D&P II.3.9a: Affine map on a convex subset

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Abstract

Exercise II.3.9a of Dodson and Poston asks us to consider an affine map on a convex subset of a space and prove that, when seen as a restriction on a map on the entire space, the larger map is unique if and only if the affine hull of the subset is the entire space. In this note, I characterize the maps that restrict to the map on the subset when the hull of the subset is a proper affine subspace of the entire set.

1 A solution to Exercise II.3.9a of D&P?

> Any affine map A between convex sets $P \subseteq X, Q \subseteq Y$ is the restriction of an affine map $\bar{A} : X \to Y$, and $\bar{A}$ is uniquely fixed by A if and only if $H(P) = X$.

At the cost of some loss of generality, let us assume that $\dim Q \leq \dim P \leq \dim X = \dim Y$.[1] In a quick look at this problem, we see that $H(P) = X$ when $\dim P = \dim X$. Clearly the "$\Leftarrow$" direction of the proof is trivial: If $H(P) = X$, there is no "wiggle room" for $\bar{A}$ once we have defined $A = \bar{A}|_P$. The "$\Rightarrow$" direction is a bit more interesting – if $\bar{A}$ is unique, what does that say about $\dim P$ with respect to $\dim X$? Let's guess that $\bar{A}$ is unique up to translations in the space $X \setminus P$, so uniqueness would imply $X \setminus P = \emptyset$, completing the proof. By looking at a concrete(ish) example, we will find that the non-uniqueness of $\bar{A}$ is a little more interesting than my initial hand-wavy guess.

2 A concrete example

Following Berger (Geometry I, Chapter 2), when we pick a basis, we can represent the affine transformation $\bar{A} : X \to Y$ with a $(n+1) \times (n+1)$ matrix, where $n = \dim X$:

$$\bar{A} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{y_0} & \bar{\mathbf{A}} \end{bmatrix}. \tag{1}$$

In this representation, $\mathbf{y_0} \in Y$ is the n dimensional translation vector, and $\bar{\mathbf{A}}$ is the "linear part" of $\bar{A}$. For some $\mathbf{x} \in X$,

$$\bar{A} \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{y_0} + \bar{\mathbf{A}}\mathbf{x} \end{bmatrix}, \tag{2}$$

[1] In fact, as pointed out by Joe and Noah, $\dim Y < \dim X$ is possible, and $\dim Y = 0$ is a special case.

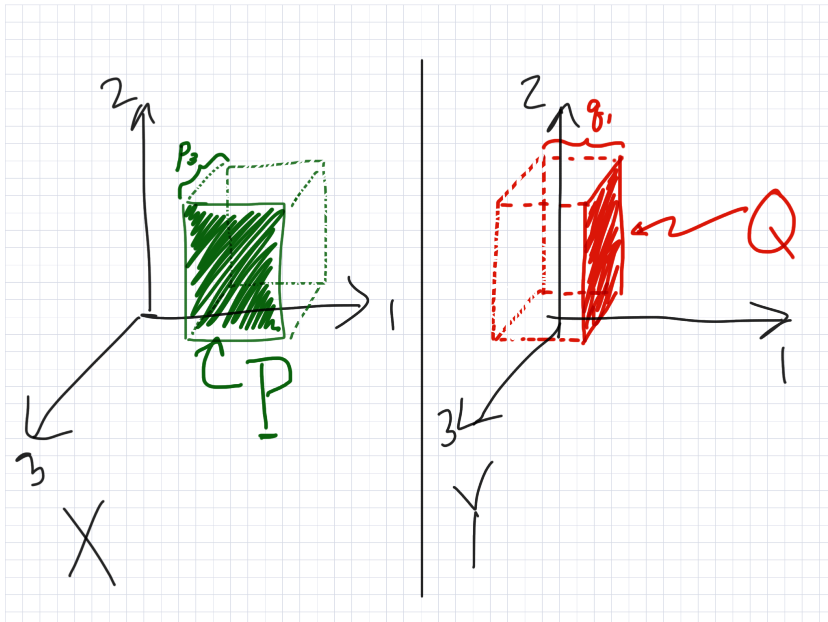

Figure 1: The convex subsets $P \subset X$ and $Q \subset Y$ (in "bird's eye axonometric" projection).

so we see explicitly that $\bar{A} : \mathbf{x} \mapsto \mathbf{y_0} + \bar{\mathbf{A}}\mathbf{x}$, in the way affine transformations do.

