Notes from D&P and related books

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Contents

`https://github.com/TechnicalBookClub/DodsonAndPoston`

1 Dodson&Poston

1.1 Contents

1.2 0.2. Functions

1.2.1 typo: In definition of g,

$\mathbb{N}$ should be $\mathbb{Z}$

1.3 II. Affine Spaces

1.3.1 1.01. Definition: *affine space*

with vector space T is a non-empty set of points and a *difference function* $d\colon X \times X \to T$

1.3.2 1.02. Tangent spaces, freeing and binding maps.

the difference function of 1.01 is related to the bijection $d_x\colon x \times X \to T$ which is called a *freeing map*.

Its inverse, $d^{\leftarrow}$ is called a *binding map*.

1.4 V.1.10. Notation.

> Vectors in $\mathrm{X^k{}_h}$ are called tensors on X, covariant of degree h and contravariant of degree k, or of type $(\mathrm{k\ h})^{\mathrm{T}}$. ... By convention $X_0^0 = \mathbb{R}$.

1.5 VI. Topological Vector Spaces

1.5.1 1. Continuity

1. VI.1.01 $\epsilon - \delta$ continuity
2. 1.06 continuity
   - a cubic is a nice function to demonstate that open $\to$ open is not necessarily true for continuous functions.
   - If your open set in the domain starts and ends between the local extrema AND includes them, then the range is the closed interval bound by the local extrema.
3. Hausdorff topology This one is important.
   - Finite intersections of open sets, any unions of open sets
   - Finite unions of closed sets, any intersections of closed sets.

1.5.2 4. Compactness and Completeness

1. 4.02 Bolzano-Weirstrauss See Stillwell NLT, Ex 8.4.1, 8.4.2
2. See Stillwell NLT excercises 8.4.3, 8.4.4

3. 4.10 theorem f(C) is compact extents. I needed a reminder that functions are not continuous at infinity. see $\epsilon - \delta$ continuity

1.6 VII. Differentiation and Manifolds

1.6.1 VII.1.01 Def: *derivative of f at x*

Here, f is a map between affine spaces, and the *derivative of f* is a map between the tangent spaces of the affine spaces.

1.6.2 VII.1.02. Higher Derivatives.

1.6.3 VII.1.03 Partial derivatives

1.6.4 VII.1.04 Theorem (Inverse Function Theorem).

$D_x f$ is an isomorphism iff $\exists$ neighborhoods of x and $f(x)$ where a local inverse $f^{\leftarrow}$ exists.

1.6.5 Ex. VII.1.6 *chain rule*

1.6.6 VII.2.xx derivative of f as a map between manifolds

$D_x f\colon T_x M \to T_{f(x)} N$

1.6.7 VII.3.02. Language: Df, df (the other d)

if f is a real-valued function, then $df\colon TM \to \mathbb{R}$ is the "collection" of all $d_{f(q)} \circ D_q f$ where $D_q f\colon T_q M \to T_{f(q)}\mathbb{R}$ and $d_{f(q)}\colon T_{f(q)} \to \mathbb{R}$

this is NOT the same as $Df\colon TM \to T\mathbb{R}$.

1.6.8 VII.3.03 Tangent bundle on a manifold, abbreviations

$(\mathrm{T}^{*}{}_{\mathrm{x}}\ \mathrm{M}) = (\mathrm{T}_{\mathrm{x}}\ \mathrm{M})^{*}$

1.6.9 A tensor field of type $(0\ 0)^{\mathbf{T}}$ is just a function,

1.6.10 VII.4.01. Covariant Vectors.

See Lee Prop 11.18 or Needham Act V. These are 1-forms.

1.6.11 VII.4.02. Contravariant Vectors

"we can idenfify $\vec{t}$ with the linear map:" $\partial_{\vec{t}}\colon f \mapsto df(\vec{t})$

so for any vector field $\vec{v}$ and function f, we have $(\vec{v}f)(x) = df(\vec{v_x})$

1.6.12 VII.6 Vector fields and flows

1. 6.02. Definition. A solution curve or integral curve of vector field v on manifold M is a curve $c\colon J \to M$ such that $c'(t) = v(c(t)) \forall t \in J$

2. 6.03. Definition. A C^k local flow...

    - $\phi(U, t)M$ where $U \subset M$, open and t is a parameter in an open interval
    - KEY POINT: by condition (ii), local flows are made up of solution curves.

3. 6.05. Corollary $\phi_{t+s} = \phi_t \circ \phi_s$ by definition, $\phi_t\colon U \to M\colon x \mapsto \phi(x, t)$, which returns the displacement from x along the flow ϕ.

4. 6.07. Lemma. Let M be a manifold on an affine space X

1.6.13 VII.7 Lie Brackets

- typo: should be $\phi((x, y), t)$, not $\phi((x + y), t)$
- typo: $\psi_1\phi_1(0, 0) = (0, 1)$
- typo: $\phi_1\psi_1(0, 0) = (1, 1)$
- ϕ and ψ are local flows (see VII.6.03 p 197)
- There are many objects in the preface to this section:

    1. (x, y): points on a manifold (or coordinates by abuse of language)
    2. (s, t): parameters
    3. (ϕ, ψ): flows
    4. (v, w): vector fields (related to the flows)
    5. f: any function on the manifold

- claim (for this example only): $[\phi \circ \psi - \psi \circ \phi](x, y) = (1, 0)$

1. 7.01. Definition. The Lie bracket or commutator

    - act on vector fields, not flows.
    - typo: (bottom of 200) v(w(fg)) - w(v(fg))
    - $= g[v, w](f) - f[v, w](g)$

"...so we have a new derivation" (namely, $[v, w]$)

2. 7.02. Theorem. iff fields commute then composition of flows also commute.
3. Fig. 7.1 is not referenced anywhere!
4. 7.04. Theorem. ("the Chart theorem") If we have a linearly independent and commuting set of vector functions that span a manifold, then they can be "realized" as the basis vectors (∂_i's) of some chart.

   These flows have constant velocity in the neighborhood of x. (see example in preface to this section.)

   (a) On Randy's discomfort with this notation We want to justify the relationship $\phi^i_t(x^1, \ldots, x^n) = (x^1, \ldots, x^i + t, \ldots, x^n)$.
   Note that $M \ni \vec{x} = (x^1, \ldots, x^n) = \theta(0, \ldots, 0)$, so the LHS can be rewritten $\phi^i_t(\theta(0, \ldots, 0)) = \phi^i_t \circ \theta(0, \ldots, 0)$
   Using 6.05 (and going through the proof backwards), this becomes $\theta(0, \ldots, t, \ldots, 0)$ with the t in the i slot, which is the same as $\vec{x} + t\vec{e}_i = (x^1, \ldots, x^i + t, \ldots, x^n)$
   In the proof, this appears with parameter s instead of t.

   (b) $\theta(t^1 \ldots t^n) \in M$ $\theta(t^1 \ldots t^n)$ is a point in M. By varying one of the t's, it describes a curve on M. $\theta(0,0,0 \ldots 0) = x \in M$. As a map, θ: $R^n \to M$, so it **might** be the inverse of a chart function.

   (c) ...by the smoothness of the ϕ^1 why only ϕ^1? Each $\phi^i{}_t(x)$ moves the point along its flow line, so the composition of all these ϕ's is a point in M. The fact that ϕ^1 is smooth means that through arbitrary points in M, ϕ^1 defines a smooth flow, so θ is also smooth through arbitrary points in M.

   (d) $D\theta$ takes the vector $e_1(t^1 \ldots t^n)\ldots$ This should be e_i, not e_1. otherwise, how does c(s) become a parameterized curve through $\theta(t^1 \ldots t^n)$ along the i direction?

   (e) typo: 6.05 "6.50" should be 6.05

   (f) typo: should be θ so we can rewrite this as $\phi^i{}_s(p)$ for $p \in M$

   (g) How does this "so c is a solution curve" come about? See 6.03. Definition. A C^k local flow...: a local flow is a solution curve by definition and $c == \phi^i{}_s$. so yes it is.

   (h) typo: remove θ should be $c^*(0) = v_i(c(0))$

(i) Dθ takes the standard basis... Dθ: $\mathrm{T}_{(0,\ldots,0)}\mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{T}_{\mathrm{x}}\mathrm{M}$: $\mathrm{e_j} \rightarrow (\mathrm{v_j})_{\mathrm{x}}$

(j) What are they saying????

> That is a linearly independent subset of the n-dimensional space $\mathrm{T_x}$ M, by assumption, so D$\theta(\mathrm{v}_1, \ldots ,\mathrm{v_n})$ is an isomorphism.

