, the median is the mean of the middle values (i.e. the 6^{th} and 7^{th} values) from the rearrange data

$\therefore$ The median = $\frac{3+4}{2}$ = 3.5

Example 2.13 (*n* is even and continuous variable is involved)

The following are the data generated from the measurement of an iron-rod; 2.5, 2.0, 2.1, 3.5, 2.5, 2.5, 1.0, 2.5, 3.1, 2.2, 3.3, 2.6. Calculate the mode and median

Solution

Mode is the most occurring number

$\therefore$ Mode = 2.5

Put the data in an ordered array i.e. 1.0, 2.0, 2.1,2.2, 2.5, 2.5, 2.5, 2.5, 2.6, 3.1, 3.3, 3.5

Median position = $\frac{n+1}{2}$th (since n is even i.e. n = 12)

Median position = $\frac{12+1}{2}$th = $\frac{13}{2}$th = 6.5^{th} position

Since the median position is 6.5^{th}, the median lies between the 6^{th} value and the 7^{th} value, then, the median is the mean of the middle values (i.e. the 6^{th} and 7^{th} values) from the rearrange data

$\therefore$ The median = $\frac{2.5+2.5}{2}$ = 2.5

Example 2.14 (Grouped data by graphical method)

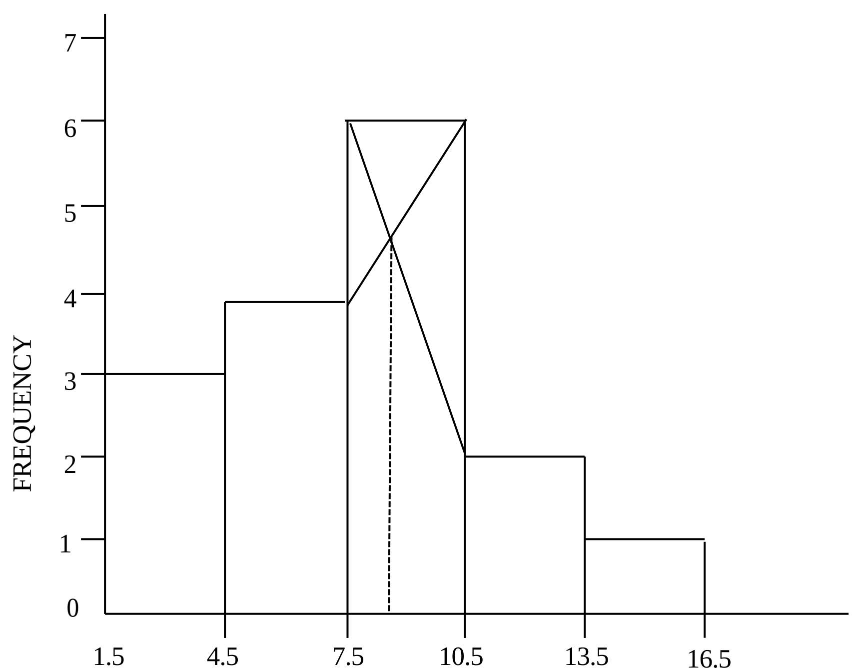
Use the graphical method to determine the mode of the following data

Tyre range	**Frequency**
2 – 4	3
5 – 7	4
8 – 10	6
11 – 13	2
14 – 16	1

Solution

Tyre range	**Class Boundaries**	**Frequency**
2 – 4	1.5 – 4.5	3
5 – 7	4.5 – 7.5	4
8 – 10	7.5 – 10.5	6
11 – 13	10.5 – 13.5	2
14 – 16	13.5 – 16.5	1

- Draw the histogram of the given table to a reasonable scale
- Interpolate on the modal class as discussed earlier in order to obtain the mode.



Class Boundaries

From the histogram, Mode = 8.5

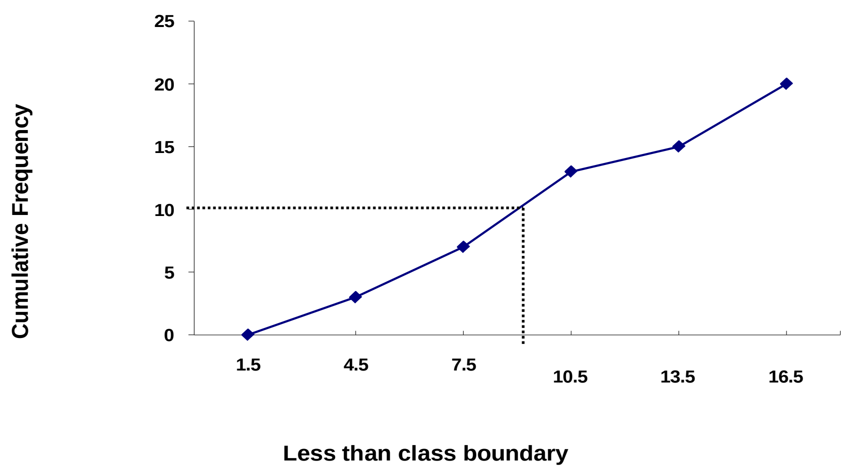
Example 2.15 **(Graphical method)**

Use graphical method to find the median using Example 2.14 data

Solution

Tyre range	frequency	cf	Less than Class Boundary
2 – 4	3	3	4.5
5 – 7	4	7	7.5
8 – 10	6	13	10.5
11 – 13	2	15	13.5
14 – 16	1	20	16.5
	$N = \sum f = 20$		

The plotted cumulative frequency curve (Ogive):



Median position = $\frac{N}{2}$th = $\frac{20}{2}$th =10th (i.e. trace 10 from the cf axis to the ogive)

From the Ogive, median = 9.0

Example 2.16

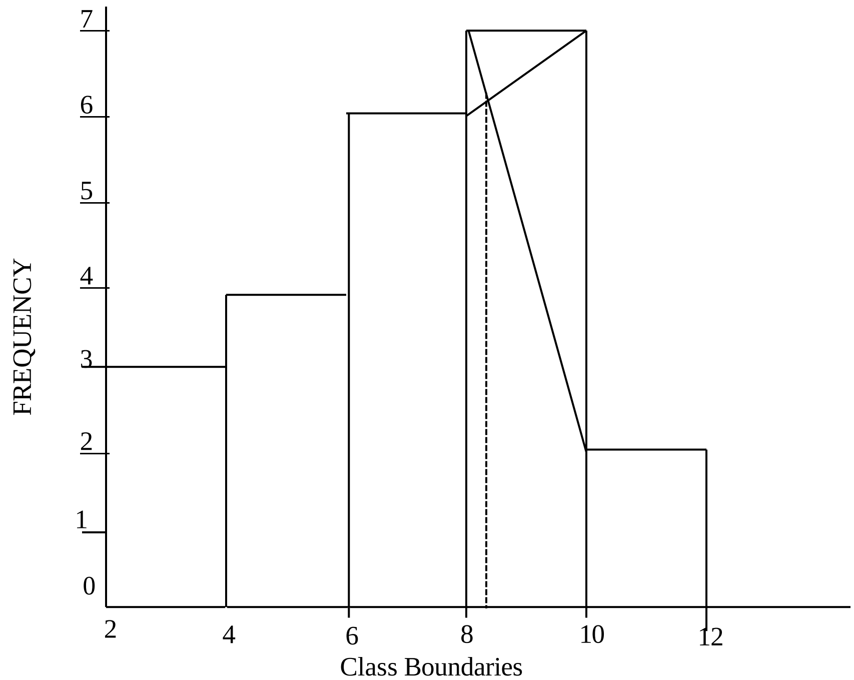
Determine the mode and median for the following data using graphical method.

Class interval	frequency(f)
2 – 4	3
4 – 6	4
6 – 8	6
8 – 10	7
10 – 12	2

Solution
To find the mode, the relevant table is as follows:

Class interval	frequency(f)
2 – 4	3
4 – 6	4
6 – 8	6
8 – 10	7
10 – 12	2

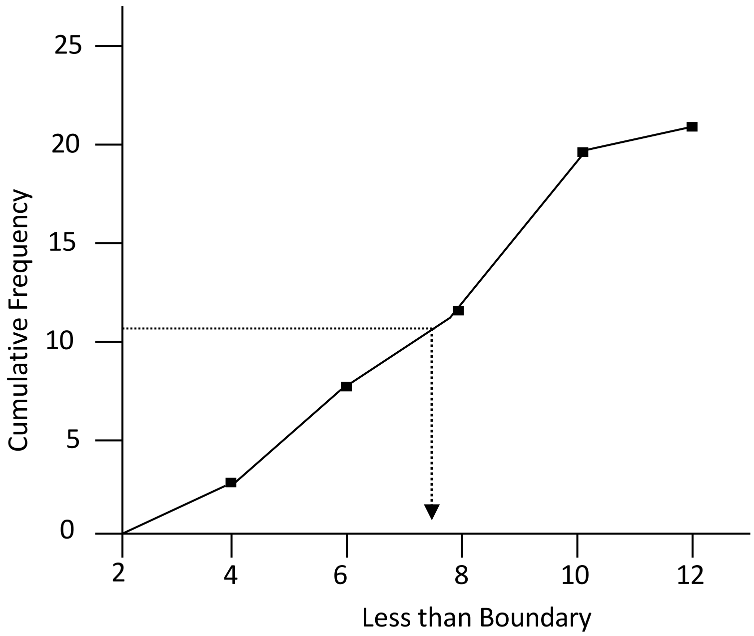
Note that the class interval is continuous.
The plotted histogram:



From the histogram, mode = 8.50

To find the Median, we proceed as follows:

Class interval	**frequency (f)**	**cf**	**Less than Boundary**
2 – 4	3	3	4
4 – 6	4	7	6
6 – 8	6	13	8
8 – 10	7	20	10
10 – 12	2	22	12
	$N = \sum f = 22$		



Median position = $\frac{N}{2}$ th = $\frac{22}{2}$ th =11th (i.e. trace 11 from the cf axis to the ogive)

From the Ogive, median = 7.8

Note: similar approach can be applied to estimate the quartiles or deciles or percentiles from the Ogive

Example 2.17 (Grouped data using formula)

The following are the data generated on the sales of electric components

Class interval	Frequency
2 – 4	3
4 – 6	4
6 – 8	6
8 – 10	7
10 – 12	2

Determine the mode and median using the formula.

Solution

Class interval	**frequency(f)**	**Cumulative Frequency(cf)**
2 – 4	3	3
4 – 6	4	7
6 – 8	6	13 → Median class
8 – 10	7 → Modal class	20
10 – 12	2	22
	$N = \sum f = 22$	

$$Mode = L_{mo} + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right) c$$

Where

L_{mo} = Lower class boundary of the modal class;

Δ_1 = Modal class frequency- frequency of the class before the modal class;

Δ_2 = Modal class frequency-frequency of the class after the modal class; and

c = Modal class size.

$$L_{mo} = 8,\ \Delta_1 = 7 - 6 = 1,\ \Delta_2 = 7 - 2 = 5,\ c = 2$$

$$Mode = 8 + \left(\frac{1}{1+5} \right) 2$$

$$Mode = 8 + \frac{1}{3}$$

$$Mode = 8 + 0.33$$

$$Mode = 8.33$$

$$Median = L_{me} + \left(\frac{\frac{N}{2} - \sum f_{me}}{f_{me}} \right) c$$

Where L_{me} = Lower class boundary of median class;

N = Total number of items in the data set;

$\sum f_{me}$ = Summation of all frequencies before the median class;

f_{me} = Frequency of the median class; and

c = Median class size or width.

$$Median\ position = \frac{N}{2}th = \frac{22}{2}th = 11^{th}$$

$L_{me} = 6$, $f_{me} = 6$, $\sum f_{me} = 3 + 4 = 7$, $c = 2$, $N = 22$

$$Median = 6 + \left(\frac{\frac{22}{2} - 7}{6} \right) 2$$

$$Median = 6 + \left(\frac{11 - 7}{6} \right) 2$$

$$Median = 6 + \left(\frac{4}{6} \right) 2 = 6 + \frac{8}{6}$$

$$Median = 6 + 1.33$$

$$Median = 7.33$$

Example 2.18

The following are the data generated on the sales of tyres:

Tyre range	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12
Frequency	3	4	6	2	5

Calculate the mode and median.

Solution

Tyre range	**Frequency**	**Cumulative Frequency**
2 – 4	3	3
5 – 7	4	7
8 – 10	6 → Modal class	13 → Median class
11 – 13	2	15
14 – 16	5	20
	$N = \sum f = 20$	

$L_{mo} = 8.5$, $\Delta_1 = 6 - 4 = 2$, $\Delta_2 = 6 - 2 = 4$, $c = 4.5 - 1.5 = 3$

$$Mode = 8.5 + \left(\frac{2}{2 + 4} \right) 3$$

$$Mode = 8..5 + \left(\frac{2}{6}\right)3 = 8.5 + \frac{6}{6}$$

$$Mode = 8.5 + 1$$

$$Mode = 9.5$$

$$Median\ position = \frac{N}{2}th = \frac{20}{2}th = 10^{th}$$

$$L_{me} = 8.5,\ f_{me} = 6,\ \sum f_{me} = 3 + 4 = 7,\ c = 3,\ N = 20$$

$$Median = 8.5 + \left(\frac{\frac{20}{2} - 7}{6}\right)3$$

$$Median = 8.5 + \left(\frac{10 - 7}{6}\right)3$$

$$Median = 8.5 + \left(\frac{3}{6}\right)3 = 8.5 + \frac{9}{6}$$

$$Median = 8.5 + 1.5$$

$$Median = 10$$

Characteristics or features of each measure

i. Mean
- It takes all observations into consideration;
- It is used for further statistical calculations; and
- It is affected by extreme values.

ii. Median
- It does not take all observation into consideration;
- It is not affected by extreme values; and
- It is not used for further statistical calculations.

iii. Mode

- It does not take all observations into consideration;
- It is not affected by extreme values;
- It is not used for further statistical calculations;
- It can be used by manufacturers to know where to concentrate production; and
- It may not be unique (i.e. it may be multimodal).

2.2 Measures of partition

These are other means of measuring location. It can be recalled that the median characterises a series of values by its midway position. By extension of this idea, there are other measures which divide a series into a number of equal parts but which are not measures of central tendency. These are the measures of partition. The common ones are the quartiles, deciles and the percentiles. They are collectively called the QUANTILES.

The method of computation for the quartiles follows the same procedure for the Median according to what is being measured. For instance, the position of the first quartile for a grouped data is located by $\frac{N}{4}$ and the value is determined graphically from Ogive or by use of formula as done in the Median.

Also the position for locating third quartile is $\frac{3N}{4}$, for the seventh decile is $\frac{7N}{10}$ and tenth percentile is $\frac{10N}{100}$. We should note that the second quartile, fifth decile and fiftieth percentile coincide with the Median.

The use of formulae to compute the quantiles are summarized below:

$$Quartile(Q_i) = L_i + \left(\frac{\frac{iN}{4} - \sum f_i}{f_i} \right) c, \qquad i = 1, 2, 3;$$

$$Decile(D_i) = L_i + \left(\frac{\frac{iN}{10} - \sum f_i}{f_i} \right) c, \qquad i = 1, 2, 3, 4, 5, 6, 7, 8, 9; \text{ and}$$

$$Percentile(P_i) = L_i + \left(\frac{\frac{iN}{100} - \sum f_i}{f_i} \right) c, \qquad i = 1, 2, 3, 4, 5, \ldots, 97, 98, 99.$$

Where

L_i = Lower class boundary of the i^th quantile class;

f_i = Frequency of the i^th quantile class;

$\sum f_i$ = Sum of the frequencies of all classes lower than i^th quantile class ;

c = Class size of the i^th quantile class; and

N = Total number of items in the distribution

Example 2.19

Calculate Q_1, Q_3, D_7 and P_{20} for the following data:

Class interval	***X***	***f***	***cf***	**Less than Class Boundaries**
0 – 2	1	2	2	2
3 – 5	4	4	6	5
6 – 8	7	8	14	8
9 – 11	10	4	18	11
12 – 14	13	2	20	14
		$\sum f = 20$		

0

Solution

Position of $Q_1 = \left(\frac{N}{4}\right)^{th} = \left(\frac{20}{4}\right)^{th} = 5^{th}$

Q_1 class $= 3 - 5$

$L_1 = 2.5,\ \frac{N}{4} = 5,\ \sum f_1 = 2,\ f_1 = 4,\ c = 3$

$$Q_1 = L_1 + \left(\frac{\frac{N}{4} - \sum f_1}{f_1}\right)c$$

$$Q_1 = 2.5 + \left(\frac{5-2}{4}\right)3$$

$$Q_1 = 2.5 + \left(\frac{3}{4}\right)3$$

$$Q_1 = 2.5 + 2.25$$.

$$Q_1 = 4.75$$

Position of $Q_3 = \left(\frac{3N}{4}\right)^{th} = \left(\frac{3\times 20}{4}\right)^{th} = 15^{th}$

Q_3 class $= 9 - 11$

$L_3 = 8.5,\ \frac{3N}{4} = 15,\ \sum f_3 = 2+4+8 = 14,\ f_3 = 4,\ c = 3$

$$Q_3 = L_3 + \left(\frac{\frac{3N}{4} - \sum f_3}{f_3}\right)c$$

$$Q_3 = 8.5 + \left(\frac{15-14}{4}\right)3$$

$$Q_3 = 8.5 + \left(\frac{1}{4}\right)3$$

$$Q_3 = 8.5 + 0.75$$.

$$Q_3 = 9.25$$

Position of $D_7 = \left(\frac{7N}{10}\right)^{th} = \left(\frac{7\times 20}{10}\right)^{th} = 14^{th}$

D_7 class $= 6 - 8$

$L_7 = 5.5$, $\frac{7N}{10} = 14$, $\sum f_7 = 2 + 4 = 6$, $f_7 = 8$, $c = 3$

$$D_7 = L_7 + \left(\frac{\frac{7N}{10} - \sum f_7}{f_7}\right)c$$

$$D_7 = 5.5 + \left(\frac{14-6}{8}\right)3$$

$$D_7 = 5.5 + \left(\frac{8}{8}\right)3$$

$$D_7 = 5.5 + 3.$$

$$D_7 = 8.5$$

Position of $P_{20} = \left(\frac{20N}{100}\right)^{th} = \left(\frac{20\times 20}{100}\right)^{th} = 4^{th}$

P_{20} class $= 3 - 5$

$L_{20} = 2.5$, $\frac{20N}{100} = 4$, $\sum f_{20} = 2$, $f_{20} = 4$, $c = 3$

$$P_{20} = L_{20} + \left(\frac{\frac{20N}{100} - \sum f_{20}}{f_{20}}\right)c$$

$$P_{20} = 2.5 + \left(\frac{4-2}{4}\right)3$$

$$P_{20} = 2.5 + \left(\frac{2}{4}\right)3$$

$$P_{20} = 2.5 + 1.5$$

$$P_{20} = 4$$

NOTE:

The quantiles can be estimated from an ogive just in the same way we estimated the median.

2.5 Chapter summary

In this chapter, the measures of central tendency, which consist of mean, mode and median, were considered for both grouped and ungrouped data. The measures of location such as Quartiles, Deciles and Percentiles were also discussed in the chapter.

2.6 Multiple-choice and short-answer questions)

1. The mean of the following set of numbers 2,4,6,8,10 is
   A. 4
   B. 5
   C. 6
   D. 7
   E. 8

2. Which of the following is not a measure of central tendency?
   A. Mean
   B. Mode
   C. Median
   D. Decile
   E. 2^{nd} quartile

3. Which of the following is not a measure of partition?
   A. Median
   B. Mode
   C. Percentile
   D. Quantiles
   E. Deciles

4. Which of the following formulae is used for the computation of quartile?
   A. $Q_1 = L_1 + \left(\dfrac{\frac{N}{2} - \sum f_1}{f_1} \right) c$
   B. $Q_1 = L_1 + \left(\dfrac{\frac{3N}{4} - \sum f_1}{f_1} \right) c$

C. $Q_1 = L_1 + \left(\dfrac{\frac{N}{4} - \sum f_1}{f_1} \right) c$

D. $Q_1 = L_1 + \left(\dfrac{\frac{N}{10} - \sum f_1}{f_1} \right) c$

E. $Q_1 = L_1 + \left(\dfrac{\frac{N}{100} - \sum f_1}{f_1} \right) c$

5. In the graphical method of obtaining the quartiles, which of the following diagrams is used?
   A. Bar chart
   B. Histogram
   C. Pie chart
   D. Ogive
   E. component bar chart.

Use the following data to answer questions 6 to 10: 6, 3, 8, 8, 5

6. Calculate the Arithmetic mean.

7. Determine the Median.

8. Determine the Mode.

9. Find the sum of the mode and the mean.

10. Find the difference between the median and the mean.

Answers

1. C
2. D
3. B
4. C
5. D
6. $Arithmetic\ mean, \bar{x} = \frac{6+3+8+8+5}{5} = \frac{30}{5} = 6$
7. 6, 3, 8, 8, 5

   Rearrange $\rightarrow$ 3, 5, 6, 8, 8

   $\therefore$ Median = 6.

8. Mode = 8

9. Mean is 6
   Mode is 8
   $\therefore$ Sum = 6+8 =14

10. Mean is 6

    Median is 6

    $\therefore$difference = 6 – 6 = 0

CHAPTER THREE

MEASURES OF VARIATION

Chapter content

(a) Introduction.

(b) Measures of Variation.

(c) Coefficient of Variation and coefficient of Skewness.

Objectives

At the end of the chapter, readers should be able to

(a) understand the meaning of measures of variation;

(b) determine various measures of variation such as mean deviation, variance, standard deviation and quantile deviation for both grouped and ungrouped data; and

(c) compute the coefficient of variation and coefficient of skewness.

3.1 Introduction

The degree to which numerical data tend to spread about an average value is referred to as measure of variation or dispersion. At times, it is called measure of spread. The purpose and significant uses of these measures are of paramount importance in statistics.

The popular ones among these measures of variation are the Range; Mean Deviation; Standard Deviation; Semi – Inter-Quartile Range or Quartile Deviation; Coefficient of Variation; and Skewness.

3.2 Measures of Variation

i) Range

The range for a set of data, is the difference between the highest number and the lowest number of the data. That is,

Range = Highest number – Lowest number (for ungrouped data); OR

Range = Upper bound of the last class – Lower bound of the first class (for grouped data)

Example 3.1

Determine the range for each of the following sets of numbers

a. 32, 6, 10, 27, 30, 5,45
b. 9, 14, 16, 13, 14, 21

Solution

The data here is ungrouped type

a. $Range$ = Highest – Lowest = 45 – 5 = 40

b. $Range$ = Highest – Lowest = 21 – 9 = 12

Example 3.2

Obtain the range for the following table

Class	Frequency
1 – 10	4
11 – 20	10
21 – 30	12
31 – 40	12
41 – 50	9

Solution

The data in this question is grouped.

Range = 50 – 1 = 49

ii) Mean Deviation (MD)

The Mean Deviation is the arithmetic mean of the absolute deviation values from the mean. For ungrouped data $(x_1, x_2, \ldots, x_n)$, the M.D is given as

$$MD = \frac{\sum |x_i - \bar{x}|}{n} = \frac{\sum |d_i|}{n} \qquad 3.1$$

where $d_i = x_i - \bar{x}$, the symbol $|\;|$ stands for modulus or absolute value.

For grouped data, it is written as:

$$MD = \frac{\sum f|x_i - \bar{x}|}{\sum f} = \frac{\sum f|d_i|}{\sum f}$$

Example 3.3

Calculate the mean deviation from number of books sold in a small bookshop given as: 15, 17, 14, 16, 18

Solution

The mean, $\bar{x} = \frac{\sum x}{n} = \frac{15+17+14+16+18}{5} = \frac{80}{5} = 16$

The following format shall be used for Mean Deviation:

x	$d_i = x_i - \bar{x}$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$
15	-1	1
17	1	1
14	-2	2
16	0	0
18	2	2
		$\sum \lvert d_i \rvert = 6$

Then, the mean deviation, $MD = \frac{\sum \lvert x_i - \bar{x} \rvert}{n} = \frac{\sum \lvert d_i \rvert}{n} = \frac{6}{5} = 1.2$

Example 3.4

The distribution of book sales (in hundreds) in a bookshop is given in the table below. Determine the mean deviation from this table

Class	Frequency
1 – 10	10
11 – 20	15
21 – 30	17
31 – 40	13
41 – 50	05

Solution

Class interval	f	x	fx	$\|d_i\| = \|x_i - \bar{x}\|$	$f\|d_i\|$
1 – 10	10	5.5	55	18	180
11 – 20	15	15.5	232.5	8	120
21 – 30	17	25.5	433.5	2	34
31 – 40	13	35.5	461.5	12	156
41 – 50	05	45.5	227.5	22	110
	$\sum f = 60$		$\sum fx = 1410$		$\sum f\|d_i\| = 600$

$$Mean\,(\bar{x}) = \frac{\sum fx}{\sum f} = \frac{1410}{60} = 23.5$$

$$Mean\;\; deviation\,(MD) = \frac{\sum f|d_i|}{\sum f} = \frac{600}{60} = 10$$

iii) Standard deviation

Standard deviation (SD) is the square root of the variance.

Variance is the square of the difference between each value in a data set and the mean of the group.

In an ungrouped data, the population standard deviation (σ) is given as

$$\sigma = \sqrt{\frac{\sum (x-\mu)^2}{N}} \qquad 3.3$$

Where,

μ is the population mean; and

N is the total number of items in the population.

NOTE: The standard deviation(s) for a sample is not exactly equal to the population standard deviation. It is given as

$$s = \sqrt{\frac{\sum (x-\bar{x})^2}{n}} \qquad 3.4$$

where $\bar{x}$ is the sample mean and n is the sample size.

Also for grouped data, the population standard deviation is

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} \qquad 3.5$$

Example 3.5

The figures 9, 5, 9, 7, 10, 14, 12, 10, 6, 17 represent the volumes of sales (₦‟000) of Adebayo Spring Water within the first ten days of operation. Calculate the population standard deviation of sales.

Solution

Let the sales be represented by *x*, then,

$$\sigma^2 = V(x) = \frac{\sum (x-\mu)^2}{N} \qquad where \quad \mu = \frac{\sum x}{N}$$

$$\mu = \frac{9+5+9+7+10+14+12+10+6+17}{10} = \frac{99}{10} = 9.9$$

$$\therefore \quad \sigma^2 = \frac{\begin{array}{c}(9-9.9)^2 + (5-9.9)^2 + (9-9.9)^2 + (7-9.9)^2 + (10-9.9)^2 + \\ (14-9.9)^2 + (12-9.9)^2 + (10-9.9)^2 + (16-9.9)^2 + (17-9.9)^2\end{array}}{10}$$

$$\therefore \quad \sigma^2 = \frac{0.81+24.01+0.81+8.41+0.01+16.81+4.41+0.01+37.21+50.41}{10} = \frac{142.9}{10}$$

$$\sigma^2 = 14.29$$

$$\sigma = SD = \sqrt{V(x)}$$

$$\sigma = SD = \sqrt{14.29} = 3.78$$

Example 3.6

The table below shows the frequency table of age distribution of 25 children in a family.

Age (x)	5	8	11	14	17
Frequency (f)	6	7	5	4	3

Determine the sample standard deviation

Solution

X	f	fx	$d_i = x_i - \bar{x}$	$d_i^{\ 2}$	$fd_i^{\ 2}$
5	6	30	-4.92	24.2064	145.2384
8	7	56	-1.9	3.61	25.27
11	5	55	1.1	1.21	6.05
14	4	56	4.1	16.81	67.24
17	3	51	7.1	50.41	151.23
	$\sum f = 25$	$\sum fx = 248$			$\sum fd_i^{\ 2} = 395.0284$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{248}{25} = 9.92$$

Where $d_i = x_i - \bar{x} = x_i - 9.92$

$$s^2 = \frac{\sum (x_i - 9.92)^2}{\sum f} = \frac{\sum fd_i^{\ 2}}{\sum f} = \frac{395.0284}{25} = 15.801$$

$$s^2 = 15.801$$

$$s = \sqrt{15.801} = 3.98$$

Remark:

From the above computations, it is clearly seen that the set of formulae used above is tedious. Hence, the call or demand for a short cut and easier method is necessary. The following short-cut formulae will be used:

For ungrouped data

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \quad or \quad \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n}} \quad or \quad \sqrt{\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n}}$$

Also for grouped data

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \quad or \quad \sqrt{\frac{\sum fx^2 - (\sum f)\bar{x}^2}{\sum f}} \quad or \quad \sqrt{\frac{\sum fx^2 - \frac{\left(\sum fx\right)^2}{\sum f}}{\sum f}}$$

Example 3.7 (Another method for ungrouped data)

The values 9, 5, 9, 7, 10, 14, 12, 10, 6, 17 represent the volume of sales (₦‟000) of Stephen Pure Water within the first ten days of operation. Calculate the population standard deviation of sales using the shortcut method.

Solution

Let the sales be represented by *x*, then

$$s^2 = \frac{\sum x^2 - n\bar{x}^2}{n}$$

$$\bar{x} = \frac{\sum x}{n} = \frac{9+5+9+7+10+14+12+10+6+17}{10} = \frac{99}{10} = 9.9$$

$$\sum x^2 = 9^2 + 5^2 + 9^2 + 7^2 + 10^2 + 14^2 + 12^2 + 10^2 + 6^2 + 17^2 = 1101$$

$$s^2 = \frac{1101 - 10(9.9)^2}{10} = \frac{1101 - 980.1}{10} = \frac{120.9}{10} = 12.09$$

$$s^2 = 12.09$$

$$s = \sqrt{12.09} = 3.48$$

Example 3.8

Using the data given in example 3.6 to find the standard deviation

x	***f***	***fx***	***x²***	***fx²***
5	6	30	25	150
8	7	56	64	448
11	5	55	121	605
14	4	56	196	784
17	3	51	289	867
	$\sum f = 25$	$\sum fx = 248$		$\sum fx^2 = 2854$

$$s = \sqrt{\frac{\sum fx^2 - (\sum f)\bar{x}^2}{\sum f}} \quad or \quad \sqrt{\frac{\sum fx^2 - \frac{\left(\sum fx\right)^2}{\sum f}}{\sum f}}$$

Using,

$$s = \sqrt{\frac{\sum fx^2 - \frac{\left(\sum fx\right)^2}{\sum f}}{\sum f}}$$

$$s = \sqrt{\frac{2854 - \frac{(248)^2}{25}}{25}} = \sqrt{\frac{2854 - \frac{61504}{25}}{25}} = \sqrt{\frac{2854 - 2460.16}{25}}$$

$$s = \sqrt{\frac{393.84}{25}} = \sqrt{15.75} = 3.97$$

Semi-Interquartile Range (SIR) or Quartile Deviation

Semi-interquartile range is obtained from quartile range and it is defined by:

$$SIR = \frac{Q_1 - Q_2}{2}$$

where Q_1 = First quartile, Q_3 = Third quartile and the numerator $Q_3 - Q_1$ is known as the quartile range.

Example 3.9

Determine the (a) standard deviation (b) semi – interquartile range for the income distribution of SAO company employees as given in the following table:

Income class in ₦‘‘000	10 – 20	20 – 30	30 – 40	40 – 50
Frequency (*f*)	7	10	8	5

Solution

Income Class	f	x	fx	fx^2	cf
10 – 20	7	15	105	1,575	7
20 – 30	10	25	250	6,250	17
30 – 40	8	35	280	3,200	25
40 – 50	5	45	225	10,125	30
	$\sum f = 30$		$\sum fx = 860$	$\sum fx^2 = 27{,}750$	

(a) $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{860}{30} = 28.667$

$$SD = \sqrt{\frac{\sum fx^2 - \left(\sum f\right)\bar{x}^2}{\sum f}}$$

$$SD = \sqrt{\frac{27{,}750 - (30)(28.667)^2}{30}} = \sqrt{\frac{27{,}750 - 24{,}653.33}{30}} = \sqrt{\frac{3{,}096.67}{30}}$$

$$SD = \sqrt{103.22} = 10.16$$

Alternatively,

$$SD = \sqrt{\frac{\sum fx^2 - \frac{\left(\sum fx\right)^2}{\sum f}}{\sum f}}$$

$$SD = \sqrt{\frac{27{,}750 - \frac{(860)^2}{30}}{30}} = \sqrt{\frac{27{,}750 - 24653.33}{30}} = \sqrt{\frac{3{,}096.67}{30}}$$

$$SD = \sqrt{103.22} = 10.16$$

(b) Position of $Q_1 = \left(\dfrac{N}{4}\right)^{th} = \left(\dfrac{30}{4}\right)^{th} = 7.5^{th}$

Q_1 class $= 20 - 30$

$L_1 = 20$, $\dfrac{N}{4} = 7.5$, $\sum f_1 = 7$, $f_1 = 10$, $c = 30 - 20 = 10$

$$Q_1 = L_1 + \left(\frac{\frac{N}{4} - \sum f_1}{f_1}\right)c$$

$$Q_1 = 20 + \left(\frac{7.5 - 7}{10}\right)10$$

$$Q_1 = 20 + \left(\frac{0.5}{10}\right)10$$

$$Q_1 = 20 + 0.50\,.$$

$$Q_1 = 20.50$$

Position of $Q_3 = \left(\frac{3N}{4}\right)^{th} = \left(\frac{3\times 30}{4}\right)^{th} = 22.5^{th}$

Q_3 class $= 30 - 40$

$L_3 = 30\,,\ \frac{3N}{4} = 22.5\,,\ \sum f_3 = 7 + 10 = 17\,,\ f_3 = 8\,,\ c = 40 - 30 = 10$

$$Q_3 = L_3 + \left(\frac{\frac{3N}{4} - \sum f_3}{f_3}\right)c$$

$$Q_3 = 30 + \left(\frac{22.5 - 17}{8}\right)10$$

$$Q_3 = 30 + \left(\frac{5.5}{8}\right)10$$

$$Q_3 = 30 + 6.88 \qquad .$$

$$Q_3 = 36.88$$

$\therefore$ semi-interquartile range $= \frac{Q_1 - Q_2}{2} = \frac{36.88 - 20.50}{2} = \frac{16.38}{2} = 8.19$

3.3Coefficient of Variation (CV) and coefficient of Skewness

Coefficient of Variation (CV)

This is a useful statistical tool. It shows the degree of variation between two sets of data. It is a dimensionless quantity and defined as:

$$CV = \frac{SD}{Mean} \times \frac{100}{1}(\%) \qquad (\,where \quad SD = \text{Standard deviation})$$

$$SIR = \frac{s}{\bar{x}} \times \frac{100}{1}(\%)$$

By a way of interpretation, the smaller the C.V of a data set, the higher the precision of that set. Also, in comparison, the data set with least C.V has a better reliability than the other.

Example 3.10

The following data consists of the ages of twelve banks during their last anniversary in Oyo state. Calculate their coefficient of variation:

15, 14, 8, 12, 11, 10, 16, 14, 12, 11, 13, 8

Solution

$$CV = \frac{SD}{Mean} \times \frac{100}{1} (\%) = \frac{s}{\bar{x}} \times \frac{100}{1} (\%)$$

$$\bar{x} = \frac{\sum x}{n} = \frac{15+14+8+12+11+10+16+14+12+11+13+8}{12} = \frac{144}{12} = 12$$

$$s^2 = \frac{\sum x^2 - n\bar{x}^2}{n}$$

$$where \quad \sum x^2 = 15^2 + 14^2 + 8^2 + 12^2 + 11^2 + 10^2 + 16^2 + 14^2 + 12^2 + 11^2 + 13^2 + 8^2 = 1800$$

$$s^2 = \frac{1800 - 12(12)^2}{12} = \frac{1800 - 1728}{12} = \frac{72}{12} = 6.0$$

$$s^2 = 6.0$$

$$s = \sqrt{6.0} = 2.45$$

$$\therefore \quad CV = \frac{s}{\bar{x}} \times \frac{100}{1} = \frac{2.45}{12} \times \frac{100}{1} = 20.42\%$$

Coefficient of skewness

If a frequency curve is not symmetrical, its skewness is the degree of asymmetry of the distribution.

The Pearsonian coefficient of skewness is given as:

$$= \frac{Mean - Mode}{SD} \qquad (\text{where} \quad SD = \text{Standard deviation})$$

OR it can also be obtained as

$$= \frac{3(Mean - Median)}{SD}$$

The coefficient of skewness is zero for a symmetry or normal distribution. For a positively skewed distribution, the mean is larger than the mode, while for a negatively skewed distribution the mean is smaller than the mode. The median takes a value between the mean and the mode.

Example 3.11

Given the following means, the medians and the standard deviations of two distributions:

A	Mean = 20,	Median = 22	and	Standard Deviation = 8
B	Mean = 20,	Median = 23	and	Standard deviation = 10

Determine which of the distributions is more skewed.

Solution

By the Pearson measure of skewness, we have

$$Skewness = \frac{3(Mean - Median)}{SD}$$

$$Skewness \ \ for \ \ distribution \ A = \frac{3(20-22)}{8} = \frac{-6}{8} = -0.75$$

$$Skewness \ \ for \ \ distribution \ B = \frac{3(20-23)}{10} = \frac{-9}{10} = -0.90$$

Since $|-0.90| > |-0.75|$, then distribution *B* is more skewed.

3.3Chapter summary

Measure of variation has been described as a measure of spread. It is commonly used to obtain the spread about the average and to make comparison of spreads for two sets of data. Among these measures of dispersion, we have the range, mean deviation, standard deviation and semi – interquartile range. Here, both ungrouped and grouped data were considered.

Also, the concepts of both coefficients of variation and skewness were discussed.

3.5 Multiple-choice and short answers questions

1. For a set of data, the difference between the highest and the lowest number is known as……………….
   A. Mean
   B. Variance
   C. Range
   D. Interquartile range
   E. Mean deviation

2. The main difference between the mean deviation and variance is ………………...
   A. That differences between the data set and mean are zero.
   B. That differences between the data set and the mean are squared before being summarized in variance.
   C. That the square roots of differences between the data set and the mean are obtained
   D. The difference in the order of arrangement.
   E. That differences between the data set and the mean are in geometric order

3. Semi – Interquartile range is determined by
   A. $\dfrac{Q_3 - Q_2}{2}$
   B. $\dfrac{Q_3 - Q_1}{2}$
   C. $\dfrac{Q_3 - Q_2}{2}$
   D. $\dfrac{Q_1 - Q_3}{2}$
   E. $\dfrac{Q_2 - Q_1}{2}$

4. For a non-skewed distribution, the coefficient of skewness is
   A. 1
   B. –1
   C. 0
   D. 2
   E. 2

Use the following set of data:3 ,7, 2, 8, 5, 6, 4 to answer questions 5 to 10.

5. Determine the range.
6. Determine the mean deviation.
7. Determine the standard deviation.
8. Determine Q_1.
9. Determine Q_3.

10. Determine Semi-interquartile range.

Solutions to multiple-choice and short answer questions

1. C
2. B
3. B
4. C
5. 5
6. 2
7. $\sqrt{\frac{40}{7}} = \sqrt{5.714} = 2.39$
8. $Q_1 = 2$
9. $Q_3 = 6$
10. $SIR = \frac{Q_3 - Q_1}{2} = \frac{6-2}{2} = \frac{4}{2} = 2$

0

CHAPTER FOUR

MEASURES OF RELATIONSHIPS

Chapter contents

(a) Introduction.

(b) Types of Correlation.

(c) Measure of Correlation.

(d) Simple Regression Line.

(e) Coefficient of Determination.

Objectives

At the end of the chapter, readers should be able to understand the

(a) concepts of univariate and bivariate;

(b) concept of scatter diagram;

(c) various types of correlation coefficients;

(d) computation of Pearson product moment correlation coefficient and make necessary interpretations;

(e) computation of Spearman's rank correlation coefficient and make necessary interpretations;

(f) concept of regression;

(g) fitting of a simple linear regression model to data;

(h) fitting of a simple linear regression line using

  (i) graphical method;

  (ii) formula (the least squares method);

(i) making of necessary interpretation of the regression constant and regression coefficient;

(j) forecasting/estimating by the use of fitted regression line; and

(k) meaning and computation of coefficient of determination.

4.1 Introduction

The type of data we have been using so far is the univariate type, i.e. one variable. But in this chapter, we shall be dealing with bivariate data, i.e. two variables.

The statistical analysis that requires the use of bivariate data is generally termed the measure of relationship and regression analysis which is the focus of this chapter. In this analysis, the interest is usually on relationship or pattern of the relationship. Here, one can look at how students' performance in one subject (Mathematics) affects another subject (Accountancy). Another good example is the consideration of how an increase in the income affects spending habits or savings.

In the measure of relationship, the usual practice is to quantify or qualify and represent the bivariate by letters. Taking for instance, the marks scored by a set of students in Mathematics and Accounts can be represented by x and y variables respectively and can be written as point (x, y), where x and y stand for independent and dependent variables respectively.

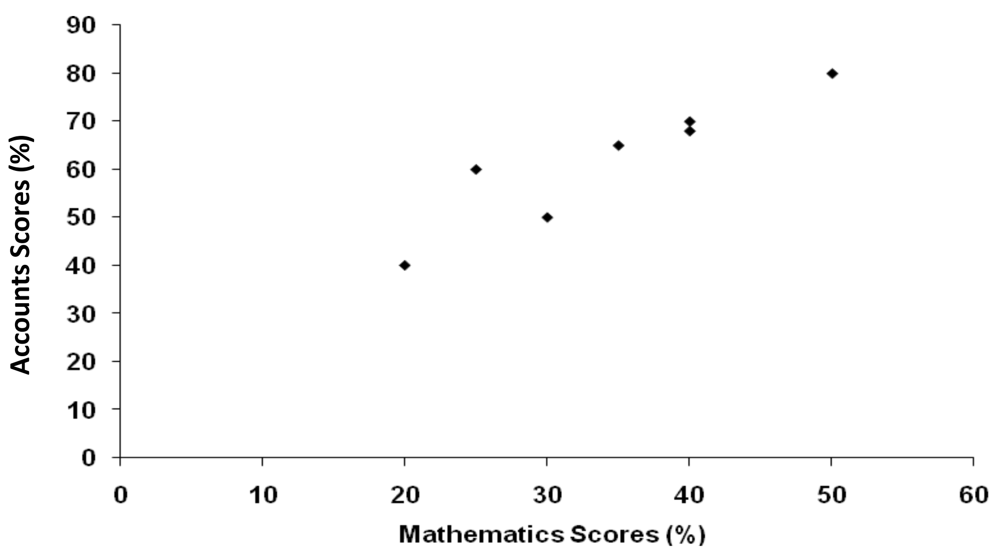
A graphical presentation of bivariate data on a two-axis coordinate graph is known as the SCATTER DIAGRAM. Here, the bivariate data are plotted on a rectangular coordinate system in order to see the existing relationship between the two variables under study. The following bivariate data will be used as an illustration:

Example 4.1
Draw the scatter diagram for the following data:

Mathematics Scores (x_i) in %	30	40	35	40	20	25	50
Accountancy scores (y_i) in %	50	70	65	68	40	60	80

Solution

Scatter diagram



4.2 Types of Correlation

Correlation measures the degree of association (relationship) between two variables. Therefore, two variables are said to be correlated or related when change in one variable results in the change of the other variable. The degree of correlation (r) between two variables x and y is expressed by a real number which lies between –1.0 and +1.0 inclusive $(i.e. -1 \le r \le 1)$ and it is called correlation coefficient or coefficient of correlation. Simple correlation is basically classified according to the value of its coefficient. Hence, we have

i. Positive correlation $(0 < r < 1)$;
ii. Perfect positive correlation $(r = 1)$;
iii. Negative correlation $(-1 < r < 0)$;
iv. Perfect negative correlation $(r = -1)$; and
v. Zero correlation $(r = 0)$.

The following scatter diagrams depict the above types of correlation:

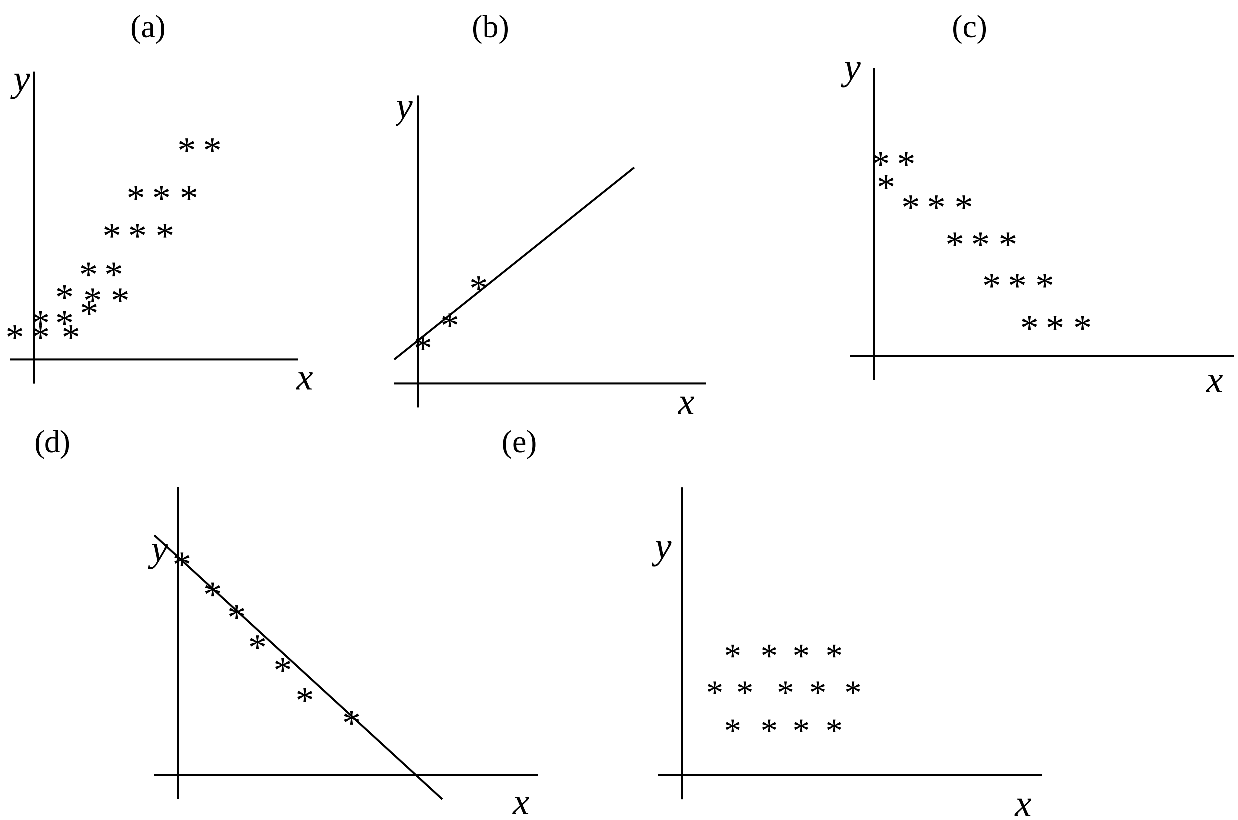


Figure (a) depicts positive correlation, where the two variables involved are directly proportional to each other. It means that when one variable increases (decreases), the other variable also increases (decreases). Therefore, there is positive correlation when two variables are increasing or decreasing together i.e. when the two variables move in the same direction. An example of paired variables that give positive correlation includes income and expenditure of a family

Figure (b) depicts a perfect positive correlation where the changes in the two related variables are exactly proportional to each other and in the same direction.

In figure(c), there is Negative correlation. Here is a situation where one variable increases (decreases), the other variable decreases (increases). Therefore, the two variables move in opposite directions. Examples of paired variables that give negative correlation include:

i. Number of labourers and time required to complete field work.

ii. Demand and price for a commodity.

Figure (d) depicts a perfect negative correlation where the changes in the two related variables are exactly proportional to each other but at reverse (opposite) directions.

Figure (e) depicts the zero correlation, where there is no correlation existing between the two variables. Here, no fixed pattern of movement can be established.

4.3 Measures of Correlation

The measure of correlation can be determined by using the following methods:

a. Product moment correlation coefficient method (Karl Pearson‟s Coefficient of correlation):

The degree of correlation between variables x and y is measured by the product moment correlation co-efficient r, defined as

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}} \qquad \left(where \quad \bar{x} = \frac{\sum x}{n} \quad and \quad \bar{y} = \frac{\sum y}{n} \right)$$

$$\left.\begin{aligned} r &= \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\sqrt{\left[\sum x^2 - \frac{(\sum x)^2}{n}\right]\left[\sum y^2 - \frac{(\sum y)^2}{n}\right]}} \\ r &= \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{\left[n\sum x^2 - (\sum x)^2\right]\left[n\sum y^2 - (\sum y)^2\right]}} \end{aligned}\right\} \qquad 4.1$$

n = the number of pairs of observations

The numerator of equation 4.1 is called <u>co-variance</u> and the denominator is the square root of the product of the variances of variables x and y i.e.

$$r = \frac{Co\text{var}\,iance(x, y)}{\sqrt{Var(x)Var(y)}}$$

Example 4.2

The marks scored by seven students in Mathematics (x) and Accounts (y) are given below. If the maximum scores obtainable in mathematics and accounts are respectively 50 and 100, determine the Pearson correlation coefficient for these scores.

X	30	40	35	40	20	25	50
Y	50	70	65	68	40	60	80

Solution:

x	y	xy	x^2	y^2
30	50	1500	900	2500
40	70	2800	1600	4900
35	65	2275	1225	4225
40	68	2720	1600	4624
20	40	800	400	1600
25	60	1500	625	3600
50	80	4000	2500	6400
240	433	15595	8850	27849

Now, Pearson Correlation Coefficient

$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\sum x^2 - \left(\sum x\right)^2\right]\left[n\sum y^2 - \left(\sum y\right)^2\right]}}$$

$$r = \frac{7(15{,}595) - (240)(433)}{\sqrt{\left[7(8{,}850) - (240)^2\right]\left[7(27{,}849) - (433)^2\right]}}$$

$$r = \frac{109{,}165 - 103{,}920}{\sqrt{(61{,}950 - 57{,}600)(194{,}943 - 187{,}489)}}$$

$$r = \frac{5{,}245}{\sqrt{(4{,}350)(7{,}454)}}$$

$$r = \frac{5{,}245}{\sqrt{32424900}}$$

$$r = \frac{5{,}245}{5694.28661}$$

$$r = 0.92$$

Example 4.3

The costs on advertisement (x) and revenues (y) generated by a company for 10 months are given below:

Advertisement (x) (₦'000)	45	70	32	24	75	16	28	43	60	15
Revenue (y) (Million)	42	51	38	39	44	20	22	46	47	35

Determine the product moment correlation coefficient for the table.

Solution

Given in the question

X	*y*	*xy*	x^2	y^2
45	42	1890	2025	1764
70	51	3570	4900	2601
32	38	1216	1024	1444
24	39	936	576	1521
75	44	3300	5625	1936
16	20	320	256	400
28	22	616	784	484
43	46	1978	1849	2116
60	47	2820	3600	2209
15	35	525	225	1225
408	384	17171	20864	15700

Now, Pearson Correlation Coefficient

$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\sum x^2 - \left(\sum x\right)^2\right]\left[n\sum y^2 - \left(\sum y\right)^2\right]}}$$

$$r = \frac{7(15{,}595) - (240)(433)}{\sqrt{\left[7(8{,}850) - (240)^2\right]\left[7(27{,}849) - (433)^2\right]}}$$

$$r = \frac{109{,}165 - 103{,}920}{\sqrt{(61{,}950 - 57{,}600)(194{,}943 - 187{,}489)}}$$

$$r = \frac{5{,}245}{\sqrt{(4{,}350)(7{,}454)}}$$

$$r = \frac{5{,}245}{\sqrt{32424900}}$$

$$r = \frac{5{,}245}{5694.28661}$$

$$r = 0.92$$

(b) **Rank correlation (Spearman's Rank Correlation)**

There are occasions when we wish to put some objects in an ordinal scale without giving actual marks to them. For instance, during a beauty contest, the judges place the contestants in some order like first, second, third etc. without actually giving specific marks to them. This process is called ranking and the ordinal scale is used.

Ranking occurs where either for lack of time, money or suitable measurements may be impossible to quantify the items. In dealing with a correlation problem where the values are in ranks, rank correlation methods are used.

The best known of such methods is the Spearman's Rank correlation co-efficient which is defined as:

$$R = 1 - \frac{6\sum d_i^{\,2}}{n(n^2 - 1)} \qquad 4.2$$

where d_i = difference in each pair of ranks; and

n = number of objects being ranked;

R is defined in such a way that when the ranks are in perfect agreement,

R equals +1 and when they are in perfect disagreement,

R equals –1; otherwise -1< R < 1.

Example 4.4

If two judges' P and Q ranked 10 contestants in a beauty show as follows:

Contestant	H	I	J	K	L	M	N	O	P	Q
Rank by P	4	2	5	6	1	8	3	7	10	9
Rank by Q	3	4	2	8	1	9	5	6	10	7

Obtain the Spearman's rank correlation coefficient for the given data.

