Notes from D&P and related books

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Contents

1 Dodson&Poston

1.1 Contents

1.2 0.2. Functions

1.2.1 typo: In definition of g,

$\mathbb{N}$ should be $\mathbb{Z}$

1.3 II. Affine Spaces

1.3.1 1.01. Definition: *affine space*

with vector space T is a non-empty set of points and a *difference function* $d\colon X \times X \to T$

1.3.2 1.02. Tangent spaces, freeing and binding maps.

the difference function of 1.01 is related to the bijection $d_x\colon x \times X \to T$ which is called a *freeing map*.

Its inverse, $d^{\leftarrow}$ is called a *binding map*.

1.4 V.1.10. Notation.

> Vectors in $\mathrm{X^k{}_h}$ are called tensors on X, covariant of degree h and contravariant of degree k, or of type $\mathrm{(k\ h)^T}$. ... By convention $X^0_0 = \mathbb{R}$.

1.5 VI. Topological Vector Spaces

1.5.1 1. Continuity

1. VI.1.01 $\epsilon - \delta$ continuity

2. 1.06 continuity

- a cubic is a nice function to demonstate that open $\rightarrow$ open is not necessarily true for continuous functions.
- If your open set in the domain starts and ends between the local extrema AND includes them, then the range is the closed interval bound by the local extrema.

3. Hausdorff topology This one is important.

    - Finite intersections of open sets, any unions of open sets
    - Finite unions of closed sets, any intersections of closed sets.

1.5.2 4. Compactness and Completeness

1. 4.02 Bolzano-Weirstrauss See Stillwell NLT, Ex 8.4.1, 8.4.2

2. See Stillwell NLT excercises 8.4.3, 8.4.4

3. 4.10 theorem f(C) is compact extents. I needed a reminder that functions are not continuous at infinity. see $\epsilon - \delta$ continuity

1.6 VII. Differentiation and Manifolds

1.7 Theorem (Inverse Function Theorem).

$D_x f$ is an isomorphism iff $\exists$ neighborhoods of x and $f(x)$ where a local inverse $f^{\leftarrow}$ exists.

1.8 VII.3.02. Language: Df, df (the other d)

if f is a real-valued function, then $df\colon TM \rightarrow \mathbb{R}$ is the "collection" of all $d_{f(q)} \circ D_q f$ where $D_q f\colon T_q M \rightarrow T_{f(q)}\mathbb{R}$ and $d_{f(q)}\colon T_{f(q)} \rightarrow \mathbb{R}$

this is NOT the same as $Df\colon TM \rightarrow T\mathbb{R}$.

1.9 VII.3.03 Tangent bundle on a manifold, abbreviations

$(\mathrm{T}^*{}_{\mathrm{x}}\ \mathrm{M}) = (\mathrm{T}_{\mathrm{x}}\ \mathrm{M})^*$

1.10 A tensor field of type $(0\ 0)^{\mathbf{T}}$ is just a function,

1.11 VII.4.01. Covariant Vectors.

See Lee Prop 11.18

1.12 VII.4.02. Contravariant Vectors

"we can idenfify $\vec{t}$ with the linear map:" $\partial_{\vec{t}}\colon f \mapsto df(\vec{t})$

so for any vector field $\vec{v}$ and function f, we have $(\vec{v}f)(x) = df(\vec{v_x})$

1.13 VII.6 Vector fields and flows

1.13.1 6.02. Definition. A solution curve or integral curve

of vector field v on manifold M is a curve $c\colon J \to M$ such that $c'(t) = v(c(t)) \forall t \in J$

1.13.2 . Definition. A C^k local flow...

- $\phi(U,t)M$ where $U \subset M$, open and t is a parameter in an open interval
- KEY POINT: by condition (ii), local flows are made up of solution curves.

1.13.3 . Corollary $\phi_{t+s} = \phi_t \circ \phi_s$

by definition, $\phi_t\colon U \to M\colon x \mapsto \phi(x,t)$, which returns the displacement from x along the flow ϕ.

1.13.4 6.07. Lemma. Let M be a manifold on an affine space X

1.14 VII.7 Lie Brackets

- typo: should be $\phi((x,y),t)$, not $\phi((x+y),t)$
- typo: $\psi_1\phi_1(0,0) = (0,1)$
- typo: $\phi_1\psi_1(0,0) = (1,1)$
- ϕ and ψ are local flows (see VII.6.03 p 197)
- There are many objects in the preface to this section:
  1. (x,y): points on a manifold (or coordinates by abuse of language)
  2. (s,t): parameters
  3. (ϕ,ψ): flows
  4. (v,w): vector fields (related to the flows)
  5. f: any function on the manifold
- claim (for this example only): $[\phi \circ \psi - \psi \circ \phi](x,y) = (1,0)$

1.14.1 7.01. Definition. The Lie bracket or commutator

- act on vector fields, not flows.
- typo: (bottom of 200) v(w(fg)) - w(v(fg))
- $= g[v,w](f) - f[v,w](g)$

"...so we have a new derivation" (namely, $[v,w]$)

1.14.2 7.02. Theorem.

iff fields commute then composition of flows also commute.

1.14.3 Fig. 7.1 is not referenced anywhere!

1.14.4 7.04. Theorem. ("the Chart theorem")

If we have a linearly independent and commuting set of vector functions that span a manifold, then they can be "realized" as the basis vectors (∂_i's) of some chart.

These flows have constant velocity in the neighborhood of x. (see example in preface to this section.)

1. On Randy's discomfort with this notation We want to justify the relationship $\phi_t^i(x^1,\ldots,x^n) = (x^1,\ldots,x^i+t,\ldots,x^n)$.

   Note that $M \ni \vec{x} = (x^1,\ldots,x^n) = \theta(0,\ldots,0)$, so the LHS can be rewritten $\phi_t^i(\theta(0,\ldots,0)) = \phi_t^i \circ \theta(0,\ldots,0)$

   Using 1.13.3 (and going through the proof backwards), this becomes $\theta(0,\ldots,t,\ldots,0)$ with the t in the i slot, which is the same as $\vec{x}+t\vec{e_i} = (x^1,\ldots,x^i+t,\ldots,x^n)$

   In the proof, this appears with parameter s instead of t.

