MODERN ALGEBRA HW4

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1. Question

Given $\alpha = (9\,3\,4\,7\,5)\,(7\,1\,3\,5\,9\,2) \in S_9$.

1.1. Calculate the order of α and its parity.

Proof. Let's write α as in c more comfortable form

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 5 & 4 & 9 & 6 & 1 & 8 & 2 \\ 4 & 5 & 9 & 7 & 3 & 6 & 1 & 8 & 2 \end{matrix}$$

And from here as disjoint cycles

$$\alpha = (1\,4\,7)\,(2\,5\,3\,9)$$
$$ord\,(\alpha) = [3,4] = 12$$

And the parity is odd because α is built of two cycles, one with even parity and the other with odd parity (3-cycle is even parity and 4-cycle is odd). Together they give an odd parity permutation. $\square$

1.2. Calculate the order of α^{22}.

Proof. We the that $ord\,(\alpha) = 12$ so $\alpha^{12} = e$. So

$$\alpha^{22} = \alpha^{12}\alpha^{10} = e\alpha^{10} = (1\,4\,7)^{10}\,(2\,5\,3\,9)^{10}$$

The last one because the cycles are disjoint.

But $ord\,((1\,4\,7)) = 3 \wedge ord\,((2\,5\,3\,9)) = 4$. So

$$(1\,4\,7)^{10} = (1\,4\,7)^{9}\,(1\,4\,7) = e^3\,(1\,4\,7) = (1\,4\,7)$$

and

$$(2\,5\,3\,9)^{10} = (2\,5\,3\,9)^{8}\,(2\,5\,3\,9)^{2} = e^2\,(2\,5\,3\,9)^{2} = (2\,5\,3\,9)^{2} = (2\,3)\,(5\,9)$$

And we get $\alpha^{22} = (1\,4\,7)\,(2\,3)\,(5\,9)$. And the order of that

$$ord\left(\alpha^{22}\right) = ord\,((1\,4\,7)\,(2\,3)\,(5\,9)) = [3,2,2] = 6$$

$\square$

1.3. Find $\beta \in S_9$ s.t. $\beta^5 = \alpha$.

Proof. We know that $(2\,5\,3\,9)^5 = (2\,5\,3\,9)$. So we only need to find a disjoin cycle for which the 5th exponent is $(1\,4\,7)$.

For a 3-cycle, the order is 3. So the 5th exponent is the same as the 2nd exponent. So if we take $(1\,7\,4)$ then we get $(1\,7\,4)^5 = (1\,7\,4)^2 = (1\,4\,7)$.

So we found

$$\beta = (2\,5\,3\,9)\,(1\,7\,4)$$

s.t. $\beta^5 = \alpha$. $\square$

2. Question

2.1. **Is the a cyclic subgroup of order 6 in S_5?**

Proof. Yes, there is. Let's look at $\alpha = (1\,2\,3)\,(4\,5)$. The $[3,2] = 6$ so $ord\,(\alpha) = 6$. Thus, $ord\,(\langle\alpha\rangle) = 6$. □

2.2. **Is the a cyclic subgroup of order 6 in A_7?**

Proof. Yes, there is. Let's look at $\beta = (1\,2\,3)\,(4\,5)\,(6\,7)$. The permutation is even because it's made of two odd permutations and an even permutation.

Also, $ord\,(\beta) = 6$ because $[3,2,2] = 6$. So $ord\,(\langle\beta\rangle) = 6$ and it is A_7 because $\beta \in A_7$ (as we mentioned before) and A_7 is closed. □

2.3. **Find all the permutations $\tau \in S_6$ s.t. $\tau\,(1\,2)\,(3\,4)\,\tau^{-1} = (5\,6)\,(1\,3)$.**

Proof. So we need to comply with $(\tau\,(1)\;\tau\,(2))\,(\tau\,(3)\;\tau\,(4)) = (5\,6)\,(1\,3) = (1\,3)\,(5\,6)$.

