Analytic continuation of solutions of Difference equations

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1. Abstract

We study the analytic continuation of solutions to $y(z+1) = R(y(z))$, where R is a rational function. In the non-integrable cases, there exist multi-valued solutions with complicated singularity structures.

2. Meromorphic solutions of $y(z+1) = R(y(z))$

The study by Kimura, Shimomura, Yanagihara, Baker and Liverpool, and Azarina [1, 2] shows that all "fundamental" meromorphic solutions must be obtained from the analytic solutions $y \to \gamma$ when $\Re z \to -\infty$, where γ is a fixed point of R either repelling or neutral with multiplicity 1. All meromorphic solutions can be obtained from the fundamental solutions by changing $z \to z + E(z)$, where E is an arbitrary entire function of period 1.
We only consider by normalization R of the form

$$R(w) = \frac{\lambda w + a_2 w^2 + \cdots + a_p w^p}{1 + b_1 w + \cdots + b_q w^q}, \quad \deg R \geq 2. \quad (1)$$

The above discussions show that y is meromorphic only when $\lambda = 1$ or $|\lambda| > 1$.

3. Local solutions and singularity structure

When $\lambda = 1$, there exists m meromorphic functions, where m is the smallest non-negative integer such that $R^{(m+1)}(0) \neq 0$. These functions are known as the Fatou functions. However, it is not known whether any similar local solutions exist when λ is a root of unity different from 1. For the remaining cases, the solution y can be found through the Inverse Schröder equation

$$y(z) = h(\lambda^z), \quad h(\lambda w) = R(h(w)), \quad (2)$$

where $h(w) = c_1 w + \sum_{k \geq 2} c_k w^k$ and c_k depends only on c_1. This function h, known as the Poincaré function, exists for all λ that are not roots of unity and outside a measure zero set of irrational θ when $\lambda = e^{2\pi i \theta}$. Using analytic continuation and Nevanlinna theory, we can show that h is either:

- meromorphic when $|\lambda| > 1$,
- meromorphic on $D(0,r)$ when $\lambda = e^{2\pi i \theta}$; $C(0,r)$ is a natural boundary,
- h has a branch point at $\hat{r}$ closest to 0, when $0 < |\lambda| < 1$ (equivalent to Fatou's theorem).

4. Riemann surface of h when $0 < |\lambda| < 1$

The Riemann surface of h has an interesting singularity structure. We will describe the case $R(w) = \lambda w + w^2$, where $0 < \lambda < 1$ following [3]. We use arguments in Mahler's work on the Mahler equation [4]. First, the branch point $\hat{r}$ yields two sheets: h_0 (continuation of $c_1 w + \sum_{k \geq 2} c_k w^k$) and h_1. We set $c_1 = -1$, then $\hat{r} > 0$. We also have

$$h_0(\lambda w) = \lambda h_i(w) + h_i^2(w), \ i = 0, 1, \quad h_0(w) + h_1(w) = -\lambda, \quad R(h_1(\lambda w)) = R^{\circ 2}(h_1(w)). \quad (3)$$

Using (2), we find that the set of branch points of h_i is $\hat{r}/\lambda^n$, $n \geq 0$. At $w = \hat{r}/\lambda$, there are two new sheets h_{1i} arising from $h_1(\lambda w) = \lambda h_{1i}(w) + h_{1i}^2(w)$, $i = 0, 1$. However, h_{1i} is not branched at $\hat{r}$. Inductively, the k-th iterative argument shows that the 2^{k-1} new Riemann sheets admit branch points exactly on the set $\{\hat{r}/\lambda^n, \ n \geq k-1\}$. The equation of the sheet H appearing in the k-th iterative argument is $R^{\circ k}(H(z)) = R^{\circ(k+1)}(H(z))$.