2. $\theta(t^1 \ldots t^n) \in$ M $\theta(t^1 \ldots t^n)$ is a point in M. By varying one of the t's, it describes a curve on M. θ(0,0,0....0) = x $\in$ M. As a map, θ: $R^n \rightarrow$ M, so it **might** be the inverse of a chart function.

3. ...by the smoothness of the ϕ^1 why only ϕ^1? Each $\phi^i{}_t$(x) moves the point along its flow line, so the composition of all these ϕ's is a point in M. The fact that ϕ^1 is smooth means that through arbitrary points in M, ϕ^1 defines a smooth flow, so θ is also smooth through arbitrary points in M.

4. $\mathrm{D}\theta$ takes the vector $\mathrm{e}_1(\mathrm{t}^1 \ldots \mathrm{t}^n)\ldots$ This should be e_i, not e_1. otherwise, how does c(s) become a parameterized curve through $\theta(\mathrm{t}^1 \ldots \mathrm{t}^n)$ along the i direction?

5. typo: 6.05 “6.50” should be 1.13.3

6. typo: should be θ so we can rewrite this as $\phi^i{}_s(\mathrm{p})$ for $\mathrm{p} \in \mathrm{M}$

7. How does this "so c is a solution curve" come about? See 1.13.2: a local flow is a solution curve by definition and $\mathrm{c} == \phi^i{}_s$. so yes it is.

8. typo: remove θ should be $\mathrm{c}^*(0) = \mathrm{v}_i(\mathrm{c}(0))$

9. $\mathrm{D}\theta$ takes the standard basis... $\mathrm{D}\theta$: $\mathrm{T}_{(0,\ldots,0)}\mathrm{R}^n \to \mathrm{T}_x\mathrm{M}$: $\mathrm{e}_j \to (\mathrm{v}_j)_x$

10. What are they saying????

    > That is a linearly independent subset of the n-dimensional space T_x M, by assumption, so $\mathrm{D}\theta(\mathrm{v}_1, \ldots ,\mathrm{v}_n)$ is an isomorphism.

    What is "that"? "That" is $\{(v_1)_x, ..., (v_n)_x\}$

    if we're in Bachman, then $\langle e_1, ..., e_n\rangle$ is a general-ish vector in T_0..) R^n

1.14.5 Exercises VII.7

1. Ex VII.7.2 parts a and c are covered by 6.4.6

2. Ex VII.7.4: Typos abound! THIS IS THE LIE DERIVATIVE!! yes, very important. See Lee. :NOTER$_{\mathrm{PAGE}}$: (220 0.42246376811594205 . 0.13516483516483516)

   The subscript to the first v should be $\phi_h(x)$: $[u, v]_x = \lim_{h\to 0} \frac{(D_x\phi_h)^{\leftarrow} v_{\phi_h(x)} - v_x}{h}$

   v_x, and $[u, v]_x$ are both tangent vectors in T_xM $v_{\phi_h(x)}$ is a tangent vector in $T_{\phi_h(x)}M$

   $\phi_h\colon M \to M$, so

   $D_x\phi_h\colon T_xM \to T_{\phi_h(x)}M\colon v_x \mapsto \lim_{h\to 0} \frac{v_{\phi_h(x)} - v_x}{h}$

3. Typo in missing f, should be at least: $[\vec{u}, f\vec{v}] = \vec{u}(f)\vec{v} + f[\vec{u}, \vec{v}]$

   Note: this agrees with Lee p 188 because $\vec{v}(1) = d(1)\vec{v} = 0$

1.15 VIII. Connections and Covariant Differentiation

1.16 VIII.2. Rolling Without Turning

1.17 VIII.3. Differentiating Sections

1.17.1 "directional derivative"

In D&P-speak, "vertical" is vertical in their picture of TM, meaning that the vector lies in T_pM for some $p \in M$.

1.17.2 "we are free to decide what properties would be nice to have"

See Tu(DG) Sec 6, p 30. Without saying so, D&P appear to be leading us to the **Riemannian** or **Levi-Civita** connection. Tu says this is the unique, torsion-free affine connection on a Riemannian manifold.

1.17.3 Typo! first f should be t

1.17.4 Need to understand the first term on the right

See Tu(DG) Prop 4.9.ii p. 27

1.17.5 3.01. Definition. A connection (∇)

Koszul connection (same as Levi-Civita connection?) See Tu, Sec 6 (tangent vector, vector field) $\rightarrow$ (tangent vector)

Ci: linear in the input tangent vector space Cii: linear in the vector field space Ciii: scaling t is the same as scaling w Civ: Liebniz property Cv: continuous vector fields have continous connections

1.17.6 3.02. Coordinates – Christoffel symbols

> First, let us see what a connection looks like in coordinates, as some proofs will be easiest that way.

1. Typo? shold be "by Ciii" "by Ci)" is wrong… maybe

1.17.7 3.03. Transformation Formula.

1.17.8 3.04. Definition: vector field along c

a vector field **along** curve c assigns a vector tangent to M at each point of c(t) for t $\in$ [a,b].

From the examples given, this appears to be in concordance with Tu's usage of **along**, in that the vectors do not necessarily have to be tangent to the curve, which we intepret as a submanifold of M when we read Tu.

1.17.9 3.05. Differentiating, Along Curves, Fields Along Curves.

D&P's funny ∇ is DV/dt in Tu – the "covariant derivative"? In IX.1.01, they call Dc^*/dt the acceleration.

In Needham, it is the *intrinsic derivative* (23.2).

Note – I'm using ~ over ∇ instead of | through ∇.

> We denote the resulting linear map, taking vector fields along c (not vector fields on M) to vector fields along c, by $\tilde{\nabla}_{c*}$. not ∇_{c*}.

1.17.10 3.06. Definition: "vertical"

a vector $\vec{v}$ in the space of derivatives of tangent vectors is *vertical* if its projection into the tangent space is zero.

1.17.11 3.07. Definition: "vertical" and "horizontal" parts of a vector

The vertical part of a vector $\vec{v}$ in the space of derivatives of tangent vectors is the part normal to the tangent space, while the horizontal part of the vector is the part in the tangent space.

The projection P_w defined in this section is the projection to the "vertical." It corresponds to $\langle\vec{v},\hat{n}\rangle\hat{n}$ in Tu Sec 6.4, Prop 6.8, p 47.