What is "that"? "That" is $\{(v_1)_x, ..., (v_n)_x\}$
if we're in Bachman, then $\langle e_1, ..., e_n\rangle$ is a general-ish vector in T_0..) $\mathrm{R^n}$

5. Exercises VII.7

(a) Ex VII.7.2 parts a and c are covered by 3.5.6

(b) Ex VII.7.4: Typos abound! THIS IS THE LIE DERIVATIVE!! yes, very important. See Lee.
The subscript to the first v should be $\phi_h(x)$: $[u, v]_x = \lim_{h\to 0} \frac{(D_x\phi_h)^{\leftarrow} v_{\phi_h(x)} - v_x}{h}$
v_x, and $[u, v]_x$ are both tangent vectors in T_xM $v_{\phi_h(x)}$ is a tangent vector in $T_{\phi_h(x)}M$
$\phi_h\colon M \to M$, so
$D_x\phi_h\colon T_xM \to T_{\phi_h(x)}M\colon v_x \mapsto \lim_{h\to 0} \frac{v_{\phi_h(x)} - v_x}{h}$

(c) Typo in missing f, should be at least: $[\vec{u}, f\vec{v}] = \vec{u}(f)\vec{v} + f[\vec{u}, \vec{v}]$
Note: this agrees with Lee p 188 because $\vec{v}(1) = d(1)\vec{v} = 0$

1.7 VIII. Connections and Covariant Differentiation

1.7.1 VIII.2. Rolling Without Turning

This section is their attempt to lay the groundwork for the Levi-Civita connection, aka the intrinsic or covariant derivative. The exposition is terrible. Better to look at Needham, chapters 21-23.

By not "turning," I assume that they mean parallel transport of the basis vectors of the tangent space, which is formally introduced in VIII.4.

1. ∇_{t} Levi-Civita connection

1.7.2 VIII.3. Differentiating Sections

We can think of T(TM) as the vector space of changes of vectors in TM, the manifold of tangent spaces.

1. w vs t

    - w(M) is an arbitrary vector field along (the curve on) M.
    - t is a vector in vector field that defines the curve. to make your life

    easy, define t as a unit vector pointing to the right (a basis vector)

    In this example, w(p) and t are parallel or antiparallel, but not necessarily equal.

    Here, we are in 1D, so the curve has no wiggle room in the manifold.

        the embedded picture to represent the tangent spaces at p
        and 1O(p) to M and TM. For a vector t tangent to Mat

2. Π, the projection map and following equation $w : M \to TM$ $D_p w : TM \to T(TM)$ $\Pi : TM \to M$ $D_{w(p)}\Pi : T(TM) \to TM$

    - t is the velocity on M ($t \in T_pM$) (so in TM it is "vertical", but in M it points to the right.)
    - $D_p w(t)$ describes the change of w(p) at velocity t. In TM, the horizontal

    component of this vector is |t| and its tail is at w(p).

    - Π is a projection that takes points in TM and maps them to M. All points in T_pM map to p, and likewise for other fibers in TM.
    - $D_{w(p)}\Pi$ maps vectors in T(TM) into vectors in TM. A vector between points w(p) and q in TM is in the tangent space $T_{w(p)}(TM)$ and maps to a vector in M from p to $\Pi(q)$.

      The difficulty for me here is that there are two "represenatations" of T_pM in Figure 3.1 - the horizontal one at the bottom of the figure, and the fiber in TM. Π projects points down into M, and $D_x\Pi$ projects vectors down into the "horizontal representation" of T_xM, not the "vertical representation."

3. "directional derivative" In D&P-speak, "vertical" is vertical in their picture of TM, meaning that the vector lies in T_pM for some $p \in M$. ~~Or maybe not~~ – see definition in 3.06.

4. "we are free to decide what properties would be nice to have" See Tu(DG) Sec 6, p 43. Without saying so, D&P appear to be leading us to the **Riemannian** or **Levi-Civita** connection. Tu says this is the unique, torsion-free affine connection on a Riemannian manifold.

5. Typo! first f should be t

6. Need to understand the first term on the right See Tu(DG) Prop 4.9.ii p. 27 Randy: ∇_t takes vector fields as an argument, but f is a scalar field, so the correct thing to do is t(f).

7. directional derivative of a vector field This may be best explained in Barret Oneill "Differential Geometry." The differential ∇ turns an n form into an (n+1) form. w and t here are vectors (one forms?) and ∇_t(fw), t(f)w(p), and f(p)∇_t(w) are all 2-forms.

   I was confused because I was thinking of w, t, and ∇_t(w) all as vectors, so I could not identify what t(f)w(p) was, and it looked to be different from the other terms in this equation.

   Why do they not use ∇_tf in place of t(f)?

8. 3.01. Definition. A connection (∇) Koszul connection (general affine connection) Levi-Civita/intrinsic/Reimannian/covariant has the additional property of being torsion-free (See Tu, Sec 6) (tangent vector, vector field) $\rightarrow$ (tangent vector)

   Ci: linear in the input tangent vector space Cii: linear in the vector field space Ciii: scaling t is the same as scaling w Civ: Leibniz property Cv: continuous vector fields have continous connections

9. 3.02. Coordinates – Christoffel symbols Tu treats this topic in Section 13.3 of DG.

   > First, let us see what a connection looks like in coordinates, as some proofs will be easiest that way.

   (a) Typo! shold be "by Ciii" "by Ci)" is wrong.

10. 3.03. Transformation Formula. Despite the suggestive index notation, Christoffel symbols are **not** tensors.

11. 3.04. Definition: vector field **along** c a vector field **along** curve c assigns a vector tangent to M at each point of c(t) for t $\in$ [a,b].

From the examples given, this appears to be in concordance with Tu's usage of **along**, in that the vectors do not necessarily have to be tangent to the curve, which we intepret as a submanifold of M when we read Tu.

Π here is the projection TM → M taking a vector to its point of attachment.

12. 3.05. Differentiating, Along Curves, Fields Along Curves. D&P's funny ∇ is DV/dt in Tu – the "covariant derivative" (DG 13.1)? In IX.1.01, they call Dc^*/dt the acceleration.

    In Needham, it is the *intrinsic derivative* (23.2).

    Note – I'm using ~ over ∇ instead of | through ∇.

    > We denote the resulting linear map, taking vector fields along c (not vector fields on M) to vector fields along c, by $\tilde{\nabla}_{c*}$. not ∇_{c*}.

13. 3.06. Definition: "vertical" a vector $\vec{v}$ in the space of derivatives of tangent vectors is *vertical* if its projection into the tangent space is zero.

    I find this definition confusing and hard to keep in my head. The picture in Fig 3.6 is likely the key (but the value is still questionable – can \ {c} be double-valued as it is under (b)?)

    $D_w(\Pi)$ definition seems incompatible with the statement in the next paragraph about finding a "path ~c". See Π discussion above. $D_w(\Pi)$ must have the "horizontal" T_pM as its range.

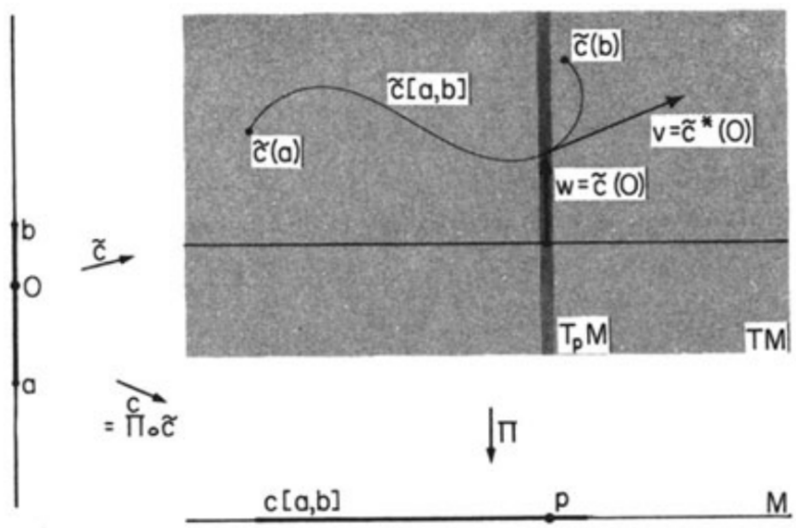


Fig. 3.6

14. 3.07. Definition: "vertical" and "horizontal" parts of a vector The vertical part of a vector $\vec{v}$ in the space of derivatives of tangent vectors is the part **in** the tangent space, while the horizontal part of the vector is the part **normal to** the tangent space.

    The projection P_w defined in this section is the projection to the "vertical," ie, the projection into the tangent space. The "horizontal" part corresponds to $\langle \vec{v}, \hat{n} \rangle \hat{n}$ in Tu Sec 6.4, Prop 6.8, p 47.

15. 3.08. Language Koszul connection - section 3.01 Ehresmann connection - split $T_w(TM)$ into horizontal and vertical parts

16. Exercises 3 & 4 ∇ and the funny ∇...

1.7.3 VIII.4. Parallel Transport

1. 4.01. Definition: parallel vector field A vector field v along c $\in$ M is *parallel* if the horizontal component of its derivative along c^* is zero. In other words, at each point along the curve, the corresponding vector from the field must maintain the same angle with respect to the tangent to the curve.

   This corresponds to Tu (DG) Sec 14.5 Defn 14.13 p 110. DV/dt == 0.

2. what does this even mean? “v, considered as a curve in TM, has c*(t) horizontal for all t”

   Just like the "rolling" business in this book, I'm not sure the "vertical"/"horizontal" distinction is pedegogically useful. In order to picture nontrivial parallel transport, we need at minimum, a 2D surface embedded in 3D. M is 2D, so v ∈ TM is also 2D, but c*(t) ∈ T(TM) is 4D – hard to picture. Are there 2 horizontal dimensions and 2 vertical dimensions? This is **much** easier to follow in Needham.

