Math446 Module Theory
Assignment 1
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1. Let R be a commutative unitary ring and let M be and R-module. Let $r \in R$ be fixed and let

$$rM = \{rx : x \in M\} \text{ and } Mr = \{x \in M : rx = 0\}$$

(a) We want to show that rM and Mr are submodules of M. Considering rM, since M is module, we know that $0 \in M$ and so for our fixed r, we know that, $0 = r0$ for $0 \in M$ and so $0 \in rM$, hence $rM \neq \emptyset$
Now since M is an R-module, we know that $\forall x \in M$ and $\forall r \in M$, $rx \in M$ and so by definition, $rM \subseteq M$.
Choose $a, b \in rM$ then $\exists x_1, x_2 \in M$ such that $a = rx_1, b = rx_2$. Now choose some $s \in R$. We want to show that $a + sb \in rM$

$$a+sb = rx_1+s(rx_2) = rx_1+(sr)x_2 = rx_1+(rs)x_2 = rx_1+r(sx_2) = r(x_1+sx_2)$$

We know that $x_1 + sx_2 \in M$ since M is a module and so $r(x_1 + sx_2)r \in M$ so we have that $r \in M$ is a submodule of M.

Considering Mr, since M is a module, we know that $0 \in M$ and by the properties of the action of the ring R on the module M, we know that $\forall r \in R$, $r0 = 0$ and so $0 \in Mr$ i.e $M \neq \emptyset$. Now choose $x_1, x_2 \in Mr$ and choose $s \in R$. We want to show that $x_1 + sx_2 \in Mr$. We know that $x_1 + sx_2 \in M$ since M is a module $r(x_1 + sx_2) = r(x_1) + r(sx_2) = r(x_1) + rs(x_2) = 0 + s(rx_2) = 0 + 0 = 0$ since $x_1, x_2 \in Mr$ and R is a commutative ring. so we have that $x_1 + sx_2 \in Mr$ Hence Mr is a submodule of M
(b)Let $R = \mathbb{Z}, M = \mathbb{Z}/n\mathbb{Z}, n = rs$ where $ra + sb = 1$ where $rM = Ms$ i.e $\{rx : x \in \mathbb{Z}/n\mathbb{Z}\} = \{x \in \mathbb{Z}/n\mathbb{Z} : sx = 0\}$
in other words $\forall rx \in rM, s(rx) = O_m$ consider $rx \in rM, rx \in \mathbb{Z}rs$ $s(rx) = sr(x)$ which is an x multiple of sr in $\mathbb{Z}/n\mathbb{Z} = Om$ thus $rM \subseteq Ms$. consider $y \in Ms$ we have that $sy = O_m$ then $y = rx, rx \in \mathbb{Z}/n\mathbb{Z}$
thus $Ms \subseteq rM$
Hence $rM = Ms$.

(c) Let $r \in R$ be forced and consider the R-module endomorphism $c\sigma_r(m) = r \cdot m$.

- $\ker(\sigma_r) = \{m \in M | rm = 0\}$ which is exactly the definition of M_r. Thus $\ker(\sigma_r) = M_r$.

- $M|M_r = \{x + m_r | m_r \in M_r\}$

- $\sigma_r(m|m_r) = \{r(x + m_r)|m_r \in M_r\}$
  $= \{rx + rm_r | m_r \in M_r\}$
  $= \{rx\} = rM$
  $\ker(\sigma_r) = \{o_m\} \subseteq$, thus (σ_r) is a bijection.

- $(\sigma_r)(m_1 + m_2) = r(m_1 + m_2)$
  $= r(x_1 + mr + x_2 + mr^2)$
  $= r(x_1 + x_2 + mr^1 + mr^2)$
  $= r(x_1 + mr^1) + r(x_2 + mr^2)$
  $= r(m_1) + r(m_2)$ where $m_1, m_2 \in M|M_r$
  thus σ is a bicycle homomorphism from $\in M|M_r \longrightarrow rM$ i.e
  $M|M_r \cong rM$.