1.17.12 3.08. Language

Koszul connection - section 3.01 Ehresmann connection - split $T_w(TM)$ into horizontal and vertical parts

1.17.13 Exercises 3 & 4

∇ and the funny ∇...

1.18 VIII.4. Parallel Transport

1.18.1 4.01. Definition: parallel vector field

A vector field v along $c \in M$ is *parallel* if the horizontal component of its derivative along c^* is zero. In other words, at each point along the curve, the corresponding vector from the field must maintain the same angle with respect to the tangent to the curve.

This corresponds to Tu (DG) Sec 14.5 Defn 14.13 p 110. DV/dt == 0.

1.18.2 what does this even mean?

"v, considered as a curve in TM, has c*(t) horizontal for all t"

Just like the "rolling" business in this book, I'm not sure the "vertical"/"horizontal" distinction is pedegogically useful. In order to picture nontrivial parallel transport, we need at minimum, a 2D surface embedded in 3D. M is 2D, so v $\in$ TM is also 2D, but c*(t) $\in$ T(TM) is 4D – hard to picture. Are there 2 horizontal dimensions and 2 vertical dimensions? This is **much** easier to follow in Needham.

See the discusion around Eqn (23.5), p 244 in Needham.

1.18.3 4.02. Theorem: uniqueness of the parallel vector field along c

for any v and c, the parallel vector field is unique.

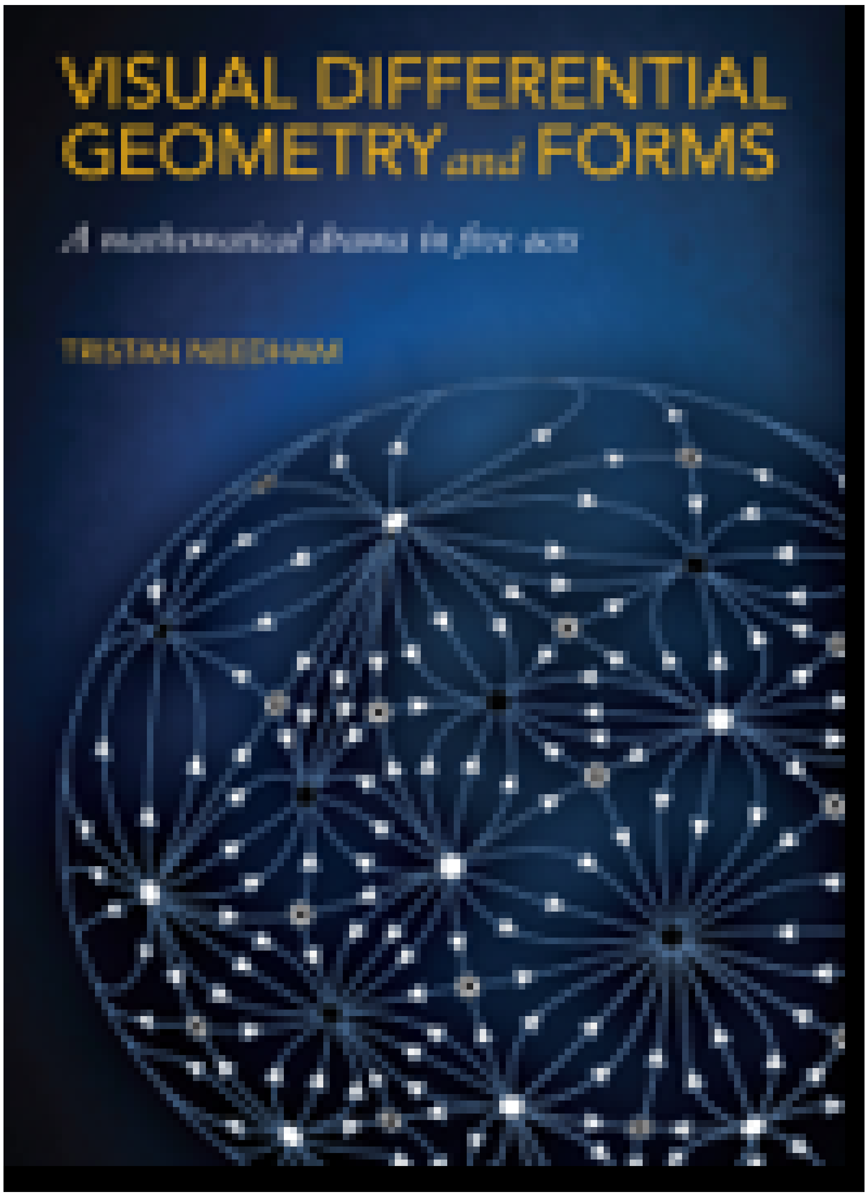
1.18.4 4.03. Definition: *parallel transport* τ

See Needham 23.2, p 243.

1.19 Bibliography

1.20 Index of Notations

2 NeedhamT VDGF



2.1 Contents

2.2 1.3 Angular excess of a spherical triangle

excess = (sum of angles) - π = A/R^2

2.3 1.5 Constructing geodesics with tape

2.4 2.1 Gaussian curvature in terms of excess angle

2.5 2.2 Circ/Area of a circle – reread this!

2.6 2.3 Local Gauss-Bonnet Theorem

angular excess is the total (area integrated) curvature.

2.7 Chapter 4 The Metric

2.7.1 4.3 Metric of a general surface

2.7.2 Gauss EFG coefficients of the first fundamental form

E = `A^2 F` = ABcosω G == B^2

2.7.3 4.4 Metric Curvature Formula

2.7.4 4.5 Conformal Maps

2.8 Chapter 5 Pseudosphere and the Hyperbolic plane

2.8.1 5.2 Tractrix and the Psuedosphere

Tractrix: object dragged along Y axis by a chord of length R. σ = distance along path of object.

2.8.2 5.3 Conformal map of the pseudosphere

(x,σ) $\rightarrow$ (x,y) = x + iy = z where x is an angle $[0,2\pi)$ of the revolution and y $\geq$ 1 (see 5.4)

2.9 8.3 Newton's curvature formula

2.10 Eqn 8.6, curvature

The area of the shaded sector in fig 8.6 explains the numerator of κ, but it doesn't really explain why the numerator should be that area. Kepler's law?

