Contents

;-- **Org** -- (org-latex-export-to-pdf)

1 PLEASE ADD YOUR STATUS HERE

(Don't worry about aligning these columns. It only takes me [PHM] a keystroke to fix it.)

Name	is working on	Notes
Peter [PHM]	read Needham Act V in lieu of VIII	
Joe V	Read thru VI.2, problems thru VI.1.4a.	
Randy [RJP]	Reading Chap VII.5&6, selected exercises	
Eric	IV.2.2c	
Barney [BKT]	reading VII.4-7, VII.4 exercises	
Anna [AHL]	Reading V, V probs	
Craig	Reading IV	
Noah [NBA]	II.3	

2 Authors' errata to 2009 printing added to Dodson-Poston folder

3 potentially useful:

3.1 scihub (website)

3.2 Frederic P Schuller (A thorough introduction to the theory of general relativity)

- lecture videos: `https://youtube.com/playlist?list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_`
- associated lecture notes: `https://github.com/lazierthanthou/Lecture_Notes_GR/blob/master/main.pdf`
- associated tutorial videos: `https://youtube.com/playlist?list=PLFeEvEPtX_0RQ1ys-7VIsKlBWz7RX-FaL`
- Note that he has another set of lectures "Lectures on the Geometric Anatomy of Theoretical Physics" with overlapping material

3.3 notes on Ch I-V

- https://faculty.etsu.edu/gardnerr/5310/notes-Dodson-Poston.htm (Bob Gardner)

3.4 I. Real Vector Spaces

- https://textbooks.math.gatech.edu/ila/complex-eigenvalues.html (Dan Margalits)

3.5 II. Affine Spaces

- https://www.cis.upenn.edu/~cis610/geombchap2.pdf (Gallier, J., Geometric Methods and Applications, Springer)
- Marcel Berger, Geometry I. (Springer)
- Marcel Berger, Geometry revealed (Springer)
- Marcel Berger, Problems in Geometry (Springer)
- Stillwell, 4 pillars, p 150 (Springer)

3.6 III. Dual Space

- Dual Spaces: https://sites.math.northwestern.edu/~scanez/courses/334/notes/dual-spaces.pdf
- Quora answer to dual space and matrix rows question: https://qr.ae/pNkXNa

3.7 IV. Metric Vector Spaces

- Metric Spaces: Set Theory and Metric Spaces by Irving Kaplansky (AMS)
- https://www.win.tue.nl/~lflorack/Extensions/2WAH0CourseNotes.pdf
- co-vectors and contours: https://www.youtube.com/watch?v=LNoQ_Q5JQMY&list=PLJHszsWbB6hrkmmq57lX8BV-o-YIOFsiG&index=7

3.8 V. Tensors and Multilinear Forms

- description of tensors and co- and contravariant vectors (from the Quora website): https://qr.ae/pN5NLi
- Basic discussion of tensor products, "multiplying" vectors: https://www.math3ma.com/blog/the-tensor-product-demystified
- Basic discussion of tensor products, intro to universal property: https://jeremykun.com/2014/01/17/how-to-conquer-tensorphobia/
- Alternate discussion of tensor products: https://www.dpmms.cam.ac.uk/~wtg10/tensors3.html
- Advanced discussion of tensor products: https://kconrad.math.uconn.edu/math5211s13/handouts/tensorprod.pdf (Keith Conrad)
- Youtube series of videos "Tensors for Beginners": https://www.youtube.com/watch?v=8ptMTLzV4-I
- Proof of Linear independence of tensor products http://mathonline.wikidot.com/linear-independence-properties-of-tensor-products-of-normed

3.9 VI. Topological Vector Spaces

- Steen and Seebach, Counterexamples in Topology (springer)
- Stillwell, Classical Topology and Combinatorial Group Theory (Springer GTM)
- Frederic Schuller Lecture on Topological Spaces - https://youtu.be/1wyOoLUjUeI
- motivating compact sets: https://youtu.be/1SpnLPcPRU0 (~10 min)

3.10 VII Differentiation and Manifolds

- Tu's text on Introduction to Manifolds - http://im0.p.lodz.pl/~kubarski/AnalizaIV/Wyklady/L-Tu-1441973990.pdf D&P VII.5-7 == Tu Sec 14.
- Manifold's primer by Littlejohn (in folder under Dodson+Poston/Chapter 7)
- Example manifolds article (in folder)

- A context for tensor bundles - https://unapologetic.wordpress.com/2011/07/06/tensor-bundles/
- Diff Geo Course & Notes - http://www.math.toronto.edu/mgualt/courses/18-367/
- Manifold videos (bright side of mathematics): (~ 10 min) https://youtube.com/playlist?list=PLBh2i93oe2qvRGAtgkTszX7szZDVd6jh1
- Thurston, Three-Dimensional Geometry and Topology, Volume 1 https://www-jstor-org.alumproxy.mit.edu/stable/j.ctt1k3s9kd There is only one volume.
- Needham, Visual Differential Geometry and Forms https://www.vdgf.space/ I haven't seen this book yet, but it sounds promising.
- Guillemin & Pollack, Differential Topology : https://math.ucr.edu/~res/math260s10/old/difftopGP.pdf
- , A Panoramic View of Riemannian Geometry (Springer, 2003)
- Nakahara
- Fortney (A Visual Introduction to Differential Forms and Calculus on Manifolds), Birkhauser (Springer)
- Bachman, D. A Geometric Approach to Differential Forms, Birkhauser (Springer) 2012

Eigenchris: Lie Bracket, Flow, Torsion Tensor - Tensor Calculus #21 Youtube)

- https://www.youtube.com/watch?v=SfOiOPuS2_U&list=PLJHszsWbB6hpk5h8lSfBkVrpjsqvUGTC
index=24

4 AGENDA (reverse chronological)

4.1 *[2023-07-23 Sun]*

- connections "This suggests...", pg 216. Where is (Xf)Y first mentioned in each of the texts? Tu, Diff Geom, Lee, Riemannian Manifolds, Lee, RM: connections Geometrically, what is (Xf)Y? Algebraically, (Xf) is a scalar, calculated as $(\partial_X f)$.

  example with X = i, Y = xi + yj, f = $(x^2 + y^2)^{-1/2}$. (see here for useful derivatives)

  - useful derivatives i (x/r) = y^2/r^3 i (y/r) = -xy/r^3 = j (x/r) j (y/r) = x^2/r^3

- Does a directional derivative (∇_v) turn a p-form into a (p+1) form? Lee (RM) Lemma 4.7 is suggestive.

