MODERN ALGEBRA HW4

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1. Question

Given $\alpha = (9\,3\,4\,7\,5)\,(7\,1\,3\,5\,9\,2) \in S_9$.

1.1. **Calculate the order of α and its parity.**

Proof. Let's write α as in c more comfortable form

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 5 & 4 & 9 & 6 & 1 & 8 & 2 \\ 4 & 5 & 9 & 7 & 3 & 6 & 1 & 8 & 2 \end{matrix}$$

And from here as disjoint cycles

$$\alpha = (1\,4\,7)\,(2\,5\,3\,9)$$
$$ord\,(\alpha) = [3,4] = 12$$

And the parity is odd because α is built of two cycles, one with even parity and the other with odd parity (3-cycle is even parity and 4-cycle is odd). Together they give an odd parity permutation. $\square$

1.2. **Calculate the order of α^{22}.**

Proof. We the that $ord\,(\alpha) = 12$ so $\alpha^{12} = e$. So

$$\alpha^{22} = \alpha^{12}\alpha^{10} = e\alpha^{10} = (1\,4\,7)^{10}\,(2\,5\,3\,9)^{10}$$

Tthe last one because the cycles are disjoint.

But $ord\,((1\,4\,7)) = 3 \wedge ord\,((2\,5\,3\,9)) = 4$. So

$$(1\,4\,7)^{10} = (1\,4\,7)^{9}\,(1\,4\,7) = e^3\,(1\,4\,7) = (1\,4\,7)$$

and

$$(2\,5\,3\,9)^{10} = (2\,5\,3\,9)^{8}\,(2\,5\,3\,9)^{2} = e^2\,(2\,5\,3\,9)^{2} = (2\,5\,3\,9)^{2} = (2\,3)\,(5\,9)$$

And we get $\alpha^{22} = (1\,4\,7)\,(2\,3)\,(5\,9)$. And the order of that

$$ord\left(\alpha^{22}\right) = ord\,((1\,4\,7)\,(2\,3)\,(5\,9)) = [3,2,2] = 6$$

$\square$

1.3. **Find $\beta \in S_9$ s.t. $\beta^5 = \alpha$.**

Proof. We know that $(2\,5\,3\,9)^5 = (2\,5\,3\,9)$. So we only need to find a disjoin cycle for which the 5th exponent is $(1\,4\,7)$.

For a 3-cycle, the order is 3. So the 5th exponent is the same as the 2nd exponent. So if we take $(1\,7\,4)$ then we get $(1\,7\,4)^5 = (1\,7\,4)^2 = (1\,4\,7)$.

So we found

$$\beta = (2\,5\,3\,9)\,(1\,7\,4)$$

s.t. $\beta^5 = \alpha$. $\square$

2. Question

2.1. **Is the a cyclic subgroup of order 6 in S_5?**

Proof. Yes, there is. Let's look at $\alpha = (1\,2\,3)\,(4\,5)$. The $[3,2] = 6$ so $ord\,(\alpha) = 6$. Thus, $ord\,(\langle\alpha\rangle) = 6$. $\square$

2.2. **Is the a cyclic subgroup of order 6 in A_7?**

Proof. Yes, there is. Let's look at $\beta = (1\,2\,3)\,(4\,5)\,(6\,7)$. The permutation is even because it's made of two odd permutations and an even permutation.

Also, $ord\,(\beta) = 6$ because $[3,2,2] = 6$. So $ord\,(\langle\beta\rangle) = 6$ and it is A_7 because $\beta \in A_7$ (as we mentioned before) and A_7 is closed. $\square$

2.3. **Find all the permutations $\tau \in S_6$ s.t. $\tau\,(1\,2)\,(3\,4)\,\tau^{-1} = (5\,6)\,(1\,3)$.**

Proof. So we need to comply with $(\tau\,(1)\;\tau\,(2))\,(\tau\,(3)\;\tau\,(4)) = (5\,6)\,(1\,3) = (1\,3)\,(5\,6)$.

Let's look at the possible options (each column is an option):

$$\begin{array}{cccccccc}
\tau(1)=5 & \tau(1)=6 & \tau(1)=5 & \tau(1)=6 & \tau(1)=5 & \tau(1)=6 & \tau(1)=5 & \tau(1)=6 \\
\tau(2)=6 & \tau(2)=5 & \tau(2)=6 & \tau(2)=5 & \tau(2)=6 & \tau(2)=5 & \tau(2)=6 & \tau(2)=5 \\
\tau(3)=1 & \tau(3)=1 & \tau(3)=3 & \tau(3)=3 & \tau(3)=1 & \tau(3)=1 & \tau(3)=3 & \tau(3)=3 \\
\tau(4)=3 & \tau(4)=3 & \tau(4)=1 & \tau(4)=1 & \tau(4)=3 & \tau(4)=3 & \tau(4)=1 & \tau(4)=1 \\
\tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4 \\
\tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2
\end{array}$$

