Dodson Poston, Exercise V.1.1

Peter H. Mao

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Abstract

"Define addition and scalar multiplication of multilinear maps, by analogy with IV. Exercise 1.4 for the bilinear case, and prove that $L(X_1, ..., X_n; Y)$ is then a vector space."

During our Jan 10, 2021 meeting Joe mentioned that we need to show closure under addition of functions in this vector space.

Consider multilinear functions

$$L \ni F : X_1 \times X_2 \times \cdots \times X_n \to Y \tag{1}$$
$$L \ni G : X_1 \times X_2 \times \cdots \times X_n \to Y \tag{2}$$

where the X's and Y's are vector spaces and L is the function space $L(X_1, ..., X_n; Y)$

We define addition of functions so that for given $x_1, \ldots, x_n$,

$$(F +_L G)(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) +_Y G(x_1, \ldots, x_n) \tag{3}$$

where addition has been subscripted by the space that it takes place in. As a vector space, Y is closed under addition, so the addition of functions produces a map that also sends $X_1 \times \cdots \times X_n$ to Y, ie

$$F +_L G : X_1 \times X_2 \times \cdots \times X_n \to Y + Y = Y \tag{4}$$

so $F +_L G \in L$.

Joe rightfully reminds me that it is not enough to show that the mapping $(F +_L G)$ is in L – we must also show that it is multilinear. To that end, we want to show that

$$(F +_L G)(x_1, \ldots, x_i + x_i', \ldots, x_n) = (F +_L G)(x_1, \ldots, x_i, \ldots, x_n) +_Y (F +_L G)(x_1, \ldots, x_i', \ldots, x_n). \tag{5}$$

By invoking the definition of $+_L$ from Equation 3, and then the multilinearity of F and G, the left side of Equation 5 becomes

$$\begin{aligned} & F(x_1, \ldots, x_i, \ldots, x_n) +_Y F(x_1, \ldots, x_i', \ldots, x_n) \\ +_Y\ & G(x_1, \ldots, x_i, \ldots, x_n) +_Y G(x_1, \ldots, x_i', \ldots, x_n). \end{aligned}$$

Invoking Equation 3 again to combine the terms containing x_i and, respectively, x_i', we arrive at the right side of Equation 5:

$$(F +_L G)(x_1, \ldots, x_i, \ldots, x_n) +_Y (F +_L G)(x_1, \ldots, x_i', \ldots, x_n) \qquad \square$$