Let's look at the possible options (each column is an option):

$$\begin{array}{cccccccc}
\tau(1)=5 & \tau(1)=6 & \tau(1)=5 & \tau(1)=6 & \tau(1)=5 & \tau(1)=6 & \tau(1)=5 & \tau(1)=6\\
\tau(2)=6 & \tau(2)=5 & \tau(2)=6 & \tau(2)=5 & \tau(2)=6 & \tau(2)=5 & \tau(2)=6 & \tau(2)=5\\
\tau(3)=1 & \tau(3)=1 & \tau(3)=3 & \tau(3)=3 & \tau(3)=1 & \tau(3)=1 & \tau(3)=3 & \tau(3)=3\\
\tau(4)=3 & \tau(4)=3 & \tau(4)=1 & \tau(4)=1 & \tau(4)=3 & \tau(4)=3 & \tau(4)=1 & \tau(4)=1\\
\tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4\\
\tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2
\end{array}$$

And also

$$\begin{array}{cccccccc}
\tau(1)=1 & \tau(1)=1 & \tau(1)=3 & \tau(1)=3 & \tau(1)=1 & \tau(1)=1 & \tau(1)=3 & \tau(1)=3\\
\tau(2)=3 & \tau(2)=3 & \tau(2)=1 & \tau(2)=1 & \tau(2)=3 & \tau(2)=3 & \tau(2)=1 & \tau(2)=1\\
\tau(3)=5 & \tau(3)=6 & \tau(3)=5 & \tau(3)=6 & \tau(3)=5 & \tau(3)=6 & \tau(3)=5 & \tau(3)=6\\
\tau(4)=6 & \tau(4)=5 & \tau(4)=6 & \tau(4)=5 & \tau(4)=6 & \tau(4)=5 & \tau(4)=6 & \tau(4)=5\\
\tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4\\
\tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2
\end{array}$$

Or in a better form

$$\begin{pmatrix}1&2&3&4&5&6\\5&6&1&3&2&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\6&5&1&3&2&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\6&5&3&1&2&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\5&6&3&1&2&4\end{pmatrix}$$
$$\begin{pmatrix}1&2&3&4&5&6\\5&6&1&3&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\6&5&1&3&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\6&5&3&1&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\5&6&3&1&4&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3&4&5&6\\1&3&5&6&2&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\1&3&6&5&2&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\3&1&6&5&2&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\3&1&5&6&2&4\end{pmatrix}$$
$$\begin{pmatrix}1&2&3&4&5&6\\1&3&5&6&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\1&3&6&5&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\3&1&6&5&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6\\3&1&5&6&4&2\end{pmatrix}$$

At total 8 options.

Here's an example of the first two permutations in cyclic form (just to prove I know how you know)

$$(1\,5\,2\,6\,4\,3)$$
$$(1\,6\,4\,3)\,(2\,5)$$

□

2.4. **Find all the permutations that are commutative in multiplication with $(1\,2\,3)$ in S_5.**

Proof. Meaning $\tau\,(1\,2\,3) = (1\,2\,3)\,\tau \Longrightarrow \tau\,(1\,2\,3)\,\tau^{-1} = (1\,2\,3)$. So

$$\begin{pmatrix}1&2&3&4&5\\1&2&3&4&5\end{pmatrix},\begin{pmatrix}1&2&3&4&5\\3&1&2&4&5\end{pmatrix},\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}$$
$$\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5\\3&1&2&5&4\end{pmatrix},\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}$$

Total 6 permutations. □

3. Question

3.1. Given $N, K \trianglelefteq G^{(1)}$. RTP: $N \cap K \trianglelefteq G$.

Proof. Let $g \in G$ and $\alpha \in N \cap K$. **RTP:** $g\alpha g^{-1} \in N \cap K$**:**

Thus $\alpha \in N$ and $\alpha \in K$. From (1) we get that $g\alpha g^{-1} \in N$ as well as $g\alpha g^{-1} \in K$. Thus $g\alpha g^{-1} \in N \cap K$.

Let's also note that $N \cap K$ is a subgroup of G.

Q.E.D $\square$

3.2. Given $H \leq G$ and $N \trianglelefteq G$. RTP: $N \cap H \trianglelefteq H$.

Proof. Let $\alpha \in N \cap H$ (which is a subgroup of G).

So $\alpha \in N^{(1)}$ and $\alpha \in H^{(2)}$.

Let $h \in H$ permutation.

$h \in G$ ($H \leq G$). From (1) we get that $h\alpha h^{-1} \in N$. But H is closed so $h\alpha h^{-1} \in H$. So $h\alpha h^{-1} \in N \cap H$.

And we got that $\forall h \in H \forall \alpha \in N \cap H : h\alpha h^{-1} \in N \cap H$ (meaning that $N \cap H \trianglelefteq H$).