- Exterior derivatives – need to discuss this vs. Covariant derivatives Lee (SM) chapter 14, Needham VDGF act V

- Π, projection operator resolution on "horizontal/vertical"? p 213

- Stackexchange, Michael Albanese compatability: "Yes, =()()=. –Michael Albanese Mar 31 at 21:43" DiffGeo derivatives discussion: "In differential geometry, there are several notions of differentiation, namely:

  - Exterior Derivative,
  - Covariant Derivative/Connection,
  - Lie Derivative, L."

4.2 *[2023-06-03 Sat]*

4.2.1 index D&P with references to Tu (DG), Needham, and whatever else we come across.

4.2.2 for Craig – see

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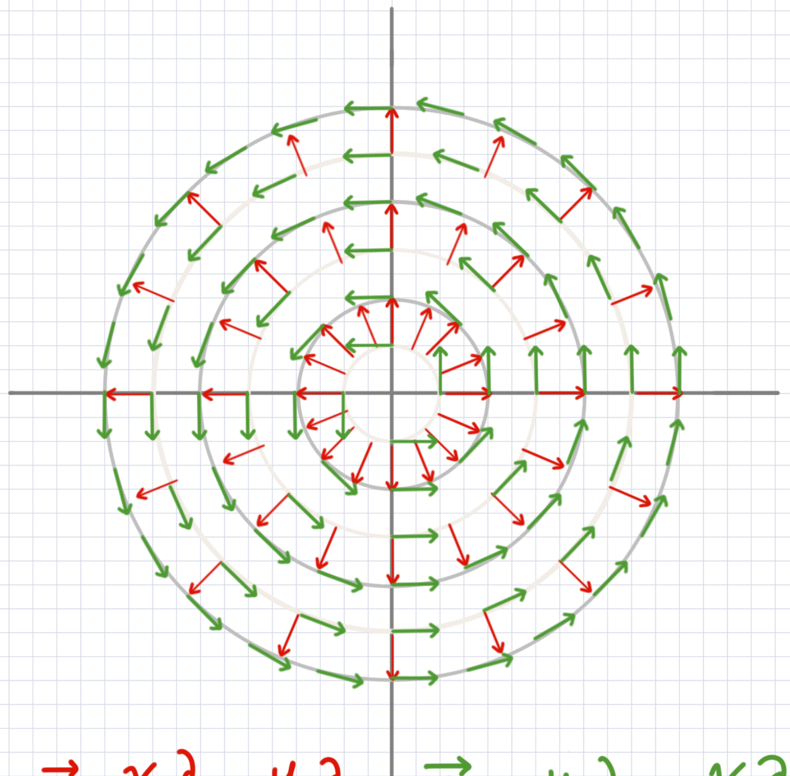
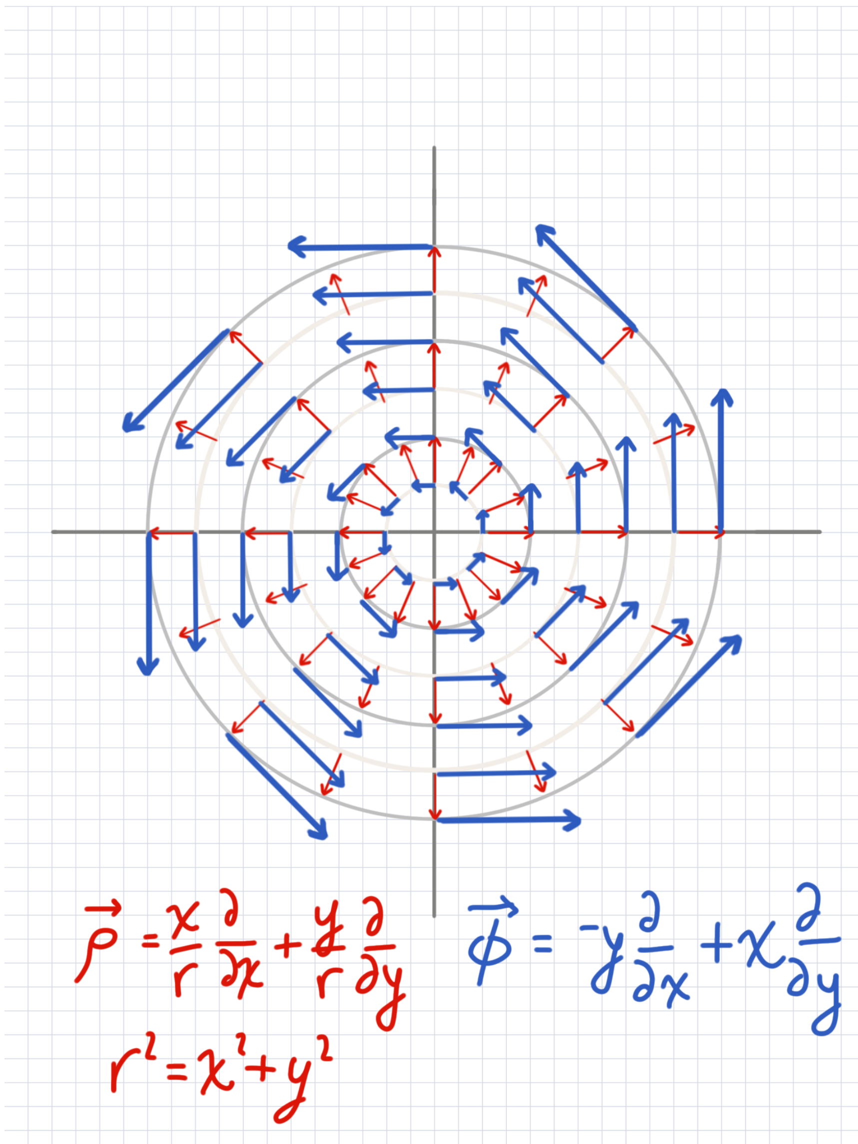


