Dodson & Poston Exercise VII.1.4

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Abstract

In-progress solution. Feel free to add/comment/disparage.

Let A be a linear map $T \to L(T; T')$ and define

$$\begin{aligned} A' \colon T \times T &\to T' \\ (x, y) &\mapsto (A(x))y. \end{aligned}$$

Prove:

(a) A' is bilinear

Solution Show that $A'(x + x', y + y') = A'(x, y) + A'(x, y') + A'(x', y) + A'(x', y')$, and scalar multiples of the arguments will follow.

(b) The map $\Phi \colon L(T; L(T; T')) \to L^2(T; T') \colon A \mapsto A'$ is an isomorphism.

Solution Let $\dim T = n$ and $\dim T' = m$. Then A is a linear map from n to $m \times n$ dimensions, while A' is a bilinear map from $n \times n$ dimensions to m dimensions. Therefore $\dim A = \dim A' = n^2 m$, which implies that Φ is surjective. Now, if we can show that $\ker \Phi = 0$ (which would imply injectivity)then Φ is an isomorphism.

Lemma/Definition: if k-linear map $Ax = 0 \forall x \in X$, then A is the zero map.

The kernel of Φ is 0 if $A' = 0 \implies A = 0$. If $A' = 0$ then $A'(x, y) = 0 \forall x, y \in T$. Using the definition of A', we also have $(A(x))y = 0 \forall x, y \implies A(x) = 0 \forall x$; therefore, $A = 0$. This shows that $\ker \Phi = 0$, so Φ is an isomorphism.

(c) (extend to higher orders)

Solution Proof is exaclty the same with $(x_2, \ldots, x_k)$ standing in for y.