

Chapter 4

Subspaces

Subspaces are a special type of subset of a vector space \mathbf{R}^n . They arise naturally in connection with linear transformations and systems of linear equations. Section 4.1 provides an introduction, and includes examples of subspaces and a general procedure for determining if a subset is a subspace. Section 4.2 introduces an important type of collection of vectors and a means for measuring (roughly) the size of a subspace. Section 4.3 connects the concept of subspaces to matrices.

4.1 Introduction to Subspaces

In Section 2.2, we introduced the hypothetical VecMobile II in \mathbf{R}^3 , the vehicle that can move only in the direction of vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Recall that this model of the VecMobile II can travel to any location in $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, the set of all linear combinations of \mathbf{u}_1 and \mathbf{u}_2 , which forms a plane in \mathbf{R}^3 (Figure 1). This subset of \mathbf{R}^3 is an example of a *subspace*. In many ways, a subspace can be viewed as a vector space contained within a vector space.

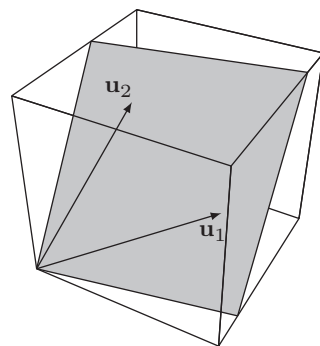


Figure 1: Subspace traversed by the VecMobile II

► The vectors in \mathbf{R}^n , together with vector arithmetic, form a vector space. (See Definition 2.1, Section 2.1)

Definition 4.1. A subset S of \mathbf{R}^n is a **subspace** if S satisfies the following three conditions:

- (a) S contains $\mathbf{0}$, the zero vector;
- (b) If \mathbf{u} and \mathbf{v} are in S , then $\mathbf{u} + \mathbf{v}$ is also in S ;
- (c) If r is a real number and \mathbf{u} is in S , then $r\mathbf{u}$ is also in S .

Def: Subspace

A subset of \mathbf{R}^n that satisfies condition (b) above is said to be **closed under addition**, and if it satisfies condition (c), then it is **closed under scalar multiplication**. Closure under addition and scalar multiplication insures that arithmetic performed on vectors in a subspace produce other vectors in the subspace.

Def: Closed Under Addition, Scalar Multiplication

Is S a Subspace?

To determine if a given subset S is a subspace, an easy place to start is with condition (a) of Definition 4.1, which states that every subspace must contain $\mathbf{0}$. A moment's thought reveals that this is equivalent to the statement

If $\mathbf{0}$ is not in a subset S of \mathbf{R}^n , then S is not a subspace of \mathbf{R}^n .

This provides a way to show that a subset S is *not* a subspace: If $\mathbf{0}$ is not in S , then S is not a subspace.

Note that the converse is not true: just because $\mathbf{0}$ is in S does not guarantee that S is a subspace, because conditions (b) and (c) must also be satisfied. For example, despite containing $\mathbf{0}$, the subset of \mathbf{R}^2 consisting of the x -axis and y -axis shown in Figure 2 is not a subspace of \mathbf{R}^2 , because the set is not closed under addition. (However, it is closed under scalar multiplication.)

Here is another quick example:

Example 1. Let S consist of all solutions $\mathbf{x} = (x_1, x_2)$ to the linear system

$$\begin{aligned} -3x_1 + 2x_2 &= 17 \\ x_1 - 5x_2 &= -1 \end{aligned}$$

Is S a subspace of \mathbf{R}^2 ?

Solution: We know that S is a subset of \mathbf{R}^2 . However, note that $\mathbf{x} = (0, 0)$ is *not* a solution to the given system. Hence $\mathbf{0}$ is not in S , and so S cannot be a subspace of \mathbf{R}^2 . ■

This section opened with the statement that the span of two vectors forms a subspace of \mathbf{R}^3 . This claim generalizes to the span of any finite set of vectors in \mathbf{R}^n , and provides a useful way to determine if a set of vectors is a subspace.

Theorem 4.2. Let $S = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a subset of \mathbf{R}^n . Then S is a subspace of \mathbf{R}^n .

Proof: To show that a subset is a subspace, we need to verify that the three conditions given in the definition are satisfied.

- (a) Since $\mathbf{0} = 0\mathbf{u}_1 + \dots + 0\mathbf{u}_m$, it follows that S contains $\mathbf{0}$.
- (b) Suppose that \mathbf{v} and \mathbf{w} are in S . Then there exist scalars r_1, r_2, \dots, r_m and s_1, s_2, \dots, s_m such that

$$\begin{aligned} \mathbf{v} &= r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \dots + r_m\mathbf{u}_m, \\ \mathbf{w} &= s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \dots + s_m\mathbf{u}_m. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \dots + r_m\mathbf{u}_m) + (s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \dots + s_m\mathbf{u}_m) \\ &= (r_1 + s_1)\mathbf{u}_1 + (r_2 + s_2)\mathbf{u}_2 + \dots + (r_m + s_m)\mathbf{u}_m, \end{aligned}$$

which shows that $\mathbf{v} + \mathbf{w}$ is in $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ and hence is in S .

► Geometrically, condition (a) says that the graph of a subspace must pass through the origin.

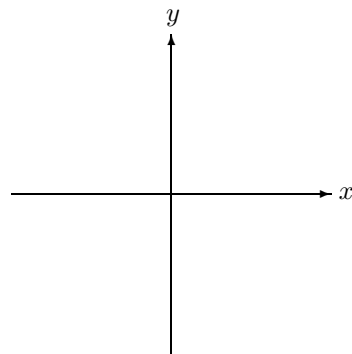


Figure 2: The coordinate axes are not a subspace of \mathbf{R}^2 .

► Theorem 4.2 also holds for the span of an infinite set of vectors. However, we do not require this case and including it would introduce additional technical issues, so we leave it out.

(c) If t is a real number, then taking \mathbf{v} as in part (b), we have

$$\begin{aligned} t\mathbf{v} &= t(r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_m\mathbf{u}_m) \\ &= tr_1\mathbf{u}_1 + tr_2\mathbf{u}_2 + \cdots + tr_m\mathbf{u}_m, \end{aligned}$$

so that $t\mathbf{v}$ is in S .

Since parts (a)–(c) of the definition hold, S is a subspace. ■

If $S = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$, then it is common to say that S is the **subspace spanned** (or **subspace generated**) by $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Def: Subspace Spanned,
Subspace Generated

To determine if a subset S is a subspace, try following these steps:

- Step 1. Check if $\mathbf{0}$ is in S . If not, then S is not a subspace.
- Step 2. If you can show that S is generated by a set of vectors, then by Theorem 4.2 S is a subspace.
- Step 3. Try to verify that conditions (b) and (c) of the definition are met. If so, then S is a subspace. If you cannot show that they hold, then you are likely to uncover a counterexample showing that they do not hold, which demonstrates that S is not a subspace.

Let's try this out on some examples.

Example 2. Determine if $S = \{\mathbf{0}\}$ and $S = \mathbf{R}^n$ are subspaces of \mathbf{R}^n .

Solution: Since $\mathbf{0}$ is in both $S = \{\mathbf{0}\}$ and $S = \mathbf{R}^n$, Step 1 is no help, so we move to Step 2. Since $\{\mathbf{0}\} = \text{Span}\{\mathbf{0}\}$, by Theorem 4.2 the set $S = \{\mathbf{0}\}$ is a subspace. We also have $\mathbf{R}^n = \text{Span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \quad (1)$$

Thus \mathbf{R}^n is a subspace of itself. (These are sometimes called the **trivial subspaces** of \mathbf{R}^n .) ■

Example 3. Let ℓ_1 denote a line through the origin in \mathbf{R}^2 (Figure 3), and let ℓ_2 denote a line that does not pass through the origin in \mathbf{R}^2 (Figure 4). Do the points on ℓ_1 form a subspace? Do the points on ℓ_2 form a subspace?

Solution: Since ℓ_1 passes through the origin, $\mathbf{0}$ is on ℓ_1 , so Step 1 is not helpful. Moving to Step 2, suppose that we pick any nonzero vector \mathbf{u} on ℓ_1 . Then all points on ℓ_1 have the form $r\mathbf{u}$ for some scalar r . Thus $\ell_1 = \text{Span}\{\mathbf{u}\}$, so ℓ_1 is a subspace.

On the other hand, the line ℓ_2 does not contain $\mathbf{0}$, so ℓ_2 is not a subspace. ■

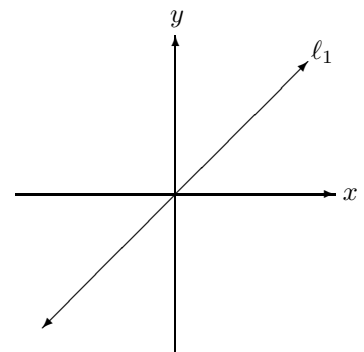


Figure 3: ℓ_1 is a subspace.

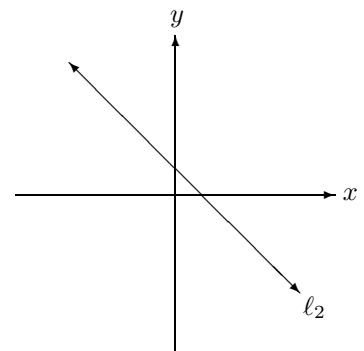


Figure 4: ℓ_2 does not pass through the origin, so is not a subspace.

Example 4. Let S be the subset of \mathbf{R}^3 consisting of all vectors of the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

such that $v_1 + v_2 + v_3 = 0$. Is S a subspace of \mathbf{R}^3 ?

Solution: Starting with Step 1, we see that setting $v_1 = v_2 = v_3 = 0$ implies $\mathbf{0}$ is in S , so we still cannot conclude anything. It is not hard to come up with examples of vectors in S , but is more challenging to find a set of vectors that spans S , so we cannot easily apply Theorem 4.2. Proceeding to Step 3, let's check to see if conditions (b) and (c) of the definition are satisfied:

(b) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be in S . Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix},$$

and since

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0,$$

it follows that $\mathbf{u} + \mathbf{v}$ is in S .

(c) With $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in S as above, for any scalar r we have $r\mathbf{v} = \begin{bmatrix} rv_1 \\ rv_2 \\ rv_3 \end{bmatrix}$. Since

$$rv_1 + rv_2 + rv_3 = r(v_1 + v_2 + v_3) = 0,$$

$r\mathbf{v}$ is also in S .

