VECTORIAL ANALYSIS HW 3

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1. ~~Question~~

2. Question

Calculate the volume of:

$$V = \{(x, y, z) \mid 2 - z \leq x + y \leq 5 - z,\, 1 - y \leq x \leq 3 - y,\, x \leq y \leq 2x\}$$

Proof. From $2 - z \leq x + y \leq 5 - z \Longrightarrow 2 \leq x + y + z \leq 5$.
From $1 - y \leq x \leq 3 - y \Longrightarrow 1 \leq x + y \leq 3$.
BWoC $x < \frac{1}{3}$. Thus $y < \frac{2}{3}$. And thus $x + y < 1$. In contradiction to $1 \leq x + y$. So $x, y \geq \frac{1}{3} > 0$. So from $\frac{1}{3} \leq x \leq y \leq 2x \Longrightarrow 1 \leq \frac{y}{x} \leq 2$
To conclude

$$V = \left\{(x, y, z) \mid 2 \leq x + y + z \leq 5,\, 1 \leq \frac{y}{x} \leq 2,\, 1 \leq x + y \leq 3\right\}$$

Let's mark $u = x + y + z$, $v = \frac{y}{x}$, $w = x + y$.

$$J = \frac{1}{J^{-1}} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{\begin{vmatrix} u'_x & u'_y & u'_z \\ v'_x & v'_y & v'_z \\ w'_x & w'_y & w'_z \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} & 0 \\ 1 & 1 & 0 \end{vmatrix}} = \frac{1}{-\frac{y}{x^2} - \frac{1}{x}} = \frac{1}{-\frac{y+x}{x^2}} = -\frac{x^2}{y + x}$$

Note that we expanded according to the last column rather than first row.
Since $x^2 \geq \frac{1}{9} > 0$ and $y > x > 0 \Longrightarrow x + y > 0$. Then $|J| = \left|-\frac{x^2}{y+x}\right| = \frac{x^2}{y+x} = \frac{}{w}$.
So $V = \{(u, v, w) \mid 2 \leq u \leq 5,\, 1 \leq v \leq 2,\, 1 \leq w \leq 3\}$

$$\iiint_V 1 d\,(x, y, z) = \iiint_V |J|\, d\,(u, v, w) = \iiint_V \frac{x^2}{y + x} d\,(u, v, w)$$

But we need to express $\frac{x^2}{y+x}$ with u, v, w. Note that $x + y = x\left(1 + \frac{y}{x}\right) \Longrightarrow \frac{x+y}{\left(1+\frac{y}{x}\right)} = \frac{w}{1+v} = x$. Thus

$$\iiint_V \frac{x^2}{y + x} d\,(u, v, w) = \iiint_V \frac{\left(\frac{w}{1+v}\right)^2}{w} d\,(u, v, w) = \iiint_V \frac{w}{(1 + v)^2} d\,(u, v, w) = \int_2^5 \int_1^2 \int_1^3 \frac{w}{(1 + v)^2} dw dv du =$$

$$= \int_2^5 \int_1^2 \frac{\overbrace{3^2 - 1^2}^{\cancel{8}4}}{\cancel{2}\,(1 + v)^2} dv du = \int_2^5 4 \int_1^2 \frac{1}{(1 + v)^2} dv du = \int_2^5 -4 \frac{1}{1 + v}\, \Big|_{v=1}^{2}\, du = \int_2^5 4 \cdot \left(\underbrace{\frac{1}{2} - \frac{1}{3}}_{1/6}\right) du =$$

$$= \int_2^5 \frac{2}{3} du = 2$$

$\square$

3. Question

Find the center mass of the quarter single-sided-cone with a uniform density of $\rho \equiv 1$ trapped between

$$z = 0,\ z = 1,\ y = 0,\ x = 0\, x^2 + y^2 = z^2$$

Proof. Since we're working with a cone, it's more convenient to work with radial tube coordinates.
So $x = r\cos\theta$, $y = r\sin\theta$.
So $J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$
So for x coordinate:

$$\iiint_V x \cdot \underbrace{|J|}_{\text{r}} \cdot 1 dv = \int_0^1 \int_0^{\pi/2} \int_0^z r^2 \cos\theta \, dr \, d\theta \, dz = \frac{1}{3} \int_0^1 \int_0^{\pi/2} z^3 \cos\theta \, d\theta \, dz =$$

$$= \frac{1}{3} \int_0^1 z^3 \left(\underbrace{\sin\frac{\pi}{2} - \sin 0}_{1} \right) dz = \frac{1}{3} \int_0^1 z^3 \, dz = \frac{1}{12} 1^4 = \frac{1}{12}$$

