MACS-332A - Linear Algebra

Homework #1: 1.3 - 1.5

True or False. You must justify your answer.

1. The weights $c_1, \ldots, c_n$ in a linear combination $c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$ cannot all be zero.
   False - see page 32 (following the subsection Linear Combinations)

2. If the augmented matrix $[A \ \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.
   False - see the warning on page 44.

3. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of A.
   True - page 41. Saying that $\mathbf{b}$ is not in the set spanned by the columns of A is the same as saying that $\mathbf{b}$ is not a linear combination of the columns of A.

4. The solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = 0$.
   False - The statement is true only when the solution set of $A\mathbf{x} = 0$ is nonempty. The theorem addressing this only applies to a consistent system.

5. The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.
   True - If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.

6. If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.
   True - See parts c) and a) in the Theorem regarding logically equivalent statments.

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1. Let $\mathbf{v_1} = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$.

   Does $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ span $\mathbb{R}^3$? Justify your answer.
   $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ spans $\mathbb{R}^3$ if A has a pivot position in every row:

$$\begin{bmatrix} 0 & 3 & -4 \\ 9 & -4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & -\frac{76}{3} \end{bmatrix}$$

   Since there is a pivot in every row of A, we know that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ spans $\mathbb{R}^3$.
   Note: You may also row reduce the augmented matrix (w/ $\mathbf{b}$ as the last column) but this is unnecessary.

2. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.
Row reducing the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & 6 & 3b_1 + b_2 \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}$$

This system will only have a solution for those values of $\mathbf{b}$ such that $b_1 + 2b_2 + b_3 = 0$ or $b_1 = -2b_2 - b_3$ and not for all possible $\mathbf{b}$.

3. Given the following non-homogeneous linear system,

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= \text{-6} \end{aligned}$$

Describe the solutions of the system in parametric vector form, and provide a geometric comparison with the solution to the corresponding homogeneous system.
Computing the solution to the homogeneous system:

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which, with the free variable x_3, yields the corresponding solution, $\mathbf{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$. Now computing the solution to the non-homogeneous system:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{cases} x_1 &= -5 - 4x_3 \\ x_2 &= 3 + 3x_3 \\ x_3 &= x_3 \end{cases} \Rightarrow \mathbf{x} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

This represents a translation or shift of $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ from the homogeneous solution, $\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.
Geometrically, this a line in $\mathbb{R}^3$ parallel to $\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ passing through the origin and the point $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$.

4. Let A be a 2×4 matrix with two pivot positions.

(a) Does $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution?
If A has 2 pivot positions, then the equation $A\mathbf{x} = \mathbf{0}$ has 2 basic variables and 2 free variables. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(b) Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?
With 2 pivot positions and only 2 rows, A has a pivot position in every row. Thus by Theorem 4, $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b} \in \mathbb{R}^2$.