Now, suppose that the convex subset P has lower dimensionality than X. More explicitly, let P be a convex subset of the X-hyperplane at some constant $x_3 = p_3$, and let Q be a convex subset of the Y-hyperplane at $y_1 = q_1$, as in Figure 1. Finally, let the mapping $A : P \to Q$ be represented by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ v_2 & a & b \\ v_3 & c & d \end{bmatrix} \tag{3}$$

so for any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in P$,

$$\begin{bmatrix} v_2 + ax_1 + bx_2 \\ v_3 + cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} \in Q \tag{4}$$

Given A, what does $\bar{A}$ look like? With no constraints, $\bar{A}$ maps elements in X to Y by the rule:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ \bar{v}_1 & \bar{r} & \bar{s} & \bar{t} \\ \bar{v}_2 & \bar{a} & \bar{b} & \bar{u} \\ \bar{v}_3 & \bar{c} & \bar{d} & \bar{w} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{5}$$

Restricting $\bar{A}$ to P means that we only feed the matrix $\bar{A}$ with arguments of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ p_3 \end{bmatrix}, \tag{6}$$

where p_3 is a constant. For A to be the restriction of $\bar{A}$ to P, the mapping must send every element of P to an element of Q, which looks like

$$\mathbf{y} = \begin{bmatrix} q_1 \\ y_2 \\ y_3 \end{bmatrix}, \tag{7}$$

where q_1 is the constant that defines the the location of Q in Y.

Now, we are in a position to see characterize the nonuniqueness of $\bar{A}$ when $\dim P < \dim X$. Given the restriction to P and the requirement that $\bar{A}|_P : P \to Q$, we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \bar{v}_1 & \bar{r} & \bar{s} & \bar{t} \\ \bar{v}_2 & \bar{a} & \bar{b} & \bar{u} \\ \bar{v}_3 & \bar{c} & \bar{d} & \bar{w} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ q_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ q_1 \\ v_2 + ax_1 + bx_2 \\ v_3 + cx_1 + dx_2 \end{bmatrix}. \tag{8}$$

Since q_1 is a constant and x_1 and x_2 are not, $r = s = 0$. Also, by comparing the coefficients of x_1 and x_2 in the equations for y_2 and y_3, we make the identifications: $\bar{a} = a$, $\bar{b} = b$, $\bar{c} = c$, and $\bar{d} = d$. The constants in Equation 8 give us the following relationships for the remaining parameters in $\bar{A}$:

$$\begin{bmatrix} \bar{v}_1 + \bar{t}p_3 \\ \bar{v}_2 + \bar{u}p_3 \\ \bar{v}_3 + \bar{w}p_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{9}$$

So the translation vector of $\bar{A}$ is, in a sense, completely arbitrary because t, u, and w of $\bar{\mathbf{A}}$ are available as extra degrees of freedom to place the output in the correct affine subspace of Y, $y_1 = q_1$, with the required translation vector of A, $\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}$.

3 Further remarks

I was rather surprised by this result because I had incorrectly assumed that only translations orthogonal to P would not affect its mapping into Q. Running through the example, we find that the nonuniqueness of $\bar{A}$ lies in all components of the translation vector **and** the part of $\bar{A}$ that couples to $X \setminus P$. So much for hand waving!

4 Trouble in paradise

Noah pointed out that for $\dim Y = 0$, $\bar{A}$ is unique no matter what the dimensions of P and X are. In the matrix representation for $\dim P = 2$ and $\dim X = 3$, we have

$$A = [1 \quad 0 \quad 0]$$

and

$$\bar{A} = [1 \quad 0 \quad 0 \quad 0],$$

so $\bar{A}$ is uniquely defined. This appears to be the only exception to the statement of the problem.