   See the discusion around Eqn (23.5), p 244 in Needham.

3. 4.02. Theorem: uniqueness of the parallel vector field along c for any v and c, the parallel vector field is unique.

4. 4.03. Definition: *parallel transport* τ See [BROKEN LINK: ade54283-c817-4d4d-bdc8-fdb1d8df37c8]

5. 4.04. Lemma. parallel transport is independent of reparameterization The statement of this lemma seems too obvious to prove.

6. 4.05. Theorem.

   (a) typo: should be \ c $\Pi \circ \Lambda$

7. 4.06. Corollary. (connection) $\nabla_{\vec{t}}\vec{w} = \lim_{h\to 0} \frac{\tau_h^{\leftarrow}\vec{w}_{f(h)} - \vec{w}_p}{h}$

8. 4.07. Corollary. (intrinsic derivative) $\nabla_{c*}\vec{w}(t) = \lim_{h\to 0} \frac{\tau_h^{\leftarrow}\vec{w}_{c(t+h)} - \vec{w}_{c(t)}}{h}$

1.7.4 VIII.5. Torsion and Symmetry

1.7.5 5.01: T=0 ==> ∇ is symmetric

but is the reverse true? In the next section, D&P appear to use symmetry implies zero torsion.

1. Lie bracket in terms of a torsion free connection How does this square with the definition of torsion? because it is defined at a point?

1.7.6 VIII.6. Metric Tensors and Connections

1. 6.01. Definition: Compatible A connection ∇ on a manifold M with a metric tensor G is *compatible* if all parallel transports τ_t that it defines are isometries.

2. 6.03. Corollary: Compatibility criteria See Tu, Differential Geometry for confirmation that this is correct.

   My [PHM] confusion with this: What object is w(u · v)? Naively, the RHS looks like scalars, but the LHS looks like a vector. This confusion has led me into Needham, Act V since it looks like I need to understand forms deeply to resolve this confusion.

3. 6.05. Definition. (∇) Levi-Civita connection for G "The unique symmetric connection compatible with G is called the Levi-Civita connection for G. From now on, ∇ (on a manifold for which we have a metric tensor field) will always refer to this connection unless we explicitly state otherwise."

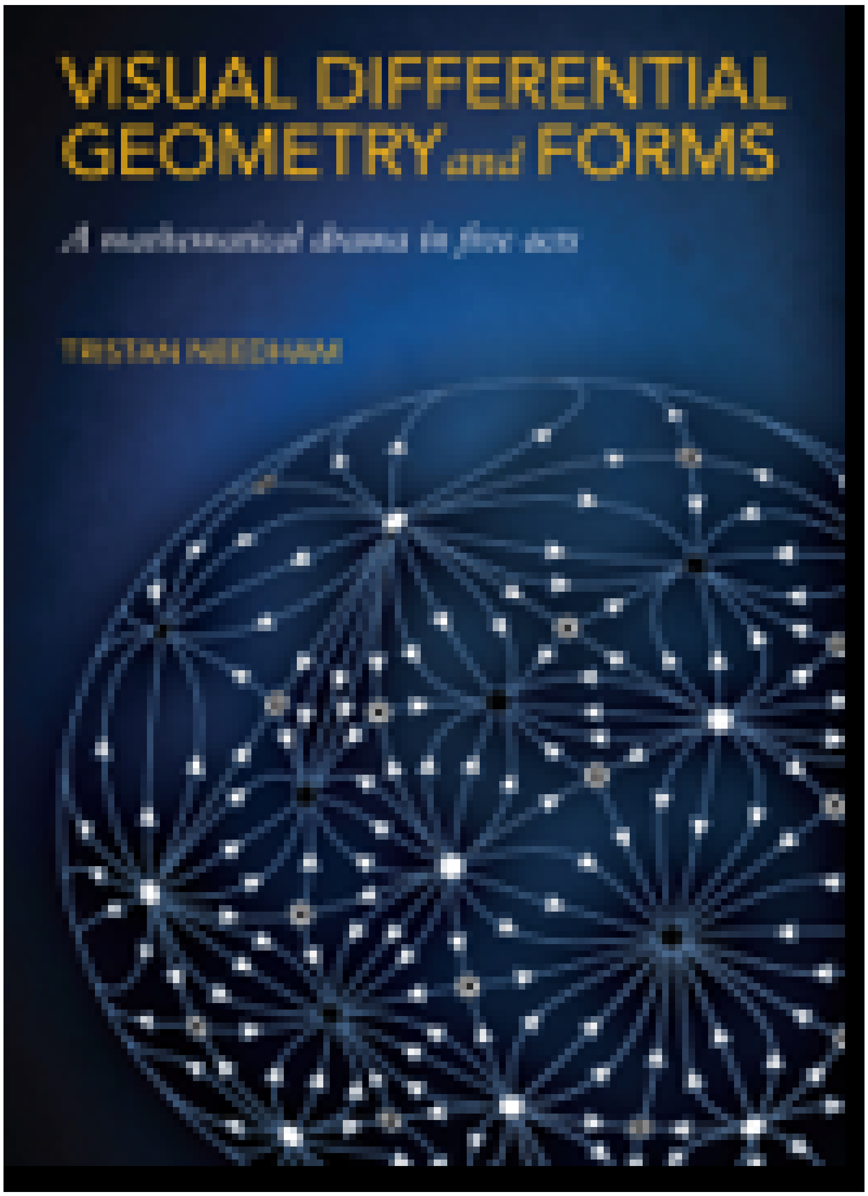
1.7.7 VIII.7. Covariant Differentiation of Tensors

Derivatives ($\nabla_{\mathbf{v}}$) and differentials (∇)of tensors.

1.8 Bibliography

1.9 Index of Notations

2 NeedhamT VDGF



2.1 Contents

2.2 1.3 Angular excess of a spherical triangle

excess = (sum of angles) - π = A/R^2

2.3 1.5 Constructing geodesics with tape

2.4 2.1 Gaussian curvature in terms of excess angle

2.5 2.2 Circ/Area of a circle – reread this!

2.6 2.3 Local Gauss-Bonnet Theorem

angular excess is the total (area integrated) curvature.

2.7 Chapter 4 The Metric

2.7.1 4.3 Metric of a general surface

2.7.2 Gauss EFG coefficients of the first fundamental form

E = `A^2 F` = ABcosω G == B^2

2.7.3 4.4 Metric Curvature Formula

2.7.4 4.5 Conformal Maps

2.8 Chapter 5 Pseudosphere and the Hyperbolic plane

2.8.1 5.2 Tractrix and the Psuedosphere

Tractrix: object dragged along Y axis by a chord of length R. σ = distance along path of object.

2.8.2 5.3 Conformal map of the pseudosphere

(x,σ) $\rightarrow$ (x,y) = x + iy = z where x is an angle $[0,2\pi)$ of the revolution and y $\geq$ 1 (see 5.4)

2.9 8.3 Newton's curvature formula

2.10 Eqn 8.6, curvature

The area of the shaded sector in fig 8.6 explains the numerator of κ, but it doesn't really explain why the numerator should be that area. Kepler's law?

$\kappa =$ (area of swept velocity vector)$/|\mathrm{v}|^3$
In Kepler's law, the position vector sweeps out equal areas in equal time.

2.11 Chapter 9 Curves in 3-Space

2.11.1 definition: Torsion

"rate of rotation of the osculating plane is called the torsion, denoted τ."

2.11.2 definition: binormal of a curve

normal vector to the osculating plane

2.12 Chapter 10 The Principal Curvatures of a Surface

2.12.1 (10.1) Euler's Curvature Formula:

$\kappa(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$

2.12.2 Dupin's indicatrix

nearby slices parallel to the tangent plane are, to first order, conic sections.

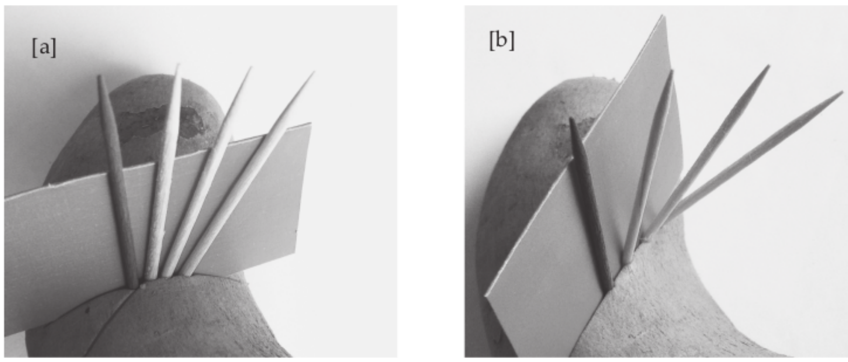
2.13 Chapter 11 Geodesics and Geodesic Curvature

2.13.1 Fig [11.2] $\mathbf{K = K_g + K_n}$

$\mathrm{K_g}$ is "geodesic curvature" IN the surface $\mathrm{K_n}$ is "normal curvature" OF the surface

2.14 Chapter 12 The Extrinsic Curvature of a Surface

2.14.1 Figure 12.5: normal vectors on principal and general curves



[12.5] **[a]** *If we move in a principal direction then the normal vector tips in that direction and initially stays within the normal plane;* **[b]** *If we move in a general direction then the normal vector immediately tips out of the normal plane.*

2.15 Chapter 15 The Shape Operator, S, and $\nabla_{\mathbf{v}}$

2.15.1 15.1 Directional Derivatives ($\nabla_{\mathbf{v}}$)

2.15.2 15.2 The Shape Operator S

"is simply defined to be the negative of the directional derivative of n along v" $S(v) = -\nabla_v n$

This encodes the same information as the Second Fundamental Form (see 15.9 and Tu (DG) Proposition 5.5: Curvature is given by the second fundamental form)

2.15.3 15.3 Geometric effect of S

> The principal directions are the eigenvectors of the Shape Operator S, and the principal curvatures are the corresponding eigenvalues: $\mathrm{S}(\mathrm{e_i}) = \kappa_\mathrm{i}\ \mathrm{e_i}$.

