ANSWERS TO PRACTICE PROBLEMS CHAPTER 5

Section 5.1:

A1. (a) $\mathbf{p}=\frac{5}{9}[2, -3, 1, 2]^T$ **(b)** $\mathbf{v}-\mathbf{p}=-\frac{1}{9}[1, 3, 5, 1]^T$ $[\mathbf{v}-\mathbf{p}]^T\mathbf{p}=0$ thus $\mathbf{v}-\mathbf{p}$ is orthogonal to $\mathbf{p}$.

(c) distance = $\|\mathbf{v}-\mathbf{p}\| = 2/3$

A2. $\mathbf{p}=[0, 1/2, -1/2]^T$

A3. Q = (1/3, 1/3, −1/3).

A4. 5/3

Section 5.2:

B1:

Basis for R($\mathbf{A}^T$) : $\{[1, 0]^T, [0, 1]^T\}$.

Basis for R(**A**) : $\{[4, 1, 2, 3]^T, [-2, 3, 1, 4]^T\}$.

N(**A**) is comprised of the zero vector only, thus it has no basis.

Basis for N($\mathbf{A}^T$): $\left\{\left[-\frac{5}{14}, -\frac{4}{7}, 1, 0\right]^T, \left[-\frac{5}{14}, -\frac{11}{7}, 0, 1\right]^T\right\}$.

B2. $\{[2, 1, 0]^T, [1, 0, 1,]^T\}$.

B3.

(a) W is a plane through the origin in $\mathbb{R}^3$ containing the two given vectors.

(b) $\{[-5, 3, 1]^T\}$.

(c) $W^{\perp}$ is a line through the origin perpendicular to the plane W.

B4: No because the vectors are not orthogonal (their scalar product is not zero).
If $[3, 1, 2]$ is in the row space of A, then $[3, 1, 2]\in R(A^T)$ and, by the Fundamental Subspaces Theorem, $N(A)^{\perp}=R(A^T)$. Thus $[2, 1, 1]\in N(A)$ if and only if it is orthogonal to $[3, 1, 2]$.

B5: $\{[1, -1/2, 1, 0]^T, [1, -1, 0, 1]^T\}$.

Section 5.3:

C1: (a) $\mathbf{P}=\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ **(b)** $\mathbf{p}=[9/2, -3, 9/2]^T$. **(c)** distance = $\|\mathbf{v}-\mathbf{p}\| = \frac{\sqrt{2}}{2}$

C2. (a) $\{[-1, 1, 0]^T, [-1, 0, 1]^T\}$ is a possible basis

(b) $\mathbf{p}=[-1/3, 2/3, -1/3]^T$.

C3.

(a) $\hat{\mathbf{x}}=[2, 1]^T$ **(b)** $\mathbf{p}=A\hat{\mathbf{x}}=[3, 1, 0]^T$

(c) $r(\hat{\mathbf{x}})=\mathbf{b}-\mathbf{p}=[0, 0, 2]^T$

(d) Since $A^T r(\hat{\mathbf{x}})=[0, 0]^T$ we have that $r(\hat{\mathbf{x}})\in N(A^T)$, that is, the residual $\mathbf{b}-\mathbf{p}$ is orthogonal to R(A).

C4. $y = 0.55 + 1.65\,x + 1.25\,x^2$

C5. $f(x)=\frac{5}{4}+\frac{3}{2}\sin(x)-6\cos(x)$

Section 5.4:

D1. $\mathbf{x}=[1,0]^T$

D2. 1.92753 radians.

Section 5.5

E1. (a) Since the function $\cos x \sin x$ is odd, we have $\langle \cos x, \sin x\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos x \sin x \; dx=0$ So the two vectors are orthogonal.

$$\|\cos x\|^2=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x \; dx$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(1+\cos(2x))\; dx \quad \text{by the double angle formula}$$
$$=\frac{1}{2\pi}\left|x+\frac{\sin(2x)}{2}\right|_{-\pi}^{\pi}=\frac{1}{2\pi}(\pi-(-\pi))=1$$

A similar calculation applies for $\sin x$ by using the identity $\sin^2 x=\frac{1}{2}(1-\cos(2x))$. Both vectors have norm 1 and therefore they form an orthonormal set.