And also

$$\begin{array}{cccccccc}
\tau(1)=1 & \tau(1)=1 & \tau(1)=3 & \tau(1)=3 & \tau(1)=1 & \tau(1)=1 & \tau(1)=3 & \tau(1)=3 \\
\tau(2)=3 & \tau(2)=3 & \tau(2)=1 & \tau(2)=1 & \tau(2)=3 & \tau(2)=3 & \tau(2)=1 & \tau(2)=1 \\
\tau(3)=5 & \tau(3)=6 & \tau(3)=5 & \tau(3)=6 & \tau(3)=5 & \tau(3)=6 & \tau(3)=5 & \tau(3)=6 \\
\tau(4)=6 & \tau(4)=5 & \tau(4)=6 & \tau(4)=5 & \tau(4)=6 & \tau(4)=5 & \tau(4)=6 & \tau(4)=5 \\
\tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=2 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4 & \tau(5)=4 \\
\tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=4 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2 & \tau(6)=2
\end{array}$$

Or in a better form

$$\begin{pmatrix} 1&2&3&4&5&6\\5&6&1&3&2&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\6&5&1&3&2&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\6&5&3&1&2&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\5&6&3&1&2&4 \end{pmatrix}$$

$$\begin{pmatrix} 1&2&3&4&5&6\\5&6&1&3&4&2 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\6&5&1&3&4&2 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\6&5&3&1&4&2 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\5&6&3&1&4&2 \end{pmatrix}$$

$$\begin{pmatrix} 1&2&3&4&5&6\\1&3&5&6&2&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\1&3&6&5&2&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\3&1&6&5&2&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\3&1&5&6&2&4 \end{pmatrix}$$

$$\begin{pmatrix} 1&2&3&4&5&6\\1&3&5&6&4&2 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\1&3&6&5&4&2 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\3&1&6&5&4&2 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5&6\\3&1&5&6&4&2 \end{pmatrix}$$

At total 8 options.

Here's an example of the first two permutations in cyclic form (just to prove I know how you know)

$$(1\,5\,2\,6\,4\,3)$$

$$(1\,6\,4\,3)\,(2\,5)$$

$\square$

2.4. **Find all the permutations that are commutative in multiplication with $(1\,2\,3)$ in S_5.**

Proof. Meaning $\tau\,(1\,2\,3) = (1\,2\,3)\,\tau \implies \tau\,(1\,2\,3)\,\tau^{-1} = (1\,2\,3)$. So

$$\begin{pmatrix} 1&2&3&4&5\\1&2&3&4&5 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5\\3&1&2&4&5 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5\\2&3&1&4&5 \end{pmatrix}$$

$$\begin{pmatrix} 1&2&3&4&5\\1&2&3&5&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5\\3&1&2&5&4 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5\\2&3&1&5&4 \end{pmatrix}$$

Total 6 permutations. $\square$

3. Question

3.1. Given $N, K \trianglelefteq G^{(1)}$. RTP: $N \cap K \trianglelefteq G$.

Proof. Let $g \in G$ and $\alpha \in N \cap K$. **RTP:** $g\alpha g^{-1} \in N \cap K$**:**

Thus $\alpha \in N$ and $\alpha \in K$. From (1) we get that $g\alpha g^{-1} \in N$ as well as $g\alpha g^{-1} \in K$. Thus $g\alpha g^{-1} \in N \cap K$.

Let's also note that $N \cap K$ is a subgroup of G.

Q.E.D $\square$

3.2. Given $H \leq G$ and $N \trianglelefteq G$. RTP: $N \cap H \trianglelefteq H$.

Proof. Let $\alpha \in N \cap H$ (which is a subgroup of G).

So $\alpha \in N^{(1)}$ and $\alpha \in H^{(2)}$.

Let $h \in H$ permutation.

$h \in G$ ($H \leq G$). From (1) we get that $h\alpha h^{-1} \in N$. But H is closed so $h\alpha h^{-1} \in H$. So $h\alpha h^{-1} \in N \cap H$.

And we got that $\forall h \in H \forall \alpha \in N \cap H : h\alpha h^{-1} \in N \cap H$ (meaning that $N \cap H \trianglelefteq H$).

Q.E.D $\square$

3.3. Give an example for a group G,$H \leq G$ and $N \trianglelefteq G$ but $N \cap H \ntrianglelefteq G$.

Proof. For $G = S_3, H = \{(1\,2), e\} = \langle (1\,2) \rangle$ and $N = \{(1\,2), (1\,3), (2\,3), e\}$. It is clear that N is normal under G because it includes all the permutations of the form $(\bullet\bullet) \in G$. It's also clear that $H \leq G$. But $N \cap H = H$ for which $N \cap H \ntrianglelefteq G$.

The latter is because H does **not** include all the permutations in G of the from $(\bullet\bullet)$.

And for example for $(2\,3) \in G$, $(2\,3) = (1\,3) \notin H$.

Q.E.D $\square$

4. Question