Meromorphic solutions of a spinning top system

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The spinning top equations are given by

$$A\frac{dp}{dt} = (B - C)qr + (y_0 h - z_0 g), \tag{1a}$$
$$B\frac{dq}{dt} = (C - A)rp + (z_0 f - x_0 h), \tag{1b}$$
$$C\frac{dr}{dt} = (A - B)pq + (x_0 g - y_0 f), \tag{1c}$$
$$\frac{df}{dt} = rg - qh, \tag{1d}$$
$$\frac{dg}{dt} = ph - rf, \tag{1e}$$
$$\frac{dh}{dt} = qf - pg. \tag{1f}$$

where $A, B, C, x_0, y_0, z_0 \in \mathbb{C}$.

Conserved quantities

There are 3 conserved quantities

$$\begin{aligned}
&Ap^2 + Bq^2 + Cr^2 - 2(x_0 f + y_0 g + z_0 h) = \ell_2, \\
&Afp + Bgq + Chr = \ell_3, \\
&f^2 + g^2 + h^2 = \ell_4.
\end{aligned}$$

Fourth conserved quantity exists in some cases: symmetric, Euler, Lagrange and Kowalevskaya's top.

Objective

In this talk, I would like to show that given

$$(A-B)(B-C)(C-A)ABCx_0y_0z_0 \neq 0,$$

All meromorphic solutions in general are one of below (known as Class W functions)

- Elliptic functions
- functions of the form $R(e^{\theta t})$ where R is rational and $\theta \in \mathbb{C}$
- Rational functions.

Kowalevskaya's work

Kowalevskaya looked for the general solutions of (1) in the case

$$A = B = 2C, \quad z_0 = 0$$

and showed the following

- general solutions are meromorphic
- general solutions solvable (a fourth conserved quantity exists)
- solutions can be written in terms of the Riemann-theta functions
- local series expansion of the top equations and resonances.

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Uniqueness and periodicity

About t_0,

$$p(t) = \frac{p_{1-k}}{(t-t_0)^k} + \sum_{i=1-k}^{\infty} p_i (t-t_0)^{i-1}. \tag{2}$$

About $\tilde{t}_0 \neq t_0$,

$$p(t) = \frac{p_{1-k}}{(t-\tilde{t}_0)^k} + \sum_{i=1-k}^{\infty} p_i (t-\tilde{t}_0)^{i-1}.$$

- Then $p(t) = p(t + \tilde{t}_0 - t_0)$ i.e. p is periodic.
- If p has the same Laurent series about t_0' which is not collinear with $t_0, \tilde{t}_0$, then $p(t) = p(t + t_0' - t_0)$ i.e. p is elliptic.

So we are left with when p is simply-periodic or p has the unique expansion (2).

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Finiteness property

Eremenko (1982, 2005) showed the following

Theorem 1 (Eremenko's finiteness property)

Given a meromorphic function $y(t)$. If $y(t)$ has just finitely many Laurent series (finiteness property)

$$y(t) = \frac{y_{1-k}}{(t-t_0)^k} + \sum_{i=1-k}^{\infty} y_i (t-t_0)^{i-1},$$

and in addition y has slow growth condition, then $y(t)$

- *is elliptic*
- *or of the form $R(e^{\theta t})$*
- *or rational.*

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Notations

- **Convention**: $cyc(\cdot)$ acts as the cyclic permutation of the ordered sets $\{A, B, C\}, \{x_0, y_0, z_0\}, \{p_i, q_i, r_i\}, \{f_i, g_i, h_i\}$

$$cyc(A) = B, \quad cyc(B) = C, \quad cyc(C) = A.$$

- **Example**: $cyc(Az_0p_2q_3) = Bx_0q_2r_3$
- **Convention**: $\sum_{cyc} Az_0p_2q_3 = Az_0p_2q_3 + Bx_0q_2r_3 + Cy_0r_2p_3$
- **Denote**:

$$k_0 = \sum_{cyc} A, \quad k_1 = \sum_{cyc} BC, \quad k_2 = ABC,$$

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Local series expansion

About a singularity t_0, we have the following expansions

$$p = (t - t_0)^{n_1}(p_0 + p_1(t - t_0) + \cdots), \tag{3a}$$

$$q = (t - t_0)^{n_2}(q_0 + q_1(t - t_0) + \cdots), \tag{3b}$$

$$r = (t - t_0)^{n_3}(r_0 + r_1(t - t_0) + \cdots), \tag{3c}$$

$$f = (t - t_0)^{n_4}(f_0 + f_1(t - t_0) + \cdots), \tag{3d}$$

$$g = (t - t_0)^{n_5}(g_0 + g_1(t - t_0) + \cdots), \tag{3e}$$

$$h = (t - t_0)^{n_6}(h_0 + h_1(t - t_0) + \cdots), \tag{3f}$$

where at least one of n_i is negative and at least one of $p_0, \ldots, h_0 \neq 0$.

Substituting (3) into the top equations

$$
\begin{aligned}
n_1 = n_2 = n_3 &= -1, & n_4 = n_5 = n_6 &= -2,\\
-Ap_0 &= (B-C)q_0 r_0 + (y_0 h_0 - z_0 g_0), & -2f_0 &= r_0 g_0 - q_0 h_0,\\
-Bq_0 &= (C-A)r_0 p_0 + (z_0 f_0 - x_0 h_0), & -2g_0 &= p_0 h_0 - r_0 f_0,\\
-Cr_0 &= (A-B)p_0 q_0 + (x_0 g_0 - y_0 f_0), & -2h_0 &= q_0 f_0 - p_0 g_0.
\end{aligned}
\tag{4}
$$

Moreover,

$$
\begin{pmatrix}
(n-1)A & (C-B)r_0 & (C-B)q_0 & 0 & z_0 & -y_0\\
(A-C)r_0 & (n-1)B & (A-C)p_0 & -z_0 & 0 & x_0\\
(B-A)q_0 & (B-A)p_0 & (n-1)C & y_0 & -x_0 & 0\\
0 & h_0 & -g_0 & n-2 & -r_0 & q_0\\
-h_0 & 0 & f_0 & r_0 & n-2 & -p_0\\
g_0 & -f_0 & 0 & -q_0 & p_0 & n-2
\end{pmatrix}
\begin{pmatrix} p_n\\ q_n\\ r_n\\ f_n\\ g_n\\ h_n \end{pmatrix}
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Leading order terms

Assumption: $(A-B)(B-C)(C-A)ABCx_0y_0z_0 \neq 0$

Kowalevskaya showed that

$$p_0 = \sqrt{\frac{2B+\lambda}{B_1}}\sqrt{\frac{2C+\lambda}{C_1}},\; q_0 = \sqrt{\frac{2C+\lambda}{C_1}}\sqrt{\frac{2A+\lambda}{A_1}},\; r_0 = -\sqrt{\frac{2A+\lambda}{A_1}}\sqrt{\frac{2B+\lambda}{B_1}},$$

$$f_0 = -A_1 q_0 r_0 \frac{\lambda}{\mu}, \quad g_0 = -B_1 r_0 p_0 \frac{\lambda}{\mu}, \quad h_0 = -C_1 p_0 q_0 \frac{\lambda}{\mu},$$

where $\mu = \sum_{cyc} Ap_0x_0$ and λ could be obtained from

$$(A+\lambda)p_0x_0 + (B+\lambda)q_0y_0 + (C+\lambda)r_0z_0 = 0.$$

Here $p_0q_0r_0f_0g_0h_0\lambda\mu \neq 0$. We call this singularity, the K-singularity.

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Remark 1

Exceptional case of the K-singularity which corresponds to $\mu = \lambda = 0$ and

$$p_0 = \sqrt{\frac{2B}{B_1}}\sqrt{\frac{2C}{C_1}}, q_0 = \sqrt{\frac{2C}{C_1}}\sqrt{\frac{2A}{A_1}}, r_0 = -\sqrt{\frac{2A}{A_1}}\sqrt{\frac{2B}{B_1}},\ f_0 = \frac{-2Ap_0}{\sum_{cyc} p_0 x_0},$$

and $\sum_{cyc} Ap_0x_0 = 0$. This will be called the K_e-singularity.

