Diagonalization

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Diagonalization of a matrix

Recall AM $=$ algebraic multiplicity, GM $=$ geometric multiplicity of an eigenvalue

❶ If AM $=$ GM for all eigenvalues of an $n \times n$ matrix A then

- The eigenvectors form a basis of Euclidean $\mathbb{R}^n$
- The matrix $X = \left[\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array}\right]$ of eigenvectors is nonsingular
- $AX = \left[\begin{array}{c|c|c} A\mathbf{x}_1 & \cdots & A\mathbf{x}_n \end{array}\right] = \left[\begin{array}{c|c|c} \lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n \end{array}\right] = \left[\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = X\Lambda$

  with Λ diagonal, i.e., $A = X\Lambda X^{-1}$

Definition

The factorization $A = X\Lambda X^{-1}$ with Λ diagonal and X nonsingular is called a **diagonalization** of the matrix A

- Under the conditions ❶ the diagonalization exists and A is said to be **diagonalizable**

❷ If GM $<$ AM for at least one eigenvalue A is said to be **defective**

Theorem

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

The diagonalizing matrix X is not unique:

- The ordering of eigenpairs does not affect the product, as long as the ordering of the columns of X (eigenvectors) matches the ordering of the diagonal coefficients of Λ (eigenvalues)

$$\left[\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \left[\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array}\right]^{-1} = \left[\begin{array}{c|c|c} \mathbf{x}_n & \cdots & \mathbf{x}_1 \end{array}\right] \begin{bmatrix} \lambda_n & & \\ & \ddots & \\ & & \lambda_1 \end{bmatrix} \left[\begin{array}{c|c|c} \mathbf{x}_n & \cdots & \mathbf{x}_1 \end{array}\right]^{-1}$$

  It may be convenient to order the diagonal coefficients of Λ (eigenvalues) by decreasing magnitude

- The scaling of the columns of X (eigenvectors) does not affect the product

$$\left[\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \left[\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array}\right]^{-1} = \left[\begin{array}{c|c|c} \frac{\mathbf{x}_1}{\alpha_1} & \cdots & \frac{\mathbf{x}_n}{\alpha_n} \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \left[\begin{array}{c|c|c} \frac{\mathbf{x}_1}{\alpha_1} & \cdots & \frac{\mathbf{x}_n}{\alpha_n} \end{array}\right]^{-1}$$

  It may be convenient to rescale the columns of X to avoid fractions or to normalize them

Example 1. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **has eigenpairs** $\left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ **and** $\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$**.**

Determine a diagonalization of A.

$$\boxed{A = X\Lambda X^{-1} \quad \text{with} \quad X = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & \\ & 1 \end{bmatrix}}$$

A is diagonalizable

Check $AX = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} = X\Lambda \quad \checkmark$

Properties of Eigenvalues and Eigenvenctors of symmetric matrices:

1. **A symmetric matrix has real eigenvalues and real orthogonal eigenvectors** (w.r.t. Euclidean inner product)
2. **A symmetric matrix is diagonalizable**, even with repeated eigenvalues
3. The diagonalization of $A = A^T$ becomes

$$A = X\Lambda X^{-1} = Q\Lambda Q^T \quad \text{with} \quad \Lambda \text{ real} \quad \text{and} \quad Q^T = Q^{-1}$$

For instance, in Example 1, X can be turned into an orthogonal matrix by normalizing its columns:

$$X = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow[\text{columns}]{\text{normalize}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = Q \quad (Q^{-1} = Q^T)$$

Example 2. $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ **has a single eigenpair** $\left(2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ $(\textbf{GM} = 1 < 2 = \textbf{AM})$.

Determine a diagonalization of A**.**

A cannot be written $A = X\Lambda X^{-1}$, A is defective

Example 3. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **has eigenpairs** $\left(2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ **and** $\left(2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Determine a diagonalization of A**.**

$$\boxed{A = X\Lambda X^{-1} \text{ with } X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,\ \Lambda = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix} = A}$$

A is diagonalizable

Example 4. $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ **has eigenpairs** $\left(2+i, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$ **and** $\left(2-i, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$

$\Rightarrow \quad A = X\Lambda X^{-1}$ with $X = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 2+i & \\ & 2-i \end{bmatrix}$ $\quad$ A is diagonalizable

Check

$$AX = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-2i & 1+2i \\ 2+i & 2-i \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2+i & \\ & 2-i \end{bmatrix} = X\Lambda \quad \checkmark$$