$$\vec{\rho} = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} \qquad \vec{\theta} = \frac{-y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}$$

$$r^2 = x^2 + y^2$$



4.3 *[2023-05-07 Sun]*

- Tu, Differential Geometry
- Needham, VDGF

4.3.1 Tu def 4.8, on vs along

example of inconsisent usage:... (sec 5.2 p30) "let N be a C^∞ unit normal vector field on M (Figure 5.1)." N is not in the tangent space (and is constructed to be normal to it), so should't the phrasing be "normal vector field **along** M"?

Barney proposes that on is intrinsic and along is extrinsic

4.3.2 Shape operator

- Tu 5.2 (p 30)
- Needham 15 (p 149)

4.3.3 first and second fundamental forms

- Tu 5.4 (p 35)
- Needham 15.9 (p 164)

4.3.4 EFG vs ABω

- Tu 5.4 (p 35)
- Needham 4.3 (p 34-37)

4.3.5 is torsion the same in both books?

Maybe not..

Tu: 6.2 (p 44) torsion of the connection

Needham: ch 9 (p 106) torsion of a curve

4.3.6 D&P sections in VII

1. Curves and Tangent Vectors
2. Rolling without turning (pre Levi-Civita connection)
    - without slipping?
    - reference Needham Chapter 22.2, 23.2 (Levi-Civita, intrinsic)
    - reference Tu DG, section 14, p 103

- Barney: how do we calculate a geodesic algebraically? it is a curve with covariant derivative 0 (intrinsic).

3. Differentiating sections (Christoffel symbols)
   - Christoffel symbols
   - The funny ∇: the intrinsic or covariant derivative $\mathrm{D_V}$ (Needham) or DV/Dt (Tu)
   - Needham (still 22 and 23?)
   - Tu (??)
4. Parallel transport
   - Needham, Act IV, esp chapters 22,23
5. Torsion and symmetry
   - Needham: ch 9 (p 106)
   - Tu: 6.2 (p 44)
6. Metric tensors and connections (∇)
7. Covariant differentiation of tensors

4.4 *[2023-04-02 Sun]*

4.4.1 Find exercises, or write up the point of the D&P ex.

4.5 *[2023-03-05 Sun]* start VIII

4.5.1 Let's have solutions for 7.2 and 7.4

4.5.2 Start into VIII.

4.6 *[2023-02-05 Sun]*

4.6.1 Read through VII.7 including the proof of 7.04

4.6.2 All exercises are ok except 7.2 and 7.4

4.7 *[2023-01-08 Sun]*

4.7.1 Sec VII.6

4.7.2 Ex VII.6.1

see solution discussed

4.7.3 Ex VII.6.2

- typo, should be $c(t) = (t/3)^3$ and $c(t) = 0$.

  if $c(t) = (1/3)t^3$ then $c'(t) = t^2$

  want $c'(t) = v(c) = c^{2/3} = (1/3)^{2/3}t^2 \neq c'(t)$

  whereas with the correction $c'(t) = (t/3)^2$ $v(c) = c^{2/3} = (t/3)^2 = c'(t)$

- reparameterization is mentioned in VII.5.02, $t \to t+m$ for any constant m

  so in this case, another curve that is a reparameterization of c(t) could be $h(s) = c(t+s)$ $h(s) = (t+s)^3/3^3$

- v is not differentiable because it goes to ∞ at 0

4.7.4 Sec VII.7

- discussed, see Notes.org file (D&P section)
- Theorem 7.04 proof needs to be looked at carefully
- We will try to have solutions to all problems in this section by next meeting.

4.8 *[2022-12-11 Sun]*

4.8.1 good book: Berger: Panoramic View of Riemannian Geometry

4.8.2 Ex VII.4.6

- homomorphism: [RP] linear for vector spaces (?)
- isomorphism: bijective homomorphism (I.2.03 p25 "linear bijection", for vector spaces.)
- homeomorphism: bijection, continuous in both directions
- diffeomorphism: homeomorphism, C^k in both directions

1. G_{up} and G_{down} are inverses of each other. Linearity => C^k? Ex VII.1.3e: if f is an affine map, $D_x f$ is the linear part of f. ie, the derivative of a linear map is itself.
2. coordinate formalae $G_{down}(x) = g_{ij}x^j$ etc.

4.8.3 Sec VII.5 Curves

1. going back to pg 154-155
   - f: $R^n \to R$, scalar function
   - $D_x f = [\partial_1 f, \partial_2 f, \ldots, \partial_n f]$ Jacobian, and here a covector
   - f^*: $D_x f$ above. and it's a vector.
2. p 192 (**) $ds^2 = \ldots$ [1 1/4 0] G = [1/4 1 0] [0 0 $(x^1)^2+1$]
3. Ex VII.5.1b: typo MINIMUM, not maximum

4.8.4 Ex VII.5.5, esp part b

- what do they mean by "even if f is not affine" (not linear/translational)

4.9 *[2022-11-06 Sun]*

4.9.1 [BKT] D&P, Ex. V.1.1 - Is Peter's soln. showing subspace, not vector space?

yes, because of laziness.

4.9.2 Randy's Halloween missive on tangent spaces

4.9.3 VII.4 readings: Lee, Chpt 3; Tu, Sections 2-6,8,12,17,18.

Tu's sections are helpful even just to skim over. Just a taste (p. 35): "If x,y, and z are the coordinates on R3, then dx, dy, and dz are 1-forms on R3. In this way, we give meaning to what was merely a notation in elementary calculus."

