Math 220 Exam 3 Solutions Spring '99

1. Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

   a) What is the characteristic polynomial $\mathrm{ch}_A(\lambda)$?

   Solution: $\mathrm{ch}_A(\lambda) = \begin{vmatrix} \lambda - 3 & -1 \\ 0 & \lambda - 3 \end{vmatrix} = (\lambda - 3)^2$.

   b) For each eigenvalue of A, find a basis for the associated eigenspace.

   Solution: The only eigenvalue is 3. The eigenspace for 3 is the nullspace of $A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which is already reduced. There is one free variable: x_1, so we set $x_1 = t$. The first row of $A - 3I$ gives $x_2 = 0$. Thus, a vector x is in the eigenspace of 3 if

   $$x = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

   so the eigenspace is one-dimensional with basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   c) Is A diagonalizable? If so, find a matrix P such that $P^{-1}AP$ is diagonal, and display the diagonal matrix $P^{-1}AP$.

   Solution: The geometric multiplicity of 3 is 1 and the algebraic multiplicity is 2. Since these are not equal, A is not diagonalizable.

2. Let $A = \begin{bmatrix} 7 & 0 & -10 \\ 5 & 2 & -10 \\ 5 & 0 & -8 \end{bmatrix}$. Then the characteristic polynomial of A is $(\lambda - 2)^2(\lambda + 3)$.

a) For each eigenvalue of A, find a basis for the associated eigenspace.

SOLUTION: The eigenspace of 2 is the nullspace of

$$A - 2I = \begin{bmatrix} 5 & 0 & -10 \\ 5 & 0 & -10 \\ 5 & 0 & -10 \end{bmatrix}.$$

This reduces to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two free variables: x_2 and x_3. We set $x_2 = s$ and $x_3 = t$. The first row of the reduction gives $x_1 = 2x_3 = 2t$, so x is in the eigenspace if

$$x = \begin{bmatrix} 2t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, a basis for the eigenspace of 2 is given by $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The eigenspace for -3 is the nullspace of $A+3I = \begin{bmatrix} 10 & 0 & -10 \\ 5 & 5 & -10 \\ 5 & 0 & -5 \end{bmatrix}$.

This reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which has one free variable: x_3. We set $x_3 = t$. The first row then gives $x_1 = x_3 = t$, and the second row gives $x_2 = x_3 = t$, so x is in the eigenspace if

$$x = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace is one-dimensional with basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

b) Is A diagonalizable? If so, find a matrix P such that $P^{-1}AP$ is diagonal, and display the diagonal matrix $P^{-1}AP$.

SOLUTION: The algebraic and geometric multiplicities are equal for each of the two eigenvalues, so A is diagonalizable. A basis of R^3 consisting of eigenvectors is given by $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We make these the columns of the matrix P:

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, the diagonal matrix whose diagonal entries are the eigenvalues corresponding to the eigenvectors in the same-numbered columns of P.

3. Let $T : R^2 \to R^2$ be the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}.$$

a) What is the matrix for T with respect to the standard basis $\mathcal{B} = \{e_1, e_2\}$ of R^2?

SOLUTION: The matrix of T with respect to the standard basis is the standard matrix of T: $[T(e_1)|T(e_2)] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

b) Let $\mathcal{B}' = \{\mathbf{v}_2, \mathbf{v}_2\}$ be the basis given by $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is the matrix $[T]_{\mathcal{B}'}$ of T with respect to $\mathcal{B}'$? (Use the transition matrix $P = [I]_{\mathcal{B},\mathcal{B}'}$ from $\mathcal{B}'$ to $\mathcal{B}$ to find it.)

SOLUTION: We have $P = [I]_{\mathcal{B},\mathcal{B}'} = [\,[\mathbf{v}_1]_{\mathcal{B}}\,|\,[\mathbf{v}_2]_{\mathcal{B}}\,] = [\mathbf{v}_1\,|\,\mathbf{v}_2]$, because $\mathcal{B}$ is the standard basis. Thus $P = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$. Since

$\det P = -1$, $P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$. We get

$$\begin{aligned}
[T]_{\mathcal{B}'} &= P^{-1}[T]_{\mathcal{B}}P \\
&= \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 3 \\ -7 & -3 \end{bmatrix}
\end{aligned}$$

4. Let $T : P_2 \to P_2$ be given by $T(p) = p(3x+2)$.

a) What is the matrix of T with respect to the standard basis $\mathcal{B} = \{1, x, x^2\}$?

SOLUTION:

$$\begin{aligned}
[T]_{\mathcal{B}} &= [[T(1)]_{\mathcal{B}} \,|\, [T(x)]_{\mathcal{B}} \,|\, [T(x^2)]_{\mathcal{B}}] \\
&= [[1]_{\mathcal{B}} \,|\, [3x+2]_{\mathcal{B}} \,|\, [(3x+2)^2]_{\mathcal{B}}] \\
&= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}
\end{aligned}$$

b) What is $\det(T)$?

SOLUTION: $\det T = \det[T]_{\mathcal{B}}$. Since $[T]_{\mathcal{B}}$ is upper triangular, its determinant is the product of its diagonal entries. Thus, $\det T = 27$.

c) What are the rank and nullity of T?

SOLUTION: The rank and nullity of T are the rank and nullity of $[T]_{\mathcal{B}}$. Because $\det[T]_{\mathcal{B}} \neq 0$, $[T]_{\mathcal{B}}$ is invertible, so its rank is 3 and its nullity is 0.

d) What is the characteristic polynomial $\mathrm{ch}_T(\lambda)$?

SOLUTION: $\mathrm{ch}_T(\lambda) = \mathrm{ch}_{[T]_{\mathcal{B}}}(\lambda) = \begin{vmatrix} \lambda - 1 & -2 & -4 \\ 0 & \lambda - 3 & -12 \\ 0 & 0 & \lambda - 9 \end{vmatrix}$. Again, the matrix is upper triangular, so $\mathrm{ch}_T(\lambda) = (\lambda-1)(\lambda-3)(\lambda-9)$.

e) What are the eigenvalues of T?

SOLUTION: The eigenvalues are 1, 3, 9.

f) Is T diagonalizable?

SOLUTION: Yes: Each eigenvalue has algebraic multiplicity 1. Eigenvalues always have geometric multiplicity at least 1, so the two multiplicities must be equal for each eigenvalue.