Beyond Conceptual Change: Using Representations to Integrate Domain-Specific Structural Models in Learning Mathematics

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ABSTRACT—Effective teaching should focus on representational change, which is fundamental to learning and education, rather than conceptual change, which involves transformation of theories in science rather than the gradual building of knowledge that occurs in students. This article addresses the question about how to develop more efficient strategies for promoting representational change across cognitive development. I provide an example of an integrated structural model that highlights the underlying cognitive structures that connect numbers, mathematical operations, and functions. The model emphasizes dynamic multiple representations that students can internalize within the number line and which lead to developing a dynamic mental structure. In teaching practice, the model focuses on a counting task format, which integrates a variety of activities, specifically addressing motor, visual, and verbal skills, as well as various types of learning transfer.

INTRODUCTION

Recent research in cognitive science and neuroscience revealed what could be called the paradox of cognitive development: Although children are endowed with capabilities that assure the passage to higher stages or levels of cognition, being able to resolve incommensurabilities between old and new knowledge by their own wits, they also develop misconceptions that prove to be very resistant to change (e.g., Feldman, 2004; Gardner, 2004). At least three issues come together to shape this paradox: infants' innate endowments for processing domain-specific information, children's cognitive achievements with development, and students' failure to overcome misconceptions. A short overview of research literature focused on these three aspects emphasizes the necessary connections for the argument of this article—that children learn through representational change rather than conceptual change. They learn by building representations, not by moving through changes in theories analogous to scientific revolutions. A dynamic approach in teaching that emphasizes representational change can contribute to optimize learning under the constraints of time and information overload, moving beyond the difficulties of traditional approaches that inevitably stress conceptual change. In order to bring representational change to schools as an intrinsic phenomenon of learning, it is necessary to develop structural models that build relevant connections within the domain of study and to make them part of the teaching–learning design. An adequate training based on these models may activate dynamic mental structures in students. This article describes the functioning of such a model, which highlights the underlying cognitive structures that connect numbers, mathematical operations, and functions. In teaching practice, the model focuses on a counting task format in which a variety of activities are integrated, which specifically address motor, visual, and verbal skills, as well as various types of learning transfer. The counting task format patterns cycles of transfers with subsequent roles: on the one hand, to

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internalize a variety of representations and, on the other hand, to build ways to move from one representation to another one. The target is to help children develop dynamic mental structures that they can self-develop and generalize across new tasks in an adequate context. Thus—as experimental work shows—representational change becomes a tool that students use trustfully in problem solving.

INFANTS' ENDOWMENTS FOR PROCESSING DOMAIN-SPECIFIC INFORMATION

During the past three decades, a large body of research has been devoted to analyzing infants' cognitive capacities. Important findings concern the roots of inductive reasoning and categorical exclusion, even if the mechanisms underlying categorization are not entirely known. Thus, Quinn, Eimas, and Rosenkartz (1993); Quinn and Eimas (1996); and Quinn, Eimas, and Tarr (2001) provided evidence that infants can form a category representation for cats that includes novel cats but excludes exemplars from other related basic categories, such as dogs. Behl-Chadha (1996) has extended these findings to human-made artifacts by showing that 3- to 4-month-olds can also form representations for chairs and couches, each of which excludes instances of the other as well as beds and tables. These examples bring evidence that the mechanism of categorical learning—essential in building concepts—is active in preverbal infants.

Another category of experimental findings relevant for the focus of this article investigates the number sense in infants. Thus, Wynn (1990, 1992) and Starkey (1992) showed that 5-month-old infants are able to compare two sets of up to three objects and able to react when the result of putting together or taking away one object is falsified. These experiments were followed by many replications and extensions. Using the infant's gaze patterns, it was possible to show that babies as young as 5 months are able to identify differences in numbers of objects up to three (Canfield & Smith, 1996). Infants showed longer looking at arrays presenting the wrong number of objects, even when the shapes, colors, and spatial location of the objects in both displays were new (Koechlin, Dehaene, & Mehler, 1997; Simon, Hespos, & Rochat, 1995; Wynn, 1995). Experiments with older infants using different response systems, such as manual search and locomotor choice (Feigenson, Carey, & Spelke, 2002; Van de Walle, Carey, & Prevor, 2000), have led to the same conclusion. In the study by Feigenson, Carey, and Hauser (2002), infants who have just begin to locomote independently were shown two cookies placed in succession into one opaque box and one cookie placed into a second box; then, they were allowed to crawl toward one or the other box. When comparing 1 versus 2 and 2 versus 3, infants crawled preferentially to the box with the greater number of cookies—which shows that preverbal children are able to compare small sets using as criteria their cardinal numbers. Yet, other replications confirm the assumption that infants can represent numerosity of sets and their equivalence in different modalities, such as visual or auditory (e.g., Mix, Levine, & Huttenlocher, 1997).

A series of experiments suggested that number representation in humans has at least three components (Dehaene, 1997; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Spelke, 2003): one for recognizing numerosity limited to up to four items at a glance, without counting—subitizing (e.g., Benoit, Lehalle, & Jouen, 2004; Gallistel & Gelman, 1991; Mandler & Shebo, 1982), a second for approximate numerosities (Dehaene, 1997; Gallistel & Gelman, 1992), and a third for large exact numerosities, in which the natural language interferes (e.g., Gelman, 1990). This area of neuroscience is important from an educational perspective because it shows that, far from being "tabula rasa" at birth, children have predispositions that allow them later to construct domain-specific knowledge.

