Commentary on Dodson & Poston Exercise VII.6.1

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December 18, 2022

Abstract

Demonstrate the connection between curves and vector fields in the examples of section VII.6.01.

1 1a: $c(t) = \frac{t}{1-t^2}$

The velocity, as a function of t is simple to calculate, as it is just the derivative $\frac{dc}{dt} = \frac{1+t^2}{(1-t^2)^2}$. The problem is to write the velocity as a function of position, rather than time. In this case, we need to invert $c(t)$ into $t(c)$.

Suppressing the function notation on c, we can rewrite the expression for the curve as

$$ct^2 + t - c = 0. \tag{1}$$

By the quadratic formula, we have t as a function of c:

$$t = \frac{-1 + \sqrt{1+4c^2}}{2c}. \tag{2}$$

Only the positive square root is valid, as the negative branch gives times outside of the domain $]-1, 1[$.

From here, we just plug in the above results:

$$\begin{aligned}
v(c(t)) = c^*(t)/\vec{e_1} &= \frac{1+t^2}{(1-t^2)^2} && (3)\\
&= \frac{t^2}{(1-t^2)^2} + \frac{1}{(1-t^2)^2} && (4)\\
&= c^2 + \frac{c^2}{t^2} && (5)\\
&= c^2 + \frac{4c^4}{(-1+\sqrt{1+4c^2})^2} && (6)\\
&= c^2 + \frac{4c^4}{1+1+4c^2-2\sqrt{1+4c^2}} && (7)\\
v(c) &= c^2 + \frac{2c^4}{1+2c^2-\sqrt{1+4c^2}} && (8)
\end{aligned}$$

as desired.

2 1b, TBD.