Dodson & Poston Exercise VII.1.1

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Abstract

In-progress solution. Feel free to add/comment/disparage.

(a) [Given] Hausdorff topological spaces X, Y and any map (not necessarily continuous of everywhere defined) $f : X \to Y$, define

$lim_{x \to p}(f(x)) = q$ if and only if for any neighborhood $N(q)$ we can find a neighborhood $N(p)$ such that, if $x \in N(p)$ and $f(x)$ is defined, then $f(x) \in N(q)$.

Comment What is there to prove or show here? Are there any counterexamples that show how this can go wrong?

(b) If X is the set of natural numbers $1, 2, 3, \ldots$ together with one extra element which we label ∞, find a topology on X which makes Definition VI.2.01 (pg 125-126, sequence of points and the limit of a sequence) a special case of the one above (a).

Solution Without loss of generality, we shall assume a metric topology for X. In VI.2.01, a sequence is defined as a mapping $S : X \to Y$, so we want to find a metric (distance function) as defined in VI.1.02 (pg 116) that makes the sequence definition a special case of (a).

Define the metric

$$d\colon X \times X \to \mathbb{R} \tag{1}$$

$$(m, n) \mapsto \left| \frac{1}{m} - \frac{1}{n} \right| \tag{2}$$

with $\frac{1}{\infty} = 0$. This satifies the conditions for a metric:

i) $d(m, n) = d(n, m)$

ii) $d(m, n) = 0 \iff m = n$

iii) $d(m, n) \leq d(m, r) + d(r, n)$

The third condition requires a quick check. If $m < r < n$, we have equality because the sign of the r term will differ in the two terms on the right side when the absolute-value is "removed." If $m < n < r$ (the case $r < m < n$ proceeds similarly), then we use the fact that $d(m, r) = d(m, n) + d(n, r)$, so that the triangle inequality now reads

$$d(m, n) \leq d(m, r) + d(r, n) = d(m, n) + d(n, r) + d(r, n) \tag{3}$$

$$\leq d(m, n) + 2d(n, r) \tag{4}$$

which holds since $d(n, r) > 0$.

Since d maps into $\mathbb{R}$, we can use the usual topology on $\mathbb{R}$. Relating this back to part (a), $p = \infty$, $f = S$, and q is the limit of S.