Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer continue on the back of the page. No notes, books, or other aids may be used on the exam.

Student Id: Answer Key

1. (10 points) ____________
2. (10 points) ____________
3. (10 points) ____________
4. (10 points) ____________
5. (10 points) ____________
6. (10 points) ____________
7. (10 points) ____________
8. (10 points) ____________
9. (10 points) ____________
10. (10 points) ____________

Total (100 points) ____________

1. (10 points) Determine whether each of the following statements is true or false. No justification is required.

(a) If matrices A and B commute, then matrices A^T and B^T must commute as well.

Solution: True; $A^T B^T = (BA)^T = (AB)^T = B^T A^T$.

(b) If matrix A is symmetric and matrix S is orthogonal, then matrix $S^{-1}AS$ must be symmetric.

Solution: True; $(S^{-1}AS)^T = S^T A^T (S^{-1})^T = S^{-1}AS$ since $A^T = A$ and $S^T = S^{-1}$.

(c) There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$.

Solution: False; $\dim(V) + \dim(V^\perp) = 5$, so the dimension of one is even and the other odd.

(d) If two $n \times n$ matrices A and B are similar, then the equation $\det(A) = \det(B)$ must hold.

Solution: True; $\det(B) = \det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \det(A)$.

(e) If a real matrix A has only the eigenvalues 1 and -1, then A must be orthogonal.

Solution: False; let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, for example.

2. (10 points) Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Solution: We must solve $A^T A\vec{x}^* = A^T\vec{b}$:

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad A^T\vec{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

Solving for $\vec{x}^*$, we see that $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

3. (10 points) Find the orthogonal projection of $\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ onto the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Solution: The three vectors given are orthogonal, but not orthonormal, so we must first normalize them:

$$\vec{u}_1 = \frac{1}{2}\vec{v}_1, \quad \vec{u}_2 = \frac{1}{2}\vec{v}_2, \quad \vec{u}_3 = \frac{1}{2}\vec{v}_3$$

The orthogonal projection of $\vec{x}$ is given by:

$$(\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + (\vec{u}_3 \cdot \vec{x})\vec{u}_3 = \frac{1}{2}\vec{u}_1 + \frac{1}{2}\vec{u}_2 + \frac{1}{2}\vec{u}_3 = \frac{1}{4}(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

4. (10 points) Consider an $n \times m$ matrix A. Find $\dim(\mathrm{im}(A)) + \dim(\ker(A^T))$ in terms of m and n.

Solution: We know that $\mathrm{rank}(A) = \mathrm{rank}(A^T)$. If we use this fact, then $\dim(\ker(A^T)) = n - \mathrm{rank}(A^T) = n - \mathrm{rank}(A) = n - \dim(\mathrm{im}(A))$. Therefore, $\dim(\mathrm{im}(A)) + \dim(\ker(A^T)) = n$.

Alternatively, we know that $\dim(\ker(A^T)) = \dim(\mathrm{im}(A)^\perp)$. And, $\dim(\mathrm{im}(A)) + \dim(\mathrm{im}(A)^\perp) = n$.

5. (10 points) Find the QR-factorization of the following matrix:

$$A = \begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix}$$

Solution: We can use the Gram-Schimdt process:

$$r_{11} = \|\vec{v}_1\| = 5, \quad \vec{u}_1 = \frac{1}{5}\vec{v}_1$$

$$r_{12} = (\vec{u}_1 \cdot \vec{v}_2) = \frac{1}{5}\vec{v}_1 \cdot \vec{v}_2 = 5, \quad \vec{v}_2^\perp = \vec{v}_2 - r_{12}\vec{u}_1 = \begin{bmatrix} 21 \\ 0 \\ -28 \end{bmatrix}$$

$$r_{22} = \|\vec{v}_2^\perp\| = 35, \quad \vec{u}_2 = \frac{1}{35}\vec{v}_2^\perp = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$r_{13} = r_{23} = 0, \quad r_{33} = 2, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 4/5 & 3/5 & 0 \\ 0 & 0 & -1 \\ 3/5 & -4/5 & 0 \end{bmatrix} \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

6. (10 points) Consider the linear space P_1 with the following inner product:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

(a) Find an orthonormal basis of this space.

Solution: We can use the standard basis $(1, t)$ for P_1 and perform the Gram-Schmidt process:

$$\|1\| = \sqrt{\langle 1, 1\rangle} = \sqrt{1} = 1, \quad g_1(t) = 1$$

$$\langle 1, t\rangle = \frac{1}{2}, \quad t - \langle 1, t\rangle = t - \frac{1}{2}$$

$$\left\|t - \frac{1}{2}\right\| = \sqrt{\left\langle t - \frac{1}{2}, t - \frac{1}{2}\right\rangle} = \sqrt{\int_0^1 \left(t^2 - t + \frac{1}{4}\right) dt} = \sqrt{\frac{1}{12}}$$

$$g_2(t) = \frac{t - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = \sqrt{3}(2t - 1)$$

Therefore, an orthonormal basis is $(1, \sqrt{3}(2t - 1))$.

