Functions with unbounded slow growth Nevanlinna characteristic in the upper halfplane

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1. Abstract

Motivated by applications to differential equations, we consider Nevanlinna theory in the upper half-plane. We give examples of meromorphic functions in the upper half-plane for which the Nevanlinna characteristic is unbounded but slowly growing.

2. Nevanlinna characteristic

[1, 2]: Let f be meromorphic on $\mathbb{H}$ and $n(r,f)$ be # poles of f counting with multiplicity inside the set $\{z : |z - ir/2| \le r/2, |z| \ge 1\}$. We define

$$N(r,f) = \int_1^r \frac{n(t,f)}{t^2} dt,$$

$$m(r,f) = \frac{1}{2\pi} \int_{\kappa(r)}^{\pi-\kappa(r)} \frac{\ln^+ |f(r \sin v e^{iv})| dv}{r \sin^2 v},$$

$$T(r,f) = m(r,f) + N(r,f),$$

where $\kappa(r) = \arcsin 1/r$.

3. Lemma on the logarithmic derivative

[1]: Let f be meromorphic on $\mathbb{H} \cup \{0\}$, then we obtain for all $1 < r < R$,

$$m\left(r, \frac{f'}{f}\right) = O\left(\ln^+ T(R,f) + \ln^+ \frac{1}{R-r} + \ln R\right)$$

which is useful when

$$m\left(r, \frac{f'}{f}\right) = S(r,f) \qquad (1)$$

where $S(r,f) = o(T(r,f))$ outside a set of finite linear measure. Relation (1) holds when

$$\ln r = o(T(r,f)). \qquad (2)$$

In the proof meromorphicity at 0 implies (by scaling) meromorphicity on $\{|z| \le 1\}$ (meromorphicity on $\{|z| = 1, \Im \ge 0\}$ is crucial for the proof).

4. Applications of (1)

Relation (1) is used in the application of differential equations. Let y be a solution meromorphic on $\mathbb{H} \cup \{0\}$ of the equation

$$y' = \frac{a_0 + a_1 y + \cdots + a_p y^p}{b_0 + b_1 y + \cdots + b_q y^q},$$

such that (2) holds for y and the coefficients a_i, b_j are meromorphic on $\mathbb{H} \cup \{0\}$ and satisfy $T(r,a_i) = S(r,y), T(r,b_j) = S(r,y)$. Then

$$y' = a_0 + a_1 y + a_2 y^2.$$

- Above generalizes the classical case as we may allow a_i, b_j at 0 to be branched or essential.
- The solution y does not need to be meromorphic on $\mathbb{C}$.

5. Characteristics which is Bounded or of order $\ln r$

All branches are chosen to be principal.

1. Since $T(r, 1/z) = O(1)$, then for any rational function $R(z)$, we obtain $T(r, R(z)) = O(1)$.
2. $T(r, \ln z) \le \int_{\kappa(r/e)}^{\pi/2} \frac{\ln(r \sin v) dv}{\pi r \sin^2 v} + O(1) = O(1)$ (by integral by part with $u = \ln(r \sin v)$).
3. $T(r, e^z) = \frac{1}{2\pi} \int_{\kappa(r)}^{\pi/2} \cot v dv = \frac{\ln r}{2\pi} + o(\ln r)$.

6. Unbounded characteristics but grow slower than $\ln r$

There exists functions such that $T(r,f)$ is unbounded and $o(\ln r)$. This feature is false in the classical Nevanlinna theory ([3], Ch.1) e.g. $T(r, \exp\{z \ln^{-1/2}(ez)\}) = \pi^{-1}\sqrt{\ln r} + o(\sqrt{\ln r})$ since

$$\int_{\kappa(r)}^{\pi/2} \frac{\cot v}{\sqrt{\ln(er \sin v) + \pi^2/4}} \frac{dv}{2\pi} \le T(r, \exp\{z \ln^{-1/2}(ez)\}) \le \int_{\kappa(r)}^{\pi/2} \frac{\cot v}{\sqrt{\ln(er \sin v)}} \frac{dv}{2\pi} + O(1).$$

