Orthonormal Sets

Arizona State University
School of Mathematical & Statistical Sciences

Orthogonal sets

Definition

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of nonzero vectors in a vector space V forms an **orthogonal set** w.r.t. the inner product $\langle \cdot, \cdot \rangle$ if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $i \neq j$.

Example 1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$

Do the columns $\mathbf{a}_j$ of A form an orthogonal set in Euclidean $\mathbb{R}^4$ w.r.t. the standard Euclidean inner product?

$\boxed{\textbf{YES}}$ since $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \mathbf{a}_j^T \mathbf{a}_i = 0$ whenever $i \neq j$. This is equivalent to

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{diagonal.}$$

Property Orthogonal sets are linearly independent.

Proof: If $x_1 \cdot \mathbf{v}_1 + \cdots + x_n \cdot \mathbf{v}_n = \mathbf{0}$ then, for $j = 1, \ldots, n$,

$0 = \langle \mathbf{0}, \mathbf{v}_j \rangle = \langle x_1 \cdot \mathbf{v}_1 + \cdots + x_n \cdot \mathbf{v}_n, \mathbf{v}_j \rangle = x_1 \langle \mathbf{v}_1, \mathbf{v}_j \rangle + \cdots + x_n \langle \mathbf{v}_n, \mathbf{v}_j \rangle = x_j \|\mathbf{v}_j\|^2 \Rightarrow x_j = 0$ ✓

Orthonormal Sets

Definition

An **orthonormal set** (ON) of vectors is an **orthogonal** set of **unit/normalized** vectors (w.r.t. to the norm induced by the inner product)

Example 2. Let $A = \left[\begin{array}{c|c} \mathbf{a}_1 & \mathbf{a}_2 \end{array}\right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$**.**

Do the columns $\{\mathbf{a}_1, \mathbf{a}_2\}$ **of** A **form an ON set w.r.t. the Euclidean inner product in** $\mathbb{R}^2$**?**

$\boxed{\textbf{YES}}$ since

- $\mathbf{a}_2^T \mathbf{a}_1 = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$
- $\|\mathbf{a}_1\|_2 = \|\mathbf{a}_2\|_2 = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

The orthonormality of $\{\mathbf{a}_1, \mathbf{a}_2\}$ is equivalent to

$$A^T A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Example 3. Does $\{p_0, p_1, p_2\} = \left\{\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}\,t, \sqrt{\frac{5}{2}}\left(\frac{3}{2}t^2 - \frac{1}{2}\right)\right\}$ **form an ON basis of** $\mathbb{P}_3$ **(equipped with standard operations), w.r.t.**

$$(\star) \qquad \langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt\,\textbf{?}$$

$$\|p_0\|^2 = \left\|\sqrt{\tfrac{1}{2}}\right\|^2 = \int_{-1}^{1}\left(\sqrt{\tfrac{1}{2}}\right)^2 dt = 1 \quad \checkmark$$

$$\langle p_0, p_1 \rangle = \left\langle \sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{3}{2}}\,t \right\rangle = \int_{-1}^{1} \frac{\sqrt{3}}{2}\,t\,dt = 0 \quad \checkmark$$

$$\langle p_0, p_2 \rangle = \left\langle \sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{5}{2}}\left(\tfrac{3}{2}t^2 - \tfrac{1}{2}\right) \right\rangle = \int_{-1}^{1} \frac{\sqrt{5}}{2}\left(\tfrac{3}{2}t^2 - \tfrac{1}{2}\right)dt = 0 \quad \checkmark$$

$$\|p_1\|^2 = \left\|\sqrt{\tfrac{3}{2}}\,t\right\|^2 = \int_{-1}^{1} \tfrac{3}{2}\,t^2\,dt = 1 \quad \checkmark$$

$$\langle p_1, p_2 \rangle = \left\langle \sqrt{\tfrac{3}{2}}\,t, \sqrt{\tfrac{5}{2}}\left(\tfrac{3}{2}t^2 - \tfrac{1}{2}\right) \right\rangle = \int_{-1}^{1} \frac{\sqrt{15}}{2}\,t\left(\tfrac{3}{2}t^2 - \tfrac{1}{2}\right)dt = 0 \quad \checkmark$$

$$\|p_2\|^2 = \left\|\sqrt{\tfrac{5}{2}}\left(\tfrac{3}{2}t^2 - \tfrac{1}{2}\right)\right\|^2 = \int_{-1}^{1} \tfrac{5}{2}\left(\tfrac{3}{2}t^2 - \tfrac{1}{2}\right)^2 dt = 1 \quad \checkmark$$

$\Rightarrow$ $\boxed{\textbf{YES}, \{p_0, p_1, p_2\}\ \textbf{IS}\text{ an ON basis of } \mathbb{P}_3 \text{ w.r.t. } (\star)}$