Since all conditions of the definition are satisfied, we conclude that S is a subspace of \mathbf{R}^3 . ■

It is not hard to extend the result in Example 4 to \mathbf{R}^n . Let S be the set of all vectors of the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

such that $v_1 + \cdots + v_n = 0$. Then S is a subspace of \mathbf{R}^n . (See Exercise 15.)

Homogeneous Systems and Null Spaces

The set of solutions to a homogeneous linear system forms a subspace. For instance, let A be the 3×4 matrix

$$A = \begin{bmatrix} 3 & -1 & 7 & -6 \\ 4 & -1 & 9 & -7 \\ -2 & 1 & -5 & 5 \end{bmatrix}.$$

Using our usual row operation algorithm, we can show that all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = r \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

where r and s can be any real numbers. Put another way, the set of solutions to $A\mathbf{x} = \mathbf{0}$ is equal to

$$\text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and so the set of solutions is a subspace of \mathbf{R}^4 . In fact, it turns out that the set of solutions to any homogeneous linear system forms a subspace.

Theorem 4.3. *If A is an $n \times m$ matrix, then the set of solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ forms a subspace of \mathbf{R}^m .*

Proof: There is no obvious set of vectors whose span equals the set of solutions, so we cannot easily apply Theorem 4.2 to show that the set forms a subspace. So instead we go straight to the definition:

- (a) Since $\mathbf{x} = \mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{0}$, the zero vector $\mathbf{0}$ is in the set of solutions.
- (b) Suppose that \mathbf{u} and \mathbf{v} are both solutions to $A\mathbf{x} = \mathbf{0}$. Then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0},$$

so that $\mathbf{u} + \mathbf{v}$ is in the set of solutions.

- (c) Let \mathbf{u} be a solution as in (b) and let r be a scalar. Then

$$A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0},$$

and so $r\mathbf{u}$ is also in the set of solutions.

Since all three conditions of the definition are met, the set of solutions is a subspace of \mathbf{R}^n . ■

A subspace given by the set of solutions to a homogeneous linear system goes by a special name:

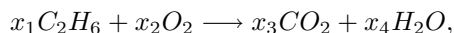
Definition 4.4. *If A is an $n \times m$ matrix, then the set of solutions to $A\mathbf{x} = \mathbf{0}$ is called the **null space** of A , denoted by $\text{null}(A)$.*

Def: Null Space

From Theorem 4.3 it follows that a null space is a subspace.

Subspaces arise naturally in a variety of applications, such as balancing chemical equations. This topic is discussed in detail in Section 1.4.

Example 5. Ethane burns in oxygen to produce carbon dioxide and steam. The chemical reaction is described using the notation



where the subscripts on the elements indicate the number of atoms in each molecule. Describe the subspace of values that will balance this equation.

Solution: To balance the equation, we need to find values for x_1 , x_2 , x_3 , and x_4 so that the number of atoms for each element is the same on both sides of the equation. Doing so yields the linear system

$$\begin{array}{rclcl} 2x_1 & - & x_3 & = & 0 & (\text{Carbon atoms}) \\ 6x_1 & & & - & 2x_4 = 0 & (\text{Hydrogen atoms}) \\ 2x_2 - 2x_3 - & x_4 = 0 & & & & (\text{Oxygen atoms}) \end{array}$$

Applying our usual methods, we find that the general solution to this system is

$$\begin{array}{l} x_1 = 2s \\ x_2 = 7s \\ x_3 = 4s \\ x_4 = 6s \end{array} \quad \text{or} \quad \mathbf{x} = s \begin{bmatrix} 2 \\ 7 \\ 4 \\ 6 \end{bmatrix},$$

where s can be any real number. Put another way, the set of solutions is equal to

$$\text{Span} \left\{ \begin{bmatrix} 2 \\ 7 \\ 4 \\ 6 \end{bmatrix} \right\},$$

which makes it clear that the set is a subspace of \mathbf{R}^4 . ■

Kernel and Range of a Linear Transformation

Two sets associated with any linear transformation T are subspaces. Recall that the range of T is the set of all vectors \mathbf{y} such that $T(\mathbf{x}) = \mathbf{y}$ for some \mathbf{x} , and is denoted by $\text{range}(T)$. The **kernel** of T is the set of vectors \mathbf{x} such that $T(\mathbf{x}) = \mathbf{0}$. The kernel of T is denoted by $\ker(T)$. Theorem 4.5 shows that the range and kernel are subspaces.

Theorem 4.5. Let $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$ be a linear transformation. Then the kernel of T is a subspace of the domain \mathbf{R}^m and the range of T is a subspace of the codomain \mathbf{R}^n .

Proof: Because $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$ is a linear transformation, it follows (Theorem 3.8, Section 3.1) that there exists an $n \times m$ matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_m]$ such that $T(\mathbf{x}) = A\mathbf{x}$. Thus $T(\mathbf{x}) = \mathbf{0}$ if and only if $A\mathbf{x} = \mathbf{0}$. This in turn implies that

$$\ker(T) = \text{null}(A),$$

and therefore by Theorem 4.3 the kernel of T is a subspace of the domain \mathbf{R}^m .

► Ethane is a gas similar to propane. Its primary use in the chemical industry is to make polyethylene, a common form of plastic.

Def: Kernel

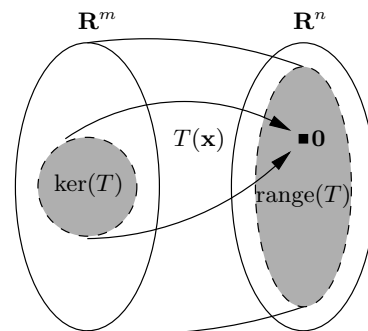


Figure 5: The kernel and range of T

Now consider the range of T . By Theorem 3.3(b), we have

$$\text{range}(T) = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_m\}.$$

Hence by Theorem 4.2, since $\text{range}(T)$ is equal to the span of a set of vectors, the range of T is a subspace of the codomain \mathbf{R}^n . ■

Example 6. Suppose that $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -3x_1 + 6x_2 \\ 2x_1 - 4x_2 \end{bmatrix}.$$

Find $\ker(T)$ and $\text{range}(T)$.

Solution: We have $T(\mathbf{x}) = A\mathbf{x}$ for

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \\ 2 & -4 \end{bmatrix}.$$

To find the null space of A , we solve the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. We have

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 6 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is equivalent to the single equation $x_1 - 2x_2 = 0$. Since $\ker(T) = \text{null}(A)$, it follows that if we let $x_2 = s$, then $x_1 = 2s$ and thus

$$\ker(T) = s \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (s \text{ real}) \quad \text{or} \quad \ker(T) = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Because the range of T is equal to the span of the columns of A , we have

$$\text{range}(T) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\},$$

because $\mathbf{a}_1 = -2\mathbf{a}_2$. ■

In Theorem 3.5 in Section 3.1 we showed that a linear transformation T is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. The next theorem formulates this result in terms of $\ker(T)$.

Theorem 4.6. Let $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$ be a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

The proof is covered in Exercise 71. As a quick application, in Example 6 we saw that $\ker(T) \neq \{\mathbf{0}\}$, so we can conclude from Theorem 4.6 that T is not one-to-one.

► Remember that the kernel is a subspace of the *domain*, while the range is a subspace of the *codomain*.

The Big Theorem – Version 4

Theorem 4.6 allows us to add another condition to The Big Theorem.

Theorem 4.7 (The Big Theorem – Version 4). *Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be a set of n vectors in \mathbf{R}^n , let $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, and let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Then the following are equivalent:*

► This updates TBT-V3 from Section 3.3.

- (a) \mathcal{A} spans \mathbf{R}^n ;
- (b) \mathcal{A} is linearly independent;
- (c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all \mathbf{b} in \mathbf{R}^n ;
- (d) T is onto;
- (e) T is one-to-one;
- (f) A is invertible;
- (g) $\ker(T) = \{\mathbf{0}\}$.

Proof: From TBT-V3 we know that (a) through (f) are equivalent. From Theorem 4.6 we know that T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$, so (e) and (g) are equivalent. Thus (a)–(g) are all equivalent. ■

Exercises

In Exercises 1–16, determine if the described set is a subspace. If so, give a proof. If not, explain why not. Unless stated otherwise, a , b , and c are real numbers.

1. The subset of \mathbf{R}^3 consisting of vectors of the

form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$.

2. The subset of \mathbf{R}^3 consisting of vectors of the

form $\begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$.

3. The subset of \mathbf{R}^2 consisting of vectors of the

form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $a + b = 1$.

4. The subset of \mathbf{R}^3 consisting of vectors of the

form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a = b = c$.

5. The subset of \mathbf{R}^4 consisting of vectors of the

form $\begin{bmatrix} a \\ 1 \\ 0 \\ b \end{bmatrix}$.

6. The subset of \mathbf{R}^4 consisting of vectors of the

form $\begin{bmatrix} a \\ a + b \\ 2a - b \\ 3b \end{bmatrix}$.

7. The subset of \mathbf{R}^2 consisting of vectors of the

form $\begin{bmatrix} a \\ b \end{bmatrix}$, where a and b are integers.

8. The subset of \mathbf{R}^3 consisting of vectors of the

form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $c = b - a$.

9. The subset of \mathbf{R}^3 consisting of vectors of the

form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $abc = 0$.

10. The subset of \mathbf{R}^2 consisting of vectors of the

form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $a^2 + b^2 \leq 1$.

11. The subset of \mathbf{R}^3 consisting of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a \geq 0$, $b \geq 0$, and $c \geq 0$.

12. The subset of \mathbf{R}^3 consisting of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where at most one of a , b , and c is nonzero.

13. The subset of \mathbf{R}^2 consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $a \leq b$.

14. The subset of \mathbf{R}^2 consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $|a| = |b|$.

15. The subset of \mathbf{R}^n consisting of vectors of the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

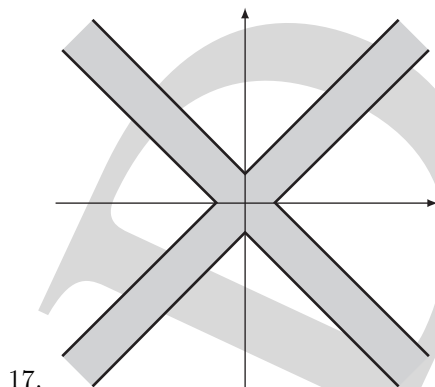
such that $v_1 + \cdots + v_n = 0$.