For y coordinate:

$$\iiint_V y \cdot \underbrace{|J|}_{\text{r}} \cdot 1 dv = \int_0^1 \int_0^{\pi/2} \int_0^z r^2 \sin\theta \, dr \, d\theta \, dz = \int_0^{\pi/2} \sin\theta \, d\theta \cdot \int_0^1 \int_0^z r^2 dr \, dz = -\left(0 - 1\right) \frac{1}{3} \cdot \int_0^1 z^3 dz =$$

$$= -\left(0-1\right) \frac{1}{12} \cdot 1^4 = \frac{1}{12}$$

Note that this isn't surprising, it's what we expected from symmetry considerations.
For z coordinate:

$$\iiint_V z \cdot \underbrace{|J|}_{\text{r}} \cdot 1 dv = \int_0^1 \int_0^{\pi/2} \int_0^z rz \, dr \, d\theta \, dz = \int_0^1 \int_0^{\pi/2} \frac{z^3}{2} \, d\theta \, dz = \int_0^1 \frac{z^3}{2} \frac{\pi}{2} \, dz = \frac{\pi \cdot 1^4}{16} = \frac{\pi}{16}$$

So the center mass is

$$\frac{1}{\text{mass}} \cdot \left(\frac{1}{12}, \frac{1}{12}, \frac{\pi}{16} \right)$$

But the mass of a cone is $\pi r^2 \frac{h}{3} = \frac{\pi}{3}$. So for a quarter of a cone it's $\frac{\pi}{12}$ so the centroid is

$$\left(\frac{1}{\pi}, \frac{1}{\pi}, \frac{3}{4} \right)$$

□

4. Question

Calculate the volume of the object trapped between the spheres

$$x^2 + y^2 + z^2 = 4$$

$$x^2 + (y+2)^2 + z^2 = 4$$

Proof. From those we get $y = -1$.
Meaning the intersection of the shells of the spheres is $y = -1$; $x^2 + z^2 = 3$.
So All we need it to calculate half of the intersection area and multiply by 2 (the magic of symmetry).
Let's look at the cut parallel to plane XZ (from $-2 \le y \le -1$).
The radius of the circle of the cut at y_0 is $\sqrt{4 - y_0^2}$ so the area in the cut is $\left(4 - y_0^2\right)\pi$. So we can integrate over all the cuts:

$$\int_{-2}^{-1} 4\pi - \pi y^2 dy = 4\pi - \frac{\pi}{3}\left((-1)^3 - (-2)^3\right) = 4\pi - \frac{\pi}{3}(-1+8) = \frac{5\pi}{3}$$

And now we multiply but 2 and get the entire area trapped

$$10\pi/3$$

□

5. Question

Let $T:\mathbb{R}^3\to\mathbb{R}^3$

$$u\left(x,y,z\right)=x^2+y^2$$
$$v\left(x,y,z\right)=x-y$$
$$w\left(x,y,z\right)=z$$

$\forall\varepsilon>0$ given that T copies the cube $[3,3+\varepsilon]\times[2,2+\varepsilon]\times[1,1+\varepsilon]$ to an object with volume $V\left(\varepsilon\right)$.
RTP:

$$\lim_{\varepsilon\to0}\frac{V\left(\varepsilon\right)}{\varepsilon^3}=L$$

Proof. The Jacobean is

$$J=\frac{1}{\begin{matrix}2x & 2y & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{matrix}}=\frac{1}{-2x-2y}=-\frac{1}{2\left(x+y\right)}$$

Thus $\left|J^{-1}\left(3,2,1\right)\right|=\left|-2\left(3+2\right)\right|=\left|-10\right|=10$.
So $V\left(\varepsilon\right)=10\cdot\varepsilon^3$ (because we can take out $\left|J^{-1}\right|$ when we do a reverse variable interchange in the integral to calculate the volume in the new space).
And thus $\lim_{\varepsilon\to0}\frac{V(\varepsilon)}{\varepsilon^3}=\lim_{\varepsilon\to0}\frac{10\varepsilon^3}{\varepsilon^3}=\lim_{\varepsilon\to0}10=10$. $\square$