Math 342, Spring 2012
Solutions to problems from §5.4

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 3\mathbf{d}_1 - 5\mathbf{d}_2, \quad T(\mathbf{b}_2) = -\mathbf{d}_1 + 6\mathbf{d}_2, \quad T(\mathbf{b}_3) = 4\mathbf{d}_2.$$

Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{D}$.

Answer. $\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$

3. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V, and let $T : \mathbb{R}^3 \to V$ be a the linear transformation with

$$T(x_1, x_2, x_3) = (2x_3 - x_2)\mathbf{b}_1 - (2x_2)\mathbf{b}_2 + (x_1 + 3x_3)\mathbf{b}_3.$$

a. Compute $T(\mathbf{e}_1), T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$.
Solution. $T(\mathbf{e}_1) = T(1,0,0) = \mathbf{b}_3$, $\quad T(\mathbf{e}_2) = T(0,1,0) = -\mathbf{b}_1 - 2\mathbf{b}_2$, and $T(\mathbf{e}_3) = T(0,0,1) = 2\mathbf{b}_1 + 3\mathbf{b}_3$.

b. Compute $[T(\mathbf{e}_1)]_{\mathcal{B}}, [T(\mathbf{e}_2)]_{\mathcal{B}}$, and $[T(\mathbf{e}_3)]_{\mathcal{B}}$.
Answers. $[T(\mathbf{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, [T(\mathbf{e}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$, and $[T(\mathbf{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

c. Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{B}$.
Answer. $\begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

4. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V and let $T : V \to \mathbb{R}^2$ be a linear transformation with the property that $T(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3) = \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ -2x_1 + 5x_3 \end{bmatrix}$. Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

Solution. 1st column: $T(\mathbf{b}_1) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ 2nd column: $T(\mathbf{b}_2) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ 3rd column: $T(\mathbf{b}_3) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Answer: $\begin{bmatrix} 2 & -3 & 1 \\ -2 & 0 & 5 \end{bmatrix}$

6. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $\mathbf{p}(t) + 2t^2\mathbf{p}(t)$.

a. Find the image of $\mathbf{p}(t) = 3 - 2t + t^2$.
Solution. $T(3 - 2t + t^2) = 3 - 2t + t^2 + 2t^2(3 - 2t + t^2) = 3 - 2t + 7t^2 - 4t^3 + 2t^4$.

b. Show that T is a linear transformation.
Solution. $T(\mathbf{p}+\mathbf{q}) = \mathbf{p}+\mathbf{q}+2t^2(\mathbf{p}+\mathbf{q}) = \mathbf{p}+2t^2\mathbf{p}+\mathbf{q}+2t^2\mathbf{q} = T(\mathbf{p})+T(\mathbf{q})$ for any $\mathbf{p}, \mathbf{q} \in \mathbb{P}_2$, and $T(c\mathbf{p}) = c\mathbf{p} + 2t^2c\mathbf{p} = c(\mathbf{p} + 2t^2\mathbf{p}) = cT(\mathbf{p})$ for any $c \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{P}_2$.

c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

Solution.

1st column: $T(1) = 1 + 2t^2$ 2nd column: $T(t) = t + 2t^3$ 3rd column: $T(t^2) = t^2 + 2t^4$

Answer: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$ is linear. Find the matrix for T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

Solution.

1st column: $T(1) = 3 + 5t$ 2nd column: $T(t) = -2t + 4t^2$ 3rd column: $T(t^2) = t^2$

Answer: $\begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

8. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V. Find $T(4\mathbf{b}_1 - 3\mathbf{b}_2)$ where T is the linear transformation from V to V whose matrix relative to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}.$$

Solution.

$$\begin{aligned} T(4\mathbf{b}_1 - 3\mathbf{b}_2) &= 4T(\mathbf{b}_1) - 3T(\mathbf{b}_2) && \text{(since } T \text{ is linear)} \\ &= 4(2\mathbf{b}_2 + \mathbf{b}_3) - 3(\mathbf{b}_2 + 3\mathbf{b}_3) && \text{(reading the 1}^{\text{st}} \text{ and 2}^{\text{nd}} \text{ columns of } [T]_{\mathcal{B}}) \\ &= 5\mathbf{b}_2 - 5\mathbf{b}_3 \end{aligned}$$

14. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

Solution. $[T]_{\mathcal{B}}$ is diagonal if and only if $\mathcal{B}$ is a basis of $\mathbb{R}^2$ containing eigenvectors of A, so let's find some eigenvectors. Since $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 9 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$. Hence the eigenvalues of A are 5 and -1. To find an eigenvector with eigenvalue 5 notice

$$A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

From that, I see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 5. Also,

$$A + I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

From that, I see that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 5. Hence, one basis of eigenvectors is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Notice that any scalar multiples of my chosen eigenvectors will give a different basis of eigenvectors.

22. If A is diagonalizable and B is similar to A, then B is also diagonalizable.

Proof. Assume A is diagonalizable and B is similar to A. Then, by the definitions of the terms diagonalizable and similar, $A = PDP^{-1}$ and $B = QAQ^{-1}$ for some invertible matrices P and Q and some diagonal matrix D. Hence

$$B = QAQ^{-1} = Q(PDP^{-1})Q^{-1} = (QP)D(QP)^{-1},$$

which implies that B is diagonalizable (by definition). □

23. If $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding also to eigenvalue λ.

Proof. Assume $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue λ. Then, by definition, $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$. Since P^{-1} is invertible and $\mathbf{x} \neq 0$, it follows that $P^{-1}\mathbf{x} \neq \mathbf{0}$ (because the null space of an invertible matrix is $\{\mathbf{0}\}$). Moreover,

$$B(P^{-1}\mathbf{x}) = P^{-1}AP(P^{-1}\mathbf{x}) = P^{-1}A(PP^{-1})\mathbf{x} = P^{-1}(A\mathbf{x}) = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x}).$$

Hence, by definition, $P^{-1}\mathbf{x}$ is an eigenvector of B with eigenvalue λ. □