Remark 2

There is an additional singularity corresponding to

$$f_0 = g_0 = h_0 = 0, p_0 = \sqrt{\frac{B}{B_1}}\sqrt{\frac{C}{C_1}}, q_0 = \sqrt{\frac{C}{C_1}}\sqrt{\frac{A}{A_1}}, r_0 = -\sqrt{\frac{A}{A_1}}\sqrt{\frac{B}{B_1}}.$$

This will be called the K^c-singularity and can be shown to satisfy $\ell_4 = 0$ or $\sum_{cyc} Ap_0x_0 = 0$.

In this talk, we assume only the existence of the K-singularity.

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Resonances

Let

$$\Omega = \begin{pmatrix} p_0 & q_0 & r_0 & 0 & 0 & 0 \\ Ap_0 & Bq_0 & Cr_0 & 0 & 0 & 0 \\ BCp_0 & CAq_0 & ABr_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_0 & g_0 & r_0 \\ f_0 & g_0 & h_0 & Ap_0 & Bq_0 & Cr_0 \\ -p_0 & -q_0 & -r_0 & x_0 & y_0 & z_0 \end{pmatrix} \tag{6}$$

Let

$$\sum_{cyc} p_0 p_n = u_{1,n}, \quad \sum_{cyc} Ap_0 p_n = u_{2,n}, \quad \sum_{cyc} A^2 p_0 p_n = u_{3,n},$$

$$\implies p_0 p_n = -B_1^{-1} C_1^{-1} \{BCu_{1,n} - (B + C)u_{2,n} + u_{3,n}\}.$$

Set $F_{i,n} = 0$,

$$\begin{pmatrix} (n-1)A & (C-B)r_0 & (C-B)q_0 & 0 & z_0 & -y_0 \\ (A-C)r_0 & (n-1)B & (A-C)p_0 & -z_0 & 0 & x_0 \\ (B-A)q_0 & (B-A)p_0 & (n-1)C & y_0 & -x_0 & 0 \\ 0 & h_0 & -g_0 & n-2 & -r_0 & q_0 \\ -h_0 & 0 & f_0 & r_0 & n-2 & -p_0 \\ g_0 & -f_0 & 0 & -q_0 & p_0 & n-2 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \\ r_n \\ f_n \\ g_n \\ h_n \end{pmatrix}$$

$$= \mathbf{0}.$$

Then $\Omega\times$ (5) becomes

$$-\lambda u_{1,n} + (n-3)u_{2,n} + \sum_{cyc}(r_0 y_0 - q_0 z_0) f_n = 0,$$

$$-\lambda u_{2,n} + (n-3)u_{3,n} + \sum_{cyc}(C r_0 y_0 - B q_0 z_0) f_n = 0,$$

$$(n+3)k_2 u_{1,n} - (4k_1 + \lambda k_0)u_{2,n} + 2(k_0+\lambda)u_{3,n} + \sum_{cyc} A(B r_0 y_0 - C q_0 z_0) f_n = 0\ ,$$

$$\sum_{cyc}(n-4) f_0 f_n = 0,$$

$$\sum_{cyc}(n-3) A f_0 p_n + \sum_{cyc}(n-3) A p_0 f_n = 0,$$

$$-\sum_{cyc}(n-2) A p_0 p_n + \sum_{cyc}(n-2) x_0 f_n = 0.$$

$$
\begin{pmatrix}
(n+3)k_2 & -4k_1-\lambda k_0 & 2(k_0+\lambda) & ABr_0y_0-ACq_0z_0 & cyc(a_{14}) & cyc(a_{15}) \\
0 & \frac{\lambda^2}{\mu} & \frac{2\lambda}{\mu} & Ap_0 & Bq_0 & Cr_0 \\
0 & -1 & 0 & x_0 & y_0 & z_0 \\
0 & 0 & 0 & f_0 & g_0 & h_0 \\
-\lambda & n-3 & 0 & r_0y_0-q_0z_0 & cyc(a_{54}) & cyc(a_{55}) \\
0 & -\lambda & n-3 & Cr_0y_0-Bq_0z_0 & cyc(a_{64}) & cyc(a_{65})
\end{pmatrix}
$$