16. The subset of \mathbf{R}^n (n even) consisting of vectors of the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

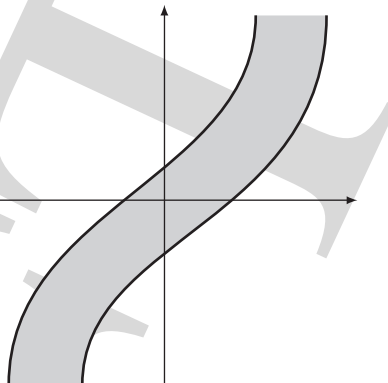
such that $v_1 - v_2 + v_3 - v_4 + v_5 - \cdots - v_n = 0$.

In Exercises 17–20, the shaded region is not a subspace of \mathbf{R}^2 . Explain why.

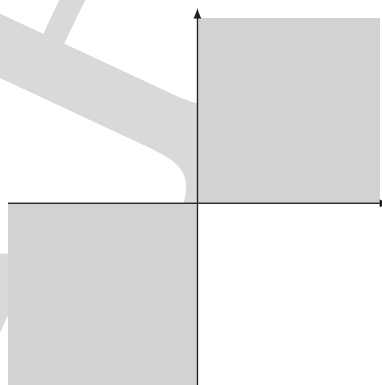


17.

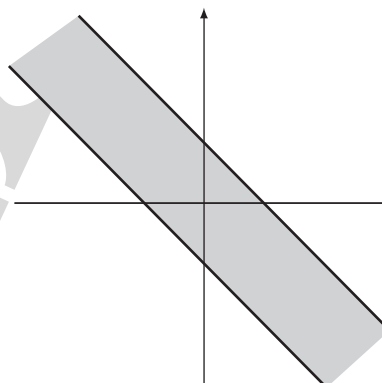
18.



19.



20.



In Exercises 21–32, find the null space for A .

21. $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

$$25. A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -7 \end{bmatrix}$$

$$26. A = \begin{bmatrix} 3 & 0 & -4 \\ -1 & 6 & 2 \end{bmatrix}$$

$$27. A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 2 & -10 \\ -3 & 15 \\ 1 & -5 \end{bmatrix}$$

$$29. A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$$30. A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -4 & -1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$31. A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$32. A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

In Exercises 33–36, let $T(\mathbf{x}) = A\mathbf{x}$ for the matrix A . Determine if the vector \mathbf{b} is in the kernel of T and if the vector \mathbf{c} is in the range of T .

$$33. A = \begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$34. A = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 4 \\ 11 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$$

$$35. A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$36. A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

Find an Example: For Exercises 37–44, find an example that meets the given specifications.

37. An infinite subset of \mathbf{R}^2 that is not a subspace of \mathbf{R}^2 .

38. Two subspaces S_1 and S_2 of \mathbf{R}^3 such that $S_1 \cup S_2$ is not a subspace of \mathbf{R}^3 .

39. Two nonsubspace subsets S_1 and S_2 of \mathbf{R}^3 such that $S_1 \cup S_2$ is a subspace of \mathbf{R}^3 .

40. Two nonsubspace subsets S_1 and S_2 of \mathbf{R}^3 such that $S_1 \cap S_2$ is a subspace of \mathbf{R}^3 .

41. A linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ such that $\text{range}(T) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

42. A linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ such that $\text{range}(T) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

43. A linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ such that $\text{range}(T) = \mathbf{R}^3$.

44. A linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ such that $\text{range}(T) = \text{Span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$.

True or False: For Exercises 45–60, determine if the statement is true or false, and justify your answer.

45. If A is an $n \times n$ matrix and $\mathbf{b} \neq \mathbf{0}$ is in \mathbf{R}^n , then the solutions to $A\mathbf{x} = \mathbf{b}$ do not form a subspace.

46. If A is a 5×3 matrix, then $\text{null}(A)$ forms a subspace of \mathbf{R}^5 .

47. If A is a 4×7 matrix, then $\text{null}(A)$ forms a subspace of \mathbf{R}^7 .

48. Let $T : \mathbf{R}^6 \rightarrow \mathbf{R}^3$ be a linear transformation. Then $\text{ker}(T)$ is a subspace of \mathbf{R}^6 .

49. Let $T : \mathbf{R}^5 \rightarrow \mathbf{R}^8$ be a linear transformation. Then $\text{ker}(T)$ is a subspace of \mathbf{R}^8 .

50. Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^7$ be a linear transformation. Then $\text{range}(T)$ is a subspace of \mathbf{R}^2 .

51. Let $T : \mathbf{R}^3 \rightarrow \mathbf{R}^9$ be a linear transformation. Then $\text{range}(T)$ is a subspace of \mathbf{R}^9 .

52. The union of two subspaces of \mathbf{R}^n forms another subspace of \mathbf{R}^n .

53. The intersection of two subspaces of \mathbf{R}^n forms another subspace of \mathbf{R}^n .

54. Let S_1 and S_2 be subspaces of \mathbf{R}^n , and define S to be the set of all vectors of the form $\mathbf{s}_1 + \mathbf{s}_2$, where \mathbf{s}_1 is in S_1 and \mathbf{s}_2 is in S_2 . Then S is a subspace of \mathbf{R}^n .
55. Let S_1 and S_2 be subspaces of \mathbf{R}^n , and define S to be the set of all vectors of the form $\mathbf{s}_1 - \mathbf{s}_2$, where \mathbf{s}_1 is in S_1 and \mathbf{s}_2 is in S_2 . Then S is a subspace of \mathbf{R}^n .
56. The set of integers forms a subspace of \mathbf{R} .
57. A subspace $S \neq \{\mathbf{0}\}$ can have a finite number of vectors.
58. If \mathbf{u} and \mathbf{v} are in a subspace S , then every point on the line connecting \mathbf{u} and \mathbf{v} are also in S .
59. If S_1 and S_2 are subsets of \mathbf{R}^n but *not* subspaces, then the union of S_1 and S_2 cannot be a subspace of \mathbf{R}^n .
60. If S_1 and S_2 are subsets of \mathbf{R}^n but *not* subspaces, then the intersection of S_1 and S_2 cannot be a subspace of \mathbf{R}^n .
-
61. Show that every subspace of \mathbf{R} is either $\{\mathbf{0}\}$ or \mathbf{R} .
62. Suppose that S is a subspace of \mathbf{R}^n and c is a scalar. Let cS denote the set of vectors $c\mathbf{s}$ where \mathbf{s} is in S . Prove that cS is also a subspace of \mathbf{R}^n .
63. Prove that if $\mathbf{b} \neq \mathbf{0}$, then the set of solutions to $A\mathbf{x} = \mathbf{b}$ is not a subspace.
64. Describe the geometric form of all subspaces of \mathbf{R}^2 .
65. Describe the geometric form of all subspaces of \mathbf{R}^3 .
66. Some texts use just conditions (b) and (c) in Definition 4.1 as the definition of a subspace. Explain why this is equivalent to our definition.
67. Let A be an $n \times m$ matrix, and suppose that $\mathbf{y} \neq \mathbf{0}$ is in \mathbf{R}^n . Show that the set of all vectors \mathbf{x} in \mathbf{R}^m such that $A\mathbf{x} = \mathbf{y}$ is *not* a subspace of \mathbf{R}^m .
68. Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4]$, and suppose that $\mathbf{x} = (2, -5, 4, 1)$ is in $\text{null}(A)$. Write \mathbf{a}_4 as a linear combination of the other three vectors.
69. Let A be a matrix and $T(\mathbf{x}) = A\mathbf{x}$ a linear transformation. Show that $\ker(T) = \{\mathbf{0}\}$ if and only if the columns of A are linearly independent.
70. If T is a linear transformation, show that $\mathbf{0}$ is always in $\ker(T)$.
71. Prove Theorem 4.6: If T is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.
- (C) In Exercises 72–75, use Example 5 as a guide to find the subspace of values that balances the given chemical equation.
72. Glucose ferments to form ethyl alcohol and carbon dioxide:
- $$x_1\text{C}_6\text{H}_{12}\text{O}_6 \longrightarrow x_2\text{C}_2\text{H}_5\text{OH} + x_3\text{CO}_2$$
73. Methane burns in oxygen to form carbon dioxide and steam:
- $$x_1\text{CH}_4 + x_2\text{O}_2 \longrightarrow x_3\text{CO}_2 + x_4\text{H}_2\text{O}$$
74. An antacid (calcium hydroxide) neutralizes stomach acid (hydrochloric acid) to form calcium chloride and water:
- $$x_1\text{Ca}(\text{OH})_2 + x_2\text{HCl} \longrightarrow x_3\text{CaCl}_2 + x_4\text{H}_2\text{O}$$
75. Ethyl alcohol reacts with oxygen to form vinegar and water:
- $$x_1\text{C}_2\text{H}_5\text{OH} + x_2\text{O}_2 \longrightarrow x_3\text{HC}_2\text{H}_3\text{O}_2 + x_4\text{H}_2\text{O}$$
-
- (C) In Exercises 76–79, find the null space for the given matrix.
76. $A = \begin{bmatrix} 1 & 7 & -2 & 14 & 0 \\ 3 & 0 & 1 & -2 & 3 \\ 6 & 1 & -1 & 0 & 4 \end{bmatrix}$
77. $A = \begin{bmatrix} -1 & 0 & 0 & 4 & 5 & 2 \\ 6 & 2 & 1 & 2 & 4 & 0 \\ 3 & 2 & -5 & -1 & 0 & 2 \end{bmatrix}$
78. $A = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 5 & 0 & 2 & -1 \\ 2 & 2 & 2 & 2 \\ -1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

79. $A = \begin{bmatrix} 2 & 0 & 5 \\ -1 & 6 & 2 \\ 4 & 4 & -1 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$

4.2 Basis and Dimension

In this section we combine the concepts of linearly independent sets and spanning sets to learn more about subspaces. Let $S = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a subspace of \mathbf{R}^n . Then every element \mathbf{s} of S can be written as a linear combination

$$\mathbf{s} = r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_m\mathbf{u}_m.$$

If $\mathbf{u}_1, \dots, \mathbf{u}_m$ is a linearly dependent set, then by Theorem 2.13 we know that one of the vectors in the set — say \mathbf{u}_1 — is in the span of the remaining vectors. Thus it follows that every element of S can be written as a linear combination of $\mathbf{u}_2, \dots, \mathbf{u}_m$, so that

$$S = \text{Span}\{\mathbf{u}_2, \dots, \mathbf{u}_m\},$$

If after eliminating \mathbf{u}_1 the remaining set of vectors is still linearly dependent, then we can repeat this process to eliminate another dependent vector. We can carry out this process over and over, and since we started with a finite number of vectors the process must eventually lead us to a set that both spans S and is linearly independent. Such a set is particularly important and goes by a special name.

