Solutions to Assignment 6

Math 217, Fall 2002

3.3.18 Suppose that all entries in A are integers and $\det A = 1$. Explain why all the entries in A^{-1} are integers.

We know that $A^{-1} = \frac{1}{\det A}\text{adj } A$. Because $\det A = 1$ we only need to show that adj A has all integer entries (in this particular situation ... it won't be true in general). The i,jth entry of adj A is $(-1)^{i+j}\det A_{j,i}$. So now it is enough to show that the determinate of $A_{j,i}$ is an integer.

Here $A_{i,j}$ is a matrix with integer entries. If we can prove that any matrix with integer entries has an integer determinate, we will be done. So let's prove a little lemma.

Lemma 1. *If* $A \in M_n(\mathbb{Z})$ *then* $\det A$ *is an integer.*

Proof. The proof is by induction on n. If $n = 2$ (we didn't define determinates for 1×1 matrices), then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some $a, b, c, d \in \mathbb{Z}$, and the determinate of A is $ad - bc$, clearly an integer. So suppose $n > 2$. A formula for the determinate of A is

$$\sum_{j=1}^{n} (-a)^{j+1} a_{1,j} \det A_{1,j}.$$

By induction, $\det A_{1,j}$ is an integer for all $j = 1, \ldots, n$ (because $A_{1,j}$ is an $(n-1) \times (n-1)$ matrix). We know that sums of multiples of integers are integers, so

$$\sum_{j=1}^{n} (-a)^{j+1} a_{1,j} \det A_{1,j}$$

is an integer, completing the result. $\square$

3.3.28 Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \to A\mathbf{x}$.

We have the formula $\{\text{area of } T(S)\} = |\det A|\{\text{area of } S\}$ (see theorem 10). We also know that the area of the parallelogram determined by S is $\left|\det \begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix}\right|$ (see theorem 9). So we have that $\{\text{area of } T(S)\} = \left|\det \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}\right| \cdot \left|\det \begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix}\right| =$ 20 square units!

3.3.32 Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. See the figure in the book on page 210.

(a) Describe a linear transformation that maps S onto S'.

Suppose we call this transformation T. We will describe the action of T by giving its standard matrix. We know that $T(\mathbf{e}_i) = \mathbf{v}_i$ for $i = 1, 2, 3$, so we need to find the matrix A such that $A\mathbf{e}_i = \mathbf{v}_i$ for $i = 1, 2, 3$. Because $A\mathbf{e}_i$ is the ith column of A, we see that $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. Thus the standard matrix of T is $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$

(b) Find a formula for the volume of the tetrahedron S' using the fact that $\{\text{volume of } S\} = (1/3)\{\text{area of base}\} \cdot \{\text{height}\}$.

We use the formula given in the problem along with theorems 9 and 10 from the text. So

$$\{\text{vol. of } T(S)\} = |\det A|\{\text{vol. of } S\}$$

$$= |\det A| \cdot (1/3)\{\text{area of base}\} \cdot \{\text{height}\} = |\det A| \cdot (1/3) \cdot (1/2) \cdot (1)$$
$$= (1/6)\left|\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}\right|$$

units cubed!

4.1.12 Let W be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

Note that

$$W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \mid s, t \in \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$$
$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

We proved in class that the Span of a set of vectors is a subspace, thus W is a subspace.

4.1.20 The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

(a) What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed?

Well, we need to know that 0 is a continuous function on $[a, b]$, that given any two continuous functions f and g on $[a, b]$, then their sum $f + g$ is a continuous function on $[a, b]$, and given any $c \in \mathbb{R}$ and $f \in C[a, b]$, then cf is a continuous function on $[a, b]$. Each of these facts is true and typically discussed in a calculus class.

(b) Show that $\{f \in C[a, b] \mid f(a) = f(b)\}$ is a subspace of $C[a, b]$.