4.9.4 Sec VII.4.01 covariant vectors

Barney's favorite section in all of D&P because of the connection to differential forms

basis covectors: $\{dx^i\}$

$dx^i = d_{x^i(u)}$ ^ $D_u x^i$

4.9.5 Sec VII.4.02 contravariant vectors (aka, vectors)

basis vectors: $\{\frac{\partial}{\partial x^i}\}$

$\frac{\partial}{\partial x^i}\hat{\imath}$?

4.9.6 Sec VII.4.03 tensors of higher degree

form basis from combinations of basis vectors

4.9.7 Sec VII.4.04 transformation formulae

- vector components: $t'^i = t^j \frac{\partial x'^i}{\partial x^j}$
- covector components: $v'_i = v_j \frac{\partial x^j}{\partial x'^i}$
- warnings about "one-overs" vs matrix inverses.

4.9.8 Sec VII.4.05 Raising and lowering indices

- see V.1.13

4.9.9 Ex VII.4.1

1. components of a smooth function are smooth (ex??) chain rule proof in D&P Ex VII.1.6 (if vs iff?)
    - given: ϕ_2 and ϕ_1 are smooth because the manifold is defined as such.
    - (Tu) if function is smooth, then components are smooth (also maybe a D&P exercise)
2. same thing...

4.9.10 Ex VII.4.2

nothing to talk about

4.9.11 Ex VII.4.3 wtf - is this even a problem?

circular reasoning?

4.9.12 Ex VII.4.4 Leibniz rule

nothing to talk about

4.9.13 Ex VII.4.5 transformations

- $({}^1,{}_2)$: $\frac{\partial x'^i}{\partial x^i} \otimes \frac{\partial x^j}{\partial x'^j} \otimes \frac{\partial x^k}{\partial x'^k}$
- $({}^0,{}_2)$:
- $({}^3,{}_1,{}^1,{}_2)$:

4.9.14 Ex VII.4.7 Sure, but we don't care...

4.10 *[2022-10-09 Sun]* VII.3 DONE

- VII.3.01: Theorem
    1. terrible proof
    2. Tu (Intro to Manifolds) p. 129
    3. Lee (Intro to Smooth Manifolds) p. 66
    4. youtube vids (Frederic Schuller - Construction of the Tangent Bundle - Lec.10 from Lectures on the Geometric Anatomy of Theoretical Physics) `https://youtu.be/XZcKSoI17r0`
- VII.3.1 product, direct sum, direct product, cartesian product
    1. is a vector space
    2. dimension
    3. "Manhattan distance" is a difference function
    4. isomorphism from (s,t) $\mapsto s{+}tDoweeverhaveasituationwhereS+ Tdoesn'tlooklikeP(x) + kerP?Topologyproof(ugh.)(Munkres?)$

5. direct sum of tangent spaces and derivative thereof notationally, this is a mess, but it's fairly straight-forward.

- VII.3.2 direct product of manifolds
    1. charts, dimensions
    2. smoothness (typo?) should be M $\to$ M $\times$ N
    3. show tangent space of product manifold is isomorphic to direct sum of tangent spaces (use problem 1 + charts)
    4. example, torus. give charts
    5. product tensor field is metric if the components are
- VII.3.3

1. continuity proof
2. show "by considering what it means to be a member of each set" where does the missing paren go? BT: after (W) PM: before (W) maybe it doesn't matter bc they are matrices anyways *** we are not sure about the essenence of this problem ***
3. pathalogical case with discontinuous derivative Choice of charts matters. [BT]

- VII.3.4 $T^k{}_hM$ is an $(n + n^{k+h})$ manifold modelled on $X\times(T^k{}_h)$
- [BKT] Tu, p.24: Lemma 3.11 question.

4.11 *[2022-08-28 Sun]* VII.2 DONE

- VII.2.7 commentary

1. use determinants
   - Randy: On pg. 63 (Sect 6.3), Tu establishes that the coordinate (chart) functions and on smooth manifolds are smooth. The proof follows basically the approach described in the 1st reference, namely, that
     (a) A map F between manifolds N and M, F : N → M, is smooth if F $^{-1}$ is smooth and vice versa, where and are chart functions in M and N, respectively.
     (b) Let (U) be the identity chart function for (U) to itself, i.e., (U) : (U) (U). This gives the chart ((U), (U)) on ^{n}. For the chart (U,) on N, we have the map (U) $^{-1}$: (U) (U) which is of course smooth. Using 1) above, can then be seen to be smooth.
   - above, use Tu, pg 63, sec 6.3 to show that ϕ is differentiable (ϕ is smooth)
   - use "inverse function theorem" to show that $D\phi$, $D\psi$ are injective
2. f: $(\theta) \to (\theta, z)$ **Peter - fix typo before eqn 3.**
3. done
4. done!

- VII.2.8 commentary

1. (θ, p3) -> (θ)xR

2. c: M is rod in R^2. f: (rod in R^2) to (length of rod) (x1,y1,x2,y2): f: (p1,p2,p3,p4)-> ||(|p3-p1|, |p4-p2|)|| "and f^←(1) is the set in question which we now know is a manifold."

4.12 *[2022-08-07 Sun]*

- VII.2.5:

  1. mostly algebraic
  2. Is Randy's solution sufficient?
     - subspace: closed, ...
     - iso: linear, bijective.
  3. can we do better than the hint?