Solution

Contestant	H	I	J	K	L	M	N	O	P	Q	
Rank by P (R_x)	4	2	5	6	1	8	3	7	10	9	
Rank by Q(R_y)	3	4	2	8	1	9	5	6	10	7	
$d = R_x - R_y$	1	-2	3	-2	0	-1	-2	1	0	2	
d^2	1	4	9	4	0	1	4	1	0	4	$\sum d^2 = 28$

where $n = 10$

$$R = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$$

$$R = 1 - \frac{6(28)}{10(10^2 - 1)}$$

$$R = 1 - \frac{6(28)}{10(99)}$$

$$R = 1 - \frac{168}{990}$$

$$R = 1 - 0.1697$$

$$R = 0.83$$

Example 4.5

The following are the marks obtained by five students in Mathematics (x) and Accounts (y). Determine the rank correlation coefficient for the data.

x	11	12	13	15	19
y	16	12	14	20	18

Solution

Since the marks are not given in ranks, we need to first rank the marks

x	y	Rank of x (R_x)	Rank of y (R_y)	$d = R_x - R_y$	d^2
11	16	5	3	2	4
12	12	4	5	-1	1
13	14	3	4	-1	1
15	20	2	1	1	1
19	18	1	2	-1	1
					$\sum d^2 = 8$

Note that ranking can be done in ascending order or descending order. What is important is that the order chosen to use must be the same for the two variables.

where $n = 5$

$$R = 1 - \frac{6\sum d_i^{\,2}}{n(n^2 - 1)}$$

$$R = 1 - \frac{6(8)}{5(5^2 - 1)}$$

$$R = 1 - \frac{6(8)}{5(24)}$$

$$R = 1 - \frac{48}{120}$$

$$R = 1 - 0.4$$

$$R = 0.6$$

Remark

Note that when ties of ranks occur (i.e. when two or more of the ranks are the same), each value of the ties is given the mean of those ranks. For example, if two places are ranked second, then each place takes the value of $\frac{2+3}{2} = \frac{5}{2} = 2.5$, since for the ties, one place would have been second while the other one would have been third. Similarly, if three places are ranked sixth, then each one is ranked $\frac{6+7+8}{3}=\frac{21}{3}=7$. Having done this, the procedure for getting R is the same as outlined before.

Example 4.6

The following are marks obtained in Mathematics (x) and Accounts (y) by 10 students. Calculate the rank correlation coefficient of the data.

Students	H	I	J	K	L	M	N	O	P	Q
x	14	12	18	14	17	16	20	11	13	19
y	13	11	15	18	15	19	15	12	14	20

Solution

X	*y*	Rank of *x* (R_x)	Rank of *y* (R_y)	$d = R_x - R_y$	d^2
14	13	6.5	8	-1.5	2.25
12	11	9	10	-1	1
18	15	3	5	-2	4
14	18	6.5	3	3.5	12.25
17	15	4	5	-1	1
16	19	5	2	3	9
20	15	1	5	-4	16
11	12	10	9	1	1
13	14	8	7	1	1
19	20	2	1	1	1
					$\sum d^2 = 48.5$

Note that in variable *x*, there are two persons with the same mark 14 and the positions to be taken by the two of them are 6th and 7th positions. Hence, the average position is $\frac{6+7}{2} = \frac{13}{2} = 6.5$.

For the variable *y*, there are three persons with the same mark of 15 and the positions to be taken by the three of them are 4th, 5th and 6th positions. Hence, the average position is $\frac{4+5+6}{3} = \frac{15}{3} = 5$.

Where $n = 10$

$$R = 1 - \frac{6\sum d_i^{\,2}}{n(n^2-1)}$$

$$R = 1 - \frac{6(48.5)}{10(10^2 - 1)}$$

$$R = 1 - \frac{291}{10(99)}$$

$$R = 1 - \frac{291}{990}$$

$$R = 1 - 0.2939394 = 0.7060806$$

$$R = 0.71$$

Example 4.7

Use the table in example 4.3 to compute its rank correlation coefficient.

x	y	Rank of x (R_x)	Rank of y (R_y)	$d = R_x - R_y$	d^2
45	42	7	6	1	1
70	51	9	10	-1	1
32	38	5	4	1	1
24	39	3	5	-2	4
75	44	10	7	3	9
16	20	2	1	1	1
28	22	4	2	2	4
43	46	6	8	-2	4
60	47	8	9	-1	1
15	35	1	3	-2	4
					$\sum d^2 = 30$

Where $n = 10$

$$R = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$$

$$R = 1 - \frac{6(30)}{10(10^2 - 1)}$$

$$R = 1 - \frac{180}{10(99)}$$

$$R = 1 - \frac{180}{990}$$

$$R = 1 - 0.18$$

$$R = 0.82$$

Remark: By comparing the two coefficients in examples 4.6 and 4.7, it could be seen that the Spearman's Rank Correlation Coefficient is on the high side compared with the Product Moment Correlation Coefficient. It therefore implies that the Spearman's Rank Correlation Coefficient is not as accurate as the Product Moment Correlation, but it is much easier to calculate.

Interpretation of Correlation

1. Interpretation of product moment correlation is as follows:
   - (a.) if *r* is close to ± 1, there is a perfect relationship between the two variables *x* and *y*. i.e. there is good correlation.
   - (b.) when *r* is close to zero (0), there is no relationship, that is, there is poor or non – existence of correlation.

   For example:
   $r = +0.92$, this implies good correlation/ strong positive correlation
   $r = -0.96$, this implies good correlation/strong negative correlation
   $r = -0.12$, this implies poor or non-existent correlation
   $r = 0$, this implies non-existent correlation or no correlation at all
   $r = +0.26$, this implies poor or non-existent correlation
   The same interpretation(s) apply to rank correlation *R.*

4.4 Simple Regression Line

In the previous section, we discussed how to measure the degree of association. In this section, the nature of the relationship that exists between two variables x and y will be looked at, the interest may be about the students' performance who offered mathematics and accounts. We want to know whether their performance in a subject like accounts is affected by how well they do in mathematics. In this case, our variable y represents the accounts (the dependent or response variable), while mathematics represents the independent or explanatory variable. The relationship between these two variables is characterized by mathematical model called a Regression Equation.

For a simple linear regression and for regression of y on x, we have:

$$y = a + bx$$

where y is the dependent variable and x is the independent variable;

a is the intercept of the line on the y-axis (i.e. the point where the line meets the y-axis); and

b is known as the regression coefficient. This is, the slope or gradient of the regression line and it indicates the type of correlation which exists between the two variables.

Note: It is important to distinguish between the independent and dependent variables. This is not necessary for correlation.

Methods of obtaining or fitting regression line

We have two methods of fitting the regression line. These are

(i) Graphical; and

(ii) Algebraic.

(c) **Graphical method:** The following steps are to be taken:

i. Draw the scatter diagram for the data.;

ii. Look at two points that a straight line will pass through on the diagram. One of the points ought to be $(\bar{x}, \bar{y})$;

iii. Estimate constants a and b from the graph; and

a = intercept on the y – axis of the drawn straight line.

B = slope or gradient of the line drawn i.e. slope $= \dfrac{vertical\ length}{horizontal\ length}$

(d) Regression line $y = a + bx$ is stated.

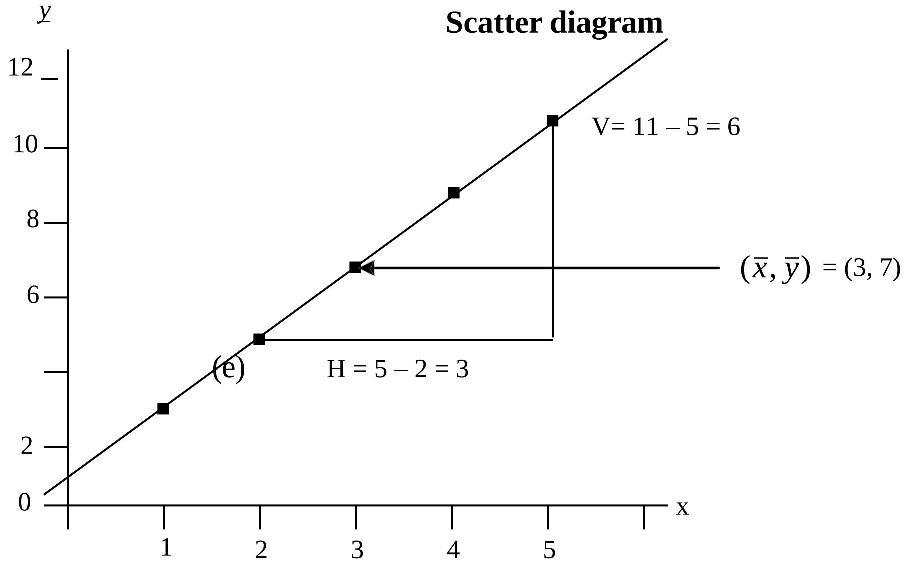
Example 4.8

Find the relationship between two variables y and x with the following data:

x	1	2	3	4	5
y	3	5	7	9	11

Solution

Scatter diagram



$$\text{Slope } (b) = \frac{Vertical\ length}{Horizontal\ length} = \frac{V}{H}$$

$$a = 1, b = \frac{V}{H} = \frac{6}{3} = 2$$

$$\therefore \quad y = a + bx \quad \Rightarrow \quad y = 1 + 2x$$

(f) **Algebraic method**: In the algebraic method, we use the "normal equation" which is derived by the Least Squares method. The said normal equations are:

$$an + b\sum x = \sum y \qquad 4.3$$

$$a\sum x + b\sum x^2 = \sum xy \qquad 4.4$$

which are used to fit the regression line of y on x as $y = a + bx$

It should be noted that when equations 4.3 and 4.4. are solved simultaneously, we have the following estimates of a and b:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} \qquad 4.5$$

$$a = \bar{y} - b\bar{x} = \frac{\sum y}{n} - b\frac{\sum x}{n} \qquad 4.6$$

Example 4.9

Use the table in Example 4.8 to calculate or fit the regression line of *y* on *x*.

Solution

x	y	xy	x^2
1	3	3	1
2	5	10	4
3	7	21	9
4	9	36	16
5	11	55	25
$\sum x = 15$	$\sum y = 35$	$\sum xy = 125$	$\sum x^2 = 55$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$b = \frac{5(125) - (15)(35)}{5(55) - (15)^2}$$

$$b = \frac{625 - 525}{275 - 225}$$

$$b = \frac{100}{50}$$

$$b = 2$$

$$a = \frac{\sum y}{n} - b\frac{\sum x}{n}$$

$$a = \frac{35}{5} - 2\left(\frac{15}{5}\right)$$

$$a = 7 - 2(3) = 7 - 6$$

$$a = 1$$

$\therefore \quad y = a + bx \quad \Rightarrow \quad y = 1 + 2x$ is the regression line of y on x

Note that the results of examples 4.8 and 4.9 are the same.

Example 4.10

The table below shows the income and expenditure (in ₦'000) of a man for 10 months.

Income (x)	8	18	52	38	26	60	40	50	82	75
Expenditure (y)	2	4	5	7	9	11	13	15	20	23

Fit simple linear regression line $y = a + bx$ to the data.

Solution

x	y	xy	x^2	y^2
8	2	16	64	4
18	4	72	324	16
52	5	260	2704	25
38	7	266	1444	49
26	9	234	676	81
60	11	660	3600	121
40	13	520	1600	169
50	15	750	2500	225
82	20	1640	6724	400
75	23	1725	5625	529
$\sum x = 449$	$\sum y = 109$	$\sum xy = 6143$	$\sum x^2 = 25261$	$\sum y^2 = 1619$

From the model

$$y = a + bx$$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$b = \frac{10(6143) - (449)(109)}{10(25261) - (449)^2}$$

$$b = \frac{61430 - 48941}{252610 - 201601}$$

$$b = \frac{12489}{51009}$$

$$b = 0.2448$$

$$a = \bar{y} - b\bar{x}$$

$$\bar{y} = \frac{\sum y}{n} = \frac{109}{10} = 10.9$$

$$\bar{x} = \frac{\sum x}{n} = \frac{449}{10} = 44.9$$

$$a = 10.9 - (0.2448)(44.9)$$

$$a = 10.9 - 10.9915$$

$$a = -0.0915\,\text{s}$$

$\therefore$ The fitted simple linear regression model is

$$y = a + bx \quad \Rightarrow \quad y = -0.0915 + 0.2448x$$

Computation or Fitting Regression Line of Variable (x) on Variable (y)

In section 4.4, fitting the regression of variable y on variable x was discussed. But in this section, we present the regression of variable x on variable y which gives the regression line.

$$x = a' - b'y$$

Where, $$b' = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - \left(\sum y\right)^2}$$

$$a' = \bar{x} - b'\bar{y}$$

We are to note that variables *x* and *y* are now dependent and independent variables respectively.

Example 4.11

Use the data in the following table to determine the regression line of variable *x* on variable y.

Income (*x*)	8	18	52	38	26	60	40	50	82	75
Expenditure (*y*)	2	4	5	7	9	11	13	15	20	23

Solution

x	y	xy	x^2	y^2
8	2	16	64	4
18	4	72	324	16
52	5	260	2704	25
38	7	266	1444	49
26	9	234	676	81
60	11	660	3600	121
40	13	520	1600	169
50	15	750	2500	225
82	20	1640	6724	400
75	23	1725	5625	529
$\sum x = 449$	$\sum y = 109$	$\sum xy = 6143$	$\sum x^2 = 25261$	$\sum y^2 = 1619$

$$b' = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - \left(\sum y\right)^2}$$

$$b' = \frac{10(6143) - (449)(109)}{10(1619) - (109)^2}$$

$$b' = \frac{61430 - 48941}{16190 - 11881}$$
$$b' = \frac{12489}{4309}$$
$$b' = 2.8984$$
$$a' = \frac{\sum x}{n} - b'\frac{\sum y}{n}$$
$$a = \frac{449}{10} - 2.8984\left(\frac{109}{10}\right)$$
$$a' = 44.9 - 2.8984(10.9)$$
$$a' = 44.9 - 31.5926$$
$$a' = 13.3074$$

$\therefore$ The regression model of x on y *is* $x = 13.3074 + 2.8984y$

- If this result is compared with the one obtained in example 4.10, it could be seen that the two results have nothing in common; it is not a question of change of subjects. Thus, regression of y on x is DEFINITELY different from regression of x on y.

Application and Interpretation of Linear Regression

(a) Since regression coefficient (*b*) is the slope of the regression line, its magnitude gives an indication of the steepness of the line and it can be Interpreted as thus-

i. If *b* = 0, it means the line is parallel to x axis;

ii. If *b* is high and positive, it gives very steep and upward slopping regression; and

iii. If *b* is negative, it gives downward sloping of the regression line.

(b) The regression line can be used for prediction. When the regression equation or regression line is used to obtain *y* value corresponding to a given *x* value, we say that *x* is used to predict *y*. We can equally use *y* to predict *x*.

Example 4.12

If $y = 16.94 + 0.96x$, what is y when $x = 50$.?

Solution

S*ubstituting $x = 50$ into the regression equation given in the question*

$$y = 16.94 + 0.96(50)$$

$$y = 16.94 + 48$$

$$y = 64.94$$

Example 4.13

Use the result obtained in Example 4.10 to find the expenditure when income is ₦29,000

Solution

$$y = - 0.0915 + 0.2448x$$

Income of 29,000 $\Rightarrow$ $x = 29$

$$\therefore \quad y = -0915 + (02448)(29)$$

$$= 7.0077$$

i.e. = ₦7,007.70

4.5 Chapter summary

Treatment of bivariate data by the use of correlation and regression analyses is presented. Correlation measures the degree of association while regression gives the pattern of the relationship between the two variables. Two types of correlation coefficients, namely: the Pearson's product moment correlation and Spearman's rank correlation were considered.

Regression line fitting by graphical and calculation methods were considered. The use of regression to predict was also presented.

4.6 MULTIPLE-CHOICE AND SHORT- ANSWER QUESTIONS

1. Correlation measures
   - A. Pattern of relationship
   - B. Degree of association
   - C. Degree of statistical analysis
   - D. Pattern of statistical analysis
   - E. Nature of statistical analysis

2. When the rank correlation is –1, it means
   - A. Perfect agreement
   - B. Directly proportional relationship
   - C. Perfect disagreement
   - D. Indirect relationship
   - E. Unstable relationship

3. Which of the following is the regression coefficient in $y = a + bx$, where x and y are variables?
   - A. $a + b$
   - B. $a - b$
   - C. ab
   - D. a
   - E. b

Use of the following table to answer questions 4 and 5

x	1	2	3	4
y	2	3.5	1	3.5

4. Assuming variables x and y are marks, the product moment correlation coefficient is__

5. Assuming variables x and y are ranks, the rank correlation coefficient is____

Use the following table to answer questions 6, 7 and 8

x	2	3	4
y	6	4	2

6. Using the regression line $y = a + bx$, the value of b is ______

7. Using the regression line $y = a + bx$, the value of a is_____

8. In the equation, $y = a + bx$, y is ___________ variable while x is ________ variable.

Use the following information to answer questions 9 and 10. Given the regression line $y = 5 + 1.5x$,

9. the value of y when $x = 6$ is ______________.

10. the value of x when $y = 11$ is _____________.

Answers

1. B
2. C
3. E
4. $$r=\frac{n\sum xy-\left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\sum x^2-\left(\sum x\right)^2\right]\left[n\sum y^2-\left(\sum y\right)^2\right]}}=\frac{4(26)-(10)(10)}{\sqrt{\left[4(30)-(10)^2\right]\left[4(29.5)-(10)^2\right]}}$$

$$r=\frac{104-100}{\sqrt{(120-100)(118-100)}}=\frac{4}{\sqrt{(20)(18)}}=\frac{4}{\sqrt{360}}=0.2108$$

5. $$R=1-\frac{6\sum d_i^{\,2}}{n(n^2-1)}=1-\frac{6(7.5)}{4(4^2-1)}=1-\frac{45}{60}=1-0.75=0.25$$

6. $$b=\frac{n\sum xy-\sum x\sum y}{n\sum x^2-\left(\sum x\right)^2}=\frac{3(32)-(9)(12)}{3(29)-(9)^2}=\frac{96-108}{87-81}=\frac{-12}{6}=-2$$

7. $a=\bar{y}-b\bar{x}=4-(-2)(3)=4+6=10$

8. Dependent, independent (in that order)

9. y = 5 + 1.5(6) = 14

10. 11 = 5 + 1.5x $\Rightarrow$ 1.5x = 11 - 5 $\Rightarrow$ 1.5x = 6 $\Rightarrow$ x = 4

CHAPTER FIVE

TIME SERIES ANALYSIS

Chapter content

(a) Introduction.

(b) Basic Components of a Time Series.

(c) Time Series Analysis.

(d) Estimation of Trend.

(e) Estimation of Seasoned Variation.

Objectives

At the end of the chapter, readers should be able to understand the

a) meaning of a time series;

b) basic components of a time series;

c) estimation of the trend by the use of moving averages and least squares methods;

d) carrying out of exponential smoothening; and

e) estimation of seasonal index.

5.1 Introduction

A time series is a set of data that are successively collected at regular intervals of time. The regular interval of time can be daily, weekly, monthly, quarterly or yearly. Examples of some time series data include:

a) Monthly production of a company;

b) Daily sales at a medical store;

c) Amount of annual rainfall over a period of time; and

d) Money deposited in a bank on various working days.

Furthermore, it is essential to know that when a time series is analysed, it has the following benefits:

i. Understanding the past behaviour of a variable and be able to:

- determine direction of periodic fluctuations; and
- predict future tendencies of the variable.

ii. Determining the impact of the various forces influencing different variables which then facilitate their comparison, such as:

- the differences that may have to do with price of commodities;
- the physical quantity of goods produced, marketed or consumed in order to make a comparison between periods of time, schools, places and etc.; and

iii. Knowing the behaviour of the variables in order to iron out intra-year variations as control events.

5.2 BASIC COMPONENTS OF A TIME SERIES

In time series, the existence of fluctuations from time to time is caused by composite forces which are constantly at work.

These factors have four components viz:

e) Secular Trend or Secular Variation (*T*);

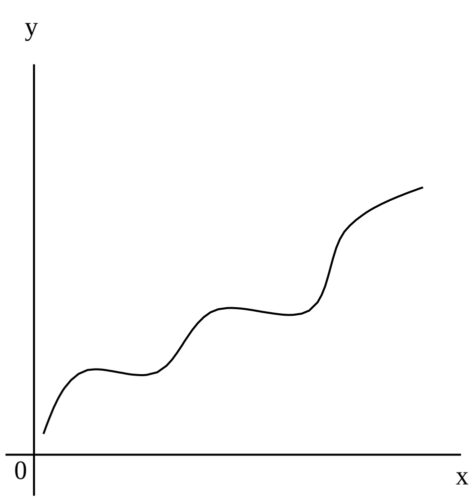
f) Seasonal Fluctuations or Variation (*S*);

g) Cyclical Variation (*C*); and

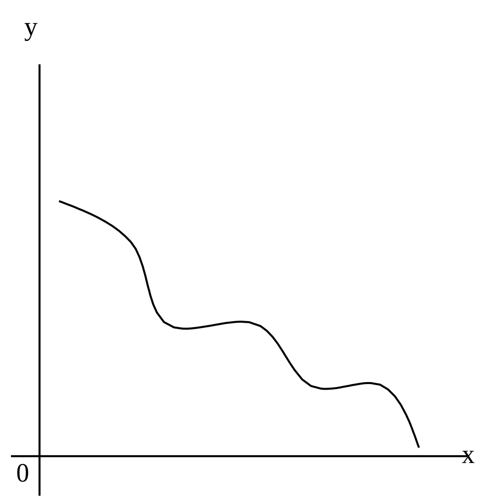
h) Irregular or Random Variations (*I*).

Secular Trend or Secular Variation (*T*)

The pattern of time series trend may be linear or non-linear. It is linear when the series values are concentrated along a straight line on the time-plot. Time-plot is known as the graph of time series values against different time points. The sketches of time plot showing typical examples of a secular trend are given below:



(a) Upward movement



(b) Downward movement

Example 5.1

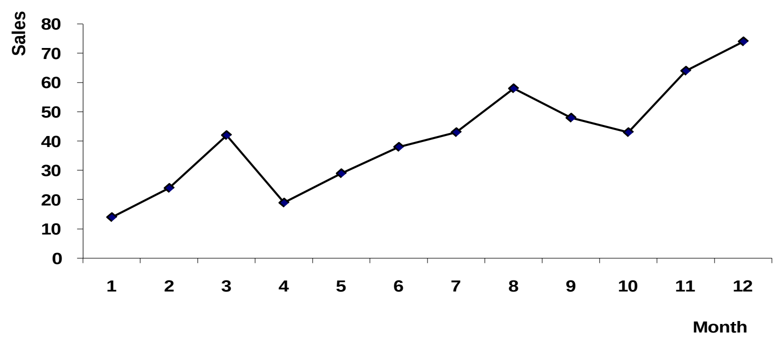
The following table shows the number of cartons of Malt drink sold by a retailer in Lagos over twelve consecutive months in a year.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Sales	14	24	42	19	29	38	43	58	48	43	64	74

Draw the time-plot of the table above.

Solution

TIME PLOT OF MALT DRINK SOLD IN TWELVE MONTHS

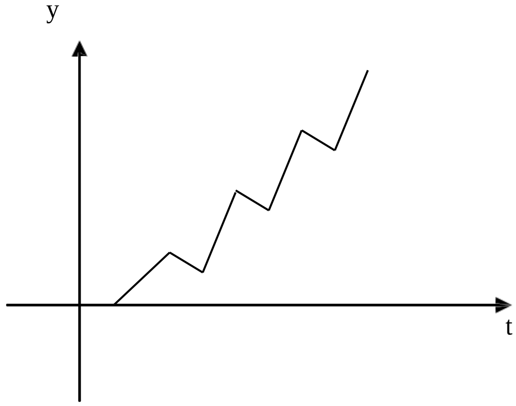


The time- plot of a time series is often referred to as histriogram.

Seasonal Fluctuations or Variations (*S*)

This is a variation that repeatedly occurs during a corresponding month or period of successive years. It is an annual reoccurring event which a time series appears to follow in a particular period or time of the year.

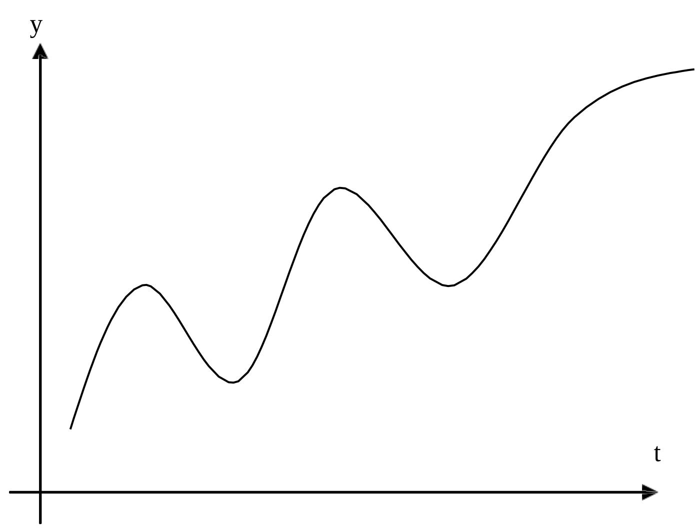
Data on climate such as rainy season, sales of goods during Christmas are good examples of time series with seasonal variations. The figure below depicts a time-plot showing the presence of seasonal variation.



Cyclical Variation (*C*)

This is a long-term oscillation or wavelike fluctuation about the trend line of a time series. It is similar to seasonal variation with a difference of reoccurring in more than one year period. Cyclical variations are called business cycles in the sense that periods of prosperity followed by recession or depression and then recovery are caused by aggregate economic conditions rather than seasonal effects. The length of the cycle varies between four and seven years.

The cyclical variation is less predictable. The figure depicted below shows the pattern of cyclical variation.

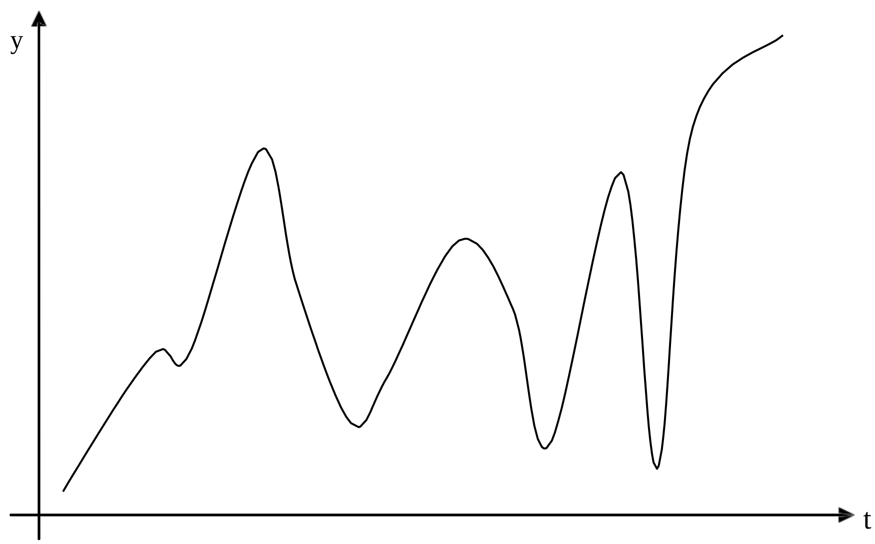


Irregular or Random Variation (*I*)

This is a variation caused by sporadic events (unpredictable events) such as floods, strikes, disasters, wars.

Random variation represents the residual variation in time series which cannot be accounted for by the three other components (i.e Trend, Seasonal, Cyclical variations).

See the figure below for a typical random variation.



5.3 TIME SERIES ANALYSIS

This is an act of analysing and interpreting time series data. It can be said to be an investigative method into the time series components; which, at times, is also referred to as decomposition of a time series.

The first step in the time series analysis is the drawing of the "time-plot". This will clearly show the pattern of the series movement.

Let Y represent the series by convention and T, S, C, I as components earlier indicated in paragraph 5.2. The two models of time series are additive and multiplicative models. For the additive, it is written as

$$Y = T + S + C + I \qquad 5.1$$

while in multiplicative model, it is written as;

$$Y = TSCI \qquad 5.2$$

Most of the time, the data available can be used to estimate only the trend and seasonal variation.

5.4 ESTIMATION OF TREND

There are various methods of estimating trend. However, this study text will consider the following methods:

- Moving Average method;
- Least Squares method; and
- Experimental Smoothening method.

Moving Average Method

This is a popular method of trend estimation. The trend is obtained in this method by smoothing out the fluctuations. The procedure of computing the trend by the Moving Average (M.A) method depends on whether the period of moving average desired, at times, called the order, is even or odd:

For a given time series $Y_1, Y_2, Y_3, \ldots, Y_n$, moving averages of order n are given by.

$$\frac{Y_1 + Y_2 + Y_3 + \ldots + Y_n}{n}, \quad \frac{Y_2 + Y_3 + Y_4 + \ldots + Y_{n+1}}{n}$$

$$\frac{Y_3 + Y_4 + Y_5 + \ldots + Y_{n+2}}{n}, \quad \text{e.t.c} \ldots \ldots \ldots \ldots \quad 5.3$$

where the moving averages are expected to be written against the middle items considered for each average and the numerators of equation 5.3 are called the moving totals.

The following are procedural steps of computing trend by moving average (M.A):

a. Determine the order to be used, whether odd or even;

b. When it is odd, directly apply equation 5.3 to get the desired moving average (M.A); and

c. If it is even, first, obtain the moving totals of order n from the given series and then obtain a 2 combined n moving totals of earlier obtained moving totals. The moving totals lastly obtained will be divided by 2n to give the desired M.A. The purpose of using two moving totals is to overcome the problem of placing the
M.A against middle items considered. By convention, M.A. must be written

against middle item of the items considered. Hence, moving totals make this possible.

The following examples illustrate the trend estimation by Moving Average (M.A) method:

Example 5.2

Use the following table to determine

(a) the moving average of order 3; and

(b) the moving average of order 4.

Month(x)	1	2	3	4	5	6	7	8	9	10	11	12
Sales(y)	14	24	42	19	29	38	43	58	48	43	64	74

Solution

a.

Month(x)	**Sales(y)**	**3 – Month moving total**	**3 – month moving average**
1	14		
2	24	80	26.67
3	42	85	28.33
4	19	90	30.00
5	29	86	28.67
6	38	110	36.67
7	43	139	46.33
8	58	149	49.67
9	48	149	51.67
10	43	155	60.33
11	64	181	
12	74		

b.

C_1	C_2	C_3	C_4	C_5
Month(x)	**Sales(y)**	**4 – Month moving total**	**2 of 4 – month moving total**	**4 – month average $= C_4 \div 8$**
1	14			
2	24			
		99		
3	42		213	26.63
		114		
4	19		242	30.25
		128		
5	29		257	31.13
		129		
6	38		297	37.13
		168		
7	43		355	44.38
		187		
8	58		379	47.38
		192		
9	48		405	50.63
		213		
10	43		442	55.25
		229		
11	64			
12	74			

Merits and Demerits of Moving Average (M.A) Method

Merits:

(i) The method is simple if compared with least squares method;

(ii) The effect of cyclical fluctuations is completely removed if the period of Moving Average (M.A) is equal to the average period of cycles; and

(iii) It is good for a time series that reveals linear trends.

Demerits

(i) The extreme values are always lost by Moving Average (M.A) method;

(ii) The method is not suitable for forecasting; and

(iii) It is not good for non-linear trend.

Least Squares Method (L.S.M)

Recall that by least squares method, fitting a linear regression of variable ($\underline{y}$) on variable ($\underline{x}$) gives $\underline{y} = \underline{a} + b\underline{x}$, where a and b are constants and are the intercept and regression coefficient respectively. This method can equally be extended to a time series data by taking the time period t as variable x and the time series value as variable y. Hence, the trend line by L.S.M is given as :

$$y = a + bx \qquad \ldots\ldots 5.4$$

$$\text{where } b = \frac{\sum xy - \sum x \sum y / n}{\sum x^2 - (\sum x)^2 / n} = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \qquad \ldots\ldots 5.5$$

$$\text{and } a = \bar{y} - b\bar{x} \qquad \ldots\ldots 5.6$$

The following examples give illustrations of Trend estimation by L.S.M.

Example 5.3

Use the data in the table below to fit the trend using the Least Squares approach.

Month($\underline{x}$)	1	2	3	4	5	6	7	8	9	10	11	12
Sales($\underline{y}$)	14	24	42	19	29	38	43	58	48	43	64	74

Solution

Month(x)	**Sales(y)**	x^2	xy
1	14	1	14
2	24	4	48
3	42	9	126
4	19	16	76
5	29	25	145
6	38	36	228
7	43	49	301
8	58	64	301
9	48	81	432
10	43	100	430
11	64	121	704
12	74	144	888
78	607	650	3693

From

$y = \text{a} + bx$

where

$$b = \frac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{(12)(3639) - (78)(607)}{(12)(650) - (78^2)}$$

$$= \frac{44316 - 47346}{7800 - 6084}$$

$$= \frac{-3030}{1716}$$

$$\therefore \quad = -1.7657$$

Also

$$a = \bar{y} - b\bar{x}$$

$$\bar{y} = \frac{\Sigma y}{n}$$

$$= \frac{607}{12} = 50.5833$$

$$\bar{x} = \frac{\Sigma x}{n}$$

$$= \frac{78}{12} = 6.5$$

Now,

$$a = \bar{y} - b\bar{x}$$

$$a = 50.5833 - (-1.7657)(6.5)$$

$$= 50.5833 + 11.47705$$

$$= 62.06035$$

$$= 62.0604$$

Recall $y = a + bx$

$$= 62.0604 + (-1.7657)x$$

$$= 62.0604 - 1.7657x$$

Month(x)	**Sales(y)**	x^2	xy	**Trend** **$\tilde{Y}$ = 62.0604 – 1.7657x**
1	14	1	14	60.2947
2	24	4	48	58.5290
3	42	9	126	56.7633
4	19	16	76	54.9976
5	29	25	145	53.2319
6	38	36	228	51.4662
7	43	49	301	49.7005
8	58	64	301	47.9348
9	48	81	432	46.1691
10	43	100	430	44.4034
11	64	121	704	42.6377
12	74	144	888	40,8720
78	607	650	3693	

REMARK: As the data in time series involve large number of periods and data, the use of L.S.M will require us to reduce the computation by coding the period aspect of the data.

The principle of the coding is achieved by the change of variable x to t using the relationship:

$t_i = x_i - x_m$, $i = 1, 2, --, n$

where x_m is the median of x_i. By this idea, the new variable t_i will give $\Sigma t_i = 0$. If this condition ($\Sigma t_i = 0$) is imposed on any time series data, then the normal equations of equations 5.5 and 5.6 will reduce to:

$$b = \frac{\Sigma y_i t_i}{\Sigma t_i^2} \qquad 5.7$$

$$a = \frac{\Sigma y_i}{n} = \bar{y} \qquad 5.8$$

for the Trend $T = a + bt$

Example 5.4

Use the Least Square Method with coding to fit the regression line to the following table:

Year	Q_1	Q_2	Q_3	Q_4
2000	20	40	25	45
2001	30	50	37	60
2002	42	60	45	65

Solution

Let's consider the following arrangement of the data:

Year	Quarter	x	$t = x - 5.5$	y	ty	t^2
2000	1	0	-5.5	20	-110	30.25
	2	1	-4.5	40	-180	20.25
	3	2	-3.5	25	-87.5	12.25
	4	3	-2.5	45	-112.5	6.25
2001	1	4	-1.5	30	-40	2.25
	2	5	-0.5	50	-25	0.25
	3	6	0.5	37	18.5	0.25
	4	7	1.5	60	90	2.25
2002	1	8	2.5	42	105	6.25
	2	9	3.5	60	210	12.25
	3	10	4.5	45	202.5	20.25
	4	11	5.5	65	357.5	30.25
			0	519	428.5	143

* Note that the mean of variable $x = 5.5$

Least squares equation is $y = a + bx$, where

$$b = \frac{\Sigma ty}{\Sigma t^2} = \frac{428.5}{143} = 2.9965$$

and $a = \bar{y} - b\bar{t}$

$$\bar{y} = \frac{519}{12} = 43.25$$

Hence, $a = 43.25 - 2.9965(5.5)$

$= 43.25 - 16.4808$

$= 26.7692$

$$\therefore y = 26.7092 + 2.9965(\underline{x} - 5.5)$$

$$= 26.7692 + 2.9965\underline{x} - 16.4808$$

$$= 10.2884 + 2.9965\underline{x}$$

a) Merits of LSM

i. No extreme values are lost in this method as in the case of the M. A;

ii. The method is free from subjective error; and

iii. The method can be used for forecasting.

b) Demerit

i. It requires more time for computation.

Exponential Trend (Exponential Smoothening)

When the trend shows an exponential function, where the x and y variables are in arithmetic and geometric progressions respectively, smoothening the trend is the appropriate approach. This is achieved as follows:

For an exponential function

$$y = ab^x \qquad \ldots\ldots 5.9$$

where a and b are constants, taking the logarithm of both sides of equation 5.9 yields

$$\text{Log } y = \text{Log } a + x \text{ Log } b \qquad \ldots\ldots 5.10$$

If Log y = z, Log a = A and Log b = B, equation 5.10 becomes

$$z = A + Bx \qquad \ldots\ldots 5.11$$

Which is now in linear form of variables z and x. The normal equations generated for the estimation of A and B are given as:

$$\Sigma z = nA + B\Sigma x \quad \} \qquad \ldots\ldots 5.12$$

$$\Sigma zx = A\Sigma x + B\Sigma x^2 \quad \} \qquad \ldots\ldots 5.13$$

Solving equations 5.12, and 5.13, we finally get a = Anti-Log A; b = antilog B.

The estimated exponential trend is obtained by putting the estimated values of a and b into equation 5.9.

Example 5.5

Given the population censuses of a country for certain number of periods as:

Census Year (x)	1950	1960	1970	1980	1990
Population in Million (y)	25.0	26.1	27.9	31.9	36.1

Fit an exponential trend $y = ab^x$ to the above data using the Least Squares Method (L.S.M).

Solution

Census (x)	**Population (y)**	**$u = \frac{x - 1970}{10}$**	**$v = \log y$**	**u^2**	**uv**
1950	25.0	-2	1.3979	4	-2.7958
1960	26.1	-1	1.4166	1	-1.4166
1970	27.9	0	1.4456	0	0
1980	31.9	1	1.5038	1	1.5038
1990	36.1	2	1.5575	4	3.1150
		$\Sigma u=0$	7.3214	10	0.4064

$y = ab^x$

$\text{Log}\, y \;=\; \log ab^x$

$\text{Log}\, y \;=\; \text{Log}\, a \;+\; x\text{Log}\, b$

$v \;=\; A \;+\; Bx$

let $\text{Log}\, y = v$, $\text{Log}\, a = A$, $\text{Log}\, b = B$. than

$\therefore\; u \;=\; (x - 1970)/10$

So that $\Sigma u = 0$

$$\therefore A = \frac{\Sigma v}{n} = \frac{7.3214}{5} = 1.4643 \;\Rightarrow\; a = \text{antilog}\, A = 29.1259$$

$$B = \frac{\Sigma uv}{\Sigma u^2} = \frac{0.4064}{10} = 0.04064 \;\Rightarrow\; b = \text{Antilog}\, B = 1.0981$$

From $y = ab^x$

$y = 29.1259(1.0981)^x$

To obtain the trend values y for different x, we use the linear trend

$v = A + Bu$

$v = 1.4643 + 0.04064u$

5.5 Estimation of Seasonal Variation

Estimation of Seasonal variation involves the use of original time series data and the trend obtained. Here, the approach depends on the model assumed. The model can be additive or multiplicative as stated in equations 5.1 and 5.2.

For the <u>additive model</u>, the following steps are to be adhered to:

- Arrange the original given time series(Y) in a column;
- The Trend T to be placed as next column;
- Obtain the seasonal variation (S) in another column by subtracting T from Y; i.e $S = Y - T - (C + I)$, from equation 5.1. Here, it is assumed that

$C + I = 0$, hence $S = Y - T$

(Note that T(trend) can be obtained by either M. A. or L. S.M);

- Obtain the average seasonal variation for each period/season (weeks, months, quarters, half-yearly, etc.). This is known as seasonal index (S.I);
- Check whether the (S.I) obtained in step 4 is a balanced one. Add the indices obtained and if the total is equal to zero, it implies balance; otherwise adjust them to balance; and

- For the adjustment of S. I., divide the difference (to zero) by the number of periods/seasons and then add to or subtract from S. I. obtained in step 4 as dictated by the sign of the difference in order to make the sum total zero.

For the multiplicative model, the following steps are to be followed:

a) The same as step (a) in the additive model;

b) The same as step (b) in the additive model;

c) Obtain the seasonal variation (S) in another column by dividing *Y* by *T*; i.e

$$S = \frac{Y}{T}$$, from equation 5.2, It is also assumed that CI = 1,

hence $S = {}^{Y}/_{T}$;

d) The same as step (d) in the additive model;

e) Check whether the S. I. obtained in step 4 is balanced or not by having the total of the indices as number of period/season. For example, a quarterly period will have total of indices as 4, for half-yearly 2, etc. Sometimes when indices are expressed in percentages, it will be 400 for quarter, 200 for half-yearly, etc. If it is not balanced, there is need to adjust the indices to the balanced form; and

f) The same as step f in the additive model with modification of difference to 4 or 2 as the period indicates.

Example 5.6

A company secretary preparing for his retirement within the next six years decided to invest quarterly in the purchase of shares either private placement or through public offers. The table below shows his quarterly investments (₦'000) in 2000, 2001, 2002 and 2003

Year	**Quarter 1**	**Quarter 2**	**Quarter 3**	**Quarter 4**
2000	15	35	40	20
2001	25	45	55	30
2002	37	55	60	40
2003	47	62	72	58

Estimate the trend by least squares method. Hence, compute seasonal variation.

Solution

Define x as follows

Quarter 1	2000,	$x = 0$
Quarter 2	2000,	$x = 1$
Quarter 3	2000,	$x = 2$
Quarter 4	2000,	$x = 3$
Quarter 1	2001,	$x = 4$ e.t.c

Computation of least square trend line (Direct method)

x	y	x^2	xy	**Trend $y = a - bx$ = 23.6692 + 2.6441x**
0	15	0	0	23.6692
1	35	1	35	26.0133
2	40	4	80	28.6574
3	20	9	60	31.3015
4	25	16	100	33.9456
5	45	25	225	36.5897
6	55	36	330	39.2338
7	30	49	210	41.8779
8	37	64	296	44.5220
9	55	81	495	47.1661
10	60	100	600	49.8102
11	40	121	440	52.4543
12	47	144	564	55.0984
13	62	169	806	57.7425
14	72	196	1008	60.3866
15	58	225	870	63.0307
120	696	1240	6119	

From

$$y = a + bx$$

$$b = \frac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma x^2 - (\Sigma y)^2}$$

$n = 16$, $\Sigma xy = 6119$, $\Sigma x = 120$, $\Sigma y = 696$

$$b = \frac{(16)(6119) - (120)(696)}{(16)(1240) - (120)^2}$$

$$= \frac{97904 - 83520}{19840 - 14400}$$

$$= \frac{14384}{5440}$$

$$= 2.6441$$

Also,

$$a = \bar{y} - b\bar{x}$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{696}{16} = 43.5$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{120}{16} = 7.5$$

$$a = 43.5 - (2.6441)(7.5)$$

$$= 43.5 - 19.8308$$

$$= 23.6692$$

Hence,

$$y = a + \mathrm{b}x$$

$$\text{Trend(T)} = y = 23.6692 + 2.6441x$$

x	***y***	**Trend T = *a* + *bx***	**Variation by multiplicative model (y/T)**
0	15	23.6692	0.6337
1	35	26.0133	1.3455
2	40	28.6574	1.3958
3	20	31.3015	0.6389
4	25	33.9456	0.7475
5	45	36.5897	1.2299
6	55	39.2338	1.4019
7	30	41.8779	0.7164
8	37	44.5220	0.8310
9	55	47.1661	1.1661
10	60	49.8102	1.2046
11	40	52.4543	0.7626
12	47	55.0984	0.8530
13	62	57.7425	1.0737
14	72	60.3866	1.1923
15	58	63.0307	0.9202

x	*y*	**Trend T = *a* + *bx***	**Variation by multiplicative model (y/T)**
0	15	23.6692	0.6337
1	35	26.0133	1.3455
2	40	28.6574	1.3958
3	20	31.3015	0.6389
4	25	33.9456	0.7475
5	45	36.5897	1.2299
6	55	39.2338	1.4019
7	30	41.8779	0.7164
8	37	44.5220	0.8310
9	55	47.1661	1.1661
10	60	49.8102	1.2046
11	40	52.4543	0.7626
12	47	55.0984	0.8530
13	62	57.7425	1.0737
14	72	60.3866	1.1923
15	58	63.0307	0.9202

Seasonal Index Table

Year	**Quarter 1(Q_1)**	**Quarter 1(Q_2)**	**Quarter 1(Q_3)**	**Quarter 1(Q_4)**
2000	0.6337	1.3455	1.3958	0.6389
2001	0.7475	1.2299	1.4019	0.7164
2002	0.8310	1.1661	1.2046	0.7626
2003	0.8530	1.0737	1.1923	0.9202
Total	**3.0652**	**4.8152**	**5.1946**	**3.0381**
(Index) Average	0.7663	1.2038	1.2987	0.7595
Adjustment	-0.007075	-0.007075	-0.007075	-0.007075
Adjusted Index	**0.759225**	**1.196725**	**1.291625**	**0.752425**

Total of Averages = 0.7663 + 1.2038 + 1.2987 + 0.7595

= 4.0283

Since the total of averages is supposed to be 4, we need to adjust.

$$\text{Adjustment by } \frac{4 - 4.0283}{4} = \frac{-0.0283}{4} = -0.007075$$

Forecasting, using the Least Squares line, is the same as discussed under the regression analysis except that we adjust the forecasted figure by using the appropriate seasonal index

Example 5.7

Use the result obtained in Example 5.6 to forecast the investment in the 3rd quarter of 2004.

Solution

$$\begin{aligned} y &= 23.67 + 2.64x \\ x &= 18 \ (3^{rd} \text{ quarter of } 2004) \\ y &= 23.67 + 2.64\,(18) \\ &= 71.19 \end{aligned}$$

The seasonal adjustment obtained for 3rd quarter using the multiplicative model is 1.29

$\therefore$ Adjusted forecast is $y = 71.19 \times 1.29 = 91.84$

5.6 Chapter Summary

The time series is described as a set of data collected at regular intervals of time. Analysis of time series reveals its principal components as Trend, Seasonal, Cyclic and Random variations. The methods of obtaining trend, as discussed in this Study Text, are the Moving Averages and Least Squares methods. The concept of seasonal index is considered from the perspective of additive and multiplicative models. Exponential smoothening is also discussed with numerical example.

5.7 Multiple-choice and short-answer questions

1. Which of the following is not a time series data?
   A. Monthly production of a company.
   B. Daily sales at a medicine store.
   C. Expenditure at home.
   D. Daily deposits in a bank.
   E. Annual rainfall.

2. For monthly time series data, the order of the moving average is_______________
   A. 2
   B. 4
   C. 6
   D. 10
   E. 12

3. Using the conventional symbol of time series components, which of the following is the additive model?
   A. $P = T + C + I + S$;
   B. $Y = T + S + C + I$
   C. $Y = TSCI$
   D. $P = TSCI$
   E. $Y = ab^X$

4. Which of the following is NOT true of moving average method?

   A. The method is simple if compared with Least Squares Method.
   B. The effect of cyclical fluctuations is completely removed by the method.
   C. The extreme values are always lost.
   D. The method is suitable for forecasting.
   E. It is not good for non-linear trend.

Use the following table to answer questions 5 – 8.

Time (t)	Value of Series (Y)	Moving Average of order 3
1	24	
2	30	a
3	25	b
4	35	c
5	33	d
6	38	

5. Find **a**
6. Find **b**
7. Find **c**

8. Find **d**

Use the following information to answer questions 9 and 10

Time (t)	**Series** (Y)	**Trend by LSM** $Y = 30 + 3t$	**Seasonal Variation assuming additive model**
1	30		p
2	32		
3	37		q

9. Find ***p***.

10. Find ***q***.

Answers

1. C
2. E
3. B
4. D
5. $a = \dfrac{24+30+25}{3} = 26.33$
6. $b = \dfrac{30+25+35}{3} = 30$
7. $c = \dfrac{25+35+33}{3} = 31$
8. $d = \dfrac{35+33+38}{3} = 35.33$
9. $p = 30 - [30 + 3(1)] = -3$
10. $q = 37 - [30 + 3(3)] = -2$

CHAPTER SIX

INDEX NUMBERS

Chapter contents

(a) Introduction.

(b) Construction Methods of Price Index Number.

(c) Unweighted Index Number.

(d) Weighted Index Number.

(e) Construction of Quantity Index Number.

Objectives

At the end of the chapter, readers should be able to understand the

a) concept of Index numbers;

b) differentiation between price indices and price relatives;

c) differences between unweighted index numbers and weighted index numbers; and

d) computation and handling problems on index numbers by the use of weighted methods such as Laspeyre, Paasche, Fisher and Marshall Edgeworth methods.

6.1 Introduction

The concept of Index Numbers is an important statistical concept which is used to measure changes in a variable or group of variables with respect to time and other characteristics.

It is a usual practice in business, economy and other areas of life to find the average changes in price, quantity or value of related group of items or variables over a certain period of time, or for geographical locations. Index number is the statistical concept or device that is usually used to measure the changes,

Spiegel defined Index numbers as "A statistical measure designed to show changes in a variable or a group of variables with respect to time, geographical location or other characteristics".

By the principle of index numbers, the statistical device measures

a) the differences in the marginal of a group of related variables;

b) the differences that may have to do with price of commodities;

c) the physical quantity of goods produced, marketed or consumed in order to make a comparison between periods of time, schools, places, etc.

Based on the above principle of the index numbers, it can be classified in terms of the variables that it tends to measure. Hence, it is broadly categorized into:

(i) Price Index Number consisting of retail price;

(ii) Quantity or Volume Index numbers; and

(iii) Value Index Numbers.

Uses / Applications of Index Numbers

It is important and of great benefits to state the following uses of Index numbers:

a) To deflate a value series in order to convert it into physical terms;

b) To keep abreast of current business condition i.e. it acts as business or economic barometer;

c) To give the trend movement in business or economy;

d) To forecast by using series of the indices;

e) To assess the worth of purchasing power of money;

f) To compare the standard of living in various areas/countries or geographical locations; and

g) To compare readers' intelligence in various schools or countries.

Problems Associated with Constructions of Index Numbers

The following are among the problems usually encountered in the construction of index numbers:

a. Definition of the purpose for which index number is being compiled or constructed;

b. Selection of commodities/items to include in the index. Here, one has to decide what type of item, what quantity and quality are to be selected;

c. Selection of sources of data. The data source must be reliable, hence, utmost care must be taken in selecting the source;

d. Method of collecting data: Once the source of data has been determined, the next line of action is to decide on an efficient and effective method of data collection. This will facilitate accurate and reliable results. Here, the method depends on the source whether it is of primary or secondary type of data;

e. Selection or choice of base year: The base year is the reference point or period that other data are being compared with. It is essentially important that the period of choice as the base year, is economically stable and free from abnormalities of economy such as inflation, depression, famines, boom etc. The period should not be too far from the current period;

f. Methods of Combining data: The use of appropriate method of combining data depends on the purpose of the index and the data available. The choice of appropriate method dictates the formula to be used; and

g. Choice of Weight: The weights in index number refer to relative importance attached to various items. It is therefore necessary to take into account the varied relative importance of items in the construction of index numbers in order to have a fair or an accurate index number.

6.2 Construction Methods of Price Index Numbers

The Price Index number measures the changes in the general level of prices for a given number or group of commodities. It could be wholesale price index or retail price index, or index for prices of manufactured products.

The construction methods of price index numbers can be broadly classified into two:

(i) the use of unweighted price index number; and

(ii) use of weighted price index numbers.

6.3 Unweighted Price Index Number Method

In the unweighted index number, it is observed that equal importance is attached to all items in the index.

The following unweighted indices shall be considered in this section:

(a) Simple Price Relative Index Number;

(b) Simple Aggregate Price Index; and

(c) Simple Average of Relative Method.

Simple Price Relative Index Numbers (*SPRI*)

This Index number is the simplest form of all index numbers. It can be simply defined as the ratio of the price of a single commodity in a current or given year to the price of the same commodity in the base year.

This can be expressed mathematically as follows:

$$SPRI = \frac{p_{ti}}{p_{0i}} \times \frac{100}{1}$$

where p_{ti} = current or given year price for item *i*; and

p_{0i} = base year price for item *i*.

Simple Aggregate Price Index (*SAPI*)

This index measures the changes in price level over time, using only the arithmetic mean and ignoring differences in the relative importance of the commodities. It is expressed as the total prices of commodities in a current or given year as a percentage of the total prices of the base year of the same commodities.

$$SAPI = \frac{\sum p_{ti}}{\sum p_{0i}} \times \frac{100}{1}$$

where $\sum p_{ti}$ = the sum of prices for item *i* at period *t*; and

$\sum p_{0i}$ = the sum of prices for item *i* at the base year.