A PARADOX: CHILDREN IN BETWEEN COGNITIVE ACHIEVEMENTS AND MISCONCEPTIONS

Systematic observations of children's behavior have shown that, in the period from infancy to adolescence, all normal children pass through several qualitatively different levels of thought. These changes happen across cultural–educational or social–economical contexts (e.g., Case, 1992; Commons, Trudeau, Stein, Richards, & Krause, 1998; Dawson-Tunik, Commons, Wilson, & Fischer, 2005; Feldman & Fowler, 1997). Gardner (2004) catches the dynamics of this process accurately, and I use two of his examples: At 8 or 9 months of age, a child observes a toy hidden a few times in a certain place. Then, in front of the infant's eyes, the toy gets moved to another place. Despite the evidence of his senses, the infant will continue to search for the toy at the initial place. The location of an object seems inextricably tied to its original hiding locus. But a few months later, without any formal training, every normal infant goes directly toward the second place. At another phase, a 5-year-old child is shown two identical beakers, each containing an equal amount of water. For these two beakers, the child affirms that they contain the same amount. But after pouring the content of one of the beakers into a thinner and taller beaker, the child will say that the second vessel contains more water "because the water is higher." A year or two later, the child appreciates correctly the conservation of matter (Gardner, 2004, pp. 50–51). Therefore, some capacities such as using the idea of conservation of matter or predicting the behavior of a balance beam do not need to be taught explicitly. Many observations of this type have led cognitive psychologists to the idea that children are crossing developmental stages. Each stage is characterized by a recurrent evolution of the child's cognitive system, in which the first part is an active construction phase that has the culminating point in what was called "taking of consciousness" of the system as a whole and the second is an active extension and elaboration phase (Bringuier, 1980; Feldman, 2004; Piaget, 1954).

On the other hand, in their effort to make sense of the world, children develop intuitive theories, some of which have proved very resistant to change, such as intuitive theories of matter ("Heavier objects fall to the ground more rapidly than lighter ones"), intuitive theories of life ("Things that are moving are alive; if something is not moving, it is dead."), intuitive theories of human relations ("Individuals who are big are powerful."), intuitive theories of mind ("All organisms have minds."). Often students trained in the best universities in the world prove to possess deep misconceptions in their domains of study, when examined outside a school context. Thus, for example, physics students cannot predict the trajectory of a pellet after it has fallen out of a curved tube or biology students continue to give Lamarkian explanations, although they studied evolution extensively (e.g., Bransford, Brown, & Cocking, 2000; Gardner, 1993, 1999, 2004).

The above-mentioned paradox entrenches two essential problems: (a) How can the emergence of the new qualitatively superior levels of understanding be explained? and (b) How can misconceptions be assessed and prevented? Both questions are relevant from an educational point of view.

The issue about how new knowledge emerges was a constant later preoccupation of Piaget; in 1971, he wrote: "For me, the real problem is novelties, how they are possible and how they are formed" (Piaget, 1971). Most of the research in this area has brought solutions in terms of information processing theories and connectionist models. Thus, a proposed strategy for simulating development is that processing capacity increases with age (Case, 1992). Other researchers have suggested the equivalent of maturational change by modeling incremental learning—gradually enlarging the inputs (e.g., Plunkett & Marchman, 1991) or by changing the learning algorithm—from contrastive Hebbian learning to backpropagation (e.g., Bechtel & Abrahamsen, 1991).

In a controversial article on neural constructivism, Quartz and Sejnowski (1997) suggested that the evolutionary emergence of neocortex in mammals is a progression toward more flexible representational structures, in contrast to the popular view of cortical evolution as an increase in innate specialized circuits. Processes that may create this flexibility in representations include differentiation and integration described by Werner (1957), Piaget (1954, 1971), and Fischer (1980). In a recent review, Johnson and Munakata (2005) brought to attention two apparently opposite proposals: increasing dissociation (differentiation) of streams of information during development and simultaneous increasing integration of those streams. The variation-centered dynamic approach developed by Fischer and his team (e.g., Fischer, 1980; Fischer & Bidell, 1998; Fischer & Pipp, 1984; Fischer & Rose, 2001; Yan & Fischer, 2002) specifies developmental mechanisms for these processes within and across domains and reshapes traditional views of learning and development, showing that human cognition is a constantly varying system.

Much of the research on domain-specific knowledge has focused on the issue of misconceptions, especially asking what characteristics would lead developmental transitions to be more or less susceptible to various misconceptions. Reformulated in this way, the above-mentioned questions may lead to answers that are more connected to education. Taking on this challenge, Carey (1985, 1999, 2001) has analyzed domain-specific conceptual changes by means of structural analogies to Kuhnian paradigmatic change (Kuhn, 1962). Kuhn claimed that the consensus in a mature science is based on a paradigm—a complex theoretical, instrumental, and methodological entity that offers models for problem solving to a scientific community. The knowledge contained in a paradigm is almost a tacit one. Paradigm change is a transition between frameworks that are incommensurable in important ways. Incommensurability between two theories essentially means that the concepts and procedures of a theory cannot be expressed by means of concepts and procedures of the other theory. For this reason, the paradigm change cannot be done step by step and constrained by a neutral experience—it needs a revolution. Why is a scientific community so resistant to changing the paradigm? Margolis (1993) explained this resistance through the habits of mind that govern scientific belief, considering that the habits of mind, like those of behavior, are neural programming that form mental barriers. Changing the paradigm implies changing habits of mind that are difficult to influence, as they are not accessible to introspection. Habits of mind are, therefore, the deep source of the attachment to a research tradition and a barrier in understanding a different research tradition.

In Carey's view, cognitive development of individuals involves theory changes, and the processes by which new domain-specific concepts and explanatory mechanisms come into being imply incommensurability. This transformation is called conceptual change and the mechanism that makes it possible is bootstrapping. Thus, "bootstrapping makes use of various modeling techniques—creating analogies between different domains, limiting case analyses, thought experiments, and so on. It also makes use of the human symbolic capacity to represent the relations among interrelated concepts directly while only partially interpreting each concept in terms of antecedently understood concepts." (Smith, Solomon, & Carey, 2005, p. 105). Development resolves some of the incommensurabilities between previous and new knowledge in individuals, but other incommensurabilities need to be solved by school instruction.

TOWARD AN EDUCATIONAL SOLUTION: REPRESENTATIONAL CHANGE

The above-discussed paradox of cognitive development can be approached fruitfully from an educational pragmatic view: How can teaching and learning be organized to make efficient use of children's natural endowment and to avoid misconceptions? As the mass schooling of contemporary societies tries to concentrate centuries of human cultural development within the school years, there is a need to solve this paradox. The failure to achieve basic literacy in domains of knowledge such as language, mathematics, or science suggests that new approaches are required to help the educational system teach more effectively.

