VECTORIAL ANALYSIS HW 4

Karin Artmonov - 205938772

Gur Telem - 206631848

1. Question

Need to calculate $\int_C x^2 y dx - y^2 x dy$ with C is the upper half circle $x^2+y^2=a^2$ counterclockwise.

Proof. Let's look at the parameterization of $r(t) = (a\cos(t), a\sin(t), 0)$ for $0 \le t \le \pi$.
So $r'(t) = (-a\sin t, a\cos t, 0)$

$$\int_C x^2 y dx - y^2 x dy = \int_{t=0}^{\pi} (a\cos t)^2 (a\sin t) \underbrace{(-a\sin t)}_{r'(t)_x} - (a\sin t)^2 (a\cos t) \underbrace{(a\cos t)}_{r'(t)_y} dt =$$

$$= \int_{t=0}^{\pi} a^4 \cos^2 t \sin^2 t \cdot (-1-1)\, dt = -\frac{1}{2}a^4 \int_{t=0}^{\pi} (2\cos t \sin t)^2\, dt = -\frac{1}{2}a^4 \int_{t=0}^{\pi} (\sin 2t)^2\, dt =$$

$$= -\frac{1}{2}a^4 \int_{t=0}^{\pi} \frac{1}{2} - \cos 4t\, dt = -\frac{1}{2}a^4 \left(\frac{\pi}{2} - \underbrace{0}_{\int_{t=0}^{\pi} \cos 4t\, dt} \right) = -\frac{1}{4}\pi a^4$$

$\square$

2. Question

Of all the smooth closed directed counterclockwise curves that are edges of closed areas on the plane. Find curve C such that the work is maximal:

$$\oint_C \left(3x^2y + x\cos x\right)dx + \left(6x - 2xy^2 + \sin y\right)dy$$

Proof. Let's look at $P(x,y) = 3x^2y + x\cos x$ and $Q(x,y) = 6x - 2xy^2 + \sin y$.
So $P'_y = 3x^2$ and $Q'_x = 6 - 2y^2$.
So $Q'_x - P'_y = 6 - 2y^2 - 3x^2$.
So according to Green's Theorem

$$\oint_C \left(3x^2y + x\cos x\right)dx + \left(6x - 2xy^2 + \sin y\right)dy = \iint_D 6 - 2y^2 - 3x^2 dxdy$$

We need to find D that maxes the above expression.
Since $6 - 2y^2 - 3x^2 \geq 0$ when $6 \geq 2y^2 + 3x^2$ then only these points will benefit us if included in D (other points will contribute negative values, lowering the sum, aka integral, expression). So we can mark $D = \left\{(x,y) \mid 6 \geq 2y^2 + 3x^2\right\}$.
Note that this is an ellipse with a border $C = \left\{(x,y) \mid 6 = 2y^2 + 3x^2\right\}$.
So C is the curve that will maximize the integral.
Q.E.D. $\square$

3. Question

Let $\gamma \in \mathbb{R}^2$ closed smooth curve.
Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ continues with continues partial derivatives in the plane.
RTP:

$$\oint_\gamma f\vec{\nabla}g \cdot \vec{d\gamma} = -\oint_\gamma g\vec{\nabla}f \cdot \vec{d\gamma}$$

Hint: Show that $\vec{\nabla}(fg)$ a conservative field.

Proof. Since f, g continues with continues partial derivatives in the plane then fg is also as such.
Let's mark $F = \vec{\nabla}(fg)$. So F (or $\vec{\nabla}(fg)$) is a conservative field since $fg \in C^1$.
We know that $\vec{\nabla}(fg) = g\vec{\nabla}f + f\vec{\nabla}g$ according to derivative rules.
Thus

$$\oint_\gamma g\vec{\nabla}f\vec{d\gamma} + \oint_\gamma f\vec{\nabla}g\vec{d\gamma} = \oint_\gamma g\vec{\nabla}f + f\vec{\nabla}g\vec{d\gamma} = \oint_\gamma \vec{\nabla}(fg)\,\vec{d\gamma} \underbrace{=}_{\vec{\nabla}(fg)\text{ is a conservative field}} 0$$

Thus we get

$$\oint_\gamma f\vec{\nabla}g \cdot \vec{d\gamma} = -\oint_\gamma g\vec{\nabla}f \cdot \vec{d\gamma}$$

Q.E.D $\square$

4. Question

Given $\vec{F}(x,y)=\left(-\frac{y}{x^2+4y^2},\frac{x}{x^2+4y^2}\right)$

4.1. RTP: $Q'_x=P'_y$ in the plane without the root $\left(\mathbb{R}^2\backslash\{(0,0)\}\right)$. So $P(x,y)=-\frac{y}{x^2+4y^2}$ and thus $P'_y=-\frac{x^2+4y^2-8y^2}{(x^2+4y^2)^2}=\frac{-(x-2y)(x+2y)}{(x^2+4y^2)^2}$

And $Q(x,y)=\frac{x}{x^2+4y^2}$ and thus $Q'_x=\frac{x^2+4y^2-2x^2}{(x^2+4y^2)^2}=\frac{(2y-x)(2y+x)}{(x^2+4y^2)^2}$

And we got $P'_y=\frac{(2y-x)(2y+x)}{(x^2+4y^2)^2}=\frac{(2y-x)(2y+x)}{(x^2+4y^2)^2}=Q'_x$.

