KEY TO PRACTICE PROBLEMS CHAPTER 5

Section 5.1:

A1.

(a)The projection **p** is given by $\mathbf{p}=\frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}=\frac{10}{18}[\,2,\,-3,\,1,\,2]^T=\frac{5}{9}[\,2,\,-3,\,1,\,2]^T$

(b) $\mathbf{v}-\mathbf{p}=-\frac{1}{9}[1,\,3,\,5,\,1]^T$ $\quad [\mathbf{v}-\mathbf{p}]^T\mathbf{p}=0$ thus **v** – **p** is orthogonal to **p**.

(c) The distance from the vector **v** to the line spanned by the vector **w** is: || **v** – **p** || = 2/3

A2. The line of intersection is the solution of the system
$$x+y+z=0$$
$$2x-y-z=0$$

Let A be the matrix of the coefficients of this system. The RREF of A is $\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}$ thus the solution is

x = $[0, -t, t]^T$ with t any real number. These are the parametric equations of the line of intersection of the two planes.

Let **w** = $[0, -1, 1]^T$ be a vector on the line. The projection of **v** onto **w** is then $\mathbf{p}=\frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}=\frac{1}{2}[0,\,1,-1]^T=\left[0,\,\frac{1}{2},\,-\frac{1}{2}\right]^T$

A3. The point Q has coordinates (x_1, x_2, x_3) where **p** = $[x_1, x_2, x_3]^T$ is the vector projection of **b** onto **a**.

$\mathbf{p}=\frac{\mathbf{b}^T\mathbf{a}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}=\frac{1}{3}[1,1,-1]^T=\left[\frac{1}{3},\,\frac{1}{3},\,-\frac{1}{3}\right]^T$ and therefore Q = (1/3, 1/3, –1/3).

A4. The distance from the point to the plane is given by the absolute value of the scalar projection of the vector **v** = $[2, 1, 1]^T$ onto the normal of the plane **N** = $[2, -1, 2]^T$: $\quad \text{distance}=|\alpha|=\frac{|\mathbf{v}^T\mathbf{N}|}{\|\mathbf{N}\|}=\frac{5}{3}$

Section 5.2:

B1: The reduced row echelon form of **A** is $\mathbf{U}=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$ The nonzero rows of **U** form a basis for the row space of **A**.

Thus, a basis for R($\mathbf{A}^T$) is $\{[1,\,0]^T,[0,\,1]^T\}$.
From **U** we can see that N(**A**) consists only of the zero vector $\{[0, 0]^T\}$, thus there is no basis for N(**A**) (note that the zero vector cannot be a basis because it is linearly dependent).
The column vectors of **A** corresponding to pivot columns of **U** are $\mathbf{a}_1$ and $\mathbf{a}_2$, thus a basis for R(**A**) is $\{[4,\,1,\,2,3]^T,[-2,\,3,\,1,\,4]^T\}$.
The reduced row echelon form of $\mathbf{A}^T$ is $\begin{bmatrix}1&0&5/14&5/14\\0&1&4/7&11/7\end{bmatrix}$ from which we can find a basis for

N($\mathbf{A}^T$): $\left\{\left[-\frac{5}{14},\,-\frac{4}{7},\,1,0\right]^T,\left[-\frac{5}{14},\,-\frac{11}{7},\,0,\,1\right]^T\right\}$.

Note that all vectors in N($\mathbf{A}^T$) are orthogonal to all vectors in R(**A**) and all vectors in N(**A**) are orthogonal to all vectors in R($\mathbf{A}^T$) (which is confirmed by the Fundamental Subspaces Theorem). Also note that, because N(**A**) is comprised only of the zero vector the dimension of N(**A**) is zero (which is confirmed by the rank nullity theorem).