In this context, I distinguish between *teaching for conceptual change* and *teaching for representational change*. Focusing on representational change means, essentially, two things: centering the educational approach on representation as a powerful tool in various disciplines and bringing powerful representational models to school very early, in a mostly informal way, to stimulate abstraction during cognitive development (e.g., Singer, 2007). Conceptual change directly translated into education seems to lead to stressing misconceptions as ways to provoke cognitive conflicts. The broad process of conceptual change as a way to enhance learning emphasizes denying a paradigm in order to replace it with a different one. Stressing the idea of incommensurability is helpful in thought experiments, in order to better understand the differences between the two paradigms. However, in educational practice, it can induce a blockage provoked by the stability of mental structures needed for building a paradigm.

In contrast to the focus on misconceptions to induce knowledge development in conceptual change, representational change focuses on organizing the inputs in a way that misconceptions are diminished. Teaching for representational change facilitates transfer by avoiding rigid connections (Singer, 2002). From an educational perspective, representational change appears as the process of natural development: Children employ various layers of understanding and bridging techniques, especially the use of multiple representations to increase problem-solving abilities, and they can be stimulated to do so by educational tools and practices. In contrast, conceptual change constrains learning (and development) to follow stages based on incommensurable concepts—step by step from the bottom up the ladder, each step being the result of overcoming an incommensurability. Subsequently, teaching for conceptual change tends toward a content-centered approach, while teaching for representational change allows an operational focus that moves beyond tasks, domains, and cross-domain connections to focus on dynamic knowledge structures (e.g., Case et al., 1996; Singer, 2001a). While conceptual change is meant at best to build students' awareness of misconceptions, representational change is meant to develop students' fluent thinking by exposing them to a variety of representations and transfers of knowledge.

Exposing children to conceptual change supposes developing a curriculum in which concepts are gradually integrated and conceptual progress is made by resolving cognitive conflicts between old and new knowledge. Teaching for representational change, instead, lacks this linearity and perhaps also lacks a certain formal coherence given by an explicit succession of information. Because of this assumed risk, a curriculum based on representational change has to meet certain criteria. First, is such a curriculum legitimate? To answer, we have to consider its advantages. Some of them have been presented above, and a more detailed view is given in the next sections, while describing a case study. Second, is it feasible? In other words, do children have the necessary capacities to process such a curriculum? I bring as arguments: children's innate propensities, their endowment for recursive processes, and their bridging ability. These are further exemplified through analysis of a concrete example in mathematics learning. Third, is a curriculum focused on representational change usable in practice in schools? In other words, can teachers use it appropriately to train students? I describe an experimental study to assess its efficacy.

CHILDREN AS ACTIVE CONSTRUCTORS OF THEIR OWN KNOWLEDGE: REPRESENTATIONAL STRATEGIES AND BRIDGING

Children often generate useful new strategies without mastering conceptual understanding of a domain (Carey & Spelke, 1994; Karmiloff-Smith, 1992; Smith et al., 2005). More specifically, they generate and master *representational strategies* without moving through conceptual revolutions. After internalizing the mechanisms of counting, for example, children process natural numbers in various ways without special learning or training because of their core knowledge, which includes the use of concepts and rules recursively (Singer & Voica, 2003). Many classroom observations demonstrate this conclusion. Thus, 6-year-old children in a school context have been shown to be able and eager to continue indefinitely a sequence of increasing numbers, even if they never formally learned such numbers. The following simple exercise was done with first-graders at the beginning of the school year: They were invited to count by twos in a relay. Each of the children from randomly selected classes was able to continue the sequence correctly, in spite of the concentration and effort required to follow the fellow students who uttered the previous numbers. Even more, when the interviewer tried to stop the process in one of the classes at the number 130, spontaneous comments appeared:

Child in the classroom: *We could continue ….*
Interviewer: How much could you continue?
Child in the classroom: *Till one million*!
Another child in the classroom: *Till one billion*!
Another child in the classroom: *Till the infinite*!
Interviewer: But we should stop somewhere, at a big number!
Child in the classroom: *Which one*?
Interviewer: For example, a billion!
Child in the classroom: *I can find a bigger one: a billion of billions*!

This fragment is an excerpt from a videotape with students in Grade 1, in the first semester, in a state school in Bucharest. The teacher was very surprised by her children's reactions: The curriculum goal for the end of the first grade states that children should know and use only the numbers smaller than 100 and do addition and subtraction without exceeding the tens. What should be noticed is students' pleasure at playing with bigger and bigger numbers. I found this pleasure as well as accuracy in counting with a given step (in twos, in threes) in other replication experiments in various first-grade classrooms.

I found repeatedly that, when asked to say a bigger number than a given one (e.g., 23 quadrillion), many first-graders do not actually use Peano's axiom (adding 1 to get the next bigger natural number), rather they use a construction that seems to be of a linguistic nature ("quadrillions of quadrillions" or as in the example given above "a billion of billions"). The connection point seems to be *recursion*. The role of recursion for language is emphasized in the Chomsky (1980) tradition as being the abstract linguistic computational system (narrow syntax) that generates internal representations and maps them into the sensory–motor interface through the formal semantic system. While the internal architecture of language is open to many debates, there is an agreement that a core property of the faculty of language in a narrow sense is recursion, attributed to narrow syntax; this takes a finite set of elements (words, sentences) and yields an array of discrete expressions (Hauser, Chomsky, & Fitch, 2002), which can be considered potentially infinite.

The embodied metaphors theory in linguistics (Lakoff, 1987; Lakoff & Johnson, 1987) extends the properties of syntax to human conceptual systems and thus helps illuminate children's behaviors in the examples with large numbers. For Lakoff and his colleagues, language is embodied, which means that its structure reflects our bodily experience, which in turn creates concepts that are then abstracted into syntactic categories. They conclude that grammar is shared (to some degree) by all humans for the simple reason that we all share roughly the same bodily experience. For example, to express the conceptual metaphor "time flies," we actually understand time in terms of space. Moreover, the core of our conceptual systems—including mathematics—is directly grounded in perception, body movement, and experience, which integrate both physical and social contexts (Lakoff & Nuñez, 2000).