Definition 4.8. A set $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a **basis** for a subspace S if

- (a) \mathcal{B} spans S ;
- (b) \mathcal{B} is linearly independent.

Figure 1 and Figure 2 shows basis vectors for \mathbf{R}^2 and \mathbf{R}^3 , respectively. Note that there is a subspace for which the above procedure will not work: $S = \{\mathbf{0}\} = \text{Span}\{\mathbf{0}\}$, the zero subspace. The set $\{\mathbf{0}\}$ is not linearly independent, and there are no vectors that can be removed. The zero subspace is the only subspace of \mathbf{R}^n that does not have a basis. (And conversely, any subspace with a basis cannot be the zero subspace.)

Each basis has the following important property.

Theorem 4.9. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a basis for a subspace S . Then every vector \mathbf{s} in S can be written as a linear combination

$$\mathbf{s} = s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m$$

in exactly one way.

Proof: Because \mathcal{B} is a basis for S , the vectors in \mathcal{B} span S , so that every vector \mathbf{s} can be written as a linear combination of vectors in \mathcal{B} in *at least* one way. To show that there can only be one way to write \mathbf{s} , let's suppose that there are two, say

$$\mathbf{s} = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m$$

and

$$\mathbf{s} = t_1\mathbf{u}_1 + \cdots + t_m\mathbf{u}_m.$$

Def: Basis

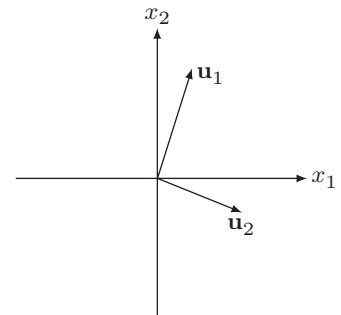


Figure 1: Any two nonzero vectors that do not lie on the same line forms a basis for \mathbf{R}^2 .

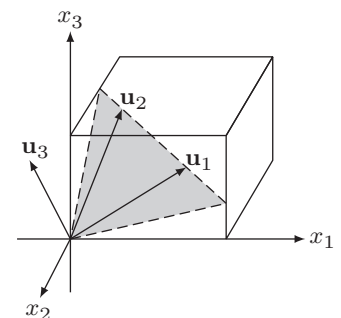


Figure 2: Any three nonzero vectors that do not lie in the same plane forms a basis for \mathbf{R}^3 .

Then $r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m = t_1\mathbf{u}_1 + \cdots + t_m\mathbf{u}_m$, so that after reorganizing we have

$$(r_1 - t_1)\mathbf{u}_1 + \cdots + (r_m - t_m)\mathbf{u}_m = \mathbf{0}.$$

Since \mathcal{B} is a basis it is also a linearly independent set, and therefore it must be that $r_1 - t_1 = 0, \dots, r_m - t_m = 0$. Hence $r_1 = t_1, \dots, r_m = t_m$, so that there is just one way to express \mathbf{s} as a linear combination of the vectors in \mathcal{B} . ■

Just to emphasize, Theorem 4.9 tells us that every vector in a subspace S can be expressed in *exactly* one way as a linear combination of vectors in a basis \mathcal{B} .

Finding a Basis

Frequently a subspace S is described as the span of a set of vectors — that is, $S = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$. Example 1 demonstrates a way to find a basis in this situation. Before getting to the example, we pause to give a theorem that we will be needing shortly. The proof is left as an exercise.

Theorem 4.10. *Let A and B be equivalent matrices. Then the subspace spanned by the rows of A is the same as the subspace spanned by the rows of B .*

► Recall that two matrices A and B are equivalent if A can be transformed into B through a sequence of elementary row operations.

Let's see how to find a basis from a spanning set.

Example 1. *Let S be the subspace of \mathbf{R}^4 spanned by the vectors*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 7 \\ -6 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}.$$

Find a basis for S .

Solution: Start by using the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to form the *rows* of a matrix:

$$A = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 7 & -6 & 5 & 2 \\ -3 & 4 & 1 & 0 \end{bmatrix}.$$

Next use row operations to transform A into the equivalent matrix B that is in echelon form:

$$B = \begin{bmatrix} 5 & 0 & 13 & 4 \\ 0 & 5 & 11 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 4.10, we know that the subspace spanned by the rows of B is the same as the subspace spanned by the rows of A , so the rows of B span S . Moreover, since B is in echelon form, the nonzero rows are linearly independent (see Exercise 37, Section 2.3). Thus the set

$$\left\{ \begin{bmatrix} 5 \\ 0 \\ 13 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 11 \\ 3 \end{bmatrix} \right\}$$

forms a basis for S . ■

Summarizing this solution method: To find a basis for $S = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$,

- (1) Use the vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ to form the rows of a matrix A ;
- (2) Transform A to echelon form B ;
- (3) The nonzero rows of B give a basis for S .

Before proceeding, we pause to state the following useful result that will be used to show a second method for finding a basis for a subspace S . (The proof is left as an exercise.)

Theorem 4.11. *Suppose that $U = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m]$ and $V = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_m]$ are two equivalent matrices. Then any linear dependence that exists among the vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ also exists among the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$.*

For example, Theorem 4.11 tells us that

$$\text{if } 3\mathbf{v}_1 - 2\mathbf{v}_4 + \mathbf{v}_6 = 5\mathbf{v}_2, \quad \text{then } 3\mathbf{u}_1 - 2\mathbf{u}_4 + \mathbf{u}_6 = 5\mathbf{u}_2.$$

Now let's look at a second way to find a basis.

Example 2. Let S be the subspace of \mathbf{R}^4 spanned by the vectors \mathbf{u}_1 , \mathbf{u}_2 , and \mathbf{u}_3 given in Example 1. Find a basis for S .

Solution: This time we start by using the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to form the columns of a matrix

$$A = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{bmatrix} 2 & 7 & -3 \\ -1 & -6 & 4 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Using row operations to transform A to echelon form, we get

$$B = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1 & 6 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The nice thing about the matrix B is that it is not hard to find the dependence relationship among the columns. For instance, we can readily verify that

$$2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3.$$

Now we apply Theorem 4.11: Since $2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3$, then we also have $2\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{u}_3$, — that is,

$$2 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -6 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}.$$

For B we have

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\},$$

and \mathbf{v}_1 and \mathbf{v}_2 are linearly independent. Hence it follows that for A ,

$$S = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\},$$

and that \mathbf{u}_1 and \mathbf{u}_2 are linearly independent. Thus the set

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -6 \\ 5 \\ 2 \end{bmatrix} \right\}$$

forms a basis for S . ■

Summarizing this solution method: To find a basis for $S = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$,

- (1) Use the vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ to form the columns of a matrix A ;
- (2) Transform A to echelon form B . The pivot columns of B will be linearly independent, and the other columns will be linearly dependent on the pivot columns.
- (3) The columns of A in the same positions as the pivot columns of B form a basis for S .

The solution method in Example 1 will usually produce a subspace basis that is relatively “simple” in that the basis vectors will contain some zeroes. The solution method in Example 2 produces a basis from a subset of the original spanning vectors, which is sometimes desirable. In general, each method will produce a different basis, so that a basis need not be unique.

Dimension

Example 1 and Example 2 show that a subspace can have more than one basis. However, note that each basis has two vectors. Although a given nonzero subspace will have more than one basis, the next theorem shows that a nonzero subspace has a fixed number of basis vectors.

Theorem 4.12. *If S is a subspace of \mathbf{R}^n , then every basis of S has the same number of vectors.*

The proof of this theorem is given at the end of the section.

Since every basis for a subspace S has the same number of vectors, the following definition makes sense.

Definition 4.13. *Let S be a subspace of \mathbf{R}^n . Then the **dimension** of S is the number of vectors in any basis of S .*

The zero subspace $S = \{\mathbf{0}\}$ has no basis, and is defined to have dimension 0. At the other extreme, \mathbf{R}^n is a subspace of itself, and in Example 2, Section 4.1 we showed that $\mathbf{e}_1, \dots, \mathbf{e}_n$ spans \mathbf{R}^n . It is also clear that these vectors are linearly independent, so that the set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ forms a basis—called the **standard basis**—of \mathbf{R}^n (see Figure 3). Thus the dimension of \mathbf{R}^n is n . It can be shown that \mathbf{R}^n is the only subspace of \mathbf{R}^n of dimension n . (See Exercise 57.)

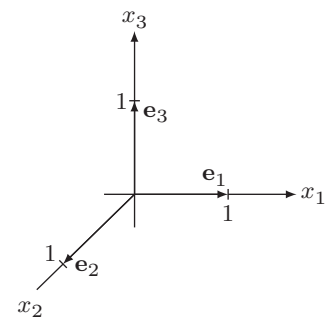


Figure 3: The standard basis for \mathbf{R}^3 .

Def: Dimension

Def: Standard Basis

Example 3. Suppose that S is the subspace of \mathbf{R}^5 given by

$$S = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -8 \\ -7 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -8 \\ 12 \\ 9 \\ -14 \end{bmatrix} \right\}.$$

Find the dimension of S .

Solution: Since our set has four vectors we know that the dimension of S will be four or less. To find the dimension we need to find a basis for S . It makes no difference how we do this, so let's use the solution method given in Example 2. Our vectors form the columns of the matrix on the left, with an echelon form given on the right:

$$\begin{bmatrix} -1 & 3 & -3 & -5 \\ 2 & 6 & 0 & -8 \\ 0 & -8 & 4 & 12 \\ -3 & -7 & -1 & 9 \\ 2 & 10 & -2 & -14 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & -3 & -5 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first two columns of the echelon matrix are the pivot columns, we conclude that the first two vectors

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -8 \\ -7 \\ 10 \end{bmatrix} \right\}$$

form a basis for S . Hence the dimension of S is 2. ■

In many instances it is handy to be able to modify a given set of vectors to serve as a basis. The following theorem gives two cases when this is possible.

Theorem 4.14. Let $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of vectors in a subspace $S \neq \{\mathbf{0}\}$ of \mathbf{R}^n .