2.15.4 (15.18) $\kappa(\mathbf{v}) = \mathbf{v} \cdot \mathbf{S}(\mathbf{v})$

where v is a unit tangent vector.

2.15.5 (15.7) sum of curvatures in perpendicular directions

> The sum of the curvatures in any two perpendicular directions is equal to the sum of the principal curvatures.

This is a surprising result, given that the curvatures are represented in earlier parts of this book by ellipses.

2.15.6 (15.26) curvature and torsion

$\$\mathrm{K}_{\mathrm{ext}} = ||[\mathrm{S}]|| = \text{-}\tau^2$

2.15.7 15.9 Classical Terminology and Notation: The Three Fundamental Forms

Note that the fundamental forms are not proper forms (see Act V, Chapter 32 and on.)

- $I(u, v) = u \cdot v$
- $II(u, v) = S(u) \cdot v$
- $III(u, v) = S(u) \cdot S(v)$

2.15.8 19.8 The Road Ahead

In the 3 chapters before this, Needham gives proofs of th

2.16 Chapter 21 An Historical Puzzle

2.16.1 GR and parallel transport

> Einstein's success was all the more remark- able, and remains all the more puzzling, because he achieved it before Levi-Civita—pictured in [21.1]—discovered the concept of parallel transport, which did not occur until 1917!

2.17 Chapter 22 Extrinsic Constructions

2.17.1 Projection (not rotation) of w into $\mathbf{T_q}$

$\mathrm{w}_{||} = \mathrm{P}[\mathrm{w}] = \mathrm{w} \quad (\mathrm{w} \cdot \mathrm{n})\mathrm{n}$

2.17.2 Rotation of w into $\mathbf{T_q}$

$\tilde{w}$. But in the limit as $\epsilon \to \infty$, rotation and projection are the same.

2.18 Chapter 23 Intrinsic (covariant) Constructions

2.18.1 23.2 The Intrinsic (aka, "Covariant") Derivative

aka Levi-Civita connection (Koszul connection?)

2.18.2 (Eqn 22.1) Instrinsic derivative, (D_v, the convariant derivative)

ϵ D_v w = w(q) - $w_{||}$(p $\to$ q) both w's on the RHS are in T_q

2.18.3 Fig 23.3

"here w is growing in length and rotating counterclockwise as it moves along G."

2.18.4 D_v is also called the Levi-Civita Connection.

2.18.5 intrinsic derivative $D_v w$

vs the definition on the previous page, here w(q) is parallel transported back to p [$w_{||}$(q $\to$ p)] and then w(p) is subtracted. (then lim $\epsilon \to \infty$)

2.18.6 condition for parallel transport

2.18.7 Here is an extrinsic way of looking at the intrinsic derivative.

2.18.8 Eqn 23.3 $D_v w$

same as Eqn 22.1, with $\nabla_v w$ dropped in for w

> In other words, to obtain $D_v w$ we take the full rate of change $_v w$ in R^3 , then subtract out the part that is not tangent to the surface, thereby leaving behind the part that is intrinsic to the surface.

2.18.9 Compare these with Ci-Cv in D&P p 217

> it is much simpler to think of **flattening onto the tabletop** the strip surrounding [curve] K, together with the vector fields x, y, z, for then D_v simply is $_v$.

2.18.10 geodesic equation

2.19 Chapter 28 Curvature as a Force between Neighbouring Geodesics

2.20 Notation change for intrinsic derivative

> we adopt the standard choice of a bold nabla symbol——to represent the intrinsic derivative,

2.21 29.5 [v , u] $_{\mathbf{v}}$ u $_{\mathbf{u}}$ v.

2.22 Chapter 32 1-Forms

AKA covariant vectors or covectors.

2.22.1 Contraction: $\langle\omega$, v$\rangle$

Needham uses v(ω) or ω(v) more than the angle bracket form of the contraction. Elsewhere, he also uses $\cdot$ for the dot product of vectors (not 1-forms and vectors).

2.22.2 32.3.1 w vs ω for work

Needham is using ω here because he writes 1-forms as lowercase Greek letters and he's making the point that work is a 1-form.

2.22.3 32.3.4 Row Vectors

First mention?

2.22.4 Typo: missing ~'s

basis vectors in [b] should be $\tilde{A}_1, \tilde{A}_2$. The poins is that when e_2 changes, ω^1 is also affected.

2.22.5 duality of sets, not elements – important!

> { 1 , 2 } is dual to the set of basis vectors {e1 , e2 }, it is simply wrong to think that 1 is dual to e1 , and that 2 is dual to e2 .

2.22.6 Typo: (32.5), not (32.6)

2.22.7 Equivalent definitions of a basis { i }:

I agree with Needham on this point. His definition, while equivalent, is more geometrically evocative. The Kronecker delta formulation *seems to* imply orthonormality which is not required.

2.22.8 32.6 (32.6) The Gradient as a 1-Form: df

In this section, the distinction between gradient (∇ f)and "exterior derivative" 1-form (df) is made.

Note the bold ∇ here – (32.10) identifies the **exterior derivative d** with the **intrinsic derivative** $D_{\mathbf{v}}$ or $\nabla_{\mathbf{v}}$

2.22.9 32.6.3 The Cartesian 1-Form Basis: {dxj }

beware of notational confusion:

- x is x^1
- y is x^2

2.22.10 Typo: missing '[' before $\partial_{\mathbf{y}}$f

Where does Needham treat the 1-form as a row vector? clearly, $[\partial_x f]dx + [\partial_y f]dx = [\partial_x f, \partial_y f]$

2.23 Chapter 33 Tensors

2.23.1 Trace is the contraction of a linear operator

ah – this is brilliant!

2.24 Chapter 34 2-Forms

p-forms are antisymmetric $\{^0{}_\mathrm{p}\}$ tensors. (swapping vector inputs reverses the sign). in R^4, $\exists$ 6 basis 2-forms

2.24.1 Only in 3D (basis of 2-forms = basis of vectors)

2.24.2 Hodge duality operator (p-form → (n-p)-form)

2.24.3 Faraday 2-form

2.24.4 Maxwell 2-form

2.24.5 35.7 Is = 0 Possible?

Ψ = dt ^ dx + dy ^ dz
Ψ ^ Ψ = (dt ^ dx + dy ^ dz) ^ (dt ^ dx + dy ^ dz) = (dt ^ dx ^ dt ^ dx + dt ^ dx ^ dy ^ dz + dy ^ dz ^ dt ^ dx + dy ^ dz ^ dy ^ dz) = (0 + dt ^ dx ^ dy ^ dz + $(-1)^4$ dt ^ dx ^ dy ^ dz + 0) = 2dt ^ dx ^ dy ^ dz

2.25 Chapter 36 Differentiation

2.25.1 Typo: (32.11) should be (32.10)

2.25.2 36.3 Leibniz rule for forms

2.25.3 First step uses (36.6)

d($[\phi_i\, dx^i]$ ^ $[\Psi_{jk}\, dx^j$ ^ $dx^k]$) = d($\phi_i\, \Psi_{jk}$ ^ dx^i ^ dx^j ^ dx^k) = $d\phi_i\, \Psi_{jk}$ ^ dx^i ^ dx^j ^ dx^k

2.26 Chapter 39 Exercises for Act V

2.26.1 1-forms used to be called covariant vectors or covectors

2.27 Further Reading

2.27.1 Intuitive Topology, by V. V. Prasolov.

2.27.2 Three-Dimensional Geometry and Topology, by William P. Thurston.

2.27.3 Differential Geometry: A Geometric Introduction, by David W. Henderson.

2.27.4 Gauge Fields, Knots and Gravity, by John Baez and Javier P. Muniain.

#**

3 LeeJM SmoothManifolds

3.1 Notes for page 9 V: 32% H: 81%

> The old joke that "differential geometry is the study of properties that are invariant under change of notation" is funny primarily because it is alarmingly close to the truth.

3.2 Contents

3.3 Proposition 2.25 (Existence of Smooth Bump Functions).

These are mentioned in Tu. Maybe the exposition here is useful (but I haven't read it yet.)

