Dodson & Poston Exercise VI.4.1a

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> (a) If the function $g : [0,1] \to \mathbb{R}$ is continuous, with $g(x) \neq 0$ for any $x \in [0,1]$, show that $\frac{1}{g} : [0,1] \to \mathbb{R} : x \to \frac{1}{g(x)}$ is also continuous.

This is pretty obviously true, but since I have no formal training in continuity proofs, I wanted to write up something semi-credible. The attempted proof that I tried in our February 13, 2022 meeting had the failing that I inadvertently assumed that g is bijective, which leaves many holes in the solution. Hopefully, this version of the proof is more solid.

Proof. For brevity, let $X = [0,1]$, and $0 < a < b \in \mathbb{R}$.[1]

By Corollary VI.1.06 (p. 120), the continuity of g ensures that for any open interval $(a,b) \subset g(X)$, $g^{\leftarrow}((a,b)) = U \subset X$ is an open set. In general, U could consist of any number of disjoint open intervals in X (see condition OC for a topology on page 121), so we denote the intervals by U_i and $U = \bigcup_i U_i$.

By construction,

$$g(U) = (a,b) \tag{1}$$
$$g(U_i) \subset (a,b). \tag{2}$$

Note that the second equation is not necessary for the proof; it merely serves as a reminder that any given $g(U_i)$ man not necessarily cover (a,b). In order for $\frac{1}{g}$ to be continuous, we need to show that the inverse mapping of any open set in $\frac{1}{g}(X)$ is also open. For convenience of notation, let us consider the interval $(\frac{1}{b}, \frac{1}{a}) \subset \frac{1}{g}(X)$. We want to show that the preimage of this interval, $\frac{1}{g}^{\leftarrow}((\frac{1}{b}, \frac{1}{a})) = V \subset X$, is open. As with U and g, we have the relations

$$\frac{1}{g}\Big(V\Big) = \Big(\frac{1}{b}, \frac{1}{a}\Big) \tag{3}$$
$$\frac{1}{g}\Big(V_i\Big) \subset \Big(\frac{1}{b}, \frac{1}{a}\Big). \tag{4}$$

Claim: $V = U$, so V is open.

Suppose there is a $u \in X$ such that $U \ni u \notin V$. By Equation 1, $g(u) \in (a,b)$, so $a < g(u) < b$. Since $u \notin V$, $\frac{1}{g}(u) = \frac{1}{g(u)} \notin (\frac{1}{b}, \frac{1}{a})$. Thus, either

$$\frac{1}{g(u)} \leq \frac{1}{b} \Longrightarrow b < g(u), \tag{5}$$

or

$$\frac{1}{g(u)} \geq \frac{1}{a} \Longrightarrow g(u) < a, \tag{6}$$

[1] This seems obvious, but aren't we using the mean value theorem here, while trying to prove a part of it?

which is a contradiction. Therefore, any $u \in U$ is also in V so $U \subseteq V$.

Likewise, by considering some $v \in X$ such that $U \not\ni v \in V$, we conclude that $V \subseteq U$.

Therefore, $V = U$, V is open, and $\frac{1}{g}$ is continuous.

□

(b) If the function $g : [0,1] \to \mathbb{R}$ is continuous, show that $|g| : [0,1] \to \mathbb{R} : x \to |g(x)|$ is also continuous.

For this proof, we find a suitable pair of intervals in $g(X)$ that have the same preimage as an arbitrary interval in $|g|(X)$. Although we know that the image of $|g|$ is strictly non-negative, let us allow the lower end of open interval to be negative in order to capture zero.

Proof. Consider an open interval interval in $\mathbb{R}$, (a,b), subject to the conditions that $a < b$ and $|a| < |b|$. We want to show that $|g|^{\leftarrow}((a,b)) = U \subset X$ is open.

Notice that $g(U) = (a,b) \cup (-b,-a)$. If $a > 0$, then $g(U)$ consists of two open intervals. If $a < 0$, then $g(U) = (-b,b)$, since $|a| < |b|$ and $b > 0$ by construction. In either case, $g(U)$ is open, so U is open by the continuity of g.

Therefore, $|g|$ is also continuous.

□