Example 4. **Does** $\{\sin t, \cos t\}$ **form an ON set in** $\mathcal{C}([-\pi,\pi])$ **w.r.t. the inner product**

$$(\star) \qquad \langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt\textbf{?}$$

$$\langle \sin t, \cos t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin t\cos t\,dt = 0$$

$$\|\sin t\|^2 = \langle \sin t, \sin t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 t\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2t)\,dt = 1$$

$$\|\cos t\|^2 = \langle \cos t, \cos t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 t\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\cos 2t)\,dt = 1$$

$\Rightarrow$ $\boxed{\textbf{YES}, \{\sin t, \cos t\}\ \textbf{IS}\ \text{an ON set in } \mathcal{C}([-\pi,\pi]) \text{ w.r.t } (\star)}$

Properties of ON bases

If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of V then

Property 1 The matrix representation C of $\langle \cdot, \cdot \rangle$ is the identity matrix

Proof: $C = \begin{bmatrix} \|\mathbf{v}_1\|^2 & \cdots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \vdots & & \vdots \\ \langle \mathbf{v}_1, \mathbf{v}_n \rangle & \cdots & \|\mathbf{v}_n\|^2 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n$

Property 2 $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{y}^T \mathbf{x}$ where $\mathbf{x} = [\mathbf{u}]_{\mathcal{B}}$ and $\mathbf{y} = [\mathbf{v}]_{\mathcal{B}}$

The inner product of two vectors is the Euclidean inner product of their representations

Proof: In general: $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{y}^T C \mathbf{x}$, where C is the matrix representation of the inner product (see previous section). By Property 1, $C = I_n$ and the statement follows.

Property 3 If $\mathbf{u} = x_1 \cdot \mathbf{v}_1 + \cdots + x_n \cdot \mathbf{v}_n$, then $x_j = \langle \mathbf{u}, \mathbf{v}_j \rangle$ for $1 \leq j \leq n$

The coefficients of the linear combination are easily determined

Proof: Since $[\mathbf{v}_j]_{\mathcal{B}} = \mathbf{e}_j$, letting $\mathbf{v} = \mathbf{v}_j$ in Property 2 yields

$$\langle \mathbf{u}, \mathbf{v}_j \rangle = \mathbf{e}_j^T \mathbf{x} = x_j \quad \text{for } 1 \leq j \leq n$$

Property 4 $\|\mathbf{u}\|^2 = \|\mathbf{x}\|_2^2$ where $\mathbf{x} = [\mathbf{u}]_{\mathcal{B}}$ (Parseval's Formula)

Proof: Letting $\mathbf{v} = \mathbf{u}$ in Property 2 yields $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|_2^2$

Property 5 In $V = \mathbb{R}^n$ with Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T \mathbf{u}$:

Let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$. If $\mathbf{u} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = A\mathbf{x}$

then $$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^T \mathbf{u} \\ \vdots \\ \mathbf{v}_n^T \mathbf{u} \end{bmatrix} = A^T \mathbf{u}$$

Proof: The result follows from the fact that if $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$, then $A^T A = AA^T = I_n$

Example 5. **Write** $\begin{bmatrix}1\\1\\1\end{bmatrix}$ **as a LC of** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}, \begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix} \right\}$ **in Euclidean** $\mathbb{R}^3$

Verify that

$$A = \left[\mathbf{v}_1 \,\middle|\, \mathbf{v}_2 \,\middle|\, \mathbf{v}_3\right] = \begin{bmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{bmatrix} \text{ satisfies } A^T A = I, \text{ i.e., } A^{-1} = A^T$$

i.e., $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms an ON basis w.r.t. $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^T\mathbf{x}$ in Euclidean $\mathbb{R}^3$. Then

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = x_1\begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix} + x_2\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix} + x_3\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix} = A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = A^T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{5}{3}\end{bmatrix}$$

i.e., $\boxed{\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix} + \frac{1}{3}\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix} + \frac{5}{3}\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}}$

Check $\begin{bmatrix}\frac{2}{9}+\frac{2}{9}+\frac{5}{9}\\ -\frac{2}{9}+\frac{1}{9}+\frac{10}{9}\\ \frac{1}{9}-\frac{2}{9}+\frac{10}{9}\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ ✓

Example 6. Use the fact that $\{\sin t, \cos t\}$ forms an ON set in $C([-\pi,\pi])$ w.r.t.

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$$

to determine $\langle f, g\rangle$, $\|f\|$, $\|g\|$ for $f(t) = 3\cos t + 2\sin t$ and $g(t) = \cos t - \sin t$

The representation vectors of f and g w.r.t. the given ON set are

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

respectively.