- VII.2.6:

  1. mostly algebraic
  2. This part asks us to show that $D_x\phi = (D_{\phi(x)}\phi^{\leftarrow})^{\leftarrow}$ f → ϕ N → X u → x

- VII.2.7: $D_x f$ injective, $\dim M \leq \dim M'$ $f\colon M \to M'$ is C^k, $x \in M$ $N \subset M$, N is a neighborhood of M containing x

  1. use Cor VII.1.05 p 156 "injective" in 1.05 : $\det(D_x\ f) \neq 0$
  2. for $U \subset M'$, $\phi\colon U \to \mathbb{R}^n$ is a chart.
  3. TYPO: should be "admissible on M" (no prime)
  4. $f|_N : \mathbb{R}^n \to \mathbb{R}^n$

- VII.2.8: $D_x f$ surjective, $\dim M > \dim M'$ $f\colon M \to M'$ is C^k, $x \in M$ $N \subset M$, N is a neighborhood of M containing x

  1. use Inverse function theorem VII.1.04 p 156 "all of VII.2.7 for surjective D_x"
  2. why $x^1 \to x^{m-n}$?
  3. are there typos? should n be m in this part?

- VII.3.1

  1.
  2.

3.
4.
5.
6. Re: Randy's ? on notation: let $Y = X \times X'$ and $y = (x, x')$ then $T_y Y = T_{x,x'}(X, X')$

4.13 *[2022-07-10 Sun]*

- VII.2.3 example [PHM]
- VII.2.5 [RJP]
- VII.3.01 [BKT]

4.14 *[2022-06-12 Sun]*

- done up to problem VII.2.04

4.15 *[2022-05-15 Sun]*

- done up to problem VII.1.06

4.16 *[2022-04-20 Wed]*

- discussion on Munkres pg 76-77, Fig 12.1, Fig 12.2. Three discrete element topology. Subsets of {a,b,c} {} {a} {b} {c} {ab} {bc} {ca} {abc}

  [...] denotes a "collection"

  X = {a,b,c} "is a topological space with topology T1", which we can write as (X,T1) T1 = [{},{a},{b},{c},{ab},{bc},{ca},{abc}] "is a topology"

  [{a},{b}] != {a,b} "The collection containing subset a and subset b is not the same as the subset containing a and b"

  There are 29 topologies out of a possible $2^{(2^3\text{-}2)} = 64$ collections of subsets that include {} and {abc}.

  For 4 elements, there are 355 topologies (`https://oeis.org/A000798`) out of $2^{(2^4\text{-}2)} = 2^{14} = 16384$ possible collections of subsets...

4.17 *[2022-03-13 Sun]* DONE WITH VI!

- Ex. VI.4.4 (Lemma 4.07) -a. Boundedness along basis vectors -b. Generalization to affine spaces
- Ex. VI.4.5: induced topology -a. Induced topology is given by the induced metric (not clearly defined in D&P) -b. Transitivity of openness, proven with metrics -c. Transitivity of closedness, proven with metrics -d. (*) Repeat b and c with topologies & induced topologies -e. Continuous function restricted to a subspace is continuous on the subspace. (This is surprising?) (barney: proof by contradiction?) -f. Sequence in subspace converges to the same point in the larger space. (This is surprising?)
- Ex. VI.4.6 impossibly badly worded -supremum: least upper bound -infimum: greatest lower bound
- Ex. VI.4.7
  - p145 Rmk: "If you are still unconvinced that topological reasoning is necessary to prove that we can diagonalise symmetric operators, do Exercise 7."
  - Not sure that this problem does (pedagogically) what it claims to do.
  - Joe: Q and R same algebraic props, but R is complete but Q is not. and completeness is a topological property.
- Ex. VI.4.8 yet another pointless sequence problem?
- Ex. VII.2 (Barney & Randy's writeups)
- EX. VII.1.1
  1. Joe's solution (see whiteboards)
- Ex. VII.1.2
  - Barney (pass)
  - Joe (wrong visualization?)
  - PHM will write this up (maybe)
- Ex. VII.1.3
  - type on part b (should be d'^{-1})

- Ex. VII.1.4
- Ex. VII.1.5
  - B: can you use topological reasoning? or are we stuck with ϵ-δ?
  - J: all of these required a norm
- Ex. VII.1.6 (generalized chain rule)
  - B: Spivak Calc on Manifolds has a proof.
- Ex. VII.1.7 (example of differential but not C1)

4.18 *[2022-02-13 Sun]*

- Compact Sets (Barney) example: f(x) = 1/x (unbounded on "]0,1]") "local/global heartbreak" finite subcovers – only needing a finite # of subsets allows. . . .
- Continuity (Barney) See TappK, Appendix A (in $\text{Stillwell}_{\text{NLT}}$ dir), Lem A.6 "The notion of convergence is topological by the second definition below, although it may not initially seem so at first:". . . Defn A.7
- Ex. VI.4.1 – see solution Chapter_{06} [PHM] Cor. VI.1.06 f_{inv}(V) for open V is open. (continuity)
- Ex. VI.4.2 no big problems
- Ex. VI.4.3 part b ERROR: Definition 1.03 (not 1.05 as stated)

4.19 *[2022-01-16 Sun]*

- [PHM] Cross references between D&P VI.4 (Compactness and Completeness) and Stillwell NLT Chapter 8
  - VI.4.02 Lemma (Bolzano-Weistrauss) == NLT Ex 8.4.1,8.4.2 (p170)
  - VI.4.03 Theorem (existence of a convergent subsequence) == NLT Ex 8.4.3, 8.4.4 (p 171)
  - VI.4.10 Theorem (image of continuous fn) == NLT sec 8.5 (p 171)

- [PHM] offered as a concrete example (not worked out, but just to think about) the mapping bewteen a 2D potential surface and a topographic surface with the same dynamical properties (a la Lagrange points in a rotating frame of two gravitating bodies). Note that height is NOT equal to potential. In particular, the max horizontal force applied by the topographic surface occurs at angle 45 deg – beyond that the vertical acceleration dominates.

4.20 *[2021-11-21 Sun]*

- (RJP) still would like to see group unpack Ex. VII.2.3
- Readings in Tu (Intro to Manifolds, Springer) sections 2 (Tangent Vectors as derivations), 5 (Manifolds), 6 (Smooth maps), 8(Tangent Space) are relevant to Chap VII of D&P

4.21 *[2021-10-24 Sun]*

- discuss RJP summary of last meeting; Peter's questions
- Ex. VII.2.3a&b: Clarify def of tangent vectors as equivalence classes
- extra bonus question: tangent vectors and tangent space; tangent to what?