$\cdot K_n = 0.$

where $K_n = (u_{1,n}\ u_{2,n}\ u_{3,n}\ f_n\ g_n\ h_n)^T$. Then

$$
\begin{pmatrix} f_n \\ g_n \\ h_n \end{pmatrix} = \begin{pmatrix}
\frac{Ap_0\lambda}{\mu} & \frac{-(n-3)Ap_0-\lambda p_0}{\mu} & \frac{(n-3)p_0}{\mu} \\
\frac{Bq_0\lambda}{\mu} & \frac{-(n-3)Bq_0-\lambda q_0}{\mu} & \frac{(n-3)q_0}{\mu} \\
\frac{Cr_0\lambda}{\mu} & \frac{-(n-3)Cr_0-\lambda r_0}{\mu} & \frac{(n-3)r_0}{\mu}
\end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \end{pmatrix}. \tag{8}
$$

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The other factor consisting of $n+1$ and

$$(n^2 - n - 6)k_2 - 2k_1\lambda - \lambda^2\mu_1/\mu, \tag{9}$$

where $\mu_1 = \sum_{cyc} A^2 p_0 x_0$. In (9) the sum of the roots is 1, then there could be at most one positive integer root $N \geq 2$ or $0, 1$ are both roots. We obtain

$$u_{1,n} = \frac{n(n-3)u_{3,n}}{\lambda^2}, \quad u_{2,n} = \frac{(n-1)u_{3,n}}{\lambda},$$
$$f_n = \frac{\{(n-3)A - 2\lambda\}p_0 u_{3,n}}{\lambda\mu}, \quad g_n = cyc(f_n), \quad h_n = cyc(g_n).$$

- If n is a root of (9), then $u_{3,n}$ is a parameter.
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Explicit form of $p_n, q_n, r_n, f_n, g_n, h_n$

Assume that 1 is not a resonance, then $p_1 = f_1 = 0$. Moreover,

$$f_2 = \frac{p_0}{\mu}\{A(u_{2,2} + \lambda u_{1,2}) - u_{3,2} - \lambda u_{2,2}\}. \tag{10}$$

- If $n \neq 2$ is a root of $(n^2 - n - 6)k_2 - 2k_1\lambda - \lambda^2\mu_1/\mu$, then

$$\begin{aligned} u_{1,2} &= \frac{(\lambda^3 + 2k_0\lambda^2 + 4k_1\lambda + 8k_2)\ell_2}{3\lambda k_2(n^2 - n - 2)} - \frac{2\ell_2}{3\lambda}, \\ u_{2,2} &= \frac{-\lambda u_{1,2}}{2},\ u_{3,2} = \frac{-\lambda(\lambda u_{1,2} + \ell_2)}{2}. \end{aligned} \tag{11}$$

- If 2 is a root (another 2 as a resonance), then

$$\ell_2 = 0,\ u_{2,2} = \frac{-\lambda u_{1,2}}{2},\ u_{3,2} = \frac{-\lambda^2 u_{1,2}}{2}. \tag{12}$$