Simple Average of Relative Price Index (*SARPI*)

This approach is to remove the shortcomings of the method of *SAPI*. The index is computed by the following formula:

$$SAPRI = \frac{\sum \left(\frac{p_{ti}}{p_{0i}} \right) \times 100}{n}$$

Where $\sum$ stands for summation;

n = the total number items/commodities; and

p_{ti} and p_{0i} are as defined in SPRI.

Example 6.1

Determine the *SPRI*, SAPI and SARPI for the following table using 2004 as base year.

Items / commodities	Price per unit (Naira/Cedi)		
	Year 2004	***Year 2005***	**Year 2006**
A	20	30	60
B	30	42	55
C	10	15	20
D	6	8	15

Solution

(i) $SPRI = \frac{p_{ti}}{p_{0i}} \times \frac{100}{1}$

$SPRI_{2005} = \frac{p_1}{p_0} \times \frac{100}{1}$, $\quad SPRI_{2006} = \frac{p_2}{p_0} \times \frac{100}{1}$

$\boldsymbol{p_{0(2004)}}$	$\boldsymbol{p_{1(2005)}}$	$\boldsymbol{p_{2(2006)}}$	$SPRI_{2005} = \frac{p_1}{p_0} \times \frac{100}{1}$ (%)	$SPRI_{2006} = \frac{p_2}{p_0} \times \frac{100}{1}$ (%)
20	30	60	150	300
30	42	55	140	183
10	15	20	150	200
6	8	15	130	250

(ii)

Items / commodities	Price per unit (Naira/Cedi)		
	Year 2004	***Year 2005***	**Year 2006**
A	20	30	60
B	30	42	55
C	10	15	20
D	6	8	15
	$\sum p_{2004i} = 66$	$\sum p_{2005i} = 95$	$\sum p_{2006i} = 150$

$$SAPI = \frac{\sum p_{ti}}{\sum p_{0i}} \times \frac{100}{1}$$

using 2004 as base year i.e. $\sum p_{0i} = \sum p_{2004i} = 66$

$$SAPI_{2005} = \frac{\sum p_{ti}}{\sum p_{0i}} \times \frac{100}{1} = \frac{95}{66} \times \frac{100}{1} = 143.9394 \equiv 144\%$$

$$SAPI_{2006} = \frac{\sum p_{ti}}{\sum p_{0i}} \times \frac{100}{1} = \frac{150}{66} \times \frac{100}{1} = 227.2727 \equiv 227\%$$

(iii) $$SAPRI = \frac{\sum \left(\frac{p_{ti}}{p_{0i}} \right) \times 100}{n}$$

Items	Year 2004 p_0	Year 2005 p_1	Year 2006 p_2	$\frac{p_1}{p_0}$	$\frac{p_2}{p_0}$
A	20	30	60	1.5	3.00
B	30	42	55	1.4	1.83
C	10	15	20	1.5	2.00
D	6	8	15	1.3	2.50
				$\sum \left(\frac{p_1}{p_0} \right) = 5.70$	$\sum \left(\frac{p_2}{p_0} \right) = 9.33$

$$SAPRI_{2005} = \frac{\sum \left(\frac{p_1}{p_0} \right) \times 100}{n}$$

$$SAPRI_{2005} = \frac{5.7 \times 100}{4} = 142.5 \equiv 143\%$$

$$SAPRI_{2006} = \frac{\sum \left(\frac{p_2}{p_0} \right) \times 100}{n}$$

$$SAPRI_{2005} = \frac{9.33 \times 100}{4} = 233.25 \equiv 233\%$$

6.4 Weighted Index Numbers

In this index, weights are attached to each item or commodity on the assumption that such weights denote the relative importance of the items. The weighted index number can be broadly categorized into:

(a) Weighted Aggregative Indices; and

(b) Weighted Average of Relative Indices.

Weighted Aggregative Indices

As weights give relative importance to the group of items, various methods of the weighted aggregative index involve different weighing techniques. Some of the methods usually encountered, include

a. Laspeyre;

b. Paasche;

c. Fisher; and

d. Marshall Edgeworth.

(a) Laspeyre's Index

In 1864, Laspeyre invented this method of weighted aggregative index by making use of the base year quantities as weights. The method is commonly and widely used because it is easy to compute being based on fixed weights of the base year. It is computed using the formula

$$I_L = \frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{100}{1}$$

where $\sum$ stands for summation;

$p_i =$ price of the current or given year;

$p_0 =$ price of the base year; and

$q_0 =$ quantity of the base year.

This method has an upward bias.

(b) Paasche Index

The German statistician, Paasche, introduced this method in the year 1874. Its focus is on the usage of current or given year quantities as weights. The formula for its construction is given by

$$I_P = \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1}$$

where $\sum$ stands for summation;

$p_i =$ price of the current or given year;

$p_0 =$ price of the base year; and

$q_i =$ quantity of the current or given year

The method is tedious to compute and of downward bias.

(c) Fisher Ideal Index

This method is based on the geometric mean of Laspeyre's and Paasche's indices. The method is theoretically better than the other methods because it overcomes the shortcomings of these other methods. It is computed by

$$I_F = \sqrt{\frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1}}$$

OR

$$I_F = \sqrt{I_L \times I_P}$$

where p_0, q_0, p_i, q_i *are as defined before*

(d) Marshall – Edgeworth Index

The method uses the average of base and current/given year quantities as weights. It constructs the index by the formula:

$$I_M = \frac{\sum p_i \frac{(q_0 + q_i)}{2}}{\sum p_0 \frac{(q_0 + q_i)}{2}} \times \frac{100}{1}$$

OR $$I_M = \frac{\sum (p_i q_0 + p_i q_i)}{\sum (p_0 q_0 + p_0 q_i)} \times \frac{100}{1}$$

OR $$I_M = \frac{\sum p_i q_0 + \sum p_i q_i}{\sum p_0 q_0 + \sum p_0 q_i} \times \frac{100}{1}$$

where p_0, q_0, p_i, q_i *are as defined before*

Example 6.2

Use the following table to calculate price index numbers for the year 2016 by taking 2011 as the base year and using the following methods:

d) Laspeyre index;

b) Paasche index

c) Fisher index; and

d) Marshall – Edgeworth index.

Commodity Item	2011		2016	
	Price	**Quantity**	**Price**	**Quantity**
A	14	45	20	35
B	13	15	19	20
C	12	10	14	10
D	10	5	12	8

Solution

Commodity Item	2011		2016	
	Price p_0	**Quantity** q_0	**Price** p_i	**Quantity** q_i
A	14	45	20	35
B	13	15	19	20
C	12	10	14	10
D	10	5	12	8

Items	p_0	q_0	p_i	q_i	p_0q_0	p_iq_0	p_iq_i	p_0q_i
A	14	45	20	35	630	900	700	490
B	13	15	19	20	195	285	380	300
C	12	10	14	10	120	140	140	100
D	10	5	12	8	50	60	96	40
					$\sum p_0q_0 = 995$	$\sum p_iq_0 = 1385$	$\sum p_iq_i = 1316$	$\sum p_0q_i = 930$

Items	$p_i q_0 + p_i q_i$	$p_0 q_0 + p_0 q_i$
A	1600	1120
B	665	495
C	280	220
D	156	90
	$\sum(p_i q_0 + p_i q_i) = 2701$	$\sum(p_i q_0 + p_i q_i) = 1925$

Using year 2011 as the base year and year 2016 as the current/given year

(a) $$I_L = \frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{100}{1}$$

$$I_{L(2006)} = \frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{100}{1} = \frac{1{,}385}{995} \times \frac{100}{1} = 139.196 \equiv 139\%$$

(b) $$I_P = \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1}$$

$$I_{P(2006)} = \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1} = \frac{1{,}316}{930} \times \frac{100}{1} = 141.505 \equiv 142\%$$

(c) $$I_F = \sqrt{\frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1}}$$

$$I_{F(2006)} = \sqrt{\frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1}} = \sqrt{\frac{1385}{995} \times \frac{1316}{930} \times \frac{100}{1}}$$

$$I_{F(2006)} = \sqrt{196.969} = 140.346 \equiv 140\%$$

(d) $$I_M = \frac{\sum p_i \frac{(q_0 + q_i)}{2}}{\sum p_0 \frac{(q_0 + q_i)}{2}} \times \frac{100}{1} = \frac{\sum (p_i q_0 + p_i q_i)}{\sum (p_0 q_0 + p_0 q_i)} \times \frac{100}{1}$$

$$I_{M\,2006} = \frac{\sum (p_i q_0 + p_i q_i)}{\sum (p_0 q_0 + p_0 q_i)} \times \frac{100}{1} = \frac{2701}{1925} \times \frac{100}{1} = 140.312 \equiv 140\%$$

6.5 Chapter Summary

The chapter covered the unweighted (price relative) and weighted indices. Among the weighted indices considered are Laspeyre, Paasche, Fisher Ideal and Marshall-Edgeworth indices. Relevant examples were solved and discussed.

6.6 Multiple-choice questions and short- answer questions

1. Which of the following is **NOT** one of the uses of an index number?
   A. To deflate a value series in order to convert it into real physical terms
   B. To compare student's intelligence in various schools or countries.
   C. To select the commodities sources.
   D. To forecast by using series of the indices.
   E. To assess the worth of purchasing power of money.

2. Which of the following is **NOT** a problem in the construction of an index number?
   A. Selection of sources of data.
   B. Definition of the purpose for which index is needed.
   C. Method of collecting data for index.
   D. Method of combining the data.
   E. Unweighted average price index.

3. Which of the following is **NOT** a weighted price index number?
   A. Laspeyre Index.
   B. Simple Aggregate Price Index
   C. Fisher Ideal Index
   D. Marshall-Edgeworth Index
   E. Paasche Index

4. The Weighted Aggregative Index with a backward bias is __
   A. Laspeyre Index
   B. Simple Aggregate Price Index
   C. Fisher Ideal Index
   D. Marshall-Edgeworth Index
   E. Paasche Index

5. Which of the following is the formula for Unweighted Price Index?

   A. $\dfrac{\sum q_{ti}}{\sum q_{0i}} \times \dfrac{100}{1}$

   B. $\dfrac{q_{ti}}{q_{0i}} \times \dfrac{100}{1}$

   C. $\dfrac{\sum p_i q_0}{\sum p_0 q_0} \times \dfrac{100}{1}$

   D. $\dfrac{\sum p_{ti}}{\sum p_{0i}} \times \dfrac{100}{1}$

E. $$\frac{\sum p_i \frac{(q_0+q_i)}{2}}{\sum p_0 \frac{(q_0+q_i)}{2}} \times \frac{100}{1}$$

Use the following table to answer Questions 6 and 7

Items	2000 price	2005 price
A	50	80
B	70	100
C	90	100
D	100	120

6. The Simple Aggregate Price Index of the above table using 2000 as base year is _____

7. The Simple Average of Relative Price Index with year 2000 as base year is __

Use the following table to answer Questions 8 to 10

Items	2001		2006	
	Price	**Quantity**	**Price**	**Quantity**
A	80	50	100	60
B	90	60	100	70
C	100	70	120	90

8. Using year 2001 as base year, the Laspeyre price index for year 2006 is ______

9. Using year 2001 as base year, the Paasche price Index for year 2006 is ______

10. Fisher's Ideal price index of the above table is _

Answers

1. C
2. E
3. B
4. E
5. D
6. $$SAPI = \frac{\sum p_{ti}}{\sum p_{0i}} \times \frac{100}{1} = \frac{400}{310} \times \frac{100}{1} = 129.03\%$$

7.

$$SAPRI = \frac{\sum\left(\frac{p_{ti}}{p_{0i}}\right)\times 100}{n} = \frac{1.6 + 1.429 + 1.1111 + 1.2}{4} = \frac{5.3401}{4} = 133.50\%$$

8. $$I_L = \frac{\sum p_i q_0}{\sum p_0 q_0} \times \frac{100}{1} = \frac{14,400}{16,400} \times \frac{100}{1} = 118.29\%$$

9. $$I_P = \frac{\sum p_i q_i}{\sum p_0 q_i} \times \frac{100}{1} = \frac{23,800}{20,100} \times \frac{100}{1} = 118.41\%$$

10. $$I_F = \sqrt{118.29 \times 118.41} = 118.35\%$$

CHAPTER SEVEN

PROBABILITY

Chapter contents

a) Introduction.

b) Concept of Probability Theory.

c) Addition Law of Probability.

d) Multiplication Law of Probability.

e) Conditional Probability and Independence.

f) Mathematical Expectation.

Objectives

At the end of the chapter, readers should be able to understand the

(a) concept of probability and be able to handle simple probability problems;

(b) computation of probabilities by using the additive and multiplicative laws; and

(c) use of mathematical expectations in discrete probability distribution.

7.1 Introduction

As various activities of business and life in general depend on chance and risks, the study of probability theory is an essential tool to make correct and right decisions.

7.2 Concept of Probability Theory

As earlier said, probability theory is mainly concerned with chance and calculated risks in the face of uncertainty. This is achieved by building a mathematical concept, based on the study of some samples of the theoretical or imaginary population.

Importance and Uses of Probability

Without any doubt, the importance of probability is felt in large percentage of human endeavour, most especially in business, commerce and other related areas where problems of risks are usually involved.

However, some of the uses of probability are itemized below:

a. Probability theory is used as quantitative analysis of some problems arising from business and other areas;

b. Probability theory is used as the basis of statistical inference;and

c. Probability theory plays vital roles in insurance and statistical quality control.

Definitions of Some Useful Terms

a. **An experiment** (a random experiment) means performing an act which involves unpredictable outcome. For example, tossing of coins, throwing a die, etc.

b. **An outcome** is one of the possible results that can happen in a trial of an experiment.

Example 7.1

i. The outcomes of a fair coin = $\{H, T\}$, where H and T stand for Head and Tail respectively.

ii. The outcomes of an unbiased die = $\{1, 2, 3, 4, 5, 6\}$.

c. **Sample space** is the list or set of all possible outcomes of an experiment, while each outcome is called SAMPLE POINT.

For example, the sample space in tossing a die is $\{1, 2, 3, 4, 5, 6\}$, while 3 is a sample point.

d. **An event** is a collection of sample points which have certain quality or characteristics in common.

In set theory, it can be defined as a subset A of the points in the sample space S.

Examples of sample spaces for random experiments

The result/ outcome of a random experiment is called a sample space. The type of experiment determines the nature of the sample space. Some of the time, more than one step may be necessary to obtain the sample space.

a) When a coin is tossed once

$S = \{H, T\}$ i.e. 2 sample points

b) When a die is tossed once

$S = \{1, 2, 3, 4, 5, 6\}$ i.e. 6 sample points

c) When a coin is tossed twice (tossing two coins together)

	H	*T*
H	*HH*	*HT*
T	*TH*	*TT*

S = {HH, HT, TH, TT}

i.e. ($2 \times 2 = 4$) sample points

d) When 3 coins are tossed (tossing a coin Thrice)

	H	*T*
H	*HH*	HT
T	*TH*	*TT*

	HH	*HT*	*TH*	*TT*
H	*HHH*	*HHT*	*HTH*	*HTT*
T	*THH*	*THT*	*TTH*	*TTT*

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

i.e. ($2 \times 2 \times 2 = 8$) sample points

e) When a coin and a die are tossed

	1	*2*	*3*	*4*	*5*	*6*
H	*H1*	*H2*	*H3*	*H4*	*H5*	*H6*
T	*T1*	*T2*	*T3*	*T4*	*T5*	*T6*

i.e. ($2 \times 6 = 12$) sample points

f) When two coins and a die are tossed

	1	2	3	4	5	6
HH	*HH1*	*HH2*	*HH3*	*HH4*	*HH5*	*HH6*
HT	*HT1*	*HT2*	*HT3*	*HT4*	*HT5*	*HT6*
TH	*TH1*	*TH2*	*TH3*	*TH4*	*TH5*	*TH6*
TT	*TT1*	*TT2*	*TT3*	*TT4*	*TT5*	*TT6*

i.e. ($2 \times 2 \times 6 = 24$) sample points

e) When two dice are tossed

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

i.e. ($6 \times 6 = 36$) sample points

Definition of Probability

Probability is a statistical concept that measures the likelihood of an event happening or not. As a measure of chance, which an event *E* is likely to occur, it is convenient to assign a number between 0 and 1. Therefore, the probability of an Event *E* written as *P(E)* is a positive number which can be formally expressed as $0 \leq P(E) \leq 1$. Note that

i. If $P(E) = 0$, then it is sure or certain that the event *E* will not occur or happen;

ii. If $P(E) = 1$, then it is sure or certain that the event *E* will occur or happen.

There are two schools of thought about the procedures of getting the estimates for the probability of an event. These are the

(a) Classical or a Priori approach: In this approach, it is assumed that all the n trials of an experiment are equally likely and the outcomes are mutually exclusive (cannot happen together). If an event E can occur in r different ways out of n possible ways, then the probability of E is written as

$$P(E) = \frac{r}{n} \quad i.e, \quad P(E) = \frac{n(E)}{n(S)}$$

Example 7.2

If a fair coin is tossed once, what is the probability of obtaining Head?

Solution:

By a fair coin, it means the coin is not loaded or biased in any way. Let *E* be the event of a Head, then $n(E) = 1$ and $n(S) = 2$.

$$P(E) = \frac{Number\ of\ times\ Head\ is\ obtained}{Number\ of\ total\ possible\ outcome} = \frac{1}{2} = 0.5$$

Example 7.3

If a fair die is rolled once, what is the probability of obtaining an even number?

Solution

$$Let\ the\ sample\ space\ be\ S = \{1, 2, 3, 4, 5, 6\} \quad then\ n(S) = 6$$

$$Let\ the\ set\ of\ even\ numbers\ be\ E = \{2, 4, 6\} \quad then\ n(E) = 3$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}\ or\ 0.5$$

b) Frequency or Posteriori approach: It is assumed in this approach that after *n* repetitions of an experiment, where n is very large, an event E is observed to occur in r of these, then the probability of E is

$$P(E) = \frac{r}{n}$$

Example 7.4

SAO Bank Plc gave out loans to 50 customers and later found that some were defaulters while only 10 repaid as scheduled. Determine the probability of repayment of loan in that bank.

Solution

$$P(repayment) = \frac{Number\ that\ repaid}{Total\ number\ of\ \ people\ that\ took\ loan} = \frac{10}{50} = 0.2$$

Remark:

However, the two above approaches are found to have difficulties because of the two underlined terms: (equally likely and very large) which are believed to be vague and even being relative terms to an individual. Due to these difficulties, a new approach tagged "Axiomatic Approach" was developed. This axiomatic approach is based on the use of set theory.

Axiomatic definition of Probability

Suppose there is a sample space S and A is an event in the sample space. Then, to every event A, there exists a corresponding real value occurrence of the event A which satisfies the following three axioms:

i. $0 \leq P(A) \leq 1$;

ii. $P(S) = 1$; and

iii. If $A_1, A_2, \ldots, A_k$ are mutually exclusive events,

$$P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_k) = P(A_1 \cup A_2 \cup \ldots \cup A_k)$$

$P(A) =$ sum of probabilities of the simple events, comprising the event A

$$P(E) = \frac{\text{Number of elements or sample points in } A}{\text{Total number of elements or sample points in } S}$$

$$P(A) = \frac{n(A)}{n(S)}$$

where n is defined as the number of sample points or elements.

Note: Two or more events are said to be mutually exclusive if they cannot occur at the same time.

Example 7.5

If a fair die is cast, determine the probability of

a. each sample point.

b. the sum total of all the sample points.

Solution:

a. Since the die is fair, each sample point is equally chanced.

Sample space (S) = {1, 2, 3, 4, 5, 6}

$$\therefore \quad P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

(This confirms the axiom (i) above)

b. $P(S) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$

$$P(S) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

(This also confirms axiom (ii) above)

Example 7.6

A ball is drawn at random from a box containing 8 green balls, 5 yellow balls and 7 red balls. Determine the probability that it is

a. green,

b. yellow,

c. red,

d. not green

e. green or yellow.

Solution

Let G, Y and R represent green, yellow, and red balls respectively.

Then, the sample points in each event is

$n(G) = 8$, $n(Y) = 5$ and $n(R) = 7$

Also, the sample space (S) consists of $8 + 5 + 7 = 20$ sample points

i.e. $n(S) = 20$

a. $P(G) = \frac{n(G)}{n(S)} = \frac{8}{20} = 0.4$

b. $P(Y) = \frac{n(Y)}{n(S)} = \frac{5}{20} = 0.25$

c. $P(R) = \frac{n(R)}{n(S)} = \frac{7}{20} = 0.35$

d. $P(not\ green) = P(G') = 1 - P(G) = 1 - 0.4 = 0.6$

e. $P(green\ or\ yellow) = P(G \cup Y) = P(G) + P(Y) = 0.4 + 0.25 = 0.65$

Alternatively, $P(green\ or\ yellow) = P(G \cup Y) = P(S) - P(R) = 1 - 0.35 = 0.65$

7.3 Addition Law of Probability

If E_1, E_2 are two events in a sample space S, then the probability of $E_1\ or\ E_2$ occurring is given by

$$P(E_1\ or\ E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

This is the addition rule of probability for any two events. We should note further that in set theory "$\cap$" is Interpreted as "and" while "$\cup$" is "or".

By extension, the addition rule for three events $E_1, E_2\ and\ E_3$ can be written as:

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) = {} & P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) \\ & - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \end{aligned}$$

Lastly, when events E_1, E_2 are mutually exclusive, $P(E_1 \cap E_2) = 0$ and the addition rule for 2 events becomes:

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

In particular, if E and E' are mutually exclusive,

$P(E \cup E') = P(E) + P(E')$.

Now, if $E \cup E' = S, then\ P(E) + P(E') = P(S) = 1$

$\Rightarrow \quad P(E) = 1 - P(E')$

i.e. **The probability of an event is one minus the probability of its complement.**

Similarly, for three mutually exclusive events $E_1, E_2\ and\ E_3$, the addition rule also becomes:

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$

Example 7.8

From the following data: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; obtain the probability of the following

(a) An even number

(b) A prime number

(c) A number not greater than 5.

(d) An even number or a prime number.

Solution

Let E_1, E_2 *and* E_3 represent even number, prime number and numbers not greater than 5 respectively

Sample Space, $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \quad n(S) = 10$

(a) Let $E_1 = \{2, 4, 6, 8, 10\}, \quad n(E_1) = 5$

Then $P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{5}{10} = \dfrac{1}{2}$ *or* 0.5

(b) Let $E_2 = \{2, 3, 5, 7\}, \quad n(E_2) = 4$

Then $P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{4}{10} = \dfrac{2}{5}$ *or* 0.4

(c) Let $E_3 = \{1, 2, 3, 4, 5\}, \quad n(E_3) = 5$

Then $P(E_3) = \dfrac{n(E_3)}{n(S)} = \dfrac{5}{10} = \dfrac{1}{2}$ *or* 0.5

(d) $P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Let $E_1 \cap E_2 = \{2\}, \quad n(E_1 \cap E_2) = 1 \quad \Rightarrow P(E_1 \cap E_2) = \dfrac{1}{10}$

$$P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = \frac{5}{10} + \frac{4}{10} - \frac{1}{10} = \frac{5+4-1}{10} = \frac{8}{10} = 0.8$$

Alternatively,

$E_1 \cup E_2 = \{2, 3, 4, 5, 6, 7, 8, 10\}, \quad n(E_1 \cup E_2) = 8$

$\therefore \quad P(E_1 \cup E_2) = \dfrac{8}{10} = 0.8$

Example 7.9

There are 24 female students in a class of 60 students. Determine the probability of selecting a student who is either a male or female.

Solution

Let E_1 *and* E_2 represent male and female respectively,

where $n(S) = 60, \; n(E_2) = 24$ *and* $n(E_1) = 60 - 24 = 36$

Then, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ since E_1 *and* E_2 are mutually exclusive

$\therefore \quad P(E_1 \cap E_2\} = 0$

$$P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} = \frac{24}{60} + \frac{36}{60} = 1$$

7.4 Conditional Probability and Independence

Let E_1 *and* E_2 be two events such that there is an intersection between E_1 *and* E_2, and that $P(E_1) > 0$. then, we denote the conditional probability by $P(E_2/E_1)$; and it can be read "the probability of E_2 given that E_1 has occurred or happened. The symbol (/) represents given. Here, event E_1 is known or assumed to have occurred or happened and it affects the sample space. By definition, the conditional probability is expressed as

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}, \quad or$$

$$P(E_1 \cap E_2) = P(E_1)P(E_2/E_1)$$

A practical example of conditional probability is seen in situations where two events are involved and must be satisfied; e.g probability of a student offering physics given that he must be a male.

The concepts of dependent and independent events go a long way to explain how the occurrence of an event affects another one. Here, if the occurrence or non-occurrence of events E_1 *and* E_2 does not affect each other, the events are said to be independent; otherwise they are dependent events.

Examples of independent events are:

(a) Obtaining Head and Tail in tossing of a coin; and

(b) Passing of QA and Economics in ATS examination.

Also, examples of dependent are:

(a) Weather condition and sales of minerals; and

(b) drug taken and rate of recovery from illness.

By applying the idea of independent event to our conditional probability $P(E_2/E_1)$, it becomes $P(E_2/E_1) = P(E_2)$ because the probability of E_2 occurring or happening is not affected by the occurrence or non-occurrence of E_1.

Hence, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

By extension for independent events E_1, E_2 *and* E_3

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2) \cdot P(E_3)$$

Example 7.10

If the probability that Dayo will be alive in 20 years is 0.8 and the probability that Toyin will be alive in 20 years is 0.3, what is the probability that they will both be alive in 20 years?

Solution

P(that Dayo will be alive in 20 years) = 0.8

P(that Toyin will be alive in 20 years) = 0.3

P(that they will both be alive in 20 years) = 0.8 x 0.3 = 0.24

Note: Both are independent events.

Example 7.11

Suppose that a box contains 5 white balls and 4 black balls. If two balls are to be drawn randomly one after the other without replacement, what is the probability that both balls drawn are black.

Solution

Let E_1 be the event "first ball drawn is black" and E_2 be the event "second ball drawn is black", where the balls are not replaced after being drawn. Here E_1 and E_2 are dependent events and conditional probability approach is an appropriate method.

The probability that the first ball drawn is black is

$$P(E_1) = \frac{4}{5+4} = \frac{4}{9}$$

The probability that the second ball drawn is black, given that the first ball drawn was black, is $P(E_2/E_1)$

$$P(E_2/E_1) = \frac{3}{5+3} = \frac{3}{8}$$

Thus, the probability that both balls drawn are black is

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1) = {}^{4}/_{9} \cdot {}^{3}/_{8} = {}^{12}/_{72} = {}^{1}/_{6}$$

7.5 Multiplication Law of Probability

Let us recall that when combining events in the addition rule, we dealt with the case of "OR" where we use the set symbol $\cup$ (union) to represent it. In a similar vein, the multiplication rule of probability is dealing with "AND" when combining events; and this is represented by the set symbol $\cap$ (intersection).

From conditional probability of events $E_1$and E_2,

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)},$$

or $$P(E_1 \cap E_2) = P(E_2/E_1) \cdot P(E_1)$$

it is also true that $$P(E_1 \cap E_2) = P(E_1/E_2) \cdot P(E_2)$$

This expression represents the multiplication rule of the probability where two events E_1 and E_2 are involved.

Note : If E_1 and E_2 are independent events $P(E_2/E_1) = P(E_2)$ $P(E_1/E_2) = P(E_1)$ $P(E_1 \cap E_2) = P(E_1) . P(E_2/E_1) = P(E_2) P(E_1/E_2) = P(E_1). P(E_2)$

Example 7.12

The probability that an employee comes late to work in any given day is 0.14. The probability that employee is a female in the company is 0.275. Obtain the probability that an employee selected from the company is a female late comer?

Solution:

Let E_1 = event of employee comes late

E_2 = event of employee is a female

Then, $P(E_1 \cap E_2)$ is required

$$\begin{aligned} P(E_1 \cap E_2) &= p(E_2) \cdot p(E_1/E_2) \\ &= 0.275 \times 0.14 \\ &= 0.0385 \end{aligned}$$

Note: E_1, E_2 are independent events.

Example 7.13

A box contains 6 black and 8 yellow balls. If two successive draws of one ball are made, determine the probability that the first drawn ball is black and the second drawn ball is yellow if

(a) the first drawn ball is replaced before the second draw;

(b) the first drawn ball is not replaced.

Solution

Let B and Y represent black and yellow balls respectively.

(a) This is a case of selection with replacement, and the events involved are independent.

$$Pr(B\cap Y) = P(B) \times P(Y)$$

where $P(B) = \frac{6}{6+8} = \frac{6}{14} = \frac{3}{7}$, and

$P(Y) = \frac{8}{6+8} = \frac{8}{14} = \frac{4}{7}$, and

$\therefore \quad P(B\cap Y) = {}^{3}/_{7} \times {}^{4}/_{7} = {}^{12}/_{49}$

(b) This is selection without replacement;

$$Pr(BY) = P(B) \times P(Y/B)$$

where $P(B) = \frac{6}{6+8} = \frac{6}{14} = \frac{3}{7}$, and

$P(Y/B) = \frac{8}{5+8} = \frac{8}{13}$

$\therefore \quad P(BY) = {}^{3}/_{7} \times {}^{8}/_{13} = {}^{24}/_{91}$

7.6 Mathematical Expectation

Mathematical expectation is an important concept in the study of statistics and probability in the sense that it can be used to determine some statistics like the mean and the variance,

Let X be a discrete random variable having the possible values $x_1, x_2, x_3, \ldots, x_n$ with the probabilities $P_1, P_2, P_3, \ldots, P_n$ respectively where $\Sigma p_i = 1$. Then the mathematical expectation, at times called "expected value" or "expectation", of X is denoted by E(X) and is defined as:

$$E(X) = x_1P_1 + x_2P_2 + \text{----} + x_nP_n$$

$$= \quad \sum_{i=1}^{n} \frac{x_i f(x_i)}{\sum f_i}$$

or

simply as $= \quad \dfrac{\sum xfx}{\sum f} \quad -$ …… 7.1

where $P_i = \frac{f(x_i)}{\sum f}$

A special class of equation 7.1 is where the probabilities are all equal to one which gives

$E(X) = \frac{x_1 + x_2 + ---- + x_n}{n}$, which is the arithmetic mean.

Similarly for a continuous random variable X having density function $f(x)$, the expected value is defined by:

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

We have to note the following expectation as defined below:

(a) $E(X) = \Sigma xf(x) = \mu$, the mean.

(b) $E(X^2) = \Sigma x^2 f(x)$

(c) SD of $(X) = \sigma = \sqrt{\frac{\sum(X-\mu)^2}{\sum f}} = \sqrt{E(X^2) - [E(X)]^2}$

where (i) is the expectation of variable X and (ii) is the expectation of the square of X.

The following are some of the properties of expectation

a. If C is any constant, then

$$E(CX) = CE(X)$$

b. IF X and Y are any random variables, then

$$E(X + Y) = E(X) + E(Y)$$

c. If X and Y are independent random variables,

$$E(XY) = E(X)\,E(Y).$$

Example 7.14

Find: a. $E(X)$, b. $E(X^2)$ and c. SD(X) for the probability distribution shown in the following table:

x	6	10	14	18	22
$P(x)$	1/6	5/36	¼	1/3	1/9

Solution

a. E(X) = $\Sigma x \cdot P(x)$

= $6 \times {}^{1}/_{6} + 10 \times {}^{5}/_{36} + 14 \times {}^{1}/_{4} + 18 \times {}^{1}/_{3} + 22 \times {}^{1}/_{9}$

= 14.3333

This represents the mean of the distribution.

b. $E(X^2)$ = $\Sigma x^2 \cdot P(x)$

= $6^2 \times {}^{1}/_{6} + 10^2 \times {}^{5}/_{36} + 14^2 \times {}^{1}/_{4} + 18^2 \times {}^{1}/_{3} + 22^2 \times {}^{1}/_{9}$

= 230.6667

c. $\sigma = \sqrt{E(X^2) - [E(X)]^2}$

= $\sqrt{230.6667 - [14.3333]^2}$

= $\sqrt{25.22}$

= 5.02 this represents the standard deviation of the distribution.

Example 7.15

A company organised a lottery where there are 200 prizes of ₦50, 20 prizes of ₦250 and 5 prizes of ₦1000. If the company is ready to issue and sell 10,000 tickets, determine a fair price to pay for a ticket

Solution: Let *X* and f(*x*) represent the amount of money to be won and density function respectively. Then the following table depicts the distribution:

X(₦)	50	250	1000	0
Frequency	200	20	5	10,000 – 225 = 9,775
f(*x*)	200/10,000 = 0.02	20 / 10,000 = 0.002	5 / 10,000 = 0.0005	9,775/10,000= 0.9775

Note: $f(x) = \dfrac{\text{Number of prizes in a category}}{\text{Total Number of available tickets}}$

Then, the fair price

= E(X) = 50(0.02) + 250(0.002) + 1000(0.0005) + 0(0.9775)

= ₦2

Example 7.16

Determine the expected value of the sum of points in tossing a pair of fair dice.

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

$$E(X) = \sum x.P(x)$$

$$= 2\times 1/36+3\times 2/36+4\times {}^3/_{36}+5\times {}^4/_{36}+6\times {}^5/_{36}+7\times {}^6/_{36}+8\times {}^5/_{36}+9\times {}^4/_{36}+10\times {}^3/_{36}+11\times {}^2/_{36}+12\times {}^1/_{36}$$

$$= {}^1/_{18}+{}^1/_6+{}^1/_3+{}^5/_9+{}^5/_6+{}^7/_6+{}^{10}/_9+1+{}^5/_6+{}^{11}/_{18}+{}^1/_3$$

$$= 7$$

Alternatively,

Let X and Y represent the points showing on the two dice. Then

$E(X) = E(Y) = 1({}^1/_6) + 2({}^1/_6) + 3({}^1/_6) + 4({}^1/_6) + 5({}^1/_6) + 6({}^1/_6) = {}^7/_2$

By the expectation property (b), $E(X + Y) = E(X) + E(Y) = {}^7/_2 + {}^7/_2 = 7$

7.7 Chapter Summary

The principle of elementary probability and its applications were treated. The additive and multiplicative laws, and even the conditional probabilities were considered with numerical examples.

7.8 Multiple-choice and short- answer questions

Use the following information to answer the next three questions. Given a set of numbers S = {2, 4, 7, 10, 13, 16, 22, 27, 81, 102}
If a number is picked at random, determine the probability that the number is

1. even
   A. 0.4
   B. 0.2
   C. 0.6
   D. 0.8
   E. 0.9

2. odd
   A. 0.4
   B. 0.8
   C. 0.2
   D. 0.9
   E. 0.6

3. prime
   A. 0.2
   B. 0.5
   C. 0.3
   D. 0.6
   E. 0.4

Use the following information to answer the next 2 questions:

Supposing there is a lottery game in which 2000 tickets are issued and that there are 5 major and 50 minor prizes to be won. If I buy a ticket, the probability that it will win

4. a major prize is
   A. 0.25
   B. 0.025
   C. 0.0025
   D. 0.5
   E. 0.05

5. a minor prize is
   A. 0.25
   B. 0.025

C. 0.0025

D. 0.5

E. 0.05

6. Using the item of question 4, the probability of winning a prize in the lottery game is ....

7. The list of all possible outcomes in a random experiment is called its ....

Use the following statement to answer questions 8 to 10:

If a ball is selected randomly from a box containing 6 white balls, 4 blue balls and 5 red balls, obtain the probability that:

8. White ball is selected.........................

9. Not white ball is selected....................

10. Blue or red ball is selected.......................

Answers

1. C

2. A

3. C

4. C

5. B

6. 0.0275 i.e. $\frac{5}{2000} + \frac{50}{2000} = 0.0275$

7. Sample space

8. Pr(white) = $^{6}/_{15}$ = $\underline{0.4}$,

9. Pr(Not white) = 1 – Pr(white) = 1 – 0.4

   = 0.6

10. Pr(Blue or Red) = $^{4}/_{15}$ + $^{5}/_{15}$ = $^{9}/_{15}$ = 0.6

CHAPTER EIGHT

HYPOTHESIS TESTING

Chapter contents

(a) Introduction.

(b) Useful Concepts in Hypothesis Testing.

(c) Testing Hypothesis About a Population Parameter.

(d) Testing Hypothesis About a Population Proportion.

Objectives

At the end of the chapter, readers should be able to understand the

(a) concept of hypothesis;

(b) difference between two types of errors in hypothesis testing;

(c) level of significance;

(d) concepts of one-tailed and two-tailed tests; and

(e) testing of hypothesis on population mean and population proportion for both large and small samples.

8.1 Introduction

An important study of statistical theory, which is commonly used in decision making, is the concept of hypothesis testing. It assists in taking decisions concerning propositions i.e. how valid is the proposition.

8.2 Useful Concepts in Hypothesis Testing

The following terms are useful in the principle of Hypothesis testing.

Hypothesis: The concept of hypothesis in statistics aids decision making. Hypothesis can be defined as the assumption or guess about the population parameters involved.

There are two types of hypothesis; namely, the

a) Null hypothesis; and

b) Alternative hypothesis.

A hypothesis which states that there is no difference between the procedures, results from samples and other phenomenon is called Null Hypothesis and it is denoted by $H_{0,}$ while any hypothesis which differs from the null hypothesis or given hypothesis is known as Alternative Hypothesis and it is denoted by H_1. For
instance, if the population mean, $\mu = 5.0$ and the sample mean, $\bar{x} = 6.0$, we set our hypothesis as thus:

$$H_0 : \mu = \bar{x} \quad i.e. \quad H_0 : 5.0 = 6.0$$

$$H_0 : \mu \neq \bar{x} \quad i.e. \quad H_0 : 5.0 \neq 6.0$$

$$H_0 : \mu < \bar{x} \quad i.e. \quad H_0 : 5.0 < 6.0$$

Errors in hypothesis testing

There are two types of errors in hypothesis testing, namely:

(a) Type I error; and

(b) Type II error.

When a hypothesis, which is supposed to be accepted is rejected, we call that error a type I error. On the other hand, if we accept a hypothesis which is supposed to be rejected, a type II error is committed.

The above statements can be summarized in the following table:

Decisions	H_0 true	H_0 False
Accept H_0	Correct decision	Type I error
Reject H_0	Type II error	Correct decision

Level of Significance

In hypothesis testing, the maximum probability of the willingness to risk a type I error is called the level of significance or simply significance level and it is denoted by α. It therefore implies that the investigator has $1-\alpha$ confidence that he/she is making right decision. In practice, this must be stated first in any hypothesis testing. The common level of usage in practice are 1% (or 0.01) and 5% (or 0.05).

Test Statistic

A test statistic is the computed value, which, when compared with the tabulated value, enables one to decide whether to accept or reject hypothesis and hence determine whether the figures of the observed samples differ significantly from that of the population. It is also called *test of hypothesis*, test of significance or rules of decision.

Critical Region

This is a region or critical value of one side of the distribution of the case study, with area equal to the level of significance. It is used to decide whether to accept a hypothesis or not.

One-tailed and Two-tailed Tests

The type of tailed test depends on the stated alternative hypothesis (H_1). It is one-tailed test or one-sided test when the alternative hypothesis is one directional. For example, we have the following hypothesis for one-tailed test:

$$H_0 : \mu = \bar{x} \quad vs \quad H_1 : \mu > \bar{x}$$

$$H_0 : \mu = \bar{x} \quad vs \quad H_1 : \mu \geq \bar{x}$$

$$H_0 : \mu = \bar{x} \quad vs \quad H_1 : \mu < \bar{x}$$

$$H_0 : \mu = \bar{x} \quad vs \quad H_1 : \mu \leq \bar{x}$$

For a two-tailed test or two-sided test, the hypothesis is of two directions e.g.

$$H_0 : \mu = \bar{x} \quad vs \quad H_1 : \mu \neq \bar{x}$$

The resultant effect of the type of tailed test is on the significance level to be used in obtaining the critical value from the table. If the significance level is α, then critical values for α and $\frac{\alpha}{2}$ will be obtained respectively for one-tailed and two-tailed tests.

For instance, if $\alpha = 0.05$, we check critical values at 0.05 and $\frac{0.05}{2} = 0.025$ respectively for the one-tailed and two-tailed tests.

8.3 Testing Hypothesis About a Population Parameter

The procedure for carrying out significance test about a population parameter is given as follows:

c) Set up the hypothesis about the parameter of interest and state its significance level.;

d) Set the hypothesis by using appropriate test statistic; and

e) Draw conclusion by making decision on the result of your computed value in step (b) above. Here, table value at a significance level is compared with computed value.

It should be noted the alternative hypothesis determine the decision rule to be applied in taking a reasonable decision in testing hypothesis about the population parameter. The following are the decision rules for different alternative hypothesis which can be applied for both small and large samples

(i) for a right one-tailed alternative hypothesis (i.e. $H_1 : \mu > \mu_0$), reject H_0 if $z_{cal} > z_{tab}$ *or* $t_{cal} > t_{tab}$ otherwise do not reject H_0.

(ii) for a left one-tailed alternative hypothesis (i.e. $H_1 : \mu < \mu_0$), reject H_0 if $z_{cal} < -z_{tab}$ *or* $t_{cal} < -t_{tab}$ otherwise do not reject H_0.

(iii) for a two-tailed alternative hypothesis (i.e. $H_1 : \mu \neq \mu_0$), reject H_0 if $|z_{cal}| > z_{tab}$ *or* $|t_{cal}| > t_{tab}$ otherwise do not reject H_0 (where

NOTE:

z_{cal} *or* t_{cal} is the calculated value i.e. computed test statistic value and

z_{tab} *or* t_{tab} is the table value i.e. value from test statistic distribution table

8.4 Testing Hypothesis About the Population Mean

This section will present testing hypothesis under two conditions namely; (i) when we have large samples and (ii) when we have small samples.

For large samples ($n \geq 30$), it is usually assumed that the sampling distribution of the desired statistic is normally distributed or approximately normal. It is for this reason that tests concerning large samples assumed normal and used the z – test. Therefore, the test statistic for large sample is

$$Z_{cal} = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

Where $\bar{x} =$ sample mean

$\mu_0 =$ hypothesis value of μ

$\sigma =$ the given value of the standard deviation; but when this not given, one can use sample standard deviation (s) for large samples

$n =$ sample size

When we have small samples ($n < 30$), the test – statistic is the t – test and it is defined thus: where $\bar{x}$, μ_0, n are as defined above and

$$t_{cal} = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

where $\bar{x}$, μ_0, n are as defined above and

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

The final decision on the tests is made by comparing the computed values (z_{cal} or t_{cal}) with the table values. The normal table is used for the z – test (for large samples) while t – distribution table at $n – 1$ degrees of freedom is used for small samples.

When $z_{cal} > z_{tab}$, and $t_{cal} > t_{tab}$ we reject the null hypothesis (H_0); otherwise, do not reject H_0.

Example 8.1

In a University, a sample of 225 male students was taken in order to find the average height. From the samples, the computed average height was 184.0cms while the mean of actual population height was 178.5cms with a standard deviation of 120cms. You are required to show if the sample mean height is significantly different from the population mean at 5% significant level.

Solution:

Hypothesis

$$H_0 : \mu = \bar{x} \Rightarrow H_0 : 178.5 = 184.0 \quad vs \quad H_1 : \mu \neq \bar{x} \Rightarrow H_1 : 178.5 \neq 184.0$$

$$Z_{cal} = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{184.0 - 178.5}{\frac{120}{\sqrt{225}}} = \frac{5.5}{\frac{120}{15}} = \frac{5.5}{8} = 0.6875$$

Table value of z at 5% level of significance (for two tailed test) = 1.96

Decision: since $z_{cal} < z_{tab}$ (0.6875 <1.96), we do not reject H_0 (i.e. accept H_0) and concluded that no significance difference between the sample mean and population mean.

Example 8.2

SAO is a manufacturing company with 5,000 workers. The company is interested in knowing the average number of items sold per week per person by the company's workers. While the quality control manager thinks that the average number of items sold per worker per week is 11, the company secretary thinks that the true value should be more. The quality control manager subsequently selected 11 workers at random and got the following results as number of items sold in a week, 13, 4, 17, 9, 3, 20, 16, 12, 8, 18,12.

You are required to set up a suitable hypothesis and test it at 5% level of significance.

Solution:

$H_0 : \mu = 11 \quad vs \quad H_1 : \mu > 11$

$\alpha = 5\% = 0.05, \quad \mu_0 = 11, \quad n = 11$

Since the sample size is small (i.e. $n = 11 < 30$), we use the t – test

$$t_{cal} = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

Where $\bar{x} = \frac{\sum x}{n} \quad and \quad s^2 = \frac{\sum (x - \bar{x})^2}{n-1}$

$$mean, \bar{x} = \frac{13 + 4 + 17 + 9 + 3 + 20 + 16 + 12 + 8 + 18 + 12}{11} = \frac{132}{11} = 12$$

The computation of s^2 is set out in the table below

x	$x - \bar{x}$	$(x - \bar{x})^2$
13	1	1
4	-8	64
17	5	25
9	-3	9
3	-9	81
20	8	64
16	4	16
12	0	0
8	-4	16
18	6	36
12	0	0
		$\sum (x - \bar{x})^2 = 312$

$$s^2 = \frac{312}{11-1} = \frac{312}{10} = 31.2$$

$$\therefore s^2 = 31.2, \rightarrow \quad s = \sqrt{31.2}$$

$$= 5.5857$$

$$\approx 5.6$$

using all these pieces of information, our test statistic becomes

$$t_{cal} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

$$= \frac{12-11}{5.6/\sqrt{11}}$$

$$= 0.5923$$

$$\approx 0.59$$

Table value of t at 5% significance level (for one-tailed test) = 1.81

Decision: the problem is of one-tail test (see the alternative hypothesis), and $t_{cal} < t_{tab}$ i.e 0.59 < 1.81, we accept H_0 and conclude that no significance difference between the sample mean and the population mean.

8.5 Testing Hypothesis About the Population Proportion

In the significance test for a proportion, there is the need to know the population proportion (P_0) and then compute the sample proportion (P) from the sample drawn. The sample proportion is obtained by taking a number of observations (x) (possessing an attribute) out of the total number n so that $P = x/n$

As we have done for the mean, we set up the hypothesis as thus:

$H_0 : P_0 = P$

$H_1 : P_0 \neq P$ ($P_0 > P$ or $P_0 < P$)

and the significance level given is α.

Then, the test statistic is

$$Z_{cal} = \frac{P - P_0}{\sqrt{\frac{P(1-P)}{n}}} \quad \text{or} \quad \frac{x/n - p_0}{\sqrt{\frac{x/n \cdot \left(1 - x/n\right)}{n}}}$$

and we decide by comparing the Z_{cal} with the table value Z_{tab} at the given α

Example 8.4

A demographer claims that pupils in all the primary schools in a State constitute 30% of the total population of the state. A random sample of 400 pupils from all primary schools in the Local Government Areas of the State shows that 25% of them are pupils of primary school.

Test at 5% level of significance the validity or otherwise of the demographer's claim.

Solution:

Let P_0 be the population proportion of pupils in the primary schools in the State and P the estimated proportion.

$H_0 : P_0 = P$ i.e. $H_0 : 0.30 = 0.25$

$H_1 : P_0 \neq P$ i.e. $H_1 : 0.30 \neq 0.25$

$$\therefore \quad Z_{cal} = \frac{P - P_0}{\sqrt{\frac{P(1-P)}{n}}}$$

$$\Rightarrow \quad Z_{cal} = \frac{0.25 - 0.30}{\sqrt{\frac{0.25(1-0.25)}{400}}} = \frac{-0.05}{\sqrt{\frac{0.25(0.75)}{400}}} = \frac{-0.05}{\sqrt{0..00046675}} = \frac{-0.05}{0.02165}$$

$$Z_{cal} = -2.309$$

It is a two-sided test

$\therefore$ for $\alpha = 0.05$, then $^{\alpha}/_{2} = 0.025$ and table value of Z = 1.96

Decision: since $|Z_{cal}| > Z_{tab}$ i.e. 2.309 > 1.96,

H_0 is rejected and it is concluded that the data cannot support the demographer's claim.

Differences between means

Test involving difference of means

i.e. $H_0 : \mu_1 - \mu_2 = 0$

$H_1 : \mu_1 - \mu_2 \neq 0$

Test statistic is

i) $Z_{cal} = \dfrac{\mu_1 - \mu_2}{\sigma_{\mu_1 - \mu_2}}$ for large samples

ii) $t_{cal} = \dfrac{\bar{x}_1 - \bar{x}_2}{S_{x_1 - x_2}}$

where $\sigma_{\mu_1 - \mu_2} = \sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$

μ_1, σ_1, n_1 are values for population 1 and

μ_2, σ_2, n_2 are values for population 2.

Similarly,

$$S_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

$\bar{x}_1$, S_1, n_1 are values for sample 1 and

$\bar{x}_2$, S_2, n_2 are values for sample 2.

Example 8.4

The same set of examination was given to two groups of students – group A and group B. Group A has 40 students and the mean score was 74 marks with a standard deviation of 8. Group B has 50 students and the mean score was 78 marks with a standard deviation of 7. Is there any significant difference between the performance of the two groups at 5% significance level?

(Hint: 5% significance level, Z_{tab} = 1.96)

Solution

$$H_0: \quad \mu_1 - \mu_2 = 0$$

$$H_1: \quad \mu_1 - \mu_2 \neq 0$$

$$Z_{cal} = \frac{\mu_1 - \mu_2}{\sigma_{\mu_1 - \mu_2}}, \qquad \sigma_{\mu_1 - \mu_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$= \sqrt{\frac{8^2}{40} + \frac{7^2}{50}}$$

$$= 1.606$$

$$\therefore \quad Z_{cal} = \frac{74 - 78}{1.606} = -2.49$$

Since |-2.49| > 1.96 (Z_{cal}), H_0 is rejected i.e. there is a significant difference between the performance of the two groups of students

8.6 Chapter Summary

Important concept such as hypothesis, errors, significance level, test statistic, one-tailed and two-tailed tests are discussed in the hypothesis testing. Tests concerning the mean and proportion were finally presented. Numerical examples were solved and discussed to illustrate all d principles involved.

8.7 Multiple-choice and short-answered questions

1. Rejection of the null hypothesis when it should have been accepted is known as ....
   A. Type II error
   B. Standard error
   C. Percentage error
   D. Hypothesis error
   E. Type I error

2. Significance level is referred to as the risk of committing _
   A. Sampling error
   B. Non-sampling error
   C. Bias error
   D. Type I error
   E. Type II error

3. Which of the following hypotheses is not for one-tailed test?
   A. H_1: $\mu > 0$
   B. H_1: $\mu \geq 0$
   C. H_1: $\mu < 0$
   D. H_1: $\mu \leq 0$
   E. H_1: $\mu \neq 0$.

4. The test-statistic for a large sample in the hypothesis testing of mean is ....

5. The degree of freedom for t-test with n samples is…………………….

6. The test-statistic for proportion is ………………….

Use the following information to answer questions 7 to 10

A sample of 4 items are taken randomly from a population sample, their weights (in Kgs) are recorded as follows: 6, 8, 12, and 14.

7. Determine the mean weight of the sample.

8. Determine the standard deviation of sample.

9. If the mean weight of increasing the sample to 5 is 10kg, what is the weight of the fifth item?

10. If the population mean μ = 9, compute the test-statistic for the data

Answers

1. E
2. D
3 E
4. $Z_{cal} = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$,
5. $\underline{n} - 1$
6. $Z_{cal} = \dfrac{P - P_0}{\sqrt{\dfrac{P(1-P)}{n}}}$,
7. 10,
8.

$$= \sqrt{\frac{(6\text{-}10)^2 + (8\text{-}10)^2 + (12\text{-}10)^2 + (14\text{-}10)^2}{4}}$$

$$= \sqrt{\frac{16 + 4 + 4\ 16}{4}}$$

$= 3.16$

9. $\dfrac{6+8+12+14+x}{5} = 10$

x + 40 =50 $\Rightarrow$ x = 10kg

10. $t_{cal} = \dfrac{\bar{x} - \mu_0}{S / \sqrt{n}} = \dfrac{10-9}{1.58} =$ 0.63

SECTION B

BUSINESS MATHEMATICS

CHAPTER NINE

PROFIT AND LOSS BASED ON SALES

Chapter contents

(a) Introduction.
(b) Concept of profit and loss.
(c) Discounting.
(d) Marked price.

Objectives

At the end of the chapter, readers should be able to understand the:

(a) meaning of cost price, selling price, profit and loss;
(b) calculation of profit and loss percentages;
(c) concept of discounting and its calculation;
(d) concept of marked price and its calculation; and
(e) relationship among selling price, discount price and marked price.

9.1 Introduction

In the business world, the ability of a traders/businessmen/businesswomen/business organisations to be able to know if it is running at a profit or loss, is very essential for the life span of the business.

The branch of business mathematics that deals with the study of profit and loss in any business transaction is known as profit and loss. In accounting world, the summary of the trading transactions of business that shows whether the business has made a profit or loss during a certain period of account can be found in the profit and loss account book. Profit is the amount gained by selling an item more than its cost price while loss is the amount incurred by the seller for selling an item less than its cost price.

The final selling price of a product is the difference between the marked price and discount.

9.2 Concept of profit and loss.

The fundamental objective of any business is to make a profit know how profitable it is. In order to know whether a profit is gained or loss is incurred for a business/trading transaction, two terms are very important and these are cost price and selling price of an item/product.