- (a) If \mathcal{U} is linearly independent, then either \mathcal{U} is a basis for S or additional vectors can be added to \mathcal{U} to form a basis for S ;
- (b) If \mathcal{U} spans S , then either \mathcal{U} is a basis for S or vectors can be removed from \mathcal{U} to form a basis for S .

Proof: Taking part (a) first, if \mathcal{U} also spans S then we are done. If not, then select a vector \mathbf{s}_1 from S that is not in the span of \mathcal{U} , and form a new set

$$\mathcal{U}_1 = \{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{s}_1\}.$$

Then \mathcal{U}_1 must also be linearly independent, for if not then \mathbf{s}_1 would be in the span of \mathcal{U} . If \mathcal{U}_1 spans S , then we are done. If not, select a vector \mathbf{s}_2 that is not in the span of \mathcal{U}_1 , and form the set

$$\mathcal{U}_2 = \{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{s}_1, \mathbf{s}_2\}.$$

As before, \mathcal{U}_2 must be linearly independent. If \mathcal{U}_2 spans S , then we are done. If not, repeat this procedure again and again, until we finally have a linearly independent set that also spans S , giving a basis.

For part (b), we start with a spanning set. All we need to do is employ the solution method from Example 2, which will give a subset of \mathcal{U} that forms a basis for S . (Or we can use the method described at the beginning of the section, removing one vector at a time until reaching a basis.) ■

Example 4. Expand the set $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$ to a basis for \mathbf{R}^3 .

Solution: Since \mathbf{R}^3 has dimension 3, we know that \mathcal{U} is not already a basis. We can see that the two vectors in \mathcal{U} are linearly independent, so by Theorem 4.14(a) we can expand \mathcal{U} to a basis of \mathbf{R}^3 . We know that the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms a basis for \mathbf{R}^3 , so that

$$\mathbf{R}^3 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Now we form the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -2 & -4 & 0 & 0 & 1 \end{bmatrix},$$

and then apply the solution method from Example 2, which will give us a basis for \mathbf{R}^3 . Since we placed the vectors that we want to include in the left columns, we are assured that they will end up among the basis vectors. Employing our usual row operations, we find an echelon form equivalent to A is

$$B = \begin{bmatrix} 2 & 0 & -4 & 0 & -3 \\ 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

Since the pivots are in the 1st, 2nd, and 4th columns of B , referring back to A we see that the vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

must be linearly independent and span \mathbf{R}^3 , and so the set forms a basis for \mathbf{R}^3 . ■

Example 5. The vector \mathbf{x}_1 is in the null space of A

$$\mathbf{x}_1 = \begin{bmatrix} 7 \\ 3 \\ -6 \\ 4 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 3 & -6 & -6 \\ 2 & 6 & 0 & -8 \\ 0 & -8 & 4 & 12 \\ -3 & -7 & -1 & 9 \\ 2 & 10 & -2 & -14 \end{bmatrix}.$$

Find a basis for the null space that includes \mathbf{x}_1 .

► The procedure cannot go on indefinitely. Since all of the vectors are in \mathbf{R}^n , no set can have more than n linearly independent vectors (see Theorem 2.12, Section 2.3).

Solution: In Example 4, we were able to exploit the fact that we knew a basis for \mathbf{R}^3 . Here we do not know a basis for the null space, so we use our usual approach: Determine the vector form of the general solution to $A\mathbf{x} = \mathbf{0}$, and use the vectors to form the initial basis. We skip the details, and just report the news, that

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} \quad (1)$$

forms a basis for the null space of A . From this point we follow the procedure in Example 4, by forming the matrix with our given vector \mathbf{x}_1 and the two basis vectors in (1), and then finding an echelon form.

$$\begin{bmatrix} 7 & -1 & -3 \\ 3 & 3 & 1 \\ -6 & 0 & 2 \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the pivots are in the first two columns, it follows that

$$\left\{ \begin{bmatrix} 7 \\ 3 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \quad (2)$$

forms a basis for the null space of A that contains \mathbf{x}_1 . ■

Note that (2) is not the only basis containing \mathbf{x}_1 . For instance, if we reverse the order of the two vectors in (1) and follow the same procedure, we end up with the basis

$$\left\{ \begin{bmatrix} 7 \\ 3 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

The **nullity** of a matrix A is the dimension of the null space of A and is denoted by $\text{nullity}(A)$. For instance, in Example 5 we have $\text{nullity}(A) = 2$.

If we happen to know the dimension of a subspace S , then the following theorem makes it easier to determine if a given set forms a basis.

Def: Nullity

Theorem 4.15. Let $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of m vectors in a subspace S of dimension m . If \mathcal{U} is either linearly independent or spans S , then \mathcal{U} is a basis for S .

Proof: First suppose that \mathcal{U} is linearly independent. If \mathcal{U} does not span S , then by Theorem 4.14 we can add additional vectors to \mathcal{U} to form a basis for S . But this gives a basis with more than m vectors, contradicting the assumption that the dimension of S equals m . Hence \mathcal{U} also must span S and so is a basis.

A similar argument can be used to show that if \mathcal{U} spans S then \mathcal{U} is again a basis. The details are left as an exercise. ■

Example 6. Suppose that S is a subspace of \mathbf{R}^3 of dimension two containing the two vectors in the set

$$\mathcal{U} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} \right\}.$$

Show that \mathcal{U} is a basis for S .

Solution: Since S has dimension two and \mathcal{U} has two vectors, by Theorem 4.15 all we need to do to show that \mathcal{U} is a basis for S is verify that \mathcal{U} is linearly independent or spans S . We do not know enough about S to show that \mathcal{U} spans S , but since the two vectors are not multiples of each other, \mathcal{U} is a linearly independent set. Hence we can conclude that \mathcal{U} is a basis for S . ■

Theorems 4.16 and 4.17 present more properties of the dimension of a subspace that are useful in certain situations. The proofs are left as exercises.

Theorem 4.16. Suppose that S_1 and S_2 are both subspaces of \mathbf{R}^n , with S_1 a subset of S_2 . Then $\dim(S_1) \leq \dim(S_2)$, and $\dim(S_1) = \dim(S_2)$ only if $S_1 = S_2$.

Theorem 4.17. Let $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of vectors in a subspace S of dimension k .

- (a) If $m < k$, then \mathcal{U} does not span S .
- (b) If $m > k$, then \mathcal{U} is not linearly independent.

The Big Theorem – Version 5

The results of this section give us another condition for The Big Theorem.

Theorem 4.18 (The Big Theorem – Version 5). Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be a set of n vectors in \mathbf{R}^n , let $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$, and let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Then the following are equivalent:

- (a) \mathcal{A} spans \mathbf{R}^n ;
- (b) \mathcal{A} is linearly independent;
- (c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all \mathbf{b} in \mathbf{R}^n ;
- (d) T is onto;
- (e) T is one-to-one;
- (f) A is invertible;
- (g) $\ker(T) = \{\mathbf{0}\}$;
- (h) \mathcal{A} is a basis for \mathbf{R}^n .

► This updates The Big Theorem — Version 4, given in Section 4.1.

Proof: From TBT-V4, we know that (a) through (g) are equivalent. By Definition 4.8, (a) and (b) are equivalent to (h), completing the proof. ■

Example 7. Let x_1, x_2, \dots, x_n be real numbers. The Vandermonde matrix arises in signal processing and coding theory, and is given by

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

Show that if x_1, x_2, \dots, x_n are distinct, then the columns of V form a basis for \mathbf{R}^n .

Solution: By TBT-V5, we can show that the columns of V form a basis for \mathbf{R}^n by showing that the columns are linearly independent. Given real numbers a_0, a_1, \dots, a_{n-1} , we have

$$a_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + a_1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \cdots + a_{n-1} \begin{bmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_n^{n-1} \end{bmatrix} = \begin{bmatrix} a_0 + a_1x_1 + \cdots + a_{n-1}x_1^{n-1} \\ a_0 + a_1x_2 + \cdots + a_{n-1}x_2^{n-1} \\ \vdots \\ a_0 + a_1x_n + \cdots + a_{n-1}x_n^{n-1} \end{bmatrix} \quad (3)$$

If the polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$, then the right side of (3) is equal to

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

This is the zero vector only if each of x_1, x_2, \dots, x_n are roots of the polynomial f . But since each of these are distinct and f has degree at most $n-1$, the only way this can happen is if $f(x) = 0$, the identically zero polynomial. Hence $a_0 = \cdots = a_{n-1} = 0$, and so the columns of V are linearly independent. Therefore the columns of V form a basis for \mathbf{R}^n . ■

► *The Fundamental Theorem of Algebra* (proved by Gauss) states that a polynomial of degree m can have at most m distinct roots.

Proof of Theorem 4.12

We state the theorem again:

Theorem 4.12. *If S is a subspace of \mathbf{R}^n , then every basis of S has the same number of vectors.*

Proof: Suppose that we have a subspace S with two bases of different sizes. The argument that follows can be generalized (this is left as an exercise), but to simplify notation we assume that S has bases

$$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\} \quad \text{and} \quad \mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Since \mathcal{U} spans S , it follows that $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 can each be expressed as linear combinations of \mathbf{u}_1 and \mathbf{u}_2 :

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{12}\mathbf{u}_2, \\ \mathbf{v}_2 &= c_{21}\mathbf{u}_1 + c_{22}\mathbf{u}_2, \\ \mathbf{v}_3 &= c_{31}\mathbf{u}_1 + c_{32}\mathbf{u}_2. \end{aligned} \quad (4)$$

Now consider the equation

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 = \mathbf{0}. \quad (5)$$

Substituting into (5) from (4) for $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 gives

$$\begin{aligned} \mathbf{0} &= a_1(c_{11}\mathbf{u}_1 + c_{12}\mathbf{u}_2) + a_2(c_{21}\mathbf{u}_1 + c_{22}\mathbf{u}_2) + a_3(c_{31}\mathbf{u}_1 + c_{32}\mathbf{u}_2) \\ &= (a_1c_{11} + a_2c_{21} + a_3c_{31})\mathbf{u}_1 + (a_1c_{12} + a_2c_{22} + a_3c_{32})\mathbf{u}_2. \end{aligned}$$