- A is diagonalizable on $\mathbb{C}$ but not on $\mathbb{R}$
- Real matrices with complex (conjugate) eigenvalues (and eigenvectors) always involve rotation

$$A = \sqrt{5}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \sqrt{5}\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{matrix of rotation of angle } \theta} \quad \text{with} \quad \begin{cases} \cos\theta = \frac{2}{\sqrt{5}} \\ \sin\theta = -\frac{1}{\sqrt{5}} \end{cases}$$

$\theta = \tan^{-1}\frac{-1}{2} \approx -0.464$

$\theta \approx -26.6^o$

Matrix powers

A diagonalization makes it easier to evaluate powers of a matrix:

$$A = X\Lambda X^{-1} = X \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} X^{-1} \quad \Rightarrow \quad A^k = X\Lambda^k X^{-1} = X \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} X^{-1}$$

Proof: $A^k = \underbrace{(X\Lambda X^{-1})\cdots(X\Lambda X^{-1})}_{k \text{ times}} = X\Lambda(X^{-1}X)\Lambda(X^{-1}X)\cdots(X^{-1}X)\Lambda X^{-1} = X\Lambda^k X^{-1}$

Example 5. Find a 2×2 matrix B s.t. $B^2 = A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

From Example 1: $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$. Of the 4 possible choices

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \pm\sqrt{3} & \\ & \pm 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

satisfying $B^2 = A$ the matrix

$$A^{\frac{1}{2}} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & \\ & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} \end{bmatrix}}$$

is the only one to have positive eigenvalues.

Example 6. Find a general expression of A^n in terms of n for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

From Example 1: $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$. Then

$$A^n = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & \\ & 1^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{1+3^n}{2} & \frac{1-3^n}{2} \\ \frac{1-3^n}{2} & \frac{1+3^n}{2} \end{bmatrix}}$$

In particular:

- $\boxed{\textbf{Check}}$ For $n = 1$ we recover $A^1 = \begin{bmatrix} \frac{1+3}{2} & \frac{1-3}{2} \\ \frac{1-3}{2} & \frac{1+3}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = A$ ✓
- For $n = -1$ we get $A^{-1} = \begin{bmatrix} \frac{1+\frac{1}{3}}{2} & \frac{1-\frac{1}{3}}{2} \\ \frac{1-\frac{1}{3}}{2} & \frac{1+\frac{1}{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$
- For $n = \frac{1}{2}$ we get $A^{\frac{1}{2}} = \begin{bmatrix} \frac{1+3^{\frac{1}{2}}}{2} & \frac{1-3^{\frac{1}{2}}}{2} \\ \frac{1-3^{\frac{1}{2}}}{2} & \frac{1+3^{\frac{1}{2}}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} \end{bmatrix}$ (see Example 5)

Similar matrices

Definition

Two matrices A and B are **similar** iff $A = SBS^{-1}$ for a nonsingular matrix S

- Similar matrices represent the same linear transformation w.r.t. two different bases. S and S^{-1} are the transition matrices between the two bases.
- If A is diagonalizable, then the matrices $A = X\Lambda X^{-1}$ and Λ are similar

Properties of Eigenvalues and Eigenvectors of Similar Matrices:

1. If A and B are similar, then they have the same eigenvalues.

   Proof: Let $B = S^{-1}AS$, then

   $$\begin{aligned}\det(B - \lambda I) &= \det(S^{-1}AS - \lambda I) = \det(S^{-1}AS - \lambda S^{-1}S) = \det(S^{-1}(A - \lambda I)S) \\ &= \det(S^{-1})\det(A - \lambda I)\det(S) = \frac{1}{\det(S)}\det(A - \lambda I)\det(S) = \det(A - \lambda I).\end{aligned}$$

   A and B have the same characteristic polynomial and therefore the same eigenvalues.

2. If A is diagonalizable and B is similar to A, then B is diagonalizable and the diagonalizing matrix is $S^{-1}X$ (i.e. the eigenvectors of B are S^{-1} times the eigenvectors of A).

   Proof: Let $A = X\Lambda X^{-1}$, then $B = S^{-1}AS = S^{-1}X\Lambda X^{-1}S = T\Lambda T^{-1}$, with $T = S^{-1}X$.

Example 7. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **and** $S = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$**.**

Determine a diagonalization of $B = S^{-1}AS$**.**

From Example 1: $A = X\Lambda X^{-1}$ with $X = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$

Then $B = S^{-1}X\Lambda X^{-1}S = T\Lambda T^{-1}$ with $T = S^{-1}X = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Check $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$ ✓