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In the case when 1 is resonance, then we take $u_{1,1}$ to be the parameter. However, the resonance condition at 3 yields

$$\sum_{cyc}(n-3)Af_0p_n+\sum_{cyc}(n-3)Ap_0f_n=\sum_{cyc}f_0F_{1,3}+\sum_{cyc}Ap_0F_{4,3} \qquad (13)$$

$$\implies 0=\frac{-\lambda^6 {u_{1,1}}^3}{2\mu}$$

and so

$$u_{1,1}=0.$$

The Laurent series of p, q, r, f, g, h about the singularities are known

$$p=\frac{p_0}{t-t_0}+\sum_{i=2}^{\infty}p_i(t-t_0)^{i-1}.$$

- ℓ_2, ℓ_3, ℓ_4 are used to replace the resonance parameters at $2, 3, 4$
- When $n = 1$ is a resonance, then there are no parameters
- When $n = 2$ is a resonance (another 2), then $u_{1,2}$ is the parameter*
- When $n = N \geq 3$ is a resonance, then $u_{3,N}$ is the parameter.
- Then about each singularity, we have

$$p = \frac{p_0}{t - t_0} + \sum_{i=2}^{\infty} p_i (t - t_0)^{i-1}.$$

- For Eremenko's finiteness property to work here, we need
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Auxiliary function

Lemma 1

Let

$$K_1 = \left(\sum_{cyc} Ap^2\right)^2 + 2\sum_{cyc} p^2 \sum_{cyc} A^2 p^2 + 2\left(\sum_{cyc} A^2 p^2\right)'' - 4\ell_2 \sum_{cyc} Ap^2 - 8\ell_3 \sum_{cyc} px_0$$

and if there is only the K-singularity, then K_1 is entire.

- If $n = N \neq 2, 3, 4$, then

$$K_1 = K_{1,0}(\lambda) + \cdots + \{4(1-N)u_{3,N} + K_{1,N-4}(\lambda)\}(t-t_0)^{N-4} + \cdots$$

- Our objective is to show that K_1 is a constant i.e. $u_{3,N}$ is unique.

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A brief of Nevanlinna theory

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Given a meromorphic function $y(t)$. Let $n(s, y)$ be the number of poles (counting with multiplicities) of y in the set $D_s = \{t : |t| \leq s\}$. We define

Proximity function (Growth): $$m(s, y) = \frac{1}{2\pi} \int_0^{2\pi} \max\{0, \ln |y(se^{i\theta})|\} d\theta,$$

Counting function: $$N(s, y) = \int_0^s \frac{n(\rho, y) - n(0, y)}{\rho} d\rho + n(0, y) \ln s,$$

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Isobaricity

$$
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr + (y_0 h - z_0 g), & \frac{df}{dt} &= rg - qh,\\
B\frac{dq}{dt} &= (C-A)rp + (z_0 f - x_0 h), & \frac{dg}{dt} &= ph - rf,\\
C\frac{dr}{dt} &= (A-B)pq + (x_0 g - y_0 f), & \frac{dh}{dt} &= qf - pg.
\end{aligned}
$$

invariant under the following transformation

$$p \to \omega p, \quad p^{(n)} \to \omega^{n+1} p^{(n)}, \quad f \to \omega^2 f, \quad f^{(n)} \to \omega^{n+2} f^{(n)}.$$

This is isobaricity with p of weight 1 and f of weight 2.
The ODE of p is still isobaric with the form $F(p, p', \dots, p^{(6)}) = 0$

$$a_{p0} p^{M_1} + a_{p1} p^{M_1 - 2} p' + \cdots = 0.$$

Slow growth condition

Application of Clunie's Lemma from Nevanlinna theory:

Lemma 2

Given

$$a_{p0}p^{M_1} + a_{p1}p^{M_1-2}p' + \cdots = 0.$$

If at least one of a_{y0} and a_{y1} is non-zero, then

$$m(s,p) = S(s,p).$$

$$a_{q0}q^{M_2} + a_{q1}q^{M_2-2}q' + \cdots = 0,$$
$$a_{r0}r^{M_3} + a_{r1}r^{M_3-2}r' + \cdots = 0.$$

So we put the following conditions

General parameters: $(a_{p0}, a_{p1}), (a_{q0}, a_{q1}), (a_{r0}, a_{r1}) \neq (0,0).$

We have shown Slow growth condition. Need to show K_1 constant.

