Analytic continuation of the inverse of the Koenig function

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(Based on joint work[1] with Rod Halburd, Risto Korhonen, Yan Liu)

Topics in Complex Dynamics 2025, 2-6 June, University of Barcelona

[1] R. Halburd, R. Korhonen, Y. Liu, and T. M. , On the extension of analytic solutions of first-order difference equations, *arXiv:2502.03955*.

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- The results of Kimura[2], Shimomura, Yanagihara, Baker and Liverpool[3], and Azarina show that meromorphic solutions y can be obtained from analytic continuation of local solutions near a fixed point γ of R which is either repelling or neutral with multiplier 1.

[2] T. Kimura, Iteration of analytic functions, *Funkcial. Ekvac 14, 1971*.
[3] I.N. Baker, and L.S.O. Liverpool, The entire solutions of a polynomial difference equation, *Aequationes Mathematicae 27, 1984*.

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- The results of Kimura, Shimomura, Yanagihara, Baker and Liverpool, and Azarina show that meromorphic solutions y can be obtained from analytic continuation of local solutions near a fixed point γ of R which is either repelling or neutral with multiplier 1.
- All meromorphic solutions $y(z)$ are either above or $y(z+E(z))$, for any entire function E of period 1.

Inverse Schröder equation

- Assume that R is normalized of the form

$$R(w) = \frac{\lambda w + a_2 w + \cdots + a_p w^p}{1 + b_1 w + \cdots + b_q w^q}, \ d = \max\{p, q\} \geq 2.$$

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- So the fixed point $\gamma = 0$. When 0 is repelling i.e. $|\lambda| > 1$, the meromorphic solution y is obtained by solving the Inverse Schröder equation:

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- There admits a Taylor series solution

$$h(w) = c_1 w + \sum_{k \geq 2} c_k w^k, \quad \text{where } c_1 \text{ is a parameter.}$$

h^{-1} is called the Koenig function, h is also known as the Poincaré or Schröder function.

Local solution near attracting fixed point

- When 0 is attracting i.e. $0 < |\lambda| < 1$, there also exists

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- It is possible to analytic continue on $\mathbb{C}$ with possible appearance of branch points.

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[2] P.G. Becker, and W. Bergweiler, Hypertranscendency of conjugacies in complex dynamics, *Mathematische Annalen 301(3), 1995*.

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Theorem 1

- *The Riemann surface of h has infinitely many branch points.*
- *We can determine the surface explicitly.*
  Hence, the special multi-valued function h is nice.

Existence of Branch points

- Assume that h is meromorphic, then applying the Maximum modulus function on $h(\lambda w) = R(h(w))$:

$$\begin{aligned} M(r,h) &\geq M(r|\lambda|,h) \\ &= M(r,h(\lambda w)) = M(r,R(h(w))) = M^d(r,h) \end{aligned}$$

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- There exists $\hat{r}$ such that

  $$h(\lambda w) - h(\lambda\hat{r}) = R(h(w)) - h(\lambda\hat{r}) = (h(w) - \alpha_1)^{p_1} \cdots, p_1 > 1$$

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- This actually implies a particular case of Fatou's theorem

  An immediate basin of attraction of R has a critical point.

Riemann surface of $h(\lambda w) = \lambda h(w) + h^2(w)$

- Recall
$$h(w) = c_1 w + \sum_{k \geq 2} c_k w^k.$$
Assume that $0 < \lambda < 1$ and by changing $w \to aw$, we set $c_1 = -1$. Using recurrence relation, we find that $c_n < 0$.

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- Again $\hat{r}$ is a branch point. Hence,

$$\hat{r} > 0, \quad h(\lambda \hat{r}) = -\frac{\lambda^2}{4}, \quad h(\hat{r}) = -\frac{\lambda}{2}$$

$$h(\lambda w) + \frac{\lambda^2}{4} = \left(h(w) + \frac{\lambda}{2}\right)^2.$$

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- Let h_0 be the Riemann sheet corresponding to $c_1 w + \sum_{k \geq 2} c_k w^k$, and h_1 be the other sheet generating from the branch point at $\hat{r}$.