3.4 Chapter 3 Tangent Vectors

3.5 Chapter 8 Vector Fields

> In the [third] section we introduce the Lie bracket operation, which is a way of combining two smooth vector fields to obtain another. Then we describe the most important application of Lie brackets: the set of all smooth vector fields on a Lie group that are invariant under left multiplication is closed under Lie brackets, and thus forms an algebraic object naturally associated with the group, called the Lie algebra of the Lie group.

3.5.1 Vector Fields on Manifolds

3.5.2 Example 8.12. cylindrical orthonormal frame on $\mathbb{R}^2\{0\}$

3.5.3 Vector Fields and Smooth Maps

first mention of "-related".

> Suppose $F : M \to N$ is smooth and X is a vector field on M; and suppose there happens to be a vector field Y on N with the property that for each $p \in M$; $dF_p(X_p) = Y_{F(p)}$. In this case, we say the vector fields X and Y are **F-related** (see Fig. 8.3). The next proposition shows how F-related vector fields act on smooth functions.

3.5.4 Lie Brackets

3.5.5 Lemma 8.25. Lie bracket of smooth vector fields...

"Lemma 8.25. The Lie bracket of any pair of smooth vector fields is a smooth vector field."

Geometric interpretation comes in Chapter 9

"of limited usefulness for computations": $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$

3.5.6 Proposition 8.26 (Coordinate Formula for the Lie Bracket).

"an extremely useful coordinate formula for the Lie bracket": $[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$ or more concisely, $[X, Y] = (XY^j - YX^j)\frac{\partial}{\partial x^j}$

3.5.7 Example 8.27

Decent concrete example.

3.5.8 see D&P 5c

$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$

fXgY - gYfX = fXg Y + fg XY - gYf X - gfYX = (fXg)Y - (gYf)X + (fg XY - gf YX)

3.6 Chapter 9 Integral Curves and Flows

3.6.1 Flowouts

1. Theorem 9.22 (Canonical Form Near a Regular Point). Randy, read this.

3.6.2 Lie Derivatives

1. Fig. 9.13 The Lie derivative of a vector field

2. Theorem 9.38 Lie derivative == Lie bracket

3.6.3 Commuting Vector Fields

1. Theorem 9.42. Equivalent statements on vector fieldsLieBracket:polar

For smooth vector fields V and W on a smooth manifold M; the following are equivalent: (a) V and W commute. (b) W is invariant under the flow of V . (c) V is invariant under the flow of W .

Consider ρ = x/r i + y/r j # radial vec field "E1" p179 θ = -y/r i + x/r j # unit rotation "E2" p179 ϕ = -y i + x j # disc rotation "W" p207

[E1, E2] = -E2 [E1, W] = 0

(a) follow E1 out in r for E2, the vector at positions in the flow of E1 are constant Q: Is E2 invariant under the flow of E1 ?? [A: no]
for W, the vector at positions in the flow of E1 are increasing in magnitude. Q: Is W invariant under the flow of E1 ?? [A: yes]

(b) follow a streamline in E2 or W The change in ρ is tangential to the flow. In W, dρ is constant, but in E2, dρ is smaller at larger radial positions.

- useful derivatives i (x/r) = y^2/r^3 i (y/r) = $-xy/r^3$ = j (x/r) j (y/r) = x^2/r^3

[E1,W] = (x/r i + y/r j)(-y i + x j) - (-y i + x j) (x/r i + y/r j) = (x/r) i(x) j - (y/r) j(y) i - [-y i(x/r) i - y i(y/r) j + xj(x/r) i + xj(y/r) j] = [(x/r) j - (y/r) i] - [-$(y/r)^3$ - x^{2y}/r^3]i - [xy^2/r^3 + $(x/r)^3$]j = [(x/r) j - (y/r) i] - (y/r)[-$(y/r)^2$ - x^2/r^2]i - (x/r)[y^2/r^2 + $(x/r)^2$]j = [(x/r) j - (y/r) i] - [(x/r) j + (y/r) i] = E2 - E2

[E1,E2] = 0 - [-y/r i + x/r j][x/r i + y/r j] = 0 - [(-y/r) i(x/r) + (x/r) j(x/r)]i -[(-y/r) i(y/r) + (x/r) j(y/r)]j = 0 - [(-y/r)y^2/r^3 i - (x/r)xy/r^3 i + xy^2/r^4 j + x^3/r^4 j] = 0 - [-y^3/r^3 i - x^{2y}/r^3 i + xy^2/r^3 j + x^3/r^3 j]/r = 0 - [-y/r i + x/r j]/r = - E2/r

2. Theorem 9.44. Smooth vector fields commute if and only if their flows commute. D&P VII.7.02

3. Example 9.45 (Commuting and Noncommuting Frames).

Example 8.12. The standard coordinate frame is a global orthonormal frame on $\mathbb{R}^n$. For a less obvious example, consider the smooth vector fields defined on $\mathbb{R}^2 \smallsetminus \{0\}$ by

$$E_1 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}, \qquad E_2 = -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}, \tag{8.3}$$

where $r = \sqrt{x^2 + y^2}$. A straightforward computation shows that (E_1, E_2) is an orthonormal frame for $\mathbb{R}^2$ over the open subset $\mathbb{R}^2 \smallsetminus \{0\}$. Geometrically, E_1 and E_2 are unit vector fields tangent to radial lines and circles centered at the origin, respectively. //

4. Theorem 9.46 (Canonical Form for Commuting Vector Fields). D&P Thm VII.7.04

3.7 Chapter 11 The Cotangent Bundle

we define the differential of a real- valued function as a covector field (a smooth section of the cotangent bundle); it is a coordinate-independent analogue of the gradient.

3.7.1 Prop 11.8 $V \cong V^{**}$

"there is no canonical isomorphism $V \cong V^*$."

3.7.2 Tangent Covectors on Manifolds

Thus it became customary to call tangent covectors **covariant vectors** because their components transform in the same way as ("vary with") the coordinate partial derivatives, the Jacobian matrix multiplying the objects associated with the "new" coordinates to obtain those associated with the "old" coordinates. Analogously, tangent vectors were called **contravariant vectors**, because their components transform in the opposite way.

3.7.3 contravariant transformation

3.7.4 covariant transformation

3.7.5 Prop 11.18 (coordinate covectors are dx^{j}'s)

In other words, the coordinate covector field λ^{j} is none other than the differential dx^{j}.

See D&P VII.4.01

3.7.6 Proposition 11.23 (Derivative of a Function Along a Curve).

3.8 Chapter 12 Tensors

> We deal primarily with covariant tensors, but we also give a brief introduction to contravariant tensors and tensors of mixed variance.

3.9 Exterior Derivatives (14)

#***************************************

4 TuLW Introduction to Manifolds

4.1 2.1 Directional derivative

In D&P, this is denoted $D_{pf}(v)$ for the directional derivative in direction v at point p of function f.

4.1.1 Defn: germ

Equivalence class of functions on an neighborhood $C^{inf}{}_p$ is the set of all germs of C^{inf} functions on R^n at p.

4.1.2 Defn: derivation

Linear map $C^{inf}{}_p \rightarrow R$ satisfying the Leibniz rule is a derivation

4.1.3 2.4 Vector fields

See D&P VII.4.02 Contravariant vectors. The 2nd paragraph above encapsulates the corresponding section in D&P in a nutshell.

4.1.4 2.5 Vector Fields as Derivations

4.2 §3 The Exterior Algebra of Multicovectors

4.2.1 3.3 k-covectors

"A 1-covector is simply a covector". A k-covector appears to be a tensor of covariant degree k

4.2.2 Lemma 3.11 $\tau(\sigma\ \mathbf{f}) = (\tau\sigma)\mathbf{f}$

$(\sigma\ \mathrm{f})(\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{k}) = \mathrm{f}(\mathrm{v}_{\sigma(1)}, \ldots, \mathrm{v}_{\sigma(\mathrm{k})})$ if $\sigma = (1,\mathrm{m},\mathrm{n})$, then $\sigma(1) = \mathrm{m}$, $\sigma(\mathrm{m}) = \mathrm{n}$, $\sigma(\mathrm{n}) = 1$

- concretely, if σ=(1,3,2)(4,5) and τ=(1,5,3,2) then (σ f)(v1,v2,v3,v4,v5) = f(v3,v1,v2,v5,v4)

  and τ(σ f)(v1,v2,v3,v4,v5) = (σ f)(v5,v1,v2,v4,v3) = f(v2,v5,v1,v3,v4)

  $\tau\ \sigma$ = (1,5,3,2) (1,3,2) (4,5) = (1,2,5,4,3)

  so this checks out.

4.2.3 3.7 Wedge product

Tu motivates the definition of the wedge product by wanting to have a product that preserves the property of being alternating on alternating functions (on a vector space). Does this make the wedge product the group multiplication on alternating functions? No – the identity is not alternating, so the wedge can't get you there.

4.2.4 Defn: graded algebra over field K

The algebra "can be written as a direct sum of vector spaces over K such that $\mathrm{A}^\mathrm{k} \times \mathrm{A}^\mathrm{l} \mapsto A^{\mathrm{k+l}}$.