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$$m(s,p) = S(s,p).$$

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$$a_{r0}r^{M_3} + a_{r1}r^{M_3-2}r' + \cdots = 0.$$

So we put the following conditions

General parameters: $(a_{p0}, a_{p1}), (a_{q0}, a_{q1}), (a_{r0}, a_{r1}) \neq (0,0)$.

We have shown Slow growth condition. Need to show K_1 constant.

Table of Contents

Showing K_1 is constant

- Recall that K_1 is entire and

$$K_1 = \left(\sum_{cyc} Ap^2\right)^2 + 2\sum_{cyc} p^2 \sum_{cyc} A^2p^2 + 2\left(\sum_{cyc} A^2p^2\right)'' - 4\ell_2 \sum_{cyc} Ap^2 - 8\ell_3 \sum_{cyc} px_0$$

then $T(s, K_1) = m(s, K_1) = S(s, p)$.

- Also for $N \neq 2, 3, 4$

$$K_1 = K_{1,0}(\lambda) + \cdots + \{4(1-N)u_{3,N} + K_{1,N-4}(\lambda)\}(t-t_0)^{N-4} + \cdots \quad (15)$$

Lemma 3

Given function y such that $T(s, y) = S(s, p)$ and y has the Laurent series (15), then either p is constant or y is constant.

- Hence K_1 is constant.

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Summary

Theorem 2

Assume that p,q,r have Slow growth condition

- *if there is only the K-singularity, then p, q, r are elliptic or rational or of the form $R(e^{\theta t})$.*
- *if $\sum_{cyc} x_0^2 \neq 0$, then p, q, r are elliptic or rational or of the form $R(e^{\theta t})$.*

The main ideas of the proof are

- local series coefficients and resonances
- the role of finiteness property with the resonance parameters
- Specific cases of isobaric system imply Slow growth condition
- Slow growth condition and auxiliary function result in Eremenko's finitness property which concludes our proof.

Future works

There are a few interesting things we would like to pursue next

- Finding all meromorphic solutions
- Going through the exceptional parameters (the union of the hyper surfaces)

  $$a_{p0}p^{M_1} + a_{p1}p^{M_1-2}p' + \cdots = 0$$

  where $(a_{p0}, a_{p1}) = (0, 0)$
- Looking at the case $(A-B)(B-C)(C-A)ABCx_0y_0z_0 = 0$.

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Thank you for your attention!

Resonance condition

Resonance condition at 4

- if 2 is the root of (9), then

$$u_{1,2}(k_0 + 3\lambda) = 0.$$

- otherwise,

$$\ell_2 u_{1,2}(\ell_2 + 3\lambda u_{1,2} + k_0 u_{1,2}) = 0.$$

Auxiliary function (2)

Theorem 3

We define

$$K_2 = 3\left(\sum_{cyc} Ap^2\right)^2 + 2\left(\sum_{cyc} A^2p^2\right)'' - 4\ell_2 \sum_{cyc} Ap^2 - 4\sum_{cyc} pf \sum_{cyc} Apx_0.$$

Then K_2 is entire.

Compare with

$$K_1 = \left(\sum_{cyc} Ap^2\right)^2 + 2\sum_{cyc} p^2 \sum_{cyc} A^2p^2 + 2\left(\sum_{cyc} A^2p^2\right)'' - 4\ell_2 \sum_{cyc} Ap^2 - 8\ell_3 \sum_{cyc} px_0$$

which is entire only on the $K \cup K_e$-singularity.

Dealing with $N = 2, 3, 4$

The main idea stays the same (showing that $u_{1,2}, u_{2,3}, u_{3,4}$ have finitely many choices). For $N = 4$:

$$L_1 = \left(\sum_{cyc} A^2 p x_0 - 2k_1 \sum_{cyc} p x_0\right) \sum_{cyc} A p x_0 - 5k_2 \left(\sum_{cyc} p x_0\right)^2,$$

