Math 314H — Solutions to Quiz 3

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 3x_3 \\ x_2 \\ x_1 + x_2 - 3x_3 \\ x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

1. Find the standard matrix for T.

   Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Then

   $$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} -3 \\ 0 \\ -3 \\ -3 \end{bmatrix}.$$

   Therefore, the standard matrix for T is

   $$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 1 & 1 & -3 \\ 1 & -2 & -3 \end{bmatrix}$$

2. Is $\mathbf{b} = \begin{bmatrix} 9 \\ 4 \\ 13 \\ 1 \end{bmatrix}$ in the range of T? If so, find a vector $\mathbf{u} \in \mathbb{R}^3$ such that $T(\mathbf{u}) = \mathbf{b}$.

   The vector $\mathbf{b}$ is in the range of T if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent. The reduced row echelon form of the augmented matrix $[A\ \mathbf{b}]$ is:

   $$R = \begin{bmatrix} 1 & 0 & -3 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   Thus, the system is consistent. In fact, we see (since x_3 is free) that there are infinitely many solutions. A particular solution is $\mathbf{u} = \begin{bmatrix} 9 \\ 4 \\ 0 \end{bmatrix}$.

3. Is T one-to-one? Justify your answer.

   T is one-to-one if and only if the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. But we saw in the previous problem that $A\mathbf{x} = b$ has infinitely many solutions for at least one vector $\mathbf{b}$. Therefore, $A\mathbf{x} = \mathbf{0}$ must have infinitely many solutions.

4. Is T onto? Again, justify your answer.

   T is onto if and only if the columns of A span $\mathbb{R}^4$. This happens if and only if every row of A has a pivot element. But from R we see that A has only two pivots. Thus, T is not onto.