D&P Lemma V.1.07

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April 5, 2021

Abstract

This note is an attempt to fill in the expository gaps in the proof of D&P Lemma V.1.07, the "natural" isomorphism

$$(X_1 \otimes \cdots \otimes X_n)^* \cong X_1^* \otimes \cdots \otimes X_n^*.$$

Remark: When I first read this section, I focused too much on what the mappings Ψ and Φ "look like.' The general strategy of this proof is to recognize that the two objects in question, $X_1^* \otimes X_2^* \otimes \cdots \otimes X_n^*$ and $(X_1 \otimes X_2 \otimes \cdots \otimes X_n)^*$, are both dual vector spaces, so mappings between the two spaces exist. The task, then, is to find a mapping that is bijective, hence isomorphic.

Proof. We start by defining a map and its direction between the two spaces:

$$\Phi \colon (X_1 \otimes \cdots \otimes X_n)^* \to X_1^* \otimes \cdots \otimes X_n^*. \tag{1}$$

The objects on the left side are dual vectors, i.e., linear forms (see Chapter III), and the objects on the right side are multilinear forms in $L(X_1, \cdots, X_n; \mathbb{R})$ (see V.1.03 p. 101).

In keeping with the notation of property Tii, we denote a linear form on the left side as

$$\hat{f} \colon X_1 \otimes \cdots \otimes X_n \to \mathbb{R} \tag{2}$$

and a multilinear map on the right side as

$$f = \hat{f}' \circ \bigotimes \colon X_1 \times \cdots \times X_n \to \mathbb{R}, \tag{3}$$

for some unique linear form $\hat{f}$. The linear forms $\hat{f}$ and $\hat{f}'$ are not necessarily the same, but we are free to define Φ such that $\hat{f} = \hat{f}'$, in which case

$$\Phi \colon \hat{f} \mapsto \hat{f} \circ \bigotimes \tag{4}$$

and the inverse map Ψ has the rule

$$\Psi \colon \hat{f} \circ \bigotimes \mapsto \hat{f}. \tag{5}$$

Now, Tii says that if f is multilinear (which it is), then $\hat{f}$ is a *unique* linear map (and, in this case, a unique linear form). This means that Ψ is one-to-one (injective) by definition. I'm a bit more paranoid than Joe with regards to Φ also being one-to-one,[1]

[1] Specifically, my (unfounded) fear is that Φ has a nontrivial kernel.

but I think this establishes it as an injective map as well: Let $\Phi(\hat{f}) = \Phi(\hat{g})$, so we can define the multilinear **0** map by:

$$\begin{aligned} \mathbf{0} &= \Phi(\hat{f}) - \Phi(\hat{g}) && (6) \\ &= \hat{f} \circ \bigotimes - \hat{g} \circ \bigotimes && (7) \\ &= (\hat{f} - \hat{g}) \circ \bigotimes \implies \hat{f} = \hat{g}. && (8) \end{aligned}$$

Yet another way to establish that Ψ and Φ are bijections is recognize that the dimensions of the two dual vector spaces are equal, so Ψ must be both injective and surjective.

D&P write that, "Evidently Φ is linear..." We can see this by considering $\Phi(a\hat{f} + b\hat{g})$ where $a, b \in \mathbb{R}$ and $f, g \in (X_1 \otimes \cdots \otimes X_n)^*$:

$$\begin{aligned} \Phi(a\hat{f} + b\hat{g}) &= (a\hat{f} + b\hat{g}) \circ \bigotimes && (9) \\ &= a\hat{f} \circ \bigotimes + b\hat{g} \circ \bigotimes && (10) \\ &= a\Phi(\hat{f}) + b\Phi(\hat{g}). && (11) \end{aligned}$$

We have thus shown that Φ is a linear bijection, i.e., an isomorphism. □