Commentary on Dodson & Poston Exercise VII.2.7

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Abstract

Not a solution to this problem, just comment and sketches of concrete examples to illustrate the problem. Feel free to add/comment/disparage.

1 Commentary on Ex. VII.2.7

D&P use x both as the point in the manifold M and as the coordinate labels in the affine spaces that the charts in M and M' map to. This leads to some confusion in reading the problem, so in this document, I take $p \in M$, N is a neighborhood of p and leave the $x^i \in X, X'$ as coordinate in the affine spaces.

> Suppose that $f\colon M \to M'$ is a C^k map between manifolds and that for some $p \in M$, the derivative $\mathbf{D}_p f$ is injective. Let $\dim M = m \leq n = \dim M'$.

(a) Deduce from Corollary VII.1.05 that p has a neighborhood N such that $f|_N$ is injective.

Comment Corollary VII.1.05 establishes that $f|_N$ is injective when the domain and range of f are affine spaces. We need show that this result extends to manifolds as well. The homeomorphic chart-maps connect the manifolds to affine spaces, but we need to show that $\mathbf{D}_{\psi(p)}(\phi \circ f \circ \psi^{\leftarrow})$ is injective.

By the chain rule,

$$\mathbf{D}_{\psi(p)}(\phi \circ f \circ \psi^{\leftarrow}) = \mathbf{D}_{f(p)}\phi \circ \mathbf{D}_p f \circ \mathbf{D}_{\psi(p)}\psi^{\leftarrow}, \tag{1}$$

so we need to show that $\mathbf{D}_{\psi(p)}\psi^{\leftarrow}$ and $\mathbf{D}_{f(p)}\phi$ are both injective.

Recall that since we are talking about linear transformations here, "injective" means that the determinants of these derivatives are non-zero.

(b) Construct a chart $\phi\colon U \to \mathbb{R}^n$ around $f(x)$ such that

$$\phi(f(N)) = \phi(U) \cap \{(x^1, \ldots, x^n) | x^{m+1} = x^{m=2} = \cdots\ x^n = 0\}. \tag{2}$$

Comment Note that (U, ϕ) is a chart on M' and $f(N) \subset U$. From part (a), we know that $f|_N$ is injective. It is useful at this point to consider what this looks like, concretely.

Example: Let $M = S^1$, the unit circle, and $M' = S^1 \times \mathbb{R}$, a unit cylinder. Let $f\colon (p^1, p^2) \mapsto (p^1, p^2, q^3)$, so the unit circle is mapped by f to a particular cut of the cylinder. In M', let $U = \{(x^1, x^2, x^3) | x^2 > 0\}$ and $\phi(U) = \{(x^1, x^2 = 0, x^3)\}$. To satisfy the condition, define the chart map as

$$\phi\colon (x^1, x^2, x^3) \mapsto (x^1, 0, x^3 - q^3) \tag{3}$$

(c) Show that if $B\colon \mathbb{R}^n \to \mathbb{R}^m\colon (x^1, \ldots, x^m, \ldots, x^n) \mapsto (x^1, \ldots, x^m)$, then $B \circ \phi \circ f$ is a chart map admissible on M.

Comment Since B is simply a projection out of the extra dimensions carried by M',

$$\psi = B \circ \phi \circ f\colon M \to \mathbb{R}^m \tag{4}$$

is C^k and is a chart map admissible on an open set of M containing N.

(d) Deduce that x and $f(x)$ have C^k charts around them which give f the local coordinate form

$$(x^1, \ldots, x^m) \mapsto (y^1, \ldots, y^m, 0, \ldots, 0) \tag{5}$$

Comment The chart maps ψ and ϕ from parts a-c give f the desired local coordinate form $\phi \circ f \circ \psi^{\leftarrow}$.