MATHEMATICS 2270. Solutions for Homework # 4.

1. [10 points] Which of the following are subspaces of $\mathbb{R}^3$? Justify your answers.

(a) $V = \{(x,y,z) : xyz = 0\}$.

(b) $W = \{(x,y,z) : x+y+z = 0\}$.

(c) $U = \{(x,y,z) : x,y,z \text{ are integers } \}$.

(Recall that integers are the numbers: $0, \pm 1, \pm 2, \pm 3, ...$)

Solution. (a) V is not a subspace. To see this consider the vectors $\overrightarrow{u} = (1,0,1)$, $\overrightarrow{v} = (0,1,0)$ from V. Their sum is $(1,1,1)$, which does not belong to V.

(b) W is a subspace: if $\overrightarrow{u} = (x,y,z)$, $\overrightarrow{v} = (a,b,c)$ are vectors in W then for the vector $\overrightarrow{u} + \overrightarrow{v}$ we get:

$$(x+a)+(y+b)+(z+c) = (x+y+z)+(a+b+c) = 0+0 = 0,$$

hence $\overrightarrow{u} + \overrightarrow{v}$ is in W. If α is a real number then for the vector $\alpha\overrightarrow{u}$ we get:

$$\alpha x + \alpha y + \alpha z = \alpha(x+y+z) = 0,$$

hence $\alpha\overrightarrow{u}$ is in W.

An alternative solution. Note that $W = Ker(T)$, where $T(x,y,z) = x+y+z$ is a linear transformation. Since kernel of a linear transformation is always a subspace, we conclude that W is a subspace.

(c) U is not a subspace since, for instance, taking $\alpha = 0.5$ and a vector $(1,0,0)$ in U we get: $\alpha(1,0,0) = (0.5,0,0)$ which is not in U. $\square$

2. [5 points] Decide if the given vectors in $\mathbb{R}^3$ are linearly independent:

$$(1,2,2), (1,1,1).$$

Solution. Let's find all the numbers x and y such that

$$x\begin{bmatrix}1\\2\\2\end{bmatrix} + y\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The above equation is actually a linear system:

$$\begin{cases} x+y=0 \\ 2x+y=0 \\ 2x+y=0 \end{cases}$$

Now, let's solve this system:

$$\begin{cases} x+y=0 \\ 2x+y=0 \end{cases} \iff \begin{cases} x+y=0 \\ y=0 \end{cases} \iff \begin{cases} x=0 \\ y=0 \end{cases}$$

Thus $x = y = 0$ which means that the vectors are linearly independent. $\square$

3. [5 points] Decide if the following vectors span $\mathbb{R}^3$ (use Summary 3.3.11), justify your answer.

$$\vec{u} = (1,1,1), \vec{v} = (2,3,1), \vec{w} = (3,4,2).$$

Solution. The easy solution is to note that $\vec{u} + \vec{v} - \vec{w} = \vec{0}$, and hence these vectors are linearly dependent. By Summary 3.3.11, three linearly dependent vectors cannot span $\mathbb{R}^3$. Here is another solution.

Let's compute the rank of the matrix A whose columns are the above vectors:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore the matrix A has rank 2, which by Summary 3.3.11 means that the above vectors cannot span $\mathbb{R}^3$.

4. [10 points] Find a basis of the kernel and the nullity of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Justify your answer.

Solution. Let's compute the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the equation $A\vec{x} = \vec{0}$ (for $\vec{x} = (x,y,z)$) has solutions

$$x = -t - 2s, y = t, z = s,$$

where t and s are arbitrary numbers. To find basis of the kernel we therefore first take $t = 1, s = 0$, which yields:

$$\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

and then take $t = 0, s = 1$, which yields:

$$\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence a basis of the kernel consists of the vectors $\vec{u}$ and $\vec{v}$ above. The nullity of A is the dimension of the kernel, i.e. the number of vectors in a basis, which is 2. Hence nullity equals 2.

5. [10 points] Find a basis of the image and the dimension of the image of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution. Let's compute the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -0 \end{bmatrix}.$$

Thus the first and second columns of the reduced echelon form have leading entries and the last column does not. Therefore, the first and second columns of the matrix A form a basis of the image:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

The dimension of the image is the number of vectors in the basis, i.e. it equals to 2. Anotheer way to see this is to note that dimension of the image is the rank of A, which equals 2 in this case, since the reduced echelon form has exactly two nonzero rows. □

6. §3.4, # 10. [10 points] Find coordinates of the vector $\vec{v} = (-4, 4)$ with respect to the basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$$

in $\mathbb{R}^2$.

Solution. The transition matrix is

$$S = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}.$$

Then

$$S^{-1} = \frac{1}{6-10} \begin{bmatrix} 6 & -5 \\ -2 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -6 & 5 \\ 2 & -1 \end{bmatrix}.$$

The coordinates of $\vec{v}$ in the above basis are given by the vector:

$$S^{-1}\vec{v} = \frac{1}{4} \begin{bmatrix} -6 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}.$$ □