4.22 *[2021-09-26 Sun]*

- Sec. VI.1.02: [PHM] I don't understand the space of the higher derivatives ($L^k(T;T)$). The mapping for $\hat{D}f$ makes sense to me, but I don't see how to extend that to higher derivatives.
- Sec. VI.1.03, pp 154-155: [PHM] I can't follow the discussion surrounding $f^*(x)$. If $f : \mathbf{R} \to X'$, then isn't $f(x)$ a vector? I'd like to hear some discussion about the "Three entries" on pg 155.
- Fig VI.2.4 pg 162, sec 2.01: [PHM] How is this showing that chart ϕ_b is gives a differentiable map and that ϕ_a does not? (See text on pg 163)

4.23 *[2021-08-29 Sun]*

- VII.1.x

4.24 *[2021-08-01 Sun]*

- VI.3.6b
- VI.3.7
- VI.3.8
- VI.4.1
- VI.4.2
- VI.4.3
- VI.4.4
- VI.4.5
- VI.4.6
- VI.4.7
- VI.4.8

4.25 *[2021-07-11 Sun]* (10A-12P)

- VI.3.2
  - b: **any** collection of subsets?
  - d: see Randy's solution
- VI.3.3
- VI.3.4
- VI.3.5 square, diamond, Euclidean metrics
  - c: Lemma 1.10? Is he referencing the right section? (Use Lem 1.12 (Joe))
- VI.3.6
  - a: Munkres Thm 31.2 for topological subspce, see D&P p 143.

4.26 *[2021-06-20 Sun]* (10A-12P)

- For next time: Review Randy's writeup on Ex VI.2.2c
- [RJP]need help w/ Ex.VI.3.2d

4.27 *[2021-05-23 Sun]*

4.27.1 VI.1.5: boundary, closure, continuity

1. can this be done using parts a & b?

4.27.2 VI.1.6: Hausdorff, open/close

1. Where does the Hausdorff axiom come into play here? If the

topology is not Hausdorff, we can have x != y and d(x,y) = 0, but {x} is still a boundary point in that every open ball contains poinsts inside (eg, x) and outside (eg, y).

4.27.3 VI.1.7: unit circle map: X->Y is a continuous bijection but not a homeomorphism.

the open neighborhood around (1,0) in Y of arclength 2*pi*d maps back to the intersection of]0,d[and]1-d,1] in X. Is this an open set in X?

4.27.4 VI.2.2c: sequence in 2nd part of Lemma 2.02 converges to x

How to do this formally?

4.28 *[2021-05-02 Sun]*

4.28.1 VI.1.1: eps/delta excercise - joe (whiteboarded)

4.28.2 VI.1.2: metric/semimetric

4.28.3 VI.1.3: open/close – Expect spirited discussion on d

`https://en.wikipedia.org/wiki/Ball_(mathematics)` `https://math.stackexchange.com/questions/2231706/boundary-of-any-open-ball` `https://math.stackexchange.com/questions/108010/when-is-the-closure-of-an-open-ball-equal-to-the-closed-ball/108017#108017`

4.28.4 VI.1.4: metric/pseudometric space

4.29 *[2021-04-11 Sun]*

4.29.1 clean up on chap V

PHM - 6b (discussed), 7b (discussed), 8a Joe - 4, 6b (discussed) Eric - 9 (discussed) Randy - 7b (discussed), 10a, 11b, 11c

4.29.2 TODO!!: All: read Randy's solutions and comment.

4.29.3 VI.1

1. beware ex VI.1.3d `https://math.stackexchange.com/questions/2231706/boundary-of-any-open-ball`

4.30 *[2021-03-21 Sun]* End of Chapter V!

to do: write up 6b, 7b, 8a, 10a

(RJP) What is difference between definitions of tensor product in V.1.03 (pg. 100) and Def. V.1.04 (pg. 102)? (RJP) What does the notation $L(X_1;X_2)$—>$X^*{}_1X_2$ mean exactly?

4.30.1 Ex V.1.8

- (a) associativity
- (b) permutation

4.30.2 Ex V.1.9 contraction is well defined

4.30.3 Ex V.1.10 $\Psi A(x, y) = Ax \cdot y$

4.30.4 Ex V.1.11 skew symmetric multilinear maps and n-dim volumes

4.30.5 Ex V.1.12 volume independent of basis (no one has looked at this yet)

4.31 *[2021-02-28 Sun]*

4.31.1 Ex V.1.5/Sec 1.06 (straightforward [PHM,])

4.31.2 Ex V.1.6/Sec 1.06

part b is one of...

1. Barney: pick basis, work out algebra. do not assume that x' and y' are scalar multiples of x and y.

2. Poorly stated (need x' and 'y are scalar multiples of x and y respectively) and trivial.

3. Incorrect.

4.31.3 Sec 1.07 - PHM will write up proof of lemma

4.31.4 Ex V.1.7/Sec 1.08

a - easy b - break into basis components?

4.31.5 Ex V.1.8

1. see Conrad Thm. 5.2 and voila.

2. see Conrad Thm. 5.1 and voila.

[BKT] Essentially, a, b proved by repeatedly invoking the respective theorems.

4.32 *[2021-02-07 Sun]*

4.32.1 what is $a \otimes b$?

- Joe's paper
- Conrad, p5 and p12

4.32.2 [BKT] Sec 1.08 Lemma/ p.104

- why does $\mathrm{f_{hat}}(\mathrm{g} \otimes \mathrm{x_2})) = 0$?

4.32.3 Ex V.1.4/Sec 1.03 (heavily referenced throughout this chapter)

1. In showing A spans B, I [PHM] thought I should show things in A

are linearly independent and that dimA=dimB, but the dimensionality part seems to be a result of part c of this problem.

