**Math 340, Fall 2008**
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## Final Review Problem Solutions

1. Consider the matrix
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

   (1) **(5 pts)** Find the characteristic polynomial and eigenvalues of $A$.
   (2) **(10 pts)** Find an orthonormal basis for the eigenspace corresponding to each eigenvalue.
   (3) **(5 pts)** Find an orthogonal matrix $P$ such that $P^T AP$ becomes a diagonal matrix.

**Solution.**
(1).

$$\det(\lambda I - A) = (\lambda - 2) \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & \lambda - 2 \end{vmatrix} - \begin{vmatrix} -1 & \lambda - 2 \\ -1 & -1 \end{vmatrix}$$

$$= (\lambda - 2)^3 - (\lambda - 2) + (-(\lambda - 2) - 1) - (-1 + (\lambda - 2))$$
$$= (\lambda - 2)^3 - 3(\lambda - 2) - 2 = (\lambda - 4)(\lambda - 1)^2.$$

The characteristic polynomial is $p_A(\lambda) = (\lambda - 4)(\lambda - 1)^2$ and so the eigenvalues of $A$ are $\lambda = 4$, 1(double).
(2). For $\lambda = 4$, we solve the equation

$$\vec{0} = (4I - A)\vec{x} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We find the solution $x_1 = t$, $x_2 = t$, $x_3 = t$, $t$ arbitrary. Therefore an eigenvector of eigenvalue 4 is $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Normalizing it, we obtain an eigenvector of unit length is given by

$$\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.$$

For $\lambda = 1$, we solve the equation

$$\vec{0} = (I - A)\vec{x} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We find the solution $x_1 = -t - s$, $x_2 = t$, $x_3 = s$, $s$, $t$ arbitrary. Two independent eigenvectors are

$$\vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \vec{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

but they are **not** orthonormal. So we apply the Gram-Schimit process to $\{\vec{u}_2, \vec{u}_3\}$ and obtain

$$v_2 = u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ v_3 = u_3 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Normalizing them, we get an orthonormal basis of the eigenspace of $\lambda = 1$,

$$\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \right\}$$

(3). The matrix $P$ is just

$$P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}.$$

2. **(5 pts each)** Consider the matrix

$$B = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix}$$

(1) Find the inverse of the matrix $B$.
(2) Consider the following matrix equation

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Find the matrix $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$.

**Solution.**

(1). $B^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -7 & 3 & -5 \\ 3 & -1 & 2 \end{pmatrix}$.

(2). We multiply $B^{-1}$ to the right of the equation

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

and obtain

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -7 & 3 & -5 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -20 & 8 & -19 \\ -4 & 2 & -3 \end{pmatrix}$$

3. **(5 pts each)** Consider the equation $A\vec{x} = \vec{b}$ with

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}$$

Find the orthogonal projection of $\vec{b}$ onto Col $A$.

**Solution.** We note that the two column vectors, denoted by $u_1$, $u_2$, **are orthogonal to each other**. Therefore the orthogonal projection of $\vec{b}$ is given by

$$\frac{(b, u_1)}{(u_1, u_1)} u_1 + \frac{(b, u_2)}{(u_2, u_2)} u_2$$

which is

$$\frac{4}{14} u_1 + \frac{6}{42} u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

4. **(5 pts each)** Consider the matrix

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

(1) Find an invertible matrix $P$ such that $P^{-1}AP$ becomes diagonal. Avoid fractional numbers in finding $P$.
(2) Compute $A^n$ for positive integer $n$.

**Solution.**

(1). We go through the diagonalization procedure. We find the eigenvalues of $A$ are $\lambda = 8, 2$. For $\lambda = 8$, we find an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 2$, we find $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (avoiding fractional numbers). Therefore we have

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

(2). From the above diagonalization, we have

$$P^{-1}AP = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}.$$

If we denote the diagonal matrix by $D = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$, we can write

$$A = PDP^{-1}.$$

Therefore we have $A^n = (PDP^{-1})^n = PD^nP^{-1}$. We compute

$$D^n = \begin{pmatrix} 8^n & 0 \\ 0 & 2^n \end{pmatrix}$$

and

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Therefore we compute

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 8^n & 0 \\ 0 & 2^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 8^n + 2^n & {\cdot}8^n - 2^n \\ 8^n - 2^n & 8^n + 2^n \end{pmatrix}.$$

5. **(5pts each)** Let $\mathrm{Sym}_n(\mathbb{R})$ be the subset of $n \times n$ symmetric matrices in $M_{n\times n}(\mathbb{R})$ (i.e., the subset of $n \times n$ real matrices $A$ satisfying $A = A^T$).
   (1) Prove that $\mathrm{Sym}_n(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.
   (2) Find the dimension of $\mathrm{Sym}_3(\mathbb{R})$.