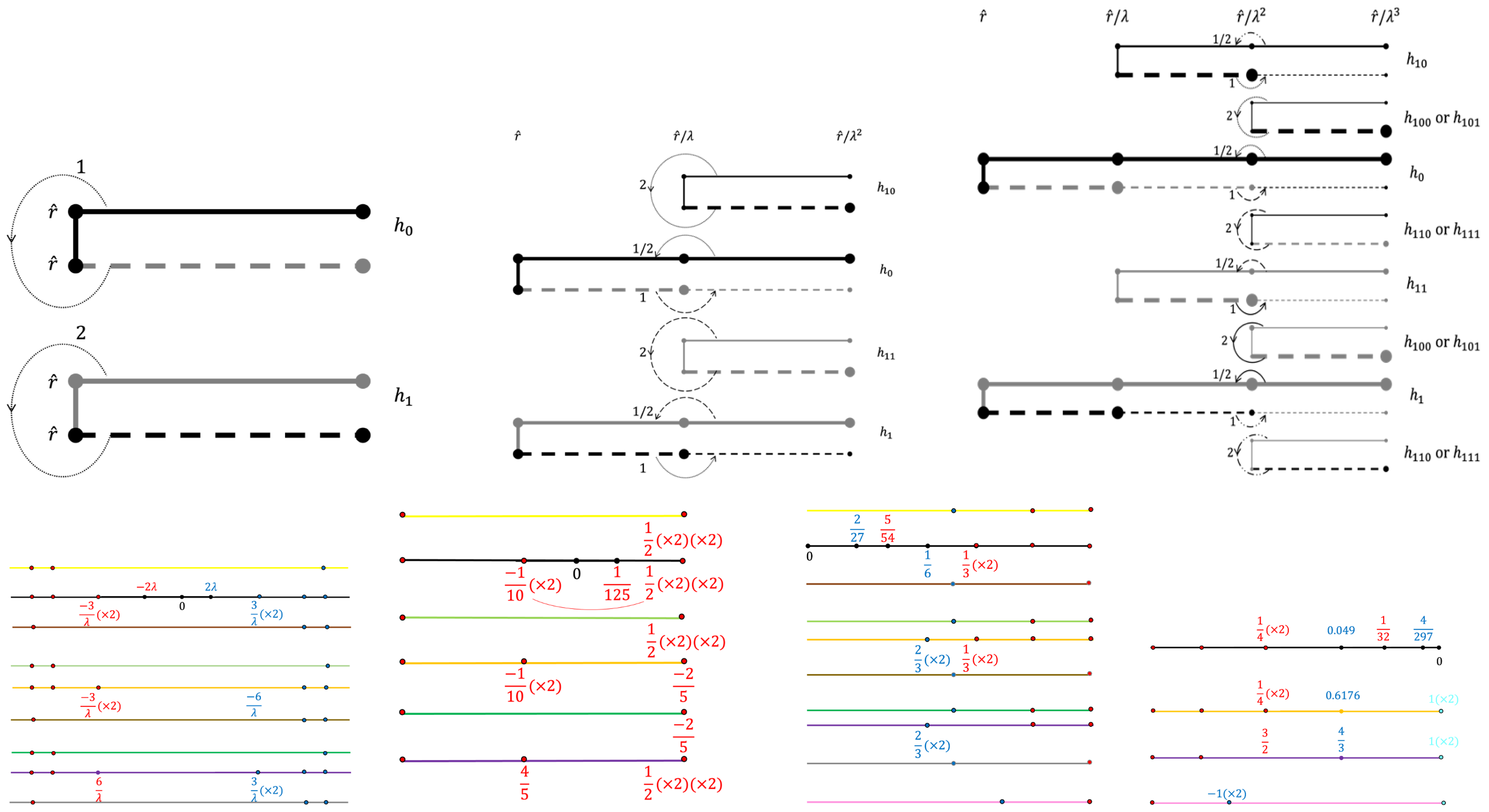

Figure 1: Second iterative argument: $R(w) = \lambda^2 w - \lambda^4 w^3/27$, $\quad -3w/20 - 12w^2/20 + w^3$, $\quad 5w/27 - w^2/2 + 27w^3$, $\quad w(w-1)^2/[3(1+2w)]$ respectively

5. Further questions

1. Is it possible to find examples where a branch point first appears on the n-th iterative sheets? We observe that in cases where the branch points appear first on the 0-th or 1-st sheet, correspond to when A_0^0 the immediate basin of attraction is simply-connected; meanwhile, in the other cases, A_0^0 has infinite genus. The result will classify the singularity structure of the Poincaré functions.
2. When $\lambda = 0$, there exists a local solution from the inverse Böttcher equation. There are meromorphic solutions as well as solutions with natural boundary (unit circle) whose interior may be infinitely multi-valued. However, it is not clear on how to separate these cases.

7. References

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