$\kappa = $ (area of swept velocity vector)$/|\mathrm{v}|^3$
In Kepler's law, the position vector sweeps out equal areas in equal time.

2.11 Chapter 9 Curves in 3-Space

2.11.1 definition: Torsion

"rate of rotation of the osculating plane is called the torsion, denoted τ."

2.11.2 definition: binormal of a curve

normal vector to the osculating plane

2.12 Chapter 10 The Principal Curvatures of a Surface

2.12.1 (10.1) Euler's Curvature Formula:

$\kappa(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$

2.12.2 Dupin's indicatrix

nearby slices parallel to the tangent plane are, to first order, conic sections.

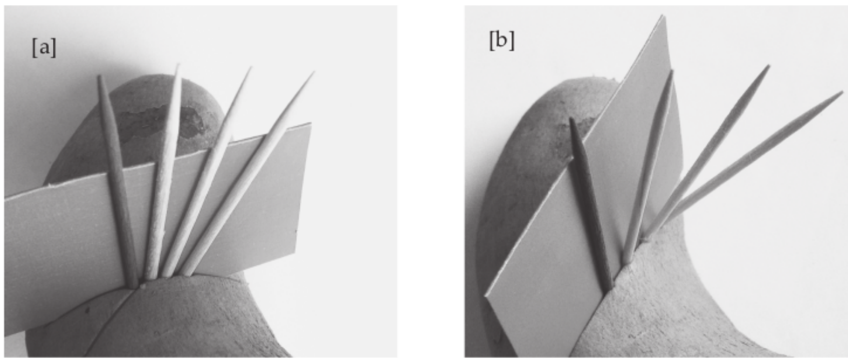
2.13 Chapter 11 Geodesics and Geodesic Curvature

2.13.1 Fig [11.2] $\mathbf{K} = \mathbf{K_g} + \mathbf{K_n}$

$\mathrm{K_g}$ is "geodesic curvature" IN the surface $\mathrm{K_n}$ is "normal curvature" OF the surface

2.14 Chapter 12 The Extrinsic Curvature of a Surface

2.14.1 Figure 12.5: normal vectors on principal and general curves



[12.5] [a] *If we move in a principal direction then the normal vector tips in that direction and initially stays within the normal plane;* **[b]** *If we move in a general direction then the normal vector immediately tips out of the normal plane.*

2.15 Chapter 15 The Shape Operator, S, and $\nabla_{\mathbf{v}}$

2.15.1 15.1 Directional Derivatives ($\nabla_{\mathbf{v}}$)

2.15.2 15.2 The Shape Operator S

"is simply defined to be the negative of the directional derivative of n along v" $S(v) = -\nabla_v n$

2.15.3 15.3 Geometric effect of S

> The principal directions are the eigenvectors of the Shape Operator S, and the principal curvatures are the corresponding eigenvalues: $\mathrm{S}(\mathrm{e_i}) = \kappa_\mathrm{i}\ \mathrm{e_i}$.

2.15.4 (15.18) $\kappa(\mathbf{v}) = \mathbf{v} \cdot \mathbf{S}(\mathbf{v})$

where v is a unit tangent vector.

2.15.5 (15.7) sum of curvatures in perpendicular directions

> The sum of the curvatures in any two perpendicular directions is equal to the sum of the principal curvatures.

This is a surprising result, given that the curvatures are represented in earlier parts of this book by ellipses.

2.15.6 (15.26) curvature and torsion

$\$\mathrm{K}_{\mathrm{ext}} = ||[\mathrm{S}]|| = \text{-}\tau^2$

2.15.7 15.9 Classical Terminology and Notation: The Three Fundamental Forms

Note that the fundamental forms are not proper forms (see Act V, Chapter 32 and on.)

- $I(u, v) = u \cdot v$
- $II(u, v) = S(u) \cdot v$
- $III(u, v) = S(u) \cdot S(v)$

2.15.8 19.8 The Road Ahead

In the 3 chapters before this, Needham gives proofs of th

2.16 Chapter 21 An Historical Puzzle

2.16.1 GR and parallel transport

> Einstein's success was all the more remark- able, and remains all the more puzzling, because he achieved it before Levi-Civita—pictured in [21.1]—discovered the concept of parallel transport, which did not occur until 1917!

2.17 Chapter 22 Extrinsic Constructions

2.17.1 Projection (not rotation) of w into $\mathbf{T_q}$

$\mathrm{w}_{||} = \mathrm{P}[\mathrm{w}] = \mathrm{w} \;\; (\mathrm{w} \cdot \mathrm{n})\mathrm{n}$

2.17.2 Rotation of w into $\mathbf{T_q}$

$\tilde{w}$. But in the limit as $\epsilon \to \infty$, rotation and projection are the same.

2.18 Chapter 23 Intrinsic (covariant) Constructions

2.18.1 23.2 The Intrinsic (aka, "Covariant") Derivative

aka Levi-Civita connection (Koszul connection?)

2.18.2 (Eqn 22.1) Instrinsic derivative, ($\mathbf{D_v}$, the convariant derivative)

ϵ D_v w = w(q) - $w_{||}$(p $\rightarrow$ q) both w's on the RHS are in T_q

2.18.3 Fig 23.3

"here w is growing in length and rotating counterclockwise as it moves along G."

2.18.4 $\mathbf{D_v}$ is also called the Levi-Civita Connection.

2.18.5 intrinsic derivative $\mathbf{D_v w}$

vs the definition on the previous page, here w(q) is parallel transported back to p [$w_{||}$(q $\rightarrow$ p)] and then w(p) is subtracted. (then lim $\epsilon \rightarrow \infty$)

2.18.6 Here is an extrinsic way of looking at the intrinsic derivative.

2.18.7 Eqn 23.3 $\mathbf{D_v w}$

same as Eqn 22.1, with ∇_vw dropped in for w

> In other words, to obtain D_vw we take the full rate of change $_v$w in R^3 , then subtract out the part that is not tangent to the surface, thereby leaving behind the part that is intrinsic to the surface.

2.18.8 Compare these with Ci-Cv in D&P p 217

> it is much simpler to think of **flattening onto the tabletop** the strip surrounding [curve] K, together with the vector fields x, y, z, for then D_v simply is $_v$.