Well, there are some things we need to check. Let $S = \{f \in C[a, b] \mid f(a) = f(b)\}$. It is true that the constant function 0 is in S (because 0 evaluated at any c such that $a \leq c \leq b$ is 0). If $f, g \in S$, then we know that $f(a) = f(b)$ and $g(a) = g(b)$. Thus $(f + g)(a) = f(a) + g(a) = f(b) + g(b) = (f + g)(b)$ as required. Finally, if $f \in S$, then $f(a) = f(b)$, so for any $c \in \mathbb{R}$, $(cf)(a) = c(f(a)) = c(f(b)) = (cf)(b)$ (for the first and last equality I am using the fact that cf is the function that takes x to $c(f(x))$, that is, that $(cf)(x) = c(f(x))$).

4.1.32 Let H and K be subspaces of a vector space V. The intersection of H and K, written as $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

Well, there are some things we have to check. It is clear that $\mathbf{0} \in (H \cap K)$, because $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $H \cap K$. Then $\mathbf{u}, \mathbf{v} \in H$, and $\mathbf{u}, \mathbf{v} \in K$. This implies that $\mathbf{u} + \mathbf{v} \in H$ (because H is a subspace), and similarly for K. We conclude that $\mathbf{u} + \mathbf{v} \in (H \cap K)$ as required. Finally, if $\mathbf{u} \in (H \cap K)$, then $\mathbf{u} \in H$ and $\mathbf{u} \in K$. Thus for each $c \in \mathbb{R}$, $c\mathbf{u} \in H$ (again because H is a subspace), and similarly for K. We conclude that $c\mathbf{u} \in (H \cap K)$ for all $c \in \mathbb{R}$ as required. We have proved that $(H \cap K)$ is a subspace of V.

In general unions do not give subspaces. Let $H = \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$, and $K = \left\{ b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$. Then $H \cup K$ is the set of all things whose form is either $a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for some $a \in \mathbb{R}$ or of the form $b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for some $b \in \mathbb{R}$. This is not a subspace, because $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $H \cup K$ and subspaces must be closed under addition ($\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not of the form $a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for some $a \in \mathbb{R}$ or of the form $b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for some $b \in \mathbb{R}$).

4.2.32 Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$. Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T and describe the range of T.

Suppose that p is in the kernel of T and write $p(t) = a + bt + ct^2$. Then $T(p) = \begin{bmatrix} a \\ a \end{bmatrix}$. So it must be the case that $a = 0$. This means that any polynomial of the form $bt + ct^2$ for some $b, c \in \mathbb{R}$ is in the kernel of T. A spanning set for these two vectors is the set $\{t, t^2\}$.

To describe the range, let $p(t) = a + bt + ct^2$ be any polynomial in $\mathbb{P}_2$, and note that $T(p) = \begin{bmatrix} a \\ a \end{bmatrix}$. We see right away that the range is contained in the set $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$. For each $\begin{bmatrix} b \\ b \end{bmatrix} \in \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$ the polynomial $p(t) = b + t$ has $T(p) = \begin{bmatrix} b \\ b \end{bmatrix}$. This implies that the set $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$ is contained in the range. We conclude that the range is $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$. (You might note that this is isomorphic to $\mathbb{R}^1$).

4.2.34 Define $T : C[0,1] \to C[0,1]$ as follows: For f in $C[0,1]$, let $T(f)$ be the antiderivative F of f such that $F(0) = 0$. Show that T is a linear transformation, and describe the kernel of T.

I remind you that the antiderivative F of f which has $F(0) = 0$ is

$$F(x) = \int_0^x f(t)\mathrm{dt}.$$

This means that $T(f)$ is the function

$$\int_0^x f(t)\mathrm{dt}.$$

So if $f, g \in C[0,1]$ then

$$T(f+g) = \int_0^x (f(t) + g(t))\mathrm{dt} = \int_0^x f(t)\mathrm{dt} + \int_0^x g(t)\mathrm{dt} = T(f) + T(g).$$

Furthermore, for all $c \in \mathbb{R}$,

$$T(cf) = \int_0^x cf(t)\mathrm{dt} = c\int_0^x f(t)\mathrm{dt} = cT(f).$$

Thus T is a linear transformation.