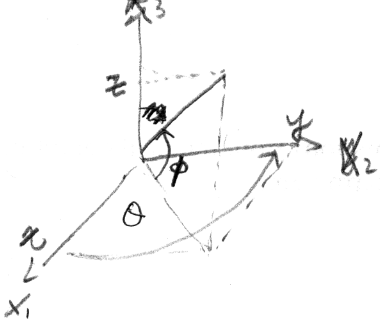
VII.1.4

Example



$x \in X_1$

$y \in X_2$

$z \in X_3$

$x^2+y^2+z^2 = 1$

$x = \cos\phi \cos\theta \qquad \theta = \arctan\left(\frac{y}{x}\right)$

$y = \cos\phi \sin\theta \qquad \phi = \arcsin(z)$

$z = \sin\phi$

$$\Phi_1 : (\theta,\varphi) \longmapsto (x,y) = (\cos\phi\cos\theta, \cos\phi \sin\theta)$$

$$\Phi_2 : (\theta,\varphi) \longmapsto (y,z) = (\cos\phi \sin\theta, \sin\phi)$$

$$\Phi_1^{\leftarrow} : (x,y) \longmapsto (\theta,\varphi) = \left(\arctan\left(\frac{y}{x}\right), +\arccos\left(\sqrt{x^2+y^2}\right)\right)$$

$$\Phi_2^{\leftarrow} : (y,z) \longmapsto (\theta,\varphi) = \left(\arctan\frac{y}{\sqrt{1-y^2-z^2}}, \arcsin z\right)$$

$$\cancel{\frac{\partial\Phi_1}{\partial\theta} = (-\cos\phi\sin\theta, \cos\phi\cos\theta)}$$

$$\frac{\partial\Phi_1^1}{\partial\theta} = -\cos\phi\sin\theta = -y \qquad \frac{\partial\Phi_1^2}{\partial\theta} = \cos\phi\cos\theta = x$$

$$\frac{\partial\Phi_1^1}{\partial\phi} = -\sin\phi\cos\theta = -z\cos\theta \qquad \frac{\partial\Phi_1^2}{\partial\phi} = -\sin\phi\sin\theta = -z\sin\theta$$

$$\frac{\partial\Phi_2^1}{\partial\theta} = \cos\phi\cos\theta = x \qquad \frac{\partial\Phi_2^2}{\partial\theta} = 0$$

$$\frac{\partial\Phi_2^1}{\partial\phi} = -\sin\phi\sin\theta = -z\sin\theta \qquad \frac{\partial\Phi_2^2}{\partial\phi} = \cos\phi = \sqrt{1-z^2}$$

$$\frac{\partial\Phi_1^{\leftarrow 1}}{\partial x} = \frac{-y}{x^2+y^2} \qquad \frac{\partial\Phi_1^{\leftarrow 2}}{\partial x} = \frac{-x}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}}$$

$$\frac{\partial\Phi_2^{\leftarrow 1}}{\partial y} = \frac{x}{x^2+y^2} \qquad \frac{\partial\Phi_1^{\leftarrow 2}}{\partial y} = \frac{-y}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}}$$

$$\frac{\partial\Phi_2^{\leftarrow 1}}{\partial y} = \frac{1}{\sqrt{1-z^2-y^2}} \checkmark \qquad \frac{\partial\Phi_2^{\leftarrow 2}}{\partial y} = 0$$

$$\frac{\partial\Phi_2^{\leftarrow 1}}{\partial z} = \frac{z \cdot y}{\sqrt{(1-z^2-y^2)}(1-z^2)} \qquad \frac{\partial\Phi_2^{\leftarrow 2}}{\partial z} = \frac{+1}{\sqrt{1-z^2}}$$

$$\left(D_{(\theta\phi)}\Phi_2 = \begin{bmatrix} x & 0 \\ -z\sin\theta & \sqrt{1-z^2} \end{bmatrix} \right) \quad D_{(x,y)}\Phi_1^{\leftarrow} = \begin{bmatrix} \frac{-y}{x^2+y^2} & \frac{-x}{z\sqrt{x^2+y^2}} \\ \frac{x}{x^2+y^2} & \frac{-y}{z\sqrt{x^2+y^2}} \end{bmatrix} \cdot \qquad \text{let } \rho^2 = x^2+y^2$$

$$D_{(x,y)}(\Phi_2 \circ \Phi_1^{\leftarrow}) = \begin{pmatrix} x & 0 \\ -\frac{z\cdot y}{\rho} & \rho \end{pmatrix} \begin{pmatrix} -y/\rho^2 & \frac{-x}{z\cdot\rho} \\ x/\rho^2 & \frac{-y}{z\rho} \end{pmatrix}$$

$$= \begin{pmatrix} -xy/\rho^2 & -x^2/z\rho \\ +\frac{zy^2}{\rho^3} + x/\rho & \frac{xy}{\rho^2} - y/z \end{pmatrix} \vec{t}_{xy} = \vec{t}_{yz}$$

$$D_{y,z}(\Phi_1 \circ \Phi_2^{\leftarrow}) = \begin{pmatrix} -y & x \\ -z\frac{x}{\rho} & -z\frac{y}{\rho} \end{pmatrix} \begin{pmatrix} \frac{1}{x} & 0 \\ \frac{zy}{x\rho^2} & \frac{1}{\rho} \end{pmatrix}$$

$$t_{yz} = \begin{pmatrix} \Delta y \\ \Delta z \end{pmatrix} \quad t_{xy} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad = \begin{pmatrix} -y/x + \frac{zy}{\rho^2} & x/\rho \\ -z/\rho - \frac{z^2y^2}{x\rho^3} & -zy/\rho^2 \end{pmatrix} \vec{t}_{yz} = \vec{t}_{xy}$$

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sanity check $D_{xy}(\Phi_2\Phi_1^{\leftarrow})\, D_{yz}(\Phi_1\Phi_2^{\leftarrow}) \stackrel{?}{=} I$

$$\begin{pmatrix} -\frac{xy}{\rho^2} & -\frac{x^2}{z\rho} \\ \frac{zy^2}{\rho^3} + \frac{x}{\rho} & \frac{xy}{\rho^2} - \frac{y}{z} \end{pmatrix} \begin{pmatrix} -\frac{y}{x} + \frac{zy}{\rho^2} & \frac{x}{\rho} \\ -\frac{z}{\rho} - \frac{z^2y^2}{x\rho^3} & -\frac{zy}{\rho^2} \end{pmatrix} \qquad \frac{x^2+y^2}{\rho^2} = 1$$

$$= \begin{pmatrix} \frac{y^2}{\rho^2} - \cancel{\frac{xy^2z}{\rho^4}} + \frac{x^2}{\rho^2} + \cancel{\frac{xy^2z}{\rho^4}} = 1 & \cancel{-\frac{x^2y}{\rho^3}} + \cancel{\frac{x^2y}{\rho^3}} = 0 \\ = 0 & \cancel{\frac{xy^2z}{\rho^4}} + \frac{x^2}{\rho^2} - \cancel{\frac{xy^2z}{\rho^4}} + \frac{y^2}{\rho^2} = 1 \end{pmatrix}$$

$$\cancel{-\frac{y^3z}{x\rho^3}} + \cancel{\frac{y^3z^2}{\rho^5}} - \cancel{\frac{y}{\rho}} + \cancel{\frac{xyz}{\rho^3}} - \cancel{\frac{xyz}{\rho^3}} - \cancel{\frac{y^3z^2}{\rho^5}} + \cancel{\frac{y}{\rho}} + \cancel{\frac{y^3z}{x\rho^3}}$$