Dodson & Poston Exercise VII.1.2

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Abstract

In-progress solution. Feel free to add/comment/disparage.

Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} |x| \exp\left(\dfrac{(y-2x^2)^2}{4x^4((y-2x^2)^2-x^4)}\right) & \text{if } x^2 < y < 3x^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Draw a picture of f.

Solution First, notice that between the two parabolas $y = x^2$ and $y = 3x^2$, f scales as $|x|$. I find the non-zero part of the equation easier to look at with the substitution $z = x^2 \geq 0$. Let $g(z,y)$ be the exponential part of f:

$$g(z,y) = \exp\left(\frac{(y-2z)^2}{4z^2((y-2z)^2-z^2)}\right) \quad (1)$$

$$= \exp\left(\frac{(y-2z)^2/(4z^2)}{(y-2z)^2-z^2}\right), \quad (2)$$

which for any value of z, is defined over the open interval $(z, 3z)$. Consider $g(z,y)$ at some fixed value of z. The independent parameter is y and we can treat z as the scaling of y-axis (see Figure 1(i)). The numerator of the exponential is a parbola in y centered at $y = 2z$, rising from 0 to 1/4 at the limits of the range. The denominator is also a parabola rising from -1 at $y = 2z$ to 0 at the limits of the range.

Notice that the numerator is non-negative and the denominator is negative-definite, so their quotient is 0 at $y = 2z$, negative everywhere else, and approaching $-\infty$ at the limits of the range (see Figure 1(ii)). Thus, for any value of z, $g(z, 2z) = 1$ and $g(z,z) = g(z,3z) = 0$. Figure 1(iii) is the y-direction cross section of f for any non-zero value of x.

The salient features of f are that it takes on a value of $|x|$ along the parabola $y = 2x^2$ and that it transitions smoothly to zero at $y = x^2$ and $y = 3x^2$.

(b) Show that for any vector $\vec{v} \in \mathbb{R}^2$, $lim_{h\to 0} \frac{f(h\vec{v})-f(0)}{h}$ exists and is zero.

Solution Consider two classes of vectors in $\mathbb{R}^2$: Those with $y \leq 0$ and those with $y > 0$. In the first case, $f(h\vec{v}) = 0$, so the limit in question is always identically zero with no additional mental work. For vectors with $y > 0$, recall that $f = 0$ when $y \geq 3x^2$. Since we are taking $\lim_{h\to 0}$, $f(h\vec{v}) = 0$ when