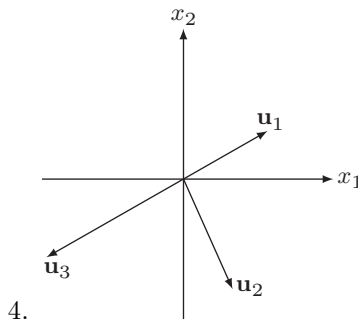
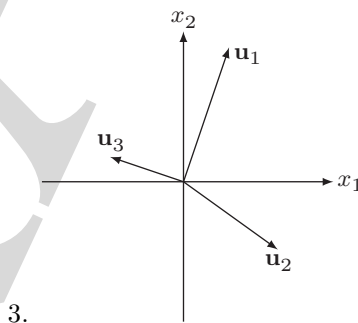
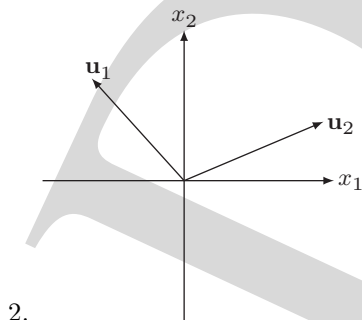
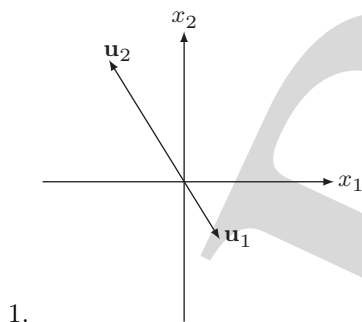
Since \mathcal{U} is linearly independent, we must have

$$\begin{aligned} a_1c_{11} + a_2c_{21} + a_3c_{31} &= 0, \\ a_1c_{12} + a_2c_{22} + a_3c_{32} &= 0. \end{aligned}$$

Now view a_1, a_2 and a_3 as variables in this homogeneous system. Since there are more variables than equations, the system must have infinitely many solutions. But this means that there are nontrivial solutions to (5), which implies that \mathcal{V} is linearly dependent, a contradiction. (Remember that \mathcal{V} is a basis.) Hence our assumption that there can be bases of two different sizes is incorrect, so all bases for a subspace must have the same number of vectors. ■

Exercises

In Exercises 1–4, determine if the vectors shown form a basis for \mathbf{R}^2 . Justify your answer.



In Exercises 5–10, use solution method from Example 1 to find a basis for the given subspace and give the dimension.

5. $S = \text{Span} \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 20 \end{bmatrix} \right\}$

$$6. S = \text{Span} \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 9 \\ -2 \end{bmatrix} \right\}$$

$$7. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix} \right\}$$

$$8. S = \text{Span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}$$

$$9. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ -3 \end{bmatrix} \right\}$$

$$10. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

In Exercises 11–16, use the solution method from Example 2 to find a basis for the given subspace and give the dimension.

$$11. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \end{bmatrix} \right\}$$

$$12. S = \text{Span} \left\{ \begin{bmatrix} 2 \\ -6 \end{bmatrix}, \begin{bmatrix} -5 \\ 15 \end{bmatrix} \right\}$$

$$13. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \right\}$$

$$14. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

$$15. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -1 \end{bmatrix} \right\}$$

$$16. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 13 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \end{bmatrix} \right\}$$

In Exercises 17–22, find a basis for the given subspace by deleting linearly dependent vectors, and give the dimension. Very little computation should be required.

$$17. S = \text{Span} \left\{ \begin{bmatrix} 2 \\ -6 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix} \right\}$$

$$18. S = \text{Span} \left\{ \begin{bmatrix} 12 \\ -3 \end{bmatrix}, \begin{bmatrix} -18 \\ 6 \end{bmatrix} \right\}$$

$$19. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$$

$$20. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\}$$

$$21. S = \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

$$22. S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

In Exercises 23–24, expand the given set to form a basis for \mathbf{R}^2 .

$$23. \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$$

$$24. \left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$$

In Exercises 25–28, expand the given set to form a basis for \mathbf{R}^3 .

$$25. \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$26. \left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \right\}$$

$$27. \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$$

$$28. \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \right\}$$

In Exercises 29–32, find a basis for the null space of the given matrix and give the dimension.

$$29. A = \begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix}$$

$$30. A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$31. A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$32. A = \begin{bmatrix} 1 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Find an Example: For Exercises 33–40, find an example that meets the given specifications.

33. A set of four vectors in \mathbf{R}^2 such that, when two are removed, the remaining two are a basis for \mathbf{R}^2 .
34. A set of three vectors in \mathbf{R}^4 such that, when one is removed and then two more are added, the new set is a basis for \mathbf{R}^4 .
35. A subspace S of \mathbf{R}^n with $\dim(S) = m$, where $0 < m < n$.
36. Two subspaces S_1 and S_2 of \mathbf{R}^5 such that $S_1 \subset S_2$ and $\dim(S_1) + 2 = \dim(S_2)$.
37. Two two-dimensional subspaces S_1 and S_2 of \mathbf{R}^4 such that $S_1 \cap S_2 = \{\mathbf{0}\}$.
38. Two three-dimensional subspaces S_1 and S_2 of \mathbf{R}^5 such that $\dim(S_1 \cap S_2) = 1$.
39. Two vectors \mathbf{u}_1 and \mathbf{u}_2 in \mathbf{R}^3 that produce the same set of vectors when the methods of Example 1 and Example 2 are applied.
40. Three vectors \mathbf{u}_1 and \mathbf{u}_2 in \mathbf{R}^3 that produce the same set of vectors when the methods of Example 1 and Example 2 are applied.
45. If a set of vectors \mathcal{U} spans a subspace S , then vectors can be removed from \mathcal{U} to create a basis for S .
46. If a set of vectors \mathcal{U} is linearly independent in a subspace S , then vectors can be removed from \mathcal{U} to create a basis for S .
47. Three nonzero vectors that lie in a plane in \mathbf{R}^3 might form a basis for \mathbf{R}^3 .
48. If S_1 is a subspace of dimension 3 in \mathbf{R}^4 , then there cannot exist a subspace S_2 of \mathbf{R}^4 such that $S_1 \subset S_2 \subset \mathbf{R}^4$ but $S_1 \neq S_2 \neq \mathbf{R}^4$.
49. The set $\{\mathbf{0}\}$ forms a basis for the zero subspace.
50. \mathbf{R}^n has exactly one subspace of dimension m for each of $m = 0, 1, 2, \dots, n$.
51. Let $m > n$. Then $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ in \mathbf{R}^n can form a basis for \mathbf{R}^n if the correct $m - n$ vectors are removed from \mathcal{U} .
52. Let $m < n$. Then $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ in \mathbf{R}^n can form a basis for \mathbf{R}^n if the correct $n - m$ vectors are added to \mathcal{U} .
53. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for \mathbf{R}^3 , then $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is a plane.
54. The nullity of a matrix A is the same as the dimension of the subspace spanned by the columns of A .
55. Suppose that S_1 and S_2 are nonzero subspaces, with S_1 contained inside S_2 . Suppose that $\dim(S_2) = 3$.
 - (a) What are the possible dimensions of S_1 ?
 - (b) If $S_1 \neq S_2$, then what are the possible dimensions of S_1 ?

True or False: For Exercises 41–54, determine if the statement is true or false, and justify your answer.

41. If S_1 and S_2 are subspaces of \mathbf{R}^n of the same dimension, then $S_1 = S_2$.
42. If $S = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\dim(S) = 3$.
43. If a set of vectors \mathcal{U} spans a subspace S , then vectors can be added to \mathcal{U} to create a basis for S .
44. If a set of vectors \mathcal{U} is linearly independent in a subspace S , then vectors can be added to \mathcal{U} to create a basis for S .
56. Suppose that S_1 and S_2 are nonzero subspaces, with S_1 contained inside S_2 . Suppose that $\dim(S_2) = 4$.
 - (a) What are the possible dimensions of S_1 ?
 - (b) If $S_1 \neq S_2$, then what are the possible dimensions of S_1 ?
57. Show that the only subspace of \mathbf{R}^n that has dimension n is \mathbf{R}^n .

58. Explain why \mathbf{R}^n ($n > 1$) has infinitely many subspaces of dimension 1.

59. Prove the converse of Theorem 4.9: If every vector \mathbf{s} of a subspace S can be written uniquely as a linear combination of the vectors $\mathbf{s}_1, \dots, \mathbf{s}_m$ (all in S), then the vectors form a basis for S .

60. Complete the proof of Theorem 4.15: Let $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of m vectors in a subspace S of dimension m . Show that if \mathcal{U} spans S , then \mathcal{U} is a basis for S .

61. Prove Theorem 4.16: Suppose that S_1 and S_2 are both subspaces of \mathbf{R}^n , with S_1 a subset of S_2 . Then $\dim(S_1) \leq \dim(S_2)$, and $\dim(S_1) = \dim(S_2)$ only if $S_1 = S_2$.

62. Prove Theorem 4.17: Let $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of vectors in a subspace S of dimension k .

- (a) If $m < k$, show that \mathcal{U} does not span S .
 (b) If $m > k$, show that \mathcal{U} is not linearly independent.

63. Suppose that a matrix A is in echelon form. Prove that the nonzero rows of A are linearly independent.

64. If the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans \mathbf{R}^3 and

$$A = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3],$$

what is $\text{nullity}(A)$?

65. Suppose that S_1 and S_2 are subspaces of \mathbf{R}^n , with $\dim(S_1) = m_1$ and $\dim(S_2) = m_2$. If S_1 and S_2 have only $\mathbf{0}$ in common, then what is the maximum value of $m_1 + m_2$?

66. Prove Theorem 4.10: Let A and B be equivalent matrices. Then the subspace spanned by the rows of A is the same as the subspace spanned by the rows of B .

67. Prove Theorem 4.11: Suppose that $U = [\mathbf{u}_1 \cdots \mathbf{u}_m]$ and $V = [\mathbf{v}_1 \cdots \mathbf{v}_m]$ are two equivalent matrices. Then any linear dependence that exists among the vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ also exists among the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$.

68. Give a general proof of Theorem 4.12: If S is a subspace of \mathbf{R}^n , then every basis of S has the same number of vectors.