B2. Let $\mathbf{A}=\begin{bmatrix}-1\\2\\1\end{bmatrix}$. Then span($[-1, 2, 1]^T$)= R(**A**) and $R(\mathbf{A})^{\perp}=N(\mathbf{A}^T)$

$\mathbf{A}^T = [-1, 2, 1]$ and a basis of N($\mathbf{A}^T$) is given by { $[2, 1, 0]^T$, $[1, 0, 1]^T$ } . This basis is also a basis for the orthogonal complement of span($[-1, 2, 1]^T$). Note that the orthogonal complement of the line is the plane with normal **N** equal to the given vector: $\mathbf{N} = [-1, 2, 1]^T$. The equation of this plane is $-x + 2y + z = 0$ (check that the vectors in the basis we found belong to this plane).

B3.

(a) W is a plane through the origin in $\mathbb{R}^3$ containing the two given vectors.

(b) If we let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$ then R(**A**) = W and, by the Fundamental Subspaces Theorem, the orthogonal complement is N($\mathbf{A}^T$). The reduced row echelon form of $\mathbf{A}^T$ is $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \end{bmatrix}$. Thus a basis for $W^{\perp}$ is $[-5, 3, 1]^T$.

(c) $W^{\perp}$ is a line through the origin perpendicular to the plane W.

B4. No it is not possible because the vectors are not orthogonal.
If [3, 1, 2] is in the row space of **A**, then $[3, 1, 2] \in R(\mathbf{A}^T)$. .
By the Fundamental Subspaces Theorem, $N(\mathbf{A})^{\perp} = R(\mathbf{A}^T)$. So $[2, 1, 1] \in N(\mathbf{A})$ if and only if it is orthogonal to [3, 1, 2]. But the scalar product of these vectors is nonzero, therefore the answer is no.

B5. Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ Then span($[1, 2, 0, 1]^T$, $[3, 4, -1, 1]^T$) = R(**A**) and $R(\mathbf{A})^{\perp} = N(\mathbf{A}^T)$. Thus we need to find a basis for N($\mathbf{A}^T$). The rref of $\mathbf{A}^T$ is $\mathbf{A}^T = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1/2 & 1 \end{bmatrix}$ and a basis for N($\mathbf{A}^T$) is $\{[1, -1/2, 1, 0]^T, [1, -1, 0, 1]^T\}$. Note that, as expected, all the vectors in this basis are orthogonal to the given vectors.

Section 5.3:

C1. (a) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ (the columns of **A** are the basis of the subspace).

The projection matrix is given by $\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

(b) The projection is given by $\mathbf{p} = \mathbf{P}\,\mathbf{b} = [9/2, -3, 9/2]^T$.

C2. (a) The plane is the nullspace of the matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and a basis is given by $\{[1, 0, -1]^T, [0, -1, 1]^T\}$.

(b) If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$, then the projection of **b** onto the plane is given by is given by
$\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = [-1/3, 2/3, -1/3]^T$.

C3.

(a) The least squares solution is given by the solution of $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\,\mathbf{b}$

which in this case reduces to $\begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ A calculation shows that: $\hat{\mathbf{x}} = [2, 1]^T$

(b) $\mathbf{p} = \mathbf{A}\,\hat{\mathbf{x}} = (3, 1, 0)^T$

(c) $r(\hat{\mathbf{x}}) = \mathbf{b} - \mathbf{A}\,\mathbf{x} = [0, 0, 2]^T$

(d) Since $\mathbf{A}^T r(\hat{\mathbf{x}}) = [0, 0]^T$ we have that $r(\hat{\mathbf{x}}) \in N(\mathbf{A}^T)$.

C4. If $y = c_0 + c_1 x + c_2 x^2$ is the desired quadratic fit, then $\mathbf{c} = [c_0, c_1, c_2]$ is the least squares solution of $\mathbf{y} = \mathbf{X}\,\mathbf{c}$, where

$$\mathbf{X}=\begin{bmatrix}1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2\\ 1 & x_3 & x_3^2\\ 1 & x_4 & x_4^2\end{bmatrix}=\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 2 & 4\end{bmatrix} \quad \text{and} \quad \mathbf{y}=\begin{bmatrix}0\\1\\3\\9\end{bmatrix},$$