$$hy \geq 3(hx)^2,$$

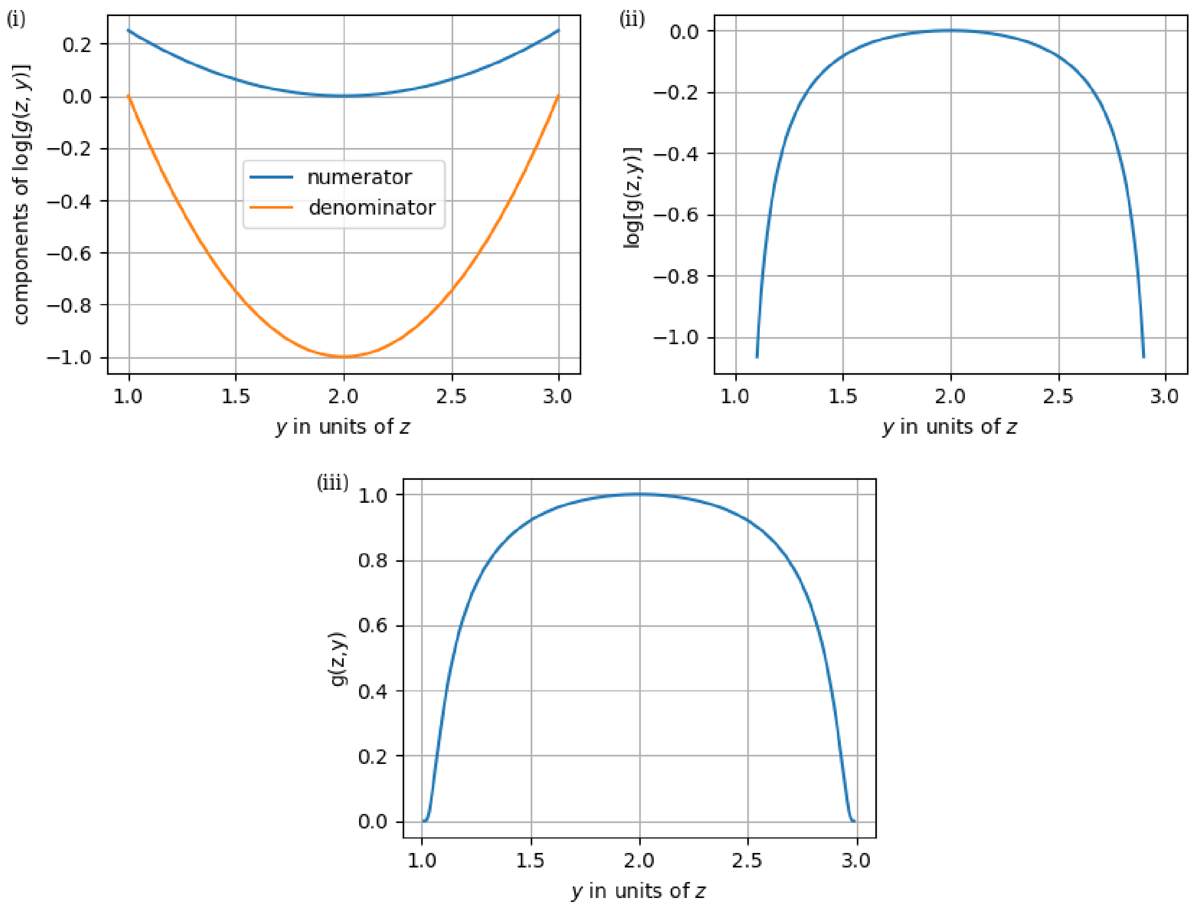

Figure 1: **(i)** Numerator and denomintor of the argument to the exponential in $g(z, y)$ from Equation 2. **(ii)** Argument to the exponential in $g(z, y)$. **(iii)** $g(z, y)$: This is the shape of $f(x, y)/|x|$ along sections in the y-direction. Notice the flatness of the function at the limits of its range ($y = z$ and $y = 3z$).

or equivalently,

$$y \geq 3hx^2.$$

Note that for *any* positive y and *any* x, there is always an h below which this relation will hold. Therefore, $lim_{h\to 0}\frac{f(h\vec{v})-f(0)}{h} = 0$ in this case as well. Geometrically, we find that when we scale down any vector from the origin far enough, it will land in a region of of the plane where $f = 0$.

(c) Show that if $\vec{v_i} = (\frac{1}{i}, \frac{2}{i^2})$ we have $\lim_{i\to\infty} \vec{v_i} = 0$, but that in the notation of Ex.VII.1.01 if $x = 0$ we have $d'(f(x + \vec{v_i}), f(x)) = \frac{1}{i}$.

Solution $\lim_{i\to\infty} \vec{v_i} = 0$ is obvious by inspection. We are given $x = 0$ and $f(\vec{0}) = 0$, so

$$d'(f(x + \vec{v_i}), f(x)) = f(\vec{v_i}).$$

Since $v_y = 2v_x^2$, for any $\vec{v_i}$, $g(\vec{v_i}) = 1$ (the exponential part of f), so we have the desired result $f(\vec{v_i}) = |v_x| = \frac{1}{i}$.

