Algo HW 2

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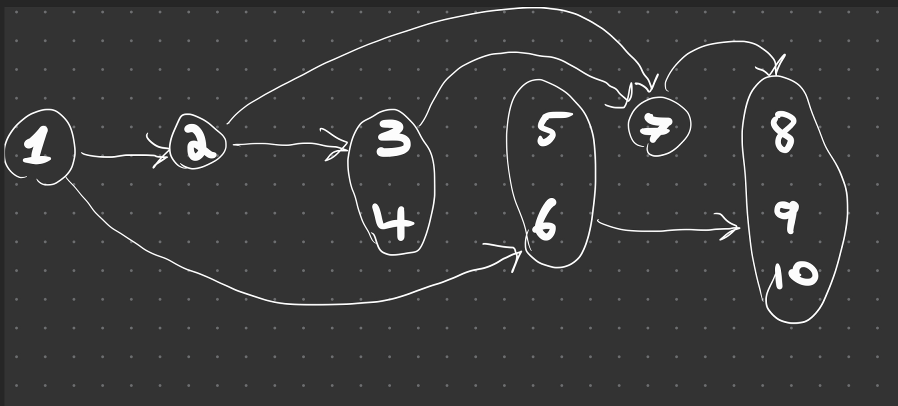
Exercise 1.

Section Aleph

Connected components: $V = \{\{8,9,10\},\{3,4\},\{1\},\{2\},\{5,6\},\{7\}\}$

Components graph: The graph is:

$E = \{(\{1\},\{2\}),(\{1\},\{5,6\}),(\{2\},\{3,4\}),(\{2\},\{7\}),(\{3,4\},\{7\}),(\{5,6\},\{8,9,10\}),(\{7\},\{8,9,10\})\}$



Section Bet

$$
\begin{array}{ccc}
 & 2 & \\
3 & 4 & 1 \\
 & & | \\
 & & 5
\end{array}
$$

Exercise 2.

1. True
2. False
3. False
4. True
5. False
6. False
7. False

Exercise 3.

Algorithm:

1. Run the `SCC` algorithm from the lecture.
2. Starting from the smallest SCC, check for each vertex `v` in the component if there is an edge in `e` from `v` to a `v'` in another component. if there isn't return the component as `U` and exit.
3. Return `NULL`.

Correctness:

Lemma 1. *If there is close group U in graph G, then it's one of the strongly connected components.*

Proof. Lets consider in contradicting that U is not one of the SCC. meaning it has a mix of vertices form different SCC or it lacks a number of vertices from his own SCC or there is multiple SCC's in U.

In the first case, let `v` be the vertex form the SCC which is't present at his fullest. because `v` is part of the SCC there's an edge form `v` to some other vertex in his SCC which isn't part of U - in contradiction to U.

In the second case, let `v` be one of the vertices in SCC. by the definition of this case, there is at list one vertex form `v`'s SCC which isn't part of U - lets say `w`. there is a path from `v`to `w` (because they are part of the same SCC), let `u` be the last vertex in this path that is part of U. meaning there is an edge from `u` to some other vertex outside of U - in contradiction to U.

In the last case, let `A` be the first SCC and `B` be the second SCC. we know that the SCC's are not well connected meaning there isn't a path from `A` to `B` and from `B` to `A`. meaning there is at list one edge that crosses the two SCC's - especially there isn't any two edge's form A to B and from B to A (because if there were, A and B would have been one SCC). and because both A and B are smaller then U - there is a contradiction to the minimality of U. □

Using Lemma 1. we know that one of the SCC is the smallest Closed Group.

By iterating over the SCC's from the smallest one to the largest, we are guaranteeing that we will receive the smallest Closed Group.

Complexity: Executing the SCC algorithm is $O(|V+E|)$. For each SCC we will check each of the vertices in the SCC if one of it edges is crossing the SCC.

meaning, we will iterate on the graph G checking each vertex and each edge only once - time complexity of $O(|E+V|)$.

In conclusion, the time complexity is $O(|E+V|)$.

Exercise 4.

Let $G = (V, E)$ undirected graph, connected with weights on the edges.

Let T, T' different minimal spanning trees of G.

Let $e \in T \backslash T'$.

RTP: $\exists e' \in T' \backslash T$ s.t. is a minimal spanning tree of G with weights.

Proof. We will refer to T as the blue tree and T' as the red tree (this red/blue names are completely unrelated to the algorithms we learned in class).

After removing the blue edge e from T. We get a forest of two trees. Let's look at the cut separating the two connected components. The blue edge e crosses the cut ofc because this is the edge that used to connect the subtrees.

Now let's look at $T' \cup \{e\}$, this is ofc a graph with a cycle because it has an extra edge (because of combinators).

The edge e is part of the cycle (also combi). Thus, there must be another red edge that crosses the cut (at least one).

BWOC: Let's assume all the red edges that crosses the cut have higher weight than the weight of e.

Thus, we can remove one of those red edges, let's mark it l, and get a connected graph $(T \backslash \{e\}) \cup \{l\}$ with $|V| - 1$ edges (it must be connected because of combi again: if it weren't, then there is only a single simple path between the nodes of l).

For this new graph, the sum of weights is $w(T) + w(e) - w(l) \underbrace{<}_{w(e)<w(l)} w(T)$ in contradiction to T being MST.

Thus, exists a red edge e' that crosses the cut and $w(e') = w(e)$.

Note the $e' \notin T$ because it crosses the cut and $e' \neq e$ (the whole idea of the cut it separating the two connected components in $T \backslash \{e\}$ which means there are no edge connecting them).

So $e' \in T' \backslash T$ and $w() = w(T) - w(e) + w(e') \underbrace{=}_{t=w(e)=w(e')} w(T)$.

Since is built from two connected components with an edge connecting those two connected components. Also, T has $|V| - 1$ edges (because we removed an edge and added another one so we stayed with $|V| - 1$).

From those two claims, we get that is an MST. $\square$