2.19 1-forms used to be called covariant vectors or covectors

2.20 Further Reading

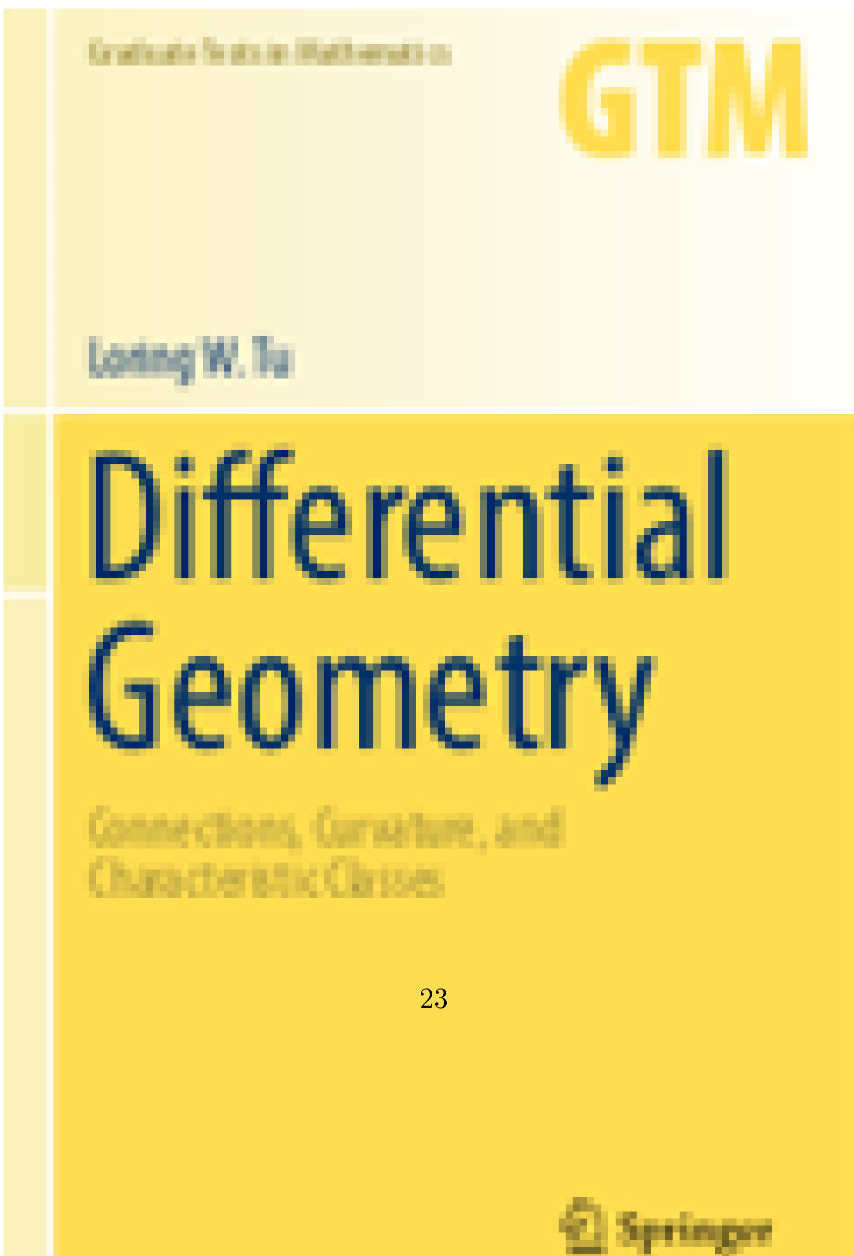
2.20.1 Intuitive Topology, by V. V. Prasolov.

2.20.2 Three-Dimensional Geometry and Topology, by William P. Thurston.

2.20.3 Differential Geometry: A Geometric Introduction, by David W. Henderson.

2.20.4 Gauge Fields, Knots and Gravity, by John Baez and Javier P. Muniain.

3 TuLW DifferentialGeometry



3.1 §1 Riemannian Manifolds

3.1.1 1.3 Riemannian Metrics

A Riemannian metric maps a continuous vector field onto a continuous function.

3.1.2 Example 1.11: Torus

On a torus in R^3 vs a torus as "the quotient space of a square with the opposite edges identified"

> We will show later that there is no isometry between these two Riemannian manifolds with the same underlying torus.

3.1.3 partition of unity

What is this? The definition is unhelpful in describing the meaning of this term.

3.1.4 Theorem 1.12. On every manifold M there is a Riemannian metric.

3.2 §2 Curves

3.2.1 2.3 Signed Curvature of a Plane Curve

$T' = \gamma''(s) = \kappa\vec{n}$, so $\kappa = \langle T', \vec{n}\rangle = \langle \gamma'', \vec{n}\rangle$

3.2.2 Curvature formulae

see Needham,

3.3 §4 Directional Derivatives in Euclidean Space

3.3.1 Torsion of the directional derivative D

If $D_X Y - D_Y X = [X, Y]$, then the torsion $T(X, Y) = D_X Y - D_Y X - [X, Y]$ is 0.

This only applies to $\mathbb{R}^n$.

3.3.2 Curvature of the directional derivative

3.3.3 Definition 4.8. "On" vs "Along" a submanifold

Along includes On, where On is tangent to the submanifold.

3.3.4 Notation X(M) and Γ(TM$|_N$)

- $\mathfrak{X}(M)$ is the set of all C^∞ vector fields **on** manifold M
- $\Gamma(TM|_N)$ is the set of all C^∞ vector fields **along** a submanifold N in a manifold M.
- $\mathcal{F} = C^\infty(M)$ is the ring of C^∞ functions on M

3.3.5 In a submanifold of $\mathbf{R}^n$, torsion "no longer makes sense"

because: X is a vector field **on** M Y is a vector field **along** M

3.4 §5 The Shape Operator

See Needham Chapter 15.

3.4.1 Defn: regular point – need a picture

3.4.2 5.2 The Shape Operator

$L_p(X_p) = -D_{X_p}N$, which is in the tangent plane T_pM and points opposite the direction in which N changes wrt X_p.