There is converging evidence from different clinical interviews that many Grade 1 and Grade 2 students can articulate the principle that there is no biggest integer (e.g., Hartnett & Gelman, 1998) by developing a dynamic rhythmic activation, as in the classroom example of recursive use of large numbers. A dynamic rhythmic activation is identified even in infants (e.g., Sansavini, Bertoncini, & Giovanelli, 1997; Hannon & Johnson, 2005). Children come spontaneously to create new entities, previously unknown, starting from entities already known. They show a kind of readiness to start, an implicit *motivation to learn.* These observations concord with the model for "neural constructivism" in which the representational features of cortex are built from the dynamic interaction between neural growth mechanisms and environmentally and bodily derived neural activity that generates a progressive increase in the representational properties of cortex (Quartz & Sejnowski, 1997, and, from another perspective, Greenough, Black, & Wallace, 1987). The interaction between the environment and the neural growth results in a flexible type of learning: "constructive learning," which is similar to what I am calling representational change. Beyond the innateness debate, what can be emphasized from the study by Quartz and Sejnowski (1997) is the fact that the representational properties of cortex are built by the nature of the problem domain confronting it. From an educational view, it is less important whether the representational configuration is innately domain specific or a domain-specificity predisposition is activated through the problem domain that confronts the developing brain; what is significant is the dual mechanism of increasing and specializing representational properties of cortex shaped through active experience.

This mechanism is probably at the basis of *cognitive bridging*, in which a person uses a conceptual shell to guide his or her own activity in order to solve a problem or learn something. The phenomenon of bridging is visible in many occasions, even with very weak inputs. Bridging is a transition mechanism that people use spontaneously at a wide range of ages. Bridging occurs through self-scaffolding as well as other scaffolding within individuals and between people in social interaction. It takes several forms with a similar underlying mechanism, all of which are characterized by setting tentative targets for an unknown skill to be constructed at a developmental level higher than the level of the person's current activity (Granott, Fischer, & Parziale, 2002; Granott & Parziale, 2002).

One example from our interviews with primary children uncovers bridging techniques that allow the child to solve problems far beyond his or her level of understanding. The interviewer asked Andrei (second-grade, 7.5-year-old) to color one half and then one tenth of a rectangle drawn on a grid paper. This question was asked in spite of the fact that the curriculum for second grade does not require division or fractions, not even multiplication.

Andrei successfully divided the rectangle in half. For the second task, he tried first successive halving, getting one fourth, one eighth. This is not surprising. For example, Mix, Levine, and Huttenlocher (1999) have shown that even preschool children can manipulate models of physical quantities based on parts and ratios (e.g., they know that half of a circle added to one fourth of a circle yields three fourths of a circle). Seeing that he was not able to succeed getting a tenth, Andrei started (by himself) to count the squares of the grid paper inside the rectangle.

> A: *I finished counting ... they are 78.*
> I: So, can you draw a tenth?
> A: *Yes, but I have to divide ...* (At that moment, children in grade 2 have not yet formally studied division.)
> I: To divide ... but do you think it's possible to draw a tenth?
> A: *Hmm ....I don't think it's possible.*

At this moment, his limits of understanding seem to be clearly set up. Next, the interviewer drew a similar (congruent) rectangle on a white paper, without a grid, and asked the same thing—to color one tenth. Andrei estimated a division in equal parts and separated "by eye" one tenth of the rectangle. It seems that he has found a "marker shell." The metaphor of marker shells was used to indicate targets for development and learning in bridging as a process that helps generate new abilities on the basis of the existing ones. The marker shells serve as placeholders that people use to direct their own learning and development toward achieving new, more powerful mental structures (Granott, 1994; Granott et al., 2002). For Granott, the shells serve as scaffolds that guide the construction of new knowledge by providing a perspective for processing new experiences.

The interviewer tried to push things further. When questioned if he could draw a thousandth of that rectangle, Andrei answered:

> Andrei: *Well, let's say it is this one* (he showed a small part of the drawing, of a rectangular form)
> Interviewer: ... that one is a thousandth, yes? But a millionth, do you think it's possible?
> A: (He drew a point inside the rectangle.)
> I: You made a small point ...
> A: *Which we almost cannot see.* (He is laughing.)
> I: (...) Do you think you could draw like that any part (fraction) of the rectangle?
> A: *N...o...*
> I: Where do you think you must stop? ...
> A: *Err... at....*
> I: At ...?
> A: *At a billionth part ...*
> I: At a billionth part .... Why exactly there?
> A: *That is roughly .... I think I could color ....*
> I: This is what you think you can, but do you think that anybody could color a smaller part of the rectangle ... could anybody color a trillionth part?
> A: *I don't know, only if* **bacteria had pencils** ....
> I: If bacteria ... and they could color ... and is there anything smaller than bacteria?
> A: *N...o.... I don't know for sure ... maybe the cells ....*

Changing the paper sheet on which the rectangle was drawn, from a grid paper to a white paper, determined a radical change of perspective: The child moved from a discrete numerical perception to a topological one (Singer & Voica, in press). Topological operations deal with margins, continuity, neighborhoods, approximations, and limits; they account for the convergence of thinking and global perception. Using this topological endowment, Andrei was able to solve the task without any help and far beyond the curriculum requirements.

Other children's patterns for solving this type of task have also shown that a small change in representational strategy leads to a higher level of performing the task. What is relevant here is that scaffolding involved a representational change that allowed the child to better cope with the task. The passage from a discrete numerical approach (counting the squares) to a topological approach (assessing globally the image and its limits) led him to use different layers of understanding (Singer, 2001b). Evidence for this perspective taking in childhood comes from various empirical studies, including, for example, the well-known study on mathematics performance of street children in Brazil (e.g., Nuñes, Schliemann, & Carraher, 1993).

