D&P Exercise V.1.6b

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Prove that if $\boldsymbol{x} \otimes \boldsymbol{y} = \boldsymbol{u} \otimes \boldsymbol{v}$, then for some $a \in \mathbb{R}$, $\boldsymbol{x} = a\boldsymbol{u}$ and $a\boldsymbol{y} = \boldsymbol{v}$.

Proof. Let $\{\boldsymbol{b}_i, i = 1 \ldots m\}$ be a basis for vector space X and $\{\boldsymbol{c}_j, j = 1 \ldots n\}$ be a basis for vector space Y. Let $\boldsymbol{x}, \boldsymbol{y} \in X$ and $\boldsymbol{y}, \boldsymbol{v} \in Y$ such that $\boldsymbol{x} \otimes \boldsymbol{y} = \boldsymbol{u} \otimes \boldsymbol{v}$.

In the given bases, the given relationship between the vectors is the Einstein double sum

$$x^i y^j \boldsymbol{b}_i \otimes \boldsymbol{c}_j = u^i v^j \boldsymbol{b}_i \otimes \boldsymbol{c}_j.$$

Without loss of generality, let us assume that $x^1 y^1 \neq 0$, so $u^1 v^1 \neq 0$; therefore, there is a nonzero scalar $a \in \mathbb{R}$ such that $x^1 = au^1$, which then forces the relationship $ay^1 = v^1$. In fact, for every $j = 1 \ldots n$, we know that $ay^j = v^j$.

Similarly, $ay^1 = v^1 \Rightarrow x^i = au^i$ for all $i = 1 \ldots m$.

Thus, $\boldsymbol{x} = a\boldsymbol{u}$ and $a\boldsymbol{y} = \boldsymbol{v}$. $\square$