1. straightforward [PHM]

2.

3. existence and uniqueness of f-hat

4.33 *[2021-01-10 Sun]*

4.33.1 Highlights on the Tensor Product

JV - upper and lower mappings AHL - eigenchris videos - (BT: very clear)

4.33.2 Reviews of other readings

- Gardner
- Bradley
- Conrad - PHM to find reference to other mathematician…
- Kun - tensorphobia (recommended by BT)

4.33.3 Ex V.1.1/Sec 1.01

- [PHM] Seems trivial, but…

do we have to be careful in checking all the properties on pg 18? Closure proof in Chapter 5 directory.

4.33.4 Ex V.1.2/Sec 1.02

- [PHM] Easy to hand wave on these two examples.

Does anyone have rigorous proofs?

4.33.5 Ex V.1.3/Sec 1.03

- [PHM] discuss multilinearity of the range (to-set) vs. multilinearity of the map.

discussion on p 101: "Tensor products…in $L^2(R^2;R)$ do not consitute a vector space" … "sits naturally inside, and spans $L(X_{1,\dots,X_n};R)$"

[PHM] error in my reading. the to-set here is a vector space (not multilinear)

4.34 *[2020-10-25 Sun]*

4.34.1 Sec IV.3

[BKT] Lemma 3.14: go over proof by Gardner/D&P Ex. 5 & 7: walk-thru the basis construction, orthogonality of A

4.34.2 Sec IV.3

finished discusssion of problems

4.34.3 Sec IV.4

discussed Ex IV.4.1 remaining problems not so hard.

4.35 *[2020-09-20 Sun]*

4.35.1 Sec III.1.07, p.61

[BKT] "Evidently $I_X* = (I_X)*$, ..." how did they arrive at this equation? equation reads as "Identity on the dual space is the dual of the identity." see defn of dual bottom of pg 57

4.35.2 Sec IV.1 & IV.3

[BKT] Why doesn't standard inner product on R follow summation convention? When does the convention apply? (pg 67 IV.1.03)

4.35.3 Sec IV.2.08

- [BKT] On p. 81, the Commutative Diagram is a bit confusing. How to understand it? Also, the dot product drops away in derivation, for a reason I don't understand.
- What is the significance of adjoint and self-adjoint operators?

4.35.4 Sec IV.3

1. Ex IV.3.5 Lorentz metric

   [RJP] What exactly is the operation of $F_{downarrow}$?

4.36 *[2020-08-30 Sun]*

4.36.1 Sec IV.1

1. Ex IV.1.7a a norm is a partial norm [PHM] discussed in email
2. Ex IV.1.7b inner product is a norm
3. Ex IV.1.7c indefinite metrics have partial norms For Ni, an indefinite metric G(x,x) is neither positive nor negative definite. This does not force G(x,x) = 0 for some x != 0, but G(x,x) **could** be zero for some nonzero x.

   QUESTION: Does G(x,x) have to be continuous?+
4. Ex IV.1.7d polarization identity
5. Ex IV.1.8 F* properties wrt F properties

4.36.2 Sec IV.2

1. Ex IV.2.1a orthogonal complement
2. Ex IV.2.1b basis
3. Ex IV.2.1c dimensions
4. Ex IV.2.1d projection in affine space
5. Ex IV.2.2a det of A^T
6. Ex IV.2.2b orthogonal ==> det = +-1
7. Ex IV.2.2c an isometry is an isomorphism
8. Ex IV.2.2d isom into is injective and surjective
9. Ex IV.2.3 isometries in H^2
10. Ex IV.2.4 preserving dot squares ==> orthogonality Find a good counterexample where ||x|| is preserved, but polarization identity doesn't work out.

4.37 *[2020-08-09 Sun]*

4.37.1 Sec IV.1

1. Ex IV.1.1 [PHM] geometric vs. algebraic solutions

2. Ex IV.1.2 implications of bilinear properties

	i	ii	iii	iv	v	vi
i	-	no	no	no	no	no
ii	no	-	no	X	X	=>
iii	no	no	-	no	no	no
iv	no	X	=>	-	X	X
v	no	X	=>	X	-	X
vi	no	no	no	X	X	-

   - no = no implication
   - X = contradiction
   - => = implied
   - i. symmetric
   - ii. antisymmetric
   - iii. nondegenerate examples of nondegenerate and degenerate
   - iv. positive definite
   - v. negative definite
   - vi. indefinite

3. Ex IV.1.3 Dot product is an inner product

4. Ex IV.1.4 Bilinear forms are a vector space [PHM] do these relations follow from anything?

5. Ex IV.1.5 Properties that apply to subspace [PHM] all but indefinite? [JV] nondegeneracy does not work either

   Follow-up to our discussion today on IV.1.5: Consider the bilinear form on R^2 defined by F((x1, x2), (y1, y2)) = x2y1 - x1y2.

   This is nondegenerate: for any (x1, x2) not equal to zero, we can find a (y1, y2) such that F((x1, x2), (y1, y2)) is not zero. (If x1 is not zero, let y1 be zero and y2 nonzero. If x2 is not zero, let y2 be zero and y1 nonzero.)

However, restricted to the 1-dimensional subspace spanned by (1, 0), F is identically zero and therefore degenerate.

Hence, it is possible for a bilinear form to be non-degenerate on a vector space X but not on every subspace of X.

6. Ex IV.1.6 null vectors of an indefinite metric do not form a subspace [PHM] 1. is there a geometric solution? [PHM] 2. I needed an additional condition (details for discussion)

4.38 *[2020-07-19 Sun]*

4.38.1 Sec II.3

1. Ex II.3.4b see notes from RJP

2. Ex II.3.9a see proposed solutions by PHM and NBA

4.38.2 Chap III - dual space

see notes by JV see notes by NBA ($\text{duality}_{\text{daily}}$)

4.39 *[2020-06-28 Sun]*

4.39.1 sec III.1

- sec 1.02: linear functionals [PHM] do there always "look like" dot products?