INTERNALIZING MATHEMATICAL OPERATIONS: A STRUCTURAL MODEL

Children show these capabilities to anticipate skills and concepts that will be fully constructed at later stages of development. Building on these early capabilities, instruction can use structural models to help students develop representational changes that lead to effective mathematics learning. I describe next an integrated structural model that emphasizes the underlying structures that connect numbers, mathematical operations, and functions. This model is meant to develop a *dynamic mental structure* in children, based on the number line. Like the model of Griffin and Case (1997) for early learning of number, it can be used in primary grades to construct the concept of number in children. The concept of dynamic mental structure is related to the "central knowledge structures" described by Griffin and Case, who call it a central conceptual structure (Case et al., 1996; Griffin & Case, 1997; Singer, 1995). I prefer the name dynamic mental structure because it distinguishes this kind of structure from the conceptual theories described in the conceptual change framework.

Recent neuropsychological and neuroimaging studies strongly suggest the existence of a language-independent spatial representation of numbers in the human brain (Dehaene, 1997; Dehaene et al., 1999). This spatial representation may rely on visuospatial cerebral networks (Dehaene, 2001; Dehaene, Piazza, Pinel, & Cohen, 2003). As the parietal cortex plays a crucial role in these networks, it seems that the parietal cortex is a candidate area that may contribute to the representation of the number line (Dehaene, 2007; Gobel, Walsh, & Rushworth, 2001). Many studies using functional imaging techniques like positron emission tomography or functional magnetic resonance imaging show that different components of processing numbers activate different brain areas within the parietal cortex. Moreover, "mental arithmetic relies on a highly composite set of processes, many of which are probably not specific to the number domain" (Piazza & Dehaene, 2004). With learning and development, the number line moves from initial innate representations in the cortex to internalization of the model proposed in this article.

I first describe the model from the mathematics perspective and then present it in action, in experimental teaching. As the domain-specific and pedagogical components of the model as well as their interactions are not obvious within the mathematical model, both descriptions are necessary. To make the model more explicit, I use as examples the sequence of even numbers: 0, 2, 4, 6 …, and explain it by decomposing its significant parts. The focus is on the underlying connections.

An increasing sequence of even numbers can be generated starting from 0 (or another even number) by adding the same number, +2 (Figure 1).

In this way, when counting by twos in increasing order, the child implicitly does addition; but addition with the same number means multiplication (Figure 2). Thus, a passage from addition to multiplication and vice versa is created through this sequence.

Symmetrically, the decreasing sequence of even numbers can be generated starting from an even number by subtracting the same number, −2 (Figure 3).

Further, subtracting the same number until 0 can be written as division. Thus, a passage from subtraction to division and vice versa is created through the sequence of even numbers (Figure 4).

In conclusion, an increasing sequence of natural numbers (e.g., 0, 2, 4, 6, …) contains a synthesis of addition (e.g., $2 = 0 + 2$, $4 = 2 + 2$, $6 = 4 + 2$) and multiplication (e.g. $2 = 2 \times 1$, $4 = 2 + 2 = 2 \times 2$, $6 = 2 + 2 + 2 = 3 \times 2$), while a decreasing one (e.g., 12, 10, 8, 6, 4, 2, 0) contains a synthesis of subtraction (e.g., $8 = 10 - 2$, $6 = 8 - 2$) and division (e.g., 10 is 5 times 2, then $10:5 = 2$).

This is why a sequence of natural numbers (in this example, the sequence of even numbers) appears as an *integrating concept* of all four arithmetical operations (Singer, 2003). Moreover, each number in the sequence can be generated by any of the four arithmetical operations, and each operation, in turn, can be derived from another one using the appropriate numbers in the sequence. All these interconnections are represented in Figure 5 into a scheme (in the Piagetian sense of the word).

This scheme highlights a multitude of representational transfers from number sequences to operations and vice versa, from addition to subtraction and vice versa, and from multiplication to division and vice versa. What makes this scheme reliable is the fact that it can be objectified in children's minds. Children do not learn the scheme by memorizing the diagram but by doing activities that embody it. Following the algorithm of the counting tasks that is described in the next section, it can be internalized in its operational complexity.

This scheme is a prototype; the same construction is implied in other sequences: that of odd numbers, or in the sequence generated by counting "in threes," "in fours," "in tens," etc., starting from 0, or starting from any other "point"—natural number on the number line. Practically, this scheme allows the extrapolation to any kind of sequence of natural numbers.

The capacity of this scheme to engage transfer is enormous: It can be applied, as we have already seen, to any sequence of natural numbers; it underlies the connections between the arithmetical operations, simplifying access to future understanding of the properties of operations; it also facilitates understanding "priority rules" (e.g., multiplication first, then addition) in computing chains of operations. Even more, it can be transferred analogously to decimal numbers. Figure 6 shows the previous scheme, with the same procedural functionality, but in which the integers have been replaced with decimal numbers.

The second scheme is an example of a structural analogy that facilitates students' learning decimals through representational change, without any need to go through conceptual theoretical change. Learning the decimal numbers in a strategically analogous way allows stressing similarities as well as differences. Smith et al. (2005) recorded deep conceptual difficulties in understanding *rational numbers* by students in Grades 4–6. I hypothesize that this dynamic structural model allows learning operations with decimal numbers as natural extensions of the algebraic operations with integers, avoiding the otherwise necessary conceptual change. Accustomed with the flexibility of various representational changes induced by this structural model, children arrive at understanding rational numbers in a natural way, which strengthen the already existing dynamic network.

This learning has at least two advantages: On the one hand, applying extant knowledge in a broader context has a feedback effect of reinforcing that extant knowledge and, on the other hand, naturally connecting new to existing knowledge has a feed-forward effect of focusing on the essential aspects of the new knowledge. Other consequences result from these two: better understanding of

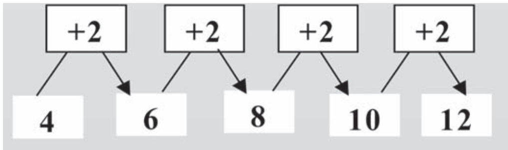

Fig. 1. Generating increasing sequences through addition.