3.4.3 Is the usage of "on" here the same as before?

"let N be a C^∞ unit normal vector field on M (Figure 5.1)."

3.4.4 Proposition 5.3. The shape operator is self-adjoint:

3.4.5 Proposition 5.5: Curvature is given by the second fundamental form

$\kappa(X_p) = \langle L(X_p), X_p\rangle = II(X_p, X_p)$

3.4.6 Proposition 5.6: Principal directions and curvatures

- Principal directions: eigenvectors of L
- Principal curvatures: eigenvalues of L

3.4.7 5.4 The First and Second Fundamental Forms

Defined for smooth surfaces in $\mathbb{R}^3$.

- first fundamental form: inner product $\langle X, Y \rangle$ E := $\langle$ e_1, e_1 $\rangle$ F := $\langle$ e_1, e_2 $\rangle$ G := $\langle$ e_2, e_2 $\rangle$
- second fundatmental form: $II(X_p, Y_p) = \langle L(X_p), Y_p \rangle$ e := II(e_1, e_1) f := II(e_1, e_2) g := II(e_2, e_2)

3.5 §6 Affine Connections

> We will see in a later section that there are infinitely many affine connections on any manifold. On a Riemannian manifold, however, there is a unique torsion-free affine connection compatible with the metric, called the **Riemannian** or **Levi-Civita** connection.

3.5.1 No canonical basis for $\mathrm{T_pM}$ not embedded in a Euclidean space

Formula 4.2 here is the directional derivative.

3.5.2 Definition 6.1. An affine connection

3.5.3 6.2 Torsion and Curvature

"There does not seem to be a good reason for calling T(X,Y) the torsion" Is this not the same torsion that appears in Needham Chapter 9?

3.5.4 Definition 6.4: Riemannian or Levi-Civita connection

"...is and affine connection that is torsion-free and compatible with the metric."

3.5.5 Theorem 6.6. On a Riemannian manifold there is a unique Riemannian connection.

3.5.6 6.4 Orthogonal Projection on a Surface in R3

$\mathrm{pr_p}$ is the projection from $\mathrm{T_p}$ R^3 to the tangent space of M at p.

3.6 §7 Vector Bundles

"Thus the set X(M) has two module structures, over R and over F."

"We will try to understand F-linear maps from the point of view of vector bundles. The main result (Theorem 7.26) asserts the existence of a one-to-one correspondence between F-linear maps : (E) → (F) of sections of vector bundles and bundle maps : E → F."

3.6.1 7.1 Definition of a Vector Bundle

"A vector bundle, intuitively speaking, is a family of vector spaces that locally "looks" like U × R^r ."

We need a picture of the terms defined here.

3.7 13.1 Theorem

> We call DV /dt the covariant derivative (associated to) of the vector field V along the curve c(t) in M.

3.8 14.5 Parallel Translation

4 Littlejohn Manifolds

4.1 Notes for page 3

4.2 (typo)

should be: $x'^{\mu} = \psi_{ij}(x^{\nu})$

4.3 (typo)

should be: $x^{\nu} = \psi^{\leftarrow}_{ij}(x'^{\mu})$

4.4 8. Tangent Vectors

4.4.1 Eqn 11

4.5 9. Equivalence Classes of Curves

4.6 10. Tangent Vectors in Coordinates

"convective derivative" (see 4.4.1) "scalar" == "scalar field" in this context?

4.6.1 Eqn 16

4.6.2 tangent vector at p

Tangent vector as a `=first order, linear, partial differential operator=`
$\vec{X} = \Sigma_i X^i \frac{\partial}{\partial x^i}|_p$

4.7 12. Covectors

4.7.1 Eqn 21 $\mathbf{df|_p}$

$df|_p\colon T_pM \to \mathbb{R}\colon X \mapsto Xf$ where Xf is defined in 4.6.1

4.7.2 $\mathbf{df|_p}$ is an operator on vectors – it is not small or "infinitesimal"

"The most confusing thing for novices about this definition is that there is nothing small or "infinitesimal" about $\mathrm{df}|_\mathrm{p}$. In traditional theoretical physics the notation df usually denotes a small increment in the function f."

4.7.3 13. Differential of Coordinates

5 BachmanD: A Geometric Approach to Differential Forms

5.1 1.2 Generalizing the integral

5.2 1.3 differential form in a nutshell

"A differential form is precisely a linear function which eats vectors, spits out numbers and is used in integration."

5.3 3 Forms

5.3.1 3.1 Coordinates for vectors

"The key to understanding the difference between L and T_pL is their coordinate systems."

1. notation $<.,.>$

> We have switched to the notation "$<\cdot, \cdot>$" to indicate that dt we are not talking about points of P anymore, but rather vectors in $\mathrm{T_p\ P}$.

5.3.2 3.2 1-Forms

> Evaluating a 1-form on a vector is the same as projecting onto some line and then multiplying by some constant.
>
> Evaluating a 1-form on a vector is the same as projecting onto each coordinate axis, scaling each by some constant and adding the results.

5.3.3 3.3 Multiplying 1-forms

The wedge product is a product operation on 1-forms that is closed, ie, produces another 1-form.

"we will use the symbol "" (pronounced "wedge") to denote multiplication"

1. Notation (),<>,[] (): x,y plane <>: dx,dy plane []: ω,ν plane

2. wedge product justification or reasoning

    > Do we know of a way to take these vectors and get a number? Actually, we know several, but the most useful one turns out to be the area of the parallelogram that the vectors span. This is precisely what we define to be the value of (V1 , V2)

    Still – why the skew-symmetric product vs any other one? What is special about a/the skew-symmetric operator?

3. 1-form in a nutshell

    > Evaluating on the pair of vectors (V1 , V2) gives the area of parallelogram spanned by V1 and V2 projected onto the plane containing the vectors $\langle\ \rangle$ and $\langle\ \rangle$, and multiplied by the area of the parallelogram spanned by $\langle\ \rangle$ and $\langle\ \rangle$.

4. 2-form in a nutshell

    > Every 2-form projects the parallelogram spanned by V1 and V2 onto each of the (2-dimensional) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results.
    >
    > Henceforth, we will define a 2-form to be a bilinear, skew-symmetric, real- valued function on $T_p\mathbb{R}^n \times T_p\mathbb{R}^n$.