Riemann surface (continued)

- So $\hat{r}$ is a branch point of h_0 and h_1. From

$$h(\lambda w) + \frac{\lambda^2}{4} = \left(h(w) + \frac{\lambda}{2} \right)^2,$$

$$\Longrightarrow h_0(w), h_1(w) = -\frac{\lambda}{2} \pm \sqrt{h_0(\lambda w) + \frac{\lambda^2}{4}}.$$

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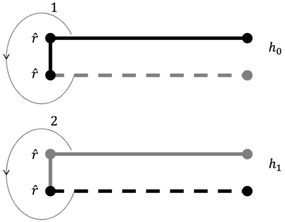
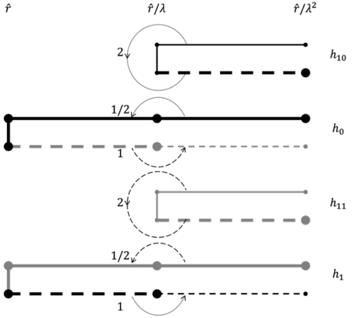
- Hence, $\{\hat{r}\lambda^{-n},\ n \geq 0\}$ is the set of branch points of h_i.
- At $w = \hat{r}$:



Riemann surface (continued)

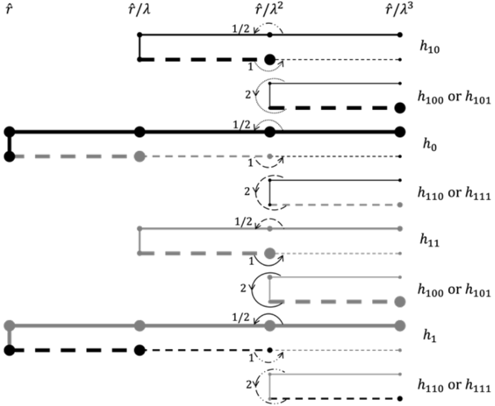
- At $w = \hat{r}/\lambda$, there are four sheets h_0, h_1 and other two sheets obtained from solving $h_1(\lambda w) = \lambda h_{1i}(w) + h_{1i}^2(w)$.



- However, on sheets h_{1i}, the point $\hat{r}$ is not branched.
- The set of branch points of h_{1i} is $\{\hat{r}\lambda^{-n},\ n \geq 1\}$.

Riemann surface (continued)

- At $w = \hat{r}/\lambda^2$:



- At $w = \hat{r}/\lambda^k$, the new 2^{k-1} sheets has one less branch point than the previous ones. The set of branch points of those 2^{k-1} sheets is $\{\hat{r}\lambda^{-n},\ n \geq k\}$.

Higher degree (In progress)

- For $h(\lambda w) = h(w)/2 + 2h^2(w) - h^3(w)$, only the critical point $(50 - 11\sqrt{22})/54 \in A_0^0$ and the branch point pattern is the same as previous example.

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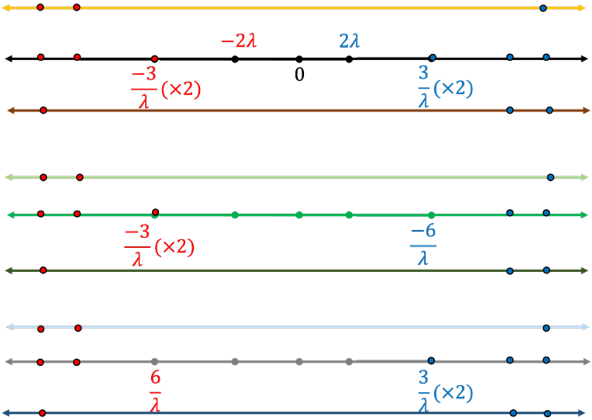
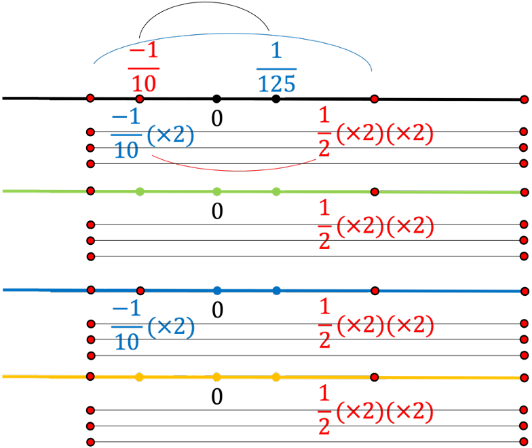
- Set $h_0 = w + \sum_{k \geq 2} c_k w^k$ and h_0 is the black line.
- Example $h(\lambda^2 w) = \lambda^2 h(w) - \lambda^4 h^3(w)/27,\ 0 < \lambda < 1$.