(d) Find a neighborhood N of the zero map $\mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbf{A} \in N \implies \mathbf{A}\left(\frac{1}{i}, \frac{2}{i^2}\right) \neq \frac{1}{i}$$

Solution First, note that $\mathbf{A}$ is a 1×2vector (co-vector?) in the dual space to $\vec{v_i}$'s $\mathbb{R}^2$, so we can represent it as a vector or point in $\mathbb{R}^2$. By inspection, we can see that for $\mathbf{A} = [1, 0]$ and $\mathbf{A} = [0, i/2]$, $\mathbf{A}\vec{v_i} = 1/i$, so any $\mathbf{A}$ on the line passing through those two points work. Therefore, any neighborhood of $\mathbf{A} = [0, 0]$ that satisfies the condition

$$A^y < -\frac{i}{2}(A^x - 1)$$

will only contain maps that satisfies the inequality $\mathbf{A}\left(\frac{1}{i}, \frac{2}{i^2}\right) \neq \frac{1}{i}$. The case $i = 1$ is the most restrictive (see Figure 2), so we find specifically that maps within radius $\frac{1}{\sqrt{5}}$ of the origin always satisfy the desired inequality.

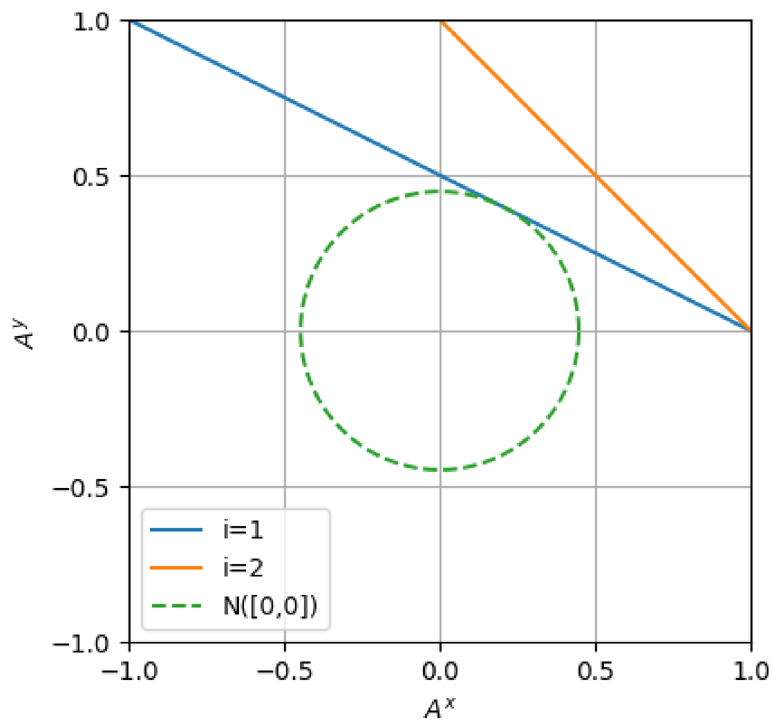

Figure 2: The space of $\mathbf{A} \colon \mathbb{R}^2 \to \mathbb{R}$. Maps that fall on the solid lines map $\vec{v_i} \mapsto 1/i$. The dashed green circle indicates an open neighborhood of the zero map for which $\mathbf{A}\vec{v_i} \neq 1/i, \forall i$, since the $i = 1$ case is the most restrictive one. As i tends to ∞, the radius of $N(\vec{0})|_i$, for which the inequality holds, tends to 1.

(e) Deduce that f has no derivative at $(0,0)$.

Solution In order for the derivative to exist at $(0,0)$, any neighborhood of the zero linear map must contain a map A that satisfies

$$d'(f(\vec{v_i}), f(\vec{0})) = d'_{f(\vec{0})}(\mathbf{D}_x f(\vec{v_i}) + \mathbf{A}\vec{v_i}). \tag{3}$$

Summarizing the prior parts of this exercise, we have:

(b) $\mathbf{D}_x f(\vec{v_i}) = 0$ for all vectors in $\mathbb{R}^2$.

(c) $d'(f(\vec{v_i}), f(\vec{0})) = 1/i$ for all i.

(d) $\exists$ neighborhoods of the zero map for which A $\neq 1/i$.

The existence of conditions under which Equation 3 does not hold means that there is no derivative of f at $(0,0)$.