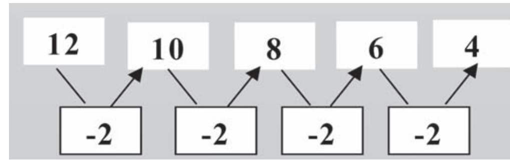

Fig. 3. Generating decreasing sequences through subtraction.

both integers and decimal numbers, stimulating the ability to transfer from one context to another, shortening the time for comprehensible learning, facilitating understanding fractions and the equivalence of number writings, facilitating operations with decimal numbers by transferring already known algorithms, and finally building on a fundamental concept related to both integers and decimals—countable infinity.

Such a pattern is complex, and its formation triggers special teaching problems because it is necessary to train the mobility of concepts explicitly and systematically and to shape their structure toward new ones, which are to be learned later. This is actually the challenge of teaching for representational change. The dynamics of the mental structures generated by internalizing this scheme (Singer, 2001b) requires special attributes of mobility: relating a point within the structure to any of the others; focusing the whole structure on a given task in order to set new information in as many points of the structure as possible; linking the structure to another one nearby or at a distance; giving freedom to each point of the structure so that it might multiply or migrate into another structure; reorganizing the structure according to a certain working hypothesis while performing a creative task; transferring the structure from one level of abstraction to another; and reconstructing the structure starting from any of its points.

The present section has shown this type of dynamic mental structure from the mathematics perspective. The following section is focused on describing how such a dynamic structure can be internalized by children.

EXPERIMENTAL TEACHING: A SNAPSHOT INTO THE METHODOLOGY AND RESEARCH

The dynamic mental structures underlying a topic or a discipline as well as the models and theories for explaining cognitive transitions should be known by teachers in order to tune their teaching according to their students' age and ability to learn domain-specific concepts and procedures. While brain science cannot give direct input into education yet (at least because existing scanning techniques do not allow visualization of mental processes at the speed that they take place; e.g., Bruer, 1997, 1999; Gardner, 1999), this input can be mediated by cognitive science, which displays an array of instruments that are potentially applicable in the teaching practice. The model proposed in this article relies on empirical evidence based on cognitive science and on indirect assumptions that relate mind, brain, and education, while its effectiveness in school acts as the main argument for its sustainability in practice.

Dynamic modeling of developmental processes is particularly relevant to linking cognitive science with learning in schools (e.g., Case et al., 1996; Fischer & Bidell, 1998; Fischer & Pipp, 1984; Fischer & Rose, 2001; van Geert, 1991). Applications of dynamic systems theory to the study of development suggest processes of self-organization within a developing system that serve as a mechanism of developmental change. Within this framework, cognitive development in both childhood and adulthood is seen as a dynamic system in which a person's activities in context vary and grow from the mutual influence of multiple specified factors interacting over time. According to Yan and Fischer (2002), four key aspects of a dynamic system—multiple factors, complex interactions, multilevel contexts, and multilevel time scales—work together to generate changes that are complex, emergent, and self-organized.

Is the dynamic mental structural model for representational change in early mathematics sustainable in practice? To obtain evidence, a 4-year longitudinal study of teaching and learning mathematics in primary school was developed: In Year 1, two classes of children from a school in Bucharest (Romania) were taught according to the experimental methodology; two further classes were added in Year 2; in Year 3,

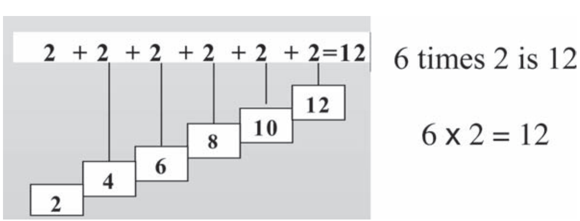

Fig. 2. Generating multiplication through addition.

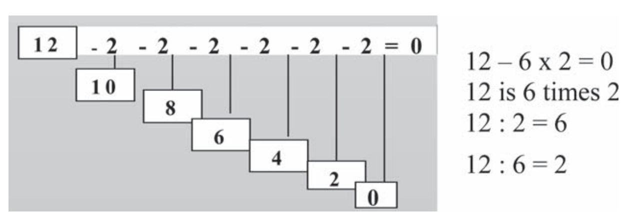

Fig. 4. Generating division through subtraction until 0.

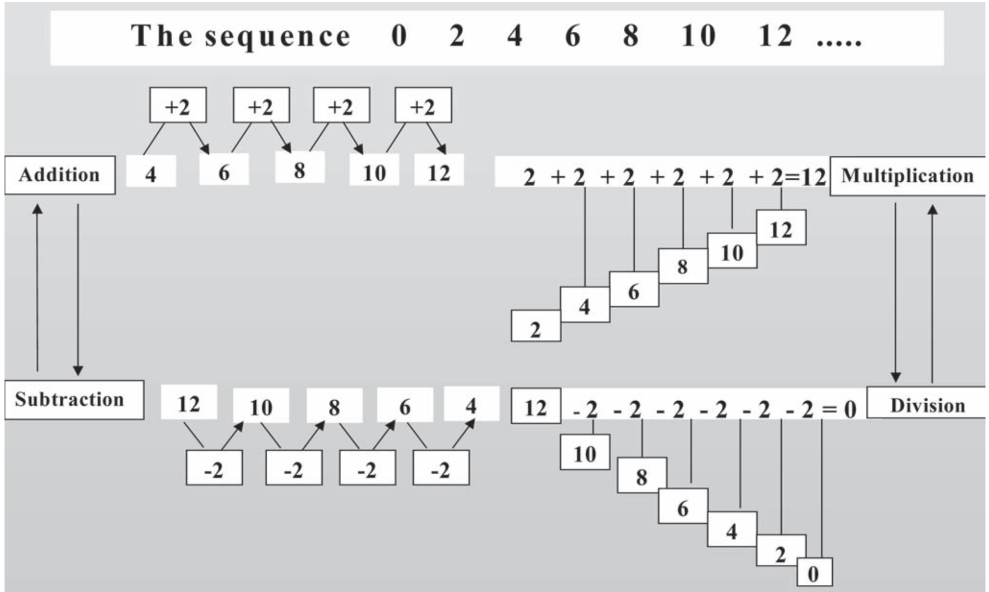

Fig. 5. A scheme for a dynamic mental structure: connecting natural numbers and operations.

five classes from the Republic of Moldova also joined the program. The number of children in the classes varied between 26 and 34, and in total, 232 children in nine experimental classes were involved. The experimental program tracked cohorts of children from Grade 1 to Grade 4 (aged 6–7 to 10–11 years). Preparatory meetings and feedback sessions were conducted with the classes' teachers. The teachers received detailed description of the tasks that they were going to offer to students, and the teaching periods were followed by discussions, on a weekly base. The students were tested 15 times per year and their teachers interviewed at the end of each semester. The data reported here are from the third year of the study.