Q.E.D

4.2. RTP: $\vec{F}$ is not a conservative vector field in $\mathbb{R}^2\backslash\{(0,0)\}$.

Proof. Let's look at the closed elliptic path $x^2+4y^2=0.01^2$ going counterclockwise. Let's mark it γ
Parameterized we get $\vec{r}(t)=\left(0.01\cos t,0.01\cdot\frac{1}{2}\sin t\right)$ with $0\le t\le 2\pi$.
And $\vec{r}(t)'=\left(-0.01\sin t,0.01\cdot\frac{1}{2}\cos t\right)$.
So

$$\oint_\gamma \vec{F}d\vec{r}=\int_{t=0}^{2\pi}\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\cdot dt=\int_{t=0}^{2\pi}-\frac{-\frac{1}{2}(0.01\sin t)^2}{(0.01\cos t)^2+4\left(\frac{1}{2}\cdot 0.01\sin t\right)^2}+\frac{\frac{1}{2}(0.01\cos t)^2}{(0.01\cos t)^2+4\left(\frac{1}{2}\cdot 0.01\sin t\right)^2}\cdot dt=$$

$$=\int_{t=0}^{2\pi}\frac{1}{2}\cdot\frac{\cancel{0.01^2}\cdot\cancel{\left((\sin t)^2+(\cos t)^2\right)}}{\cancel{0.01^2}\cdot\cancel{\left((\cos t)^2+(\sin t)^2\right)}}dt=\int_{t=0}^{2\pi}\frac{1}{2}dt=\frac{1}{2}\cdot 2\pi=\pi\neq 0$$

So the field isn't conservative in $\mathbb{R}^2\backslash\{(0,0)\}$.
Note: Remember this result for later. □

4.3. **Need to check: Is the field conservative in the following area.** Solution:

ⓔ הפעם נתון המישור:

$$D = \{ (x,y) \mid 1 \le (x-10)^2 + (y-5)^2 \le 4 \}$$

להוו התחום פשוט קשר ולכן $Q'_x = P'_y$

ולכן ממשפט נובע כי במישור כזה F

שדה משמר. נמצא את פונק׳ הפוטנציאל:

$$(\phi_x, \phi_y) = \frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}$$

$$\phi_x = \frac{-y}{x^2+4y^2}$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a}$$

$$\phi = \int \frac{-y}{x^2+4y^2}dx =$$

$$= -y\int \frac{1}{x^2+\underbrace{4y^2}_{a^2}}dx = -y\left(\frac{1}{2y}\arctan\left(\frac{x}{2y}\right)\right)$$

$$\phi = -\frac{1}{2}\arctan\frac{x}{2y} + C(y)$$

$$\phi_y = -\frac{1}{2}\cdot\frac{1}{1+\left(\frac{x}{2y}\right)^2}\cdot\left(\frac{-2x}{4y^2}\right) + C'(y)$$

ושווה ל- $\frac{x}{x^2+4y^2}$:

$$Qy = -\frac{1}{2}\cdot\frac{1}{1+\left(\frac{x}{2y}\right)^2}\cdot\left(\frac{-2x}{4y^2}\right) + C'(y) = \frac{x}{x^2+4y^2}$$

$$-\frac{1}{\cancel{2}}\cdot\frac{\cancel{4y^2}}{4y^2+x^2}\cdot-\frac{\cancel{2}x}{\cancel{4y^2}} + C'(y) = \frac{x}{x^2+4y^2}$$

$$\Rightarrow \cancel{\frac{x}{4y^2+x^2}} + C'(y) = \cancel{\frac{x}{x^2+4y^2}}$$

$$\Rightarrow C'(y) = 0$$

ולכן $\boxed{\phi = -\frac{1}{2}\arctan\frac{x}{2y} + k}$

שמוגדרת ב- D היא פונק׳ הפוטנציאל.