5.3.4 3.4 2-Forms on $T_p\mathbb{R}^3$

5.3.5 3.5 2-Forms and 3-forms on $T_p\mathbb{R}^4$

5.3.6 3.6 n-Forms

length, area and volume are all signed quantities?

5.3.7 3.7 Algebraic computation of products

5.4 4 Differential Forms

6 LeeJM _SmoothManifolds (+18)

6.1 Contents

6.2 Proposition 2.25 (Existence of Smooth Bump Functions).

These are mentioned in Tu. Maybe the exposition here is useful (but I haven't read it yet.)

6.3 Chapter 3 Tangent Vectors

6.4 Chapter 8 Vector Fields

> In the [third] section we introduce the Lie bracket operation, which is a way of combining two smooth vector fields to obtain another. Then we describe the most important application of Lie brackets: the set of all smooth vector fields on a Lie group that are invariant under left multiplication is closed under Lie brackets, and thus forms an algebraic object naturally associated with the group, called the Lie algebra of the Lie group.

6.4.1 Vector Fields on Manifolds

6.4.2 Example 8.12. cylindrical orthonormal frame on $\mathbb{R}^2\{0\}$

6.4.3 Vector Fields and Smooth Maps

first mention of "-related".

> Suppose F : M $\rightarrow$ N is smooth and X is a vector field on M; and suppose there happens to be a vector field Y on N with the property that for each p $\in$ M; $\mathrm{dF_p}\ (\mathrm{X_p}) = \mathrm{Y_{F(p)}}$. In this case, we say the vector fields X and Y are **F-related** (see Fig. 8.3).

The next proposition shows how F-related vector fields act on smooth functions.

6.4.4 Lie Brackets

6.4.5 Lemma 8.25. Lie bracket of smooth vector fields...

"Lemma 8.25. The Lie bracket of any pair of smooth vector fields is a smooth vector field."

Geometric interpretation comes in Chapter 9

"of limited usefulness for computations": $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$

6.4.6 Proposition 8.26 (Coordinate Formula for the Lie Bracket).

"an extremely useful coordinate formula for the Lie bracket": $[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$ or more concisely, $[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j}$

6.4.7 Example 8.27

Decent concrete example.

6.4.8 see D&P 3

$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$

fXgY - gYfX = fXg Y + fg XY - gYf X - gfYX = (fXg)Y - (gYf)X + (fg XY - gf YX)

6.5 Chapter 9 Integral Curves and Flows

6.5.1 Flowouts

1. Theorem 9.22 (Canonical Form Near a Regular Point). Randy, read this.

6.5.2 Lie Derivatives

1. Fig. 9.13 The Lie derivative of a vector field

2. Theorem 9.38 Lie derivative == Lie bracket

6.5.3 Commuting Vector Fields

1. Theorem 9.42. Equivalent statements on vector fields

   For smooth vector fields V and W on a smooth manifold M; the following are equivalent: (a) V and W commute. (b) W is invariant under the flow of V . (c) V is invariant under the flow of W .

   Consider ρ = x/r i + y/r j # radial vec field "E1" p179 θ = -y/r i + x/r j # unit rotation "E2" p179 ϕ = -y i + x j # disc rotation "W" p207

   [E1, E2] = -E2 [E1, W] = 0

   (a) follow E1 out in r for E2, the vector at positions in the flow of E1 are constant Q: Is E2 invariant under the flow of E1 ?? [A: no]
       for W, the vector at positions in the flow of E1 are increasing in magnitude. Q: Is W invariant under the flow of E1 ?? [A: yes]

   (b) follow a streamline in E2 or W The change in ρ is tangential to the flow. In W, dρ is constant, but in E2, dρ is smaller at larger radial positions.

   - useful derivatives i (x/r) = y^2/r^3 i (y/r) = -xy/r^3 = j (x/r) j (y/r) = x^2/r^3

   [E1,W] = (x/r i + y/r j)(-y i + x j) - (-y i + x j) (x/r i + y/r j) = (x/r) i(x) j - (y/r) j(y) i - [-y i(x/r) i - y i(y/r) j + xj(x/r) i + xj(y/r) j] = [(x/r) j - (y/r) i] - [-(y/r)^3 - x^{2y}/r^3]i - [xy^2/r^3 + (x/r)^3]j = [(x/r) j - (y/r) i] - (y/r)[-(y/r)^2 - x^2/r^2]i - (x/r)[y^2/r^2 + (x/r)^2]j = [(x/r) j - (y/r) i] - [(x/r) j + (y/r) i] = E2 - E2

   [E1,E2] = 0 - [-y/r i + x/r j][x/r i + y/r j] = 0 - [(-y/r) i(x/r) + (x/r) j(x/r)]i -[(-y/r) i(y/r) + (x/r) j(y/r)]j = 0 - [(-y/r)y^2/r^3 i - (x/r)xy/r^3 i + xy^2/r^4 j + x^3/r^4 j] = 0 - [-y^3/r^3 i - x^{2y}/r^3 i + xy^2/r^3 j + x^3/r^3 j]/r = 0 - [-y/r i + x/r j]/r = - E2/r

2. Theorem 9.44. Smooth vector fields commute if and only if their flows commute. D&P VII.7.02

3. Example 9.45 (Commuting and Noncommuting Frames).

Example 8.12. The standard coordinate frame is a global orthonormal frame on $\mathbb{R}^n$. For a less obvious example, consider the smooth vector fields defined on $\mathbb{R}^2 \smallsetminus \{0\}$ by

$$E_1 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}, \qquad E_2 = -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}, \tag{8.3}$$

where $r = \sqrt{x^2 + y^2}$. A straightforward computation shows that (E_1, E_2) is an orthonormal frame for $\mathbb{R}^2$ over the open subset $\mathbb{R}^2 \smallsetminus \{0\}$. Geometrically, E_1 and E_2 are unit vector fields tangent to radial lines and circles centered at the origin, respectively. //

4. Theorem 9.46 (Canonical Form for Commuting Vector Fields). D&P Thm VII.7.04

6.6 Chapter 11 The Cotangent Bundle

we define the differential of a real- valued function as a covector field (a smooth section of the cotangent bundle); it is a coordinate-independent analogue of the gradient.

