Singularity structure of some Discrete equations

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1. Abstract

We investigate the singularity structure of an infinite-sheeted Riemann surface of a particular solution of a discrete equation. The singularity structure has some complicated features reflecting the non-integrability of the equation.

2. Difference equation

Given a difference equation

$$y(z+1) = R(y(z)) = \lambda y(z) + y^2(z). \tag{1}$$

When $|\lambda| \neq 0, 1$, there exists a unique solution $y(z)$ expressing in terms of the solutions of the inverse Schröder equation:

$$y(z) = h(\lambda^z), \quad h(\lambda w) = \lambda h(w) + h^2(w), \tag{2}$$

where h is analytic at $w = 0$ and satisfies

$$h(w) = w + a_2 w^2 + \cdots. \tag{3}$$

The inverse function of h^{-1} is called the Koenig function or Poincaré function.

When $|\lambda| > 1$, 0 is a *repelling* fixed point of R, and $y(z)$ is *meromorphic*.

When $0 < |\lambda| < 1$, 0 is an *attracting* fixed point of R. Given any $\sigma > 0$, the solution $y(z)$ exists in a domain $D(\sigma, \rho)$ for some ρ, where

$$D(\sigma, \rho) = \{z : \Re z > \rho, |\Im z| < \sigma\}. \tag{4}$$

3. Fatou's theorem

From now on, we assume that $0 < \lambda < 1$.

In this case, h admits a branch point. This is equivalent to Fatou's theorem, which states: "*The immediate basin of attraction at 0 of R contains a critical point*".

Hence, there is a quadratic branch point $w = \hat{r}$ of h, which satisfies

$$h(\lambda \hat{r}) = -\frac{\lambda^2}{4}, \quad h(\hat{r}) = -\frac{\lambda}{2}. \tag{5}$$

Let h_0 be the Riemann sheet of the analytic continuation of (3), and h_1 the other sheet.

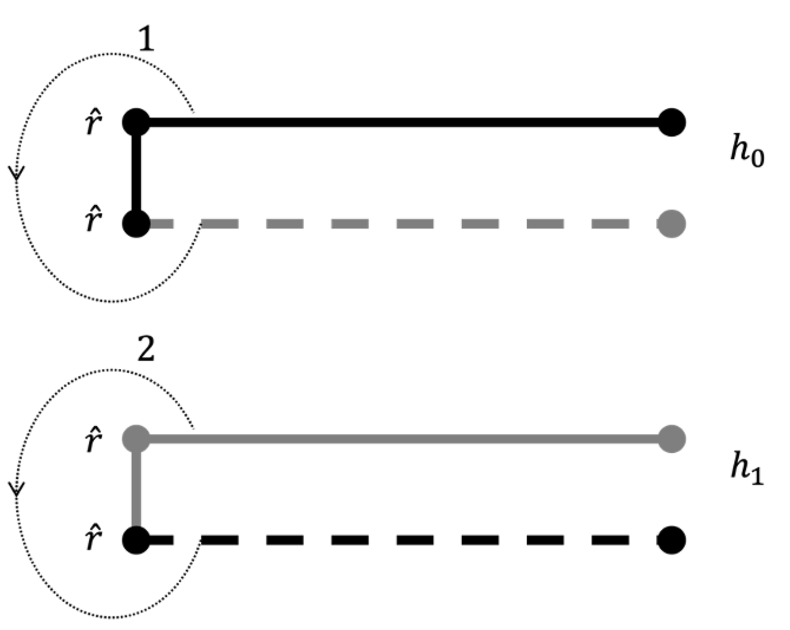

Figure 1: Branch point at $w = \hat{r}$

4. Singularity structure of the Riemann surfaces of $h(w)$ and $y(z)$

Using (2), we obtain:

$$\begin{aligned} &h_0(\lambda w) = \lambda h_i(w) + h_i^2(w), \ i = 0, 1, \\ &h_0(w) + h_1(w) = -\lambda. \end{aligned} \tag{6}$$

Successive arguments can be used to show that the set of quadratic branch points of $h_0, \ h_1$ is given by $\{\hat{r}, \ldots, \widehat{r/\lambda^n}, \ldots\}, \ n \geq 1$.

We investigate the branch points $\hat{r}/\lambda^n, \ n \geq 1$ and the analytic continuation of the Riemann surface from these points. About $w = \hat{r}/\lambda$, there are four Riemann sheets: two of them are $h_0, \ h_1$, and the other two, $h_{10}, \ h_{11}$, are obtained from the relations:

$$h_1(\lambda w) = \lambda h_{1i}(w) + h_{1i}^2(w), \ i = 0, 1. \tag{7}$$

However, it can be shown that h_{1i} is analytic at $\hat{r}$. The set of branch points of h_{1i} is $\{\hat{r}/\lambda^n, \ n \geq 1\}$. The k-th iterative argument shows that the 2^{k-1} new Riemann sheets admit branch points exactly on the set $\{\hat{r}/\lambda^n, \ n \geq k\}$. The equation of the sheet H appearing in the k-th iterative argument is $R^{\circ(k-1)}(H(z)) = R^{\circ k}(H(z))$. A full description can be found in [1].

