Dodson & Poston Exercise VII.1.3

Peter Mao, . . .

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Abstract

In-progress solution. Feel free to add/comment/disparage.
This problem aims to fill in some gaps in the book regarding the derivative map $D_x f$.

Show that if a map $f: X \to X'$ between affine spaces has a derivative $D_x f$ at $x \in X$:

(a) $D_x f$ is **unique**

Solution Consider a (possibly different) linear map $\mathcal{D}_x f$ which satisfies the same definition as $D_x f$, so that for any neighborhood N of the zero map in $L(T_x X; T_{f(x)} X')$, there is a neighborhood $N'(0) \subset T_x X$ such that if $t \in N'(0)$, then for some $B \in N$,

$$d'(f(x), f(x+t)) = d'_{f(x)}(\mathcal{D}_x f(t) + B(t)). \tag{1}$$

Combining this definition with the definition for $D_x f$, we arrive at the relation

$$\mathcal{D}_x f(t) + B(t) = D_x f(t) + A(t) \tag{2}$$

for some $A, B \in N(0)$ if $t \in N'(0)$. A and B are arbitrarily close to 0, $\mathcal{D}_x f(t) = D_x f(t)$; therefore, $D_x f$ is the unique linear map with the stated properties.

(b) $D_x f(t) = \lim_{h\to 0} d'^{\leftarrow}_{f(x)}(\frac{d'(f(x), f(x+ht))}{h})$

Solution First we need a small lemma: Given affine space X, $x, y \in X$ and scalar $a \in \mathbb{R}$,

$$[\mathbf{d}_{\mathbf{x}}^{\leftarrow}(\mathbf{d}(x,y))]a = \mathbf{d}_{\mathbf{x}}^{\leftarrow}[(\mathbf{d}(x,y))a] \tag{3}$$

Proof. Geometrically, this identity says we get the same resulting bound vector whether we multiply a bound vector by a scalar (LHS of Eq. 3) or if we multiply the associated free vector by the same scalar and then bind the resulting free vector to x (RHS of Eq. 3). From Section II.1.01 (Definition: *affine space*), Equation Aii, the restricted difference map is bijective:

$$\mathbf{d}_{\mathbf{x}}(x,y) = \mathbf{d}(x,y) \tag{4}$$

From Section II.1.02 (Tangent spaces, p. 44), we have:

$$\begin{aligned}
(x,y)a &= \mathbf{d}_{\mathbf{x}}^{\leftarrow}[(\mathbf{d}(x,y))a] && \text{given} \\
[\mathbf{d}_{\mathbf{x}}^{\leftarrow}(\mathbf{d}_{\mathbf{x}}(x,y))]a &= \mathbf{d}_{\mathbf{x}}^{\leftarrow}[(\mathbf{d}(x,y))a] && \mathbf{d}_{\mathbf{x}} \text{ is a bijection} \\
[\mathbf{d}_{\mathbf{x}}^{\leftarrow}(\mathbf{d}(x,y))]a &= \mathbf{d}_{\mathbf{x}}^{\leftarrow}[(\mathbf{d}(x,y))a] && \text{by Equation 4}
\end{aligned}$$

□

To find the expression for $D_x f$, we start from the defining equation for the derivative:

$$d'_{f(x)}(D_x f(t) + A(t)) = d'(f(x), f(x+t)) \tag{5}$$

Apply $d'^{\leftarrow}_{f(x)}$ to both sides of the above equation and isolate $D_x f(t)$:

$$\begin{aligned}
D_x f(t) &= d'^{\leftarrow}_{f(x)}[d'(f(x), f(x+t)] - A(t) && \\
D_x f(ht) &= d'^{\leftarrow}_{f(x)}[d'(f(x), f(x+ht)] - A(ht) && t \to ht \\
hD_x f(t) &= d'^{\leftarrow}_{f(x)}[d'(f(x), f(x+ht)] - A(ht) && \text{Linearity of } D_x f \\
D_x f(t) &= \frac{d'^{\leftarrow}_{f(x)}(d'(f(x), f(x+ht))) - A(ht)}{h} && \\
D_x f(t) &= \lim_{h\to 0} \frac{d'^{\leftarrow}_{f(x)}(d'(f(x), f(x+ht))) - A(ht)}{h} && \lim_{h\to 0} \text{ has no effect on the LHS} \\
D_x f(t) &= \lim_{h\to 0} \frac{d'^{\leftarrow}_{f(x)}(d'(f(x), f(x+ht)))}{h} && \lim_{h\to 0} \frac{A(ht)}{h} = 0\text{, by design} \\
D_x f(t) &= \lim_{h\to 0} d'^{\leftarrow}_{f(x)}\left(\frac{d'(f(x), f(x+ht))}{h}\right) && \text{by the lemma (Eqn 3) with scalar } \frac{1}{h}
\end{aligned}$$

Note: $\lim_{h\to 0} A(ht)$ goes to 0 faster than linear. Otherwise, we have the wrong $D_x f$ (which *is* linear). In particular, $A(ht) \neq (A(t))h$.

(c) Construct an example in the style of Ex 2 to show that Theorems 1.04 and 1.05 become false if we use $\tilde{D}_x f$ ($D_x f$ without the binding map $d'^{\leftarrow}_{f(x)}$).

Solution No idea, although this would give us a good understanding of the importance of the binding map.

(d) If f is differentiable at x, it is continuous at x.

Solution f differentiable means that $D_x f(t) = \lim_{h\to 0} d'^{\leftarrow}_{f(x)}(\frac{d'(f(x), f(x+ht))}{h})$ exists for all t, while continuity requires that for every open set in $B' \subset X'$, there exists an open set in $B \subset X$ such that $f(B) = B'$.

Let the open ball $B'(f(x), |d'_{f(x)}(f(x+ht))|)$ be the open set in X' and $B(x, |ht|)$ be the open set in X. If $f(B) = B'$ for any value of h, then f is continuous. It is not completely clear to me that this should hold, but since we are taking $\lim_{h\to 0}$, it applies to arbitrarily small open sets around $f(x)$ and x.

(e) If f is an affine map, then $\hat{D}_x f$ is the linear part of f.

Solution An affine map has the form $f(\vec{x}) = \mathbf{A}\vec{x} + \vec{b}$, where $\mathbf{A}$ is the linear part and $\vec{b}$ is a constant translation. By high-school calculus,

$$\hat{D}_x f = \frac{df}{dx} = \mathbf{A}, \tag{6}$$

so $\hat{D}_x f$ is the linear part of f.