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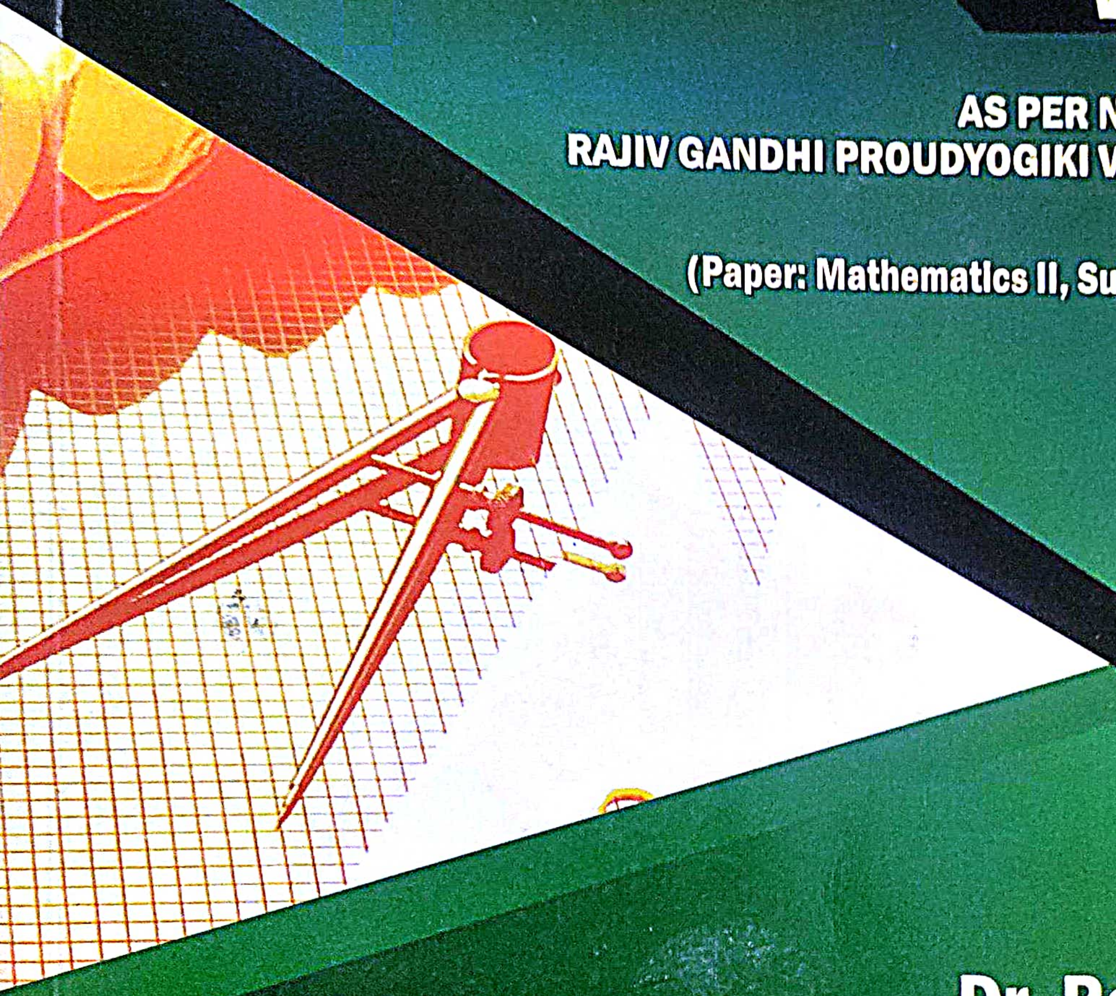

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UNIT – III: Ordinary Differential Equation-II

8 SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

CHAPTER

8.1 INTRODUCTION

No general procedure exists for solving second and higher order linear differential equation with variable coefficient. However, by transformation we can solve the second order differential equation with variable coefficients by the methods given in the next Article.

The method of variation of parameters to solve such equation is given in the next chapter.

8.2 SECOND ORDER DIFFERENTIAL EQUATIONS

A differential equation which involves the maximum second order derivative is known as second order differential equation.

e.g. $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Where P, Q and R are the functions of x only. The coefficient of $\frac{d^2y}{dx^2}$ is always one.

