VECTORIAL ANALYSIS HW 2

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1. Question

Proof. We can look at the 3 points

t	$x(t)$	$y(t)$	$z(t)$
0	0	0	1
1	2	0	2
2	6	2	3

Since all 3 points need to be on the plane then $Ax + By + Cz + D = 0$ must apply to these 3 points.

We know that the two vectors $(2-0, 0-0, 2-1) = (2,0,1)$ and $(6-0, 2-0, 3-1) = (6,2,2)$ are parallel to the plane so $(2,0,1) \times (6,2,2)$ is a normal to the plane.

Meaning that $\begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 6 & 2 & 2 \end{vmatrix} = -2\hat{i} - (4-6)\hat{j} + (4-0)\hat{k} = -2\hat{i} + 2\hat{j} + 4\hat{k}$ is a normal to the plane. Thus $A = -2, B = 2, C = 4$.

Since $(0,0,1)$ is on the plane then $-2 \cdot 0 + 2 \cdot 0 + 4 \cdot 1 + D = 0 \implies D = -4$

So the plane is $-2x + 2y + 4z - 4 = 0$ is the (only) plane that contains the 3 points.

Let's assign $x(t), y(t), z(t)$:

$$-2\left(t^2 + t\right) + 2\left(t^2 - t\right) + 4(t+1) - 4 = \cancel{-2t^2} \cancel{-2t} + \cancel{2t^2} \cancel{-2t} + \cancel{4t} + 4 - 4 = 4 - 4 = 0 = 0$$

So we got that the plane's equation is true for all $(x(t), y(t), z(t))$. So the entire curve is on the plane.

Q.E.D. □

2. Question

Vector $\vec{v} \neq \vec{0}$ in $\mathbb{R}^3$. And

$$f(x,y,z) = \begin{cases} \frac{|(x,y,z)\times\vec{v}|}{|(x,y,z)|} & (x,y,z) \neq \vec{0} \\ 0 & else \end{cases}$$

We'll note that $\frac{|(x,y,z)\times\vec{v}|}{|(x,y,z)|} = \frac{|(x,y,z)\cdot|\vec{v}|\cdot\sin\theta\cdot\vec{n}|}{|(x,y,z)|} = \vec{v}\cdot\sin\theta$

2.1. **Aleph.**

So taking a different vector (say $\vec{u}$) is equal to rotating the function by an angle and multiplying by a constant ($\frac{|\vec{u}|}{|\vec{v}|}$). In other words, the function is isometric.

2.2. **Bet.**

Proof. From previous section, we get that only the angle between (x,y,z) and $\vec{v}$ affects the value of f. So for each value of value k of f, all the vectors with an angle of $\sin^{-1}\left(\frac{k}{|\vec{v}|}\right)$ get the same value with f.

So let's cross product a vector $\vec{u} = (a,b,c)$ with $\vec{v} = (0,0,1)$:

$$(a,b,c)\times(0,0,1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = (b-0)\,\hat{i} - (a-0)\,\hat{j} + (0-0)\,\hat{k} = b\hat{i} - a\hat{j} + 0\hat{k}$$

And the magnitude of the cross product is $\sqrt{b^2+a^2}$.

So

$$f(a,b,c) = \frac{\sqrt{b^2+a^2}}{|(a,b,c)|} = \frac{\sqrt{b^2+a^2}}{\sqrt{a^2+b^2+c^2}} = k \implies \sqrt{b^2+a^2} = k\sqrt{a^2+b^2+c^2} = b^2+a^2 = k^2\left(a^2+b^2+c^2\right) \implies$$
$$\implies b^2+a^2 = k^2a^2+k^2b^2+k^2c^2 \implies \left(k^2-1\right)a^2 + \left(k^2-1\right)b^2 + k^2c^2 = 0 \implies$$
$$\implies \left(k^2-1\right)a^2 + \left(k^2-1\right)b^2 + k^2c^2 = 0$$

