Commentary on Dodson & Poston Exercise VII.2.8

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Abstract

Not a complete solution to this problem, just comments and sketches of concrete examples to illustrate the problem. Feel free to add/comment/disparage.

1 Commentary on Ex. VII.2.8

D&P use x both as the point in the manifold M and as the coordinate labels in the affine spaces that the charts in M and M' map to. This leads to some confusion in reading the problem, so in this document, I take $p \in M$, N is a neighborhood of p and leave the $x^i \in X, X'$ as coordinate in the affine spaces.

Suppose that $f\colon M \to M'$ is a C^k map between manifolds and that for some $p \in M$, the derivative $\mathbf{D}_p f$ is surjective with $\dim M = m > n = \dim M'$.

(a) Show similarly to Ex VII.2.7 that p and $f(p)$ have charts around them giving f the local form

$$(x^1, \ldots, x^{m-n}, x^{m-n+1}, \ldots, x^m) \mapsto (x^{m-n+1}, \ldots, x^m). \tag{1}$$

Comment As with Ex VII.2.7, this is easier to see with a concrete example. Taking the same spaces used in VII.2.7b-d, let $M = S^1 \times \mathbb{R}$ (unit cylinder) and $M' = S^1$ (unit circle). Let $f\colon (p^1, p^2, p^3) \mapsto (q^1, q^2)$, and per the hint, $F\colon (p^1, p^2, p^3) \mapsto (q^1, q^2, q^3)$.

A less obvious example is a mapping from $M = S^2 \backslash \{x^3 = \pm 1\}$ (unit sphere with poles taken out) to $M' = S^1$ (unit circle) where f takes each slice in the (x^1, x^2)-plane to the corresponding azimuthal position on the unit circle. The corresponding F mapping would take M into a unit-radius cylinder spanning the range $(-1, 1)$ in the third coordinate.

In both cases, the local form of f then drops the extra dimension.

(b) Deduce that if for some $q \in M'$, every $x \in f^{\leftarrow}(q)$ has $\mathbf{D}_p f$ surjective, then a chart giving coordinates $(x^1, \ldots, x^{m-n})$ of $f^{\leftarrow}(q)$ may be constructed around each $p \in f^{\leftarrow}(q)$. Prove that these make $f^{\leftarrow}(q)$ into a C^k manifold by satisfying Mi – Miii.

Comment Note that $f^{\leftarrow}(q)$ is a subset of M – the set of points in M that f maps to $q \in M'$. In the examples proposed in part (a), we are left with $\mathbb{R}$ or $(-1, 1) \subset \mathbb{R}$, which are clearly 1-dimensional charts as desired.

In the general case, we need to make use of the fact that the derivative is surjective so show that the slices of M formed by $f^{\leftarrow}(q)$ satisfy Mi – Miii. If $\mathbf{D}_p f$ were *not* surjective, what would it look like, and how would that change the result? I suspect that a deep understanding of this question requires understanding how it breaks if the key assumption is taken away, but I do not have that understanding right now.

(c) Deduce in particular that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is C^∞ and has $\mathbf{D}_p f \neq 0$ for every $p \in f^{\leftarrow}(1)$, then $f^{\leftarrow}(1)$ has the structure of a smooth $(n-1)$-manifold. Construct such functions f to deduce with less work than in Ex VII.2.1 that the sets there given in a,b,c are manifolds.

Comment If we believe part b, the deduction is elementary.

a. Let $M = \mathbb{R}^3$ and $f\colon (p^1, p^2, p^3) \mapsto \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}$.

b. Let $M = \mathbb{R}^3$ and $f\colon (p^1, p^2, p^3) \mapsto \sqrt{-(p^1)^2 + (p^2)^2 + (p^3)^2}$.

c. (unit rod in plane) Let $M = \mathbb{R}^4$ and $f\colon (p^1, p^2, p^3, p^4) \mapsto \sqrt{(p^3)^2 + (p^4)^2}$.

c. (unit rod in $\mathbb{R}^3$) Let $M = \mathbb{R}^6$ and $f\colon (p^1, p^2, p^3, p^4, p^5, p^6) \mapsto \sqrt{(p^4)^2 + (p^5)^2 + (p^6)^2}$.