(b) Since cos x and sin x are orthogonal we can use the Pythagorean Law to find the distance between them: $\|\sin x-\cos x\|^2=\|\sin x\|^2+\|\cos x\|^2=1+1=2$ which gives $\|\sin x-\cos x\|=\sqrt{2}$.

(c) (i) By Corollary 5.5.3, since cos x and sin x form an orthonormal set on C[-π , π], the inner product $\langle f, g\rangle$ can be found by taking the scalar product of the coordinate vector of f with the coordinate vector of g:
$\langle f, g\rangle=5(-1)+(-2)(3)=-11$
(ii) From Parseval's formula: $\|f\|=\sqrt{5^2+(-2)^2}=\sqrt{29}$ and $\|g\|=\sqrt{(-1)^2+(3)^2}=\sqrt{10}$

E2.

(a) $\langle 1, 2x-1\rangle=\int_0^1(2x-1)\;dx=x^2-x|_0^1=0$ Hence 1 and $2x-1$ are orthogonal relative to this inner product.

(b) $\|1\|=<1,1>^{1/2}=\left(\int_0^1 1^2 dx\right)^{1/2}=1$

$\|2x-1\|=\langle 2x-1, 2x-1\rangle^{1/2}=\left(\int_0^1(2x-1)^2 dx\right)^{1/2}=1/\sqrt{3}$.

(c) $\mathbf{p}=\frac{4}{5}x+\frac{4}{15}$.

E3. **(a)** Note that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal, but not orthonormal, thus an orthonormal basis is:

$$\mathbf{u}_1=\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}=\left[\frac{3}{\sqrt{11}},-\frac{1}{\sqrt{11}},\frac{1}{\sqrt{11}}\right] \qquad \mathbf{u}_2=\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}=\left[\frac{-2}{\sqrt{94}},\frac{3}{\sqrt{94}},\frac{9}{\sqrt{94}}\right]$$

(b) $\mathbf{p}=\left[\frac{518}{47},-\frac{448}{47,}-\frac{968}{47}\right]^T$ **(c)** distance = $\|\mathbf{b}-\mathbf{p}\|=\sqrt{\frac{352}{47}}$

E4. **(a)** <**x** , **y**> 0. **(b)** $\|\mathbf{x}\|=3$.

E5. $\mathbf{w}=\frac{2}{7}\mathbf{v}_1+\frac{1}{5}\mathbf{v}_2-9\mathbf{v}_3$

Section 5.6

F1.

(a) Possible answer: $\mathbf{x}_1 = [1, 0, -2]^T$ and $\mathbf{x}_2 = [0, 1, 1)]^T$.

(b) $\mathbf{u}_1 = \left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right]^T$ $\quad\mathbf{u}_2 = \left[\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right]^T$

F2.

$$A^T A\,\mathbf{x} = A^T\,\mathbf{b}$$
$$(QR)^T(QR)\,\mathbf{x} = (QR)^T\mathbf{b}$$
$$R^T Q^T Q R\,\mathbf{x} = R^T Q^T\,\mathbf{b}$$
$$R^T R\,\mathbf{x} = R^T Q^T\,\mathbf{b} \quad \text{since } Q^T Q = I$$
$$R\mathbf{x} = Q^T\,\mathbf{b} \quad \text{since } R^T \text{ is nonsingular}$$

F3. (a)The QR decomposition is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -\sqrt{2}/6 \\ 1/3 & 4\sqrt{2}/6 \\ 2/3 & -\sqrt{2}/6 \end{bmatrix} \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}$

(b) The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the solution of $R\mathbf{x} = Q^T\,\mathbf{b}$ (see problem F2). We have $Q^T\mathbf{b} = \begin{bmatrix} 22 \\ -\sqrt{2} \end{bmatrix}$ so the system $R\mathbf{x} = Q^T\,\mathbf{b}$ reduces to $\begin{aligned} 3x_1 + \frac{5}{3}x_2 &= 22 \\ \frac{\sqrt{2}}{3}x_2 &= -\sqrt{2} \end{aligned}$ <u>Using back substitution</u> we find the solution $\hat{\mathbf{x}} = [9, -3]^T$.

F4: $\mathbf{u}_1 = \frac{1}{5}[4, 2, 2, 1]^T$ $\quad\mathbf{u}_2 = \frac{1}{5}[1, -2, -2, 4]^T$ $\quad\mathbf{u}_3 = \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]^T$

F5: $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ $\qquad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$