Math 415 Spring 2011 Homework 6 Solutions

Sec 4.1, No 5: Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$:

(a) $L(\mathbf{x}) = (x_2, x_3)^T$

Solution:

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= L((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)^T) \\ &= (\alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)^T \\ &= \alpha(x_2, x_3)^T + \beta(y_2, y_3)^T \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

so this one is linear

(b) $L(\mathbf{x}) = (0, 0)^T$

Solution:

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= L((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)^T) \\ &= (0, 0)^T \\ &= \alpha(0, 0)^T + \beta(0, 0)^T \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

so this one is linear

(c) $L(\mathbf{x}) = (1 + x_1, x_2)^T$

Solution: Since linearity implies, in particular, that $L(\alpha\mathbf{x}) = \alpha L(\mathbf{x})$, if we set $\alpha = 0$ we see that $L(\mathbf{0}) = \mathbf{0}$. But for the L we are given we see that $L(0) = (1, 0)^T \neq (0, 0)^T$, so this one is not linear.

(d) $L(\mathbf{x}) = (x_3, x_1 + x_2)^T$

Solution:

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= L((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)^T) \\ &= (\alpha x_3 + \beta y_3, (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2))^T \\ &= (\alpha x_3 + \beta y_3, \alpha(x_1 + x_2) + \beta(y_1 + y_2))^T \\ &= \alpha(x_3, x_1 + x_2)^T + \beta(y_3, y_1 + y_2)^T \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

so this one is linear

Sec 4.2, No 3: For each of the following linear operators L on $\mathbb{R}^3$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

(a) $L((x_1, x_2, x_3)^T) = (x_3, x_2, x_1)^T$

Solution: The matrix we need has columns

$$\begin{aligned} \mathbf{a}_1 &= L(\mathbf{e}_1) = L((1, 0, 0)^T) = (0, 0, 1)^T \\ \mathbf{a}_2 &= L(\mathbf{e}_2) = L((0, 1, 0)^T) = (0, 1, 0)^T \\ \mathbf{a}_3 &= L(\mathbf{e}_3) = L((0, 0, 1)^T) = (1, 0, 0)^T \end{aligned}$$

so

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{Check} \qquad A\mathbf{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = L(\mathbf{x})$$

(b) $L((x_1, x_2, x_3)^T) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$

Solution:The matrix we need has columns

$$\begin{aligned} \mathbf{a}_1 &= L(\mathbf{e}_1) = L((1,0,0)^T) = (1,1,1)^T \\ \mathbf{a}_2 &= L(\mathbf{e}_2) = L((0,1,0)^T) = (0,1,1)^T \\ \mathbf{a}_3 &= L(\mathbf{e}_3) = L((0,0,1)^T) = (0,0,1)^T \end{aligned}$$

so

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\text{Check} \qquad A\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix} = L(\mathbf{x})$$

(c) $L((x_1, x_2, x_3)^T) = (2x_3, x_2 + 3x_1, 2x_1 - x_3)^T$

Solution:The matrix we need has columns

$$\begin{aligned} \mathbf{a}_1 &= L(\mathbf{e}_1) = L((1,0,0)^T) = (0,3,2)^T \\ \mathbf{a}_2 &= L(\mathbf{e}_2) = L((0,1,0)^T) = (0,1,0)^T \\ \mathbf{a}_3 &= L(\mathbf{e}_3) = L((0,0,1)^T) = (2,0,-1)^T \end{aligned}$$

so

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\text{Check} \qquad A\mathbf{x} = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ 3x_1 + x_2 \\ 2x_1 - x_3 \end{pmatrix} = L(\mathbf{x})$$

Sec 4.2, No 13: Let L be the linear mapping of P_2 into $\mathbb{R}^2$ defined by

$$L(p(x)) = \begin{pmatrix} \int_0^1 p(x)dx \\ p(0) \end{pmatrix}$$

Find a matrix A such that

$$L(\alpha + \beta x) = A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Solution: Take as a basis of P_2 the polynomials $p_1(x) = 1, p_2(x) = x$. Then we compute

$$\mathbf{a}_1 = L(p_1) = \begin{pmatrix} \int_0^1 p_1(x)dx \\ p_1(0) \end{pmatrix} = \begin{pmatrix} \int_0^1 dx \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{a}_2 = L(p_2) == \begin{pmatrix} \int_0^1 p_2(x)dx \\ p_2(0) \end{pmatrix} = \begin{pmatrix} \int_0^1 xdx \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Therefore the matrix we want is:

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\text{Check} \qquad A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \frac{1}{2}\beta \\ \alpha \end{pmatrix}$$

$$L(\alpha + \beta x) = \begin{pmatrix} \int_0^1 (\alpha + \beta x)dx \\ (\alpha + \beta x)|_{x=0} \end{pmatrix} = \begin{pmatrix} \alpha + \frac{1}{2}\beta \\ \alpha \end{pmatrix}$$