The cost of price of an item is the price at which it is purchased by the buyer or the amount paid by a consumer to the wholesaler or the manufacturer to buy goods and it can be abbreviated as *CP*. The cost price is very important in the calculation of profit and loss percentages.

Cost price can be classified into two categories: fixed cost (The fixed cost is constant and doesn't vary under any circumstance) and variable cost(the variable cost vary and the variation depends on other factors and number of units).

The selling price of an item is the price at which an item is sold to the buyer by the seller and it can be abbreviated as *SP*. The selling price, in actual sense, is the sum of the cost price and the target gross profit. When the *SP* is greater than *CP* (*SP*>*CP*) or (*CP*<*SP*),then the seller is said to have made a profit or gain and when the *SP* is less than *CP* (*SP*<CP) or (*CP*>*SP*),then the seller is said to have incurred a loss.

It should be noted that the comparison of *CP* with *SP* is always considered first in order to know whether the transaction will lead to a profit or a loss.

The profit percentage formula is given as

$$profit\% = \frac{SP - CP}{CP} \times \frac{100}{1} = \frac{profit}{CP} \times \frac{100}{1}\%$$

The loss percentage formula is given as

$$loss\% = \frac{CP - SP}{CP} \times \frac{100}{1} = \frac{loss}{CP} \times \frac{100}{1}\%$$

Note:

If an item is sold at a profit of say x%, then *SP* = (100+x)% of *CP* and if an item is sold at a loss of say x%, then *SP* = (100-x)% of *CP*.

Example 9.1

A petty trader bought an article for ₦1,500 and sold it for ₦1,800. Calculate the trader's profit/loss percent.

Solution.

Cost price, *CP* = ₦1,500

Selling price, *SP* = ₦1,800

Since *SP*>*CP* then trader make a profit.

Profit = *SP*-*CP* = ₦1,800 - ₦1,500 = ₦300

Example 9.2

A fruit seller bought 5 baskets of oranges at ₦1,200 per basket and sold them for ₦1,100 per basket. Calculate the percentage loss incurred by the fruit seller.

Solution

CP = 5× ₦1200 = ₦6,000

SP = 5 × ₦1100 = ₦5,500

Since CP>SP, then the fruit seller incurred a loss i.e.

Loss = CP - SP = ₦6,000 - ₦5,500 = ₦500

Loss Percent $= \frac{500}{6,000} \times \frac{100}{1} = 8.33\%$

Example 9.3

A motor spare parts dealer buys a cooling fan for ₦12,000 and sells it at a loss of 7.5%. What is the selling price of the cooling fan?

Note:

(i) *CP* --------- 100%

SP --------- (100 + *x*)% where *x* is the profit %

$CP \times (100 + x) = SP \times 100$

(ii) *CP* --------- 100%

SP --------- (100 - *k*)% where *k* is the loss %

$CP \times (100 - k) = SP \times 100$

Solution

CP = ₦12,000

Loss% = 7.5%

SP =?

CP ------- 100%

SP ------- (100 – 7.5)% = 92.5%

$CP \times 92.5 = SP \times 100$

$$SP = \frac{CP \times 92.5}{100}$$

$$SP = \frac{12000 \times 92.5}{100} = ₦11,100$$

9.3 Discounting

This is a term used when describing a situation where a seller gives a reduction in price of an item i.e. discount. Discount is the amount of rebate given on the price of product and it is usually given by the seller to attract customers so as to increase sales. Discount is also given by the retailer to clear out old inventory and create space for new collections. Reduction in price of goods and services offered by a retailer also leads to early payment by the buyer. Discount is always a reduction given on the market price (i.e. marked price which will be discussed in the next section. It should be noted that the price of a product after giving a discount is always considered as the selling price of the product. The following formulas are used for computation of discount:

(i) *Discount = Discount% of Marked price*

(ii) *Discount = Marked price (MP) –Actual Selling price (SP)*

(iii) $\text{Discount percentage} = \frac{\text{Discount}}{\text{Marked price}} \times \frac{100}{1}$

Example 9.4
Mr. WAOFAO sold a bicycle for ₦23,000 which has a market price of ₦25,000. Calculate the discount percent given by the seller on the bicycle.

Solution
Selling price = ₦23,000, *Marked price* = ₦25,000
Discount = Marked price (MP) – Selling price (SP)= ₦25,000 – ₦23000 = ₦2,000
$\text{Discount percent} = \frac{2,000}{25,000} \times \frac{100}{1} = \frac{200}{25} = 8\%$

Example 9.5
A wedding gown was sold at a discount of 15%. Calculate the discount given if the gown was marked at ₦45,000.

Solution
Discount percent = 15% of the Marked price = ₦45,000
Discount = Discount percent of Marked price (MP)

$$Discount = \frac{15}{100} \times \frac{45{,}000}{1} = ₦67{,}500$$

9.4 Marked price

Marked price is the price quoted on a product which appears in form of a label. It is also referred to as market price or retail price or list price. It should be noted that the marked price is the price on which the discount is normally given. Also, the marked price of a product may or may not be the same as its selling price since selling price is the actual price at which a product is sold. It should be of note that an item/ a product may not necessarily be sold at the given marked price. If a product is sold at the market price, then there is no difference between the marked price and the selling price i.e. they are the same. This simply implies that no discount is offered on the product. Marked price is the price at a specific percentage above the cost price of a product. There is a relationship that exists among the Marked Price, Selling Price and Discount. It is given as

Marked price (MP) = Selling price (SP) + Discount

Example 9.6

A dress is sold for ₦9,000. Calculate its marked price if a discount of 10% is allowed on the dress

Solution

Selling price = ₦9,000, Discount% = 10%

Marked price (MP) = Selling price (SP) + Discount = 9,000 + 10% of MP

$\Rightarrow \quad MP = 9{,}000 + 0.1MP$

$\Rightarrow \quad MP - 0.1MP = 9{,}000$

$\Rightarrow \quad 0.9MP = 9{,}000$

$$\therefore \quad MP = \frac{9{,}000}{0.9} = ₦10{,}000$$

Example 9.7

A retailer allows a discount of 15% on a particular product to his customers and still makes a profit of 25%. Calculate the marked price of the product which costs ₦2,500 to the retailer.

Solution

Cost price = ₦2,500, Discount % = 15%, Profit% = 25%

$$Selling\ price,\ SP = \frac{CP \times (100 + profit\%)}{100} = \frac{2{,}500 \times (100 + 25)}{100} = \frac{2{,}500 \times 125}{100} = ₦3{,}125$$

Selling price (SP) = Marked price (MP) - Discount
Selling price (SP) = Marked price (MP) – 15% of $MP = MP - 0.15MP = 0.85MP$

$$\Rightarrow \quad SP = 0.85MP$$

$$\Rightarrow \quad MP = \frac{SP}{0.85} = \frac{3{,}125}{0.85} = ₦3{,}676.47$$

9.5 Chapter summary

The chapter treated the concept of profit and loss with basic definitions of cost price, selling price, profit, loss, profit percent and loss percent. The principle of discounting and marked price, the relationship that exist among selling price, marked price and discount and its application to sales problems were discussed.

9.6 Multiple choice and short-answer questions

1. A storekeeper bought a used car for ₦90,000 and sold it for ₦81,000. What is the storekeeper profit or loss%?
   A. 9% loss
   B. 9% profit
   C. 10% loss
   D. 10% profit
   E. 15% profit
2. Which of the following represents the relationship among marked price (*MP*), selling price (*SP*) and discount (*D*)?
   A. *SP = MP - D*
   B. *MP = SP - D*
   C. *SP = MP + D*
   D. *MP = SP - D*
   E. *D = MP - SP*
3. A fruit seller sold a basket of oranges for ₦3,000 at a profit of 10%. What is the cost price of the basket of oranges?
   A. ₦300
   B. ₦700
   C. ₦2,700
   D. ₦2,727
   E. ₦3,300
4. The formula to find the selling price (*SP*) of an item if the cost price (*CP*) and the loss% (say *x*%) are given is
   A. *SP = (100 + x)% of CP*
   B. *SP = (100 – x)% of CP*
   C. *SP = x% of CP*

D. *SP = CP + (100 – x)% of CP*
E. *SP = CP - (100 + x)% of CP*

5. A dozen crates of egg at marked price ₦8,000 are available at a discount of 10%. How many crates of eggs can be bought for ₦2,400?
   A. 2
   B. 4
   C. 6
   D. 8
   E. 10
6. The marked price of an item is also known as ………………………
7. A trader is said to have incurred a loss by selling a product if the selling price is …………………. the cost price
8. Discount is the reduction given on the…………………………… of an item
9. The selling price of a product is the difference between the ………. and ………
10. When the marked price of a product equals its selling price then …………… was given on the product.

ANSWERS
1. C
2. A
3. E
4. B
5. B
6. Market price or retail price or list price
7. Less than
8. Marked price or Market price or retail price or list price
9. Marked price, discount (in that order)
10. No discount

CHAPTER TEN

SET THEORY AS APPLIED TO BUSINESS

Chapter contents

(a) Introduction.

(b) Definition of basic terms in set theory.

(c) Application of concept of set theory to business-oriented problems using Euler-Venn diagrams.

Objectives

At the end of the chapter, readers should be able to understand the

a) concept of set theory;

b) basic terms in set theory; and

c) Solution of business-oriented problems based on set theory using Euler-Venn diagrams.

10.1 Introduction

In real life situations, arrangement and collections of objects or people according to their common properties or characteristics are part of day-to-day activities. For example, arrangement of chartered accountants in an accounting firm register based on their membership number, types of animals in a zoo and so on. While each of this collection is known as a set, then the objects in the set are called elements or members

10.2 Definition of basic terms in set theory

A set is a collection of well-defined objects, things or units. The objects which are called members or elements, must be well defined in order to determine and ascertain whether a particular object belongs to the set or not.

Examples of set are set of all integer numbers, set of accountancy students in a University, set of ATS candidates in Ghana, etc.

By convention, a set is represented or denoted by a capital letter while an element by small letter.

The symbols used in sets are:

i. $\{\ldots.\}$ to represent a set;

ii. $b \in B$ implies "*b* belongs to *B*";and

iii. $b \notin B$ implies "*b* does not belong to *B*".

A set is completely specified in the following ways:

a. By actually listing all its elements (This is called the roaster method) e.g.
$A = \{1,2,3,4,5,6\}$
$B = \{a, b, c, d\}$; and

b. By describing some property held by all elements in the set (This is called the property method) e.g.
$A = \{x: x \text{ is an integer}\}$.

Note: Curly bracket { } is always used to represent a set.

10.3 Types of sets

i. **Finite and Infinite sets:** When the number of elements in a set is countable, then the set is said to be finite; otherwise it is infinite.
Examples: A = {Odd numbers between 0 and 12} is a finite set; while
$B = \{1, 2, 3, \ldots\ldots\}$ is an infinite set

ii. **Cardinality or Number of elements in a set:** for any finite set, the number of elements in a set A is denoted by $n(A)$.
Example: $A = \{1,4,7,8,5,6\}$, $n(A)=6$

iii. **Empty or Null Set:** This is a set with no element. It is denoted by the symbol { }
Example: The set of integers that are both odd and even is an empty set

iv. **Universal set:** This is the collection of all conceivable objects under consideration. "U" is the symbol used to denote the universal set.
Examples: U = {All English Alphabets};
U = {All students in a college}; and
U = {All candidates writing a particular diet of ATS examination}.

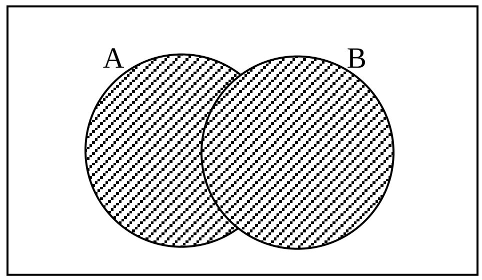
v. **Subset:** If all the elements of set A are contained in another set B, then set A is a subset of set B. In symbolic form, it is written $A \subset B$, where $\subset$ represents subset. At times, it can be .written as $B \supset A$. The symbol $\supset$ denotes superset. Note that if all the elements of A are also the elements of B, then A and B are equal sets. The following examples illustrate the above relationships:

i. $A = \{1,2,3\}$ and $B = \{3,2,1\}$
then $A = B$

ii. $D = \{a, b, c\}$ and $F = \{a, e, c, b\}$.
D is subset of F, i.e. $D \subset F$.

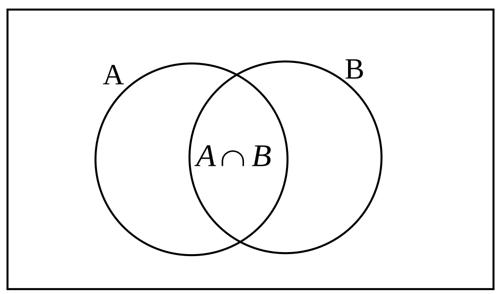
vi. **Union:** The union of two sets A and B is the set which consists of all elements or points of sets belonging to either A or B or both A and B. It is denoted by $A \cup B$, where $\cup$ is the union symbol

$A \cup B$ is shaded



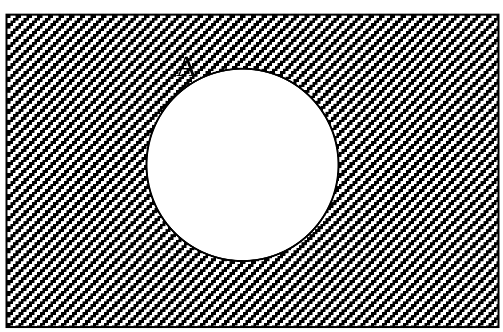
vii. **Intersection:** The intersection of two sets A and B is the set of all elements or points of sets belonging to both A and B. It is denoted by $A \cap B$, where symbol $\cap$ represents intersection.

$A \cap B$ is the indicated part



viii. **Complement:** Complement of a set A is a set containing all elements in Universal set but not in A. It is denoted by A' or A^c.

The shaded part is A' or A^c



Example 10.1

List the elements of each of the following sets:

a. X = {odd numbers between 0 and 10};

b. Y = {odd numbers less than 20}; and

c. Z = {Even numbers between 9 and 31}.

Solution:

a. $X = \{1,3,5,7,9\}$

b. $Y = \{1,3,5,7,9,11,13,15,17,19\}$

c. $Z = \{10,12,14,16,18,20,22,24,26,28,30\}$

10.4 Applications to business-oriented problems using Euler-Venn diagrams

Euler-Venn diagram in set theory was developed by both Euler and John Venn. Euler-Venn diagram can simply be defined as a pictorial/diagrammatical representation of sets. It serves as a relationship provider between or among sets most especially when set operations are involved. By convention, a universal set U is represented by the set of points inside a rectangle while the other subsets by sets of points inside circles. The concept of Euler-Venn diagram will now be applied to business-oriented problems with the following examples.

Example 10.2

In a market consisting of 50 traders, 25 sell either Tubers of yam(T)or Onions(O). Of these, 4 sell both, there are 6 traders that sell Onions. Find how many traders sell

(i) Tubers of yam;

(ii) Tubers of yam but not Onions; and

(iii) Neither Tubers of yams nor Onions.

Solution

Let U stand for all the traders

$n(U) = 50, \quad n(O) = 6, \quad n(T \cap O) = 4 \quad and \quad n(T \cup O) = 25$

(i) $n(T \cup O) = n(T) + n(O) - n(T \cap O)$

$$25 = n(T) + 6 - 4$$

$$25 = n(T) + 2$$

$$n(T) = 25 - 2 = 23$$

(ii) $n(T\ only) = n(T \cap O') = n(T) - n(T \cap O)$

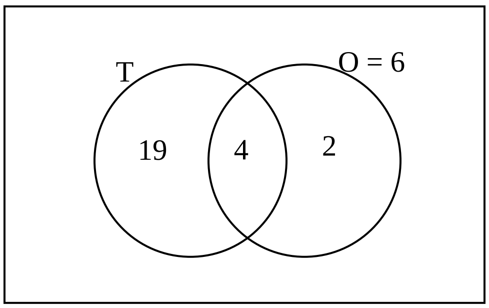
$$n(T\ only) = n(T \cap O') = 23 - 4$$

$n(T\ only) = n(T \cap O') = 19$

(iii) $n(T \cup O)' = n(U) - n(T \cup O)$

$n(T \cup O)' = 50 - 25 = 25$

With the use of Euler-Venn diagrams, we have



Then $n(O\ only) = n(T' \cap O) = 6 - 4 = 2$

(i) $n(T) = 19 + 4 = 23$

(ii) $n(T\ only) = n(T \cap O') = 25 - (4 + 2) = 19$

(iii) $n(T \cup O)' = n(U) - n(T \cup O)$

$n(T \cup O)' = 50 - (19 + 4 + 2)$

$n(T \cup O)' = 50 - 25 = 25$

Example 10.3

Each member of staff in a company with staff strength of 40 operates no account or at least one type of account in FAO microfinance Bank limited. If 15 members operate savings account only, 9 operates current account only and 5 operates neither savings nor current account in the bank.

(a) Draw Euler-Venn Diagram to illustrate the given information above.

(b) Calculate the number of staff that operates

  (i) the two types of account

  (ii) savings account

  (iii) current account

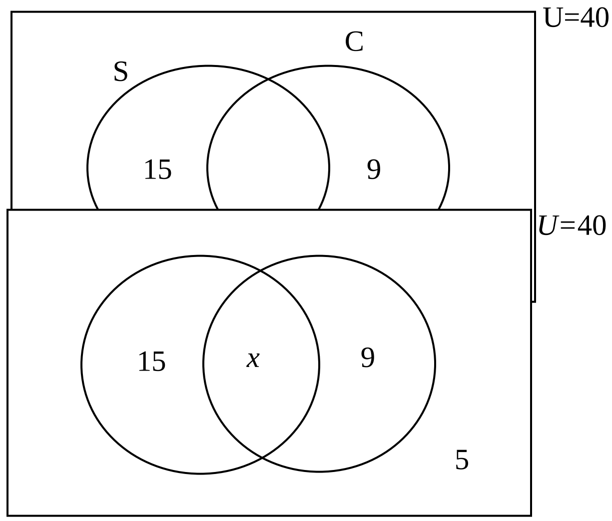
  (iv) at least one type of account.

Solution

Let *S* denote the set of members that operate savings account and *C* denotes the set of members that operate current account. From the piece of information given in the question:

$n(U)=40, \quad n(S\cap C')=15, \quad n(S'\cap C)=9 \quad and \quad n(S\cup C)' \, or \, n(S'\cap C')=5$

(a) Euler-Venn diagram



b(i)

Let $n(S\cap C)=x$

$n(U)=n(S\cup C)+n(S\cup C)'$

$n(U)=n(S\cap C')+n(S\cap C)+n(S'\cap C)+n(S\cup C)'$

$40=15+x+9+5$

$40=x+29$

$x=40-29=11$

$n(S)=n(S'\cap C)+n(S\cap C)$

Hence, 11 members operate both accounts

(ii) Let $n(S)=n(S\cap C')+n(S\cap C)$

$n(S)=15+11=26$

Hence, 26 members operate saving account

(iii) Let $n(C)=n(S\cap C)+n(S'\cap C)$

$n(C)=11+9=20$

Hence, 20 members operate current account

(iv) Let $n(\textit{at least one type of account})=n(\textit{only one account})+n(\textit{both accounts})$

$n(\textit{at least one type of account})=n(\textit{savings account only})+n(\textit{current account only})$
$+n(\textit{both accounts})$

$n(S\cup C)=n(S\cap C')+n(S'\cap C)+n(S\cap C)$

$n(S\cup C)=15+11+9=35$

Hence, 35 members operate at least one type of account

Example 10.4

A company has 120 employees whose records show that each of the members have bought no insurance policy or at least one of the three insurance policies namely health (H), vehicle (V) and life (L) from WAFAO insurance company plc. Suppose 45 employees bought health insurance policy, 49 vehicle insurance policy and 54 life policy. 15 employees bought health and vehicle policies, 18 both health and life policies, while 12 bought both vehicle and life policies. If 10 did not buy any of the 3 insurance policies.

(a) Draw the Euler-Venn Diagrams to illustrate the information given above

(b) Determine the number of employees who bought

 (i) All the 3 insurance policies

 (ii) Health policy only

 (iii) Health and life policies only.

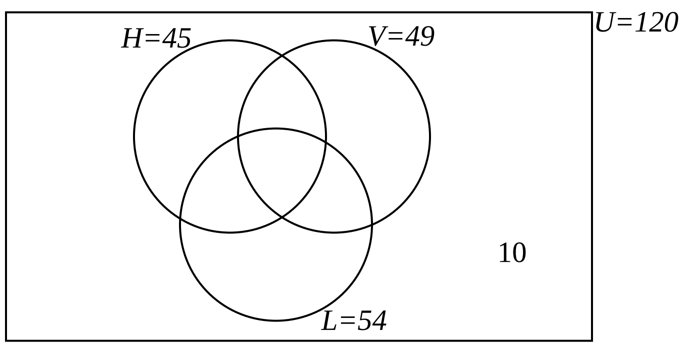
Solution

Let H denote the set of employees that bought health insurance policy, V denotes the set of employees that bought vehicle insurance policy and L denotes the set of employees that bought life insurance policy. From the piece of information given in the question:

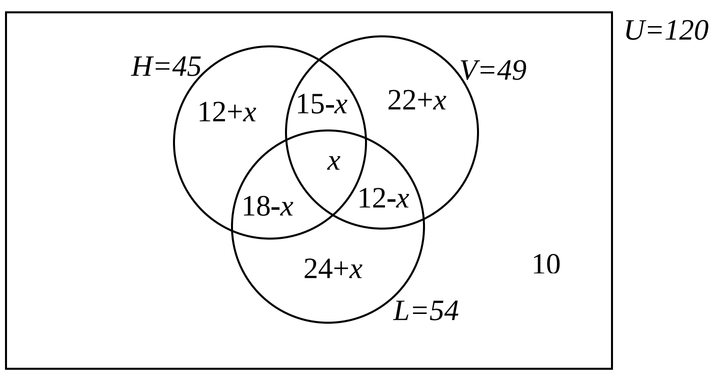
$n(U) = 120,\ n(H) = 45,\ n(V) = 49,\ n(L) = 54,\ n(H \cap V) = 15,\ n(H \cap L) = 18,$

$n(V \cap L) = 12 \text{ and } n(H \cup V \cup L)' \text{ or } n(H' \cap V' \cap L') = 10$

(a) Euler-Venn diagram



(bi) Let $n(H \cap V \cap L) = x$



$$n(H\ and V\ only) = n(H \cap V \cap L') = n(H \cap V) - n(H \cap V \cap L) = 15 - x$$
$$n(H\ and\ L only) = n(H \cap V' \cap L) = n(H \cap L) - n(H \cap V \cap L) = 18 - x$$
$$n(V\ and\ L only) = n(H' \cap V \cap L) = n(V \cap L) - n(H \cap V \cap L) = 12 - x$$
$$n(H\ only) = n(H \cap V' \cap L') = n(H) - \left[n(H \cap V \cap L') + n(H \cap V \cap L) + n(H \cap V' \cap L)\right]$$
$$= 45 - (15 - x + x + 18 - x) = 12 + x$$
$$n(V\ only) = n(H' \cap V \cap L') = n(V) - \left[n(H \cap V \cap L') + n(H \cap V \cap L) + n(H' \cap V \cap L)\right]$$
$$= 49 - (15 - x + x + 12 - x) = 22 + x$$
$$n(L\ only) = n(H' \cap V' \cap L) = n(L) - \left[n(H' \cap V \cap L) + n(H \cap V \cap L) + n(H \cap V' \cap L)\right]$$
$$= 54 - (12 - x + x + 18 - x) = 24 + x$$
$$n(U) = n(H \cup V \cup L) + n(H \cup V \cup L)'$$
$$n(U) = n(H \cap V \cap L) + n(H \cap V \cap L) + n(H \cap V \cap L) + n(H \cap V \cap L) + n(H \cap V \cap L) +$$
$$n(H \cap V \cap L) + n(H \cap V \cap L)$$
$$120 = 12 + x + 22 + x + 24 + x + 15 - x + 18 - x + 12 - x + x$$
$$x + 113 = 120$$
$$x = 120 - 113 = 7$$

Hence, 7 employees bought all the 3 insurance policies

(ii) $n(Health\ policy\ only) = n(H\ only) = n(H \cap V' \cap L') = 12 + x$

$n(Health\ policy\ only) = 12 + 7 = 19$

Hence, 19 employees bought health policy only

(iii) $n(Health\ and\ life\ policies\ only) = n(H\ and\ L\ only) = n(H \cap V' \cap L) = 18 - x$

$n(C) = 18 - 7 = 11$

Hence, 11 employees bought health and life policies only

10.5 Chapter summary

The chapter treated the principle and concept of set theory, types of sets and basic operations of sets. Solving business-oriented problems using set theory including drawing of appropriate Venn-Diagrams was discussed.

10.6 Multiple- choice and short-answer questions

1. Which of the following does not constitute a set?
   - A. an aggregate of books in a library
   - B. a collection of tools in a carpentry shop
   - C. a collection of historical artifacts in a museum
   - D. a collection of undefined items
   - E. a group of all vowels in the English alphabet
2. Which of the following sets is infinite?
   - A. D = {all the days in a week}
   - B. S = {all the ICAN students in a Tuition house}
   - C. T = {all the letters of alphabet}
   - D. V = {all even numbers}

E. P = $\{x : 1 \le x \le 7\}$

3. The following pieces of information were obtained from the records of candidates' enrollment in Principle of Accounting (*A*) and Quantitative Analysis (*Q*) for a particular examination diet: *n(A)* = 120, $n(A \cap Q) = 40$ and $n(A \cup Q) = 240$. Calculate *n(Q)*
   A. 80
   B. 160
   C. 200
   D. 280
   E. 400
4. A survey conducted on the types of bank accounts operated by staff of a particular company revealed that 35 operate savings and current accounts, 45 operate savings account, 20 operate current account only while 5 operate neither of the two accounts. Determine the number of staff in the company.
   A. 60
   B. 65
   C. 70
   D. 80
   E. 105
5. The selected amount (in thousands of Naira) of withdrawals from two different paying points of a particular branch of a bank are as follows: *A* = {3, 5, 10, 15, 17, 21} and *B* = {11, 15, 20, 21, 30}, then $n(A \cup B)$ is
   A. 7
   B. 8
   C. 9
   D. 10
   E. 11
6. Any given set is a subset of the universal set, the set of points which are not in the given set is known as………………………
7. In Euler-Venn diagram, the circle represents a……………while the rectangle represents a ………………….
8. Given $n(X) = 33, n(Y) = 40$ *and* $n(X \cap Y) = 40$ *then* $n(x \cup Y)$ *is* …………
9. The set that contains all the elements of set *E* or set *F* or both sets is called the …………….of set *E* and set *F*
10. A set with countable members is known as …………… while a set with uncountable members is known as ………………

ANSWER

1. D
2. D

3. B
4. C
5. C
6. Complement
7. Any set, universal set (in that order)
8. 49
9. Union
10. Finite set, Infinite set (in that order)

CHAPTER ELEVEN

FUNCTIONAL RELATIONSHIPS

Chapter contents

Objectives

At the end of this chapter, readers should be able to understand the

(a) concept of functions;

(b) different types of function;

(c) solution to different types of equations by algebraic and graphical methods;

(d) concept of linear inequalities and their solutions; and

(e) application of all the concepts above to business and economic problems.

11.1 Introduction

A function is a mathematical way of describing a relationship between two or more variables. In other words, it is a mathematical expression involving one or more variables. Functions are useful in business and commerce where fragments of information can be connected together by functional relationships.

For example, a function of x can be written as $f(x) = 4x + 3$; $f(x)$ is read as "f of x".

Example 11.1

If (a) $f(x) = 4x + 3$, Find (i) $f(0)$, (ii) $f(1)$, (iii) $f(14)$.

(b) $f(x) = 2x^2 + 5x + 7$, Find (i) $f(0)$, (ii) $f(2)$, (iii) $f(15)$.

Solutions

(a) $f(x) = 4x + 3$

(i) $f(0) = 4 \times 0 + 3 = 3$

(ii) $f(1) = 4 \times 1 + 3 = 7$

(iii) $f(14) = 4 \times 14 + 3 = 59$

(b) $f(x) = 2x^2 + 5x + 7$

(i) $f(0) = 2 \times 0^2 + 5 \times 0 + 7 = 7$

(ii) $f(2) = 2 \times 2^2 + 5 \times 2 + 7 = 25$

(iii) $f(15) = 2 \times 15^2 + 5 \times 15 + 7 = 532$

Some of the time, y is used to represent $f(x)$

i.e. $f(x) = 7x^2 + 2$ can be written as
$y = 7x + 2$; in such a case, y is said to be a function of x.

Explicit and Implicit Functions

- **An explicit function** is a function where one variable is directly expressed in terms of the other variable(s).

  For example, $y = 5x + 9$; and $y = 3x^2 + 8$ are explicit functions.

- **An implicit function** is a function where the relationship between the variables is expressed as an equation involving all the variables.

  For example, $2x^2 + 3xy + 3y^2 + 10 = 0$ is an implicit function.

11.2 TYPES OF FUNCTIONS

Linear Functions

A linear function is one in which the variables are of first degree and is generally of the form

$f(x) = a + bx$ or $y = a + bx$, where a and b are constants and the power of variable x is always 1. *e.g.* $y = 18 - 2x$; $y = 6x + 9$

Graph of a Linear Function

The graph of a linear function ($y = a + bx$) is a straight line; a is the intercept on the y-axis (*i.e.* the value of y when $x = 0$) and b is the gradient or slope of the line.

The gradient of a line shows the increase in y for a unit increase in x. It may be positive or negative and is unique to that line (*i.e.* a line has only one gradient).

e.g. for the line $y = 5x + 3$, the intercept on the y-axis is 3 and the gradient is 5.

Example 11.2

Draw the graph of each of the following functions

(a) $y = 3x + 12$

(b) $y = 46 – 5x$

Solution

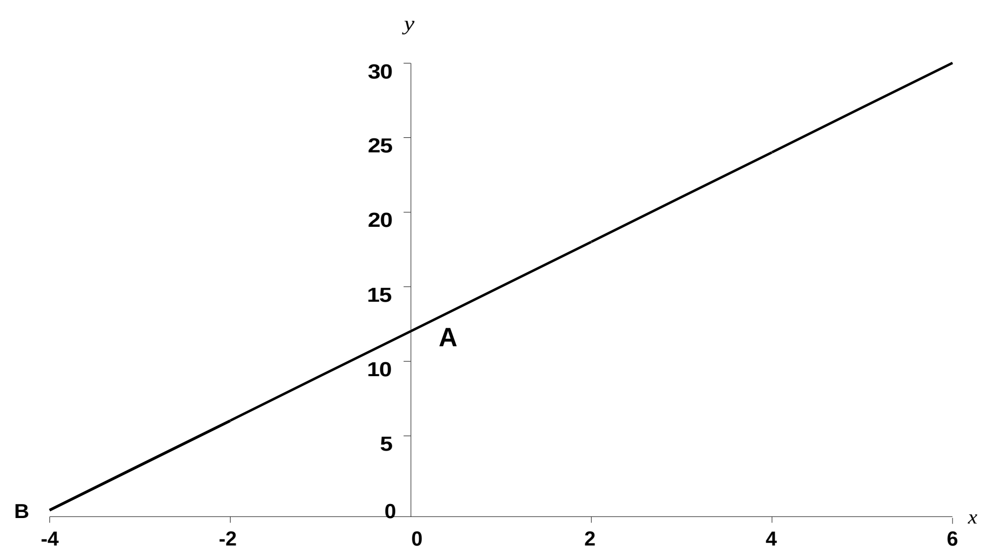
Any two points are enough to draw the graph of a straight line and these are usually the intercept on the y-axis, where $x = 0$; and the intercept on the x-axis, where $y = 0$.

Once the two points are obtained, the line can be drawn and then extended as desirable.

(a) $y = 3x + 12$

When $x = 0$, $y = 12$, *i.e.* (0, 12) is a point on the line; i.e point A

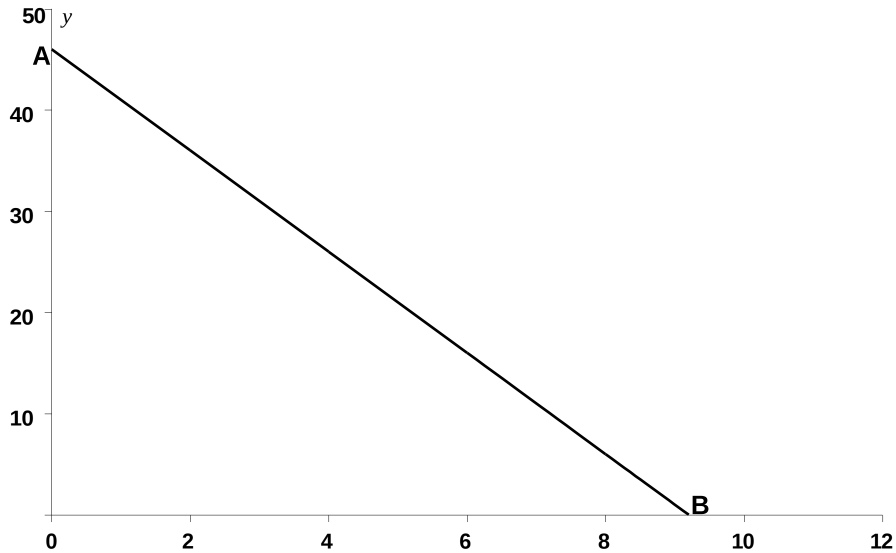
When $y = 0$, $x = –4$, *i.e.* (–4, 0) is a point on the line; i.e point B



b. $y = 46 - 5x$

When $x = 0$, $y = 46$, *i.e.* (0, 46) is a point on the line i.e. point A;

When $y = 0$, $x = 9.2$, *i.e.* (9.2, 0) is a point on the line i.e. point B.



Quadratic Functions

A quadratic function is one in which the variable x is of second degree and is generally of the form $y = ax^2 + bx + c$, where a, b and c are constants and ***a* must not be zero.**

Examples of quadratic functions are: $y = x^2 + 5x + 6$;

$y = 16x - 3x^2 + 1$; and

$y = 9x^2$

Graph of a Quadratic Function

The graph of a quadratic function $y = ax^2 + bx+c$ is either cup-shaped (U) when a is positive or cap-shaped (∩) when a is negative. Usually, the range of values of x is known.

Example 11.3

Draw the graph of the following for $-4 \le x \le 3$

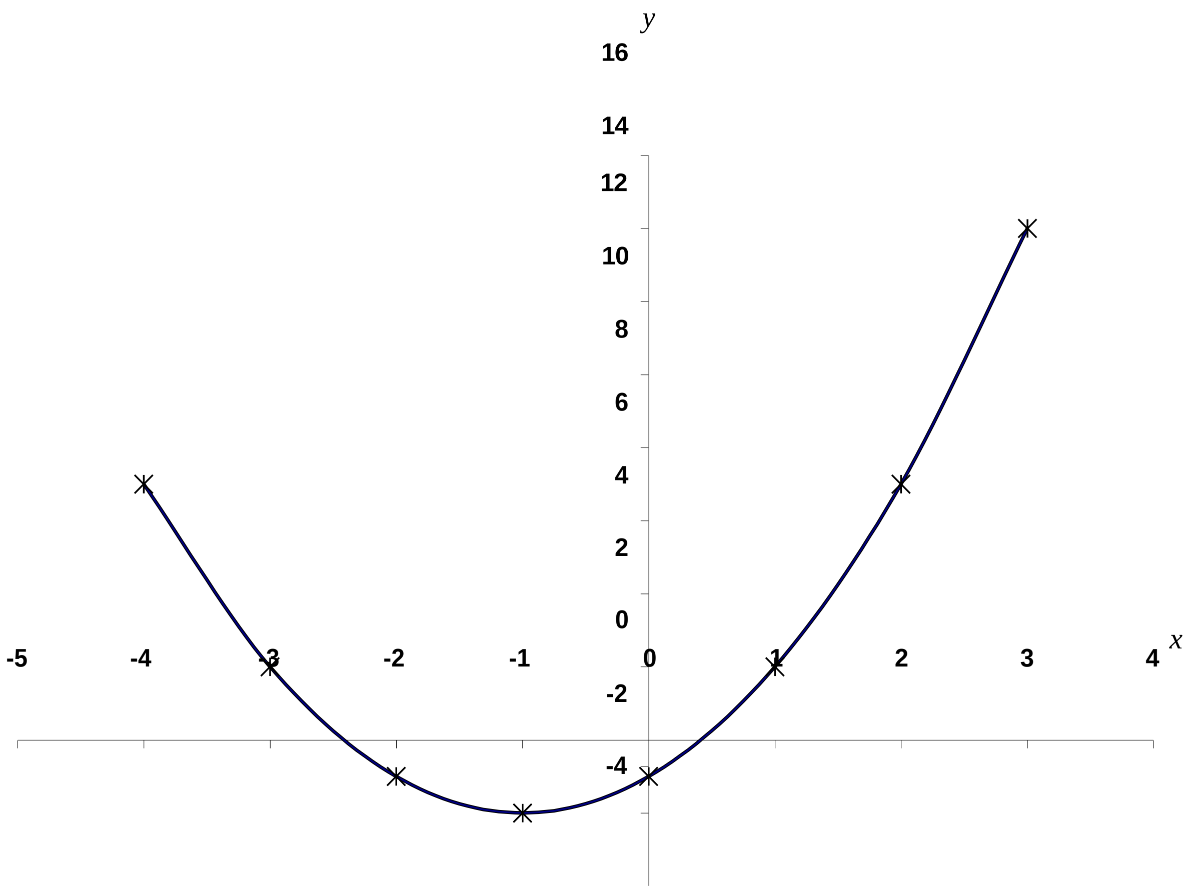
(a) $y = x^2 + 2x - 1$

(b) $y = 5 - 2x - 3x^2$

Substitute values of x within the given range in the quadratic function to obtain the corresponding values of y. This gives a table of values, which are plotted on the graph.

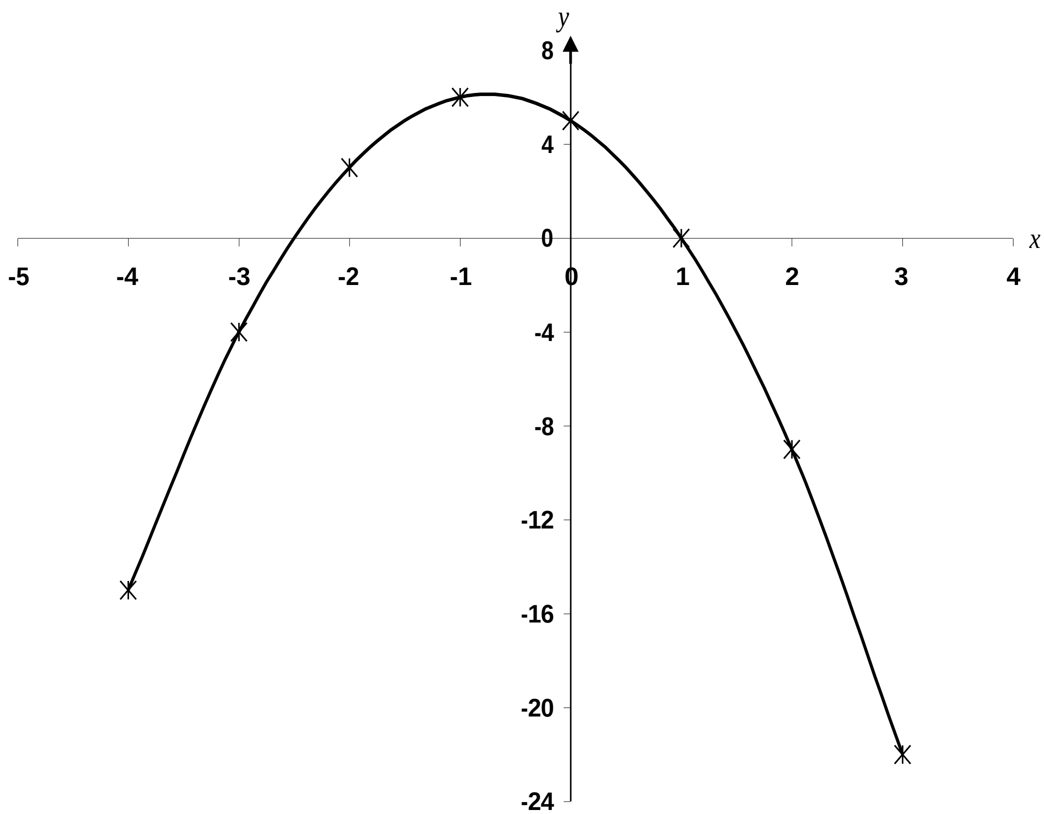
(a) $y = x^2 + 2x - 1$;

x	-4	-3	2	-1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
$2x$	-8	-6	-4	-2	0	2	4	6
-1	-1	-1	-1	-1	-1	-1	-1	-1
Y	7	2	-1	-2	-1	2	7	14



b) $y = 5 - 2x^2 - 3x$;

x	-4	-3	-2	-1	0	1	2	3
5	5	5	5	5	5	5	5	5
$-2x^2$	-32	-18	-8	-2	0	-2	-8	-18
$-3x$	12	9	6	3	0	-3	-6	-9
y	15	-4	3	6	5	0	-9	-22



Exponential Functions

An exponential function is a function, which has a constant base and a variable exponent. For example, if $y = a^x$, then y is said to be an exponential function of x; 'a' is the base and x is the exponent. It is non-linear function.

If 'a' exceeds one, there is an exponential growth but if 'a' is less than one there is an exponential decay.

Most exponential functions used in economic theory have the base 'e' (*i.e.* $y = e^x$). Values for exponential functions with base 'e' are easily obtainable from statistical tables or by the use of a calculator.

Graphs of Exponential Functions

The shape of the graph of an exponential function depends largely on the value of the base.

Example 11.4

Plot the graph of the following for $-2 \leq x \leq 2$

(a) $y = 2^x$

(b) $y = \left(\frac{1}{3}\right)^x$

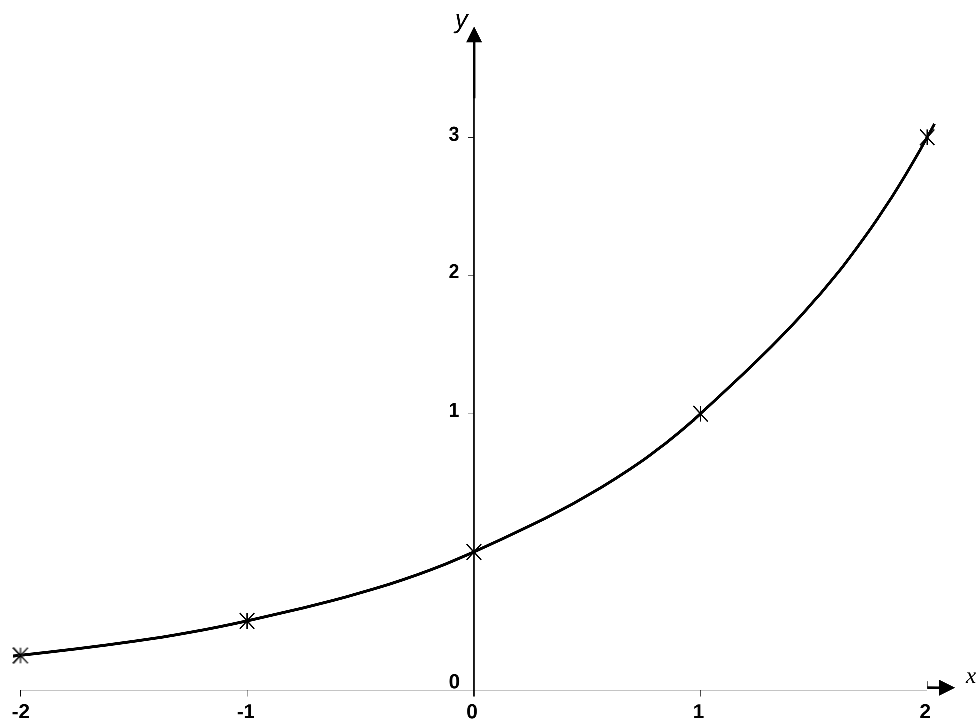
(c) $y = 3e^{\frac{x}{2}}$

Solutions

As usual, we substitute values of x within the range to obtain table of values

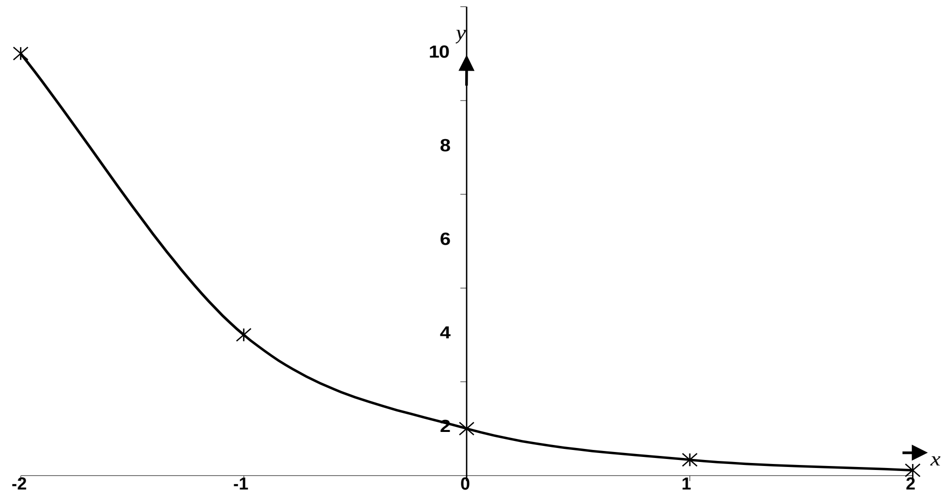
(a) $y = 2^x$

x	-2	-1	0	1	2
y	0.25	0.5	1	2	4



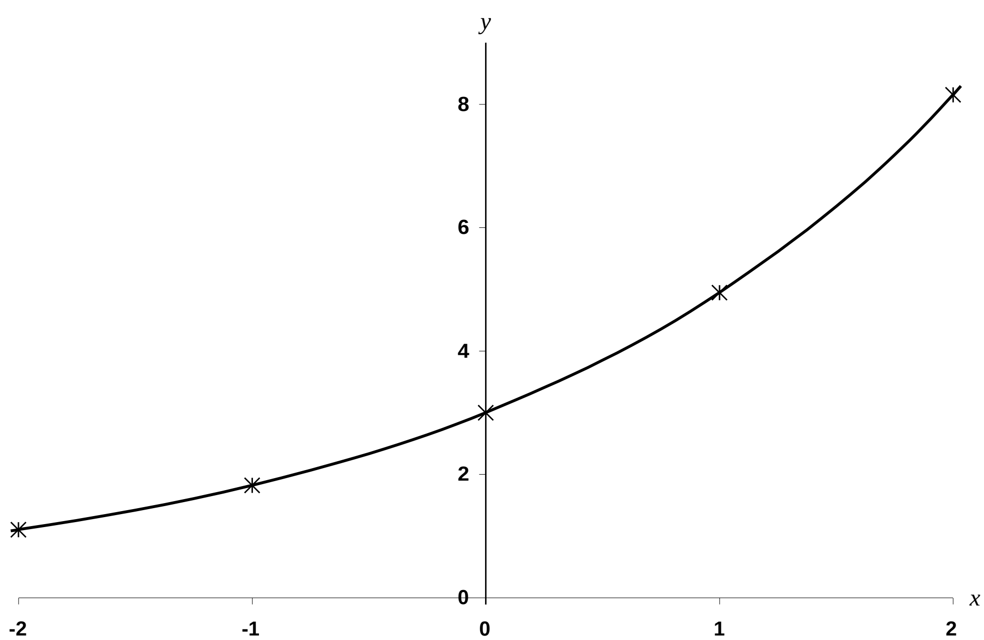
(b) $y = \left(\frac{1}{3}\right)^x$

x	-2	-1	0	1	2
y	9	3	1	0.33	0.11



(c) $y = 3e^{\frac{x}{2}}$

x	-2	-1	0	1	2
y	1.1	1.82	3	4.95	8.15



Logarithmic function

Logarithmic function is another non-linear function and is of the form

$$y = ax^b$$

If the logarithms of both sides are taken, we have

$$log y = log a + b log x$$

y is said to be a logarithmic function of x

where x is the time periods

The graph of logarithmic function cannot be easily drawn except if the values of a, b and x are known.

Example 11.5

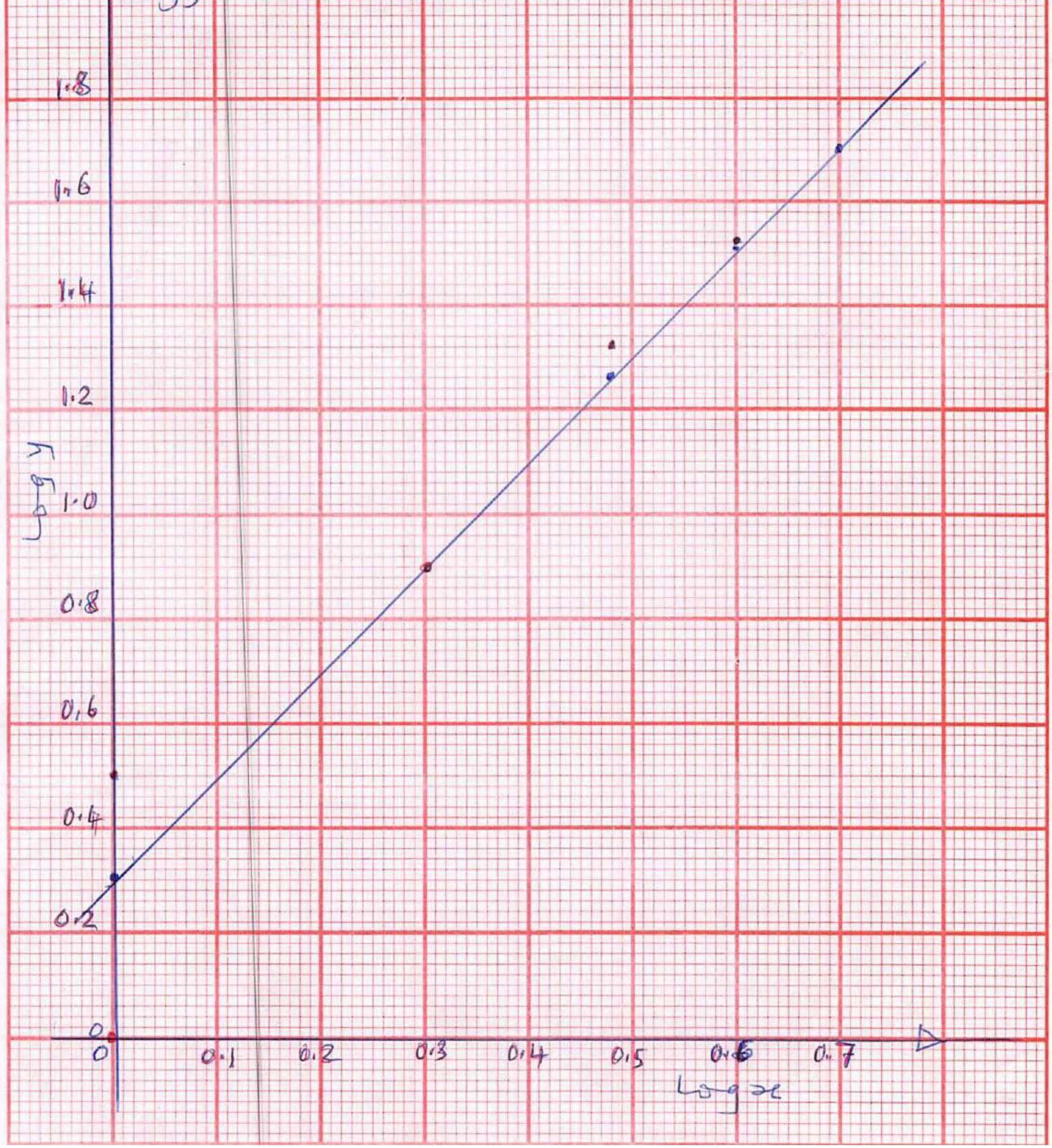
If $y = 2x^2$, plot the relevant logarithmic graph within the range $1 \leq x \geq 5$

Solution

X	1	2	3	4	5
Y	2	8	18	32	50
log x	0	0.30	0.48	0.60	0.70
log y	0.30	0.90	1.26	1.51	1.70

$y = 2x^2$

x	1	2	3	4	5
y	2	8	18	32	50
$\log x$	0	0.30	0.48	0.60	0.70
$\log y$	0.30	0.90	1.26	1.51	1.70



NOTE:

y is actually a quadratic function but its logarithmic graph is a straight line as can be seen on the graph.

11.3 EQUATIONS

An equation is a mathematical expression of two equal quantities. It consists of variables which are referred to as unknowns and numbers which are called constants.

For example,

(a) $x + 5 = 2$

(b) $2x + 15 = x + 8$

(c) $x^2 + 6x + 9 = 0$

Rules for handling equations:

(i) An equation is unchanged if

- the same number or expression is added to (or subtracted from) each side of the equation; and
- each side of an equation is multiplied (or divided) by the same number or expression.