6.6.1 Prop 11.8 $V \cong V^{**}$

"there is no canonical isomorphism $V \cong V^*$."

6.6.2 Tangent Covectors on Manifolds

Thus it became customary to call tangent covectors **covariant vectors** because their components transform in the same way as ("vary with") the coordinate partial derivatives, the Jacobian matrix multiplying the objects associated with the "new" coordinates to obtain those associated with the "old" coordinates. Analogously, tangent vectors were called **contravariant vectors**, because their components transform in the opposite way.

6.6.3 contravariant transformation

6.6.4 covariant transformation

6.6.5 Prop 11.18 (coordinate covectors are dx^{j}'s)

In other words, the coordinate covector field λ^{j} is none other than the differential dx^{j}.

See D&P VII.4.01

6.6.6 Proposition 11.23 (Derivative of a Function Along a Curve).

6.7 Chapter 12 Tensors

> We deal primarily with covariant tensors, but we also give a brief introduction to contravariant tensors and tensors of mixed variance.

7 TuLW Introduction to Manifolds

7.1 2.1 Directional derivative

In D&P, this is denoted $D_{pf}(v)$ for the directional derivative in direction v at point p of function f.

7.2 Defn: germ

Equivalence class of functions on an neighborhood ${C^{inf}}_p$ is the set of all germs of C^{inf} functions on R^n at p.

7.3 Defn: derivation

Linear map ${C^{inf}}_p \rightarrow R$ satisfying the Leibniz rule is a derivation

7.4 2.4 Vector fields

See D&P VII.4.02 Contravariant vectors. The 2nd paragraph above encapsulates the corresponding section in D&P in a nutshell.

7.5 3.3 k-covectors

"A 1-covector is simply a covector". A k-covector appears to be a tensor of covariant degree k

7.6 Lemma 3.11 $\tau(\sigma\ \mathrm{f}) = (\tau\sigma)\mathrm{f}$

$(\sigma\ f)(v_1\ ,\ldots,\ v_k) = f(v_{\sigma(1)},\ldots,v_{\sigma(k)})$ if σ = (1,m,n), then $\sigma(1)$ = m, $\sigma(m)$ = n, $\sigma(n)$ = 1

- concretely, if σ=(1,3,2)(4,5) and τ=(1,5,3,2) then (σ f)(v1,v2,v3,v4,v5) = f(v3,v1,v2,v5,v4)

  and τ(σ f)(v1,v2,v3,v4,v5) = (σ f)(v5,v1,v2,v4,v3) = f(v2,v5,v1,v3,v4)

$\tau\ \sigma$ = (1,5,3,2) (1,3,2) (4,5) = (1,2,5,4,3)

so this checks out.

7.7 3.7 Wedge product

Tu motivates the definition of the wedge product by wanting to have a product that preserves the property of being alternating on alternating functions (on a vector space). Does this make the wedge product the group multiplication on alternating functions? No – the identity is not alternating, so the wedge can't get you there.

7.8 Defn: graded algebra over field K

The algebra "can be written as a direct sum of vector spaces over K such that $\mathrm{A}^{\mathrm{k}} \times \mathrm{A}^{\mathrm{l}} \mapsto A^{\mathrm{k+l}}$.

7.9 4 Differential forms on $\mathbf{R}^{\mathbf{n}}$

7.10 4.6 Applications to Vector Calculus

related to exact sequences

grad: scalar → vector curl: vector → vector div : vector → scalar

7.11 4.7 Convention on Subscripts and Superscripts

vector fields: subscripts	e_1 ,. . . , e_n	contravariant
differential forms: superscripts	ω^1 ,. . . , ω^n	covariant
coordinate functions (0-forms):	x^1 ,. . . , x^n	covariant
differentials of coord functs (1-forms):	dx^1 ,. . . , dx^n	covariant
coordinate vector fields	$\mathrm{d/dx}^1$,. . . , $\mathrm{d/dx}^\mathrm{n}$	contravariant
coordinate vector fields (alt form)	δ_1 ,. . . , δ_n	contravariant

- coefficient functions

of a vector field	a^i
of a differential form	b_j

7.12 Sec 5: Manifolds

smooth manifolds are the focus of this book

"maximal Cˆ∞ atlas" make a topological manifold into a smooth manifold.

7.13 5.1 Topological Manifolds

defn: `second countable` A topological space with a countable basis.

7.14 5.3 Smooth Manifolds - maximal atlas defn

A maximal atlas on a locally Euclidean space is not conatined in any larger atlas.

7.15 Definition 5.9 smooth manifold

A smooth manifold is a topological manifold with a maximal atlas. A maximal atlas is also called a `differentiable structure` "In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do"

7.16 Smooth manifold conditions

to show that a topological space M is a C manifold, it suffices to check that:

1. M is Hausdorff and `second countable`,
2. M has a C atlas (not necessarily maximal).

7.17 5.4 Examples of Smooth Manifolds

- Euclidean space
- Open subset of a manifold
- Manifolds of dimension zero
- Graph of a smooth function
- General linear groups (nonzero determinant matrices)
- Unit circle in the (x,y) plane
- Product manifold

7.18 §6 Smooth Maps on a Manifold

7.19 6.3 Diffeomorphisms

7.20 6.6 Partial Derivatives

7.21 6.7 The Inverse Function Theorem

A function on an affine space or manifold is invertible at p iff the Jacobian determinant at p is nonzero

7.22 Chapter 3: The tangent space

7.23 bump functions?

These are covered in section 13, p 140

7.24 Theorem 20.4.

> If X and Y are C vector fields on a manifold M, then the Lie deriva- tive LX Y coincides with the Lie bracket [X ,Y].

7.25 A.4 First and Second Countability

7.25.1 Lemma A.10. Every open set in $\mathbf{R^n}$ contains a rational point.

7.25.2 Prop A.11 collection of open balls w/ rational centers/radii is a basis for $\mathbf{R^n}$

7.25.3 Defn A.12 second countable <== countable basis

7.25.4 Defn A.15 basis of neighborhoods, first countable

- A `basis of neighborhoods at p` is a collection of of neighborhoods of p such that $p \in B_\alpha \subset U$
- Topological space S is `first countable` if it has a countable basis of neighborhoods at every point $p \in S$.
- Every second countable space is first countable.