The least squares solution is given by $\mathbf{X}^T\mathbf{X}\,\mathbf{c} = \mathbf{X}^T\,\mathbf{y}$ which in this case is $\begin{pmatrix}4 & 2 & 6\\ 2 & 6 & 8\\ 6 & 8 & 18\end{pmatrix}\begin{pmatrix}c_0\\ c_1\\ c_2\end{pmatrix}=\begin{pmatrix}13\\21\\39\end{pmatrix}$

A calculation gives $[c_0, c_1, c_2]$= [0.55, 1.65, 1.25]. Hence the least-squares parabola is $y = 0.55 + 1.65\,x + 1.25\,x^2$

C5. If we require the function $f(x)$ to go through the points we obtain the system:

$$\begin{aligned} 0 &= c_0 - c_1\\ -5 &= c_0 + c_2\\ 3 &= c_0 + c_1\\ 7 &= c_1 - c_2 \end{aligned}$$

The coefficient matrix is given by $\mathbf{A}=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 0 & -1\end{bmatrix}$ and solving the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{y}$ gives

$\mathbf{c}$ = [5/4, 3/2, −6]. Thus the function is $f(x)=\frac{5}{4}+\frac{3}{2}\sin x-6\cos x.$

Section 5.4:

D1`. Possible answer: $\mathbf{x} = [1, 0]^T$

D2.

$$\langle f,g\rangle=\int_0^1 f(x)g(x)\,dx=\int_0^1 2(9x^2-4)\,dx=-2$$

$$\|f\|=\sqrt{\int_0^1 2^2\,dx}=2 \qquad \|g\|=\sqrt{\int_0^1 (9x^2-4)^2\,dx}=\sqrt{\frac{41}{5}}$$

$$\theta=\arccos\left(\frac{\langle f,g\rangle}{\|f\|\|g\|}\right)=\arccos\left(-\sqrt{\frac{5}{41}}\right)=1.92753 \quad \text{radians.}$$

Section 5.5

E1:

(a) Since the function $\cos x\sin x$ is odd, we have $\langle\cos x,\sin x\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos x\sin x\,dx=0$ So the two vectors are orthogonal.

$$\begin{aligned}\|\cos x\|^2&=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx\\ &=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(1+\cos(2x))\,dx \quad \text{by the double angle formula}\\ &=\frac{1}{2\pi}\left|x+\frac{\sin(2x)}{2}\right|_{-\pi}^{\pi}=\frac{1}{2\pi}(\pi-(-\pi))=1\end{aligned}$$

A similar calculation applies for $\|\sin x\|$ by using the identity $\sin^2 x=\frac{1}{2}(1-\cos(2x))$. Both vectors have norm 1 and therefore they form an orthonormal set.

(b) Since cos x and sin x are orthogonal we can use the Pythagorean Law to find the distance between them:
$\|\sin x - \cos x\|^2 = \|\sin x\|^2 + \|\cos x\|^2 = 1 + 1 = 2$ which gives $\|\sin x - \cos x\| = \sqrt{2}$.

(c) (i) Since cos x and sin x form an orthonormal set on C[-π , π], the inner product <f, g> can be found by taking the

scalar product of the coordinate vector of f with the coordinate vector of g:
$\langle f, g\rangle = 5(-1)+(-2)(3) = -11$

(ii) From Parseval's formula: $\|f\| = \sqrt{5^2+(-2)^2} = \sqrt{29}$ and $\|g\| = \sqrt{(-1)^2+(3)^2} = \sqrt{10}$

E2.

(a) We have $\langle 1, 2x-1\rangle = \int_0^1 (2x-1)\,dx = x^2 - x|_0^1 = 0$ Hence 1 and $2x-1$ are orthogonal relative to this inner product.

(b) We have $\|1\| = \langle 1,1\rangle^{1/2} = \left(\int_0^1 1^2\,dx\right)^{1/2} = 1$

$\|2x-1\| = \langle 2x-1, 2x-1\rangle^{1/2} = \left(\int_0^1 (2x-1)^2\,dx\right)^{1/2} = 1/\sqrt{3}$.

