ALGO HW 3

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1. Question

The weights of the lightest paths to the nodes from s are

$$
\begin{aligned}
w\left(s\to s\right) &= 0\\
w\left(s\to a\right) &= -4\\
w\left(s\to b\right) &= 2\\
w\left(s\to c\right) &= -1\\
w\left(s\to d\right) &= 0
\end{aligned}
$$

S
|
B
|
C
|
D
|
A

2. Question

(1) True
(2) True
(3) False
(4) False
(5) False

3. Question

Proof. We'll use Bellman-Ford on a modified graph.

Let us modify the graph so there will be 3 "layers" of the graph (3 copies of each node, each representing a different $\mod 3$ result).

The directed graph $G' = (V', E')$ s.t. $V' = \{v_0, v_1, v_2 \mid v \in V\}$ (note that does s we don't need duplicates but it won't hurt).

For each edge $(u, v) \in E$ with weight $w((u, v)) \mod 3 = m$, then $(u_0, v_m), \left(u_1, v_{(1+m) \mod 3}\right), \left(u_2, v_{(2+m) \mod 3}\right) \in E'$.

We'll also expand the definition of w s.t. $w((u_j, v_i)) = w((u, v))$.

We got $|E'| = 3|E|$ and $|V'| = 3|V|$.

Before we run the Bellman-Ford, let us prove that if we go from s_0 (s_1, s_2 are unreachable) in a certain path and end up in node v_m (for $m \in \{0, 1, 2\}$) then the total path's length modulu 3 was m.

Lemma 1. *Starting from s_0. The length of the path is* 0.

Proof. Let's assume we're in u_m and the total path length modulu 3 is m. We'll mark the length of the path as l.

Now for an edge $(u_m, v_t) \in E'$ (for some v), but from the definition of the graph (how defined its edges) $t = (m + w((u, v))) \mod 3$, meaning that continuing the path to v_t will give us a path of length $l + w((u, v))$ and with $\mod 3$ we get t. And that concludes the induction. □

Also, for a shortest path of length l from s to u in G, the same path exists (using the marked edges we create) and it will necessarily end in $u_{l \mod 3}$ from the lemma we just proved.

So each path exists and ends in the correct node replica.

And now let us run the Bellman-Ford starting from s_0 in G'.

For each node $v_m \in V'$, we now have the shortest path from s_0 to v_m, and because all of the paths from the original graph G exist in G', when the node v_m will not contain the shortest path of length with a remainder of m when divided by 3.

If we just look at v_0 $(\forall v \in V)$ we get all the shortest paths with a length dividable by 3 with no remainder.

Q.E.D □

4. Question

Proof. We'll prove in induction on the amount of edges in the path. The node with the longest shortest path is with at most $|L| + |R|$ edges because there are no negative weight cycles.

The idea: each path can be divided into at most $\frac{|L|+|R|}{2}$ pairs of edges with the first being in E_1 and the second in E_2 (optionally with an additional edge from E_1 at the end).

We'll prove for an even number of edges in the path. For an odd number, we can add an extra edge to a dummy node in the L set, and use this dummy edge with the unpaired edge (that caused the path to be of an odd number of edges).

For a path with 0 edges

Then no iterations needed.

Let's assume that we got all the paths with $2k$ edges (or less) until the kth iteration

For a shortest path with $2k + 2$ edges, the subpath which includes the first $2k$ edges (which is also shortest because if it wasn't we could use the better path to that node and concat the last two edges and get a shorter path), was discovered already.

The last two edges are from L to R and then from R to L. We'll call the last two nodes in the path v, u respectively.

Since the given algorithm does $L \to R$ first and then $R \to L$, we'll find the shortest paths to v and u both because of the order in which they're updated. Thus, until the $(k + 1)$th iteration, we got all the paths with at most $2k + 2$ edges.

In conclusion, at most we have paths of $|L| + |R| - 1$ edges, so we'll get even the longest path until at most the iteration $\frac{|L|+|R|}{2}$. Again, for an odd amount of edges, we'll use a dummy node which won't matter due to extra iteration we have (also, during the proof, we explained that we get all odd paths which are subpaths of even paths).

Q.E.D $\square$