$$\begin{aligned} L_2 =& \alpha_1 \left(\sum_{cyc} A p x_0 \sum_{cyc} p^2 - 2 \sum_{cyc} p x_0 \sum_{cyc} A p^2\right) \\ &+ \alpha_2 \left(2 \sum_{cyc} A^2 p^2 \sum_{cyc} p x_0 - \sum_{cyc} A p^2 \sum_{cyc} A p x_0\right) \\ &+ \alpha_3 \sum_{cyc} p x_0 + \alpha_4 \sum_{cyc} A p x_0 + \alpha_5 \sum_{cyc} A^2 p x_0 \end{aligned}$$

$$L_3 = \sum_{cyc} A^2 p^2 + \beta_1 \sum_{cyc} A p^2 + \beta_2 \sum_{cyc} p^2,$$

$$E_p = p_0^2 p'' - 2p^3 + 6 p_0 p_2 p,$$

Dealing with $N = 2, 3, 4$

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\begin{aligned}
L_1 &= \left(\sum_{cyc} A^2 p x_0 - 2k_1 \sum_{cyc} p x_0\right) \sum_{cyc} A p x_0 - 5k_2 \left(\sum_{cyc} p x_0\right)^2, \\
L_2 &= \alpha_1 \left(\sum_{cyc} A p x_0 \sum_{cyc} p^2 - 2 \sum_{cyc} p x_0 \sum_{cyc} A p^2\right) \\
&\quad + \alpha_2 \left(2 \sum_{cyc} A^2 p^2 \sum_{cyc} p x_0 - \sum_{cyc} A p^2 \sum_{cyc} A p x_0\right) \\
&\quad + \alpha_3 \sum_{cyc} p x_0 + \alpha_4 \sum_{cyc} A p x_0 + \alpha_5 \sum_{cyc} A^2 p x_0 \\
L_3 &= \sum_{cyc} A^2 p^2 + \beta_1 \sum_{cyc} A p^2 + \beta_2 \sum_{cyc} p^2, \\
E_p &= p_0^2 p'' - 2p^3 + 6 p_0 p_2 p,
\end{aligned}
$$

For $N = 2, 3$:

$$L_4 = \sum_{cyc} A^2 px_0 - 2k_1 \sum_{cyc} px_0,$$

$$L_5 = \left(\sum_{cyc} Apx_0 - k_0 \sum_{cyc} px_0 \right) \left(3 \sum_{cyc} Apx_0 - k_0 \sum_{cyc} px_0 \right),$$

$$L_6 = \sum_{cyc} A^2 px_0 - \frac{6(k_0 k_1 - 6k_2)}{k_0^2} \sum_{cyc} Apx_0,$$

$$L_7 = \left(\sum_{cyc} A^2 px_0 - 2k_1 \sum_{cyc} px_0 \right) \left(3 \sum_{cyc} Apx_0 - k_0 \sum_{cyc} px_0 \right).$$

Examples of Class W solutions (1)

Example 1

Set

$$f = v(t)x_0, g = v(t)y_0, h = v(t)z_0, \sum_{cyc} x_0^2 = 0, \ell_4 = 0$$

where

$$\frac{v'(t)}{v(t)} = \frac{y_0 r - z_0 q}{x_0} = \frac{z_0 p - x_0 r}{y_0} = \frac{x_0 q - y_0 p}{z_0}, \ v(t)\sum_{cyc} Apx_0 = \ell_3.$$

Choose

$$p = \frac{p_0}{t - t_0}, q = \frac{q_0}{t - t_0}, r = \frac{r_0}{t - t_0}$$

then

$$\sum_{cyc} p_0 x_0 = \sum_{cyc} Ap_0 x_0 = \ell_3 = \ell_2 = 0, \quad v(t) = \frac{K}{t - t_0}.$$

Examples of Class W solutions (2)

Example 2

Find p, q, r from

$$p_0^2 p'' - 2p^3 + 6p_0 p_2 p = E_p,$$
$$q_0^2 q'' - 2q^3 + 6q_0 q_2 q = E_q,$$
$$r_0^2 r'' - 2r^3 + 6r_0 r_2 r = E_r.$$

Here E_p, E_q, E_r relate to ℓ_2, ℓ_3, ℓ_4.