1. For $k<0$ there is no solution because the fraction of two square root will never be negative.
2. For $k=0$ we get $a^2+b^2=0$ but in $\mathbb{R}$ this is possible only if $a=b=0$. And c can be anything. So all the vectors $(0,0,c)$.
3. For $1>k>0$ we get $\left(k^2-1\right)<0$ and thus $-\left(k^2-1\right) = \left(1-k^2\right)>0$. So $\frac{a^2}{\frac{k^2}{(1-k^2)}} + \frac{b^2}{\frac{k^2}{(1-k^2)}} = c^2$ is an elliptical cone (equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$ with $a=b=\frac{k}{\sqrt{(1-k^2)}}$). Note that for each k in this range, the level set is a different elliptical cone.
4. For $k=1$ we need $c=0$ for the nominator and the denominator to equal. So all $\vec{u}=(a,b,0)$ would solve the equation $f(\vec{u}) = k = 1$.
5. For $k>1$ there are no solutions since $\sqrt{b^2+a^2} \leq \sqrt{a^2+b^2+c^2}$.

And the answer is yes. There's a level set $\{(a,b,0) \mid a,b\in\mathbb{R}\}$ a plane which is perpendicular to the line level set $\{(0,0,c) \mid c\in\mathbb{R}\}$. It's easy to see that $(a,b,0)\cdot(0,0,c) = a\cdot 0 + b\cdot 0 + 0\cdot c = 0+0+0 = 0$. $\square$

(3) א

$$\begin{cases} x^2+y^2\neq 0 \Rightarrow (0,0,z)\notin A \\ -1\leq \cos t\leq 1 \end{cases}$$

$$\Downarrow$$

$$-1\leq \frac{z}{\sqrt{x^2+y^2}}\leq 1 \quad \leftarrow$$ תחום ההגדרה

$$-\sqrt{x^2+y^2}\leq z\leq \sqrt{x^2+y^2}$$

$$\Downarrow$$

$$z^2=x^2+y^2 \quad \leftarrow$$ חרוט דו-כיווני

A פתוחה כיוון שאינה מכילה את הראשית אך; מכילה נקודה ששואפת לראשית.

A לא חסומה, מח"כ לכל $xy\neq 0$ קיים z כך: ש- $(x,y,z)\in A$.

A קשירה כי לכל שתי נקודות ב-A קיים עקום שמחבר אותן ומוכל ב-A.

ב $S_2=\{(x,y,z)\mid x^2+(y-1)^2+z^2=1\}$

ספרה ברדיוס 1 סביב נקודה $(0,1,0)$.

S_1 שפת A ולכן הנקודות המקיימות $z^2=x^2+y^2$

$S_1=\{(x,y,z)\mid x^2+y^2-z^2=0\}$

נמצא את חיתוך המשטחים S_1,S_2.

$$x^2+y^2-2y+z^2=x^2+y^2-z^2$$
$$2z^2=2y$$
$$z^2=y$$

נציג:

$$x^2+y^2-y=0 \Rightarrow x^2+\left(y-\tfrac{1}{2}\right)^2-\tfrac{1}{4}=0 \Rightarrow x^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$

במישור xy, זה מעגל סביב $(0,\frac{1}{2})$ עם רדיוס $\frac{1}{2}$.

$$x(t)=\tfrac{1}{2}\cos t,\quad y(t)=\tfrac{1}{2}\sin t+\tfrac{1}{2},\quad z(t)=\sqrt{\tfrac{1}{2}\sin t+\tfrac{1}{2}} \qquad (z^2=y)$$

$$\gamma(t)=\left\{\left(\tfrac{1}{2}\cos t,\ \tfrac{1}{2}\sin t+\tfrac{1}{2},\ \sqrt{\tfrac{1}{2}\sin t+\tfrac{1}{2}}\right)\ \middle|\ t\in[0,2\pi]\right\}$$