Math 217, Linear Algebra, Fall 2002

Exam 1, October 4, 2002

Name: **Solutions**

1.(5pts) Let A be a 6×5 matrix. What must a and b be in order to define $T : \mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

If we are trying to compute $A\mathbf{x}$ then $\mathbf{x}$ must be a length 5 vector. The result of $A\mathbf{x}$ is a length 6 vector. So $a = 5$ and $b = 6$.

2.(5pts) Give an example of a 2×2 matrix A which has the following three properties: 1) $A \neq \mathbf{0}$, 2) $A \neq I_2$, and 3) $A^{\mathrm{T}} = A$.

Try $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ on for size. In fact, any matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ will work as long as if $b = 0$ then both a and d are not both 1 or both zero.

3.(10pts) Note that the matrix $A = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 8 & 7 & 6 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ is similar to the matrix $B = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{bmatrix}$. Write down an elementary matrix E such that $EA = B$. What is E^{-1}?

You can find an elementary matrix E such that $EA = B$ by performing the same row operation on the identity as you did on A. That is, replace Row(2) of the identity with Row(2) $-$ 2Row(1). Doing this yields the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix E^{-1} is such that $E^{-1}E = I_4$. So we find E^{-1} by performing the same row operation on the identity as we would perform on E to obtain I_4. That is, Row(2)$\to$Row(2)+2Row(1) transforms E to I_4, thus Row(2)$\to$Row(2)+2Row(1) transforms I_4 to

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.(5pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & 20 \\ 2 & 3 & 4 & \cdots & 21 \\ & \vdots & & & \vdots \\ 19 & 20 & 21 & \cdots & 39 \end{bmatrix}.$$

Are the columns of A linearly independent? (Explain).

We have a theorem which states that a set of p vectors in $\mathbb{R}^n$ for $p > n$ is linearly dependent. Note that A has 20 columns and only 19 rows. That means that the 20 column vectors form a set of vectors $\mathbb{R}^{19}$, and we conclude that they must be linearly dependent.

5.(10pts) Suppose that $A = LU$ for $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Find $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$. (Explain).

Let $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$. Now because $A = LU$, we can set $U\mathbf{x} = \mathbf{y}$, solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. To do this, first row reduce the augmented matrix $[L\ \mathbf{b}]$.

$$[L\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

So $\mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}$. Now row reduce the augmented matrix $[U\ \mathbf{y}]$.

$$[U\ \mathbf{y}] = \begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

So we conclude that $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 1 \end{bmatrix}$.

6.(10pts) Show that T is invertible and find a formula for T^{-1} when $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{bmatrix}$. (Explain).

Note that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Thus T is a linear transformation with standard matrix $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$. We have a theorem which says that T will be invertible if A is invertible, in which case $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$. We also have a theorem which says that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A is invertible if and only if $ad-bc \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. In this particular problem, $ad-bc = (-5)(-7)-(4)(9) = -1$ is not zero so $A^{-1} = \frac{1}{-1}\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$. A formula for T^{-1} is thus $T^{-1}(\mathbf{x}) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 + 9x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$.

7.(5pts) Ever striving for the perfect breakfast, you decide to mix Cracklin' Oat Bran, glue, and toothpaste in your cereal bowl. I remind you that Cracklin' Oat Bran contained 110 calories, 3 g of protein, 21 g of carbohydrates, and 3 g of fat per 28 g serving. Elmer's glue contained 110 calories, 2 g of protein, 25 g of carbohydrates, and 0 g of fat per 28 g serving. Finally, toothpaste contains 110 calories, 0 g of protein 23 g of carbohydrates and 1.5 g of fat per 28 gram serving. Set up a matrix equation which you could use to decide if some mixture of these three delectables could supply a serving with 110 calories, 2.25 g of protein, 23 g of carbohydrates, and 1 g of fat. Tell me what each indeterminate you use stands for. You do not need to solve this equation.

A matrix equation you can use to decide if some mixture of these three ingredients could supply a serving with the desired properties is

$$\begin{bmatrix} 110 & 110 & 110 \\ 3 & 2 & 0 \\ 21 & 25 & 23 \\ 3 & 0 & 1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 110 \\ 2.25 \\ 23 \\ 1 \end{bmatrix},$$

where x_1 is the number of servings of Cracklin' Oat Bran, x_2 is the number of servings of glue, and x_3 is the number of servings of toothpaste.

8.(10pts) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ gives a linearly dependent set. Is the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ a linearly dependent set? (Explain).

Because the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, there exist $a, b, c \in \mathbb{R}$ not all zero such that $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$. This means that $T(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3) = T(\mathbf{0})$. Because T is a linear transformation, we know that $T(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3) = aT(\mathbf{v}_1) + bT(\mathbf{v}_2) + cT(\mathbf{v}_3)$ and $T(\mathbf{0}) = \mathbf{0}$ (we proved this second assertion in class). Thus $aT(\mathbf{v}_1) + bT(\mathbf{v}_2) + cT(\mathbf{v}_3) = \mathbf{0}$ for some $a, b, c \in \mathbb{R}$ not all zero, so $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$ are linearly dependent.

9.(10pts) Can a square matrix with two identical columns be invertible? (Explain).

The Invertible Matrix Theorem tells us that if a square matrix is invertible, then its columns must be linearly independent. If two of the columns are the same, then the columns are clearly dependent. We conclude that the matrix cannot be invertible.

E. C. (5pts) Suppose that A is a 4×4 matrix, $\mathbf{v}$ and $\mathbf{u}$ are the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and the general solution to the equation $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_4 + x_3 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix}.$$

If $A\mathbf{v} = \mathbf{u}$, then is $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix}$ a solution to $A\mathbf{x} = \mathbf{u}$? (Explain).

We showed in class that any solution to the equation $A\mathbf{x} = \mathbf{u}$ can be written as $\mathbf{x} = \mathbf{p} + \mathbf{h}$ where $\mathbf{p}$ is any solution to $A\mathbf{x} = \mathbf{u}$ and $\mathbf{h}$ is a solution to $A\mathbf{x} = \mathbf{0}$. So if $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{u}$, we can write it as

$$\begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} x_4 + x_3 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix} + \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} x_4 + x_3 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

for some x_3 and x_4. Rearranging things, this means that there are x_3 and x_4 such that $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_4 + x_3 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix}$. If this were true, however, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ would be a solution to $A\mathbf{x} = \mathbf{0}$. This cannot happen because we know that $A\mathbf{v} = \mathbf{u} \neq \mathbf{0}$. Another way to see this is that there cannot possible be x_4 and x_3 making $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_4 + x_3 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix}$ (by the last two equations, $x_4 = 1$ and $x_3 = 2$, but then $x_4 + x_3 = 3 \neq 1$).