(d) Let $M = M_1 \bigoplus M_2$ show that $rM = rM_1 \bigoplus rM_1$ and $M_r = (M_1)_r \bigoplus (M_2)_r$
Let $M = M_1 \bigoplus M_2$ we know that $M_1 \cap M_2 = \{0\}$ and $\forall x \in M, \exists, x_1, x_2 \in M_1.M_2$ respectively such that $x = x_1 + x_2$ thus $\forall rx \in rM, rx = r(x_1 + x_2) = rx_1 + rx_2$ i.e $rM \subseteq rM_1 \bigoplus rM_2$. Conversely suppose we have $rx_1 + rx_2 \in rM_1 \bigoplus rM_2$.
$rx_1 + rx_2 = r(x_1 + x_2)$
$= r(x)$(since $M = M_1 \bigoplus M_2$)
Thus $rM_1 \bigoplus rM_2 \subseteq rM$
since $M = M_1 \bigoplus M_2$
$\forall x \in M_r \exists! x_1, X_2 \in M_1, M_2 : x = x_1 + x_2$
thus we know that $rx = r(x_1 + x_2)$
$O = r(x_1 + x_2)$.Since M_r is a submodule, we have $x = x_1 + x_2 \in M_r \Leftrightarrow x_1, x_2 \in M_r$
Thus $rx_1, rx_2 = O$. Hence $M_r = M_1r + M_2r$

2. Let R be a ring
(a) Let M be an R-module. Suppose I is a two sided Ideal of R with the property that $IM = 0$. We want to show that the rule

$$\bar{r} * m = rm + Im$$

gives a well defined action of R/I on M
We see that * : R/I x $M \longrightarrow M$. Also, suppose that $\bar{r}_1 = \bar{r}_2$ and that $m_1 = m_2$ we want to show that $\bar{r}_1 * m_1 = \bar{r}_2 * m_2$
$\bar{r}_1 * m_1 = r_1m_1 + Im_1 = r_1m_1 + 0 = r_1m_1$ but we know that $\bar{r}_1 = \bar{r}_2$ and $m_1 = m_2$ so $r_1m_1 = r_2m_2 = r_2m_2 + 0 = r_2m_2 + Im_2 = \bar{r}_2 * m_2$
i.e. $\bar{r}_1 * m_2$. Hence we have that $*$ gives a well defined action of R/I on M.
We want to show that M is an R/I-module with the ring action $*$
We know that M is an R-module and so $(M, +)$ is an abelian group.
We now show that the rest of the axioms of a module are satisfied.
Let $\bar{r}_1, \bar{r}_2 \in R/I$ and $m_1, m_2 \in M$

- $(\bar{r}_1 + \bar{r}_2) * m_1 = (r_1 + r_2)m_1 + Im_1$
  $= r_1m_1 + r_2m_1 + Im_1$
  $= r_1m_1 + Im_1 + r_2m_1 + Im_1$
  $= (\bar{r}_1 * m_1) + (\bar{r}_2 * m_1)$

- $(\bar{r}_1\bar{r}_2) * m_1 = (\bar{r}_1\bar{r}_2)m_1 + Im_1$
  $= r_1(r_1m_1) + Im_1$
  $= r_1((r_2m_1) + Im_1) + Im_1$
  $= \bar{r}_1 * (\bar{r}_1 * m_1)$

- $\bar{r}_1 * (m_1 + m_2) = r_1(m_1 + m_2) + I(m_1 + m_2)$
  $= r_1m_1n + r_1m_2 + Im_1 + Im_2$
  $= r_1m_1n + Im_1 + r_1m_2 + Im_2$
  $= (\bar{r}_1 * m_1) + (\bar{r}_1 * m_2)$