We will discuss the following types of differential equations:

1. Method of undetermined coefficients y, of the type $\frac{d^2y}{dx^2} = f(x)$.
2. By known integral
3. By removing the first derivative (Normal form).
4. By changing the independent variable by putting Either $P_1 = 0$ or Q_1 = a constant
5. Variation of parameters.

8.3 TYPE I

Now we solve the second order differential equations which do not contain y are of the type

$$\boxed{\frac{d^2y}{dx^2} = f(x)}$$

This type of exact differential equations are solved by successive integration.

Example 1. *Solve* $\frac{d^2y}{dx^2} = x^2 \sin x$.

Solution. We have, $\frac{d^2y}{dx^2} = x^2 \sin x$ ...(1)

Integrating the differential equation (1), we get

$$\frac{dy}{dx} = x^2(-\cos x) - (2x)(-\sin x) + (2)(\cos x) + c_1$$

$$\frac{dy}{dx} = -x^2 \cos x + 2x \sin x + 2 \cos x + c_1$$ **Ans.**

Example 2. *Solve* $\frac{d^2y}{dx^2} = xe^x$

Solution. Here, we have $\frac{d^2y}{dx^2} = xe^x$ ...(1)

Integrating (1), we get

$$\frac{dy}{dx} = xe^x - e^x + c_1$$ ...(2)

Again integrating (2), we get

$$y = xe^x - e^x - e^x + c_1x + c_2$$

$$\Rightarrow \quad y = (x-2)\,e^x + c_1 x + c_2$$ **Ans.**

8.4 TYPE II

Now we will solve the differential equations of the type $\boxed{\frac{d^2y}{dx^2} = f(y)}$.

Working Rule:

Step 1. Multiplying by $2\frac{dy}{dx}$, we get $2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}$ ...(1)

Step 2. Integrating (1), we have $\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c = \phi(y)$ (say)

Step 3. $\frac{dy}{dx} = \sqrt{\phi(y)} \quad \Rightarrow \quad \frac{dy}{\sqrt{\phi(y)}} = dx$

Step 4. $\int \frac{dy}{\sqrt{\phi(y)}} = x + c$

Example 3. *Solve* $\frac{d^2y}{dx^2} = \sqrt{y}$, *under the condition* $y = 1$, $\frac{dy}{dx} = \frac{2}{\sqrt{3}}$ *at* $x = 0$.

Solution. We have, $\frac{d^2y}{dx^2} = \sqrt{y}$ ...(1)

Multiplying (1) by $2\frac{dy}{dx}$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\sqrt{y}\frac{dy}{dx} \qquad ...(2)$$

Integrating (2), we get $\left(\frac{dy}{dx}\right)^2 = \frac{4}{3}y^{3/2} + c_1$ ...(3)

On putting $y = 1$ and $\frac{dy}{dx} = \frac{2}{\sqrt{3}}$, we have $c_1 = 0$

Equation (3) becomes $\left(\frac{dy}{dx}\right)^2 = \frac{4}{3}y^{3/2} \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{3}}y^{3/4} \Rightarrow y^{-3/4}\,dy = \frac{2}{\sqrt{3}}dx$

Again integrating, we get $\frac{y^{1/4}}{\frac{1}{4}} = \frac{2}{\sqrt{3}}x + c_2 \Rightarrow 4y^{\frac{1}{4}} = \frac{2}{\sqrt{3}}x + c_2$ ...(4)

On putting $x = 0$, $y = 1$, we get $c_2 = 4$

(4) becomes $4y^{1/4} = \frac{2}{\sqrt{3}}x + 4$ **Ans.**

Example 4. *Solve* $\frac{d^2y}{dx^2} = \sec^2 y \tan y$ *under the condition* $y = 0$ *and* $\frac{dy}{dx} = 1$ *when* $x = 0$.

Solution. We have, $\frac{d^2y}{dx^2} = \sec^2 y \tan y$

$$\Rightarrow \quad 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\sec^2 y \tan y \frac{dy}{dx}$$

$$\int 2\frac{dy}{dx}\frac{d^2y}{dx^2} = \int 2\sec^2 y \tan y \frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2 y + c_1 \Rightarrow \frac{dy}{dx} = \sqrt{\tan^2 y + c_1}$$

On putting $y = 0$ and $\frac{dy}{dx} = 1$, we get $c_1 = 1$

Now, $\frac{dy}{dx} = \sqrt{\tan^2 y + 1} = \sec y$

$$\Rightarrow \quad \cos y\, dy = dx$$

~~On integrating, we get~~ $\sin y = x + c$

On putting $y = 0$, $x = 0$, we have $c = 0$

$$\sin y = x \Rightarrow y = \sin^{-1} x$$ **Ans.**

Example 5. *Solve* $\frac{d^2y}{dx^2} = 2(y^3 + y)$, *under the condition* $y = 0, \frac{dy}{dx} = 1$, *when* $x = 0$.