(C) In Exercises 69–70, determine if the given set of vectors is a basis of \mathbf{R}^3 . If not, then determine the dimension of the subspace spanned by the vectors.

$$69. \left\{ \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \\ 4 \end{bmatrix} \right\}$$

$$70. \left\{ \begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -9 \end{bmatrix} \right\}$$

(C) In Exercises 71–72, determine if the given set of vectors is a basis of \mathbf{R}^4 . If not, then determine the dimension of the subspace spanned by the vectors.

$$71. \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -5 \\ 4 \end{bmatrix} \right\}$$

$$72. \left\{ \begin{bmatrix} 6 \\ 0 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \\ 6 \\ 8 \end{bmatrix} \right\}$$

(C) In Exercises 73–74, determine if the given set of vectors is a basis of \mathbf{R}^5 . If not, then determine the dimension of the subspace spanned by the vectors.

$$73. \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$74. \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

4.3 Row and Column Spaces

In Example 7, it was shown that if x_1, \dots, x_n are distinct real numbers, then the columns of the Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

form a basis for \mathbf{R}^n . But suppose that the x_i 's are not distinct. Can we tell if the columns are linear independent or linearly dependent? One result that we develop in this section will make this question easy to answer.

In this section we round out our knowledge of subspaces of \mathbf{R}^n . As we have seen, subspaces arise naturally in the context of a matrix. For instance, suppose that

$$A = \begin{bmatrix} 1 & -2 & 7 & 5 \\ -2 & -1 & -9 & -7 \\ 1 & 13 & -8 & -4 \end{bmatrix}.$$

The **row vectors** of A come from viewing the rows of A as vectors. For our given matrix A , the set of row vectors is

Def: Row Vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -9 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 13 \\ -8 \\ -4 \end{bmatrix} \right\}.$$

Similarly, the **column vectors** of A come from viewing the columns of A as vectors. For A , the set of column vectors is

Def: Column Vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 13 \end{bmatrix}, \begin{bmatrix} 7 \\ -9 \\ -8 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ -4 \end{bmatrix} \right\}.$$

Taking the span of the row or column vectors yields the following subspaces:

Definition 4.19. Let A be an $n \times m$ matrix.

- (a) The **row space** of A is the subspace of \mathbf{R}^m spanned by the row vectors of A , and is denoted by $\text{row}(A)$.
- (b) The **column space** of A is the subspace of \mathbf{R}^n spanned by the column vectors of A , and is denoted by $\text{col}(A)$.

Def: Row Space

Def: Column Space

In Section 4.2 we proved Theorem 4.10 and Theorem 4.11, which concern the rows and columns of matrices and can be used to find the basis for a subspace. Theorem 4.20 is a reformulation of those theorems, stated in terms of row and column spaces.

Theorem 4.20. *Let A be a matrix and B an echelon form of A .*

- (a) *The nonzero rows of B form a basis for $\text{row}(A)$.*
- (b) *The columns of A corresponding to the pivot columns of B form a basis for $\text{col}(A)$.*

Example 1. *Find a basis and the dimension for the row space and the column space of A .*

$$A = \begin{bmatrix} 1 & -2 & 7 & 5 \\ -2 & -1 & -9 & -7 \\ 1 & 13 & -8 & -4 \end{bmatrix}.$$

Solution: To use Theorem 4.20, we start by finding an echelon form of A , which is given by

$$B = \begin{bmatrix} 1 & -2 & 7 & 5 \\ 0 & -5 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 4.20(a), we know that a basis for the row space of A is given by the nonzero rows of B ,

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

By Theorem 4.20(b), we know that a basis for the column space of A is given by the columns of A corresponding to the pivot columns of B , which in this case are the first and second columns. Thus a basis for $\text{col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 13 \end{bmatrix} \right\}.$$

Since both $\text{row}(A)$ and $\text{col}(A)$ have two basis vectors, the dimension of both subspaces is two. ■

In Example 1, the row space and the column space of A have the same dimension. This is not a coincidence.

Theorem 4.21. *For any matrix A , the dimension of the row space equals the dimension of the column space.*

Proof: Given a matrix A , use the usual row operations to find an equivalent echelon form matrix B . From Theorem 4.20(a), we know that the dimension of the row space of A is equal to the number of nonzero rows of B . Next note that each nonzero row of B has exactly one pivot, and that different rows have pivots in different columns. Thus the number of pivot columns equals the number of nonzero rows. But by Theorem 4.20(b), the number of pivot columns of B equals the number of vectors

in a basis for the column space of A . Thus the dimension of the column space is equal to the number of nonzero rows of B , and so the dimensions of the row space and column space are the same. ■

Now let's return to the question about the Vandermonde matrix from the start of the section.

Example 2. Suppose that two or more of x_1, \dots, x_n are the same. Are the columns of

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

linearly independent or linearly dependent?

Solution: If two or more of x_1, \dots, x_n are the same, the two or more of the rows of V are the same. Hence the rows of V are linearly dependent, so by TBT-V5 (applied to the rows of V) the rows of V do not span \mathbf{R}^n . Therefore the dimension of $\text{row}(V)$ is less than n , and thus by Theorem 4.21 the dimension of $\text{col}(V)$ is less than n . Finally, again by TBT-V5 (applied to the columns of V), we conclude that the columns are linearly dependent. ■

Because the dimensions of the row and column spaces for a given matrix A are the same, the following definition makes sense.

Definition 4.22. The **rank** of a matrix A is the dimension of the row (or column) space of A , and is denoted by $\text{rank}(A)$.

Def: Rank of a Matrix

Example 3. Find the rank and the nullity for the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix}.$$

Solution: Applying the standard row operation procedure to A yields the echelon form

$$B = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since B has two nonzero rows, the rank of A is 2. To find the nullity of A , we need to determine the dimension of the subspace of solutions to $A\mathbf{x} = \mathbf{0}$. Adding a column of zeroes to A and B gives the augmented matrix for $A\mathbf{x} = \mathbf{0}$ and the corresponding echelon form (why?)

$$\begin{bmatrix} 1 & -2 & 3 & 0 & -1 & 0 \\ 2 & -4 & 7 & -3 & 3 & 0 \\ 3 & -6 & 8 & 3 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix on the right corresponds to the system

► Recall that the nullity is the dimension of the null space.

$$\begin{array}{rcl} x_1 - 2x_2 + 3x_3 & - & x_5 = 0 \\ & & x_3 - 3x_4 + 5x_5 = 0 \end{array} \quad (1)$$

For this system, x_2 , x_4 , and x_5 are free variables, so we assign the parameters $x_2 = s_1$, $x_4 = s_2$, and $x_5 = s_3$. Back substitution gives us

$$\begin{aligned} x_3 &= 3x_4 - 5x_5 = 3s_2 - 5s_3, \\ x_1 &= 2x_2 - 3x_3 + x_5 = 2s_1 - 3(3s_2 - 5s_3) + s_3 = 2s_1 - 9s_2 + 16s_3. \end{aligned}$$

In vector form, the general solution is

$$\mathbf{x} = s_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

The three vectors in the general solution form a basis for the null space, which shows that $\text{nullity}(A) = 3$. ■

Let's look at another example, and see if a pattern emerges.

Example 4. Determine the rank and nullity for the matrix A given in Example 5 of Section 4.2.

Solution: In Example 5, Section 4.2, we showed that a basis for the null space of A is given by

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\},$$

so that $\text{nullity}(A) = 2$. Since we did not show an echelon form for A earlier, we report it at right:

$$A = \begin{bmatrix} -3 & 3 & -6 & -6 \\ 2 & 6 & 0 & -8 \\ 0 & -8 & 4 & 12 \\ -3 & -7 & -1 & 9 \\ 2 & 10 & -2 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the echelon form, we see that the rank of A is 2. ■

Let's review what we have seen:

- Example 3: $\text{rank}(A) = 2$, $\text{nullity}(A) = 3$, total number of columns is 5.
- Example 4: $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$, total number of columns is 4.

In both cases, $\text{rank}(A) + \text{nullity}(A)$ equals the number of columns of A . This is not a coincidence.

► Once we know that the system $A\mathbf{x} = \mathbf{0}$ has 3 free variables, we can conclude that $\text{nullity}(A) = 3$. For the sake of completeness, we continue to the vector form of the solution.

Theorem 4.23 (Rank–Nullity Theorem). *Let A be an $n \times m$ matrix. Then*

$$\text{rank}(A) + \text{nullity}(A) = m.$$

Proof: Transform A to echelon form B .

- The rank of A is equal to the number of nonzero rows of B . Each nonzero row has a pivot, and each pivot appears in a different column. Hence the number of pivot columns equals $\text{rank}(A)$.
- Every non-pivot column corresponds to a free variable in the system $A\mathbf{x} = \mathbf{0}$. Each free variable becomes a parameter, and each parameter is multiplied times a basis vector of $\text{null}(A)$. (This is shown in detail in Example 3). Therefore the number of non-pivot columns equals $\text{nullity}(A)$.

Since the number of pivot columns plus the number of non-pivot columns must equal the total number of columns m , we have

$$\text{rank}(A) + \text{nullity}(A) = m.$$

Example 5. Suppose that A is a 5×13 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If the dimension of the kernel of T is 9, what is the dimension of the range of T ?

Solution: Since Theorem 4.23 is expressed in terms of the properties of a matrix A , we first convert the given information into equivalent statements about A . We are told that the dimension of $\ker(T)$ equals 9. Since $\ker(T) = \text{null}(A)$, then $\text{nullity}(A) = 9$. By Theorem 4.23, $m - \text{nullity}(A) = \text{rank}(A)$, so $\text{rank}(A) = 4$ because A has 13 columns. Recall that $\text{range}(T)$ is equal to the span of the columns of A (Theorem 3.3), which is the same as $\text{col}(A)$. Therefore the dimension of $\text{range}(T)$ is 4. ■

Example 6. Find a linear transformation T that has kernel equal to $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

Solution: Since T is a linear transformation, we know that there exists a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. Since the kernel of T equals the null space of A , another way to look at our problem is: we need a matrix A such that $\text{null}(A) = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

To get us started, since \mathbf{x}_1 and \mathbf{x}_2 are linearly independent (why?), they form a basis for $\text{null}(A)$ and so $\text{nullity}(A) = 2$. Moreover, A must have 4 columns because \mathbf{x}_1 and \mathbf{x}_2 are in \mathbf{R}^4 . Thus $\text{rank}(A) = 4 - 2 = 2$ by the Rank–Nullity Theorem. This tells us that A must have at least 2 rows, so let's assume that A has the form

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

and see if we can solve the problem.

