PRACTICE PROBLEMS Chapter 5

Section 5.1:

A1. Given $\mathbf{v} = [1, -2, 0, 1]^T$ and $\mathbf{w} = [2, -3, 1, 2]^T$

(a) Find the vector projection, $\mathbf{p}$, of $\mathbf{v}$ onto $\mathbf{w}$.

(b) Verify that $\mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.

(c) Find the distance from the vector $\mathbf{v}$ to the line spanned by the vector $\mathbf{w}$.

A2. Find the projection of $\mathbf{v} = [1, 2, 1]^T$ on the line intersection of the two planes $x + y + z = 0$ and $2x - y - z = 0$.

A3. Find the point Q of the line spanned by $\mathbf{a} = [1, 1, -1]^T$ that is closest to $\mathbf{b} = [2, 3, 4]^T$.

A4. Find the distance from the point (2,1,1) to the plane $2x - y + 2z = 0$.

Section 5.2:

B1. Determine a basis for each of the subspaces $R(\mathbf{A}^T)$, $N(\mathbf{A})$, $R(\mathbf{A})$, and $N(\mathbf{A}^T)$: $\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$

B2. Find a basis for the orthogonal complement in $\mathbb{R}^3$ of the line spanned by the vector $[-1, 2, 1]^T$.

B3. Let $W = \text{span}([1, 1, 2]^T, [2, 3, 1]^T)$.

(a) Give a geometrical description of W

(b) Find a basis for the orthogonal complement $W^{\perp}$

(c) Give a geometrical description of $W^{\perp}$.

B4: Is it possible for a matrix to have the vector [3, 1, 2] in its row space and $[2, 1, 1]^T$ in its nullspace? Explain.

B5: Find a basis for the orthogonal complement of $\text{span}([1, 2, 0, 1]^T, [3, 4, -1, 1]^T)$

Section 5.3:

C1. Let $S = \text{span}([0, 1, 0]^T, [1, 1, 1]^T)$.

(a) Find the projection matrix P that projects vectors in $\mathbb{R}^3$ onto S.

(b) Find the orthogonal projection of $\mathbf{v} = [5, -3, 4]^T$ on S.

(c) Find the distance from $\mathbf{v}$ to S.

C2. Consider the plane in $\mathbb{R}^3$ with equation $x + y + z = 0$.

(a) Find a basis for this plane.

(b) Find the orthogonal projection of $\mathbf{b} = [1, 2, 1]^T$ on to the plane.

C3. (a) Find the least squares solution to the system $A\mathbf{x} = \mathbf{b}$:

$$x_1 + x_2 = 3$$
$$2x_1 - 3x_2 = 1$$
$$0x_1 + 0x_2 = 2$$

(b) Determine the projection $\mathbf{p}$ of $\mathbf{b}$ onto $R(A)$.

(c) Calculate the residual $r(\hat{\mathbf{x}})$.

(d) Verify that $r(\hat{\mathbf{x}}) \in N(A^T)$.

C4. Find the best least squares fit by a quadratic function to the data

x	-1	0	1	2
y	0	1	3	9

C5. Find the least squares fit by a function of the form $f(x)=c_0+c_1\sin(x)+c_2\cos(x)$ to the data

x	$-\pi/2$	0	$\pi/2$	π
y	0	-5	3	7

Section 5.4:

D1. Give an example of a nonzero vector $\mathbf{x} \in \mathbb{R}^2$ for which $\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_2 = \|\mathbf{x}\|_1$.

D2. In $C([0,1])$ with inner product defined by $\int_0^1 f(x)g(x)\,dx$, determine the angle between $f(x)=2$ and $g(x)=9x^2-4$.

Section 5.5

E1. Consider the vector space C[$-\pi$, π] with inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$

(a) Show that {cos x, sin x} form an orthonormal set.
(b) Determine the distance between cos x and sin x.
(c) If $f(x)=5\cos x-2\sin x$ and $g(x)=-\cos x+3\sin x$
   (i) Use Corollary 5.5.3 to determine the value of $\langle f,g\rangle$
   (ii) Use Parseval's Formula to determine $\|f\|$ and $\|g\|$

E2. Consider the inner product space C[0,1] with inner product defined by $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$
Let S be the subspace spanned by the vectors 1 and $2x-1$.
(a) Show that 1 and $2x-1$ are orthogonal.
(b) Determine $\|1\|$ and $\|2x-1\|$.
(c) Find the least squares approximation to $\sqrt{x}$ by a function from the subspace S.

E3. Let S= span($\mathbf{v}_1$, $\mathbf{v}_2$) where $\mathbf{v}_1 = [6, -2, 2]^T$ and $\mathbf{v}_2 = [-4, 6, 18]^T$.
(a) Find an orthonormal basis for S.
(b) Find the orthogonal projection of $\mathbf{b} = [10, -12, -20]^T$ onto S.
(c) Find the distance from **b** to S.

E4. Let { $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$} be an orthonormal basis for a three dimensional subspace S of an inner product space V and let

$$\mathbf{x} = 2\,\mathbf{u}_1 - 2\,\mathbf{u}_2 + \mathbf{u}_3 \quad \text{and} \quad \mathbf{y} = 3\,\mathbf{u}_1 + \mathbf{u}_2 - 4\,\mathbf{u}_3$$

(a) Determine the value of $<\mathbf{x}, \mathbf{y}>$.
(b) Determine the value of $\|\mathbf{x}\|$.

E5. Let { $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} be an orthogonal basis for a three dimensional subspace S of an inner product space V. Let **w** be a vector in S and let $\langle \mathbf{v}_1,\mathbf{v}_1\rangle = 7$, $\langle \mathbf{v}_2,\mathbf{v}_2\rangle = 5$, $\langle \mathbf{v}_3,\mathbf{v}_3\rangle = -1$, $\langle \mathbf{v}_1,\mathbf{w}\rangle = 2$, $\langle \mathbf{v}_2,\mathbf{w}\rangle = 1$, $\langle \mathbf{v}_3,\mathbf{w}\rangle = 9$.
Express **w** as a linear combination of { $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$}

Section 5.6

F1. a) Find a basis for the plane $2x - y + z = 0$ in $\mathbb{R}^3$

b) Convert the basis you found into an orthonormal basis by using the Gram Schmidt algorithm.

F2. Assume $A = QR$ is the QR factorization of the matrix A. Prove that the normal equations for the least squares problem reduce to $R\mathbf{x} = Q^T\mathbf{b}$.

F3. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$

(a) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.
(b) <u>Use the QR factorization</u> to solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

F4: Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = [4, 2, 2, 1]^T$, $\mathbf{x}_2 = [2, 0, 0, 2]^T$, and $\mathbf{x}_3 = [1, 1, -1, 1]^T$.

F5: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$, if the Gram-Schmidt process is applied to determine the QR factorization of A, then after the first two orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are computed, we have

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \_\_ \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \_\_ \\ 0 & -\frac{2}{\sqrt{6}} & \_\_ \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \_\_ \\ 0 & \sqrt{6} & \_\_ \\ 0 & 0 & \_\_ \end{bmatrix}$$

Finish the process: determine $\mathbf{q}_3$ and fill in the third columns of Q and R.