(b) Find the orthogonal projection onto P_1 of $f(t) = t^2$.

Solution: The orthogonal projection is given by $\langle g_1(t), f(t)\rangle g_1(t) + \langle g_2(t), f(t)\rangle g_2(t)$. We find:

$$\langle g_1(t), f(t)\rangle g_1(t) = \int_0^1 t^2 dt = \frac{1}{3}$$

$$\langle g_2(t), f(t)\rangle = \sqrt{3}\int_0^1 (2t^3 - t^2)dt = \frac{\sqrt{3}}{6}$$

Therefore, the projection is given by $\frac{1}{3} + \frac{1}{2}(2t - 1) = t - \frac{1}{6}$.

7. (10 points) Calculate the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$$

Solution: We can use Gauss-Jordan elimination to compute the determinant:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

At this point we see that $\det(A) = (-1)(-2)(-36) = -72$.

8. Given a set of n functions $f_1, \ldots, f_n$, the *Wronskian* $W(f_1, \ldots, f_n)$ is given by:

$$W(f_1, \ldots, f_n) = \det\left(\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}\right)$$

The Wronskian can be used to determine whether a set of differentiable functions is linearly independent on a given interval. If the Wronskian is nonzero at some point in an interval, then the associated functions are linearly independent.

(a) (5 points) Calculate the Wronskian of $\{x, \cos x, \sin x\}$.

Solution:

$$W = \det\left(\begin{bmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix}\right) = x(\sin^2 x + \cos^2 x) - 1(-\sin x \cos x + \sin x \cos x) = x$$

(b) (5 points) Calculate the Wronskian of $\{2x^2 + 3, x^2, 1\}$.

Solution:

$$W = \det\left(\begin{bmatrix} 2x^2 + 3 & x^2 & 1 \\ 4x & 2x & 0 \\ 4 & 2 & 0 \end{bmatrix}\right) = 1(8x - 8x) = 0$$

9. Consider a 2×2 matrix of the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are positive numbers such that $a + c = b + d = 1$. Such a matrix is called a *regular transition matrix*.

(a) (5 points) Verify that $\begin{bmatrix} b \\ c \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of A, and find the associated eigenvalues.

Solution: We can solve for each eigenvalue in turn:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} ab + bc \\ bc + cd \end{bmatrix} = \begin{bmatrix} (a+c)b \\ (b+d)c \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow \lambda_1 = 1$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a - b \\ c - d \end{bmatrix} = (a - b) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \lambda_2 = a - b$$

(b) (5 points) Using part (a), find a closed formula for the components of the following dynamical system with initial value $\vec{x}_0 = \vec{e}_1$:

$$\vec{x}(t+1) = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \vec{x}(t)$$

Solution: Using part (a), we know that our matrix has two eigenvectors: $\vec{v}_1 = \begin{bmatrix} 0.25 \\ 0.50 \end{bmatrix}$ with $\lambda_1 = 1$, and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, with $\lambda_2 = 0.25$. We now need to express $\vec{e}_1$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\left[\begin{array}{cc|c} 1/4 & 1 & 1 \\ 1/2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 4 \\ 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 4 \\ 0 & -6 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 4 \\ 0 & 1 & 2/3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & 2/3 \end{array}\right]$$

Therefore, $\vec{e}_1 = \frac{4}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2$, and $\vec{x}(t) = \frac{4}{3}\lambda_1^t \vec{v}_1 + \frac{2}{3}\lambda_2^t \vec{v}_2$. We can now write a closed formula for each component of our system:

$$x_1(t) = \left(\tfrac{4}{3}\right)(1)^t \left(\tfrac{1}{4}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{1}{4}\right)^t (1) = \tfrac{1}{3} + \tfrac{2}{3}\left(\tfrac{1}{4}\right)^t$$
$$x_2(t) = \left(\tfrac{4}{3}\right)(1)^t \left(\tfrac{1}{2}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{1}{4}\right)^t (-1) = \tfrac{2}{3} - \tfrac{2}{3}\left(\tfrac{1}{4}\right)^t$$

10. (10 points) Consider an invertible $n \times n$ matrix A. Give five different equivalent statements about A. Trivial statements such as "$3A$ is invertible" or "A^T is invertible" are not acceptable.

Solution: Any five of the following statements will suffice:

- the system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ in $\mathbb{R}^n$
- $\text{rref}(A) = I_n$
- $\text{rank}(A) = n$
- $\text{im}(A) = \mathbb{R}^n$
- $\ker(A) = \{\vec{0}\}$
- the columns of A form a basis of $\mathbb{R}^n$
- the columns of A span $\mathbb{R}^n$
- the columns of A are linearly independent
- $\det(A) \neq 0$
- 0 fails to be an eigenvalue of A