In order for \mathbf{x}_1 and \mathbf{x}_2 to be in $\text{null}(A)$, we must have $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. Computing the first entry of $A\mathbf{x}_1$ and $A\mathbf{x}_2$ and setting each equal to zero produces the linear system

$$\begin{aligned} a - 2c + d &= 0 \\ b + 3c + 2d &= 0 \end{aligned}$$

The system is in echelon form, and after back substituting we find that the general solution is given by

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s_1 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}. \quad (2)$$

There are many choices for s_1 and s_2 , but let's make it easy on ourselves by setting $s_1 = 1$ and $s_2 = 0$, so that $a = 2$, $b = -3$, $c = 1$, and $d = 0$. This gives us half of A ,

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 \\ e & f & g & h \end{bmatrix}.$$

In order to find e, f, g , and h , we could repeat the same analysis. However, we will just get the same answers, with e, f, g , and h replacing a, b, c , and d . So we can save time by setting $s_1 = 0$ and $s_2 = 1$ and using the second vector in (2) as the second row of A ,

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}.$$

Since the two rows of A are linearly independent, we know that $\text{rank}(A) = 2$. This insures that $\text{nullity}(A) = 2$, so that $\text{null}(A) = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$. ■

We wrap up this subsection with a theorem that relates row and column spaces to other topics that we previously encountered. The proofs of both parts are left as exercises.

Theorem 4.24. *Let A be an $n \times m$ matrix and \mathbf{b} a vector in \mathbf{R}^n .*

- (a) *The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} is in the column space of A .*
- (b) *The system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if \mathbf{b} is in the column space of A and the columns of A are linearly independent.*

The Big Theorem – Version 6

We can add three more conditions to The Big Theorem based on our work in this section. The Big Theorem is starting to get really big. This theorem provides great flexibility — do not hesitate to use it.

Theorem 4.25 (The Big Theorem – Version 6). Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be a set of n vectors in \mathbf{R}^n , let $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, and let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Then the following are equivalent:

► This updates The Big Theorem — Version 5, given in Section 4.2.

- (a) \mathcal{A} spans \mathbf{R}^n ;
- (b) \mathcal{A} is linearly independent;
- (c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all \mathbf{b} in \mathbf{R}^n ;
- (d) T is onto;
- (e) T is one-to-one;
- (f) A is invertible;
- (g) $\ker(T) = \{\mathbf{0}\}$;
- (h) \mathcal{A} is a basis for \mathbf{R}^n ;
- (i) $\text{col}(A) = \mathbf{R}^n$;
- (j) $\text{row}(A) = \mathbf{R}^n$;
- (k) $\text{rank}(A) = n$.

Proof: From TBT-V5, we know that (a) through (h) are equivalent. Theorem 4.21 and Definition 4.22 imply that (i), (j), and (k) are equivalent, and by definition (a) and (i) are equivalent. Hence the 11 conditions are all one big equivalent family. ■

Exercises

In Exercises 1–4, find bases for the column space of A , the row space of A , and the null space of A . Verify that the Rank-Nullity Theorem holds. (To make your job easier, an equivalent echelon form is given for each matrix.)

$$1. A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & 0 \\ -3 & 8 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 1 & 0 & -4 & -3 \\ -2 & 1 & 13 & 5 \\ 0 & 1 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3. A = \begin{bmatrix} 1 & 0 & -4 & -3 \\ -2 & 1 & 13 & 5 \\ 0 & 1 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$4. A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 4 & 1 \\ -4 & 0 & 2 \\ 1 & -2 & 0 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In Exercises 5–8, find bases for the column space of A , the row space of A , and the null space of A . Verify that the Rank-Nullity Theorem holds.

$$5. A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -2 & 3 \\ -1 & -2 & 0 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & -1 & 4 \\ 1 & -4 & 1 & 5 \end{bmatrix}$$

$$7. A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 11 & 7 & 1 \\ 1 & 1 & 4 & 0 \end{bmatrix}$$

$$8. A = \begin{bmatrix} 1 & 4 & -1 & 1 \\ 3 & 11 & -1 & 4 \\ 1 & 5 & 2 & 3 \\ 2 & 8 & -2 & 2 \end{bmatrix}$$

In Exercises 9–12, find all values of x so that $\text{rank}(A) = 2$.

$$9. A = \begin{bmatrix} 1 & -4 \\ -2 & x \end{bmatrix}$$

$$10. A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & x & 0 \end{bmatrix}$$

$$11. A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 11 \\ 4 & 3 & x \end{bmatrix}$$

$$12. A = \begin{bmatrix} -2 & 1 & 0 & 7 \\ 0 & 1 & x & 9 \\ 1 & 0 & -3 & 1 \end{bmatrix}$$

-
13. Suppose that A is a 6×8 matrix. If the dimension of the row space of A is 5, what is the dimension of the column space of A ?
14. Suppose that A is a 9×4 matrix. If the dimension of $\text{col}(A)$ is 5, what is the dimension of $\text{row}(A)$?
15. Suppose that A is a 4×7 matrix that has an echelon form with one zero row. Find the dimension of the row space of A , the column space of A , and the null space of A .
16. Suppose that A is a 6×11 matrix that has an echelon form with two zero rows. Find the dimension of the row space of A , the column space of A , and the null space of A .
17. A 8×5 matrix A has a null space of dimension 3. What is the rank of A ?
18. A 5×13 matrix A has a null space of dimension 10. What is the rank of A ?
19. A 7×11 matrix A has rank 4. What is the dimension of the null space of A ?
20. A 14×9 matrix A has rank 7. What is the dimension of the null space of A ?
21. Suppose that A is a 6×11 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If $\text{nullity}(A) = 7$, what is the dimension of the range of T ?
22. Suppose that A is a 17×12 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If $\text{rank}(A) = 8$, what is the dimension of the kernel of T ?
23. Suppose that A is a 13×5 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If T is one-to-one, then what is the dimension of the null space of A ?
24. Suppose that A is a 5×13 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If T is onto, then what is the dimension of the null space of A ?

25. Suppose that A is a 5×13 matrix. What is the maximum possible value for the rank of A , and what is the minimum possible value for the nullity of A ?

26. Suppose that A is a 12×7 matrix. What is the minimum possible value for the rank of A , and what is the maximum possible value for the nullity of A ?

In Exercises 27–32, suppose that A is a 9×5 matrix, and that B is an equivalent matrix in echelon form.

27. If B has three nonzero rows, what is $\text{rank}(A)$?

28. If B has two pivot columns, what is $\text{rank}(A)$?

29. If B has three nonzero rows, what is $\text{nullity}(A)$?

30. If B has one pivot column, what is $\text{nullity}(A)$?

31. If $\text{rank}(A) = 3$, how many nonzero rows does B have?

32. If $\text{rank}(A) = 1$, how many pivot columns does B have?

33. Suppose that A is an $n \times m$ matrix, that $\text{col}(A)$ is a subspace of \mathbb{R}^7 , and that $\text{row}(A)$ is a subspace of \mathbb{R}^5 . What are the dimensions of A ?

34. Suppose that A is an $n \times m$ matrix, with $\text{rank}(A) = 4$, $\text{nullity}(A) = 3$, and $\text{null}(A)$ a subspace of \mathbb{R}^5 . What are the dimensions of A ?

Find an Example: For Exercises 35–42, find an example that meets the given specifications.

35. A 2×3 matrix A with $\text{nullity}(A) = 1$.

36. A 4×3 matrix A with $\text{nullity}(A) = 0$.

37. A 9×4 matrix A with $\text{rank}(A) = 3$.

38. A 5×7 matrix A with $\text{rank}(A) = 4$.

39. A matrix A with $\text{rank}(A) = 3$ and $\text{nullity}(A) = 1$.

40. A matrix A with $\text{rank}(A) = 2$ and $\text{nullity}(A) = 2$.

41. A 2×2 matrix A such that $\text{row}(A) = \text{col}(A)$.
 42. A 2×2 matrix A such that $\text{row}(A) = \text{col}(A)$.

True or False: For Exercises 43–54, determine if the statement is true or false, and justify your answer.

43. If A is a matrix, then the dimension of the row space of A is equal to the dimension of the column space of A .
 44. If A is a square matrix, then $\text{row}(A) = \text{col}(A)$.
 45. The rank of a matrix A can not exceed the number of rows of A .
 46. If A and B are equivalent matrices, then $\text{row}(A) = \text{row}(B)$.
 47. If A and B are equivalent matrices, then $\text{col}(A) = \text{col}(B)$.
 48. If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then \mathbf{b} is in $\text{row}(A)$.
 49. If \mathbf{x}_0 is a solution to $A\mathbf{x} = \mathbf{b}$, then \mathbf{x} is in $\text{row}(A)$.
 50. If A is a 4×13 matrix, then the nullity of A could be equal to 5.
 51. Suppose that A is a 9×5 matrix, and that $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. Then T can be onto.
 52. Suppose that A is a 9×5 matrix, and that $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. Then T can be one-to-one.
 53. Suppose that A is a 4×13 matrix, and that $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. Then T can be onto.
 54. Suppose that A is a 4×13 matrix, and that $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. Then T can be one-to-one.

-
55. Prove that if A is an $n \times m$ matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

56. Prove that if A is an $n \times m$ matrix and $c \neq 0$ is a scalar, then $\text{rank}(A) = \text{rank}(cA)$.

57. Prove that if A is an $n \times m$ matrix and $\text{rank}(A) < m$, then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

58. Prove that if A is an $n \times m$ matrix and $\text{rank}(A) < n$, then the reduced row echelon form of A has a row of zeroes.

59. Prove Theorem 4.24: Let A be an $n \times m$ matrix and \mathbf{b} a vector in \mathbf{R}^n .

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} is in the column space of A .

(b) Show that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if \mathbf{b} is in the column space of A and the columns of A are linearly independent.

(C) In Exercises 60–63, determine the rank and nullity of the given matrix.

60. $A = \begin{bmatrix} 1 & 3 & 2 & 4 & -1 \\ 1 & 5 & -3 & 3 & -4 \\ 2 & 8 & -1 & 7 & -5 \end{bmatrix}$

61. $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 5 & 2 & 1 & -4 \\ -1 & -4 & -1 & 6 \\ -8 & -5 & -2 & 9 \end{bmatrix}$

62. $A = \begin{bmatrix} 4 & 8 & 2 \\ 3 & 5 & 1 \\ 9 & 19 & 5 \\ 7 & 13 & 3 \\ 5 & 11 & 3 \end{bmatrix}$

63. $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & -1 & 3 & 2 \\ 2 & 1 & 3 & 6 \\ 7 & 10 & 3 & 1 \\ 6 & 4 & 5 & 7 \\ -2 & -2 & 1 & 5 \end{bmatrix}$