Theorem: Let $e_n = \exp\{e_{n-1}\}$, $e_1 = e^2$, $\ln^{\circ 1} x = \ln x$ and $\ln^{\circ n} x = \ln(\ln^{\circ(n-1)} x)$. We define $g_n(z) = \exp\{z[\ln(e_n z) \cdots \ln^{\circ n}(e_n z)]^{-1}\}$. Then

$$T(r, g_n) = 2^{-1}\pi^{-1} \ln^{\circ(n+1)} r + o(\ln^{\circ(n+1)} r).$$

Sketch of proof when $n = 1$:

1. Show that $T(r, g_1) \le \int_{\kappa(r)}^{\pi/2} \frac{\cot v}{\ln(er \sin v)} \frac{dv}{2\pi} + O(1) = \frac{\ln \ln r}{2\pi} + o(\ln \ln r)$
2. Show that $T(r, g_1) \ge \int_{\kappa(r)}^{\pi/2} \frac{\cos v \ln(er \sin v)}{\sin v \{\ln^2(er \sin v) + \pi^2/4\}} \frac{dv}{2\pi} = \frac{\ln \ln r}{2\pi} + o(\ln \ln r)$.

Sketch of proof when $n \ge 2$:

1. $T(r) \le \int_{\kappa(r)}^{\pi/2} \frac{\cos v \ln |e_n z| \cdots \ln |\ln^{\circ(n-1)}(e_n z)|}{\sin v (\ln^2 |e_n z| + v^2) \cdots \left\{\ln^2 |\ln^{\circ(n-1)}(e_n z)| + \arctan^2 \frac{\Im(\ln^{\circ(n-1)}(e_n z))}{\Re(\ln^{\circ(n-1)}(e_n z))}\right\}} \frac{dv}{2\pi} + O(1)$
2. $T(r) \le \int_{\kappa(r)}^{\pi/2} \frac{\cos v}{\sin v} \frac{1}{\ln(e_n r \sin v)} \cdots \frac{1}{\ln^{\circ n}(e_n r \sin v)} \frac{dv}{2\pi} + O(1) \le \frac{\ln^{\circ(n+1)} r}{2\pi} + o(\ln^{\circ(n+1)} r)$
3. Show that $\Re\left[z\{\ln(e_n z) \cdots \ln^{\circ n}(e_n z)\}^{-1}\right] \ge 0$, for $v \in [\kappa(r), \pi/2]$
4. $T(r) + O(1) \ge \int_{\kappa(r)}^{\pi/2} \frac{\cos v}{\sin v} \frac{1}{\ln(e_n r \sin v) + \frac{\pi^2}{4}} \cdots \frac{1}{\ln(\cdots(\ln(\ln(e_n r \sin v) + \frac{\pi^2}{4}) + \frac{\pi^2}{4}) + \cdots) + \frac{\pi^2}{4}} \frac{dv}{2\pi}$
5. Deduce that $T(r) \ge 2^{-1}\pi^{-1} \ln^{\circ(n+1)} r + o(\ln^{\circ(n+1)} r)$.

7. Further questions

Although examples in section 4 and 5 contradict (2), relation (1) still holds. Here are what we would like to consider next

1. Relation (1) fails when the characteristic is bounded, however, we do not know whether it always holds if the characteristics in unbounded. When it fails, it must not satisfy (2).
2. Are there any functions whose characteristics are unbounded but grow slower than $\ln^{\circ n} r$?
3. Moreover, given an increasing unbounded function $\Phi(r)$, is there an f such that $T(r,f) = \Phi(r)$?

8. References

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[3] Walter Kurt Hayman. *Meromorphic functions*, volume 2. Oxford Clarendon Press, 1964.