D&P Exercise V.1.8a

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Prove that the tensor product operation is associative, i.e.,

$$(X_1 \otimes \cdots \otimes X_n) \otimes (Y_1 \otimes \cdots \otimes Y_n) \cong (X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes \cdots \otimes Y_n) \tag{1}$$

Proof. Consider the diagram

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \times Y_1 \times \cdots \times Y_n \arrow[r, "\bigotimes"] \arrow[dr, "g"'] & X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes \cdots \otimes Y_n \arrow[d, "\hat{g}"] \\
& (X_1 \otimes \cdots \otimes X_n) \otimes (Y_1 \otimes \cdots \otimes Y_n)
\end{tikzcd} \tag{2}$$

for which the $\bigotimes$ map is the tensor product map defined in D&P Section V.1.04, and, if g is multilinear, then $\hat{g}$ is a unique linear map such that the diagram is commutative. To prove that $\hat{g}$ is an isomorphism, we first show that g is multilinear, which makes $\hat{g}$ linear (by Tii), and then we show that $\hat{g}$ is bijective.

Multilinearity of g: By considering how elements like $(\boldsymbol{x_1}, \ldots, \boldsymbol{x_i} + \boldsymbol{x_i}', \ldots, \boldsymbol{y_n})$ and $(\boldsymbol{x_1}, \ldots, \boldsymbol{y_j} + \boldsymbol{y_j}', \ldots, \boldsymbol{y_n})$ are transformed by g, and using the multilinearity of the tensor product in each of the X and Y subspaces, we can easily show that g is multilinear. Thus, by Tii, $\hat{g}$ is a linear map.

Injectivity of $\hat{g}$: We establish injectivity by showing that $\ker \hat{g} = \mathbf{0}$. Note that

$$\hat{g} \colon \boldsymbol{x_1} \otimes \cdots \otimes \boldsymbol{x_n} \otimes \boldsymbol{y_1} \otimes \cdots \otimes \boldsymbol{y_n} \mapsto (\boldsymbol{x_1} \otimes \cdots \otimes \boldsymbol{x_n}) \otimes (\boldsymbol{y_1} \otimes \cdots \otimes \boldsymbol{y_n}). \tag{3}$$

On the left side, the tensor product is **0** if and only if at least one of the component vectors is zero. Likewise, on the right side, the tensor product of tensor products is **0** if and only if at least one of the component vectors is zero. Therefore, $\ker \hat{g} = \mathbf{0}$, so $\hat{g}$ is injective.

Surjectivity of $\hat{g}$: Given injectivity, we establish surjectivity by showing that the dimesions of the domain and range are identical. For both the domain and range of $\hat{g}$, the dimensionality is the product of the dimensions of the component vector spaces; therefore, $\hat{g}$ is surjective.

Since $\hat{g}$ is linear, injective and surjective (making it linear and bijective), it is an isomorphism.

$\square$