(c) 1 is already a unit "vector", but $2x-1$ is not. We can form an orthonormal basis by dividing $2x-1$ by its norm. That is $\{1, \sqrt{3}(2x-1)\}$ is an orthonormal basis for S.

Using the formula for the projection from Theorem 5.5.7, the least squares approximation to $\sqrt{x}$ by a function from S is $p = \langle\sqrt{x}, 1\rangle\cdot 1 + \langle\sqrt{x}, \sqrt{3}(2x-1)\rangle\cdot\sqrt{3}(2x-1) = \frac{2}{3}\cdot 1 + \frac{2}{15}\sqrt{3}\sqrt{3}(2x-1) = \frac{4}{5}x + \frac{4}{15}$.

$p = \frac{4}{5}x + \frac{4}{15}$ is the linear combination of 1 and $2x-1$ that best approximates the function $\sqrt{x}$ in the least squares sense.

E3. Note that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal but not orthonormal. We normalize them:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left[\frac{3}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right]^T \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left[\frac{-2}{\sqrt{94}}, \frac{3}{\sqrt{94}}, \frac{9}{\sqrt{94}}\right]$$

Let U be the matrix whose columns are the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, then the projection is given by

$$\mathbf{p} = UU^T\mathbf{b} = \left[\frac{518}{47}, -\frac{448}{47,} - \frac{968}{47}\right]^T$$

E4.

(a) From Corollary 5.5.3 we have <**x** , **y**> = 2(3) – 2(1) + 1(−4) = 0.

(b) From Parseval's Formula: $\|\mathbf{x}\| = \sqrt{2^2+(-2)^2+1^2} = \sqrt{9} = 3$.

E5: The coefficients of the linear combination are given by $\frac{\langle\mathbf{v}_1, \mathbf{w}\rangle}{\langle\mathbf{v}_1, \mathbf{v}_1\rangle} = \frac{2}{7}$, $\frac{\langle\mathbf{v}_2, \mathbf{w}\rangle}{\langle\mathbf{v}_2, \mathbf{v}_2\rangle} = \frac{1}{5,}$ $\frac{\langle\mathbf{v}_3, \mathbf{w}\rangle}{\langle\mathbf{v}_3, \mathbf{v}_3\rangle} = -9$,

thus $\mathbf{w} = \frac{2}{7}\mathbf{v}_1 + \frac{1}{5}\mathbf{v}_2 - 9\mathbf{v}_3$

Section 5.6

F1.

(a) The plane is the nullspace of the matrix $A = \begin{bmatrix}2 & -1 & 1\end{bmatrix}$ and a possible basis is given by $\mathbf{x}_1 = [1, 0, -2]^T$ and $\mathbf{x}_2 = [1, 2, 0]^T$.

(b) To convert the basis into an orthonormal basis we start by normalizing $\mathbf{x}_1$:

$$\|\mathbf{x}_1\| = \sqrt{5} \qquad \mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right]^T$$

$$\mathbf{p}_1 = \langle\mathbf{x}_2, \mathbf{u}_1\rangle\,\mathbf{u}_1 = \frac{1}{\sqrt{5}}\left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right]^T = \left[\frac{1}{5}, 0, -\frac{2}{5}\right]^T \qquad \mathbf{u}_2 = \frac{\mathbf{x}_2-\mathbf{p}_1}{\|\mathbf{x}_2-\mathbf{p}_1\|} = \frac{5}{2\sqrt{30}}\left[\frac{4}{5}, 2, \frac{2}{5}\right] = \left[\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right]^T$$

F2. Normal Equations: $\mathbf{A}^T\mathbf{A}\,\mathbf{x} = \mathbf{A}^T\mathbf{b}$

$(\mathbf{QR})^T(\mathbf{QR})\,\mathbf{x} = (\mathbf{QR})^T\mathbf{b}$ since $\mathbf{A} = \mathbf{QR}$

$\mathbf{R}^T\mathbf{Q}^T\mathbf{QR}\,\mathbf{x} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b}$ since $(\mathbf{QR})^T = \mathbf{R}^T\mathbf{Q}^T$

$\mathbf{R}^T\mathbf{R}\,\mathbf{x} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b}$ since $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$ since $\mathbf{R}^T$ is nonsingular we multiply both sides by $(\mathbf{R}^T)^{-1}$

F3.