Solution. We have, $\frac{d^2y}{dx^2} = 2(y^3 + y) \Rightarrow 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4(y^3 + y)\frac{dy}{dx}$

Integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = 4\left(\frac{y^4}{4} + \frac{y^2}{2}\right) + c_1 = y^4 + 2y^2 + c_1 \quad \text{...(1)}$$

On putting $y = 0$ and $\frac{dy}{dx} = 1$ in (1), we get $1 = c_1$

Equation (1) becomes $\left(\frac{dy}{dx}\right)^2 = y^4 + 2y^2 + 1 = (y^2 + 1)^2$

$$\frac{dy}{dx} = y^2 + 1 \Rightarrow \frac{dy}{1 + y^2} = dx$$

Again integrating, we get $\tan^{-1} y = x + c_2$ ...(2)

On putting $y = 0$ and $x = 0$ in (2), we have $0 = c_2$

Equation (2) is reduced to $\tan^{-1} y = x \Rightarrow y = \tan x$ **Ans.**

Example 6. *A motion is governed by* $\frac{d^2x}{dt^2} = 36x^{-2}$, *given that at* $t = 0$, $x = 8$ *and* $\frac{dx}{dt} = 0$, *find the displacement at any time t.*

Solution. We have, $\frac{d^2x}{dt^2} = 36x^{-2}$

$$\Rightarrow \quad 2\frac{d^2x}{dt^2}\frac{dx}{dt} = 2 \times 36x^{-2}\frac{dx}{dt} \quad \text{...(1)}$$

Integrating (1), we have $\left(\frac{dx}{dt}\right)^2 = -72x^{-1} + c_1$ ...(2)

Putting $x = 8$ and $\frac{dx}{dt} = 0$ in (2), we get

$$0 = -\frac{72}{8} + c_1 \Rightarrow c_1 = 9$$

(2) becomes $\left(\frac{dx}{dt}\right)^2 = -\frac{72}{x} + 9 \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{-72 + 9x}{x}$

$$\Rightarrow \quad \frac{dx}{dt} = 3\sqrt{\frac{(x-8)}{x}}$$

$$\Rightarrow \quad \int \frac{\sqrt{x}\,dx}{\sqrt{x-8}} = 3\int dt + c_2$$

$$\Rightarrow \quad \int \frac{x\,dx}{\sqrt{x^2-8x}} = 3t + C_2$$

$$\frac{1}{2}\int \frac{2x-8+8}{\sqrt{x^2-8x}}\,dx = 3t + C_2$$

$$\frac{1}{2}\int \frac{2x-8}{\sqrt{x^2-8x}}\,dx + 4\int \frac{1}{\sqrt{(x-4)^2-(4)^2}}\,dx = 3t + C_2$$

$$\sqrt{x^2-8x} + 4\cosh^{-1}\left(\frac{x-4}{4}\right) = 3t + C_2 \quad ...(3)$$

On putting $x = 8$ and $t = 0$ in (3), we get $c_2 = 0$

(3) becomes $\sqrt{x^2-8x} + 4\cosh^{-1}\left(\frac{x-4}{4}\right) = 3t$ **Ans.**

EXERCISE 8.1

Solve the following differential equations :

1. $y^3 \frac{d^2y}{dx^2} = a$ **Ans.** $c_1 y^2 = (c_1 x + c_2)^2$

2. $e^{2y} \frac{d^2y}{dx^2} = 1$ **Ans.** $c_1 e^y = \cosh(c_1 x + c_2)$

3. $\sin^3 y \frac{d^2y}{dx^2} = \cos y$ **Ans.** $\sin[(x+c_2)\sqrt{(1+c_1)}] + \sqrt{\left(\frac{1+c_1}{c_1}\right)}\cos y = 0$

TYPE III.

8.5 EQUATIONS WHICH DO NOT CONTAIN 'y' DIRECTLY

(Change of Dependent variable)

$$\text{Let } f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0 \quad ...(1)$$

We will solve, the differential equation by changing the dependent variable y by P.