To link the ideas with the example previously described, I have chosen for this article to focus on only one type of learning activity: a counting task in Grade 1. This type of counting task was systematically practiced for about 5–10

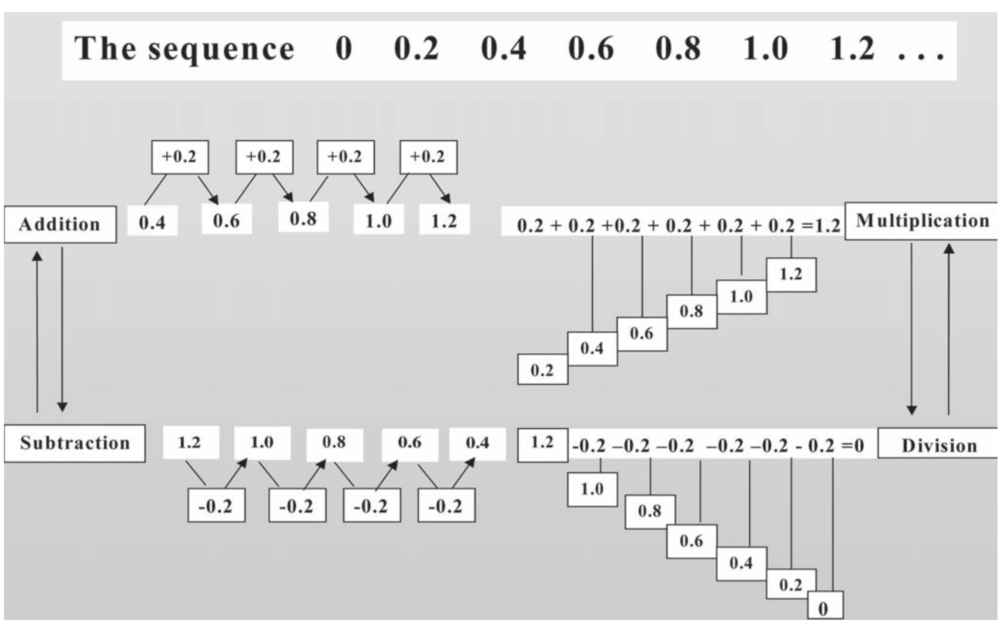

Fig. 6. A scheme for a dynamic mental structure: connecting decimal numbers and operations.

minutes in each mathematics lesson following the pattern listed below. The structure of the counting task takes into account that learning is facilitated by situations that activate multiple types of processing components of behavior and brain (e.g., Luo, Niki, & Phillips, 2004). In the activities, there is, for example, a crossing focus from motor activity to higher cognitive levels and back. Another aspect involves the emotional and relational dimensions: During the task, the child is in a situation that involves relating to his/her classmates as well as self-developing metacognitive processes (e.g., Singer, 1994–1997; Singer & Radu, 1994–1996). Some comments about what is behind each step of the task from a mind–brain perspective are mentioned in parentheses.

The basic format of the counting task covers the following steps:

1. Some children count, in turn, from 1 to 10, using sticks or marbles. (The counting routine is usually mastered by 6- to 7-year-olds at the beginning of the first grade. However, commuting the counting activity to manipulating concrete objects is important for the sake of transfer.)
2. While a child counts, the teacher or another student marks the number of objects on the blackboard, using points or lines: -; - -; - - -, etc. (Unconventional representations—such as points, small lines, stars, other drawings in regular positions—are introduced to suggest the variation of the size of sets when counting. The focus is on visualizing degrees of magnitude.)
3. One child counts loudly, without drawing. (The material support is missing now; the idea is to focus on an internal representation, stressing on both verbal and motor components.)
4. The previous three steps are repeated while counting in a decreasing order, from 10 to 1. (This activity facilitates the underlying reversibility of the process while involving successively different kinds of activities, such as visual, motor, and verbal.)
5. Children are asked to observe that the set of objects is increasing/decreasing when the counting is made in an increasing/decreasing order. (The activity is meant to create bridges to the conservation of number that is already internalized by most of the students in the first grade (Piaget & Szeminska, 1952). It also bridges to the more general ability to compare numbers and sets. It facilitates dynamic associations through the focus on increasing and decreasing.)
6. Some children, identified by the teacher as having different levels of numerical competence, count independently, in turn, from 10 to 1, without objects. (The activity has a social–relational target: first trial of independent work.)
7. Children count all together, loudly, while the teacher is tapping the rhythm. (The stress is on rhythm, with the target the recursive property of mind.)
8. Keeping the rhythm, children count mentally from 1 to 10. (Through the rhythm, the social context is harmonized, and the child feels safer within the group.)
9. The teacher asks the children in the classroom to count mentally and to raise a hand when each of them has finished. (More self-monitoring of metacognitive processes. The child can focus on independent work with more confidence. He/she might also mutually assess his/her own activity through observing classmates' reactions.)

The following extensions were developed in the same pattern within the task of counting, in the subsequent lessons:

- Extending the interval—from 10 to 20, to 30, to 100, etc.
- Modifying the starting point: from 1, from 2, from 10, from 34, etc.
- Modifying the distance between the elements of the sequences: one by one, in twos, in threes, in fours, and so on.

These counting tasks are meant to capture and objectify the dynamics of the multiple representations and connections emphasized in Figure 5. Within these tasks, children gradually advance from manipulating concrete objects to mental computing.

The format underlies cycles of transfers in order to internalize, together with the number line sequences, a variety of representations and, what is equally important, ways to move from one representation to another one. Thus, representational change becomes a trustful tool for problem solving.

