Dodson & Poston Exercise VII.2.3: concrete example

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July 10, 2022

Abstract

It may be helpful to have a concrete example to understand the equivalence relation in Exercise VII.2.3a.

1 Introduction

This excercise establishes an equivalence relation between tangent vectors of overlapping charts:

> Let (U, ϕ), (U', ϕ') be charts on a smooth manifold M, modelled on an affine space X with vector space T, $u \in U \cap U'$, and $\vec{t}, \vec{t'} \in T$. Define the relation $\sim$ by
>
> $$(U, \phi, \vec{t}) \sim (U', \phi', \vec{t'}) \iff D_{\phi(u)}(\phi' \circ \phi^{\leftarrow}\vec{t}) = \vec{t'}$$

In the following section, I concretely demonstrate symmetry of this equivalence relation using two charts on the unit circle (S^1).

2 Example with S^1

Table 1 summarizes the concrete elements and Figure 1 gives a visual representation of this example.

Table 1: Elements of this example in relation to the symbols of Exercise VII.2.3.

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M	S^1	the unit circle in $\mathbb{C}$ or $\mathbb{R}$, parameterized by θ
X	(X, Y)	$\mathbb{R}^{\nvDash}$, the affine space on which M is modelled
U	U_{Y+}	the open semicircle in $Y > 0$
ϕ	$\cos\theta = x$	projection onto X
U'	U_{X+}	the open semicircle in $X > 0$
ϕ'	$\sin\theta = y$	projection onto Y

For reference, here are some relevant not-so-often used derivatives:

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \tag{1}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}. \tag{2}$$

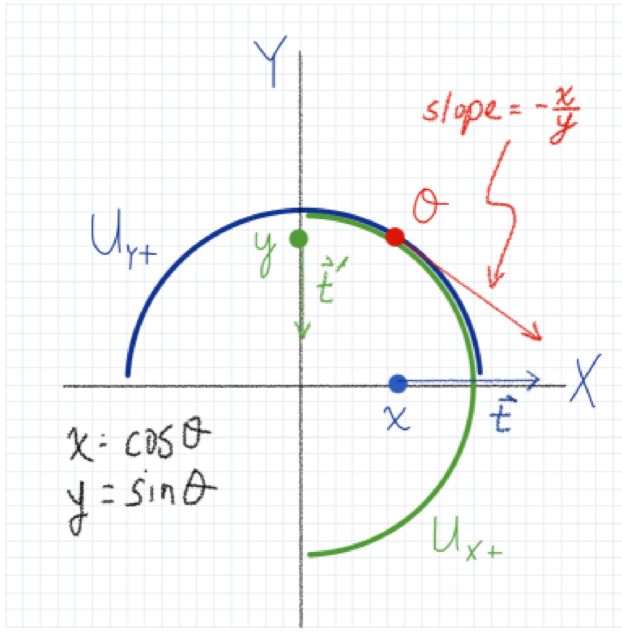

Figure 1: Charts on S^1 used in this example.

For the triplet $(U_{Y+}, \cos\theta, \vec{t} = 1)$, we can solve for $\vec{t'}$ by the definition given:

$$\begin{aligned}
\vec{t'} &= D_x(\sin \circ \arccos) && (3)\\
&= D_{\arccos x} \sin \circ D_x \arccos && (4)\\
&= \cos(\arccos x) \cdot -\frac{1}{\sqrt{1-x^2}} && (5)\\
&= -\frac{x}{\sqrt{1-x^2}} && (6)\\
&= -\frac{x}{y}. && (7)
\end{aligned}$$

In going from Equation 4 to Equation 5, recall that the composition in Equation 4 is a composition of linear maps (ie, matrices), and that in one dimension, the composition of linear maps reduces to standard mulitiplication of real numbers. For $(U_{X+}, \sin\theta, \vec{t'} = 1)$ we can solve for $\vec{t}$ in the same way:

$$\begin{aligned}
\vec{t} &= D_y(\cos \circ \arcsin) && (8)\\
&= D_{\arcsin y} \cos \circ D_y \arcsin && (9)\\
&= -\sin(\arcsin y) \cdot \frac{1}{\sqrt{1-y^2}} && (10)\\
&= -\frac{y}{\sqrt{1-y^2}} && (11)\\
&= -\frac{y}{x}. && (12)
\end{aligned}$$

So, in this concrete example, we have the equivalence relations

$$(U_{Y+}, \cos\theta, 1) \sim \left(U_{X+}, \sin\theta, -\frac{x}{y}\right) \tag{13}$$

$$(U_{X+}, \sin\theta, 1) \sim \left(U_{Y+}, \cos\theta, -\frac{y}{x}\right) \tag{14}$$

from which we can check symmetry:

$$\left(U_{X+}, \sin\theta, -\frac{x}{y}\right) \sim \left(U_{Y+}, \cos\theta, \left(-\frac{y}{x}\right)\left(-\frac{x}{y}\right)\right) \tag{15}$$

$$\sim (U_{Y+}, \cos\theta, 1). \tag{16}$$

Transitivity cannot be checked with this example because we don't have three easily overlapping charts. Using S^2, the unit sphere, we could follow the same procedures to establish transitivity in a concrete example.