Digital Image Processing HW1

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Question 1

Since the camera moves during the time interval (T), the intensity of each pixel in $g_k[n]$ is integration on the location of this pixel in the image at that time. We will mark $\tau_{p(t)}$ as the movement (translation - p) of the camera at time t, and we assume that the PSF is ideal box with ideal anti-aliasing filter, therefore:

$$h_k(x) = \int_k^{k+T} \tau_{p(t)} Box(x)\, dt$$

Where $Box(x)$ is the unit box function in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$ as seen in the lecture.

Question 2

$$H_k(\xi) = \mathcal{F}[h_k(x)](\xi) = \mathcal{F}\left[\int_k^{k+T} \tau_{p(t)}\, Box(x)\, dt\right](\xi) =$$

We can use the linearity of the transform:

$$= \int_k^{k+T} \mathcal{F}[\tau_{p(t)}\, Box(x)]\, (\xi) dt$$

We have seen in the lectures that the Fourier transform of the box is $sinc$, and we will use the Fourier transform of a translation as well:

$$= \int_k^{k+T} e^{-2i\pi\xi^T p(t)} sinc(\xi)\, dt$$

Question 3

We can use the definition of $g_k[n]$ and use the Fourier transform and the convolution theorem.

$$G_k[w] = \mathcal{F}[f(x) * h_k(x)](w) = F[w]H_k[w] = F[w] \int_k^{k+T} e^{-2i\pi w^T p(t)} sinc(w)\, dt =$$

The $sinc$ doesn't depends on t therefore we can take it out of the integral:

$$= F[w] \int_k^{k+T} e^{-2i\pi w^T p(t)}\, dt\; sinc(w) = F[w]P_k[w]sinc(w)$$

And we got that $P_k[w] = \int_k^{k+T} e^{-2i\pi w^T p(t)}\, dt$

Question 4

First we will use Cauchy-Schwarz inequality:

$$|P_k[w]| = \left|\int_k^{k+T} e^{-2i\pi w^T p(t)}\, dt\right| < \int_k^{k+T} |e^{-2i\pi w^T p(t)}|\, dt \;=\; \int_k^{k+T} \left|\left(e^{i\pi}\right)^{-2w^T p(t)}\right| dt =$$

Now we will use $e^{i\pi} + 1 = 0 \Rightarrow e^{i\pi} = -1$:

$$= \int_k^{k+T} \left|(-1)^{-2w^T p(t)}\right| dt = \int_k^{k+T} \left|((-1)^{-2})^{w^T p(t)}\right| dt = \int_k^{k+T} \left|(1)^{w^T p(t)}\right| dt = \int_k^{k+T} dt = T$$

We got that the upper bound is T.

Question 5

In this case, for our notations $p(t) = v[t-k-0.5, 0, 0]^T$ and $w = [w_1, w_2, w_3]^T$. We can assign this value into our $P_k[w]$ expression:

$$P_k[w] = \int_k^{k+T} e^{-2i\pi w_1 v(t-k-0.5)}\, dt = e^{2i\pi w_1 v(k+0.5)} \int_k^{k+T} e^{-2i\pi w_1 vt}\, dt$$

$$= e^{2i\pi w_1 v(k+0.5)} \left[\frac{e^{-2i\pi w_1 vt}}{-2i\pi w_1 v}\right]_k^{k+T} = e^{2i\pi w_1 v(k+0.5)} \frac{e^{-2i\pi w_1 vT}}{-2i\pi w_1 v} = \frac{e^{2i\pi w_1 v(k+0.5-T)}}{-2i\pi w_1 v}$$

Now we can look at $|P_k[w]|$:

$$|P_k[w]| = \left|\frac{e^{2i\pi w_1 v(k+0.5-T)}}{-2i\pi w_1 v}\right| = \frac{\left|e^{2i\pi w_1 v(k+0.5-T)}\right|}{|-2i\pi w_1 v|} = \frac{1}{|-2i\pi w_1 v|} = \frac{1}{2\pi w_1 v}$$

(The Nominator is on the unit circle, therefore it's absolute value is always 1)

We got that $|P_k[w]|$ has reverse ratio to v.

Question 6

This time, for our notations $p(t) = o(\tau)$.

$$P_k[w] = \int_k^{k+T} e^{-2i\pi w^T p(t)}\, dt = \int_k^{k+T} e^{-2i\pi \boldsymbol{w}^T(\boldsymbol{q}+t\boldsymbol{v})}\, dt = e^{-2i\pi \boldsymbol{w}^T \boldsymbol{q}} \int_k^{k+T} e^{-2i\pi t \boldsymbol{w}^T \boldsymbol{v}}\, dt$$

$$= e^{-2i\pi \boldsymbol{w}^T \boldsymbol{q}} \left[\frac{e^{-2i\pi t\boldsymbol{w}^T \boldsymbol{v}}}{-2i\pi \boldsymbol{w}^T \boldsymbol{v}}\right]_k^{k+T} = e^{-2i\pi \boldsymbol{w}^T \boldsymbol{q}} \frac{e^{-2i\pi T\boldsymbol{w}^T \boldsymbol{v}}}{-2i\pi \boldsymbol{w}^T \boldsymbol{v}} = \frac{e^{-2i\pi \boldsymbol{w}^T(\boldsymbol{q}+T\boldsymbol{v})}}{-2i\pi \boldsymbol{w}^T \boldsymbol{v}}$$

Now lets look at $|P_k[w]|$:

$$|P_k[w]| = \left|\frac{e^{-2i\pi \boldsymbol{w}^T(\boldsymbol{q}+T\boldsymbol{v})}}{-2i\pi \boldsymbol{w}^T \boldsymbol{v}}\right| = \frac{1}{2\pi \boldsymbol{w}^T \boldsymbol{v}}$$

Question 7

We can use the approach suggested in the paper. Instead of solving the inverse problem, we can use the weighted average of the Fourier transform of the images burst. We use the assumption that the direction of movement is random for each image in the burst, therefore we have images taken with movement to different directions.

We use the fact that moving to a certain direction, smears the image in that directions, thus attenuates high frequencies, but it keeps the high frequencies on the vertical direction, therefore we can combine the high frequencies from different images to reconstruct a sharp image. We take frequencies that was attenuated less from each image in the burst, and give them more weight in the Fourier transform of the reconstructed image we create, and then we can use the inverse Fourier transform to get the original image.

<u>Formally</u>:

We have a burst of images

$$\{g_1[n], \dots g_N[n]\}$$

we compute their discrete Fourier transform:

$$\{G_1[\xi] \dots G_N[\xi]\}$$

We compute the weight of each frequency in each image, as it's magnitude compared to the other images in the burst.

$$W_i[\xi] = \frac{|G_i[\xi]|^p}{\sum_{j=1}^{N} \left|G_j[\xi]\right|^p} \quad \forall i \in [1 \dots N], \xi$$

Where p is a hyperparameters – the higher it is the more weight there is to images with higher magnitude in each frequency, therefore we get sharper image but amplified noise.

We take the inverse Fourier transform of the calculated weighted sum of the burst images:

$$\hat{g}[n] = \mathcal{F}^{-1}\left\{\frac{\sum_{i=1}^{N} W_i[\xi] \cdot G_i[\xi]}{\sum_{i=1}^{N} W_i [\xi]}\right\}$$

The multiplication and division in this formula is pointwise in the frequency.