1. By Property 2

$$\langle f, g\rangle = \mathbf{y}^T\mathbf{x} = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boxed{1}$$

2. By Parseval formula

$$\|f\|^2 = \mathbf{x}^T\mathbf{x} = 2^2 + 3^2 = 13, \quad \|g\|^2 = \mathbf{y}^T\mathbf{y} = (-1)^2 + 1^2 = \boxed{2}$$

Orthogonality and Least Squares

Theorem

Let $A = \begin{bmatrix} \mathbf{a}_1 \cdots \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{m\times n}$ *s.t.* $A^T A = I_n$*. Then*

- $m \geq n$ *and* $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *is an ON set w.r.t.* $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$ *in Euclidean* $\mathbb{R}^m$
- *The least squares solution to* $A\mathbf{x} = \mathbf{b}$ *is*
$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = A^T \mathbf{b} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{b} \\ \vdots \\ \mathbf{a}_n^T \mathbf{b} \end{bmatrix}$$
- *The orthogonal projection of* $\mathbf{b}$ *onto* $R(A)$ *is*
$$\mathbf{p} = A\mathbf{x} = A(A^T \mathbf{b}) = (\mathbf{a}_1^T \mathbf{b})\mathbf{a}_1 + \ldots + (\mathbf{a}_n^T \mathbf{b})\mathbf{a}_n$$
- $\mathbf{p} = P\mathbf{b}$ *where*
$$P = A(A^T A)^{-1} A^T = AA^T = \mathbf{a}_1 \mathbf{a}_1^T + \cdots + \mathbf{a}_n \mathbf{a}_n^T$$
is the matrix of the orthogonal projection onto $R(A)$

Orthogonal projection on subspaces of general vector spaces

Theorem

Let $\mathbf{b} \in V$ *and* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *be an ON basis for a subspace* S *of* $(V, +, \cdot)$*.*

Then the orthogonal projection $\mathbf{p}$ *of* $\mathbf{b}$ *onto* S *is given by*

$$\mathbf{p} = \langle \mathbf{b}, \mathbf{u}_1 \rangle \cdot \mathbf{u}_1 + \cdots + \langle \mathbf{b}, \mathbf{u}_n \rangle \cdot \mathbf{u}_n.$$

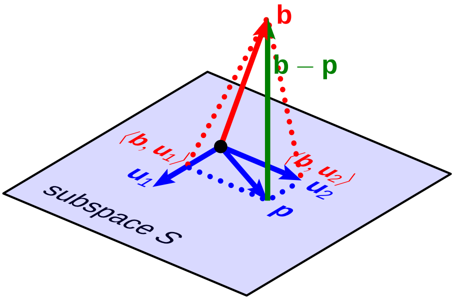
Properties:

1. $\mathbf{p} \in S$
2. $\langle \mathbf{p}, \mathbf{u} \rangle = \langle \mathbf{b}, \mathbf{u} \rangle$ for all $\mathbf{u} \in S$

   This is equivalent to

   $$\langle \mathbf{b} - \mathbf{p}, \mathbf{u} \rangle = \langle \mathbf{b}, \mathbf{u} \rangle - \langle \mathbf{p}, \mathbf{u} \rangle = 0$$

   for all $\mathbf{u} \in S$, i.e., $\mathbf{b} - \mathbf{p} \perp S$.



3. The induced norm $\|\mathbf{b} - \mathbf{u}\|$ is minimum for $\mathbf{u}$ in S when $\mathbf{u} = \mathbf{p}$

Example 7. **Use the fact that $\{\mathbf{u}_1, \mathbf{u}_2\} = \{1, \sqrt{3}(2t-1)\}$ is an ON basis of $\mathbb{P}_2$ w.r.t. $\langle p, q\rangle = \int_0^1 p(t)q(t)\,dt$ to find the $\perp$ projection p of t^2 onto $\mathbb{P}_2$**

❶ Verify $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an ON set:

$$\langle \mathbf{u}_1, \mathbf{u}_2\rangle = \int_0^1 \sqrt{3}(2t-1)dt = 0,\ \|\mathbf{u}_1\|^2 = \int_0^1 dt = 1,\ \|\mathbf{u}_2\|^2 = \int_0^1 3(2t-1)^2 dt = 1$$

❷ $p(t) = \langle t^2, \mathbf{u}_1\rangle \mathbf{u}_1 + \langle t^2, \mathbf{u}_2\rangle \mathbf{u}_2$ with

$$\langle t^2, \mathbf{u}_1\rangle = \langle t^2, 1\rangle = \int_0^1 t^2\,dt = \frac{1}{3}$$

$$\langle t^2, \mathbf{u}_2\rangle = \langle t^2, \sqrt{3}(2t-1)\rangle$$

$$= \int_0^1 t^2\,\sqrt{3}(2t-1)\,dt = \frac{1}{6}\sqrt{3}$$

$$\Rightarrow \quad p(t) = \frac{1}{3}\cdot 1 + \frac{1}{6}\sqrt{3}\cdot\sqrt{3}(2t-1)$$

$$= \frac{1}{3} + t - \frac{1}{2} = \boxed{t - \frac{1}{6}}$$