**Solution.**

(1). Recall the definition of symmetric matrix : $A$ is called symmetric if $A = A^T$.

First, if $A$ is symmetric, we know the scalar multiple $cA$ in $M_{n\times n}(\mathbb{R})$ is symmetric because $(cA)^T = cA^T = cA$. Similarly if $A_1$, $A_2$ are symmetric, we know the addition in $M_{n\times n}(\mathbb{R})$ satisfies

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2$$

Here the first equality comes from the property of the transpose and the second follows from the hypothesis. Hence $\mathrm{Sym}_n(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(2). We recall that a $3 \times 3$ symmetric matrix $A$ has the form

$$\begin{pmatrix} a & e & f \\ e & b & g \\ f & g & c \end{pmatrix}$$

We can write this as

$$\begin{pmatrix} a & e & f \\ e & b & g \\ f & g & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+e\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + g\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and so the set

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\}$$

spans $\mathrm{Sym}_n(\mathbb{R})$. You can easily check that the set is linearly independent and so forms a basis of $\mathrm{Sym}_n(\mathbb{R})$. Therefore the dimension is 6.

6. **(5 pts each)** Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ defined by

$$T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$$

(1) Find the polynomial whose value under $T$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(2) Find the matrix of $T$ relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}^2$ and the basis

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

for $\mathbb{R}^3$.

**Solution.** Write a polynomial $\mathbf{p}(t) = at^2 + bt + c$ in $\mathbb{P}_2$. Then we find

$$T(\mathbf{p}) = \begin{pmatrix} a - b + c \\ c \\ a + b + c \end{pmatrix}$$

(1). Therefore the polynomial $\mathbf{p}(t) = at^2 + bt + c$ satisfies equation

$$1 = a - b + c,\ 1 = c,\ 1 = a + b + c$$

Solving this, we find $a = b = 0$, $c = 1$ and so the corresponding polynomial is $\mathbf{p}(t) = 1$.

(2). We need to express $T(1)$, $T(t)$, $T(t^2)$ as linear combinations of the given basis of $\mathbb{R}^3$. We evaluate

$$T(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, T(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, T(t^2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We now need to express them as linear combinations of the given basis :

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

will be the matrix we look for. To find the matrix, we form the triply-augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

and reduce it to the **reduced row echelon form** :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1 \end{array}\right).$$

Then the matrix of $T$ is given by

$$\begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

7. **(5 pts each)** Answer the following equations. You do not have to give the explanations.

(1) Find the inverse of orthogonal matrix $U$ where

$$U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Is the following set orthogonal

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

with respect to the inner product on $\mathbb{R}^3$ defined by

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2 + x_3y_3, \quad \text{where } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}?$$

(3) Interpret the Cauchy-Schwarz inequality for the inner product as given in (2).

(4) Check if the vector $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}$? If so, find the corresponding eigenvalue.

**Solutions.**

(1). By definition of the orthogonal matrix, we have $UU^T = U^TU = I$, i.e.,

$$U^{-1} = U^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(2). We evaluate $(\vec{u}_1, \vec{u}_3) = 2 \times 1 \times 1 + 3 \times 0 \times 0 + 1 \times (-1) = 1 \neq 0$. Therefore they are not orthogonal.

(3). The Cauchy-Schwarz inequality becomes

$$(2x_1y_1 + 3x_2y_2 + x_3y_3)^2 \leq (2x_1^2 + 3x_2^2 + x_3^2)(2y_1^2 + 3y_2^2 + y_3^2)$$

in this case.

(4). We just compute

$$\begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Therefore it is indeed an eigenvector with eigenvalue 3.