(ii) The sign of an expression or a number changes when it crosses the equality sign.

Linear Equations

A linear equation in one variable is the simplest type of an equation and it contains one unknown, which is of the first degree. Examples (a) and (b) above are linear equations

Solutions of Linear Equations

A linear equation can be solved by applying the rules listed above

Example 11.6

Solve the following equations

(a) $2x + 3 = 5$

(b) $8a + 5 = 3a + 30$

(c) $\frac{1}{3}(y+7) = 2(y-1)$

(d) $\frac{2x}{5} - \frac{3}{14} = \frac{x}{14} + \frac{2x}{7}$

Solutions

(a) $2x + 3 = 5$

Subtract 3 from both sides to give

$2x = 2$

Divide both sides by 2 to give

$x = 1$

(b) $8a + 5 = 3a + 30$

$8a - 3a = 30 - 5$

$5a = 25$

$a = 5$

(c) $\frac{1}{3}(y + 7) = 2(y - 1)$

Multiply each side by 3 to give

$y + 7 = 6(y - 1)$

$y + 7 = 6y - 6$

$y - 6y = -6 - 7$

$-5y = -13$

$y = 2.6$

(d) $\frac{2x}{5} - \frac{3}{14} = \frac{x}{14} + \frac{2x}{7}$

Multiply each side by 70 (because the LCM of all the denominators is 70) to give

$28x - 15 = 7x + 20x$

$28x - 27x = 15$

$x = 15$

Linear Equations in Two Variables or Unknowns (Linear Simultaneous Equations)

If the solutions to the equations containing two unknowns can be found at the same time, the equations are referred to as simultaneous equations.

In order for any set of simultaneous equations to have solutions, the number of equations and unknowns must be the same.

Any letter could be used to represent the unknowns but the letters x and y are commonly used.

Example

$$2x + 3y = 13$$

$$3x + 2y = 32$$

Solutions of Simultaneous Equations

The following three methods of solving simultaneous equations are discussed below;

- Substitution method;
- Elimination method; and
- Graphical method.

The fourth method, which is the matrix method, is not within the scope of this Study Text.

Example 11.7

Solve the following simultaneous equations

(a) $2x + 3y = 13$

$3x + 2y = 32$

by substitution method

(b) $5x - 4y = -6$

$6x + 2y = 20$

by (i) elimination method

(ii) graphical method

Solutions

(a) $2x + 3y = 13$ (i)

$3x + 2y = 32$ (ii)

Express x (or y) in terms of y (or x), from equation (i)

$2x + 3y = 13$

$x = \frac{1}{2}(13 - 3y)$ (iii)

Substitute the value of x from equation (iii) in equation (ii) to get

$\frac{3}{2}(13 - 3y) + 2y = 32$

$39 - 9y + 4y = 64$

$-5y = 25$

$y = -5$ (iv)

Use the result of (iv) in (iii) to obtain x, *i.e.*

$x = \frac{1}{2}(13 - 3 \times -5)$

$x = \frac{1}{2}(13 + 15)$

$x = 14$

$\therefore$ The solutions are $x = 14$ and $y = -5$

Equation (ii) could also be used to express one variable in terms of the other. The common sense approach is to use the simpler of the two equations

(b) $5x - 4y = -6$ (i)

$6x + 2y = 20$ (ii)

(i) By the elimination method

equation (ii) × 2 gives $12x + 4y = 40$ (iii)

$5x - 4y = -6$ (i)

equations (iii) + (i) gives $17x = 34$

$x = 2$ (iv)

Put the value of x from equation (iv) in equation (iii) to get

$$6(2) + 2y = 20$$
$$2y = 8$$
$$y = 4$$

$\therefore$ The solutions are $x = 2$ and $y = 4$

or putting the value of x from equation (iv) in equation (i),

$$5x - 4y = -6$$
$$5(2) - 4y = -6$$
$$-4y = -16$$
$$y = 4 \text{ as before.}$$

(ii) Draw the graph of each of the two equations above on the same axes.

The point of intersection of the two lines is the solution

$5x - 4y = -6$ (i)

$6x + 2y = 20$ (ii)

From equation (i) $5x - 4y = -6$

$$-4y = -5x - 6$$
$$y = \frac{5}{4}x + \frac{6}{4}$$
$$y = 1.25x + 1.5$$

When $x = 0$, $y = 1.5$ => (0, 1.5)

When $y = 0$, $x = -1.2$ => (-1.2, 0)

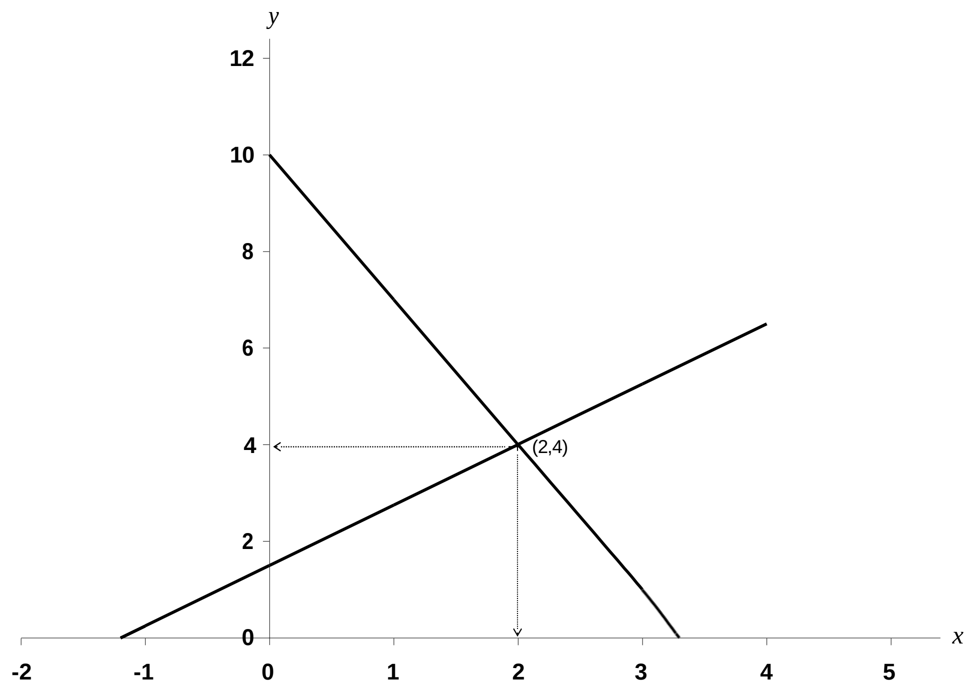
From equation (ii) $6x + 2y = 20$

$$2y = -6x + 20$$

$$y = -3x + 10$$

When $x = 0$, $y = 10$ => (0, 10)

When $y = 0$, $x = \dfrac{10}{3}$ => (3.3, 0)



From the graph, the solution is at the point (2, 4) *i.e.* $x = 2$, $y = 4$ as before

Note: some of the times, graphical solutions may not be exactly the same as the algebraic solutions due to some approximations that might have crept in - they should not be too different though. In fact, graphical solutions are regarded as estimates.

Quadratic Equations

Equations in one variable and of the second degree are called quadratic equations. It is generally of the form:

$$ax^2 + bx + c = 0, \text{ where } a\ (\neq 0),\ b \text{ and } c \text{ are constants}$$

e.g. $x^2 + x + 8 = 0$

$3x^2 + 13x = 0$

$4x^2 + 11 = 0$

Solutions of Quadratic Equations

The following three methods of solving quadratic equations are discussed below:

- Factorisation method;
- Formula method; and
- Graphical method.

Example *11.8*

Solve the following quadratic equations

(a) $x^2 + 6x + 8 = 0$ by factorisation method
(b) $2x^2+13x-16 = 0$ by formula method
(c) $x^2 +3x - 7 = 0$ by graphical method

Solutions

(a) $x^2 + 6x + 8 = 0$

Find two numbers whose sum is 6 and product is 8 *i.e.* 2 and 4 So we hav e

$x^2 + 2x + 4x + 8 = 0$

$x(x + 2) + 4(x + 2) = 0$

i.e. $(x + 2)(x + 4) = 0$

$x + 2 = 0$ or $x + 4 = 0$

$x = -2$ or -4

This method is not applicable to equations that cannot be factorised.

(b) $2x^2 + 13x - 16 = 0$

The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, $a = 2$, $b = 13$, $c = -16$

Substitute these in the formula to obtain

$$x = \frac{-13 \pm \sqrt{13^2 - 4(2)(-16)}}{2(2)}$$

$$x = \frac{-13 \pm \sqrt{169+128}}{4}$$

$$x = \frac{-13 \pm 17.23}{4}$$

i.e. $x = \frac{-13+17.23}{4}$ or $x = \frac{-13-17.23}{4}$

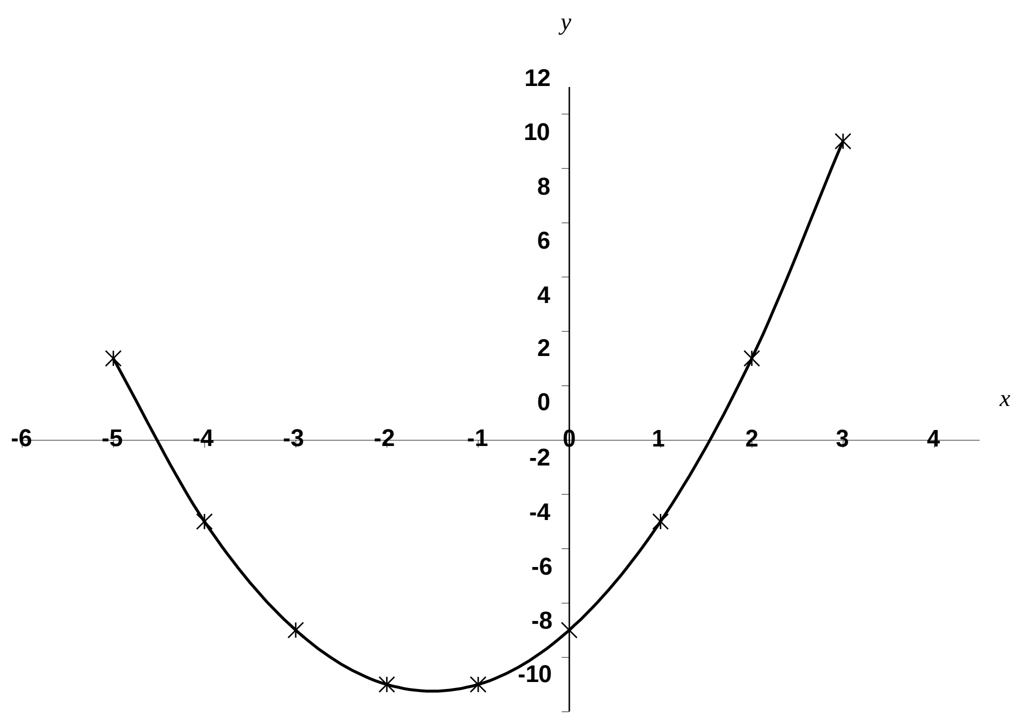
$x = -7.56$ or 1.0

This method can be applied whether the equation can be factorised or not.

(c) $x^2 + 3x - 7$

Obtain the table of values as

X	-5	-4	-3	-2	-1	0	1	2	3
x^2	25	16	9	4	1	0	1	4	9
$3x$	-15	-12	-9	-6	-3	0	3	6	9
-7	-7	-7	-7	-7	-7	-7	-7	-7	-7
Y	3	-3	-7	-9	-9	-7	-3	3	-11



The solutions are the points where the curve intersects the x-axis.

i.e. $x = -4.5$ or 1.5

This method can be applied to any type of quadratic equation.

11.4 Inequalities and Their Graphical Solutions

When two quantities or expressions are not equal, then one has to be greater or less than the other. They are known as inequalities.

The symbols used to express inequalities are

$<$ means "less than";

$\leq$ means "less than or equal to";

$>$ means "greater than";and

$\geq$ means "equal to or greater than".

For example

(i) $19 < 30$

(ii) $2x + 5 > x - 4$

(iii) $3x + 2y \leq 6$

Rules for Handling Inequalities

(a) The symbol of an inequality does not change if

- a number or an expression is added to (or subtracted from) both sides of the inequality; and
- each side of the inequality is multiplied (or divided) by a *positive* number.

(b) The symbol of an inequality changes if both sides of the inequality are multiplied (or divided) by a *negative* number.

(c) The sign of a number or an expression changes when it crosses the inequality symbol.

Generally, solutions to inequalities consist of range of values rather than point values, as is the case with equations.

Example 11.9

Indicate the region where each of the following inequalities is satisfied:

a. $2x + 5 < x + 8$

b. $4x + 7 \geq 2x + 15$

c. $3x + 10 \leq 5x - 2$

d. $x \geq 0$

e. $y \geq 0$

f. $x \geq 0$; $y \geq 0$

g. $x \geq 0$; $y \leq 3$

h. $x + 2y \leq 6$

i. $3x + 2y \geq 12$

j. $2x + 3y \leq 9$; $5x + 2y \leq 10$

k. $2x + y \leq 18$; $1.5x + 2y \geq 15$; $x \geq 0$; $y \geq 0$

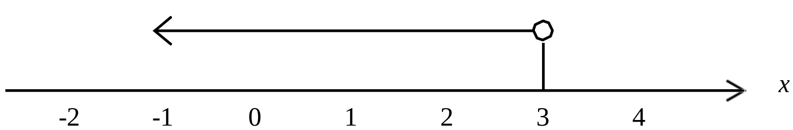
Solutions

(a) $2x + 5 < x + 8$

$2x - x < 8 - 5$

$x < 3$

This is represented on the number line as:



○ indicates that 3 is not part of the solution

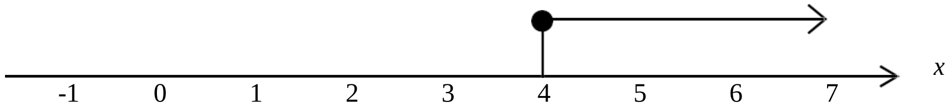
(b) $4x + 7 \geq 2x + 15$

$4x - 2x \geq 15 - 7$

$2x \geq 8$

$x \geq 4$

This is represented on the number line as:



● indicates that 4 is part of the solution

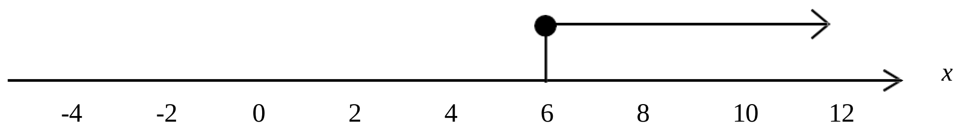
(c) $3x + 10 \leq 5x - 2$

$3x - 5x \leq -2 - 10$

$-2x \leq -12$

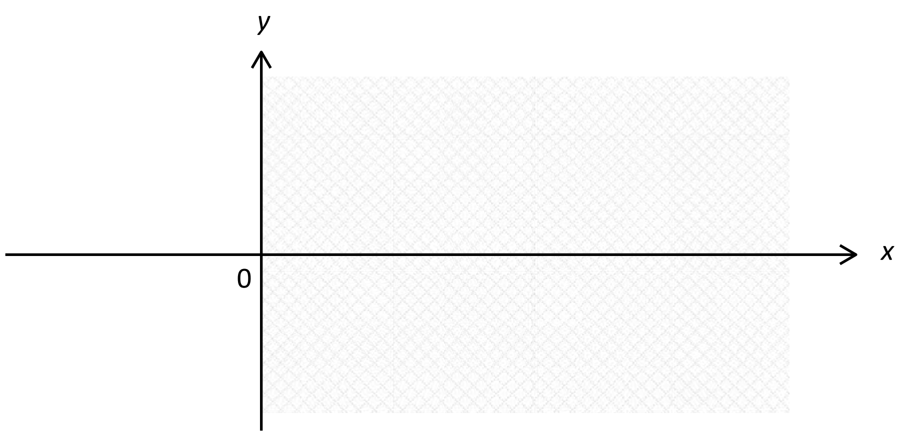
$x \geq 6$ (the sign of the inequality changes since we have divided by -2)

This is represented on the number line as:

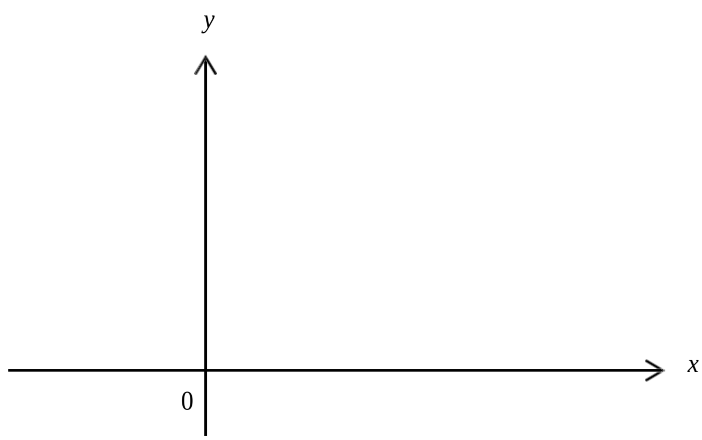


● indicates that 6 is part of the solution

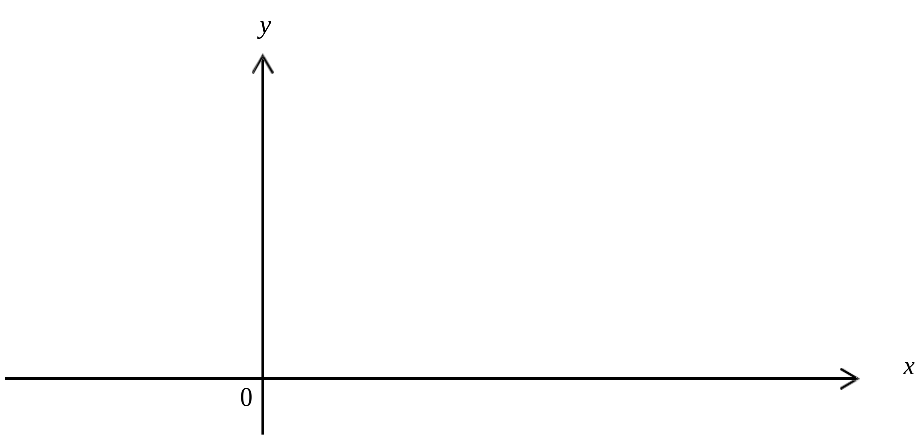
(d) $x \geq 0$



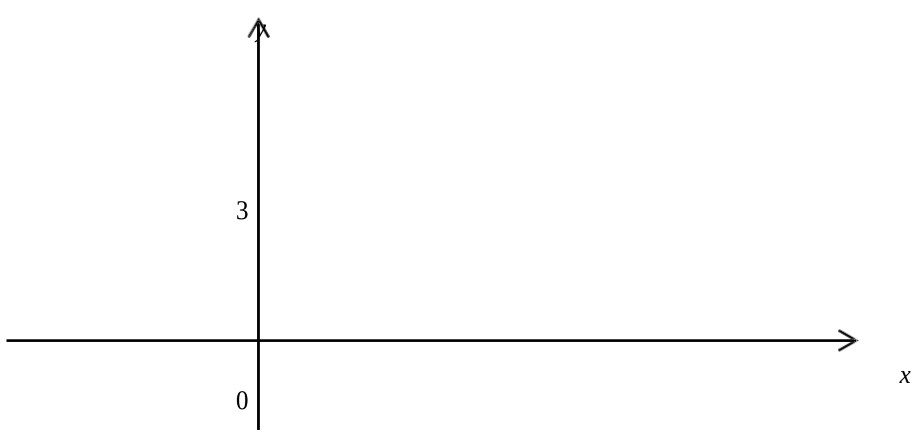
(e) $y \geq 0$



(f) $x \geq 0$ and $y \geq 0$



(g) $x \geq 0$ and $y \leq 3$



(h) $x + 2y \leq 6$

Draw the graph of $x+2y = 6$

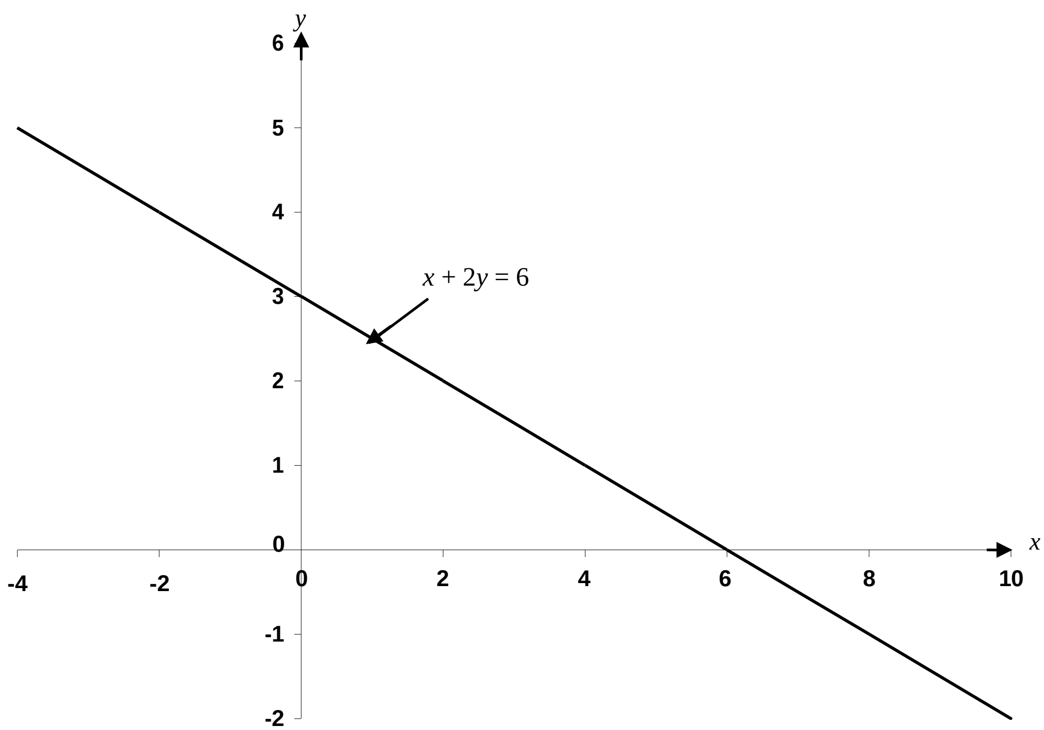
The line divides the whole region into two parts. The origin (0, 0) is in one of the parts and is used to determine the part that satisfies the inequality.

$x + 2y = 6$

When $x = 0$, y = 3; => (0, 3)

When y = 0, $x = 6$; => (6, 0)

Now substitute the origin (0, 0) in $x + 2y < 6 \Rightarrow 0 < 6$, which is true. Hence the part which satisfies the inequality contains the origin



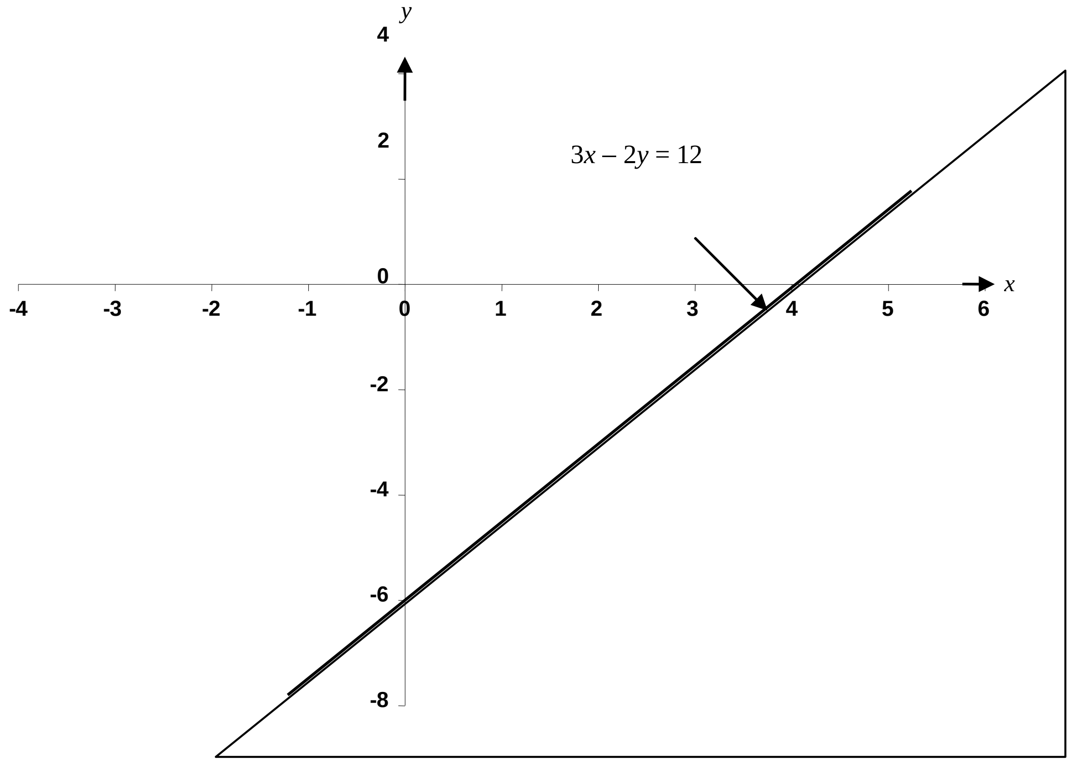
(i) $3x–2y \geq 12$

Draw the line for $3x–2y = 12$

When $x = 0$, $y = –6$ => (0, –6)

When $y = 0$, $x = 4$ => (4, 0)

Substituting the origin (0, 0) in $3x-2y \geq 12 \Rightarrow 0 \geq 12$, which is not true, hence the part in which the inequality is satisfied does not contain the origin.



(j) $2x + 3y \leq 9$; $5x + 2y \leq 10$

Draw the lines for $2x + 3y = 9$ and $5x + 2y = 10$ on the same axes

For $2x + 3y = 9$,

When $x = 0$, $y = 3$ => (0, 3)

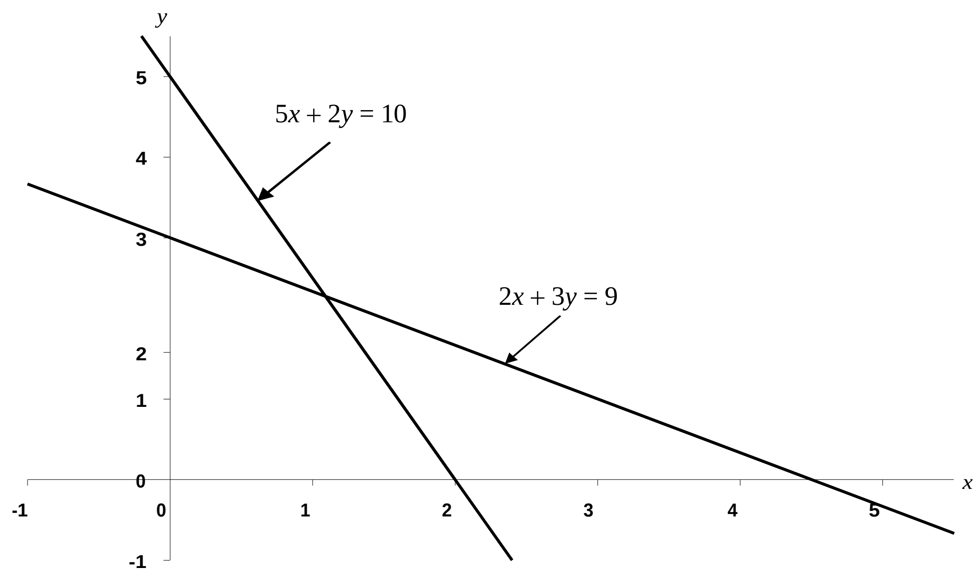
When $y = 0$, $x = 4.5$ => (4.5, 0)

For $5x + 2y = 10$,

When $x = 0$, $y = 5$ => (0, 5)

When $y = 0$, $x = 2$ => (2, 0)

The required region is where both inequalities are satisfied simultaneously.



(k) $2x + y \leq 18$; $1.5x + 2y \geq 15$; $x \geq 0$; $y \geq 0$

For $2x + y = 18$,

When $x = 0$, $y = 18$ => (0, 18)

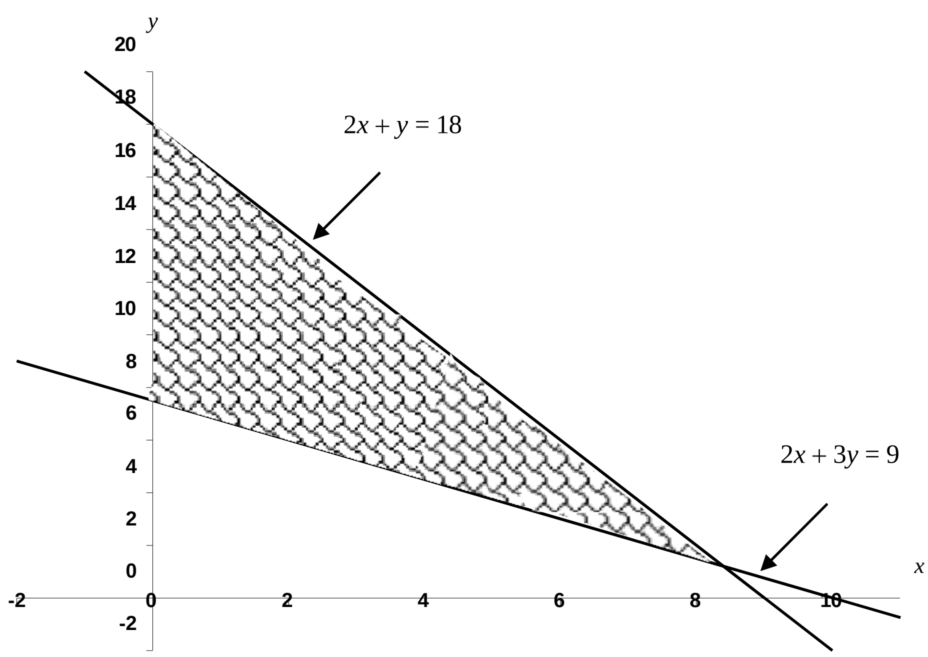
When $y = 0$, $x = 9$ => (9, 0)

For $1.5x + 2y = 15$,

When $x = 0$, $y = 7.5$ => (0, 7.5)

When $y = 0$, $x = 10$ => (10, 0)

The required region is where all the four inequalities are satisfied simultaneously, i.e the shaded region.



11.5 APPLICATIONS TO BUSINESS, ECONOMIC AND MANAGEMENT PROBLEMS

The topics discussed in this chapter can be applied to various practical problems.

Cost, Revenue and Profit Functions

The **total cost** of a commodity, project, business etc. consists of two parts, viz:

(a) the **fixed cost** (cost of machinery, infrastructure etc.), it can also be referred to as the **capital cost; and**

(b) the **variable cost** (cost of material, labour, utility etc.).

The fixed cost is always constant. It represents the "set up cost" while the variable cost depends on the number of units of items produced or purchased.

i.e. total cost = fixed cost + variable cost.

For example

In a normal situation, the transport cost to a market is constant, and is independent of whatever commodities are purchased in the market. This is the fixed cost.

The variable cost is the cost of commodities purchased and it depends on the price and number of units of the commodities. If x represents the number of units of commodity purchased or items produced, $\boldsymbol{C}(x)$ is called the **cost function, it** can be expressed as follows:

(a) $\boldsymbol{C}(x) = a + bx$

This is a linear function, a represents the fixed cost, bx the variable cost and b is the price per unit of x.

(b) $\boldsymbol{C}(x) = ax^2 + bx + c$

This is a quadratic function, a $(\neq 0)$, b and c are constants; and c, which is independent of $\boldsymbol{x}$**,** is the fixed cost.

Example 11.10

The cost of renting a shop is ₦120,000 per annum and an additional ₦25,000 is spent to renovate the shop. Determine the total cost, if the shop is to be equipped with 480 items at ₦75 per item.

Solution

The fixed cost = ₦120 000 + ₦ 25 000

= ₦145 000

$\therefore$ **C**(x) = ₦145000 + ₦75x

If x = 480

C(x) = ₦145 000 + ₦75 × 480

= ₦181, 000

The **revenue function,** ***R***(x) represents the income generated from the sales of x units of item.

R(x) totally depends on x and is always of the form:

***R*(*x*) = *px* where *p* is the sales price of an item.**

There is nothing like fixed revenue.

Example 11.11

If 2,500 items are produced and sold at ₦120 per item, calculate the revenue that will accrue from the sales

Solution

R(x) = px

= 120x

= ₦120 × 2,500

= ₦300, 000

The **profit function** of any project or business is the difference between the revenue function and the cost function.

i.e. ***P***(x) = ***R***(x) – ***C***(x)

If ***R***(x) > ***C***(x), then ***P***(x) > 0, *i.e.* positive (gain)

If ***R***(x) < ***C***(x), then ***P***(x) < 0, *i.e.* negative (loss)

If ***R***(x) = ***C***(x), then ***P***(x) = 0, *i.e.* no profit, no loss. This is the break-even case.

Example 11.12

A business man spends ₦1.5m to set up a workshop from where some items are produced. It costs ₦450 to produce an item and the sale price of an item is ₦1,450. Find the minimum quantity of items to be produced and sold for the business man to make a profit of at least ₦800,000.

Solution

It is always assumed that all the items produced are sold. Let x represent the number of items produced and sold.

Then

$$\boldsymbol{C}(x) = ₦1{,}500{,}000 + ₦450x$$

$$\boldsymbol{R}(x) = ₦1{,}450x$$

$$\Rightarrow \boldsymbol{P}(x) = ₦1{,}450x - ₦(1{,}500{,}000 + 450x)$$

$$= ₦1{,}000x - ₦1{,}500{,}000$$

To make a profit of at least ₦80, 000

$$\boldsymbol{P}(x) \geq 800{,}000$$

i.e. $1{,}000x - 1{,}500{,}000 \geq 800{,}000$

$$1{,}000x \geq 2{,}300{,}000$$

$$x \geq 2{,}300$$

i.e. at least 2,300 items must be produced and sold to make a profit of at least ₦800,000.

Break-even Analysis

A business or project breaks even when the revenue function is equal to the cost function.

i.e. $\boldsymbol{R}(x) = \boldsymbol{C}(x)$

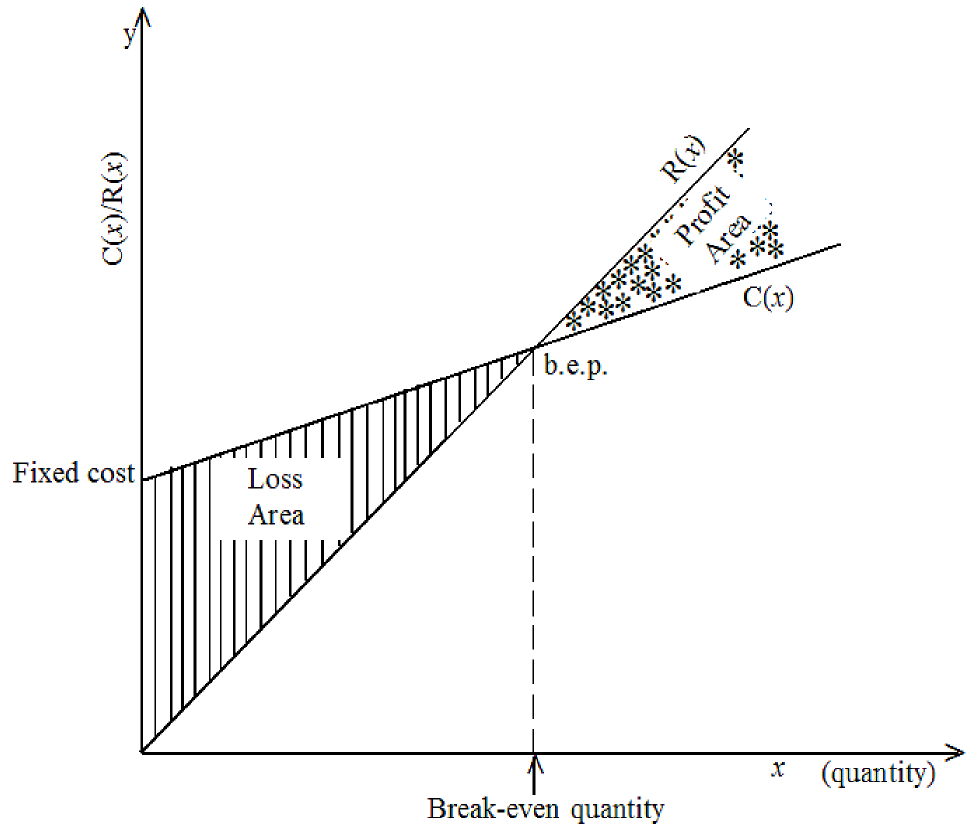
or $\boldsymbol{R}(x) - \boldsymbol{C}(x) = 0$

but $\boldsymbol{R}(x) - \boldsymbol{C}(x) = \boldsymbol{P}(\mathrm{x})$

$\boldsymbol{P}(x) = 0$

Hence, the break-even point (b.e.p) is at the point where $\boldsymbol{P}(x) = 0$ *i.e.* when profit is zero

The general outlook of a break-even graph is as shown below.



Example 11.13

Calculate the break-even quantity for example 9.11 above

Solution

$\boldsymbol{R}(x) = 1{,}450x$

$\boldsymbol{C}(x) = 1{,}500.000 + 450x$

For the break-even situation

$\boldsymbol{R}(x) = \boldsymbol{C}(x)$

i.e. $1{,}450x = 1{,}500{,}000 + 450x$

$1{,}000x = 1{,}500{,}000$

$x = 1{,}500$

i.e. the break-even quantity is 1,500

There are three possibilities

- $x < 1{,}500$ => negative profit *i.e.* loss
- $x = 1{,}500$ => break-even *i.e.* no profit, no loss
- $x > 1{,}500$ => profit

Example 11.14

The cost and revenue functions of a company are given by

$\boldsymbol{C}(x) = 400 + 4x$

$\boldsymbol{R}(x) = 24x$

Use the graphical method to determine the break-even quantity. Identify the profit and loss areas on your graph.

Solution

The cost and revenue functions are drawn on the same graph. The quantity at the point of intersection of the two lines is the break-even quantity.

For $\boldsymbol{C}(x) = 400 + 4x$,

when $\boldsymbol{C}(x) = 0$, $x = -100$ => $(-100, 0)$

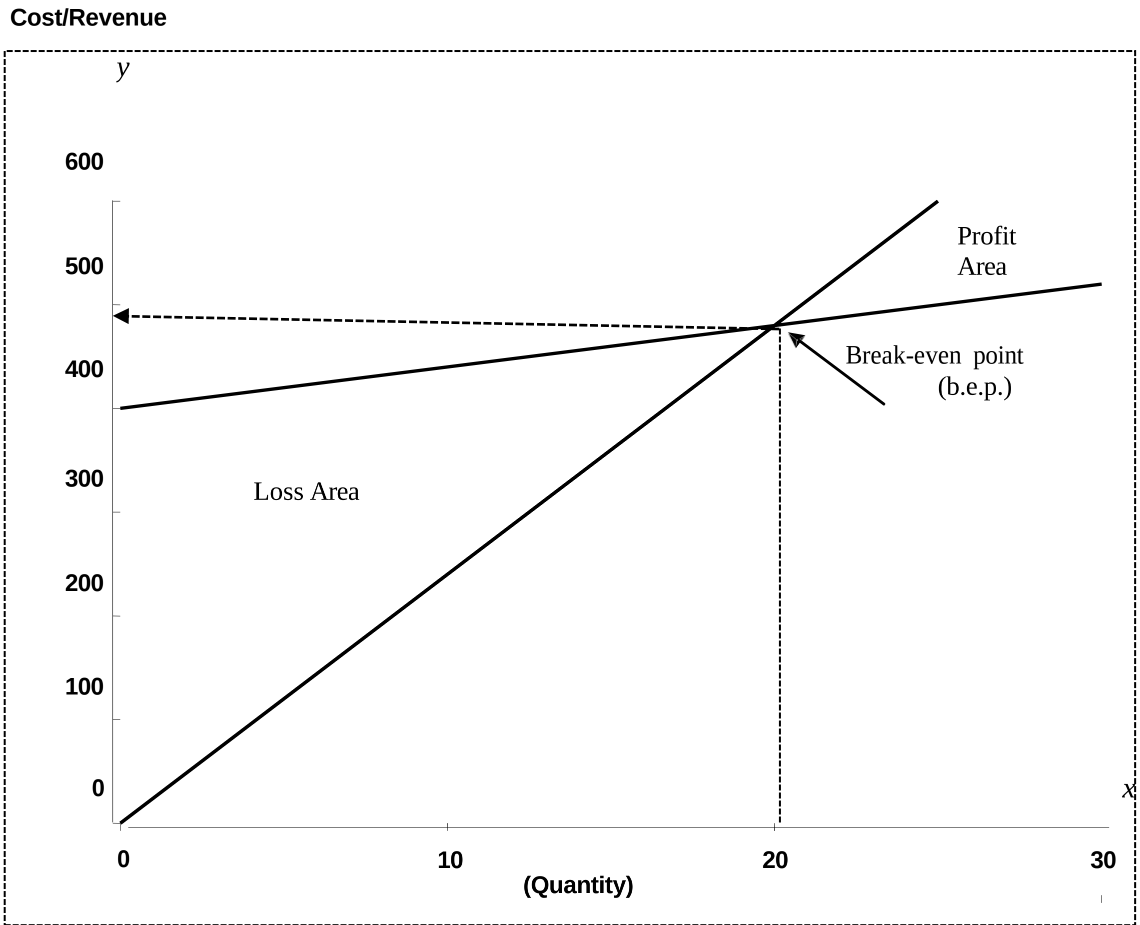
when $x = 0$, $\boldsymbol{C}(x) = 400$ => $(0, 400)$

For $\boldsymbol{R}(x) = 24x$

Since there is no fixed revenue, the graph of the revenue function always passes through the origin. So we need another point which is obtained by substituting any positive value of x in $\boldsymbol{R}(x)$.

When $x = 0$, $\boldsymbol{R}(x) = 0$ => $(0, 0)$

When $x = 25$, $\boldsymbol{R}(x) = 600$ => $(25, 600)$



The break-even point is as indicated on the graph.

i.e. the break-even quantity is 20.

The loss area is the region below the break-even point and within the lines while the profit area is the region above the break-even point and within the lines. Alternatively, the loss area is the region where the revenue line is below the cost line and the profit area is the region where the revenue line is above the cost line.

Example 11.15

SEJAYEB (Nig Ltd) consumes 8 hours of labour and 10 units of material to produce an item thereby incurring a total cost of ₦2100. When 12 hours of labour and 6 units of material are consumed, the cost is ₦2700.

Find the cost per unit of labour and material.

Solution

Let x represent the cost per unit of labour and y the cost per unit of material. Then we have the following simultaneous equations to solve:

$8x + 10y = 2,100$ ....(i)

$12x + 6y = 2,700$……..........................(ii)

equation (i) × 6 gives

$48x + 60y = 12,600$…............(iii)

equation (ii) × 10 gives

$120x + 60y = 27,000$…..........(iv)

equation (iv) – equation (iii) gives $72x = 14,400$

$x = 200$

Substitute x into (i)

i.e. $1,600 + 10y = 2,100$

$10y = 500$

$y = 50$

i.e. the cost per unit of labour is ~~N~~200 and the cost per unit of material is ~~N~~50

Example 11.16

The Planning and Research department of a firm has estimated the sale function

S(x) to be $\boldsymbol{S}(x) = 800x – 250$ and the cost function $\boldsymbol{C}(x)$ as $50,000 + 200x^2 – 500x$, where x is the number of items produced and sold.

Determine the break-even quantity for the firm.

Solution

$\mathbf{C}(x) = 50,000 + 200x^2 – 500x$

$\mathbf{R}(x) =$ (number of items sold) (sale function)

$= x.\mathbf{S}(x)$

$= x(800x – 250)$

$= 800x^2 – 250x$

For break-even quantity, $\mathbf{R}(x) = \mathbf{C}(x)$

i.e. $800x^2 – 250x = 50,000 + 200x^2 – 500x$

$600x^2 + 350x – 50,000 = 0$

$12x^2 + 7x – 1000 = 0$

Using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 12$, $b = 7$, $c = –1000$

$\therefore \quad x = \dfrac{-7 \pm \sqrt{7^2 - 4(12)(-1000)}}{2(12)}$

$= \dfrac{-7 \pm \sqrt{7^2 \ - 4(12)(-1000)}}{2(12)}$

$= -9.43$ or 8.84

Since x cannot be negative because negative items cannot be produced, $x = 8.84$ or 9

i.e. the break-even quantity is approximately 9.

Example 11.17

(a) The total monthly revenue of DUPEOLU enterprise (in Leone) is given by the equation: $R = 200{,}000\ (0.5)^{0.6x}$
where x (Le'000) is the amount spent on overheads.
Calculate the
(i) maximum revenue
(ii) total revenue if Le 5,000 is spent on overheads.

(b) The production costs of GODSOWN Company are estimated to be $\boldsymbol{C}(x) = 200 - 80e^{-0.02x}$
where x is the number of units of items produced.
Determine:
(i) the fixed costs for the company
(ii) the costs of producing 250 items.
(iii) the percentage of the production costs in (ii) which are fixed.

Solutions

(a)

(i) $R = 200{,}000(0.5)^{0.6x}$, maximum revenue occurs when nothing is spent on overheads,
i.e. $x = 0$
$R = 200{,}000(0.5)^0$
$R =$ Le200, 000

(ii) $x = \frac{5000}{1000} = 5$ since x is in thousands of naira

$R = 200{,}000(0.5)^{0.6(5)}$

$R = 200{,}000(0.5)^{3}$

R = Le25,000

(b)

(i) $C = 200 - 80e^{-0.02x}$, fixed costs are the costs incurred when no items have been produced, *i.e.* $x = 0$

$C = 200 - 80e^{0}$

C = 120 *i.e.* ₦120 000

(ii) $C = 200 - 80e^{-0.02x}$

when $x = 250$,

$C = 200 - 80e^{-0.02(250)}$

C = 199.461 *i.e.* ₦199 461

(iii) Percentage required is $\frac{120000 \times 100}{199461} = 60.16\%$

Demand and Supply Equations; Market Equilibrium.

Usually, demand and supply equations can be reasonably approximated by linear equations.

- **Demand Equation**

  Demand is inversely proportional to price i.e. quantity demanded decreases as price increases and it increases as price decreases.

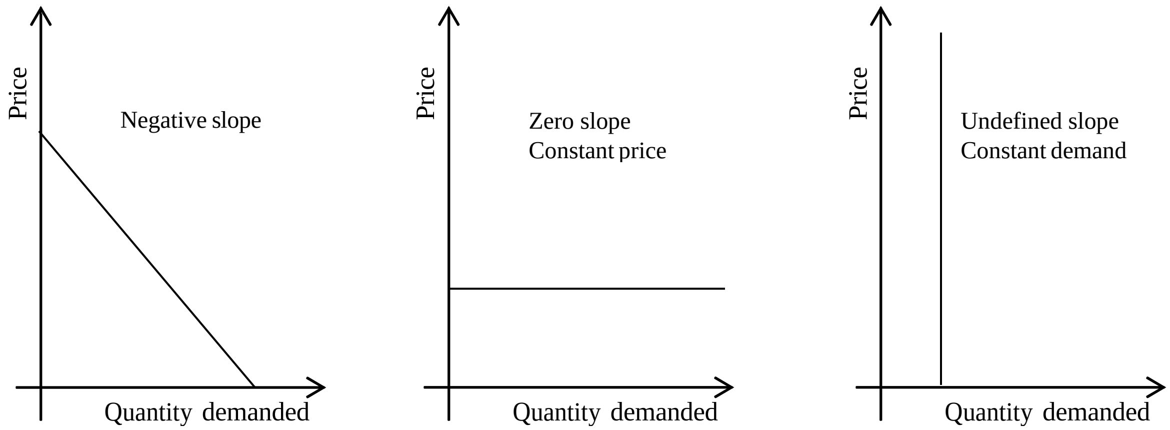
  For this reason, the slope of a demand curve is negative.

  If the slope is zero, then the price is constant irrespective of demand and if the slope is undefined, it implies constant demand irrespective of price.

  If x represents quantity demanded and y represents the price, both x and y must be positive.

  The demand equation is of the form $y = a + bx$, where b is the slope.

The three situations are shown below



Example 11.18

500 watches are sold when the price is ₦2,400 while 800 watches are sold when the price is ₦2,000

(a) obtain the demand equation.

(b) how many watches will be sold if the price is ₦3,000?

(c) what is the highest price that could be paid for a watch?

Solutions

(a) Let x be the quantity demanded and y the price, then the demand equation is $y = a + bx$

When $x = 500$, $y = 2{,}400$

i.e. $2{,}400 = a + 500b$ .... (i)

when $x = 800$, $y = 2{,}000$

i.e. $2{,}000 = a + 800b$ .... (ii)

(i) – (ii) gives

$400 = -300b$

$\therefore \quad b = \dfrac{-4}{3}$

Substitute for b in equation (i)

$$i.e.\ 2{,}400 = a - \frac{2{,}300}{3}$$

$$a = \frac{9{,}200}{3}$$

Hence, the demand equation is

$$y = \frac{9{,}200}{3} - \frac{4x}{3}$$

i.e. $3y = 9{,}200 - 4x$

(b) if $y = 3{,}000$, then

$9{,}000 + 4x = 9{,}200$

$4x = 200, \quad x = 50$

i.e. 50 watches will be sold when the price is ₦3,000

(c) the highest price for a watch is attained when demand is zero

$3y + 4x = 9{,}200$

When $x = 0$,

$3y = 9{,}200$

$y =$ ₦3,066.67

- **Supply Equation**

Supply is directly proportional to price i.e. quantity supplied increases as price increases and quantity supplied decreases as price decreases.

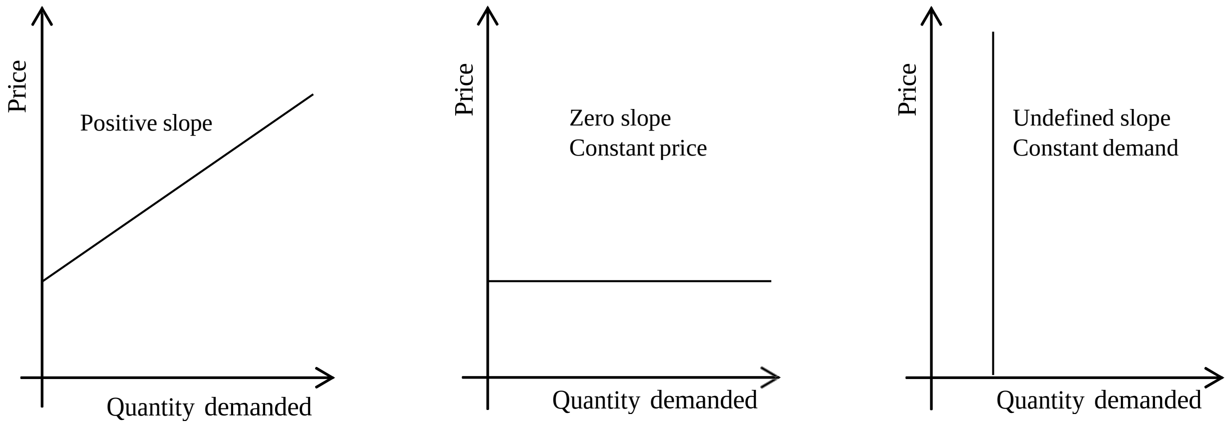
This implies that the slope of supply curve is positive. If the slope is zero, it means constant price irrespective of supply and if the slope is undefined, it implies constant supply irrespective of price.

If x represents the quantity supplied and y the price, both x and y must be positive.

The supply equation is also of the form

$y = a + bx$, where b is the slope.

The three situations are drawn below



Example 11.19

2,000 printers are available when the price is ₦15,000 while 3,200 are available when the price is ₦21,000.

(a) determine the supply equation

(b) how many printers will be available if the price is ₦30,000?

(c) if 8,000 printers are available, what is the price?

(d) what is the lowest price for a printer?

