Math 304 Answers to Selected Problems

1 Section 4.1

4. Let $L : R^2 \to R^2$ be a linear operator. If

$$L\left((1,2)^T\right) = (-2,3)^T \qquad \text{and} \qquad L\left((1,-1)^T\right) = (5,2)^T$$

determine the value of $L\left((7,5)^T\right)$.

Answer: First, we write $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$:

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

This gives us the equations

$$\begin{aligned} c_1 + c_2 &= 7 \\ 2c_1 - c_2 &= 5 \end{aligned}$$

Solving for c_1 and c_2, we get $c_1 = 4$ and $c_2 = 3$. Thus,

$$\begin{aligned} L\begin{pmatrix} 7 \\ 5 \end{pmatrix} &= L\left(4\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\ &= 4L\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3L\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 4\begin{pmatrix} -2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 18 \end{pmatrix} \end{aligned}$$

6. Determine whether the following are linear transformations from R^2 into R^3.

   (b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$

   (d) $L(\mathbf{x}) = (x_1, x_2, x_1^2 + x_2^2)^T$

   Answer:

   (b) Yes, L is a linear transformation. It is the same as $L(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

   (d) No, L is not a linear transformation. Here is why:

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$L\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$$

   Since

$$L\begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq 2L\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

   L is not a linear transformation.

7. Determine whether the following are linear operators on $R^{m \times n}$.

   (a) $L(A) = 2A$

   (c) $L(A) = A + I$

   Answer:

(a) Yes, L is a linear transformation, because

$$L(\alpha A) = 2(\alpha A) = \alpha(2A) = \alpha L(A)$$

and

$$L(A+B) = 2(A+B) = 2A + 2B = L(A) + L(B)$$

(c) No, L is not a linear transformation, because

$$L\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$L\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

Since

$$L\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq 2L\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

L is not a linear transformation.

10. For each $f \in C[0,1]$ define $L(f) = F$, where

$$F(x) = \int_0^x f(t)\,dt \qquad 0 \le x \le 1$$

Show that L is a linear operator on $C[0,1]$ and then find $L(e^x)$ and $L(x^2)$.

Answer: L is a linear operator on $C[0,1]$, because

$$L(\alpha f) = \int_0^x \alpha f(t)\,dt = \alpha \int_0^x f(t)\,dt = \alpha L(f)$$

and

$$L(f+g) = \int_0^x f(t) + g(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = L(f) + L(g)$$

We can compute $L(e^x)$ and $L(x^2)$:

$$L(e^x) = \int_0^x e^t \, dt = \left[e^t\right]_0^x = e^x - 1$$

$$L(x^2) = \int_0^x t^2 \, dt = \left[\frac{t^3}{3}\right]_0^x = \frac{x^3}{3}$$

17. Determine the kernel and the range of each of the following linear transformations on R^3.

    (a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$

    (b) $L(\mathbf{x}) = (x_1, x_2, 0)^T$

    (c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$

    Answer:

    (a) This linear transformation is the same as multiplication by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

    The kernel of this linear transformation is the same as the nullspace of the matrix. Thus, the kernel of L is $\{\mathbf{0}\}$.

    The range of this linear transformation is the same as the column space of the matrix. Thus, the range of L is R^3.

    (b) This linear transformation is the same as multiplication by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

    The kernel of this linear transformation is the same as the nullspace of the matrix. Thus, the kernel of L is spanned by the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The range of this linear transformation is the same as the column space of the matrix. Thus, the range of L is spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(c) This linear transformation is the same as multiplication by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

The kernel of this linear transformation is the same as the nullspace of the matrix. Thus, the kernel of L is spanned by the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The range of this linear transformation is the same as the column space of the matrix. Thus, the range of L is spanned by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

19. (a) Determine the kernel and range of the following linear operator on P_3:

$$L(p(x)) = xp'(x)$$

Answer: Suppose that $p(x) = ax^2 + bx + c$ is in the kernel of L.

$$L(p(x)) = 2ax^2 + bx$$

Thus, if $p(x)$ is in the kernel of L, $2ax^2 + bx = 0$ for all x, which implies that $a = 0$ and $b = 0$. Thus, every polynomial in the kernel of L is of the form $p(x) = c$. Thus,

$$\ker(L) = \mathrm{Span}(1)$$

To determine the range of L, we again consider an arbitrary polynomial $p(x) = ax^2 + bx + c$, and apply L to the polynomial. $L(p(x)) = 2ax^2 + bx$. Thus, the range of L is all polynomials of the form $2ax^2 + bx$. Thus,

$$\text{range}(L) = \text{Span}(x, x^2)$$

An alternative way to find the kernel and range of L is to find the matrix representing L with respect to the basis $[1, x, x^2]$.

$$\begin{aligned} L(1) &= 0 \\ L(x) &= x \\ L(x^2) &= 2x^2 \end{aligned}$$

Thus, the matrix representing L with respect to the basis $[1, x, x^2]$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The nullspace of this matrix is the span of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which corresponds to the polynomial 1. Thus,

$$\ker(L) = \text{Span}(1)$$

The column space of this matrix is the span of the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, which correspond to the polynomials x and $2x^2$. Thus,

$$\text{range}(L) = \text{Span}(x, 2x^2)$$

Note that $\text{Span}(x, 2x^2) = \text{Span}(x, x^2)$ so the two answers obtained for the range of L are the same.