Tensor examples...

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Abstract

examples to elucidate claims about tensors in D&P.

1 What is $\vec{a} \otimes \vec{b}$?

Following Tai-Danae Bradley's examples and Joe's examples (section 5) we represent the tensor product of two vectors as their outer product. Concretely, if

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{1}$$

then

$$\vec{a} \otimes \vec{a} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{a} \otimes \vec{b} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \tag{2}$$

$$\vec{b} \otimes \vec{a} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \vec{b} \otimes \vec{b} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3}$$

2 On D&P's note about tensor products and vector spaces

On pg 101 of D&P, they not that "...the tensor products $f \otimes g$ in $L^2(\mathbb{R}^2; \mathbb{R})$ do not constitute a vectors space, since they are not closed under addition ..." This is a correct, but misleading statement, especially when we were struggling just to know what $f \otimes g$ looks like.

During our discussion on Feb 7, 2021, someone (put your name here!) pointed out that not all matrices are outer products of vectors. I'd like to add here that *all* matrices that result from outer products are degenerate – each column or row of the resulting matrix is a multiple of every other row. Thus, in general, matrices are usually not the result of outer products, and so we can think of the ones representing simple tensors to be rather peculiar.

In addition, we can verify, in the equations (2) and (3) above, that the four matrices there form a basis for 2×2 real matrices, ie, they span the space $L^2(\mathbb{R}^2; \mathbb{R})$. So the statement under scrutiny is almost equivalent to saying that the basis vectors of a vector space are not a vector space in an of themselves, but they do span the space.

A simpler example of tensor products comes from defining

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

in which case,

$$\vec{a} \otimes \vec{a} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \vec{a} \otimes \vec{b} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

$$\vec{b} \otimes \vec{a} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \vec{b} \otimes \vec{b} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

With this example, it is plainer to see that the tensor products form a basis, but it is less glaringly obvious that they are always degenerate.