Solutions

(a) Let y be the price and x the quantity supplied, then the supply equation is

$$y = a + bx$$

when $y = 15{,}000$, $x = 2{,}000$

i.e. $15{,}000 = a + 2{,}000b$........(i)

when $y = 21{,}000$, $x = 3{,}200$

i.e. $21{,}000 = a + 3{,}200b$ ……(ii)

equation (ii) – equation (i) gives

$6{,}000 = 1{,}200b$

=> $b = 5$

Substitute for b in equation (i)

i.e. $15,000 = a + 10,000$

$a = 5,000$

hence the supply equation is

$y = 5,000 + 5x$

or $y - 5x = 5000$

(b) if $y = 30000$, then

$30,000 - 5x = 5,000$

$-5x = -25,000$

=> $x = 5,000$

i.e. 5,000 printers will be available when the price is ₦30,000

(c) if $x = 8,000$, then

$y - 40,000 = 5,000$

$y = 45,000$ i.e. if 8,000 printers are available, the price is ₦45,000

(d) The lowest price is attained when supply is zero i.e. $y - 5x = 5,000$

when $x = 0$,

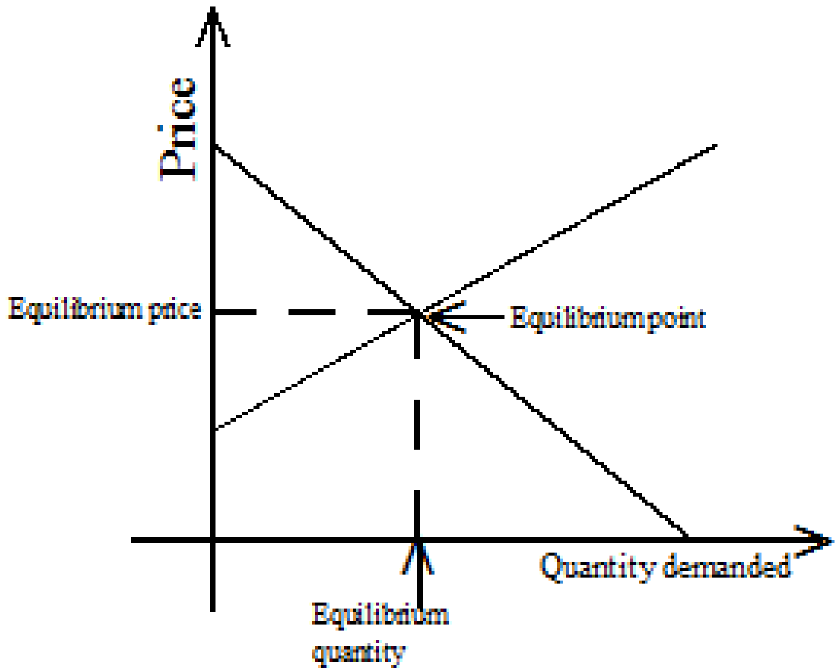
$y = 5,000$ hence the lowest price is ₦5,000

- **Market Equilibrium**

Market equilibrium occurs at the point where the quantity of a commodity demanded is equal to the quantity supplied.

The equilibrium price and quantity are obtained at that point. As mentioned earlier, neither x nor y can be negative hence equilibrium is only meaningful when the curves intersect in the first quadrant i.e. where both x and y are positive.

To find the equilibrium price or quantity, we solve the supply and demand equations simultaneously or graphically by drawing the supply and demand curves on the same graph. The point of intersection of the two curves is the equilibrium point.



Example 11.20

The demand and supply equations for a commodity are respectively given by

$$20y + 3x = 335$$

$$15y - 2x = 60$$

Determine the point of equilibrium by solving the equations

(a) simultaneously

(b) using graphical method

Solution

(a) $20y + 3x = 335$ (i)

$15y - 2x = 60$ (ii)

(i) × 2 gives $40y + 6x = 670$ (iii)

(ii) × 3 gives $45y - 6x = 180$ (iv)

(iii) + (iv) gives $85y = 850$

$y = 10$

Substitute for y in equation (ii) to get $150 - 2x = 60$

$-2x = -90$

$x = 45$

i.e. the equilibrium price is 10 and the equilibrium quantity is 45.

(b) Draw the lines of the two equations on the same axes.

For $20y + 3x = 335$, when $x = 0$, $y = 16.75$

=> (0, 16.75)

when $y = 0$, $x = 111.67$ => (111.67, 0)

For $15y - 2x = 60$,

when $x = 0$, $y = 4$ => (0, 4)

when $y = 6$, $x = 15$ => (15, 6

As could be seen from the graph, the equilibrium is at the point (44, 10) i.e. $x = 44$, $y = 10$

i.e. the equilibrium price is 10 while the equilibrium quantity is 44. As mentioned earlier, graphical solutions are estimates and this accounts for the slight difference in the value of the quantity obtained.

11.6 Chapter Summary

A function has been described as a mathematical expression involving two or more variables. Linear, quadratic and exponential functions were discussed. Also discussed are different types of equations and the methods of solving them. Inequalities and their graphical solutions were discussed as well. Finally, comprehensive applications of these concepts to Business and Economics were treated.

11.7 Multiple-choice and short-answer questions

1. If $f(x) = 3x^3 - 4x^2 + 2x + 555$, then $f(-5)$ is
   - A. 820
   - B. 290
   - C. 270
   - D. 70
   - E. 90

2. The condition for the expression $ax^2 + bx + c$ to be a quadratic expression is
   - A. $b \neq 0$
   - B. $a \neq 0$ & $b \neq 0$
   - C. $b \neq 0$ & $c \neq 0$
   - D. $c \neq 0$
   - E. $a \neq 0$

3. The cost function of a company is $\boldsymbol{C}(x) = 500 + 2x^2 + 5x$ while the revenue function is $\boldsymbol{R}(x) = 3x^2 - 10x$, where x is the number of items produced and sold. The profit that will accrue from 500 items is
   - A. 24,200
   - B. 242,000
   - C. 124,000
   - D. 224,000
   - E. 257,000

4. If the demand and supply equations for a commodity are respectively given by

   $30y + 7x = 800$

   $14y - 5x = -40$

   the equilibrium point is then

   A. (15, 50)

   B. (50, 20)

   C. (50, 15)

   D. (20, 50)

   E. (25, 25)

5. On the graph of $y = a + bx$, a is the .... on the y-axis while b is the ....

6. An exponential function is a function which has a .... base and a.................exponent

7. The sign of a variable or a constant in an equation changes when it crosses the..................

8. When two expressions are not equal, then one has to be .... or.............than the other.

9. If for a business, $R(x) - C(x) < 0$, then the business is said to be running at a ....

10. The demand for a commodity is .... proportional while supply is .... proportional to the price of the commodity

Answers

1. $f(x) = 3x^3 - 4x^2 + 2x + 555$

   $f(-5) = 3(-5)^3 - 4(-5)^2 + 2(-5) + 555$

   $= -375 - 100 - 10 + 555$

   $= 70$ (D)

2. $a \neq 0$ (E)

   because if $a = 0$, the quadratic term will vanish.

3. Profit = $R(x) - C(x)$

   $= 3x^2 - 10x - 500 - 2x^2 - 5x$

   $= x^2 - 15x - 500$

   if $x = 500$, then profit $=(500)^2 - 15(500) - 500$

   $= 242,000$ (B)

4. $30y + 7x = 800$...........(i)

$14y - 5x = -40$. ....(ii)

equation (i) x 5 gives

$150y + 35x = 4,000$......(iii)

equation (ii) × 7 gives

$98y - 35x = -280$. .....(iv)

equation (iii) + (iv) gives

$248y = 3,720$

$y = 15$

Substitute for y in equation (i)

$30(15) + 7x = 800$

$7x = 350$

$x = 50$

So the equilibrium point (x, y) is (50, 15) (C)

5. Intercept, slope of the line. (in that order)

6. Constant, variable (in that order)

7. Equality sign

8. Less, greater (or vice versa)

9. Loss

10. Inversely, directly. (in that order)

CHAPTER TWELVE

MATHEMATICS OF FINANCE

Chapter contents

a) Introduction.

b) Sequences and Series.

c) Simple Interest.

d) Compound Interest.

e) Annuities.

Objectives

At the end of this chapter, readers must be able to understand the

a) definition of sequences and series;

b) difference between Arithmetic Progression (AP) and Geometric Progression (GP);

c) calculation nth term and sum of the first *n* terms of AP and GP;

d) concept of simple interest and compound interest;

e) meaning of Annuities;

f) calculation of Present Values (PV) and Net Present Values (NPV); and

g) application of all of the above to economic and business problems.

12.1 Introduction

Concept of Sequences and Series is applied to Annuities, Net Present Values(NPV) and Internal Rate of Return(IRR).Simple and Compound Interests will be discussed.

12.2 Sequences and series

A sequence is a set of numbers **that follow a definite pattern.** e.g

(a) 7,12,17,22 ... the pattern is that the succeeding term is the preceding time plus 5

(b) 256, 64,16,4: the pattern is that the succeeding term is the preceding term divided by 4.

If the terms in a sequence are connected by plus or minus signs, we have what is called a series.

e.g. the series corresponding to (a) above is 7 + 12 + 17 + 22 + .. ,

Arithmetic progression (A.P)

A sequence in which each term increases or decreases by a constant number is said to be an A.P

The constant number is called the common difference.

The first term is usually represented by *a* while the common difference is represented by *d*

An A. P. is generally of the form

a, a + d, (a + d) + d, {(a + d) + d} + d, ...

i.e. a, a + d, a + 2d, a + 3d, ...

2^{nd} term is a + d = a + (2 -1) d

3^{rd} term is a+2d = a+(3 -1) d

4th term is a + 3d = a + (4-1) d and so on

following this pattern, the nth term will be a+(n-l)d

The sum of the first n terms of an A. P.

is given by $S_n = \dfrac{n\{2a + (n - l)d\}}{2}$ or $\dfrac{n(a+l)d}{2}$

where *l* is the last term

Example 12.1

(a) Find the 12th term of the A. P: 7,13,19,25, ...

(b) Find the difference between the 8^{th} and 52^{nd} terms of the A. P: 210, 205, 200, .

(c) Find the sum of the first 10 terms of the A.P in (a) and (b)

(d) How many terms of the series 15 + 18 + 21 +.....will be needed to obtain a sum of 870?

Solutions

(a) 7, 13, 19,25,

$d = 13 - 7$ or $25 - 19 = 6$

$a=7, n=12$

nth term $= a + (n-1)d$

$12^{th} = 7 + (12 - 1)\,6$

$= 73$

(b) 210, 205, 200, .

$a = 210,\ d = -5$

8^{th} term $= 210 + (8 - 1)(-5)$

$= 175$

52^{nd} term $= 210 + (52 - 1)(-5)$

$= \mathbf{-45}$

Difference $= 175 - (-45) = 220$

(i) 7, 13, 19,25, .

$a = 7, d=6, n=10$

$$S_{10} = \frac{10}{2}\{2(7) + (10 - 1)6\}$$

$= 340$

(ii) 210, 205, 200, .

$a=210, d=-5, n=10$

$$S_{10} = \frac{10}{2}\{2(21O) + (10 - 1)-5\}$$

$=1875$

(d) 15 +18+21+…..

$a = 15,\ d = 3,\ S_n = 870,\ n = ?$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$870 = \frac{n}{2}(30 + (n-1)3\}$$

i.e $1740 = n(27 + 3n)$

i.e $3n^2 + 27n – 1740 = 0$

$n^2 + 9n – 580 = 0$

$n^2 + 29n – 20n - 580 = 0$

i.e $n(n + 29) – 20(n + 29) = 0$

i.e $(n-20)(n+29) = 0$

.: $n = 20$ or -29

but n cannot be negative since n is number of terms, .: $n = 20$

Example 12. 2

Mr. Emeka earns N240, 000 per annum with an annual constant increment of ₦25, 000.

a. How much will his annual salary be in the eighth year?

b. What will his monthly salary be during the 15^{th} year?

c. If he retires after 30 years, and his gratuity is 250% of his terminal annual salary, how much will he be paid as gratuity?

Solution

(a) a = 240,000, d = 25,000, n = 8

n^{th} term = a + (n-1) d

8^{th} term = 240,000 + (8-1) (25,000)

= ₦415,000

(b) a = 240,000, d= 25,000, n = 15

15^{th} term = 240,000 + (15 – 1) (25,000)

= ₦590,000

i.e his annual salary in the 15^{th} year is N590,000

.: his monthly salary in the 15^{th} year is

$\frac{₦,000}{12}$ = ₦49,166.67

(c) a = 240,000, d = 25,000, n = 30

30^{th} term = 240,000 + (30 – 1) (25,000)

= ₦965,000

i.e his terminal annual salary = ₦965,000

$\therefore$ his gratuity = $\frac{250}{100}$ x 965,000

=₦2,412,500

Geometric Progression (G.P.)

A sequence is said to be a G. P. if each of its terms increases or decreases by a constant ratio.

The constant ratio is called the common ratio and is usually denoted by r while the first term is denoted by a.

The general form of a GP is a, ar, ar^2, ar^3, ar^4, ……..

The pattern is

2^{nd} term = ar^{2-1}

3^{rd} term = ar^{3-1}

4^{th} term = ar^{4-1}

n^{th} term= ar^{n-1}

The sum of the first n terms of a GP is

$$S_n = a + ar + ar^2 + ar^3 + \ldots.. + ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad (r<1) \text{ or } S_n = \frac{a(r^n-1)}{r-1}, \; r > 1$$

If r < 1, r^n decreases as n increases and in fact tends to zero as n gets very big. In the extreme case where n is close to infinity (∞), r^n is approximately zero.

In such a case, we talk about the sum to infinity of a G.P. given by

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{, if } n \to \infty \text{, then } r^n = 0$$

$$.: = S\infty = \frac{a}{1-r}, \; s_n \text{ is said to converge to } \frac{a}{1-r}$$

Example 12.3

a. Find the 7^{th} term of the G.P 8, 16, 32, …….

b. Find the 6^{th} term of the G.P 243, 81, 27, ……

c. Find the sum of the first 15 terms of the G.P in (a) and (b)

d. Find the sum to infinity for the G.P in (a) and (b)

Solutions

(a) 8, 16, 32, ………..

$r = \frac{16}{8}$ or $\frac{32}{16} = 2$

a = 8, n = 7

n^{th} term $= ar^{n-1}$

.: 7^{th} term = $8(2)^{7-1}$

= 512

(b) 243, 81, 27

$r = \frac{81}{243} = \frac{1}{3}$

a = 243, n = 6

6^{th} term $= 243\{\frac{1}{3}\}^{6-1}$

= 1

(c) (i) 8, 16, 32, ………..

a = 8, r = 2, n = 15

here r> 1, then

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{8(2^{15} - 1)}{2 - 1}$$

$= 262{,}136$

(ii) 243,81, 27, ..………..

a = 243, r = 1/3, n = 15

$$S_{15} = \frac{243(1 - (1/3)^{15})}{1 - (1/3)}$$

$= 364.5$

(d) (i) 8, 16, 32, ………..

a = 8, r = 2

S_∞ is not possible since r>1

(ii) 243, 81, 27, ………..

a = 243, r = 1/3

$$S_\infty = \frac{a}{r - 1}$$

$$= \frac{234}{1 - 1/3}$$

$= \quad 364.5$

$S_{15} = S_\infty = 364.5$ i.e. S_n has actually converged to 364.5 from $n = 15$

Example 12.4

a. A job is estimated to take 12 days. If the overhead costs were N5,000 for the first day, N7,500 for the second day, N11,250 for the third day and so on, how much will the total overhead costs be?

b. The swamp in a village is highly infested by tsetse flies. After a lot of appeals by the head of the village, the state government took necessary actions and the population of the flies starts to decrease at a constant rate of 20% per annum. If at the initial stage, there were 104 million flies in the swamp, how long will it take to reduce the population of flies to 30m?

Solutions

(a) 5000, 7500, 11250, ...

$$r = \frac{7500}{5000} = \frac{11250}{7500} = 1.5$$

:. the series is a G.P., then

a = 5000, n = 12 and common ratio = r = 1.5

$$S_n = \frac{a(r-1)}{r-1} \text{ since } r>1$$

$$= \frac{5000(1.5^{12}-1)}{1.5-1}$$

$$= ₦\,1,287,463.38$$

(b) Decrease at the rate of 20% per annum means r = 80% = 0.8

a = 104m, $ar^{n-1}=30$, n=?

$30 = ar^{n-1}$

$= 104\,(0.8)^{n-1}$

$\frac{30}{104} = (0.8)^{n-l}$

i.e (n-1) log 0.8 = log 0.28846

$$n - 1 = \frac{\log 0.28846}{\log 0.8}$$

i.e n = 6.57 years

Example 12.5

A businessman realised that returns from his business are dwindling due to the economic recession.

He decided to be saving some amount of money every month which follows the dwindling returns as follows:

240,000, 96,000, 38,400, ……………

You are required to

(a) determine

i) his total savings in 16 months

ii) the sum to infinity of his savings

(b) compare the results in a(i) and (ii) above

Solution

(a) (i) $S_n = \frac{a(1-r^n)}{1-r} \quad , where \quad a = 240{,}000, \quad r = \frac{96{,}000}{240{,}000} = \frac{38{,}400}{96{,}000} = \frac{2}{5}$

$$S_{16} = \frac{240{,}000\left(1-\left(\frac{2}{5}\right)^{16}\right)}{1-\frac{2}{5}} = \frac{240{,}000(1-(0.4)^{16})}{1-0.4} = ₦399{,}99.83$$

(ii) $S_\infty = \frac{a}{1-r} = \frac{240{,}000}{1-\frac{2}{5}} = ₦400{,}000$

(b) If S_{16} is approximated to the nearest ₦, the result will be the same, i.e. S_∞ has converted to ₦400,000 from n = 16. Additional savings from the 17th month are minimal.

Example 12.6

An investment is estimated to grow at the rate of 15% per annum. If the worth of the investment now is ₦850,000

a. What will its worth be in the 6th year?

b. What is the percentage increase in its worth after 6 years?

c. In what year will the investment worth ₦3.28m?

Solutions

(a) growth of 15% per annum means r

=1+15/100

= 1.15

a= 850, 000, n = 6

n^{th} term = ar^{n-1}

6^{th} term = $850,000(1.15)^5$

= ₦l, 709,653.61

(b) % age increase after 6 years is

$$\frac{1,709,653.61 - 850,000 \times 100}{850,000}$$

= 101.14%

(c) n^{th} term = ₦3.28m, a = 850000, r = 1.15, n = ?

$3,280,000 = 850,000\ (1.15)^{n-1}$

(n-1) log 1.15 = log 3.8588

$$n - 1 = \frac{\log 3.8555}{\log 1.15}$$

n – 1 = 9.66

i.e. n = 10.66 years

Example 12.7

GODSLOVE Enterprises purchased a machine for N930,000 which is expected to last for 25 years.

If the machine has a scrap value of N65,000,

a. how much should be provided in each year if depreciation is on the straight line method?

b. what depreciation rate will be required if depreciation is calculated on the reducing balance method?

Solutions

Note: For this type of problem, number of terms is 1 more that the number of years because the cost is the value at the beginning of the first year and the scrap value is at the end of the final year.

So, a = 930,000, n = 26, scrap value = 65000

a. This is an A. P.

$65000 = a + (n-1)d$

$= 930000 + (26 - 1)d$

$= 930000 + 25d$

$\therefore d = \frac{65000 - 930000}{25}$

$= N34,600$

b. This is a GP

$65000 = ar^{n-1}$

$= 930000\, r^{25}$

i.e. $r^{25} = \frac{65000}{930000}$

$$r = \left(\frac{65,000}{930,000}\right)^{1/25} = 0.90$$

:. Reducing balance depreciation rate is

$1 - 0.90 = 0.1$

i.e. <u>10%</u>

12.3 SIMPLE INTEREST

Simple Interest is the interest that will accrue on the principal (original money invested/borrowed) for the period for which the money is invested/borrowed.

If P is the principal, r% is the rate of interest and n is the number of periods, then the simple interest is given by

$$I = \mathbf{P.\,r.\,n}$$

n can be in years, quarters, months etc.

the amount A_n at the end of the nth period is

$A_n = P + I$

$= P + Prn$

$= P\,(1+r.n)$

Example 12.8

(a) (i) what is the interest that will accrue on ₦25000 at 12% simple interest at the end of 15 years?

(ii) how much will it amount to at the end of this period?

(b) ow long will it take a money to triple itself at 9.5% simple interest?

Solutions

(a) (i) $I = Pr.n$

$P = 25000$, $r = 0.12$, $n = 15$

$I = 25000\,(0.12)\,(15)$

$= ₦45000$

(ii) Amount $= A_{15} = P + I = 25000 + 45000$

$= ₦70,000$

If the interest is not required, then

$A_{15} = P\,(1 + r.\,n)$

$= 25000\,\{1 + (0.12)\,(15)\}$

$= ₦70,000$

(b) If P is to triple itself, it becomes 3P, so we have

$A_n = P(1 + r.n)$

i.e. $3P = P(1 + r.n)$,

$r = 0.095$

$3P = P(1 + 0.095n)$

$3 = 1 + 0.095n$

$n = \frac{2}{0.095}$

$= 21.05$ years

Note from $I = Pr.n$.

• $P = \frac{1}{r.n}$

• $r = \frac{1}{P.n}$

• $n = \frac{1}{P.r}$

The **Present Value (PV)** of any money is the current worth of its future amount based on prevailing conditions.

Example 12.9

Mohammed wants to purchase a house for ₦ 1.80m in 5 years' time. Interest rate remains constant at 10% per annum computed by simple interest method. How much should he invest now?

Solution

$A_n = P(l + r.n)$

$A_n =$ N1.80m, $r = 0.10$, $n = 5$, $P = ?$

$1.80m = P\{1 + (0.10)(5)\}$

$= P(1.5)$

$\therefore P = \frac{1.80m}{1.5}$

₦1.20m

Note

(a) ₦ 1.20m is the present value of ₦ 1.80m under the prevailing conditions.

(b) The question is the same as: What is the present value of ₦ 1.80m at

10% Simple Interest rate over 5 years?

Generally, the present value (PV) of a future amount (A_n) at r % simple interest for n periods is obtained as follows:

$A_n = P(1 + r.n)$

$$.:. P = \frac{A_n}{1 + r.n}$$

for the last example, A_n= 1.80m, r =: 0.10, n = 5

$$\text{so } P = \frac{1.80m}{1+(0.10)(5)}$$

=N1.20m

12.4 COMPOUND INTEREST

Compound interest can be referred to as multi-stage single period simple interest. It is the type of interest commonly used in banks and financial institutions. Simple interest on the initial principal for single period added to the principal itself (i.e, amount at the end of the first period) is the new principal for the second period. Amount at the end of the second period is the principal for the third period etc.

Example 12.10

How much will ₦200, 000 amount to at 8% per annum compound interest over 5 years?

Solution

Year	Principal	Interest I = P.r.n (n=1)	Amount
1	200000	16000	216000
2	216000	17280	233280
3	233280	18662.4	251942.4
4	251942.4	20155.4	272097.8
5	272097.8	21767.8	293865.6

.: required amount is ₦293,865.6

At simple interest

A_5 = 200000 (I + (0.08)5)

= ₦280,000

The accrued amount at simple interest is always less than that at compound interest. The compound interest formula is given by

$$\mathbf{A_n = P\,(1 + r)^n}$$

where A_n **is the accrued amount after the nth period**

P is the Principal

r is the interest rate per period

n is the number of periods

Example 12.11

a) Use the compound interest formula to calculate the amount for example 10.9

b) What compound interest rate will be required to obtain ₦230,000 after 6 years with an initial principal of ₦120,000?

c) How long will it take a sum of money to triple itself at 9.5% compound interest.

d) How much will ₦250,000 amount to in 3 years if interest rate is 12% per Annum compounded quarterly?

Solution

(a) $A_n = P\,(1 + r)^n$

$P = 200000$, $r = 0.08$, $n = 5$

$A_5 = 200000\,(1 + 0.08)5$

$= 200000\,(1.469328)$

= ₦293,865.60

(b) $A_n = P\,(1 + r)^n$

$A_n = 230{,}000$, $P = 120{,}000$, $n = 6$, $r = ?$

So, $230{,}000 = 120{,}000\,(1 + r)^6$

$(1 + r)^6 = 1.916667$

$1 + r = (1.916667)^{1/6}$

$= 1.1145$

i.e $\quad r = 1.1145 - 1$

$\quad = 0.1145$

i.e $\quad = 11.5$

(c) If P is to triple itself, then we have 3P

$A_n = P(1+ r)^n$

$A_n = 3P$, $r = 0.095$, $n = ?$

$3P = P(1 + 0.095)^n$

$(1.095)^n = 3$

$n\log(1.095) = \text{Log } 3$

$$n = \frac{\log 3}{\log(1.095)}$$

$= 12.11$

Recall that the number of years under simple interest is 16.67 years (Ex. 6b). So, the number of years is less under compound interest (obviously?)

(d) $A_n = P(1+r)^n$, $P=250{,}000$, $r = \frac{0.12}{4}=0.03$, $n = (4 \times 3)=:12$

$A_{12} = 250{,}000\,(1 + 0.03)^{12}$

= ₦356,440.22

Example 12.12

Calculate the sum of money that will need to be invested now at 9% compound interest to yield N320,000 at the end of 8 years.

Solution

$A_n = \mathbf{P(l+r)^n}$

A_n **= 320,000, r = 0.09, n =** 8, P = ?

320,000 = P($\boldsymbol{1+0.09)^8}$

$$P = \frac{320{,}000}{(1.09)^8}$$

= **₦ 160,597.21**

Again, ₦ 160597.21 is the present value of ₦ 320,000 under the given situation.

Generally, from

$A = P(l+r)^n$

$P = \frac{A_n}{(1 + r)^n}$

This formula forms the basis for all discounting methods and it is particularly useful as the basis of **Discounted Cash Flow (DCF)** techniques.

12.5 ANNUITIES

An annuity is a sequence of constant cash flows received or paid.

Some examples are:

(a) Weekly or monthly wages;

(b) Hire-purchase payments;and

(c) Mortgage payments.

Types of Annuity

a) An **ordinary annuity** is an annuity paid at the end of the payment periods. This type of annuity is the one that is commonly used.

b) A **due annuity or annuity due** is an annuity paid at the beginning of the payment periods (i.e. in advance)

c) A **certain annuity** is an annuity whose term begins and ends on fixed dates.

d) A **perpetual annuity** is an annuity that goes on indefinitely.

Sum of an Ordinary Annuity (Sinking fund)

The sum(s) of an ordinary annuity with interest rate of r% per annum compounded over n periods is given by

$$S = A+A(l+r) +A(l+r)^2+A(l+r)^3+ \quad +A(l+r)^{n-1}$$

This is a G.P with first term A and common ratio (l +r) as developed earlier on.

So,

$$S = \frac{A\{l-(l+r)^n\}}{l-(l+r)}$$

$$= \frac{A\{l-(l+r)^n\}}{-r}$$

$$= \frac{A\{(l+r)^n-l\}}{r}$$

where A is the amount paid at the end of each period

Example 12.13

(a) Find the amount of an annuity of ₦50,000 per year at 4% interest rate per annum for 7 years.

(b) Calculate the annual amount to be paid over 4 years for a sinking fund of N2,886,555 if the compound interest rate is 7.5% per annum

Solutions

(a) $S = \frac{A[(1 + r)^n -1]}{r}$

$A = 50{,}000$, $r = 0.04$, $n = 7$

$S = \frac{50{,}000\ [(1+0.04)^7-1]}{0.04}$

= ₦ 394,914.72

b) $S = \frac{A[(l+r)^{n-1}]}{r}$

$S = A[(l+r)^{n-1}]/r$

$S = 2886555$, $r= 0.075$, $n = 4$, $A = ?$

$2{,}886{,}555 = \frac{A[(1 + 0.075)^{4-1}]}{0.075}$

$.:A = \frac{(2886555)\ (0.075)}{(1.075)^{4\ -\ 1}}$

= ₦174,267.22

Present Value of an Annuity

The present value of an annuity is the sum of the present values of all periodical payments. Thus, the present value of an annuity *(P)* with compound interest rate of r% per annum for n years is

$$P = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + \ldots\ldots + \frac{A}{(1+r)^n}$$

This is a G.P with $\frac{A}{1+r}$ as the common ratio. Therefore,

$$P \quad = \quad \frac{\frac{A}{1+r} \cdot \left\{ 1 - \left(\frac{1}{1+r} \right)^n \right\}}{1 - \frac{1}{1+r}}$$

$$P \quad = \quad \frac{\frac{A}{1+r} \cdot \left\{ 1 - (1+r)^{-n} \right\}}{1 - \frac{1}{1+r}}$$

$$P \quad = \quad \frac{A \cdot \left\{ 1 - (1+r)^{-n} \right\}}{r}$$

Example 12.14

Determine the present value of an annuity ofN45,000 for 9 years at 5.5% compounded annually

Solution

$$P = \frac{A\left\{ 1 - (1+r)^{-n} \right\}}{r}$$

A = 45000, r = 0.055, n =9

$$P = \frac{45000\,[1 - (1 + 0.055)^{-9}]}{0.055}$$

= N312,848.79

Net Present Value (NPV)

The NPV is the sum of the present values (PV) of all future net cash flows of an investment. The net cash flow may be positive or negative.

The NPV can be used to determine the desirability of an investment. If the NPV is positive, the investment is desirable but if it is negative, the investment is not worth it.

$$NPV = A_0 + \frac{A_1}{1+r} + \frac{A_2}{(1+r)^2} + \frac{A_3}{(1+r)^3} + \ldots\ldots + \frac{A_n}{(1+r)^n}$$

A_0 is the cost of the investment at year 0 and is always recorded as negative (cash outflow) in calculating the NPV

A_1 is the expected net cash flow for year1

A_2 is the expected net cash flow for year 2

A_n is the expected net cash flow for year n

Generally, for most business projects, it is usual to have an outlay (or to invest) a sum at the start and then expect to receive income from the project (i.e revenue) at various times in the future. Supposing a company has the opportunity to buy a new machine now for ₦ 2.5m. It intends to use the machine to manufacture a large order for which it will receive ₦ 1.75m after 2 years and ₦ 1.25m after 3 years.

A project like this can be assessed by assuming a discount rate and then calculating the present values of all the flows of money, in and out. The total of the present values of money in (revenue), less the total of the present values of the money out (costs) is the net present value (NPV) as discussed above.

Example 12.15

A project is presently estimated to cost ₦ 1.1 m. The net cash flows of the project for the first 4 years are estimated respectively as ₦ 225,000, ₦ 475,000, ₦655,000 and ₦ 300,000. If the discount rate is 12%.

(a) Calculate the NPV for the project. Is the project desirable?
(b) Is the project desirable?
(c) If ₦ 95,000 and ₦ 125,000 were spent on the project during the second year and fourth year respectively, will the project be desirable?

Solutions

(a) The usual set up is

r = 0.12

Year	Net Cash Flow (A) ₦	Discounting factor $\frac{1}{(1+r)^n}$	PV= $\frac{A}{(1+r)^n}$ (₦)
0	-1.1m	$\frac{1}{(1+0.12)^0} = 1$	-1.1m
1	225000	$\frac{1}{(1+0.12)}$ = 0.8929	200,902.5
2	475000	$\frac{1}{(1+0.12)^2}$ = 0.7972	378770
3	655000	$\frac{1}{(1+0.12)^3}$ = 0.7118	466229
4	300000	$\frac{1}{(1+0.12)^4}$ = 0.6355	190650
		NPV	₦136,551.50

The tables are available for the discounting factors.

NPV = ₦1,100,000+₦200,902.5+₦378,770+₦466,229+190,650=₦136,551.50

b) Since the NPV is positive, the project is desirable.

c) The new net cash flow for 2nd year = N475,000 – N95,000 = N380,000 and for 4th year = N300,000 – N125,000 = N175,000

So, we now have

Year	Net Cash flow	Discounting factor	PV
0	-1.1m	1	-1.1m
1	225000	0.8929	200902.5
2	380000	0.7972	302936
3	655000	0.7118	466229
4	175000	0.6355	111212.5
		NPV	-18720

Since the NPV is negative, the project is not desirable.

Example 12.16

Supposing a company intends to buy a new machine now for N2.5m. It intends to use the machine to manufacture a large order for which it will receive N1.75m afte2r 2 years and N1.25m after 3 years.

The table below calculates the net present value of the machine project assuming a discount rate of 5%.

End of year	₦	Present Value	₦
0	-2,500,000		-2,500,000
2	+1,750,000	$\frac{1,750,000}{1.05^2}$	1,587,301.6
3	+1,250,000	$\frac{1,250,000}{1.05^3}$	1,079,797
		.: Net Present Value	167,098.6

As the net present value (NPV) is positive , we then conclude that purchasing a new machine is worthwhile

Example 12.17

An architectural outfit has an opportunity of buying a new office complex in Lekki selling for N75m. It will cost N25m to refurbish it which will be payable at the end of year 1. The company expects to be able to lease it out for N125m in 3 years' time. Determine the net present value of this investment if a discount rate of 6% is allowed.

Solution

The corresponding table at a 6% discount rate is given as follows:

End of year	₦	Present Value	₦
0	-75m		-75,000,000
1	-25m	$\frac{-25,000,000}{1.06}$	-23,584,905.7
2	+125m	$\frac{125,000,000}{1.06^3}$	104,952,410.4
		Net Present Value	6,367,504.70

.: The investment is profitable because the NPV is positive.

The Concept of Internal Rate of Return (IRR)

In example 12.15 above, a discount rate of 5% was used for the company that wanted to buy a new machine, we obtained a positive NPV of ₦167,098.60. If the discount rate were larger, the present value of the future revenue would reduce, which for this project would reduce the NPV.

Let‟s consider a discount rate of 7%, the calculations are tabulated as follows:

End of year	₦	Present Value	₦
0	-2.5m		-2,500,000
2	+1.75m	$\frac{1,750,000}{1.07^2}$	1,528,517.80
3	+1.25m	$\frac{1,250,000}{1.07^3}$	1,020,372.30
		Net Present Value	48,890.10

You can see that though the NPV is still positive, but it is far less that the case of 5% discount rate.

Further calculations show that when the discount rate is 8%, the NPV is obtained as follows:

End of year	₦	Present Value	₦
0	-2.5m		-2,500,000
2	+1.75m	$\frac{1,750,000}{1.08^2}$	1,500,342.90
3	+1.25m	$\frac{1,250,000}{1.08^3}$	992,290.30
		Net Present Value	-7,366.80

This reveals that the project breaks even, that is, the NPV is zero at a discount rate between 7% and 8%.

Therefore, **the discount rate at which a project has a net value of zero is called Internal rate of Return (IRR).** Suppose the discount rate is 7.8%, this means that the project is equivalent (as far as the investor is concerned), to an interest rate of 7.8% per year. So, if the investor can invest her money elsewhere and obtain a higher rate than 7.8%, or needs to pay more than 7.8% to borrow money to finance the project, then the project is not worthwhile and should not be invested on.

Example 12.18

If a financial group can make an investment of N85m now and receives N100m in 2 years" time, estimate the internal rate of return.

Solution

From the above analysis, the NPV of this project is

$$NPV = -85{,}000{,}000 + \frac{100{,}000{,}000}{(1+i)^2}$$

where *i* is the discount rate

since the IRR is the value of I which makes the NPV zero, then i can be calculated from

$$\mathbf{-85{,}000{,}000} + \frac{100{,}000{,}000}{(1+i)^2} = 0$$

$$\Rightarrow \quad \frac{100{,}000{,}000}{(1+i)^2} = \frac{100{,}000{,}000}{85{,}000{,}000}$$

$$\Rightarrow \quad (1+i)^2 = \frac{100{,}000{,}000}{85{,}000{,}000}$$

$$\Rightarrow \quad 1+i = \sqrt{\frac{100}{85}} \Rightarrow i = \sqrt{\frac{100}{85}} - 1 = 0.084652$$

giving the required IRR to be 0.084652

12.6 Chapter Summary

A sequence has been defined as a set of numbers that follow a definite pattern. Arithmetic Progression (A.P) and Geometric Progression (G.P) have been treated. Also treated are Simple and Compound Interests, Annuities and Net Present Value (NPV), when NPV is positive, it means the project is worthwhile otherwise it should not be invested on.

A special case is considered when the Net Present Value is zero. The discount rate that forces NPV to be zero is called the Internal Rate of Return (IRR). The relevance of all these concepts to Business and Economics was highlighted and applied through worked examples.

12.7 Multiple-choice and short-answer questions

1) If the 4th term of a geometric progression is – 108 and the 6th term is -972, then the common ratio is

   A. +3

   B. -3

   C. 3

   D. 9

   E. -9

2) Determine the present value of ₵450,000 in 3 years‟ time if the discount rate is 6% compounded annually.

   A. ₵377,826.68

   B. ₵386,772.68

   C. ₵268,386.68

   D. ₵338,277.68

   E.₵287,768.68

3) The lifespan of a machine which costs N2.5m is 15years and has a scrap value of N150,000. If depreciation is on the straight line method, the amount of money to be provided for each year is

   A. ₦ 157,666.67

   B. ₦ 167,666.67

   C. ₦ 156,777.67

   D. ₦156,666.67

   E. ₦167,777.67

4) Mr. Nokoe earns ₵ 50,000 per month with an annual increment of 5%. What will his annual salary be in the 4th year?

   A. ₵ 561,500

   B. ₵ 651,500

   C. ₵ 515,500

   D. ₵ 615.500

E. ¢ 661,500

5) If the discount rate of 6% in Question 2 is compounded six-monthly, the present value will be ……………

6) A project is said to be desirable if the net present value is ....

7) The present value of ₦2.8m at 15% simple interest rate over six years is ....

8) The time that a sum of money will take to triple itself using simple interest is ………than that of compound interest.

9) An annuity is a sequence of………… cash flows…………… or paid.

10) The discount rate that occurs when the net present value is zero is known as…………………

Answers

1) 4^{th} term is ar^3

i.e. $ar^3 = -108$

6^{th} term is ar^5

i.e. $ar^5 = -972$

$\frac{ar^5}{ar^3} = \frac{-972}{-108}$

i.e. $r^2 = 9$

$r = \pm 3$

but since $ar^3 = -108$, $r = -3$ (B)

2) $¢\left(\frac{450,000}{1.06^2}\right) = ¢377,828.68$ (A)

3) $a = 2.5$, $n = 16$, scrap value =150,*000*

i.e. $l50{,}000 = a + (n - l)d$

$= 2{,}500{,}000 + l5d$

d = ₦156,666.67 (D)

4) ₵*50,000* per month => ₵600,000 per annum

$a = 600{,}000$, $n = 4$, $r = 1.05$

:. $T_4 = ar^3$

$= 600{,}000\ (1.05)^3$

= ₵*661*,500 (E)

5) $PV = \left(\dfrac{450{,}000}{1.03^6}\right) = N317{,}232.24$

6) Positive

7) $A_n = P\ (1 + r.n)$

$2.8m = P\ \{\ 1 + (0.I5)(6)\ \}$

$= 1.9P$

:. $P = \dfrac{2.8m}{1.9}$

= N1.47m

8) More

9) Constant, received (in that order)

10) Internal Rate of Return (IRR)

CHAPTER THIRTEEN

DIFFERENTIAL AND INTEGRAL CALCULUS

Chapter contents

a) Introduction.
b) Differentiation.
c) Basic Rule of Differentiation.
d) Maximum and Minimum Points.
e) Integration.
f) Rules of Integration.
g) Totals from Marginals.
h) Consumers' Surplus and Producers' Surplus.

Objectives

At the end of this chapter, readers should be able to understand the

a) calculation of the first and second derivatives of functions;
b) calculation of the maximum and minimum points for functions;
c) meaning of Marginal functions;
d) determination of elasticity of demand;
e) calculation of indefinite integrals of functions;
f) evaluation of the definite integrals for functions;
g) calculation of the totals from marginals; and
h) application of all the above to economic and business problems.

13.1 Introduction

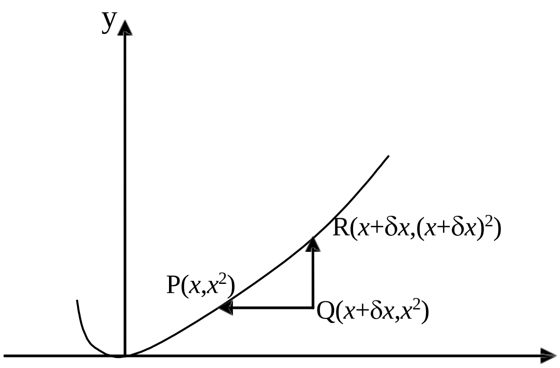
In this Chapter, principles of differentiation and integration will be discussed. The basic difference between differentiation and integration will be explained. Marginals of Revenue, Cost, Profit and Loss will be obtained via the first derivatives of the relevant parameters. Revenue, Cost, Profit and Loss will be derived from respective marginal of the relevant parameters through integration. The use of integration to assess Consumers' and Producers' surpluses will be discussed.

13.2 Differentiation

The gradient or slope of a line is constant. i.e. no matter what points are used, we obtain the same result. It is the increase in y divided by the increase in x. The gradient of a curve at a point is the gradient of the tangent to the curve at that point i.e. different points will result in different gradients. Hence, the gradient of a curve is not constant

Let us consider the simplest quadratic curve, y = x^2 shown below

X	–2	–1	0	1	2	3	4
y	4	1	0	1	4	9	16



x

Let the tangent be drawn at point P (as shown). The gradient of the tangent is

$$\frac{RQ}{PQ} = \frac{\text{increase in y}}{\text{increase in } x}$$

$$= \frac{\delta y}{\delta x}$$

$$= \frac{(x+\delta x)^2 - x^2}{x + \delta x - x}$$

$$= \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$

$$= 2x + \delta x$$

In the limit as $\delta x \to 0$

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, thus

$$\frac{dy}{dx} = 2x$$

i.e. the gradient $= 2x$

The gradient is also referred to as the rate of change.

$\frac{dy}{dx}$ is in fact called the derivative of y with respect to x

$\frac{dy}{dx}$ can be written as y' or $\frac{df(x)}{dx}$ or $f'(x)$

13.3 Basic Rule Of Differentiation

Generally, if y= x^n, then $\frac{dy}{dx} = nx^{n-1}$

i.e. multiply x by its original index and raise x to one less than its original index.

Note:

Let c & k be constants

(i) if $y = cx^n$, $\frac{dy}{dx} = nx^{n-1}$

(ii) $\frac{dc}{dx} = 0$, since $c = cx^0$, $\frac{dc}{dx} = 0.cx^{0-1} = 0$

(iii) if $y = x^n + c$, $\frac{dy}{dx} = nx^{n-1}$

(iv) if $y = cx^n + kx^m + x^r$

$\frac{dy}{dx} = ncx^{n-1} + mkx^{m-1} + rx^{r-1}$

i.e. the derivative of sum of functions is the sum of the derivative of each of the functions.

(v) if $y = (cx^n)(kx^m)$

$= ck\, x^{n+m}$

$\frac{dy}{dx} = ck(n+m)x^{n+m-1}$

i.e. if y is the product of two functions, multiply out the functions and then differentiate or you can apply the product rule. This will be discussed later.

(vi) if $y = (a+bx)^n$, $\frac{dy}{dx} = nb(a+bx)^{n-1}$

Note that this is an example of function of a function i.e.

Let $u = a + bx$, then $y = u^n$, $du/dx = b$, and $dy/du = nu^{n-1}$

$dy/dx = (dy/du).(du/dx) = (nu^{n-1})b = nbu^{n-1} = nb\,(a + bx)^{n-1}$

(vii) if $y = e^{cx}$, $\dfrac{dy}{dx} = ce^{cx}$

Example 13:.1

Find the derivative of each of the following functions:

(a) $2x^2 + 5x$

(b) $4x^3 - 7x^2 + 6x - 14$

(c) $(5x+2)(4x-9)$

(d) $3e^{4x} - 8x^6$,

(e) $(5x + 13)^7$

(f) $\dfrac{1}{(6x - 20)^4}$

Solutions

(a) Let $y = 2x^2 + 5x$

$$\frac{dy}{dx} = (2)2x^{2-1} + (1)5x^{1-1}$$
$$= 4x + 5$$

(b) $y = 4x^3 - 7x^2 + 6x - 14$

$$\frac{dy}{dx} = (3)4x^{3-1} - (2)7x^{2-1} + (1)6x^{1-1} - (0)14x^{0-1}$$
$$= 12x^2 - 14x + 6$$

(c) let $y = (5x + 2)(4x - 9)$

$$= 20x^2 - 37x - 18$$

$$\frac{dy}{dx} = 40x - 37$$

(d) let $y = 3e^{4x} - 8x^6$

$$\frac{dy}{dx} = (4)3e^{4x} - (6)8x^{6-1}$$
$$= 12e^{4x} - 48x^5$$

(e) let $y = (5x + 13)^7$

$$\frac{dy}{dx} = (7)(5)(5x + 13)^{7-1} = 35(5x + 13)^6$$

or, let $u = 5x + 13$

$$\therefore \frac{du}{dx} = 5$$

$\therefore\ y = u^7 \quad \Rightarrow \quad \frac{dy}{du} = 7u^6 = 7(5x + 13)^6$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 7(5x+13)^6 \cdot 5 = 35(5x+13)^6$$

(f) let $y = \frac{1}{(6x-20)^4} = (6x-20)^{-4}$

$\frac{dy}{dx} = (-4)6(6x-20)^{-4-1}$

$= -24(6x-20)^{-5}$

or let $u = 6x - 20 \Rightarrow \frac{du}{dx} = 6$

$\therefore\ y = u^{-4} \Rightarrow \frac{dy}{du} = -4u^{-5} = -4(6x - 20)^{-5}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = -4(6x - 20)^{-5}\ (6) = -24(6x - 20)^{-5}$

13.4 The Second Derivative

If $y = f(x)$, then $\frac{dy}{dx} = \frac{df(x)}{dx}$

If $\frac{dy}{dx}$ is differentiated again, we have the second derivative

i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ or $\frac{d^2\left[f(x)\right]}{dx^2}$, (read as d two y by d x squared) or (the squared y d x

squared). It can also be written as y″ or f″(x).

Example 13.2

Find the second derivative of each of the following functions:

(a) $16x^2 - 5x$

(b) $7x^3 + 2x^2 + 3x - 21$

(c) $(x+4)(11x+1)(3x-2)$

Solutions

(a) Let $y = 16x^2 - 5x$

$$\frac{dy}{dx} = 32x - 5$$

$$\therefore \frac{d^2y}{dx^2} = 32$$

(b) Let $y = 7x^3+2x^2+3x-21$

$$\frac{dy}{dx} = 21x^2+4x+3$$

$$\therefore \frac{d^2y}{dx^2} = 42x + 4$$

(c) $y = (x + 4)(11x + 1)(3x+2)$

$= (11x^2 + 45x + 4)(3x-2)$

$= 33x^3 + 113x^2 - 78x - 8$

$$\frac{dy}{dx} = 99x^2 + 226x - 78$$

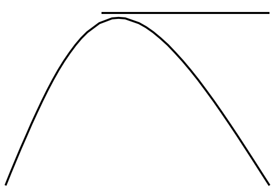
$$\therefore \frac{d^2y}{dx^2} = 198x + 226$$

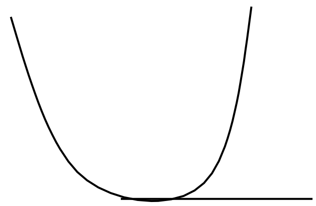
13.5 Maximum And Minimum Points, Marginal Functions And Elasticity

Maximum and Minimum Points

If the graph of y = f(x) is drawn, the turning points on the graph could indicate maximum point, minimum point, or point of inflexion.

For quadratic curves, we have the following curves:


Maximum point


Minimum point

However, these can be obtained without drawing the graph as follows:

Step 1 Find $\frac{dy}{dx}$

Step 2 Equate $\frac{dy}{dx} = 0$ and solve for x. This gives the turning point(s.)

Step 3 Find $\frac{d^2y}{dx^2}$

Step 4 Substitute the value(s) of x obtained in (step 2) above into $\frac{d^2y}{dx^2}$ obtained in Step 3

Step 5 If the result in (step 4) is negative, the point is a maximum point but if the result is positive, the point is a minimum point. If the result is

zero, it means it is neither a maximum nor a minimum point.
It is taken as a point of inflexion.

i.e. * $\frac{d^2y}{dx^2} < 0 \Rightarrow$ minimum point

* $\frac{d^2y}{dx^2} > 0 \Rightarrow$ minimum point

* $\frac{d^2y}{dx} = 0 \Rightarrow$ point of inflexion

Example 13.3

Find the maximum or minimum points for each of the following functions:

(a) $4x^2+9x-2$

(b) $12x - 3x^2 - 7$

(c) $5x-8.5x^2 - 4x^3 + 18$

Solutions

(a) Let $y = 4x^2 + 9x - 2$

$\frac{dy}{dx} = 8x + 9$

$\frac{dy}{dx} = 0 \Rightarrow 8x + 9 = 0,\ x = \frac{-9}{8}$

$\frac{d^2y}{dx^2} = 8 \Rightarrow x = \frac{-9}{8}$ is a minimum point since $\frac{d^2y}{dx^2}$ is positive (i.e. > 0)

(b) Let $y = 12x - 3x^2 - 7$

$\frac{dy}{dx} = 12 - 6x$

$\frac{dy}{dx} = 0 \Rightarrow 12 - 6x = 0,\ x = 2$

$\frac{d^2y}{dx^2} = -6 \Rightarrow x = 2$ is a maximum point since $\frac{d^2y}{dx^2}$ is negative (i.e. < 0)

s

(c) $y = 5x - 8.5x^2 - 4x^3 + 18$

$$\frac{dy}{dx} = 5 - 17x - 12x^2$$

$$\frac{dy}{dx} = 0 \Rightarrow 5 - 17x - 12x^2 = 0$$

i.e $12x^2 + 17x - 5 = 0$

factorizing, we get $(3x + 5)(4x - 1) = 0$

$$x = \frac{-5}{3} \text{ or } \frac{1}{4}$$

Now $\dfrac{d^2y}{dx^2} = -17 - 24x$

when $x = {}^{-5}/_{3}$, $\dfrac{d^2y}{dx^2} = 23$

$\Rightarrow x = {}^{-5}/_{3}$ is a maximum point.

when $x = {}^{1}/_{4}$,

$$\frac{d^2y}{dx^2} = -23$$

=> $x = {}^1/_4$ is a minimum point

Example 13.4

The gross annual profit of DUOYEJABS Ventures has been estimated to be $P(x) = 4x^3 - 10,800x^2 + 540,0000x$

where x is the number of products made and sold.

Calculate

(a) The number of products to be made for maximum and/ or minimum profit.

(b) The maximum and/or minimum profit.

Solutions

(a) $P(x) = 4x^3 - 10800x^2 + 5400000x$

$$\frac{dP(x)}{dx} = 12x^2 - 21600x + 540000$$

at the turning point, $\frac{dP(x)}{dx} = 0$

i.e. $12x^2 - 21600x + 5400000 = 0$

$x^2 - 1800x + 450000 = 0$

$x^2 - 1500x - 300x + 450000 = 0$

$x(x - 1500) - 300(x - 1500) = 0$

$(x-1500)(x-300) = 0$

i.e. $x = 1500$ or 300

$$\frac{d^2P(x)}{dx^2} = 24x - 21600$$

when $x = 1500$, $\frac{d^2P(x)}{dx^2} = 24(1500) - 21600$

$= 14400 \Rightarrow$ minimum profit.

when $x = 300$, $\frac{d^2P(x)}{dx^2} = 24(300) - 21600$

$= -13600 \Rightarrow$ maximum profit.

(b) $P(x) = 4x^3 - 10800x^2 + 5400000x$

when $x = 1500$

$P(x) = 4(1500)^3 - 10800(1500)^2 + 5400000(1500)$

$= -918 \times 10^7$, this is a loss

when x=300,

$$P(x) = 4(300)^3 – 10800(300)^2 + 5400000(300)$$

$$= 756 \times 10^6 \quad \text{is the maximum profit}$$

Though unusual, these results indicate that the profit decreases as the number of products made increases.

Marginal Functions

As before, let the cost function be C(x), revenue function be R(x) and profit function be P(x), where x is the number of items produced and sold, then

$$\frac{dC(x)}{dx} = C'(x) \text{ is the marginal cost function}$$

$$\frac{dR(x)}{dx} = R'(x) \text{ is the marginal revenue function}$$

and $$\frac{dP(x)}{dx} = P'(x) \text{ is the marginal profit function}$$

Now, profit is maximized or minimized when the first derivative of the marginal profit dP(x)/dx is less than zero or greater than zero respectively.
The same condition applies to the Maximisation or Minimisation of Revenue and Total Cost
It is also accepted that Profit is maximum when Marginal Revenue is equal to the Marginal Cost. That is

$$dR(x)/dx = dC(x)/dx$$

Example 13.5

The demand and cost functions of a COAWOS (Nig Ltd) are given as follows:

$p = 22500 - 3q^2$

$C = 5000 + 14400q$

where p is the price per item;

q is the quantity produced and

sold;and

C is the total cost.

You are required to calculate

(a) the marginal revenue

(b) the quantity and price for maximum revenue

(c) the marginal profit function and hence, the maximum profit

(d) the price for maximum profit.

Solutions

(a) $p = 22500 - 3q^2$

$\therefore$ Revenue $R = p.q$

$= 22500q - 3q^3$

Marginal Revenue $= \frac{dR}{dq}$,

$= 22,500 - 9q^2$

(b) Maximum Revenue is obtained as follows:

At the turning point,$dR/dq = 0$,then

$.9q^2 = 22500$

$q = \pm 50$

since q is the quantity produced and sold, it cannot be negative, $\therefore q = 50$

Now $\frac{d^2R}{dq^2} = -18q$

When q = 50, $\frac{d^2R}{dq^2} = -900$, this is less than zero which implies maximum revenue.