- sec 1.03: dual map A* [PHM] note that this is a map from Y* to X*

- sec 1.04: dimensionality of dual space

- sec 1.05: avoid this

- sec 1.06: "dual basis" (matrix representation of dual map)

- sec 1.07: change of basis [PHM] essentially the same message as 1.06

- sec 1.08: notation

- sec 1.09: double duals (see Spivak vol 1 p. 107 and Axler 3F for comparison) [PHM] apparently, there is no unique (or it well-defined) map $X \to X*$

- Ex 1a [PHM] See Ex. 34 in Axler's Linear Algebra, sec 3F.

- Ex 1b
- Ex 2

4.40 *[2020-06-07 Sun]*

4.40.1 Chapter 2 in a nutshell [PHM]

- points (coeff 1) and vectors (coeff 0) vectors are defined as differences of points, hence the zero-coefficient.
- affine combinations (sum of coeffs == 1)
- affine transformations (A) for L in Reals; x,y in affine space X DEFINITION:: A((1-L)x + Ly) = (1-L)Ax + LAy every affine transformation consists of a "linear part" (in practice, a matrix or linear operator) and a "constant" part (vector in T).
- Connection to Stillwell section 4.6
- [AHL] another reference on affine spaces: `https://www.cis.upenn.edu/~cis610/geombchap2.pdf`

4.40.2 Sec II.1

- Ex 1: [PHM] note that (x,y) on p 44 is not a coordinate pair. it is a pair of points. (x,y) + (x,z) has no meaning outside of RHS of equation on p 44 (Sec II.1.02)
- Ex 2a: [PHM] consider vector space from 1-D affine space. this is an example of d_x not being bijective.
- Ex 2b: [PHM] proof part 1
- Ex 2c: [PHM] proof part 2 (vec ss + translate => affine)
- Ex 2d: [PHM] this should precede (b) and (c)
- Ex 2e: [PHM] this should precede (b) and (c)

4.40.3 Sec II.2

- Ex 1a: [PHM] repeated affine combination for 3 points
- Ex 1b: [BKT] what does the inductive set-up look like? [PHM] I used $(_{i.xi})$ == repeated combination eqn, so it is independent of permutation. On the whole, not a very useful equation.
- Ex 2a: Affine hull H(S)
- Ex 2b: S if affine iff H(S) = S.
- Ex 3: Simplex: k+1 linearly independent points are needed to span a k-dimensional affine space.

4.40.4 Sec II.3

- Ex 1a: [BKT] is subtracting points allowed? [PHM] to the extent that d(x,x)=0 and d(x,y)=y-x
- Ex 1b:
- Ex 2:
- Ex 3: Use definition of affine map (3.01) to prove...
- Ex 3a:
- Ex 3b:
- Ex 3c: [BKT] what is the picture used to prove this? [PHM] use 3.01.
- Ex 4a: [AHL] why is the empty set a flat? [PHM] formally, it satisfies the axioms of an affine space/subspace.
- Ex 4b: [RJP] AX{y}=single pt {y}; begs question of inverse of A:Rn->Rq, q<n [PHM] probably cheating, but I leaned on group theory for this one [RJP] confusion due to wording of problem resolved: image of affine map is a single point => all vectors in domain live in null space
- Ex 4c:
- Ex 5a:
- Ex 5b:

- Ex 6: [BKT] want to use Lemma I.2.03, but why must B be linear?
- Ex 7:
- Ex 8: [PHM] Seems to me that they need to restrict the definition of the map M more, I can't really see what they're driving at here.
- Ex 9a:
- Ex 9b: [PHM] Is this an example of Ex 9a where H(P) $<$ X?

4.41 *[2020-05-16 Sat]*

4.41.1 [AHL]

- p. 48 Ex II.1.2d typo? Does that affect p. 46 1.05 proof part 2)?
- p. 48 Ex II.1.2a should d_x^$<$-(X' x X') be d_x({x} x X') or d(X' x X')?

4.41.2 Sec II.3

- [PHM] I excerpeted chapter 2 of Bergers Problems book, which has a nice exposition on affine spaces.

4.42 *[2020-04-19 Sun]*

4.42.1 Sec I.2

- Ex 1.2.6b: look at, talk about people's solutions
- tangential topic: given a square matrix with random values, what is the probability that it has a rotational "factor"? For 2x2 matrices, the characteristic equation is x^2 - Tr*x + Det = 0, so the condition for having a rotation "factor" is Tr^2 - 4*Det $<$ 0. For 3x3 (or any odd x odd for that matter) there is always one real root, so it's never purely rotational (Hamilton?).

  See three png's filed under chapter 1. I will elaborate if there is interest.

4.42.2 Sec I.3

- Ex I.3.2d: discuss strategies for arriving at detC
- Ex I.3.2e: [NBA] see notes "$vol_{det.pdf}$"fs
- Ex I.3.7: [PHM] The algebra is easy, as is the conclusion, but the logical thread is tricky and may bear discussion.

4.42.3 Sec II.1

- the many typos of this section.

4.43 *[2020-03-31 Tue]*

4.43.1 General discussion

- shall we move to weekly 1 hr or 40 minute meetings? (deal with after meeting.)

4.43.2 Sec I.1

- Ex I.1.3: Intersection of 2 subspaces is a subspace this comes over directly from group theory (Alekseev problems 63 and 103), but if that's too glib, we can discuss further. Essentially, the only thing you have to do is to demonstrate closure.

4.43.3 Sec I.2

- Ex I.2.6(b,c): The interpretation in this problem is a good to have in one's head.

4.43.4 Sec I.3

- Ex I.3.1a: See also, Alekseev problems 174-176. The braid method that I [PHM] mentioned at our January meeting is a nice visual way to "prove" this one. Reference: page 7 of "Office Hours with a Geometric Group Theorist" by Matt Clay & Dan Margalit (eds), Princeton Univ. Press (2017).