Hence we have that M is an R/I - module with the ring action $*$
(b)We want to show that every simple left R-module is a cyclic left left R-module
Let M be a simple module. Then, the only submodules of M are the 0 submodule and M
itself. Suppose also that M is not cyclic, that is M is not generated by each non zero element in M will be a submodule of M but this contradicts our assumption that the M is simple. Hence M must be generated by at least one of its elements and so M is cyclic.
(c)Consider $f : M \to N$ an R-module homomorphism
$ker(f) = \{m \in M : f(m) = O_N\}$
$f(O_m) = O_N$ thus $O_m \in ker(f)$ and $m + rm_2 \forall r \in R$.
$f(m_1 + m_2) = f(m_1) + rf(m_2)$
$O_N + O_N = O_N$
Thus $m_1 + rm_2 \in ker(f)$ and the submodule criterion is satisfied.
$lm(f) = \{n \in N | \exists m \in M, f(m) = n\}$
$O_N \in lm(f)$ since $f(O_m) = O_N$.Thus $lm(f)$ is not empty. Consider $n_1, n_2 \in$

$lm(f)$. and $n_1 + n_2 \forall r \in R$.
$f(m_1) + rf(m_2) = f(m) + f(rm_2) = f(m_1 + m_2) = f(m_3)$
Thus $\exists m_3 = m_1 + rm_2 : f(m_3) = n_1 + m_2$
$xn_1 + rn_2 \in lm(f)$ and the submodule criterion is satisfied.

(d) Assume M and N are cyclic simple then
$ker(f) = \{O_m\}, M$
$lm(f) = \{O_N\}, N$.

if $ker(f) = \{O_m\}$ and $lm(f) = \{O_N\}$ f is an isomorphism. if $ker(f) = \{O_m\}$ and $lm(f) = N$ f is an isomorph.
if $ker(f) = M$ and $lm(f) = \{O_N\}$, f is the zero map.
if $ker(f) = M$ and $lm(f) = N$ then f is both injective and surjective so f is an isomorphism.

3.Suppose M is a finite abelian group. We know that then M is naturally a $\mathbb{Z}$-module.

<u>Claim</u>:
This action cannot be extended to make M into a $\mathbb{Q}$-module

<u>Proof</u>:
Given any two unitary rings M, $\mathbb{S}$ and an R-module M. If $\exists$ a homomorphism $f: S \longrightarrow R$ then M is also an M-module with the actionof S on M defined by $(s, m) \longrightarrow fm$ for $s \in S$, $m \in M$.
But no such homomorphism exists for $S = \mathbb{Q}$ and $R = \mathbb{Z}$. Hence the action of $\mathbb{Z}$ on M cannot be extended to make M into a $\mathbb{Q}$-module.

4. Suppose that A $\leq$ B, then a $\in$ A such that a $\in$ B and b $\in$ B (Since A $\subseteq$ B).

$$Let\ A{+}C = B{+}C,\ then\ a{+}c \in A{+}C\ and\ b{+}c \in B{+}C\ for\ a\ \in A,\ b\ \in B\ and\ c \in C \tag{1}$$

$$Let\ A{\cap}C = B{\cap}C,\ then\ a = c\ for\ some\ a \in A\ and\ c \in C\ and\ b = c\ for\ some\ b \in B and\ c \in C. \tag{2}$$

Note that A $\subseteq$ A + C and C $\subseteq$ A + C ;
B $\subseteq$ B + C and C $\subseteq$ B + C
so we have that A $\cap$ C $\subseteq$ A + C and B $\cap$ C $\subseteq$ B + C.
That is we have that a $\in$ A $\cap$ C $\subseteq$ A + C and b $\in$ B $\cap$ C $\subseteq$ B + C.
Hence from (2) and (3), we have that
a = b.

Also, from (2) and (3), we have that A $\cap$ C $\subseteq$ B $\cap$ C since A $\leq$ B.
Thus a $\in$ A $\cap$ C such that a $\in$ B $\cap$ C.
So we can write that b = c = a
$\implies$ b = a.
therefore $A = B$.

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