$\therefore$ Maximum Revenue $= 22500\,(50) - 3(50)^3$

$= 75{,}000$

$$\text{Price} = \frac{\text{Revenue}}{\text{Quantity}} = \frac{75000}{50} = 15{,}000$$

(c) $R = 22500q - 3q^3$

$C = 5000 + 14400q$

$\therefore \quad P = 22500q - 3q^3 - 5000 - 14400q$

$= 8100q - 3q^3 - 5000$

$\therefore$ Marginal profit function is

$\frac{dP}{dq} = 8100 - 9q^2$

and $\frac{dP}{dq} = 0$ at the turning point

i.e. $8100 - 9q^2 = 0$

$q^2 = 900$

$q = \pm 30$

As explained above, q cannot be negative, therefore q = 30

$\frac{d^2P}{dq^2} = -18q = -540$ when $q = 30 \Rightarrow$ maximum profit

$\therefore$ maximum profit occurs at q = 30

$\therefore$ maximum profit = P(30) = $8100 \times 30 - 3(30)^3 - 5000 = 157{,}000$

(d) Revenue when q = 30 is

$22500(30) - 3(30)^3 = 594000$

$\therefore$ price $= \frac{594000}{30} = 19{,}800$ when profit is maximum

It should be observed that the price for maximum revenue is not the same as the price for maximum profit. This is due to the effect of the cost function.

13.6 Elasticity

The point of elasticity (or just elasticity) of the function $y = f(x)$ at the point x is the ratio of the relative change in y (i.e. the dependent variable), to the relative change in x (i.e. the independent variable).

Elasticity of y with respect to x is given by

$$E_{y/x} = \frac{relative\ change\ in\ y}{relative\ change\ in\ x}$$

$$= \frac{\frac{dy}{y}}{\frac{dx}{x}}$$

$$= \frac{xdy}{ydx}$$

Elasticity is dimensionless i.e. it has no unit, and is represented by η .

Elasticity of demand

Elasticities are, most of the time, used in measuring the responsiveness of demand or supply to changes in prices or demand.

In other words, a "business set up" will be faced with the problem of deciding whether to increase the price or not. The demand function will always show that an increase in price will cause a decrease in sales (revenue). So, to what extent can we go? Will a very small increase in price lead to increase or decrease in revenue?

Price elasticity of demand helps in answering these questions.
The price elasticity of demand is defined by

$$\eta = -\frac{\frac{p}{q}}{\frac{dp}{dq}} \qquad \text{(- sign is to make } \eta \text{ positive)}$$

$$\eta = -\left(\frac{p}{q}\right)\left(\frac{dq}{dp}\right)$$

Note:

(a) A demand curve is

(i) elastic if $|\eta| > 1$

(ii) of unit elasticity if $|\eta| = 1$

(iii) inelastic if $|\eta| < 1$

(b) If (i) $\eta > 1$, an increase in price will cause a decrease in revenue.

(ii) $\eta < 1$, an increase in price will cause an increase in revenue.

Example 13.6

The demand function for a certain type of item is

$$p = 40 - 0.03\sqrt{q}$$

Investigate the effect of price increase when

(a) 3,600 (b) 6,400

items are demanded

Solutions

$$p = 40 - 0.03\sqrt{q}$$

$$\frac{dp}{dq} = -0.03\frac{1}{2}q^{\frac{1}{2}}$$

$$\frac{dp}{dq} = \frac{-0.03}{2\sqrt{q}}$$

(a) when $q = 3{,}600$

$$\frac{dp}{dq} = \frac{-0.03}{2\sqrt{3{,}600}} = -0.00025$$

$$p = 40 - 0.03\sqrt{3{,}600} = 38.2\,.$$

$$\eta = -\left(\frac{p}{q}\right)\left(\frac{dq}{dp}\right)$$

$$\eta = -\left(\frac{38.2}{3{,}600}\right)(-0.00025)$$

$$\eta = \frac{38.2}{9}$$

$$\eta = 4.24$$

Since $\eta > 1$, there will be a decrease in revenue if there is an increase in price when 3,600 items are demanded.

(b) when $q = 6{,}400$

$$\frac{dp}{dq} = \frac{-0.03}{2\sqrt{6{,}400}} = -0.0001875$$

$$p = 40 - 0.03\sqrt{6{,}400} = 37.6\,.$$

$$\eta = -\left(\frac{p}{q}\right)\left(\frac{dq}{dp}\right)$$

$$\eta = -\left(\frac{37.6}{6,400}\right)(-0.0001875)$$

$$\eta = \frac{37.6}{1.2}$$

$$\eta = 31.33$$

Since $\eta > 1$, there will be a decrease in revenue if there is an increase in price when 6,400 items are demanded.

13.7 Integration

When $y = f(x)$, the derivative is $\frac{dy}{dx}$, the process of obtaining y back from $\frac{dy}{dx}$ will involve reversing what has been done to y i.e. anti-differentiation.

The process of reversing differentiation is called integration. i.e. Integration is the reverse of differentiation. e.g. if $y = x^2$, then $\frac{dy}{dx} = 2x$

For differentiation, we multiply by the old or the given index of x and decrease this index by 1.

Reversing this procedure means going from the end to the beginning and doing the opposite. Increasing the index by 1 gives $2x^{1+1} = 2x^2$ and dividing by the new index gives $\frac{2x^2}{2} = x^2$,which is the original function differentiated.

The integral sign is $\int$ (i.e. elongated S)

$$\therefore \quad \int \frac{dy}{dx} dx = y$$

i.e. $\int 2x dx = x^2$

Check it out

Differentiation:	decrease	old and	multiply
Integration:	increase	new and	divide

It is clear from the above that integration is the reverse of differentiation.

This integral is incomplete because x^2, $x^2 + 8$, $x^2 + 400$, $x^2 + c$ all have the same derivative $2x$ and the integral will give us just x^2 for each one of them. This is incomplete. For this reason, there is the need to add a constant of integration i.e.

$\int 2x\,dx = x^2 + c$ where c is a constant that can take any value

13.8 RULES OF INTEGRATION

Generally, if $y = x^n$ then $\int y\,dx = \int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (provided\ n \neq -1)$

Note:

If a, b, c and k are constants, then

(i) $$\int ax^n dx = a\int x^n dx = \frac{ax^{n+1}}{n+1} + c \quad (n \neq -1)$$

(ii) $$\int (ax^n + bx^m + kx^r)dx = \frac{ax^{n+1}}{n+1} + \frac{bx^{m+1}}{m+1} + \frac{kx^{r+1}}{r+1} + c \quad (n \neq -1, m \neq -1, k \neq -1)$$

(iii) $$\int (x^m)(x^n)dx = \int x^{m+n} dx = \frac{x^{m+n+1}}{m+n+1} + c \quad (\textit{provided } m+n \neq -1)$$

(iv) $$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

(v) $$\int e^{kx} dx = \frac{e^{kx}}{k} + c$$

(vi) $$\int a dx = \int ax^0 dx = \frac{ax^{0+1}}{0+1} + c = ax + c$$

Example 11.7

Integrate each of the following functions with respect to x:

(a) $2x^3 - 7x^5 + 17$

(b) $(5x^2 + 6x)(4x - 15)$

(c) $(15x + 9)^4$

(d) $\dfrac{1}{(11x+10)^{12}}$

(e) $12e^{1.5x}$

Solutions

(a) $\int(2x^3 - 7x^5 + 17)dx$

$= \int 2x^3 dx - \int 7x^5 dx + \int 17 dx$

$= \dfrac{2x^{3+1}}{3+1} - \dfrac{7x^{5+1}}{5+1} + \dfrac{17x^{0+1}}{0+1} + c$

$= \dfrac{x^4}{2} - \dfrac{7x^6}{6} + 17x + c$

(b) $\int (5x^2 + 6x)(4x - 15)dx$

$= \int(20x^3 - 51x^2 - 90x)dx$

$= \dfrac{20x^4}{4} - \dfrac{51x^3}{3} - \dfrac{90x^2}{2} + c$

$= 5x^4 - 17x^3 - 45x^2 + c$

(c) $\int (15x+9)^4\, dx = \dfrac{(15x+9)^{4+1}}{15\,(4+1)} + c$

$= \dfrac{(15x+9)^5}{75} + c$

(d) $\displaystyle\int \frac{1}{(11x+10)^{12}}\, dx = \int (11x+10)^{-12}\, dx$

$= \dfrac{(11x+10)^{-12+1}}{11(-12+1)} + c$

$= \dfrac{(11x+10)^{-11}}{-132} + c$

(e) $\int 8e^{1.5x}dx = 8\int e^{1.5x}dx = \dfrac{8}{1.5}e^{1.5x} + c$

$= \dfrac{16}{3}e^{1.5x} + c$

13.9 INDEFINITE AND DEFINITE INTEGRALS

Indefinite integral is when the integration is done without any limits i.e. $\int f(x)\,dx$ is an indefinite integral.

In fact, all the integrals we have discussed so far are indefinite integrals.

(c) **Definite integral** is when the integration is done within given limits

e.g. $\int_a^b f(x)dx$, b > a, is a definite integral

In this case, we talk about evaluating the integral and we obtain a number as our result, by subtracting the value of the integral for **a** (the lower limit) from the value of the integral for **b** (the upper limit)

Example 11.8

Evaluate each of the following integrals:

(a) $\int_2^5 15x^2 dx$

(b) $\int_{10}^{50} (3x^2 + 4x+1)dx$

(c) $\int_{-\infty}^{0} 100e^{0.05x}\, dx$

Solutions

(a) $\int_2^5 15x^2 dx = \left[\dfrac{15x^{2+1}}{2+1} + c\right]_2^5$

$= \left[5x^3 + c\right]_2^5$

$= [5(5)^3 + c] - [5(2)^3 + c]$

$= 625 + c - 40 - c$

$=585$

Observe that the constant of integration "c" cancels out; hence it is not necessary to add the constant of integration when evaluating definite integrals.

(b) $\int_{10}^{50} (3x^2 + 4x + 1)dx = [x^3 + 2x^2 + x]_{10}^{50}$

$= [(50)^3 + 2(50)^2 + 50] – [(10)^3 + 2(10)^2 + 10]$

$= 125{,}000 + 5{,}000 + 50 – 1000 – 200 – 10$

$=$ **128,840**

(c) $\int_{-\infty}^{0} 100e^{0.05x}dx = \left[\frac{100e^{0.05x}}{0.05}\right]_{-\infty}^{0}$

$= \frac{100e^0}{0.05} - \frac{100^{-\infty}}{0.05}$

$= 200$ (note that $e^0 = 1$, $e^{-\infty} = 0$)

13.10 TOTALS FROM MARGINALS

The idea of integration as the reverse of differentiation is used to reverse marginal functions to obtain the total (original) functions.

Recall the marginal functions:

$\frac{dR(x)}{dx}$ is the marginal revenue

hence $\int_a^b \frac{dR(x)}{dx}dx = R(b) – R(a)$ is the total revenue

- Note that when no quantity is produced, nothing will be sold, hence no revenue. It means when $x = 0$, $R(a) = 0$

$\therefore \int \frac{dR(x)}{dx}dx = R(x)$ always

also $\frac{dC(x)}{dx}$ is the marginal cost

$\therefore \int_m^n \frac{dC(x)}{dx}dx = C(n) – C(m)$ is the total cost

Note that $\int \frac{dC(x)}{dx}dx = C(x) + k$ where k is a constant

and $\frac{dP(x)}{dx}$ is the marginal profit

$$\therefore \int_{x_1}^{x_2} \frac{dP(x)}{dx}dx = P(x_2) - P(x_1) \text{ is the total profit}$$

Example 13.9

The marginal revenue function of a production company is given by

$3x^2 - 5x - 50$

while the marginal cost function is

$6x^2 - 500x + 400$

where x is the number of items produced and sold.

Calculate

(d) the number of items that will yield maximum or minimum revenue.

(e) the total revenue if 200 items are produced and sold

(f) the total profit for the 200 items

Solutions

(a) $\frac{dR(x)}{dx} = 3x^2 - 5x - 50 = 0$

i.e. $(3x + 10)(x - 5) = 0$

$\boldsymbol{x = -10/3}$ or 5

x can not be negative since it is the number of items produced and sold. Therefore, $x = 5$

$\frac{d^2R}{dx^2} = 6x - 5$

when $x = 5$, $\frac{d^2R}{dx^2} = 30 - 5 = 25 > 0 \Rightarrow$ minimum

i.e. minimum revenue is achievable when 5 items are produced.

(b) total revenue when 200 items are produced is obtained as follows:

$$\int_0^{200} \frac{dR(x)}{dx}dx = \int_0^{200} (3x^2 - 5x - 50)dx$$

$$= \left[x^3 - \frac{5x^2}{2} - 50x\right]_0^{200}$$

- The lower limit MUST be ZERO, because when $x = 0$ (i.e nothing is produced), there will be no revenue

$= (200)^3 - \frac{5(200)^2}{2} - 50\,(200) - 0$

$= 7{,}890{,}000$

(c) $\frac{dP(x)}{dx} = \frac{dR(x)}{dx} - \frac{dC(x)}{dx}$

$= 3x^2 - 5x - 50 - (6x^2 - 500x + 400)$

$= -3x^2 + 495x - 450$

$\therefore$ total profit for 200 items is

$$\int_0^{200} \frac{dP(x)}{dx}dx = \int_0^{200} (-3x^2 + 495x - 450)dx$$

$$= \left[x^3 - \frac{495x^2}{2} - 450x \right]_0^{200}$$

$$= (-200)^3 + \frac{495}{2}(200)^2 - 450(200) - 0$$

$$= 1,810,000$$

13.11 Consumers' Surplus And Producers' Surplus Consumers' surplus

Recall that a demand function represents the quantities of a commodity that will be purchased at various prices.

Some of the time, some consumers may be willing to pay more than the fixed price of a commodity. Such consumers gain when the commodity is purchased, since they pay less than what they are willing to pay.

The total consumer gain is known as consumers' surplus.

If the price is denoted by y_0 and the demand by x_0, then

$$\text{Consumers' surplus} = \int_0^{x_0} f(x)dx - x_0 y_0$$

where y = f(x) is the demand function

Also it could be defined as

$$\text{Consumers' surplus} = \int_{y_0}^{y_1} g(y)dy$$

where x = g(y) is the demand function and y_1 is the value of y when $x = 0$

Either of these expressions could be used to determine the **consumers'** surplus.

Example 13.10

If the demand function for a commodity is y = 128 + 5x – 2x^2, find the

Consumers' surplus when

(a) $x_o = 5$

(b) $y_o = 40$

Solutions

(a) $y = 128 + 5x - 2x^2$

When $x_o = 5$, $y_o = 128 + 25 - 50 = 103$

$$\text{Consumers' surplus} = \int_0^5 (128 + 5x - 2x^2)\,dx - x_0\, y_0$$

$$= \left[128x + \frac{5}{2}x^2 - \frac{2x^3}{3}\right]_0^5 - 5(103)$$

$$= 640 + \frac{125}{2} - \frac{250}{3} - 0 - 515$$

$$= 104.17$$

(b) $y = 128 + 5x - 2x^2$

when $y_o = 40$, $40 = 128 + 5x - 2x^2$

i.e $2x^2 - 5x - 88 = 0$

i.e $2x^2 - 16x + 11x - 88 = 0$

$\Longrightarrow$ $2x(x-8) + 11(x - 8) = 0$

$(2x + 11)(x - 8) = 0$ i.e. $x =$ **-11/2** or 8

As explained earlier, x cannot be negative, $\therefore\ x_o = 8$

$$\text{Consumers' surplus} = \int_0^8 (128 + 5x - 2x^2)\,dx - x_0\, y_0$$

$$= \left[128x - \frac{5}{2}x^2 - \frac{2x^3}{3}\right]_0^8 - 8(40)$$

$$= 1024 + 160 - 341.33 - 320$$

$$= 522.67$$

Example 13.11

The demand function for a commodity is p= $\sqrt{25-q}$, where p represents price and q is the quantity demanded. If $q_o = 9$, show that

$$\int_0^{q_0} f(q)dq - q_0 p_0 = \int_{p_0}^{p_1} g(p)dp$$

where p_1 is the value of p when q=0

Solution

$$p = \sqrt{25-q}, \quad \text{If } q_0 = 9$$

$$p_0 = \sqrt{25-q} = \sqrt{25-9} = \sqrt{16} = 4$$

$$\therefore \quad \int_0^{q_0} f(q)dq = \int_0^{9} (25-q)^{1/2} dq - 9(4)$$

$$= \left[-\frac{(25-q)^{3/2}}{3/2} \right]_0^9 - 36$$

$$= -\frac{2}{3}\left[(16)^{3/2} - (25)^{3/2} \right] - 36$$

$$= 4.67$$

Now $\quad p = \sqrt{25-q}$

$p^2 = 25 - q$

$q = 25 - p^2$

when q = 0, $p_1 = 5$, $p_o = 4$ (as before)

$$\therefore \int_{p_0}^{p_1} g(p)dp = \int_4^5 (25 - p^2)dp$$

$$= \left[25p - \frac{p^3}{3} \right]_4^5$$

$$= \left(125 - \frac{125}{3} \right) - \left(100 - \frac{64}{3} \right)$$

$$= 4.67$$

i.e. $\int_0^{q_0} f(q)dq - q_0 p_0 = \int_{p_0}^{p_1} q(p)dq$ ***=4.67 QED.***

Producers' Surplus

Recall that the supply function represents the quantities of a commodity that will be supplied at various prices.

At times, some producers may be willing to supply a commodity below its fixed price. When the commodity is supplied, such producers' gain since they will supply at a price higher than the fixed price

The total producers' gain is referred to as producers' surplus.

Let the fixed price be y_o and the supply be $\boldsymbol{x_o}$, **then**,

$$\text{Producers' surplus} = x_o y_o - \int_0^{x_0} f(x)dx$$

where the supply function is f(x).

It could also be expressed as

$$\text{Producers' surplus} = \int_{y_1}^{y_0} g(y)dy$$

where the supply function is g(y) and y_1 is the value of y when $x = 0$

Either expression could be used to determine the producers' surplus.

Example 13.12

If the supply function for a commodity is

$y = (x+3)^2$, find the producers' surplus when

a. $y_o = 16$

b. $x_o = 5$

Solution

(a) $y = (x+3)^2$

$x + 3 = \sqrt{y}$

$x = \sqrt{y} - 3$

When $x = 0$, y = 9

$$\text{Producers' surplus} = \int_{y_1}^{y_0} g(y)dy = \int_9^{16} (y^{1/2} - 3)dy$$

$$= \left[\frac{2}{3}y^{3/2} - 3y\right]_9^{16}$$

$$= \left(\frac{128}{3} - 48\right) - \left(\frac{54}{3} - 27\right) = \frac{11}{3} \text{ or } 3.67$$

Or

$y_o = 16$, $x_o = \sqrt{y} - 3 = \sqrt{16} - 3 = 4 - 3 = 1$

$$\text{Producers' surplus} = x_o y_o - \int_0^{x_0} f(x)dx = 1(16) - \int_0^1 (x+3)^2\,dx$$

$$= 16 - \left[\frac{1}{3}(x+3)^3\right]_0^1$$

$$= 3.67$$

(b) $y = (x + 3)^2$

when $x_o = 5$, $y_o = (5 + 3)^2 = 64$

$$\therefore \text{producers' surplus} = x_o y_o - \int_0^{x_0} f(x)dx = 5(64) - \int_0^5 (x+3)^2\,dx$$

$$= 320 - \left[\frac{(x+3)^3}{3}\right]_0^5$$

$$= 149.33$$

Note:

If both the demand and supply functions for a commodity are given, the idea of equilibrium could be applied.

If the supply function S(q) and demand function D(q) are given, then the idea of equilibrium will be applied to obtain the equilibrium price p_0 and quantity q_0.

Then, we have

(a) producers' surplus = $\int_0^{q_0} \{p_0 - S(q)\}dq$ and

(b) consumers' surplus = $\int_0^{q_0} \{D(q) - p_0\}dq$.

Example 13.13

The supply function S(q) and the demand function D(q) of a commodity are given by

$S(q) = 400 + 15q$

$D(q) = 900 - 5q$

where q is the quantity

Calculate

(a) the equilibrium quantity;and.

(b) the equilibrium price

(c) the producers' surplus

(d) the consumers' surplus

Solutions:

(a) at equilibrium

S(q) = D(q)

i.e. 400 + 15q = 900 – 5q

20q = 500

q = 25 which is the equilibrium quantity.

(b) to obtain equilibrium price, we substitute q = 25 into either the supply or demand functions, thus:

p = 400 + 15q

= 400 + 15(25)

= 775

or

p = 900 – 5q

= 900 – 5(25)

= 775

(c) producers' surplus $= \int_0^{q_0} \{p_0 - S(q)\}\,dq$

$$= \int_0^{25} \{775 - (400 + 15q)\}\,dq$$

$$= \int_0^{25} \{375 - 15q)\}\,dq$$

$$= \left[375q - 15q^2/2\right]_0^{25}$$

$$= 375(25) - \frac{15(25)^2}{2}$$

$$= 9375 - \frac{9375}{2}$$

$$= 4687.5$$

(d) Consumers' surplus $= \int_0^{q_0} \{D(q) - p_0\} dq$

$$= \int_0^{25} \{900 - 5q - 775\} dq$$

$$= \int_0^{25} \{125 - 5q\} dq$$

$$= \left[125q - 5q^2/2\right]_0^{25}$$

$$= 125(25) - 5(^{25}/_{2})^2$$

$$= 3125 - \frac{3125}{2}$$

$$= 1562.5$$

13.12 Chapter Summary

Differentiation and Integration (which is the reverse of differentiation) were treated extensively. Marginal functions and elasticity of demand were also discussed. These concepts were thereafter applied to Business and Economic problems.

13.13 Multiple-Choice And Short-Answer Questions

1. The second derivative of a function is used to determine its

   A) Turning point(s)

   B) Maximum point only

   C) Minimum point only

   D) Point of inflection only

   E) Minimum or maximum point or poin**t of** inflection**.**

2. The demand function for a certain item is, $p = 10 - \sqrt{0.04 q}$. Calculate the elasticity of demand when 1,600 items are demanded.

   A. 8.4

   B. –8.4

   C. 10.5

   D. –10.5

   E. 16

3. If the marginal revenue function is $x^2 - 5x$, what is the total revenue when $x = 10$

   A. 50

   B. 833.33

   C. 83

   D. 83.33

   E. 500

4. Given that the profit function is $2x^2 - 8x$, the minimum profit is

   A . 8

   B. 10

   C. 12

   D. 14

   E. 16

5. If $y = (8 + 5x)^3$, then $\frac{dy}{dx}$ = ……….

6. The maximum cost is obtainable when the first derivative of the marginal cost function is ………..

7. A demand is elastic if the elasticity (η) is such that ……………

8. Consumers' surplus arises when consumers pay.................................... than what they are ……………………. to pay

9. If the marginal profit function is $3x^2 - 2x$, then the total profit when $x = 5$ is …………

10. The equilibrium price is obtained at the point of intersection of .... and ………… functions

Answers

1. E

   Recall that the first derivative is zero at the turning point and there are three possibilities: if the second derivative is less than zero, the turning point is a maximum, if it is more than zero, the turning point is **minimum** and if it is zero, the turning point is a point of inflection.

2. $p = 10 - 0.04\sqrt{q} = 10 - 0.04q^{½}$

$$\frac{dp}{dq} = -0.04(½)\, q^{-½}$$

$$= \frac{-0.04}{2\sqrt{q}}$$

when $q = 1600$, $\frac{dp}{dq} = \frac{-0.04}{2\sqrt{1600}}$

$$= -0.0005$$

$$p = 10 - 0.04\sqrt{1600}$$

$$= 8.4$$

$\therefore \quad \eta = ({}^{-p}/_{q})\,({}^{dq}/_{dp})$

$$= \frac{-8.4}{1600(-0.0005)}$$

$$= 10.5 \qquad \text{(C)}$$

3. $\frac{dR(x)}{dx} = x^2 - 5x$

$$R(x) = \int_0^{10} \left(x^2 - 5x\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2}\right]_0^{10}$$

$$= 83.33$$

4. $P(x) = 2x^2 - 8x$

$\frac{dP(x)}{dx} = 4x - 8 = 0$ at turning points

i.e. $4x - 8 = 0$, $x = 2$

$\frac{d^2P(x)}{dx^2} = 4$ which is positive and hence minimum profit

$\therefore$ minimum profit $= 2(2^2) - 8(2) = -8$(i.e loss) (C)

5. $y = (8 + 5x)^3$

Let $u = 8 + 5x$, $\frac{du}{dx} = 5$

then $y = u^3$, $\frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$= (3u^2)(5)$$

$$= 15(8 + 5x)^2$$

6. **less than zero**

7. $|\eta| > 1$

8. Less, willing (in that order)

9. $\frac{dP(x)}{dx} = 3x^2 - 2x,$

$$P(x) = \int_0^5 \left(3x^2 - 2x\right)dx = \left[x^3 - x^2\right]_0^5 = 100$$

10. Demand, Supply (or vice-versa)

SECTION C

OPERATIONS RESEARCH

CHAPTER FOURTEEN

INTRODUCTION TO OPERATIONS RESEARCH (OR)

Chapter contents

a) Introduction.

b) Main stages of Operations Research,

c) Relevance of Operations Research in business.

Objectives

At the end of this chapter, readers should be able to understand the

a) concept of OR;

b) major steps in OR; and

c) various situations where OR can be applied.

14.1 Introduction

Decision-making is a day-to-day activity.

Individuals, societies, government, business organizations and so on do make decisions. In all these cases, decisions are made in order to benefit the decision makers and in most cases, those that the decisions will affect.

A problem would have arisen before a decision is made. In fact, decision-making is a response to an identified problem, which arises as a result of discrepancy between existing conditions and the organisation's set objectives.

When a decision is to be made, a lot of factors have to be taken into consideration in order to ensure that the decision is not only the best under the existing conditions but also for the nearest future.

Simple as it may look, decision-making is not an easy task to perform. It is a complex issue. The manager of a business organisation will have to decide a course of action to be taken when confronted with a problem. In deciding on this course of action, he has to take some risks since there is some uncertainty (however little it may be) about the consequences of such a decision, he will like to reduce such risks to the barest minimum.

Thus, there is the need to find a method that will assist in making decisions that are objective and scientific. This method is called operations research.

14.2 Main stages of OR

The main stages involved in OR are:

(a) **identification of problems and objectives**.

The problems for which decisions are being sought must first of all be defined and the objectives clearly spelt out.

(b) identification of variables

It is very important to identify both the controllable or decision variables and the uncontrollable variables of the system.

The constraints on the variables and the system should be taken into recognition.

The "bounds" of the system and the options open must also be established.

(c) construction of a model

This is the central aspect of an OR project.

A model has to be used because it would be impossible to experiment with the real life situations.

A suitable model to represent the system must first of all be established. Such a model should specify quantitative relationships for the objective and constraints of the problem in terms of controllable variables.

It must also be decided, based on the available information, whether the system is to be treated as a deterministic or a probabilistic one.

The model can be a mathematical one or a heuristic one.

Mathematical models are mostly used for OR

A major assumption is that all the relevant variables are quantifiable thus the model will be a mathematical function that describes the system under study.

Some mathematical models are:

i) **Allocation models:** these are concerned with sharing scarce resources among various competing activities. Linear programming, transportation and assignment are some examples of the allocation models.

ii) Inventory models: these deal with policies of holding stocks of items of finished goods, ordering quantities and re- order level.

iii) **Queuing models:** these are concerned with arrival at and departures from service points, and consequent development of queues of customers waiting for service.

iv) **Replacement models:** these are concerned with determination of an optimal policy for replacing “failed” items

v) **Simulation models**- simulation means to imitate or feign an original situation and so simulation models are based on probabilities of certain input values taking on (i.e. imitating) a particular value.
Random numbers are used most of the time for this type of model.

(i) and (ii) are treated in details in subsequent chapters but (iii) - (v) are beyond the scope of this study pack.

Heuristic models are essentially models that employ some intuitive rules to generate new strategies, which hopefully will yield improved solutions.

- **Solution of the model**
  Once the model has been constructed, various mathematical methods can then be used to manipulate the model to obtain a solution.

  If analytic solutions are possible we can then talk of optimal solutions but if simulation or heuristic models are used, we can only talk about "good" solution.

- **Testing the model**
  It is essential to validate the model and its solutions to determine if the model can reliably predict the actual system"s performance. It must be ensured that the model reacts to change in the same way as the real system. The past data available for the system may be used to test the validity of the model.

  The model for a system may be considered to be valid, if under similar input conditions, it can reproduce to a reasonable extent, the past performance of the system.

- **Implementation**
  It is important to have those who will implement the result obtained on the team of the operations research study. If otherwise, the team should be on hand to give any necessary advice in case any difficulties are encountered in the course of the implementation. A set of operating instructions manual may be necessary.

14.3 Relevance Of OR In Business

OR has a very wide area of application in business, engineering, industry, government and science.

OR will always be relevant in any situation where resources do not merge the needs or requirements. Even where the resources are enough, there is still the need to allocate such resources in the best way (an optimal way).

a) In production planning, OR may be used to allocate various materials to production schedules in an optimal way. In transportation problems, OR may be used to decide on best routes (i.e. routes with minimum cost).

b) An accountant may apply OR to investment decisions where the fund available is not sufficient for all available projects- capital rationing.

Also, OR can be applied by an accountant in every situation for cost benefit analysis.

14.4 Chapter Summary

The concept of Operations Research (OR) has been treated. Major steps necessary in OR and real life application were also discussed.

14.5 Multiple-Choice And Short-Answer Questions

1. Decision-making is important in order to
   A. Make profit for a business
   B. Solve an identified problem
   C. Please the customers
   D. Perform a task
   E. Please the management

2. A simulation model is a
   A. Mathematical model
   B. Probabilistic model
   C. Non – mathematical model
   D. Constant model
   E. Non-probabilistic model

3. Operations Research is relevant because
   A. Resources do not always merge the needs
   B. Resources have to be allocated
   C. All activities have to be taken care of
   D. In any operation, research is important
   E. Resources have to be allocated in an optimal way

4. One of the following is not a mathematical model
   A. Allocation model
   B. Inventory model
   C. Queueing model
   D. Additive model
   E. Replacement model

5. Operations Research is a method which assists in making decisions that are …..........…. and……………..

6. A model should specify quantitative relationships for the ………… and …………….. of the problem in terms of controllable variables.

7. In the financial circle, OR is used for..rationing

8. In OR, transportation problem can also be referred to as an............problem

9. OR assists to reduce the................involved in decision making

10. OR will always be relevant in any situation where ………………. do not merge the needs

Answers

1. B
2. A
3. E
4. D
5. Objective, Scientific (or vice-versa)
6. Objective, constraints
7. Capital
8. Allocation
9. Risks
10. Resources

CHAPTER FIFTEEN

LINEAR PROGRAMMING (LP)

Chapter contents

a) Introduction.
b) Meaning and Nature of LP,
c) Concepts and Notations of LP.
d) Graphical Solutions of LP.
e) Simplex Method solution of LP.
f) Duality Problem.

Objective

At the end of this chapter, readers should be able to understand the

a) basic concepts of LP;
b) meaning of objective function and constraints in LP;
c) concept of optimal solution to a LP problem
d) assumptions underlying the LP
e) formulation of LP problems;
f) solution of LP problems using Simplex method; and
g) solution of duality problems of L.P.

15.1 Introduction

Meaning and Nature of Linear Programming

As mentioned in chapter 14, decision concerning allocation of scarce or limited resources to competing activities is a major one which managers of business organizations or executive directors of companies will have to take from time to time. These resources could be machines, materials, men and money (4M‟s) or a combination of two or more of them.

The main objective of a manager will be to use the limited resources to the best advantage. The linear programming model can assist in achieving this very important objective.

Linear Programming as a Resource Allocation Tool

Linear Programming can be defined as a mathematical tec2nique which is concerned with the allocation of limited resources to competing activities. That is, it is a resource allocation tool.

It is mainly concerned with **optimization** of an **objective function** (e.g. to maximize profit or minimize cost) given some constraints (e.g. machine hours, labour hour, quantity of materials).

Thus, the linear programming consists of an objective function and certain constraints based on the limited resources.

15.2 Concepts And Notations Of Linear Programming Assumptions of Linear Programming (LP)

The two major assumptions are:

a) **linearity** of the objective function and the constraints;and

This guarantees the **additivity** and **divisibility** of the functions involved.

b) **non – negativity** of the decision variables

i.e negative quantities of an activity are not possible.

Note that linear programming consists of two words: the linear part as stated above; and the „programming" part, which is the solution method.

Formulation of Linear Programming Problems

The main steps involved in the formulation of linear programming problems are as follows:

a) define all variables and their units;

b) determine the objective of the problem – either to maximize or minimize the objective function;

c) express the objective function mathematically;and

d) express each constraint mathematically including the non-negativity constraints, which are the same for all linear programme problems.

The constraints are always expressed as inequalities.

15.3

15.3 GRAPHICAL SOLUTION OF LINEAR PROGRAMMING PROBLEMS

There are two methods of solving linear programming problems viz:

a) graphical method;and
b) simplex method.

Computer packages are also available for this method and it is particularly useful when the decision variables are more than two.

The graphical method is applicable only when problem involves two decision variables

The necessary steps are as follows:

a) turn the inequalities into equalities;
b) draw the lines representing the equations on the same axes;
c) identify the region where each constraint is satisfied;
d) identify the region where all the constraints are simultaneously satisfied. This is called the **feasible region;**
e) find or read off the coordinates of the corner points of the boundary of the feasible region;
f) calculate the value of the objective function for each of the points by substituting each of the coordinates in the objective function;and
g) determine the optimal solution, which is the point with the highest value of the objective function for maximization problems or the point with the lowest value of the objective function for minimization problems.

Note that when there are more than two decision variables, the Simplex method (which is beyond the scope of this study pack) is used.

Illustration 13.1

$$\begin{aligned} \text{Maximize} \quad & 800x + 600y \\ \text{Subject to} \quad & 2x + y \leq 100 \\ & x + 2y \leq 120 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

Solution

Maximize $800x + 600y$ - this is the objective function

Subject to

$$\left.\begin{array}{r} 2x + y \leq 100 \\ 2x + 2y \leq 120 \\ x \geq 0 \\ y \geq 0 \end{array}\right\} \text{These are the constraints.}$$

Draw the graphs of

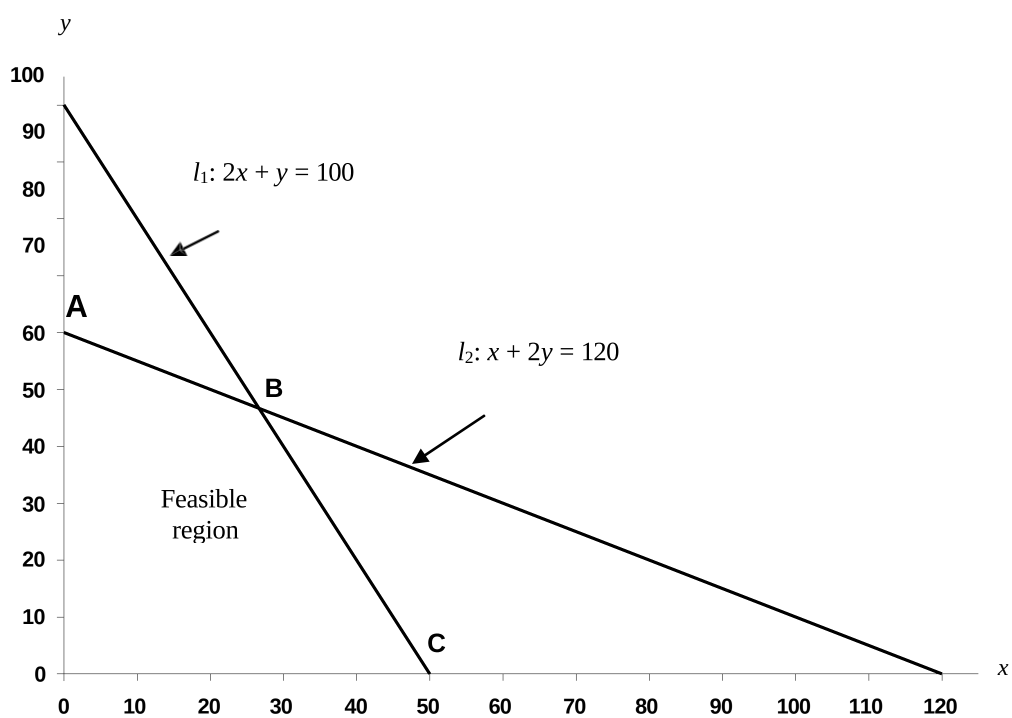
$$2x + y = 100$$

$$x + 2y = 120$$

on the same axes.

For line l_1: $2x + y = 100$, points are (0, 100) and (50, 0)

For line l_2: $x + 2y = 120$, points are (0, 60) and (120, 0)



- From the graph, corner points of the boundary of the feasible region are A, B, C, (as indicated) with coordinates A(0,60), B(26, 47), C(50,0)
- Obtain the value of the objective function for each of the points identified above.

Coordinates	Value of 800x + 600y (₦)
A(0,60),	800(0) + 600 (60) = (36,000)
B(26, 47)	800(26) + 600(47) = (49,000)
C(50,0)	800(50) + 600(0) = (40,000)

Coordinates of B (26, 47) give the highest value of the objective function (i.e. N49,000). Consequently the best (optimal) combination is x = 26 and y = 47.

Note that

i) the corner point B is just the point of intersection of the two lines and usually gives the optimal solution for the two constraints.

ii) if l_1 and l_2 are solved simultaneously, the coordinates of their point of intersection are

$$x = 26\frac{2}{3} \text{ and } y = 46\frac{2}{3}$$

Example 15.2

Maximization problem (Production Problem)

AWOSOYE furniture produces two types of chairs – executive and ordinary by using two machines M_I and M_{II}. The machine hours available per week on M_I and M_{II} are 100 and 120 respectively.

An executive chair requires 4 hours on M_I and 3 hours on M_{II}, while the requirements for an ordinary chair are 1.25 hours on M_I and 2 hours on M_{II}. The company has a standing order to supply 15 ordinary chairs a week.

Profit (contribution) on (from) an executive chair is ₵2,500 while profit on an ordinary chair is ₵1,000.

You are required to:

a) formulate the problem as a linear programming problem.

b) use graphical method to solve the linear programming problem.

Solution

Let x units of the executive chairs and y units of ordinary chairs be made.

Let P be the total profit (contribution).

a) The objective function is

$P = 2{,}500x + 1000y$ (2500 from an executive chair and 1000 from ordinary chair)

The constraints are:

$4x + 1.25y \leq 100$ (M_I constraint)

$3x + 2y \leq 120$ (M_{II} constraint)

$y \geq 15$ (standing order constraint)

$x \geq 0$ (Non-negativity constraint)

On the other hand, the information can be summarized first in tabular form, from where the objective function and the constraints will be obtained as follows:

Type of chair	Machine hours per unit		**Profit (¢)**
	M_I	**M_{II}**	
X	**4**	**3**	2500
Y	**1.25**	**2**	1000
Available machine hours	100	120	

The problem is

Maximize $P = 2500x + 1000y$

Subject to $4x + 1.25y \leq 100$-----------(l_1)

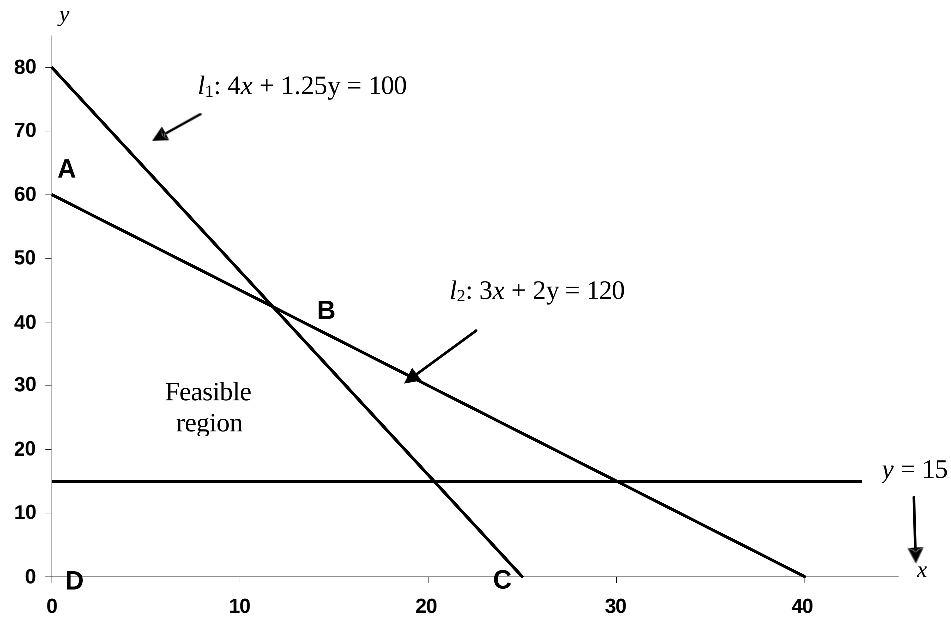
$3x + 2y \leq 120$-----------(l_2)

$y \geq 15$

$x \geq 0$

For line l_1: $4x + 1.25y = 100$, points are (0, 80) and (25, 0)

For line l_2: $3x + 2y = 120$, points are (0, 60) and (40, 0)



From the graph, the corner points of the boundary of the feasible region are A, B, C and D with the coordinates:

A	(0, 60);	B	(11.75, 42.5);
C	(20.25, 15);	D	(0, 15).

Coordinates	*Value of the objective function* (2500x + 1000y) (in ¢)	
A(0, 60)	2500(0) + 1000(60)	= 60000
B(11.75, 42.5)	2500(11.75) + 1000(42.5)	= 71875
C(20.25, 15)	2500(20.25) + 1000(15)	= 65625
D(0, 15)	2500(0) + 1000(15)	= 15000

Coordinates (11.75, 42.5) gives the highest profit of ₦71,875. Hence, the optimal combination to be produced is 12 executive chairs and 43 ordinary chairs.

Note that solving l_1 and l_2 simultaneously gives point B as (11.8, 42.4) and solving l_1 and y = 15 simultaneously gives point C as (20.3, 15).

Example 15.3

Minimization problem (Mix Problem)

A poultry farmer needs to feed his birds by mixing two types of ingredients (F_1 and F_2) which contain three types of nutrients, N_1, N_2 and N_3.

Each kilogram of F_1 costs ₦200 and contains 200 units of N_1, 400 units of N_2 and 100 units of N_3 while, each kilogram of F_2 costs ₦250 and contains 200 units of N_1, 250 units of N_2 and 200 units of N_3.

The minimum daily requirements to feed the birds are 14,000 units of N_1, 20,000 units of N_2 and 10,000 units of N_3.

a) Formulate the appropriate linear programming problem.

b) Solve the linear programming problem by graphical method.

Solution

Let x kg of F_1 and y kg of F_2 be used per day.

Let C (₦) be the total cost of daily feed mixture.

The problem is to meet the necessary nutrient requirement at minimum cost.

The objective function is

$C = 200x + 250y$

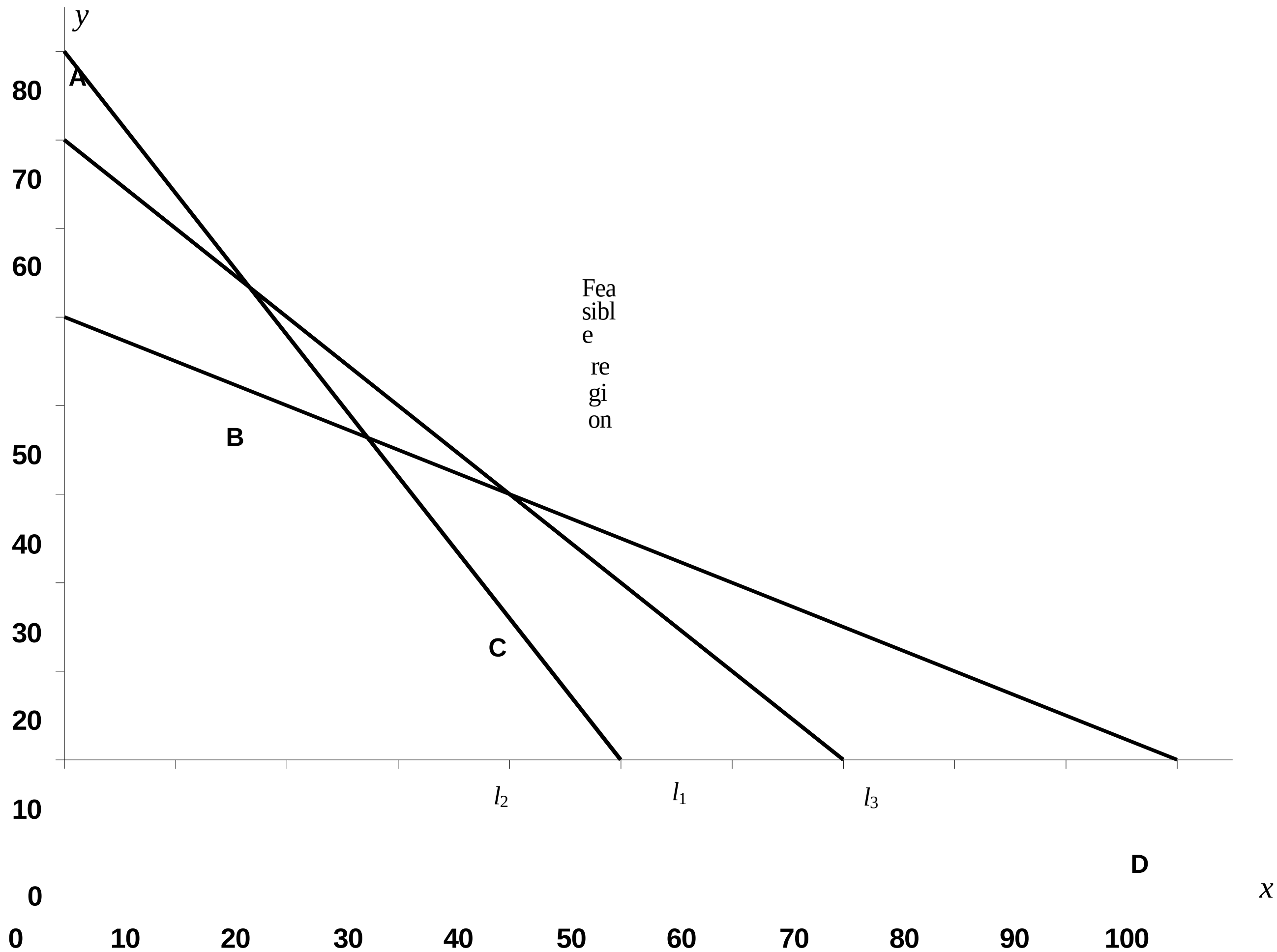
The constraints are

$200x + 200y \geq 14{,}000$ (N_1 constraint)----------l_1

$400x + 250y \geq 20{,}000$ (N_2 constraint)----------l_2

$100x + 200y \geq 10{,}000$ (N_3 constraint)----------l_3

$$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\} \text{Non-negativity constraints}$$



From the graph, the corner points of the boundary of the feasible region are A,B,C,D with the coordinates A(0, 80), B(16, 53), C(38, 31), D(100, 0)

Coordinates	**Value of the objective function = (200*x* + 250y)**
A(0, 80)	200(0) + 250(80) = 20,000
B(16, 53)	200(16) + 250(53) = 16, 450
C(38, 31)	200(38) + 250(31) = 15,350
D(100, 0)	200(100) + 250(0) = 20,000

Coordinates (38, 31) give the lowest cost hence the optimal mix is 38kg of F_1 and 31kg of F_2.

Note that (i) solving l_1 and l_2 simultaneously gives point B as $(16\frac{2}{3}, 53\frac{1}{3})$; and

(ii) solving l_1 and l_3 simultaneously gives point C as (40, 30)

Shadow Costs

Only the binding constraints have shadow costs. They are also known as shadow prices or dual prices.

The binding constraints are the two constraints which intersect at the optimal solution point.

The shadow cost of a binding constraint is the amount by which the objective function decreases (or increases) as a result of availability of one unit less or more of the scarce resource.

The solutions to the dual problem of the primal problem give the shadow costs (prices) hence the alternative term dual costs (prices).

The shadow costs help management of a business organization to carry out sensitivity analysis on the availability of scarce resources.

Solving the dual problem is the same as carrying out sensitivity analysis.

Example 15.4

ROYEJAS Sewing Institute produces two types of dresses, shirt and blouse. A shirt has a contribution of ₦800 while a blouse has a contribution of ₦.

A shirt requires 2 units of materials and 1 hour of labour while a blouse requires 1 unit of materials and 2 hour5 of labour.

If 100 units of materials and 120 labour hours are available per week;

a) formulate and solve the linear programming problem

b) find the shadow cost of a

   i) unit of materials

   ii) labour hour

c) advise the Sewing Institute accordingly.

Solutions

a) Let x units of shirts and y units of blouse be produced

Then the Linear Programming is

Maximize: $800x + 600y$

Subject to: $2x + y \leq 100$ (materials constraint)

$x + 2y \leq 120$ (labour hours constraints)

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\}$ Non-negativity constraints

This problem is the same as Example 13.1 with solution (26, 47), which gives the highest revenue of ₦49 000. i.e. the optimal combination (mix) to be produced per week is 26 shirts and 47 blouses.

b) The binding constraints are

$2x + y \leq 100$ (materials constraint)

$x + 2y \leq 120$ (labour constraints)

i) Materials

- Increase the units of materials by 1 while the labour hours remain unchanged
- Solve the resulting simultaneous equation to obtain new values for x and y
- Calculate the resulting difference in contribution. This is the shadow cost.

The binding constraints now became

$2x + y = 101$

$x + 2y = 120$

Solving these simultaneous equations (see chapter 9 section 9.3.1.2), we obtain

$x = 27.33$ and $y = 46.34$

Now, substitute these values into the objective function ($800x + 600y$) to obtain,

800 (27.33) + 600 (46.34)	=	₦49668
Original contribution	=	₦49000
Difference	=	₦668

i.e. 1 extra unit of material has resulted in an increase of ₦668 in the contribution.

Thus, the shadow cost per unit of materials is ₦668

ii) Labour hours

Here, we increase the labour hours by 1hour while the units of materials remain unchanged.

The new binding constraints will now be:

$2x + y = 100$

$x + 2y = 121$

which when solved will give

$x = 26.33$ and $y = 47.33$

and the new contribution is

800(26.33) + 600(47.33)	=	₦49,466.67
original contribution	=	₦ 49,000
Difference	=	₦ 466.67

i.e. 1 extra labour hour has resulted in an increase of ₦ 466.67 in contribution i.e. the shadow cost per hour of labour is ₦ 466.67

c) The sewing institute can consequently be advised to increase the units of materials rather than to increase the hours of labour because the increase in the contribution brought about by a unit increase in materials is bigger than that of the labour.

15.4 Simplex Method

Furtherance to graphical method of solving LP problems, a non-graphical method shall be considered, the method involves usage of simplex algorithm tagged "Simplex method". This method can handle more than 2 variables.

Useful terms in simplex method

Standard Firm: Standard form of a LP problem involves the conversion of original LP problem to algebraic form which can be easily handled for computations.

Here, one needs to utilise the slack or surplus variable in the constraint in order to make inequalities become equations.

After adding the slack or surplus variables, the expected standard form shall appear as follows:

- Objective function

  Min (or max) $Z = CX + 0S$

  Where C = the cost/profit

  S = Slack/surplus variable

  X = Vector x, i.e. $x_1, x_2, \ldots\ldots\ldots$ and

  O = Zero

- In constraints, the inequalities are converted by using slack (S) to make inequality of less than or equal to ($\leq$) equality, whilst the surplus (S) to make the inequality of type greater than or equal to ($\geq$) an equality
- The added variables: slack and surplus are restricted to non-negativity
- The right hand side of the constraints must be non-negative.

With the above explanation, let's consider the following example:

Example 15.5: Express the following linear programming problem in Standard Form:

$$\text{Min } Z = 3x_1 + 2x_2 + x_3$$

$$\begin{aligned} \text{Subject to:} \quad & x_1 + 3x_3 \leq 10 \\ & 2x_1 + x_2 + 3x_3 \leq 40 \\ & 3x_1 + 2x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The Standard Form is

$$\text{Min } Z = 3x_1 + 2x_2 + x_3 + 0S_1 + 0S_2 + 0S_3$$

$$\begin{aligned} \text{Subject to:} \quad & x_1 + 3x_2 + 2x_2 + 0x_2 + S_1 = 10 \\ & 2x_1 + x_2 + 3x_3 + 0S_1 + S_2 = 40 \\ & 3x_1 + 0x_2 + 2x_3 + 0S_1 + 0S_2 + S_3 = 20 \\ & x_1, x_2, x_3, S_1, S_2, S_3 \geq 0 \end{aligned}$$

where S_1, S_2, S_3 are the slack variables added.

Slack and Surplus Variables: These two variables are used to change the inequality in the constraint to the equality type. Slack variable (usually represented by S) is added to the inequality of type ≤ (less than or equal to). This represents the unused resources in the constraints. For the surplus variable, which is also called negative slack, is the addition of negative S. It is used for ≥ inequality and is representing the amount by which the constraint values exceed the resources.

Canonical Form: A system is in canonical form, with a distinguished set of basic variables, if the given system of *m* equations and *n* variables have coefficient 1 in one equation and zero in the others for the said basic variables.