4.3 §4 Differential forms on $\mathbf{R}^\mathrm{n}$

4.3.1 4.4 The Exterior Derivative

4.3.2 4.6 Applications to Vector Calculus

related to exact sequences
grad: scalar → vector curl: vector → vector div : vector → scalar

4.3.3 4.7 Convention on Subscripts and Superscripts

vector fields: subscripts	$\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{n}$	contravariant
differential forms: superscripts	$\omega^1, \ldots, \omega^\mathrm{n}$	covariant
coordinate functions (0-forms):	$\mathrm{x}^1, \ldots, \mathrm{x}^\mathrm{n}$	covariant
differentials of coord functs (1-forms):	$\mathrm{dx}^1, \ldots, \mathrm{dx}^\mathrm{n}$	covariant
coordinate vector fields	$\mathrm{d}/\mathrm{dx}^1, \ldots, \mathrm{d}/\mathrm{dx}^\mathrm{n}$	contravariant
coordinate vector fields (alt form)	$\delta_1, \ldots, \delta_\mathrm{n}$	contravariant

- coefficient functions

of a vector field	a^i
of a differential form	b_j

4.4 §5 Manifolds

smooth manifolds are the focus of this book

"maximal C^∞ atlas" make a topological manifold into a smooth manifold.

4.4.1 5.1 Topological Manifolds

defn: `second countable` A topological space with a countable basis.

4.4.2 5.3 Smooth Manifolds - maximal atlas defn

A maximal atlas on a locally Euclidean space is not conatined in any larger atlas.

4.4.3 Definition 5.9 smooth manifold

A smooth manifold is a topological manifold with a maximal atlas. A maximal atlas is also called a `differentiable structure` "In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do"

4.4.4 Smooth manifold conditions

to show that a topological space M is a C manifold, it suffices to check that:

1. M is Hausdorff and `second countable`,
2. M has a C atlas (not necessarily maximal).

4.4.5 5.4 Examples of Smooth Manifolds

- Euclidean space
- Open subset of a manifold
- Manifolds of dimension zero
- Graph of a smooth function

- General linear groups (nonzero determinant matrices)
- Unit circle in the (x,y) plane
- Product manifold

4.5 §6 Smooth Maps on a Manifold

4.5.1 6.3 Diffeomorphisms

4.5.2 6.6 Partial Derivatives

4.5.3 6.7 The Inverse Function Theorem

A function on an affine space or manifold is invertible at p iff the Jacobian determinant at p is nonzero

4.6 Chapter 3: The tangent space

4.7 bump functions?

These are covered in section 13, p 140

4.8 Theorem 20.4.

> If X and Y are C vector fields on a manifold M, then the Lie deriva- tive LX Y coincides with the Lie bracket [X ,Y].

4.9 A.4 First and Second Countability

4.9.1 Lemma A.10. Every open set in $\mathbf{R^n}$ contains a rational point.

4.9.2 Prop A.11 collection of open balls w/ rational centers/radii is a basis for $\mathbf{R^n}$

4.9.3 Defn A.12 second countable <== countable basis

4.9.4 Defn A.15 basis of neighborhoods, first countable

- A `basis of neighborhoods at p` is a collection of of neighborhoods of p such that $\mathrm{p} \in \mathrm{B}_\alpha \subset \mathrm{U}$
- Topological space S is `first countable` if it has a countable basis of neighborhoods at every point $\mathrm{p} \in \mathrm{S}$.
- Every second countable space is first countable.

4.10 List of Notations

#**

5 Lee Riemmanian Manifolds

5.1 Properties of Connections

product rule (Leibniz) (Xf)Y see Oneill.

5.1.1 product rule for connections

5.2 Lemma 4.7. F and (∇ F)

if F is a $(^{k}{}_{l})$ tensor, then ∇ F is a $(^{k+1}{}_{l})$ tensor

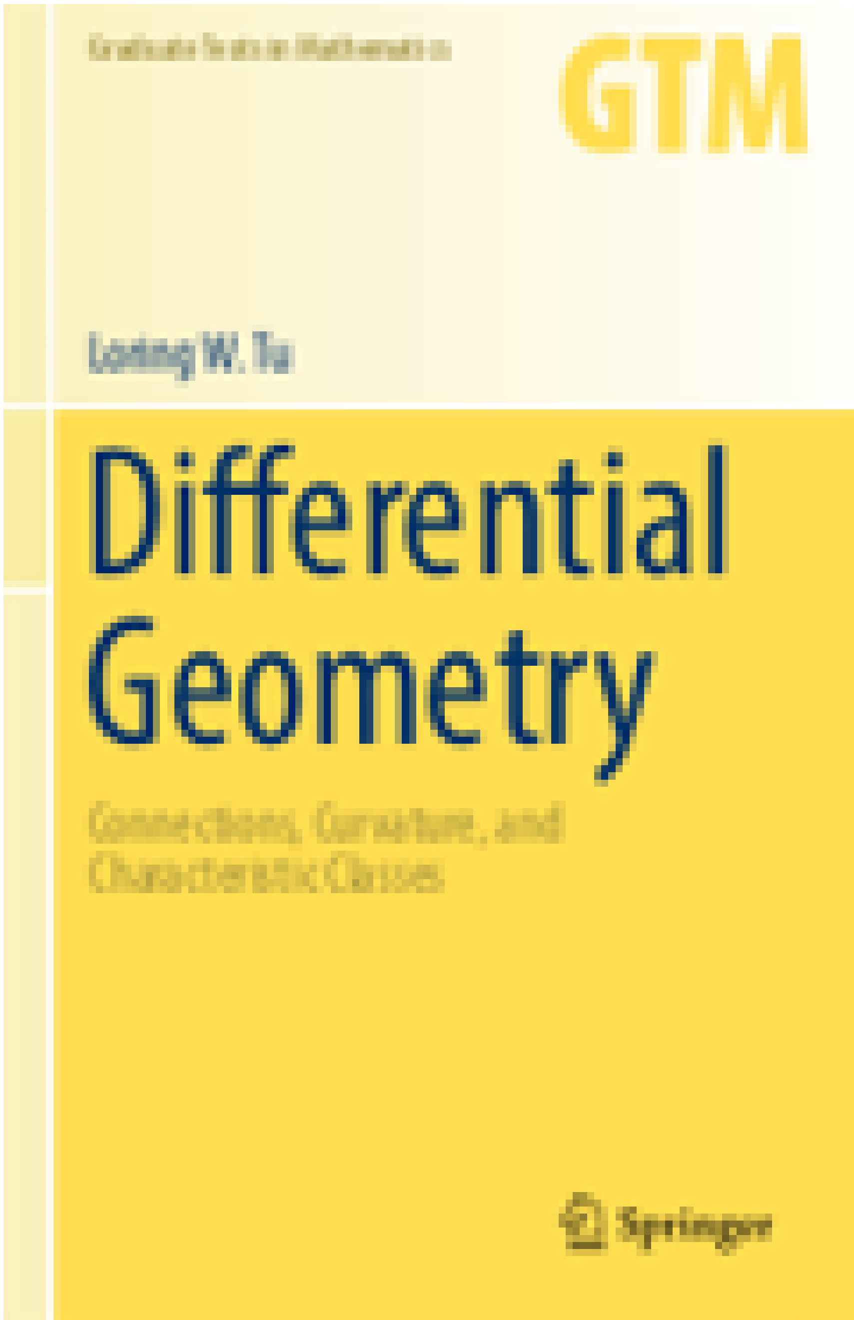
5.3 Lemma 5.1: The *tangential connection*

Same (Xf)Y term in the definition of connections

5.4 Tangential and normal projections

projections into TM or NM #**

6 TuLW DifferentialGeometry



6.1 §1 Riemannian Manifolds

6.1.1 1.3 Riemannian Metrics

A Riemannian metric maps a continuous vector field onto a continuous function.

6.1.2 Example 1.11: Torus

On a torus in R^3 vs a torus as "the quotient space of a square with the opposite edges identified"

> We will show later that there is no isometry between these two Riemannian manifolds with the same underlying torus.

6.1.3 partition of unity

What is this? The definition is unhelpful in describing the meaning of this term.

6.1.4 Theorem 1.12. On every manifold M there is a Riemannian metric.

6.2 §2 Curves

6.2.1 2.3 Signed Curvature of a Plane Curve

$T' = \gamma''(s) = \kappa\vec{n}$, so $\kappa = \langle T', \vec{n}\rangle = \langle \gamma'', \vec{n}\rangle$

6.2.2 Curvature formulae

see Needham,

6.3 §4 Directional Derivatives in Euclidean Space

6.3.1 4.1 Directional Derivatives in Euclidean Space

6.3.2 Prop 4.2: directional derivative $\mathbf{D_X Y}$

(ii) Leibniz rule: What is (Xf)Y? is this an implicit wedge product as in Barret Oneill I.6?

6.3.3 Torsion of the directional derivative D

If $D_X Y - D_Y X = [X, Y]$, then the torsion $T(X, Y) = D_X Y - D_Y X - [X, Y]$ is 0.

This only applies to $\mathbb{R}^n$.

6.3.4 Curvature of the directional derivative

6.3.5 Lie derivative == Lie bracket

6.3.6 Definition 4.8. "On" vs "Along" a submanifold

Along includes On, where On is tangent to the submanifold.