(a) We have $R_{11}=\|\mathbf{a}_1\|=3$, so

$\mathbf{q}_1=\frac{1}{3}[2,1,2]^T$ $\qquad R_{12} = \langle \mathbf{a}_2, \mathbf{u}_1\rangle = 5/3$

$\mathbf{p}_1=R_{12}\mathbf{u}_1=\frac{5}{9}[2,1,2]^T$ $\qquad R_{22} = \|\mathbf{a}_2 - \mathbf{p}_1\| = \sqrt{2}/3$

$\mathbf{q}_2=\frac{\mathbf{a}_2-\mathbf{p}_1}{\|\mathbf{a}_2-\mathbf{p}_1\|}=\frac{\sqrt{2}}{6}[-1,4,-1]^T$

The QR decomposition is $\begin{bmatrix}2 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}2/3 & -\sqrt{2}/6\\ 1/3 & 4\sqrt{2}/6\\ 2/3 & -\sqrt{2}/6\end{bmatrix}\begin{bmatrix}3 & 5/3\\ 0 & \sqrt{2}/3\end{bmatrix}$

(b) The least-squares solution of $\mathbf{Ax} = \mathbf{b}$ is the solution of $\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$ (see problem F2). We have $\mathbf{Q}^T\mathbf{b}=\begin{bmatrix}22\\ -\sqrt{2}\end{bmatrix}$ so the system $\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$ reduces to

$$3x_1+\frac{5}{3}x_2=22$$
$$\frac{\sqrt{2}}{3}x_2=-\sqrt{2}$$

Using back substitution we find the solution $\hat{\mathbf{x}}=[9,-3]^T$.

F4: The calculations are as follows ;

$\mathbf{u}_1 = \mathbf{x}_1/\|\mathbf{x}_1\| = \mathbf{x}_1/5$

$\mathbf{p}_1=\langle\mathbf{x}_2,\mathbf{u}_1\rangle\mathbf{u}_1=\frac{1}{5}[8,4,4,2]^T$

$\mathbf{u}_2=(\mathbf{x}_2-\mathbf{p}_1)/\|\mathbf{x}_2-\mathbf{p}_1\|=\frac{1}{5}[1,-2,-2,4]^T$

$\mathbf{p}_2=\langle\mathbf{x}_3,\mathbf{u}_1\rangle\mathbf{u}_1+\langle\mathbf{x}_3,\mathbf{u}_2\rangle\mathbf{u}_2=\mathbf{u}_1+\mathbf{u}_2=[1,0,0,1]^T$

$\mathbf{u}_3=(\mathbf{x}_3-\mathbf{p}_2)/\|\mathbf{x}_3-\mathbf{p}_2\|=\left[0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right]^T$

The required orthonormal basis is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

F5: $r_{13} = \langle \mathbf{a}_3, \mathbf{q}_1\rangle = \sqrt{18}$ $\qquad r_{23} = \langle\mathbf{a}_3, \mathbf{q}_2\rangle = -\sqrt{6}$

$\mathbf{p}_2=r_{13}\mathbf{q}_1+r_{23}\mathbf{q}_2=3\mathbf{q}_1+2\mathbf{q}_2=[2,-4,2]^T$

$r_{33}=\|\mathbf{a}_3-\mathbf{p}_2\|=\sqrt{3}$

$\mathbf{q}_3=\frac{\mathbf{a}_3-\mathbf{p}_2}{\mathbf{r}_{33}}=\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]^T$

Thus

$$\mathbf{Q}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix} \qquad \mathbf{R}=\begin{bmatrix}\sqrt{2} & \sqrt{2} & \sqrt{18}\\ 0 & \sqrt{6} & -\sqrt{6}\\ 0 & 0 & \sqrt{3}\end{bmatrix}$$