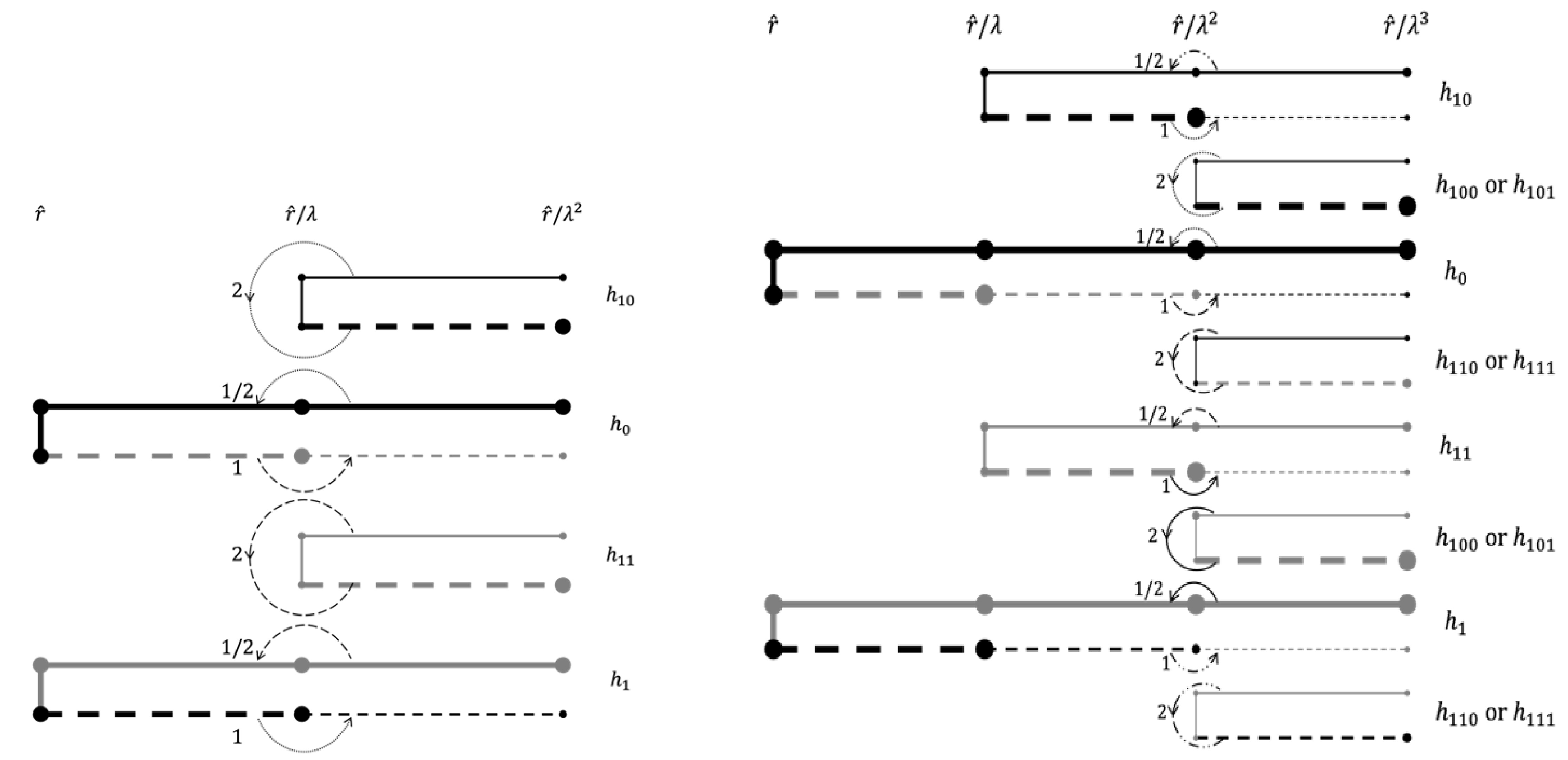

Figure 2: Second and third iterations

Using (2), it is easy to see that

$$y(z) = y(z + 2\pi i / \ln \lambda). \tag{8}$$

Hence, $y(z) = h(\lambda^z)$ is periodic and inherits the singularity structure from $h(w)$.

5. Further questions

1. The higher-degree analogue of equation (1), where the right hand side is a rational function with 0 as the attracting fixed point, exhibits the singularity structure of more diverse features. This is because many critical points could present simultaneously in the basin of attraction. It is part of our current interest to classify the higher degree cases.
2. This work would lay the ground work to study the difference equation $R_1(y(z+1)) = R_2(y(z))$, where R_i are some rational functions and $y(z)$ satisfies some asymptotic conditions. In the case where R_1 and R_2 are respectively the $(n-1)$- and n-times compositions of a polynomial $\lambda z + z^2$, the analytic continuation coincides with one of the Riemann sheets of the above description. The higher-degree case, as well as different rational functions R_i will be our future interest.
3. Given (1) and (8), it would be nice if we could characterize all *"fundamental"* solutions whose analytic continuation results in a Riemann surface over $\mathbb{C}$ in the sense that the sets of singularities, except poles, are the simplest.

6. References

[1] Rod Halburd, Risto Korhonen, Yan Liu, and Techheang Meng. On the extension of analytic solutions of first-order difference equations. *https://arxiv.org/abs/2502.03955*, 2025 (preprint).