Basic Solution: This is a solution in which there are at most *m* zero values in linear programming problem that has *n* variables and *m* constraints ($m < n$). If $m < n$, the solution degenerates (i.e. *m* unique solutions).

Artificial variables: A letter is usually used for a standard form of LP problem, which is not in canonical form. For it to be in standard canonical form, it is applied to the constraints of types ≥ (greater than or equally to) and = (equal to) in order to make them canonical

Simplex Method for Solving Liner Programming (LP) Problem

In the Simplex Algorithm for both minimisation and maximisation problems of LP, there is the need to first express the problem in standard canonical form.

By this, the original LP problem can be expressed in the tabular form for the standard form as Initial simplex table.

Table 1: Initial Simplex Tableau

Coefficient of Basic Variables C_B	Variables in Basic X_B	Cost/Profit $C_1,\ C_2,\ C_3, \ldots\ldots C_n, 0\ \ 0 \ldots\ldots 0$ Variables $x_1,\ x_2,\ x_3, \ldots\ldots x_n,\ S_1,\ S_2,\ S_3 \ldots\ldots S_m$	Bi
C_{B_1}	S_1	$a_{11},\ a_{12},\ a_{13}, \ldots\ldots a_{1n},\ 1\ \ 0 \ldots\ldots 0$	b_1
C_{B_2}	S_2	$a_{21},\ a_{22},\ a_{23}, \ldots\ldots a_{2n},\ 0\ \ 1 \ldots\ldots 0$	B_2
:	:		:
:	:		:
:	:		:
C_{B_M}	S_m	$a_{m1},\ a_{m2},\ a_{m3}, \ldots\ldots a_{mn},\ 0\ \ 0 \ldots\ldots 1$	b_m
	Ƶ * = C – Ƶ	$C_B\ A$	

As a typical example, let the standard form of Example 13.4 be as follows for the initial simplex table

Table 2

		3	2	1	0	0	0	
C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	b_i
0	S_1	1	3	0	1	0	0	10
0	S_2	2	1	3	0	1	0	40
0	S_3	3	0	2	0	0	1	20
Ƶ								

Step 1: Select the most negative number in row Ƶ (excluding last column of the row) for minimisation problem or most positive number for maximuisation problem. Refer to the column in which this number appears as the work column or pivot column. If there is a tie in the selection, a simple rule of thumb can be applied.

Step 2: Obtain ratios (i.e. b_i/a_{ij}). Pick the smallest ratio and call it pivot element. If there are more than one, chose any one.

Step 3: Apply elementary row operations to convert the pivot element to 1 and then reduce all other elements in the work column to 0 (zero) by row operations.

Step 4: Replace the variable in the pivot row into X_B to form the new current set of basic variables.

Step 5: Step 1 to 4 should be repeated until there are no negative numbers in the Ƶ row for minimisation problem or no positive numbers in the Ƶ row for maximisation problem. Hence, the problem is optimal and the optimal value of the objective function is obtained.

Example 15.6: Solve the following LP problem using simplex method

$$\text{Max Ƶ} = 40x_1 + 240x_2 + 200x_3$$

$$\text{Subject to:} \quad 2x_1 + 3x_2 + 4x_2 \leq 45$$

$$x_1 + 8x_2 + 5x_3 \leq 30$$

$$x_1,\ x_2,\ x_{3,} \geq 0$$

Solution:

- Express the problem in standard form, we have

$$\text{Max Ƶ} = 40x_1 + 240x_2 + 200x_3 + 0S_1 + 0S_2$$

$$\text{Subject to:} \quad 2x_1 + 3x_2 + 4x_2 + S_1 + 0S_2 = 45$$

$$x_1 + 8x_2 + 5x_3 + 0S_1 + S_2 = 30$$

$$x_1,\ x_2,\ x_{3,}\ S_1,\ S_2, \geq 0$$

- The initial simplex tableau is obtained as follows:

Tableau 1

		40	240	200	0	0		
C_B	X_B	x_1	x_2	x_3	S_1	S_2	b_i	Ratio
0	S_1	2	3	4	1	0	45	15
0	S_2	1	8	5	0	1	30	3¾ ←
Ƶ* = C – Ƶ		40	240	200	0	0	0	

- Here, $C_B A$ = Ƶ for column $x_1 = 0$, also zero for others.
- Highest positive is 240 (column x_2) → this is the work column (from step 1).
- Ratio 3¾ is the smallest and it should be picked. The intersection of this row on

work column will give the pivot element

- Step 3 is carried out to have the next tableau as follows:

Tableau 2

		40	240	200	0	0	
C_B	X_B	x_1	x_2	x_3	S_1	S_2	b_i
0	S_1	$1\frac{5}{8}$	0	$2\frac{1}{8}$	1	$-\frac{3}{8}$	$33\frac{3}{4}$
240	x_2	$\frac{1}{8}$	1	$\frac{5}{8}$	0	$\frac{1}{8}$	$3\frac{3}{4}$
Ƶ* = C – Ƶ		10	0	50	0	- 30	900
				↑			

- Further, step 4 is carried out
- Highest positive 50 (column x_3) – this is the work column.

Tableau 3

		40	240	200	0	0	
C_B	X_B	x_1	x_2	x_3	S_1	S_2	b_i
0	S_1	$\frac{6}{5}$	$-\frac{17}{5}$	0	1	$-\frac{3}{17}$	21
200	x_3	$\frac{1}{5}$	$\frac{8}{5}$	1*	0	$\frac{1}{5}$	6
Ƶ* = C – Ƶ		0	-80	0	0	0	1,200

Since there is no positive value in Ƶ, the problem is optional with the optimal value of $Z = 40x_1 + 240x_2 + 200x_3 = 0 + 0 + 200(6) = 1200$

Example 15.7: Solve the following LP problem using simplex method:

Min Ƶ $= -x_1 - 9x_2 - x_3$

Subject to: $x_1 + 2x_2 + 3x_2 \leq 9$

$3x_1 + 2x_2 + 2x_3 \leq 15$

$x_1, x_2, x_3, \geq 0$

Solution:

- To express the problem in standard form, we have

Min Ƶ $= -x_1 - 9x_2 - x_3 + 0S_1 + 0S_2$

Subject to: $x_1 + 2x_2 + 3x_2 + S_1 + 0S_2 = 9$

$3x_1 + 2x_2 + 3x_3 + 0S_1 + S_2 = 15$

$x_1,\ x_2,\ x_3, S_1,\ S_2,\ \geq 0$

- To form the initial simplex tableau

Tableau 1

		-1	-9	-1	0	0		
C_B	X_B	x_1	x_2	x_3	S_1	S_2	b_i	Ratio
0	S_1	1	2	3	1	0	9	4.5
0	S_2	3	2	2	0	1	15	7.5
Ƶ $= C_B A$		0	0	0	0	0	0	
Ƶ* = C – Ƶ		-1	-9	-1	0	0		
			↑					

- The most negative is – 9 (under variable x_2), therefore, x^2 becomes work column .
- 4.5 is the smaller ratio and it should be picked. Note that the intersection of this row on work column leads to the pivot element
- Next, carry out Steps 3 and 4 to obtain new table.

Tableau 2

		-1	-9	-1	0	0	
C_B	X_B	x_1	x_2	x_3	S_1	S_2	b_i
-9	x_1	½	1	$^3/_2$	½	0	$^9/_2$
0	S_2	2	0	-1	-1	1	6
Ƶ $= C_B A$		-4½	-9	- $^{27}/_2$	$^9/_2$	0	- $^{81}/_2$
Ƶ* = C – Ƶ		3½	0	$^{25}/_2$	$^9/_2$	0	

Since there is no negative value in Ƶ, the problem is optimal with the optimal value of Z = - $^{81}/_2$ = - 40.5

15.5 Duality Problem

For every linear programming problem, there is a corresponding dual form. The original LP problem is known as primal problem whilst the dual form is the dual problem. The optimal values in the two cases are the same.

The dual form is useful in reducing either large number of variables in original problem or simplifying a more complex primal.

The following points are to be noted for the formulation of a dual problem:

i. If the primal contains *n* variables and *m* constraints, the dual will contain *m* variables and *n* constraints;

ii. The maximisation problem in the primal becomes the minisation problem in the dual and vice versa;

iii. The maximisation problem has (≤)constraints while the minimisation problem has (≥) constraints;

iv. The constants C_1, C_2, ………, C_n in the objective function of the primal appear in the right hand side of the dual constraints;

v. The constants B_1, B_2, ………, B_m are the constraints of the primal and they appear in the object function of the dual;and

vi. The transpose of matrix of coefficients of the primal becomes the matrix of coefficients for the dual. Transpose means columns become rows and vice versa.

Example 15.8: Obtain the dual problem of the following primal problem: Min $\cancel{Z}$ $= 25x_1 + 15x_2$

Subject to: $x_1 + 2x_2 \geq 1$

$2x_1 + x_2 \geq 2$

$2x_1 - 3x_2 \geq -3$

$-3x_1 + 2x_2 \geq 4$

$x_1, x_2 \geq 0$

Solution:

Let the dual variables be y_1, y_2, y_3, y_4, then the dual problem is Max $\cancel{Z}$ =

$y_1 + 2y_2 - 3y_3 + 4y_4 \leq 25$

Subject to: $y_1 + 2y_2 + 2y_3 - 3y_4 \leq 15$ $y_1, y_2, y_3, y_4 \geq 0$

Example 13.9: Obtain the dual problem of the following: Max $\cancel{Z}$ = $40x_1 + 240x_2 + 200x_3$

Subject to: $2x_1 + 3x_2 + 4x_3 \leq 45$

$x_1 + 8x_2 + 5x_3 \leq 30$ $x_1, x_2, x_3 \geq 0$

Solution:

Let the dual variables be y_1, y_2, then the dual problem is Min $\cancel{Z}$ =

$$45y_1 + 30y_2$$

Subject to: $2y_1 + y_2 \geq 40$

$3y_1 + 8y_2 \geq 240$

$4y_1 + 5y_2 \geq 200$

$y_1, y_2, \geq 0$

15.6 Limitations of Linear Programming

Limitations of OR methods have been enumerated in chapter 12 (section 12.4) so, linear programming being one of the OR methods has its own limitations.

Some of these are:

a) Linear programming cannot be applied if the objective function and/or the constraint are not linear;

b) Linear programming involves a model which is just a simplified form of the reality but not the reality itself;

c) Solution of the linear programming model is only valid based on the assumptions used;

d) Linear programming is static and therefore not robust enough to accommodate changes; and

e) In particular, solution of linear programming by the graphical method cannot be applied if the decision variables are more than two although other methods such as Simplex can be adopted. Note that the Simplex method is complex.

15.7 Chapter Summary

Linear programming was described as a resource allocation tool. Methods of formulating and solving Linear programming problems using both the graphical and simplex method were discussed. Formulation of duality problems was also discussed and practical applications of Linear programming were also treated.

15.8 Multiple-Coice And Short-Answer Questions

1. The objective function for a minimisation LP problem is $500x + 700y$ and the coordinates of the corner points of the feasible region are (0, 40), (30, 50) and (60, 0). The optimal solution is

   A. 28 000

   B. 30 000

   C. (60, 0)

   D. (0, 40)

   E. (30, 50)

2. Linear programming can be applied only if the objective function and/or constraints are

   A. Linear
   B. Non-linear
   C. Complex
   D. Independent functions
   E. Dependent functions

3. Find the value of z in the following linear programming problem: Minimize $z = 50x + 60y$

   Subject to $2x + 4y \geq 80$

   $x + y \geq 30$

   $x, y \geq 0$

   A. 2000

   B. 1600

   C. 1800

   D. 1400

   E. 1200

4. If the original linear programming problem is a minimising one, then What is maximising one and vice versa?

   A. Shadow occurrence
   B. Anal formulation
   C. Simplex method
   D. Objective function
   E. Inequality constraint

5. The region where all the constraints are simultaneously satisfied is called the __
   A. Equilibrium region
   B. Maximum profit region
   C. Feasible region
   D. Minimum profit region
   E. Break-even region

6. Linear programming is concerned with the allocation of …………… to ……………….

7. The solution to a linear programming problem is always at ………………… point of the boundary of the ....region.

8. The linear programming cannot be applied when there are more than two decision variables. Yes or No?

9. If the primal problem of a linear programming is a minimizing one, the dual problem will be a ……………

10. The two major assumptions in linear programming are ……………… of the objective function and ....of the decision variables.

Answers

1 **Coordinates**	**Value of the objective function**
(0, 40)	500(0) + 700(40) = 28000
(20, 50)	500(30) + 700(50) = 50000
(60, 0)	500(60) + 700(0) = 30000

28000 is the least, hence (0, 40) is the optimal solution (D).

2. Linear (A)

3. Recall that since only two constraints are involved, the solution is at the point of intersection of the constraints.

So, solving the equations simultaneously

$2x + 4y = 80$ ....(i)

$x + y = 30$ .... (ii)

equation (ii) × 2 gives

$2x + 2y = 60$........................(iii)

equation (i) – equation (iii) gives

$2y = 20$, $y = 10$

substitute for y in equation (ii)

$x + 10 = 30$, $x = 20$

value of $z = 50x + 60y$

$= 50 (20) + 60 (10)$

$=1600$ (B)

4 Dual formulation (B)

5 Feasible region (C)

6 Scarce resources, competing activities (in that order)

7 Corner, feasible region (in that order)

8 No

9 Maximizing one

10 Linearity, non-negativity (in that order)

CHAPTER SIXTEEN

INVENTORY AND PRODUCTION CONTROL

Chapter contents

a) Introduction.

b) Meaning and Functions of Inventory.

c) Definition of Terminologies

d) General Inventory Models.

e) Basic Economic Order Quantity Model,

Objectives

At the end of this chapter, readers should be able to: understand the

a) meaning and functions of inventory;

b) basic concepts in inventory control;

c) difference between deterministic and stochastic inventory models; and

d) calculation of the Economic Order Quantity (EOQ) under various situations.

16.1 Introduction

Inventory control is an operations research model that deals with delivering right quantity of goods of the right quality to the right place as well as at the right time, it explains how to identify the order of quantity that minimizes the relevant costs for a given annual demand. Thus, the concept of economic order quantity(EOQ) is established.

16.2 Meaning and Functions of Inventory

Inventory could mean a list of items in a shop or a house or a company. It could also mean the stock of items available in an organisation. The items could be raw materials, partly finished products or finished products. Inventory taking also means stock taking.

There are three major motives for holding stocks. These are

a. Transaction motive

This is to meet the demand at any given time. The quantity demanded is known with certainty and when stock-out occurs, replenishment of stocks is immediate.

b. Precautionary motive

This is to avoid loss of sales due to some uncertainties. Buffer or safety stocks are held so as not to run out of supply.

c. **Speculative motive**

This is in anticipation of shortage from the supplier or price increase by the supplier. Current stock may be increased.

Reasons for Holding Stock

The three major objectives discussed above may be broken down into reasons why a business needs to hold stock. Some of the reasons are to

a) act as a buffer for variations in demand and usage;

b) take advantage of quantity discount by buying in bulk;

c) take advantage of seasonal and price fluctuations;

d) keep to the barest minimum the delay in production process which may be caused by lack of raw materials;

e) take advantage of inflation or possible shortages; and

f) ensure no stock-outs.

The main objective of inventory control is to maintain stock levels so as to minimise the total inventory costs. Two main factors are to be established – when to order and what quantity to order.

Inventory Costs

Inventory costs are of four types

a) **Ordering or procuring costs** – These are all the costs relating to the placement of orders for the stocks. They could be internal or external.

(i) administrative costs associated with the departments involved in placing and receiving orders;

(ii) transport costs; and

(iii) production set-up costs where goods are manufactured internally – cost associated with production planning, preparing the necessary machinery and the work force for each production run.

(i) and (ii) are external while (iii) is internal.

b) **Stock costs** – these are the suppliers' price or the direct costs of production.

These costs need to be considered especially when

i. bulk purchase discounts are available; and

ii. savings in the direct costs of production are possible with longer "batch runs".

c) **Holding costs** – these are also known as **carrying costs** and include the following:

i. cost of capital tied up including interest on such capital;

ii. handling and storage costs;

iii. insurance and security costs;

iv. loss on deterioration and /or obsolescence;

v. stock taking, auditing and perpetual inventory costs; and

vi. loss due to pilferage and vermin damage.

d) **Shortage or stock-out costs** – as a result of running out of stock, a company will normally incur some loss. Stock-out costs include:

i. loss of customers;

ii. loss of sale and contribution earned from the sale;

iii. loss on production stoppages; and

iv. loss on emergency purchase of stock at a higher price.

16.3 Definition of Terminologies

(a) **Lead time or Procurement time** is the time expressed in days, weeks or months, which elapses between ordering and eventual delivery.

A supply lead time of one week means that it will take one week from the time an order is placed until the time it is supplied.

(b) **Physical stock** is the number of items physically in stock at the time of inventory.

(c) **Free stock** is the physical stock added to awaiting orders less unfulfilled demands.

(d) **Maximum stock** is the selected stock level to indicate when stocks have risen too high

(e) **Stock-outs** refer to a situation where there is a demand for an item of stock but the warehouse is out of stock.

Four stock-outs means that there is a demand for 12 items but only eight items are available

(f) **Buffer stock or safety stock or minimum stock** is the level to indicate when stock has gone too low and is usually held to safeguard against stock-outs.

(g) **Economic Order Quantity (EOQ) or Economic Batch Quantity(EBQ)** is the ordering quantity of an item of stock which minimises the costs involved

(h) **Re-order quantity** is the number of units of item in one order.

(i) **Re-order level** is the level of stock at which a new order for more units of items should be placed.

(j) **Demand** is the number of units of stock required within a particular period of time.

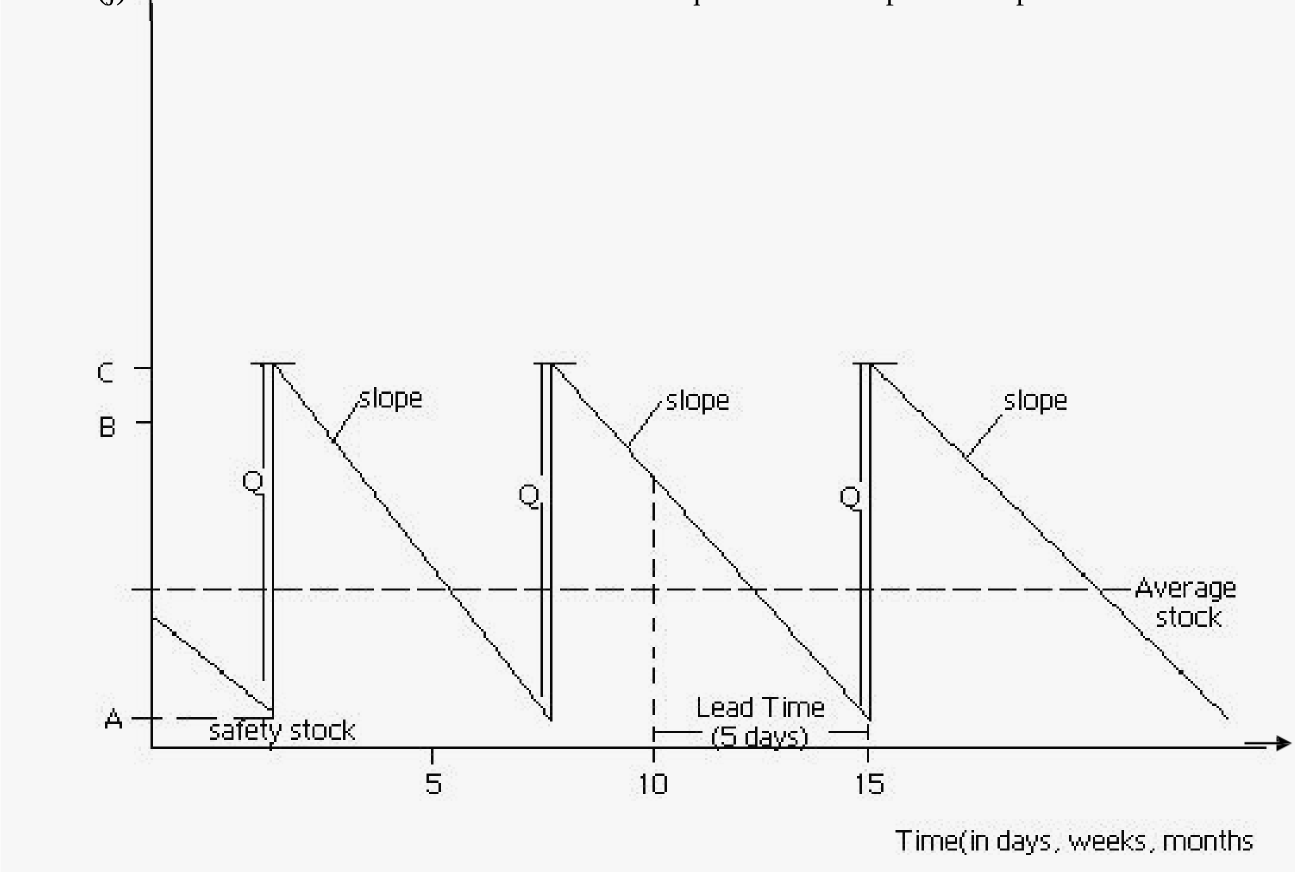


Fig. 16.1 - Illustration of Simple Stock

- Average Stock = $\frac{Q}{2}$ i.e. $\frac{C-6}{2}$
- Re – order level = B
- Re – order quantity (Q) = C – A
- The slopes show anticipated rates of demand

- Note that safety stock is allowed here.

Simple Calculations

Example 16.1

Data on a given stock item are as follows:

Normal usage	2400 per week
Minimum usage	1,600 per week
Maximum usage	3,500 per week
Lead time	20 -23 weeks
EOQ	50,000

Calculate the various control levels.

Solutions

The control levels are

Reordered level = maximum usage x maximum lead time

= 3, 500 x 23

= 80, 500

Maximum level = re-order level + EOQ – (minimum usage x minimum lead time)

= 80, 500 + 50, 000 – (1600 x 20)

= 98, 500

Minimum level = re – order level – (normal usage x average lead time)

= 80,500 – (2400 x 21.5)

= 28, 900

16.4 General Inventory Models

Inventory models help to decide how to plan and control stock in order to minimise costs. An inventory model may be a deterministic or stochastic one.

a) Deterministic model

It is one in which all factors like demand, ordering cost, holding cost etc are known with certainty. No element of risk is involved. Deterministic models may or may not allow for stock-outs.

b) Stochastic Models

A stochastic model is one in which one or all of the factors is/are not known with certainty. It is also known as probabilistic model. Because of the uncertainty in this type of model, there may be necessity for buffer stock which is not necessary in deterministic model.

Stochastic models may be analysed as

(i) A **Periodic Review System (PRS)** in which stock levels for all parts are reviewed at fixed time intervals and varying quantities ordered at each interval as necessary i.e. varying quantities ordered at fixed intervals.

Advantages

i Since stock items are reviewed periodically, obsolete items will be eliminated from time to time;

ii It allows for the spreading of the “load” of purchasing department more evenly;

iii Production planning will be more efficient since orders will be in the same sequence;and

iv Range of stock items will be ordered at the same time thus allowing for larger quantity discount and reduction in ordering costs.

Disadvantages

i Larger stocks are required;

ii EOQ not taken into consideration hence reorder quantities are not economical;

iii Changes in consumption not taken care of; and

iv May be difficult to set up especially if demands are not consistent.

(ii) **Re-order Level System (RLS)** in which a fixed or predetermined quantity, usually EOQ, is ordered at irregular time intervals. This may involve what is referred to as a “two-bin system” where the stock is segregated into two places (bins) and usage is from the first bin so that an order is placed when it becomes empty while usage continues form the second bin.

Advantages

i Lower stocks are required;

ii EOQ is used. Hence, re-order quantities are economical;

iii. It is more responsive to changes in consumption; and

iv It takes care of differing types of inventory.

Disadvantages

i Overloading of purchasing department when many items reach re-order level at the same time;

ii No particular sequence as items reach re-order level randomly; and

iii EOQ may not be appropriate some of the time.

16.5 Basic Economic Order Quantity (EOQ) Model

The EOQ has earlier been defined as the ordering quantity which minimises all the costs involved.

Assumptions of the Model

There are some assumptions on which the model is based. These are:

(a) rates of demand are known;

(b) stock holding cost is known and constant;

(c) price per unit is known and constant;

(d) no stock-outs are allowed;

(e) ordering cost is known and constant; and

(f) no part-delivery; ordered batch is delivered at once.

The symbols used are:

d: the annual demand

Q: the re-order quantity

c: the ordering cost for single order

h: the cost of holding a unit of stock for one year.

Derivation of the EOQ Formula

The total cost per annum is

$$T = \frac{cd}{Q} + \frac{Qh}{2}$$

by the definition of EOQ, we want to minimise T

$$T = cdQ^{-1} + \frac{Qh}{2}$$

$$\frac{dT}{dQ} = -cdQ^{-2} + \frac{h}{2}$$

but $\frac{dT}{dQ} = 0$ at the turning point

this implies that

$$-cdQ^{-2} + \frac{h}{2} = 0$$

$$\frac{h}{2} = \frac{cd}{Q^2}$$

$$Q^2 h = 2cd$$

$$Q^2 = \frac{2cd}{h}$$

$$\therefore Q = \sqrt{2cd/h}$$

$$\frac{d^2T}{dQ^2} = 2cdQ^{-3}$$

$$\therefore \textit{when } Q = \sqrt{2cd/h}, \textit{ then } \frac{d^2T}{dQ^2} = \frac{2cd}{\left(\frac{2cd}{h}\right)^{3/2}} = \frac{h^{3/2}}{\sqrt{2cd}} > 0 \textit{ for all values}$$

of h, c and d since *none of h, c & d can be negative,*

then $Q = \sqrt{2cd/h}$ *gives the* minimum total cost.

Note that:

(i) Number of orders in a year is $\frac{d}{Q}$

(ii) Ordering cost in a year is $c.\frac{d}{Q}$

(iii) Average stock is $\frac{Q}{2}$

(iv) Holding cost per annum is $h.\frac{Q}{2}$

(v) Length of inventory cycle is 52*/No. of orders i.e* $\frac{52Q}{d}$ weeks or $\frac{12Q}{d}$ months

(vi) Total cost per annum = ordering cost p.a+ holding cost p.a i.e. $\frac{cd}{Q} + \frac{Qh}{2}$

Example 16.2

The demand for an item is 60, 000 per annum. The cost of an order is ₦25 and holding cost per item is ₦2·00 per annum. You are required to find

(a) the number of orders per year and the associated ordering cost

(b) length of inventory cycle

(c) total cost per annum

Solutions

(a) C = 25, d = 60, 000, h = 2

$$Q = \sqrt{\frac{2cd}{h}}$$

$$Q = \sqrt{\frac{2 \times 25 \times 60000}{2}} \qquad = 1225 \text{ items}$$

No. of orders per year = $\frac{d}{Q}$

$$= \frac{60000}{1225} \qquad = 49 \text{ orders}$$

associated cost = 49×25 = ₦1225

(b) Length of inventory cycle = 52/49 = 1.06 weeks

(c) Total cost per annum = $25 \times 49 + (60000 \times 2)/2$

$= ₦61,225$

16.6 Chapter Summary

The meaning and function of inventory have been discussed. Deterministic and stochastic inventory models were also treated. Calculation of Economics Order Quantity (EOQ) and its usefulness in taking decisions were treated.

16.7 MULTIPLE-CHOICE AND SHORT-ANSWER QUESTIONS

1. Which of these is not a reason for holding stock?
   - A. To take advantage of quantity discount
   - B. To act as a buffer for variations in demand and usage
   - C. To ensure that the store is filled up at all times
   - D. To take advantage of inflation
   - E. To ensure no stock – outs

2. Physical stock is the number of items in stock at
   - A. The end of each week
   - B. The end of each month
   - C. The time after customers have been supplied
   - D. The time of inventory
   - E. The end of the year.

3. Stock-out refers to a situation when
   - A. An item is not in the store
   - B. No item is in the store
   - C. The store runs out of stock
   - D. There is no demand for an item
   - E. There is demand but the item is not in store.

4. The demand for an item is 3,600 units per annum, the cost of an order is ₦16 and

holding cost per unit of an item is ₦2 per annum. The number of orders per year is

A. 240

B. 15

C. 225

D. 220

E. 25

5.. The main objective of inventory control is to maintain stock levels to the total ....costs

6. The lead time is the time that elapses between ……………….. and ………………….

7. The reorder level is the product of ……………. usage and maximum ……………….

8. If discount rates are allowed for bulk purchase, the highest discount rate is always the best. Yes or No?

9. Economic Order Quantity is the ordering quantity which ………….. all the costs involved.

10. Given that the annual demand is 50,000, the re-order quantity is 2000, ordering cost is ₦20 per order and holding cost per item is ₦2 per annum, then total cost per annum is

Answers

1. C
2. D
3. E
4. Q $= \sqrt{\dfrac{2cd}{h}}$

$= \sqrt{\dfrac{(2)(16)(3600)}{2}}$

$= 240$

No of orders per year is

$\dfrac{\text{Demand}}{Q}$

$= \underline{3600}$

240

= 15 (B)

5. Minimise, Inventory (in that order)

6. Ordering, Delivery (in that order)

7. Maximum, Lead Time (in that order)

8. No

9. Minimises

10. Total Cost per annum is

ordering cost per annum + holding cost per annum

i.e. $\frac{cd}{Q} + \frac{Qh}{2}$

$$= \frac{(20)(50000)}{2000} + \frac{(2000)(2)}{2}$$

= ₦2,500

CHAPTER SEVENTEEN

NETWORK ANALYSIS

Chapter Contents

a) Introduction.

b) Network Diagram.

c) Earliest Start Times, Latest Start Times and Floats.

Objectives

At the end of this chapter, readers should be able to understand the

a) concept of Network Analysis;

b) concept of Activity, Event and Dummy Activity;

c) construction/drawing of a network diagram;

d) array of paths in a Network diagram and calculation of their durations;

e) critical path and the critical activities;

f) calculation of the shortest time for the completion of a project;

g) calculation of the Earliest Start Time (EST), Latest Start Time (LST), Earliest Finish time (EFT) and Latest Finish Time (LFT) for an activity; and

h) calculation of all the three floats and Interpretation of their values.

17.1 Introduction

Network analysis is another Operations Research (OR) method of approach to management problems. It involves management of large projects in an optimal way. It is a technique for planning, scheduling and controlling projects**.**

Some examples of such projects include:

(a) setting up of a new business;

(b) construction of projects;

(c) maintenance of buildings and machines; and

(d) personnel training etc.

Its primary objective is to complete the project within the minimum time

The project manager has to decide on the "flow diagram" for the project by identifying

(i) the tasks that must be done first before others can start i.e. the tasks which precede other tasks;

(ii) the tasks that can be done simultaneously; and

(iii) the tasks which are "crucial" to the project.

Some of other terms used for Network analysis are:

(i) Critical Path Analysis (CPA); and

(ii) Project Evaluation and Review Technique (PERT).

17.2 Definition of Terms

(e) The tasks are called activities

An activity is represented by an arrowed line (from left to right and not drawn to scale): it runs between two events.

It consumes time and resources e.g prepare a set of accounts

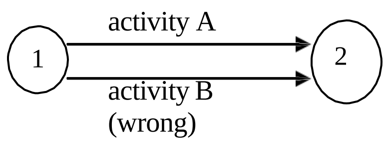
(f) **An event** is the start and/or completion of an activity and is represented by a circle called a node. It is usually numbered.

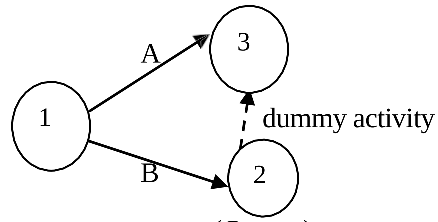
Tail/Start Event | Head/End Event

The activity is between event 1 and event 2

(g) **A dummy activity** is an activity of "circumstance". It consumes neither time nor resources and is only used to ensure non-violation of the rules for drawing a Network diagram.

It is usually represented by a dotted line ------------------- The use of a dummy activity will ensure that two different activities do not have the same starting and finishing nodes.





(Correct)

(h) **A path** of a Network is a sequence of activities which will take one from the start to the end of the Network.

(i) The **critical path** is the path of a Network with the longest duration and gives the shortest time within which the whole project can be completed. It is possible to have more than one critical path in a Network.

(j) **Critical activities** are the activities on the critical path.
They are very crucial to the completion of a project. There must not be any delay in starting and finishing these activities otherwise the duration of the whole project will be extended.

17.3 Construction/Drawing of a Network Diagram based on Activity-On-Arrow(AOA)

A Network diagram is a combination of activities and events in a logical sequence for the completion of a project.

In order to draw a Network diagram, the following must be noted:

a) all activities with their durations must be known or estimated;

b) the logical sequence of the activities must be put in place i.e which activities must be done one after the other (preceding activities) and which ones can be done simultaneously;

c) all activities must contribute to the progression of the project otherwise they should be discarded; and

d) a network diagram must have one starting event and one finishing event.

e) The Activity-On-Arrow(AOA)type of Network Diagram is the only method employed in this Study Text.

Example 17.1

The activities involved in a project are as shown below:

Activity	*Preceding Activity*
A	-
B	-
C	B
D	C
E	A
F	E

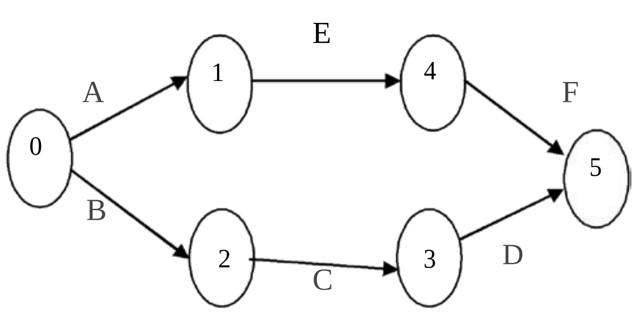
Draw the activity – on – the arrow Network diagram for the project.

Solution

When a Network diagram is to be drawn, do not look at the entire Network to start with but rather take one step after the other. It is also advisable to use a pencil (so that any mistake can easily be erased) and then trace out.

Start off with the activity (ies) that have no preceding activity(ies), then follow through the given information on the activities.

(try to draw your own Network diagram before looking at the solution)



Note: The paths of the Network are: B,C, D and A,E,F

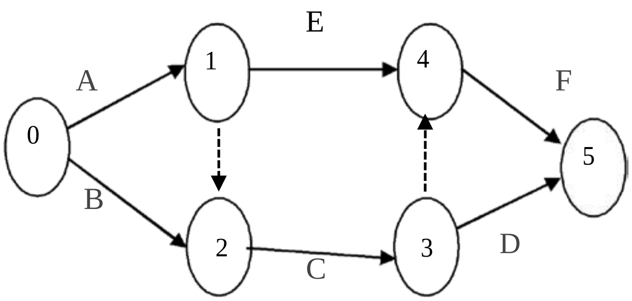
The numbers inside the nodes are the event numbers: The arrows must be directed (>) otherwise the whole diagram is meaningless.

Example 17.2

In the last example, if C is preceded by B and F is preceded by E; draw the corresponding Network diagram.

Solution

To draw the Network diagram, the use of dummy activities will be necessary.



The paths of the Network are now as follows:

A, E, F

A, Dummy, C, D

B, C, D

B, C, Dummy, F

17.4 Durations of Paths and Critical Path

The following example will be used to explain both the duration along paths of a Network and the critical path.

Example 17.3

The following activities with their durations are necessary to complete a project.

Activity	**Preceding Activity**	**Duration (Weeks)**
A	-	4
B	A	6
C	A	3
D	B, C	5
E	A	7
F	D	1
G	D	4
H	E, F	6

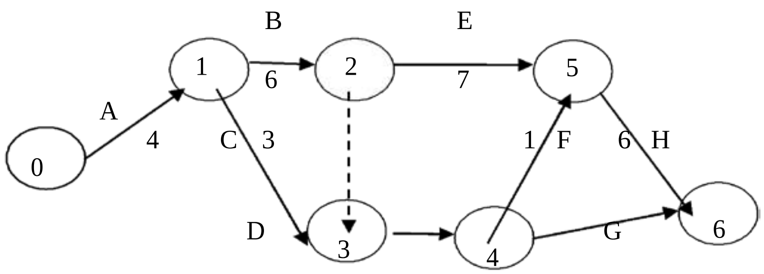
a) Draw the AOA Network diagram for the project

b) Identify all the paths and calculate the duration for each path

c) Identify the critical path and the critical activities

Solutions

a)



The activities and their durations are as shown e.g. for activity B we have

$$\xrightarrow[\text{6 (duration)}]{\text{B (activity)}}$$

b)

Path	Duration
A, B, E, H	4 + 6 + 7 + 6 = 23 weeks
A, C, D, G	4+ 3 + 5 + 4 =16 weeks
A, C, D, F, H	4 + 3 + 5 + 1 + 6 = 19 weeks
A, B, Dummy, D, G	4 + 6 + 0 + 5 + 4 = 19 weeks
A, B, Dummy, D, F, H	4 + 6 + 0 + 5 + 1 + 6 = 22 weeks

c) The critical path is A, B, E, H with duration of 23 weeks.

The critical activities are A, B, E H

d) The duration of the project is 23 weeks

17.5 Earliest Start Time (EST), Latest Start Time (LST) and Floats

The **Earliest Start Time** is the earliest possible time that a succeeding activity can start while the Latest Start Time is the latest possible time that a preceding activity must be completed in order not to increase the project duration.

The **float** of an activity is the amount of spare time associated with the activity.

Only non-critical activities have floats. These activities can start late and/or take longer time than specified without affecting the duration of the project.

The three types of float are:

a) **Total float** is the amount of time by which the duration of an activity could be extended without affecting the project duration;

b) **Free float** is the amount of time by which the duration of an activity can be extended without affecting the commencement of subsequent activities; and

c) **Independent float** is the amount of time by which the duration of an activity can be extended without affecting the time available for succeeding activities or preceding activities.

17.6 Calculation of ESTs, LSTs and Floats

The following example will be used for all the calculations

Example 17.4

Use the diagram in example 15.3 to calculate

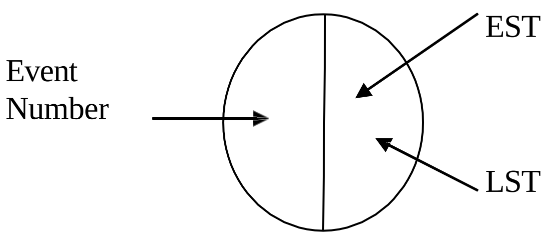
(k) (i) The ESTs

(ii) LSTs for all activities

(l) floats for all activities

Solution

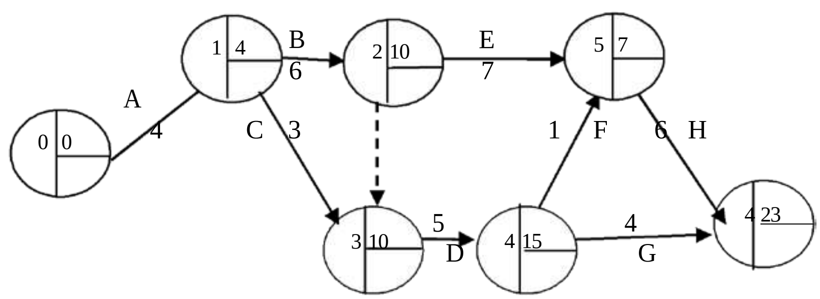
Key: Figures in the nodes are Identified as:



(a) *Calculation of ESTs (Forward pass)*

(i) Start with the event zero and work forward (hence the term "forward pass") through the Network;
(ii) The EST of a head event is the sum of the EST of the tail event and the duration of the linking activity;
(iii) Where two or more activities have the same head event, the largest time is chosen; and
(iv) The EST in the finish event gives the project duration.

Thus, the ESTs are as shown



Explanation

Event Nos	EST
1	0 + 4 = 4
2	4 + 6 = 10
3	either 4+3= 7 or 10+0 (Dummy) = 10, pick 10 because it is the larger of the two i.e. 10> 7
4	10+5 =15
5	either 10 + 7 =17 or 15 + 1 = 16, pick 17 because it is the larger of the two i.e. 17 > 16
6	either 15 + 4 = 19 or 17 + 6 = 23, pick 23 because it is the larger of the two i.e. 23 > 19

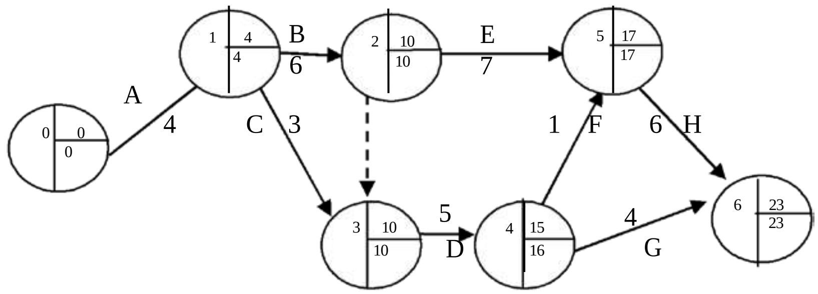
(b) Calculation of LSTs (Backward pass)

i. Start at the finish event using its EST and work backwards (hence the term "backward pass") through the network.

ii. Deduct each activity duration from the previous LST

iii. Where there are two or more LSTs, select the shortest time

Thus, the LSTs are as shown in the positions as in the key.



Explanation

Event Nos	LST
6	23 (EST)
5	23 – 6 = 17
4	either 23- 4 = 19 or 17 - 1 = 16 (the smaller)
2	10-6 = 4
2	either 16 - 5 =11 or 10 - 0 = 10, pick 10 (the smaller)
1	10 – 6 = 4
0	4 – 4 = 0

Note:

The ESTs and LSTs along the critical path are equal.

This is another way of identifying the critical path. In the last example, events 0, 1, 2, 5 & 6 are on the critical path since the relevant ESTs and LSTs are equal.

c) Calculation of floats

To calculate floats, we need to know two other values apart from the ESTs and the LSTs. These are the Earliest Finish Time (EFT) and the Latest Finish Time (LFT) which are read directly from the head node of an activity.

These values are shown below;

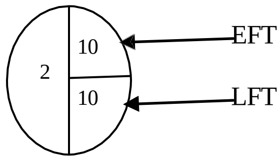
Note that the EFT is just smaller of the two values while the LFT is the larger. Consequently, the EST of an activity is the EFT of the preceding activity and the LST of an activity is the LFT of the preceding activity.

Activity	EFT	LFT
A	4	4
B	10	10
C	10	10
D	15	16
E	17	17
F	17	17
G	23	23
H	23	23

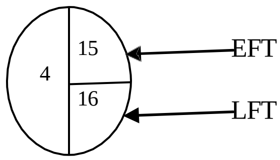
All the values are now combined to calculate the float.

Explanation: Both the EST and LST for an activity are read directly from the head node of the activity. EFT is the smaller of the two numbers.

- For activities A, B, C, E, G and H, the ESTs and LSTs are equal e.g. head node of activity B is



- For activity D, the head node is



Activity	EFT	LFT	EST	LST	Duration D	Total Float LFT-EST-D	Free Float EFT-EST-D	Independent Float EFT-LST-D
A	4	4	0	0	4	0	0	0
B	10	10	4	4	6	0	0	0
C	10	10	4	4	3	3	3	3
D	15	16	10	10	5	1	0	0
E	17	17	10	10	7	0	0	0
F	17	17	15	16	1	1	1	0
G	23	23	15	16	4	4	4	3
H	23	23	17	17	6	0	0	0

As could be seen, all floats of the critical activities are zeros confirming the fact that only non-critical activities have floats.

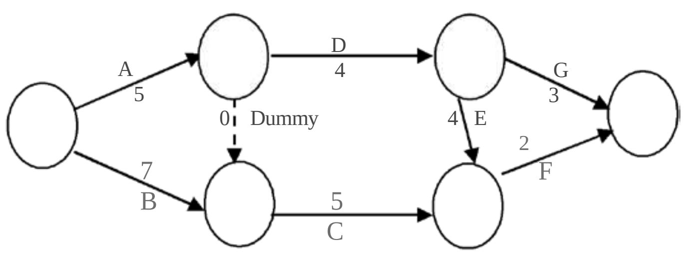
An alternative method to <u>activity-on-arrow</u> diagram is <u>activity-on node</u> diagram which is also known as precedence diagram.

17.6 Chapter Summary

The concept of Network Analysis has been discussed in details. Steps to follow in drawing the Network diagram for a project were enumerated and explained. Identification of critical path and calculation of floats were treated as well. Practical applications of Network work analysis were also treated.

17.7 Multiple-Choice And Short-Answer Questions

1. In a network diagram, two different activities must not have
   - A. The same duration
   - B. The same starting nodes
   - C. The same finishing nodes
   - D. The same starting and finishing nodes
   - E. The same preceding activity



The diagram above shows the network diagram to complete a project. The durations are in weeks. Use this to answer questions (2) and (3).

2. The shortest time within which the project can be completed is
   - A. 12 weeks
   - B. 14 weeks
   - C. 10 weeks
   - D. 11 weeks
   - E. 15 weeks

3. The number of paths of the network is
   - A. 2
   - B. 4
   - C. 3
   - D. 5
   - E. 6

4. In a network diagram, the critical activities have
   A. Free float only
   B. Independent float and free float
   C. Independent float only
   D. Total float
   E. No float.

5. The primary objective of a Network Analysis is to ………………. a project within the ………………. time

6. A dummy activity neither consumes ……………………… nor………………....

7. The critical path of a network is the path with the ....duration.

8. A Network diagram can be drawn using events on the …..........…… and event on the ………………

9. The calculation of the latest start time (LST) is also referred to as............................ pass

10. The float of an activity is the amount of ………….. time associated with the activity.

Answers

1. D

2. The paths with their durations are:

A, D, G	12 weeks
A, Dummy, C, F	12 weeks
A, D, E, F	15 weeks
B, C, F	14 weeks

(E)

3. See 2, 4 paths (B)

4. E

5. Complete, minimum (in that order)

6. Time, resources (or vice-versa)

7. Longest

8. arrow, mode

9. Backward

10. Spare

CHAPTER EIGHTEEN

REPLACEMENT ANALYSIS

Chapter Contents

Objectives

At the end of this chapter, readers should be able to understand the

a) meaning and purpose of replacement theory;

b) concept and technique of replacement;

c) replacement policy of those items that wear out gradually;

d) replacement policy of the items that fail suddenly; and

e) difference between individual and group replacement policies

18.1 Introduction

In a real life, all equipment used in industries, military, and even at homes have a limited life span. By passage of time, these equipment can fail suddenly or wear out gradually. As these machines or equipment are wearing out, the efficiency of their functions continues to decrease with time and in effect it affects the production rates or economic/social benefits of the system. When the equipment become old or when they get to the wearing-out stage, they will definitely require higher operating costs and more maintenance costs due to repairing and replacement of some parts. And at that point, replacing them completely may be a better option.

Examples of the equipment are transportation vehicles (such as cars, lorries or aircraft), machines used in industries, tyres. These wear out gradually whereas highway tube lights, electric bulbs, and contact set, used by vehicles wear out suddenly.

The two essential reasons for the study of replacement analysis are to

a) ensure efficient functioning of the equipment; and

b) know when and how best the equipment can be replaced in order to minimize the total costs of maintaining them.

The two major replacing policies are:

(i) Replacement of equipment/items that deteriorate or wear-out gradually; and

(ii) Replacement of equipment/items that fail suddenly.

18.2 Replacement Of Equipment/Items That Deteriorate Or Wear-Out Gradually

The efficiency of equipment/items that deteriorate with time will be getting low. As the items deteriorate, gradual failure sets in. The gradual failure is progressive in nature and thus affects the efficiency and it results in

c) a decrease in the equipment production capacity;

d) increasing maintenance and operating costs; and

e) decrease in the value of the re-sale (or salvage) of the item

Due to these effects, it is reasonable and economical to replace a deteriorating equipment/item with a new one. As the repair and maintenance costs are the determining factors in replacing policy, the following two policies are to be considered.

a) Replacement of items that deteriorate and whose maintenance and repair costs increase with time, ignoring changes in the value of money during the period. This is a simple case of minimizing the average annual cost of an equipment when the maintenance cost is an increasing function time but the time value of money remains constant.

In order to determine the optional replacement age of a deteriorating equipment/item under the above conditions, the following notations/symbols shall be used:

C = Capital or purchase cost of the equipment/item;

S = Scrap (or salvage) value of the equipment/item at the end of t years;

TC(t) = Total cost incurred during t years;

ATC = Average annual total cost of the equipment/item;and

n = Replacement age of the equipment/item: i.e. number of equipment years before it is replaced

Example 18.1

An owner of a grinding machine estimates from his past records that cost per year of operating his machine is as follows:

Year	1	2	3	4	5	6	7	8
Operating Cost (₦)	250	550	850	1250	1850	2550	3250	4050

If the cost price is ₦12,300 and the scrap value is ₦250, when should the machine be replaced?

Solution

C = ₦12,300 and S = ₦250

Year of Service n	Running Cost (₦) f(n)	Cumulative Running Cost (₦) ∑f(n)	Depreciation Cost Price = C - S	Total Cost (₦) TC	Average Cost (₦) ATC(n)
Col 1	Col2	Col 3	Col 4	Col 3 + Col 4 = Col 5	Col 6 = Col 5/n
1	250	250	12050	12300	12300
2	550	800	12050	12850	6425
3	850	1650	12050	13700	4566.67
4	1250	2900	12050	14950	3737.50
5	1850	4750	12050	16800	3360
6	2550	7300	12050	19350	**3225 ***
7	3250	10550	12050	22600	3228.57
8	4050	14600	12050	26650	3331.25

It is observed from the table that the average annual cost ATC (n) is minimum in the sixth year. Hence, the machine should be replaced at the end of sixth year of usage.

Example 18.2

A company is considering replacement of a machine which cost ₦70,000. The maintenance cost and resale values per year of the said machine are given below:

Table 18.1

Year	1	2	3	4	5	6	7	8
Maintenance Cost (₦)	9000	12000	16000	21000	28000	37000	47000	59000
Resale Cost (₦)	40000	20000	12000	6000	5000	4000	4000	4000

When should the machine be replaced?

Solution

Year of Service n	Resale Value (₦) (S)	Purchase Price resale value (₦) C - S	Annual Maintenance Cost f(t)	Cumulation Maintenance Cost (₦) $\sum_{0}^{n} f(t)$	Total Cost (₦) TC $\left[C - S + \sum_{0}^{n} f(t) \right]$	Average Cost (₦)
Col 1	Col2	Col 3 =70000 – Col 2	Col 4	Col 5	Col 6 = Col 3 + Col 5	Col 7 = Col 6/n
1	40000	30000	9000	9000	39000	39000
2	20000	50000	12000	21000	71000	35500
3	12000	58000	16000	37000	95000	31666.67
4	6000	64000	21000	58000	122000	30500
5	5000	65000	28000	86000	151000	**30200 ***
6	4000	66000	37000	123000	189000	31500
7	4000	66000	47000	170000	236000	33714
8	4000	66000	59000	229000	295000	36875

From the table, it is observed that the average cost ATC(n) is minimum in the fifth year. Hence, the machine should be replaced by the end of 5^{th} year.

b) Replacement of items that deteriorate and whose maintenance cost increase with time with the value of money also changing with time

This replacement policy can be seen as a value of money criterion. In this case, the replacement decision is normally based on the equivalent annual cost whenever we have time value of money effect.

Whenever the value of money decreases at constant rate, the issue of depreciation factor or ratio comes in as in the computation of present value (or worth). For example, if the interest rate on ₦100 is r percent per year, the present value (or worth) of ₦ 100 to be spent after n years will be:

$$D = \frac{100}{100 + r^{n}} \quad \text{..} \quad 16.1$$

where D is the discount rate or description value. With the principle of the discount rate, the critical age at which an item should be replaced can be determined.

Example 18.3

The yearly costs of two machines A and B when money value is neglected are is given below:

Year	1	2	3
Machine A (₦)	1400	800	1000
Machine B (₦)	24000	300	1100

If the money value is 12% per year, find the cost patterns of the two machines and find out which of the machines is more economical.

Solution

The discount rate per year = (d) = $\frac{1}{1 + 0.12}$ = 0.89

The discounted cost patterns for machines A and B are shown below:

Year	1	2	3	Total Cost (₦)
Machine A (Discounted Cost in ₦)	1400	800 x 0.89 = 712	1000 x $(0.89)^2$ = 792.1	2,904.1
Machine B (Discounted Cost in ₦)	24000	300 x 0.89 = 267	1100 x $(0.89)^2$ = 871.31	25,138.3

Decision: Machine A is more economical because its total cost is lower

18.3 REPLACEMENT OF EQUIPMENT/ITEMS THAT FAIL SUDDENLY

In a real life situation, we have some items that do not deteriorate gradually but fail suddenly. Good examples of these items are electric bulbs, contact set, plugs, resistor in radio, television, computer, etc. Majority of items in this category are not usually expensive, but a quick attention or preventive replacement should be given in order not to have a complete breakdown of the system.

The items that experience sudden failure normally give desired service at variant periods. Service periods follow some frequency distributions which may be random, progressive and retrogressive.

There are two types of policies in the sudden failure category. These are

f) The individual replacement policy; and

g) The group replacement policy