Subspaces

Assuming that some set V is a vector space, we shall look at subsets of V and determine if they are subspaces of V. A subset is a **subspace** of V if the following conditions hold.

(1) **u** + **v** ε S,
(2) k**u** ε S for any scalar k, and
(3) **0** ε S*.

*Note: Condition 3 is a special condition in that all subspaces must contain all vectors including the zero vector.

Example: Given vectors of the form (a, 0, 0), determine if they form a subspace of $\mathbf{R}^3$.

To determine if this is a subspace of $\mathbf{R}^3$, we must check the three conditions mentioned above. For convenience, we will check them in reverse order.

Analysis:

(3) **0** ε $\mathbf{R}^3$

$\vec{\mathbf{0}} = (0,0,0) = (a,0,0) \rightarrow a = 0 \rightarrow (0,0,0) \in R^3$

Hence, condition 3 holds.

(2) k**u** ε $\mathbf{R}^3$

Let **u** = (r, 0, 0). Then $k\vec{\mathbf{u}} = k(r,0,0) = (kr,0,0) \in R^3$.

Hence, condition 2 holds.

(1) **u** + **v** ε $\mathbf{R}^3$

Let **u** = (r, 0, 0) and Let **v** = (s, 0, 0). Then

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (r,0,0) + (s,0,0) = (r+s,0,0) \in R^3.$$

Hence, condition 1 holds.

Since all three conditions hold, then these vectors form a subspace in $\mathbf{R}^3$.

Example: Given vectors of the form (a, b, 1), determine if they form a subspace of $\mathbf{R}^3$.

Analysis:

(3) **0** ε $\mathbf{R}^3$

$\vec{\mathbf{0}} = (0,0,0) = (a,b,1) \rightarrow a = 0, b = 0, 0 \neq 1 \rightarrow (0,0,1) \notin R^3$

Since the zero vector is not in $\mathbf{R}^3$, then these vectors do not form a subspace in R3.

Hence, they do not form a subspace in $\mathbf{R}^3$.

Example: Given matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, determine if they form a subspace of $\mathbf{M}_{2x2}$.

Analysis:

(3) $\mathbf{0}$ ε $\mathbf{M}_{2x2}$

$$\vec{\mathbf{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \rightarrow a = 0, b = 0, c = 0 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{2\times 2}$$

Hence, condition 3 holds.

(2) k$\mathbf{u}$ ε $\mathbf{M}_{2x2}$

Let $\vec{\mathbf{u}} = \begin{bmatrix} f & g \\ 0 & h \end{bmatrix}$. Then $k\vec{\mathbf{u}} = k\begin{bmatrix} f & g \\ 0 & h \end{bmatrix} = \begin{bmatrix} kf & kg \\ 0 & kh \end{bmatrix} \in M_{2\times 2}$

Hence, condition 2 holds.

(1) $\mathbf{u} + \mathbf{v}$ ε $\mathbf{R}^3$

Let $\vec{\mathbf{u}} = \begin{bmatrix} f & g \\ 0 & h \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} r & s \\ 0 & t \end{bmatrix}$. Then

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \begin{bmatrix} f & g \\ 0 & h \end{bmatrix} + \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} = \begin{bmatrix} f + r & g + s \\ 0 & h + t \end{bmatrix} \in M_{2\times 2}$$

Hence, condition 1 holds.

Since all three conditions hold, then these matrices for a subspace of $\mathbf{M}_{2x2}$.

Example: Given polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$, where a_0, a_1, a_2, and a_3 are integers, determine if they form a subspace of $\mathbf{P}^4$.

Analysis:

(3) $\mathbf{0}$ ε $\mathbf{P}^4$

$$\vec{\mathbf{0}} = 0 + 0x + 0x^2 + 0x^3 = a_1 + a_2x + a_3x^2 + a_4x^3 \rightarrow a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0 \in P^4$$

Since zero is in the set of integers, then condition 3 holds.

(2) k$\mathbf{u}$ ε $\mathbf{M}_{2x2}$

Let $\mathbf{u}$ = b + cx + dx^2 + fx^3, where b, c, d, and f are integers. Then

$$k\vec{\mathbf{u}} = k\left(b + cx + dx^2 + fx^3\right) \rightarrow (kb) + (kc)x + (kd)x^2 + (kf)x^3 \notin P^4$$

Although this is the form of the polynomial in $\mathbf{P}^4$, the coefficients will not result in being integers. For example, the scalar multiplier k can be a rational number. It can also be an irrational number. Because of the many possibilities of k, condition 2 fails.

Hence, they do not form a subspace in $\mathbf{P}^4$.