6.3.7 Notation X(M) and Γ(TM$|_N$)

- $\mathfrak{X}(M)$ is the set of all C^∞ vector fields **on** manifold M
- $\Gamma(TM|_N)$ is the set of all C^∞ vector fields **along** a submanifold N in a manifold M.
- $\mathcal{F} = C^\infty(M)$ is the ring of C^∞ functions on M

6.3.8 4.5 Directional Derivatives on a Submanifold of Rn

"Torsion no longer makes sense"

6.3.9 In a submanifold of $\mathbf{R^n}$, torsion "no longer makes sense"

because: X is a vector field **on** M Y is a vector field **along** M

6.3.10 Proposition 4.9.

6.3.11 Prop 4.11: Differentiation along a curve

6.4 §5 The Shape Operator

See Needham Chapter 15.

6.4.1 Defn: regular point – need a picture

6.4.2 5.2 The Shape Operator

$L_p(X_p) = -D_{X_p} N$, which is in the tangent plane T_pM and points opposite the direction in which N changes wrt X_p.

6.4.3 Is the usage of "on" here the same as before?

"let N be a C^∞ unit normal vector field on M (Figure 5.1)."

6.4.4 Proposition 5.3. The shape operator is self-adjoint:

6.4.5 Proposition 5.5: Curvature is given by the second fundamental form

$\kappa(X_p) = \langle L(X_p), X_p \rangle = II(X_p, X_p)$

6.4.6 Proposition 5.6: Principal directions and curvatures

- Principal directions: eigenvectors of L
- Principal curvatures: eigenvalues of L

6.4.7 5.4 The First and Second Fundamental Forms

Defined for smooth surfaces in $\mathbb{R}^3$.

- first fundamental form: inner product $\langle X, Y \rangle$ E := ⟨ e_1, e_1 ⟩ F := ⟨ e_1, e_2 ⟩ G := ⟨ e_2, e_2 ⟩
- second fundatmental form: $II(X_p, Y_p) = \langle L(X_p), Y_p \rangle$ e := II(e_1, e_1) f := II(e_1, e_2) g := II(e_2, e_2)

6.5 §6 Affine Connections

> We will see in a later section that there are infinitely many affine connections on any manifold. On a Riemannian manifold, however, there is a unique torsion-free affine connection compatible with the metric, called the **Riemannian** or **Levi-Civita** connection.

6.5.1 No canonical basis for T_pM not embedded in a Euclidean space

Formula 4.2 here is the directional derivative.

6.5.2 Definition 6.1. An affine connection

6.5.3 6.2 Torsion and Curvature

"There does not seem to be a good reason for calling T(X,Y) the torsion" Is this not the same torsion that appears in Needham Chapter 9?

6.5.4 A connection is compatible with the metric if...

$\forall$ X,Y,Z $\in$ X(M), Z $\langle$ X,Y$\rangle$ = $\langle\nabla_Z$X,Y$\rangle$ + $\langle$ X,∇_{Znil}Y$\rangle$ This is the same as D&P VIII.6, with the same confusion for me (vector on the LHS, scalar on the RHS).

6.5.5 Definition 6.4: Riemannian or Levi-Civita connection

"...is and affine connection that is torsion-free and compatible with the metric."

6.5.6 Theorem 6.6. On a Riemannian manifold there is a unique Riemannian connection.

6.5.7 6.4 Orthogonal Projection on a Surface in R3

pr_p is the projection from T_p R^3 to the tangent space of M at p.

6.5.8 Proposition 6.8.

6.5.9 6.5 The Riemannian Connection on a Surface in R^3

note distinction bewteen ∇ and D.

6.6 §7 Vector Bundles

"Thus the set X(M) has two module structures, over R and over F."

"We will try to understand F-linear maps from the point of view of vector bundles. The main result (Theorem 7.26) asserts the existence of a one-to-one correspondence between F-linear maps : (E) $\to$ (F) of sections of vector bundles and bundle maps : E $\to$ F."

6.6.1 7.1 Definition of a Vector Bundle

"A vector bundle, intuitively speaking, is a family of vector spaces that locally "looks" like U $\times$ R^r ."

We need a picture of the terms defined here.

6.7 10.1 first appearance of "d" derivative

p 10 sec 2.2 Arc Length Parameterization bottom of p 22?

6.8 11.1 nth appearance of "d" derivative in this book?

In the list of notations "d" isn't mentioned until p 93 in the discussion of the Poincare Half-Plane.

6.9 13.1 Theorem

> We call DV /dt the covariant derivative (associated to) of the vector field V along the curve c(t) in M.

6.10 13.3 Christoffel Symbols

6.11 14.5 Parallel Translation

6.12 Index

#***

7 Carroll - An Introduction to General Relativity

7.1 1. SR and flat Spacetime

7.2 2. Manifolds

7.3 3. Curvature

7.4 4. Gravitation

7.5 5. Schwarzschild's solution

7.6 6. More general black holes

7.7 7. Perturbation theory & Gravitational radiation

7.8 8. Cosmology

7.9 9. QFT in curved spacetime

7.10 A. Maps between manifolds

7.11 B. Diffeomorphisms and Lie derivatives

7.12 C. Submanifolds

7.13 D. Hypersurfaces

7.14 E. Stokes's Theorem

7.15 F. Geodesic congruences

7.16 G. Conformal transformations

7.17 H. Conformal diagrams

7.18 I. The parallel propagator

7.19 J. Noncoordinate bases

7.20 Bibliography

7.21 Index

#**

8 Littlejohn Manifolds

8.1 Notes for page 3

8.2 (typo)

should be: $x'^{\mu} = \psi_{ij}(x^{\nu})$

8.3 (typo)

should be: $x^{\nu} = \psi_{ij}^{\leftarrow}(x'^{\mu})$

8.4 8. Tangent Vectors

8.4.1 Eqn 11

8.5 9. Equivalence Classes of Curves

8.6 10. Tangent Vectors in Coordinates

"convective derivative" (see 8.4.1) "scalar" == "scalar field" in this context?

8.6.1 Eqn 16

8.6.2 tangent vector at p

Tangent vector as a `=first order, linear, partial differential operator=`
$\vec{X} = \Sigma_i X^i \frac{\partial}{\partial x^i}|_p$

8.7 12. Covectors

8.7.1 Eqn 21 $\mathrm{df}|_{\mathrm{p}}$

$df|_p \colon T_pM \to \mathbb{R} \colon X \mapsto Xf$ where Xf is defined in 8.6.1

8.7.2 $\mathrm{df}|_{\mathrm{p}}$ is an operator on vectors – it is not small or "infinitesimal"

"The most confusing thing for novices about this definition is that there is nothing small or “infinitesimal” about $\mathrm{df}|_{\mathrm{p}}$. In traditional theoretical physics the notation df usually denotes a small increment in the function f."

8.7.3 13. Differential of Coordinates

#***************************************

9 BachmanD: A Geometric Approach to Differential Forms

9.1 1.2 Generalizing the integral

9.2 1.3 differential form in a nutshell

"A differential form is precisely a linear function which eats vectors, spits out numbers and is used in integration."

9.3 3 Forms

9.3.1 3.1 Coordinates for vectors

"The key to understanding the difference between L and T_pL is their coordinate systems."

1. notation <.,.>

   > We have switched to the notation "$<\cdot, \cdot>$" to indicate that dt we are not talking about points of P anymore, but rather vectors in $\mathrm{T_p}$ P .

9.3.2 3.2 1-Forms

> Evaluating a 1-form on a vector is the same as projecting onto some line and then multiplying by some constant.
>
> Evaluating a 1-form on a vector is the same as projecting onto each coordinate axis, scaling each by some constant and adding the results.

9.3.3 3.3 Multiplying 1-forms

The wedge product is a product operation on 1-forms that is closed, ie, produces another 1-form.

"we will use the symbol "" (pronounced "wedge") to denote multiplication"

1. Notation (),<>,[] (): x,y plane <>: dx,dy plane []: ω,ν plane
2. wedge product justification or reasoning

> Do we know of a way to take these vectors and get a number? Actually, we know several, but the most useful one turns out to be the area of the parallelogram that the vectors span. This is precisely what we define to be the value of (V1 , V2)

Still – why the skew-symmetric product vs any other one? What is special about a/the skew-symmetric operator?

3. 1-form in a nutshell

> Evaluating on the pair of vectors (V1 , V2) gives the area of parallelogram spanned by V1 and V2 projected onto the plane containing the vectors $\langle\ \rangle$ and $\langle\ \rangle$, and multiplied by the area of the parallelogram spanned by $\langle\ \rangle$ and $\langle\ \rangle$.

4. 2-form in a nutshell

> Every 2-form projects the parallelogram spanned by V1 and V2 onto each of the (2-dimensional) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results.
>
> Henceforth, we will define a 2-form to be a bilinear, skew-symmetric, real- valued function on $T_p\mathbb{R}^n \times T_p\mathbb{R}^n$.

9.3.4 3.4 2-Forms on $T_p\mathbb{R}^3$

9.3.5 3.5 2-Forms and 3-forms on $T_p\mathbb{R}^4$

9.3.6 3.6 n-Forms

length, area and volume are all signed quantities?