Q.E.D $\square$

3.3. Give an example for a group G,$H \leq G$ and $N \trianglelefteq G$ but $N \cap H \ntrianglelefteq G$.

Proof. For $G = S_3, H = \{(1\,2)\,, e\} = \langle (1\,2) \rangle$ and $N = \{(1\,2)\,, (1\,3)\,, (2\,3)\,, e\}$. It is clear that N is normal under G because it includes all the permutations of the form $(\bullet\,\bullet) \in G$. It's also clear that $H \leq G$. But $N \cap H = H$ for which $N \cap H \ntrianglelefteq G$.

The latter is because H does **not** include all the permutations in G of the from $(\bullet\,\bullet)$.

And for example for $(2\,3) \in G$, $(2\,3)\underbrace{(1\,2)}_{\in H}\underbrace{(3\,2)}_{=(2\,3)^{-1}=(2\,3)} = (1\,3) \notin H$.

Q.E.D $\square$

4. Question

Given group G and $a \in G$.

$$C_G(a) = \{g \in G \mid ga = ag\}, \, Cl_G(a) = \left\{gag^{-1} \mid g \in G\right\}$$

4.1. **RTP:** $C_G(a) \leq G$.

Proof.

4.1.1. *Closure.* Let $t, k \in C_G(a)$. $\underbrace{ta = at}_{given} \Longrightarrow (ta)k = (at)k \underbrace{=}_{associatibity + given} t(ka) = (tk)a = a(tk)$

4.1.2. *Identity.* $e \in G$ and $\forall a \in G : ae = ea = a$ and thus $e \in C_G(a)$.

4.1.3. *Inverse.* Let $g \in C_G(a)$. So $ga = ag \Longrightarrow gag^{-1} = agg^{-1} = ae = a \Longrightarrow g^{-1}gag^{-1} = g^{-1}a \Longrightarrow eag^{-1} = ag^{-1} = g^{-1}a$.
Q.E.D $\square$

4.2. **Given $G = S_5$ and $a = (1\,2\,3)$. Calculate $|C_G(a)|$ and $|Cl_G(a)|$.**

We calculated $C_G(a)$ in 2.4. So $|C_G(a)| = 6$.

Regarding $Cl_G(a)$, it's all the permutations of the form $(\bullet\ \bullet\ \bullet)$ and there are $\binom{5}{3}$ choices for different elements there. And For each choice, there are two options for different permutations: the second lowest is after the lowest value, or the highest value is after the lowest value. Either one of these choices define the order of the triplet.

So in total $2 \cdot \binom{5}{3} = 20$. Thus $|Cl_G(a)| = 20$.

And $20 = |Cl_G(a)| = \frac{|S_5|}{|C_G(a)|} = \frac{5!}{6} = \frac{120}{6}$ like we expect.

4.3. **Given G final. RTP:** $|Cl_G(a)| = \frac{|G|}{|C_G(a)|}$.

Proof. Meaning we would like to prove that the index of $C_G(a)$ in G is $|Cl_G(a)|$.

We'll define $f : \frac{G}{C_G(a)} \to Cl_G(a)$ s.t. $f(gC_G(a)) = gag^{-1}$ and we'll prove that f is a well defined bijection.

Let $gC_G(a)$ coset of G (each coset can be written as such). According to f's definition, $f(gC_G(a)) = gag^{-1}$.

Let $g_1, g_2 \in C_G(a)$ s.t. $g_1C_G(a) = g_2C_G(a)$, RTP: $g_1ag_1^{-1} = g_2ag_2^{-1}$.

We got

$$g_1C_G(a) = g_2C_G(a) \Longleftrightarrow g_1^{-1}g_2 \in C_G(a) \Longleftrightarrow g_1^{-1}g_2a = ag_1^{-1}g_2 \Longleftrightarrow$$
$$\Longleftrightarrow g_1^{-1}g_2ag_2^{-1} = ag_1^{-1} \Longleftrightarrow eg_2ag_2^{-1} = g_2ag_2^{-1} = g_1ag_1^{-1}$$

Meaning that we proved that f is both defined and injective.

Now for surjectiveness:

Let $t \in Cl_G(a)$ so $\exists g \in G$ s.t. $gag^{-1} = t$. Thus for $gC_G(a) \in \frac{G}{C_G(a)} : f(gC_G(a)) = gag^{-1} = t$.

Thus $|Cl_G(a)| = \left|\frac{G}{C_G(a)}\right| = [G : C_G(a)] = \frac{|G|}{|C_G(a)|}$.

Q.E.D $\square$