The target of the counting tasks as described above is far beyond learning counting. Actually, when starting school, most children master the counting routine—at least from 1 to 10. (The teachers interviewed each student in this respect at the beginning of the study.) The target is to help children build a dynamic mental structure that they can self-develop and generalize across new tasks in an adequate context. As the inclusion of the arithmetical operations is not obviously visible, being *implicitly* integrated into the sequences on the number line, they remain lost there if support for learning is not adequate. This loss actually happens in classical training, and, for this reason, there is a need to separately train each category of numbers, operations, and properties. Each of these transfers needs a conceptual change when taught from the traditional framework. I suggest that the missing link between the study of cognition and the educational studies is this inefficient use of mind–brain endowment in school practice, this failure to highlight the dynamic mental structures behind mathematics.

Karmiloff-Smith (1992, 1994) noted a thread that is common to several spheres of cognition and that relates to learning with the methods proposed here: The passage from procedural nonexpert to the automatic (nonprocedural) expert also involves a parallel passage from implicit to explicit knowledge (from executing mechanically to understanding how it works). Within this type of task, the child becomes an expert by processing in two ways: On the one hand, he or she internalizes the algorithms and, on the other hand, he or she gets an understanding of the internal relationships that connect the numbers and the operations (that configure the explanation about how it works). Within the experimental learning strategy used in this research, the sequence of natural numbers as an integrating concept is acquired in two ways: through processing in the framework of the sequence (identifying the rule involved in the specific pattern, adding new terms in the sequence, applying a given rule when the starting term is known, etc.) and by strengthening some operational characteristic of each element of the sequence. Thus, each number in the sequence is compared with the elements in the neighborhood (e.g., in the sequence 3, 6, 9, 12, one gets 9 by adding 3 to 6 or by subtracting 3 from 12), and with the elements that are situated further on the number line (e.g., one gets 9 by multiplying 3 by 3—or by making 3 jumps three steps long). Moreover, each element of the sequence is systematically related to concrete objects (underlying conservation) and mental processing (with or without imagery support).

A similar format is used for other tasks, involving comparison, composing and decomposing numbers, word problem solving, analyzing, and transforming word problems. Another important aspect within the school practice is the variation introduced by the teacher when selecting students to carry on specific activities. Within the format of the counting task, some learning activities become very familiar during the school year, while some new ones are added in each lesson. The teacher orchestrates the distribution of the activities among the students in the class so that all children are involved and each child gets involved at his/her level of competence and confidence. The distribution of the activities allow remedial teaching within the task format because low achievers learn from their classmates while keeping the rhythm and they manifest publicly when they are confident with their own achievements. What becomes visible after a few months of training is that children advance gradually and naturally to a maximum of individual potential within a social learning context.

CONCLUSION: SOME RESULTS OF THE EXPERIMENTS

Tracking the development of students' abilities in a longitudinal study is a difficult enterprise because of the multitude of variables acting and interfering in the teaching–learning process in the classroom. Within these constraints, written tests were used mostly as feedback for corrections rather than as a way to report or grade the students. However, they give an insight into the children's progress. Two items from Grade 1 are discussed below.

> An item from February: "Make 12 using as many additions as you can."
> Spontaneous solutions involving repeated addition (i.e. multiplication), such as 3 + 3+3 + 3 or 2 + 2+2 + 2+2 + 2, frequently appeared. There were often 10 solutions per student. Some students took into consideration the addition with 0, as a distinct situation. We noticed that usually 0 does not pose problems anymore (unlike many classes using traditional teaching), being naturally accepted as "a starting point" in the sequence and, therefore, as a number.
> Another written task, given in April: "Devise a word problem." Out of 36 students, 28 solved the task correctly, proposing coherent word problems, as follows: 7 students composed a problem to be solved doing one operation, 17 students composed a problem to be solved doing two operations, 4 students composed a problem to be solved doing four operations. The results in other classes were similar, the percent of pertinent proposals of word problems being situated around 70%, with more than 50% of students choosing to use multiple operations.

During the school year, the children were tested 15 times. Each test contained 10 items, among which four to six were open-ended questions focused on creative tasks. The average success at solving these creative tasks was more than 60% of students for each of the classes involved in the experiment. For a comparison, I refer to a national assessment in Grade 4 at the end of the same school year that involved 18,844 students from 992 schools in urban and rural areas. In this assessment, an item requiring a nonstandard creative answer was: "Compose a problem using numbers smaller than 20, which is solved using addition, multiplication and subtraction." This item was correctly solved by only 20% of the students (only the text of the word problem was scored, not its solving), and 46.1% of the students did not attempt the task at all. Although this task may be more difficult than just to generate a word problem without other constraints, it is significant, in the local cultural context, to compare that mass refusal (46%) to try a solution with children's preference for multiple operations in the experimental classes.

A few findings of the teachers involved in the experiment are quoted from the interviews:

> Compared to my experience with non-experimental classes, this method realizes an important saving of time in the process of knowledge assimilation. This fact is pointed out by the precision of the students' answers and by their promptness to respond to any question.
> The performances obtained by the students involved in the experiment when confronted with creative tasks are much better than the results obtained in the past with the classical teaching methods.
> The students show facility in devising problems or exercises with many operations.

> Children exhibit a real pleasure in working during mathematics classes and this is maybe the most important aspect of this research. To come and attend mathematics lessons with pleasure is usually quite unlikely.

This fact strongly and constantly appeared during the experiments: Children were willing to generate things, and if they were stimulated to do this, the level of associations was extensive, both in breadth and in depth.

A dynamic approach emphasizing representational change can contribute to optimize learning under the constraints of time and information overload, moving beyond the difficulties of traditional approaches based on conceptual change. Teaching that targets representational change extends learning through continuity of activities, stimulating in this way the fluidity of thinking. In order to bring representational change to schools as an intrinsic phenomenon of learning, it is necessary to develop dynamic structural models that underlie connections and perspectives of development for each discipline and to make them part of the teaching–learning design. This is a difficult process, which requires both knowledge of mind–brain science achievements and domain-specific knowledge of the discipline. Nevertheless, the rewards of this approach are significant: preparing fluent thinkers for the dynamic schools of tomorrow.

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