9.3.7 3.7 Algebraic computation of products

9.4 4 Differential Forms

#**

10 OneillB Elementary Differential Geometry

1st edition (1966)

10.1 I Calculus on Euclidean space

10.1.1 I.5 1-forms (p 22)

1. I.5.2 Def: the differential df is a 1-form (p 23) If f is a differentiable real-valued function on $\mathbb{E}^3$, $df(\vec{v}_p) = \vec{v}_p[f]$ examples:

   - $dx_i[\vec{v}_p] = v_i$
   - in $\mathbb{E}^3$, any 1-form has the form $fdx + gdy + hdz$, where f,g, and h are differentiable real valued functions

   interpretation:

   - on the RHS, the vector acting on a function is the directional derivative in the vector direction.
   - on the LHS, $df = f_x dx + f_y dy + f_z dz$.
   - in components, the expression above is $v_x \partial_x f + v_y \partial_y f + v_z \partial_z f$

10.1.2 I.6 Differential forms (p 26)

1. I.6.3 Def: exterior derivative of a 1-form if $\phi = \sum f_i dx_i$ is a 1 form on $\mathbb{E}^3$, the *exterior derivative* of ϕ is the 2-form $d\phi = \sum df_i \wedge dx_i$

10.1.3 I.7 Mappings (between spaces or manifolds) (p 32)

1. I.7.4 Def: Derivative map F$*$ of F (p 35) Let $F\colon \mathbb{E}^n \to \mathbb{E}^m$ be a mapping. If $\vec{v} \in T_{\mathbf{p}}\mathbb{E}^n$, then let $F_*(\vec{v}) \in T_{F(\mathbf{p})}\mathbb{E}^m$ be the initial velocity of the curve $t \to F(\mathbf{p} + t\vec{v})$. F_* is the *derivative map* of F.

   This is the equivalent of $D_x f$ in D&P.

2. I.7.8 Thm: F$*$ preserves the velocity of curves (p 38) Given curve α and mapping F, if curve $\beta = F(\alpha)$, then $\beta' = F_*(\alpha')$

3. I.7.9 Thm: Inverse function theorem (p 39) Statement w/o proof of the theorem with commentary

   "The proof is based on the idea that at points $\mathbf{p} + \mathbf{\Delta p}$ very near $\mathbf{p}$, $F(\mathbf{p} + \mathbf{\Delta p}) \approx F(\mathbf{p}) + F_*(\mathbf{\Delta p})$"

10.1.4 I.8 Summary (p 41)

Oneill gives a differentiation operation for each object introduced:

object	diff op
function in R	directional derivative
form	exterior derivative
curve	velocity
mapping	derivative map
...	
vector field	connection (??)

All of the differentiation operations here (first 4) reduce to ordinary or partial derivatives of real-valued coordinate functions, but the *definitions* do not involve coordinates.

10.2 II Frame Fields

10.2.1 II.1.4 Def: *frame* (p 44)

A frame at point p is a set $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_\mathrm{n}\}$ mutually orthogonal unit vectors in $\mathbb{E}^n$

10.2.2 II.2.2 Def: *vector field on a curve* (p 52)

This is NOT the same definition used in Tu or D&P – **on** here does not require that the vectors be tangent to the curve itself. In Tu and D&P, this would be **along** a curve. Oneill does not seem to use any terminology to describe the set of vectors tangent to the curve aside from calling it a parameterization of the velocity of the curve.

10.2.3 II.5 Covariant derivatives (p 77)

11 Derivatives

11.1 of function between...

- D&P: affine spaces manifolds $D_x f$
- Oneill: I.7.4 Derivative map F_* of F (p 35)

11.2 exterior derivatives (div, curl, grad)

- Tu (IM): 4 Differential forms on R^n "from C^∞ function f:U $\to$ R, we can construct a 1-form *df*, called the *differential of f*
- Tu (IM) extends this to 4.4 The Exterior Derivative
- Needham Chapter 36 Differentiation introduces exterior derivative, relating it to the intrinsic/covariant/Levi-Civita derivative and uses it to derive div, curl and grad for 1-3 forms in R^3.
- Oneill: I.6.3 Def: exterior derivative of a 1-form

11.3 "higher derivatives"

D&P: VII.1.02. Higher Derivatives. $\hat{D}f$ "D-hat"

11.4 partial derivatives

D&P: VII.1.03 Partial derivatives $\frac{\partial f^i}{\partial x^j}$ or $\partial_j f^i$

11.4.1 contravariant vectors

D&P: contravariant vectors intersection of partial derivative, vectors and "differentials" of real-valued functions

11.5 "differential" between manifolds

D&P: Df: TM $\to$ TN Dw: TM $\to$ T(TM)

11.5.1 for real valued functions

D&P: df: TM $\to$ R (not TR) not to be confused with the difference function for affine spaces.

11.6 Lie derivative/Lie bracket

- D&P: Ex VII.7.4 but don't look here!
- Tu (DG): Lie derivative == Lie bracket (sec 4.2)
- Needham: Gives the Lie bracket as 29.5 [v , u] $_v$ u $_u$ v. In the prior section (29.4), he makes a Notation change for **intrinsic derivative**, using a **bold** ∇ to indicate the intrinsic derivative.

11.7 $\nabla_{\mathbf{t}}$ Directional derivative

- First appears in D&P Ex VII.2.1 as the Levi-Civita connection?
- next in D&P as the "directional derivative"
- D&P: by VIII.6.03, $\mathbf{w}(\cdot)$ and $\nabla_{\mathbf{w}}(\cdot)$ are equivalent notations, but this leads to confusion with Torsion and the Lie Bracket.
- Tu (DG): 4.1 Directional Derivatives in Euclidean Space tangent vector at point p: $X_p = \Sigma a^i \frac{\partial}{\partial x^i}|_p$ given a C^∞ function f in neighborhood of p, the *directional derivative*, $D_{X_p} f = X_p f$
- Tu (DG): 6.5 The Riemannian Connection on a Surface in R^3 $\nabla_X Y = pr(D_X Y)$ where ∇_{X} is the Riemannian connection and D_{X} is the directional derivative $pr(D_X Y) = X - \langle X, N \rangle N$
- Needham: 15.1 Directional Derivatives (∇_{v}) $\nabla_{\mathbf{v}}\mathbf{n}$ is the "directional derivative of n along v". Not sure if **along** here holds the technical meaning that it does in D&P or Tu. When n is the normal vector to the surface, this derivative is the negative of 15.2 The Shape Operator S
- Needham relates this "extrinsic derivative" to the intrinsic derivative via the shape operator Eqn 23.3, section 23.2

11.8 ∇ (connection on a manifold)

- D&P 3.01. Definition. A connection (∇) has properties Ci: linear in the input tangent vector space Cii: linear in the vector field space Ciii: scaling t is the same as scaling w Civ: Leibniz property Cv: continuous vector fields have continous connections
- D&P: Levi-Civita connection for G when it is the symmetric connection compatible with G. (zero torsion? see 5.01: T=0 ==> ∇ is symmetric) D&P call ∇ the *Levi-Civita connection* after VIII.6.05 $\nabla\mathbf{v}$ a *covariant differential*
- Tu (DG): Definition 6.1. An affine connection the unique torsion-free, compatible affine connection is called a Riemannian or Levi-Civita connection.

11.9 $\tilde{\nabla}_{c^*}$ where c* is a velocity along the curve.

- D&P: differential *along* a curve This differs from the non-decorated ∇ by having its domain restricted from the full manifold to a curve on the manifold.
- Tu (DG): Prop 4.11: Differentiation along a curve dV/dt = $\mathrm{D_{c'(t)}}$ V(t)
- Tu (DG): 13.1 Theorem Tu calls this the **covariant derivative associated to ∇**

11.10 parallel transport is a key to identifying notational differences among texts

- D&P: VIII.4.01 $\tilde{\nabla}_{c^*}\mathbf{v} = \mathbf{0}$
- Tu (DG): 14.5 Parallel Translation $DV/dt = 0$ on the interval over which the curve is defined. (covariant derivative)
- Needham: condition for parallel transport $D_{\mathbf{v}}\mathbf{w}_{||} = \mathbf{0}$ in Needham, $\mathrm{D_v}$ is the **intrinsic derivative**, which he notes is commonly called the **covariant derivative** or the **Levi-Civita connection**. In contrast, he uses $\nabla_{\mathbf{v}}\mathbf{w}$ to denote the *extrinsically defined rate of change."

12 Torsion

`https://jollywatt.github.io/simple-torsion-example` - example of a connection with non-zero torsion.

12.1 D&P VIII.6?

5.01: T=0 ==> ∇ is symmetric according to Tu, this is an iff situation.

12.2 Tu (DG)

Torsion of the directional derivative D (A.2) Lie bracket: $[X,Y]_p f = X_p(Yf) - Y_p(Xf)$ for $p \in M, f \in C_p^\infty(M)$ $T(X,Y) = D_X Y - D_Y X - [X,Y]$ from p 25, the first two terms above are the Lie bracket, so T(X,Y) = 0 always (for Euclidean space?)

For a Submanifold of $\mathrm{R^n}$, the torsion "no longer makes sense." Why? because X is on M and Y is along M?

Appears again in the context of manifolds in 6.2 Torsion and Curvature.

12.3 Needham

Needham's definition: Torsion is a property of a curve in 3-space, so it appears to be fundamentally different from the concept of torsion in D&P and Tu.