4.4. **Given** $\vec{\gamma_1} = \left(\frac{\cos 2\pi t}{t+1}, \frac{\sin 2\pi t}{t+1}\right)$ **for** $0 \le t \le 10$ **and** $\vec{\gamma_2} = (t, 0)$ **for** $\frac{1}{11} \le t \le 1$**.** Calculate the work

$$W = \oint_{\gamma_1 \cup \gamma_2} \vec{F} \cdot \vec{dr}$$

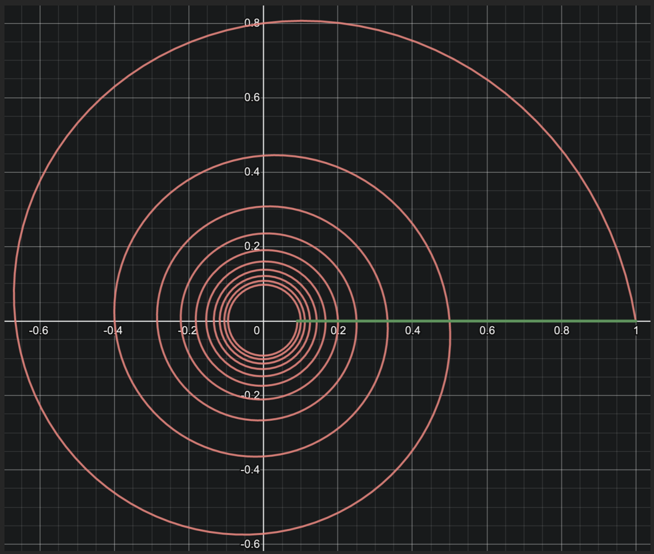
Proof. The path is a counterclockwise spiral (10 loops) from $(1,0)$ around and towards the root $(0,0)$ ending in $\left(\frac{1}{11}, 0\right)$ and then a straight line back to $(1,0)$ (this is not a simple closed path).

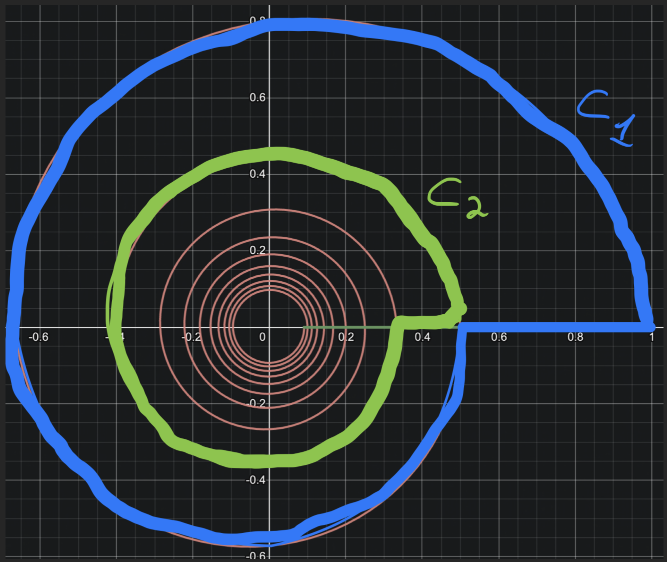
We see this from because for $t = 0$ we start doing a full cycle (note that $(\cos t, \sin t)$ is a circle around the root) except that the radius goes down with t going up.

Looking at the path we see that we get a number of closed curves. 10 to be exact. Each curve consisting of one full cycle of the spiral and the straight line from γ_2 that connects the beginning and the end. Note that the 10 curves we build use each part of γ exactly once and preserve the directions.

The curves define areas that each is fully included in the previous.

On the left a picture from Desmos (promise I thought about it before putting the equation there). On the right you see an illustration of what C_1 and C_2 are (as we later mark them)





Since we got from section A that $Q'_x = P'_y$ and since each curve is piecewise smooth. We can remove a tiny circle around $(0,0)$ (going clockwise) and safely use Green.

We'll mark the tiny circle as C_0 and the curves C_i for $1 \le i \le 10$. We'll mark the areas as D_i.

So for curve i:

$$\oint_{C_i \cup C_0} \vec{F} \cdot \vec{dr} = \iint_{D_i \setminus D_0} Q'_x - P'_y dxdy = \iint_{D_i \setminus D_0} 0 dxdy = 0$$

So the sum of the integrals on these closed curves is equal to the integral that goes over γ except that it includes 10 times the integral over the tiny circle that we added. To get the integral over just γ we need to remove the integral over the tiny circle 10 times.

$$\sum_{i=1}^{10} \oint_{C_i \cup C_0} \vec{F} \cdot \vec{dr} = \oint_{\bigcup_{i=1}^{10} C_i} \vec{F} \cdot \vec{dr} + 10 \cdot \oint_{C_0} \vec{F} \cdot \vec{dr} = 0$$

So we get that

$$\oint_{\bigcup_{i=1}^{10} C_i} \vec{F} \cdot \vec{dr} = \oint_{\gamma} \vec{F} \cdot \vec{dr} = -10 \cdot \oint_{C_0} \vec{F} \cdot \vec{dr}$$

Since $\bigcup_{i=1}^{10} C_i = \gamma$. In section B we calculated that

$$\oint_{C_0} \vec{F} \cdot \vec{dr} = -\pi$$

(the minus is because it was calculated as counterclockwise circle and we're going clockwise with C_0).

thus

$$W = \oint_{\gamma} \vec{F} \cdot \vec{dr} = 10\pi$$

Q.E.D $\square$