Similar pattern

- Critical point maps to another critical point:

$$h\left(\frac{-3}{20}w\right) = -\frac{3}{20}h(w) - \frac{12}{20}h^2(w) + h^3(w)$$



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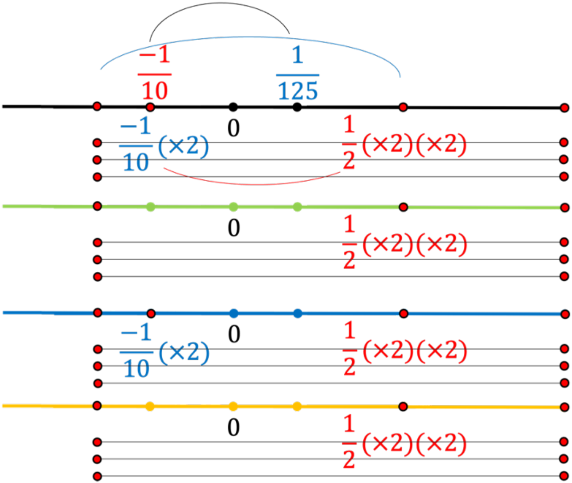
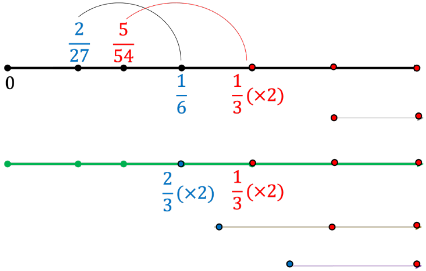


The branch point $-1/10$ is quadratic. The branch point $1/2$ is of fourth order and has similar pattern as above.

More patterns

- Critical points occur on the 1st iterative sheet:

$$h\left(\frac{5}{27}w\right) = \frac{5}{27}h(w) - \frac{1}{2}h^2(w) + 27h^3(w)$$



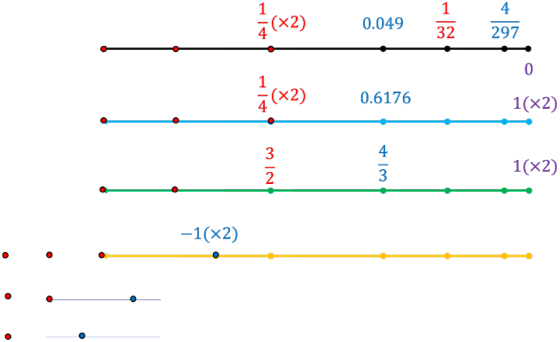
Here, the branch point corresponding to $2/3$ appears on the later sheet h_1. The pattern now is hopping on "specific" sheets instead of producing a discrete set on a line on every sheet; $h_0(\lambda w) = R(h_i(w))$ and $h_1(\lambda w) = R(h_{1i}(w))$.

More patterns (continued)

- Critical points occur on the 2nd iterative sheet:

$$h\left(\frac{1}{3}w\right) = \frac{h(w)(h(w)-1)^2}{3(1+2h(w))}.$$

There are three critical points ± 1 and $1/4$.



The branch point corresponding to -1 is hopping.

Further questions

- Is it possible to construct examples where the critical points occur on the n-th iterative sheet?
- Simply-connectedness immediate basin of attraction yields a line and first sheet hopping pattern? Infinite connectivity case corresponds to the critical points that occur on the n-th iterative sheet?
- If the n-th iterative argument pattern does appear, given a degree d of the set of polynomial or rational functions, can we determine this $n(d)$?

- For $h(\lambda w) = R(h(w))$, is it possible to find a solution having a pole or essential singularity at $w = 0$? Arguments of singularities can still be used here.
- Mahler's equation: $h(w^2) = h(w)^2 + c$. At $w = 1$, Julia's construction of a solution with essential singularity at 0 and infinity. Is that all of it?
- Back to $y(z) = h(\lambda^z)$ shows that $y(z) = y(z + 2\pi i/\ln\lambda)$:

  "Determine all fundamental solutions satisfying $y(z) = y(z + 2\pi i/\ln\lambda)$ and $y(z+1) = R(y(z))$".

Summary

- The inverse h^{-1} in dynamics plays the role of local conjugacy of immediate basin of attraction.
- We can describe the singularities of the function h in detail, which serves its role as a special solution of the λ-difference equation and difference equation (for y).

Thank you for your attention!