D&P Exercise V.1.7b

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Abstract

In Lemma V.1.08 (p 104), D&P show that for simple tensors in $X^* \otimes X$,

$$\hat{\boldsymbol{f}}(\boldsymbol{g} \otimes \boldsymbol{x}_2) = 0 \Longrightarrow \boldsymbol{g} \otimes \boldsymbol{x}_2 = \mathbf{0}.$$

In order to show that $\hat{\boldsymbol{f}}$ is injective, we need to show that this inference holds for composite tensors, not just simple tensors.

Show that any finite sum

$$\boldsymbol{t} = \sum_{i=1}^{n} \boldsymbol{g}^i \otimes \boldsymbol{x}_i$$

can be written in a form in which all the $\boldsymbol{x}_i$'s are linearly independent and infer that $\hat{\boldsymbol{f}}(\boldsymbol{t}) = 0 \Longrightarrow \boldsymbol{t} = 0$.

Proof. Suppose that in the list of n simple tensors, d of them are linearly independent. Certainly $d \leq \dim X$, and if $d = n$, then we are done. For $d < n$, arrange and label the $\boldsymbol{x}_i$'s such that $\{\boldsymbol{x}_i \mid i = 1 \ldots d\}$ are linearly independent. Then each vector in $\{\boldsymbol{x}_j \mid j = d+1 \ldots n\}$ is a linear combination of the vectors in $\{\boldsymbol{x}_i \mid i = 1 \ldots d\}$:

$$\boldsymbol{x}_j = \sum_{i=1}^{d} a_j^i \boldsymbol{x}_i \qquad \text{for } d < j \leq n. \tag{1}$$

In terms of $\{\boldsymbol{x}_i \mid i = 1 \ldots d\}$ we rewrite $\boldsymbol{t}$ as:

$$\begin{aligned}
\boldsymbol{t} &= \sum_{i=1}^{d} \boldsymbol{g}^i \otimes \boldsymbol{x}_i + \sum_{j=d+1}^{n} \boldsymbol{g}^j \otimes \boldsymbol{x}_j && \text{(finite sum with } d \text{ linearly independent } \boldsymbol{x}\text{'s)} \\
&= \sum_{i=1}^{d} \boldsymbol{g}^i \otimes \boldsymbol{x}_i + \sum_{j=d+1}^{n} \boldsymbol{g}^j \otimes \sum_{i=1}^{n} a_j^i \boldsymbol{x}_i && \text{(substituting in Equation 1)} \\
&= \sum_{i=1}^{d} (\boldsymbol{g}^i + \sum_{j=d+1}^{n} a_j^i \boldsymbol{g}^j) \otimes \boldsymbol{x}_i && \text{(collect simple tensors by } \boldsymbol{x}_i\text{)} \\
&= \sum_{i=1}^{d} \boldsymbol{g'}^i \otimes \boldsymbol{x}_i. && \text{(collect } \boldsymbol{g}\text{'s)} \qquad (2)
\end{aligned}$$

Thus we find that we can rewrite any finite sum of simple tensors as a sum of linearly independent simple tensors, and by the linearity of $\hat{\boldsymbol{f}}$ and D&P's result on simple tensors, we find the result we were looking for:

$$\hat{\boldsymbol{f}}(\boldsymbol{t}) = 0 \Longrightarrow \boldsymbol{t} = \mathbf{0}. \tag{3}$$

$\square$

This means that $\ker \hat{\boldsymbol{f}} = \{\mathbf{0}\}$; or, $\hat{\boldsymbol{f}}$ is injective.