On substituting $\frac{dy}{dx} = P$ *i.e.*, $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ in (1), we get

$$f\left(\frac{dP}{dx}, P, x\right) = 0$$

Example 7. *Solve* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

Solution. Here, we have

$$\frac{d^2y}{dx^2} - \frac{x}{(1-x^2)}\frac{dy}{dx} = \frac{2}{1-x^2} \quad ...(1)$$

Put $\frac{dy}{dx} = P$ so that $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ in (1), we get

$$\frac{dP}{dx} - \frac{x}{1-x^2} P = \frac{2}{1-x^2}$$

$$\text{I.F.} = e^{-\int\left(\frac{x}{1-x^2}\right)dx} = e^{\frac{1}{2}\log(1-x^2)} = e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Hence, its solution is

$$P\sqrt{1-x^2} = 2\int\frac{\sqrt{1-x^2}}{1-x^2}dx \Rightarrow P\sqrt{1-x^2} = \int\frac{2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow \quad P\sqrt{1-x^2} = 2\sin^{-1}x + c_1 \Rightarrow \frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} + \frac{c_1}{\sqrt{1-x^2}}$$

On integrating, we get

$$y = (\sin^{-1}x)^2 + c_1\sin^{-1}x + c_2$$ **Ans.**

EXERCISE 8.2

Solve the following differential equations:

1. $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0$ **Ans.** $y = c_2 - ax + c_1\log[x + \sqrt{(1+x^2)}]$

2. $(1+x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$ **Ans.** $y = -\frac{x}{k} + \frac{1+k^2}{k^2}\log(1+kx) + a$

TYPE IV.

8.6 EQUATIONS THAT DO NOT CONTAIN '*x*' DIRECTLY

(Change of dependent variable)

The equations that do not contain x directly are of the form

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0 \quad ...(1)$$

On substituting $\frac{dy}{dx} = P,$

and $\frac{d^2y}{dx^2} = \frac{dP}{dx} = \frac{dP}{dy}\cdot\frac{dy}{dx} = \frac{dP}{dy}P$

in (1), we get

$$f\left(P\frac{dP}{dy}, P, y\right) = 0 \quad ...(2)$$

Equation (2) is solved for P. Let

$$P = f_1(y) \Rightarrow \frac{dy}{dx} = f_1(y) \Rightarrow \frac{dy}{f_1(y)} = dx$$

$$\Rightarrow \int \frac{dy}{f_1(y)} = x + c$$

Example 8. *Solve:* $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx}$ ...(1)

Solution. Here, we have

$$\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = \frac{1}{y}\frac{dy}{dx}$$

Putting $\dfrac{dy}{dx} = P, \dfrac{d^2y}{dx^2} = \dfrac{dP}{dx} = \dfrac{dP}{dy} \cdot \dfrac{dy}{dx} = \dfrac{dP}{dy} P$ in (1), we get

$$P\frac{dP}{dy} + \frac{P^2}{y} = \frac{P}{y} \Rightarrow P\frac{dP}{dy} = \frac{P}{y}(1-P)$$

$$\Rightarrow \frac{dP}{1-P} = \frac{dy}{y} \Rightarrow -\log(1-P) = \log y + \log c_1$$

$$\Rightarrow \frac{1}{1-P} = c_1 y \Rightarrow P = 1 - \frac{1}{c_1 y} \Rightarrow \frac{dy}{dx} = \frac{c_1 y - 1}{c_1 y}$$

$$\Rightarrow \frac{c_1 y}{c_1 y - 1} dy = dx \Rightarrow \left(1 + \frac{1}{c_1 y - 1}\right) dy = dx$$

$$y + \frac{1}{c_1}\log(c_1 y - 1) = x + c_2$$ **Ans.**

TYPE V

8.7 METHOD OF REDUCTION

(Equations Whose one solution of complementary function is known)

If $y = u$ is given solution belonging to the complementary function of the differential equation. Let the other solution be $y = v$. Then $y = u.\, v$ is complete solution of the differential equation.

Let $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ ....(1), be the differential equation and u is the complementary function of (1)

$$\therefore \quad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad ...(2)$$

$$y = u.\, v \text{ so that } \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + u\frac{d^2v}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we get

$$v\frac{d^2u}{dx^2}+2\frac{dv}{dx}\frac{du}{dx}+u\frac{d^2v}{dx^2}+P\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+Qu.v=R$$

On arranging

$$\Rightarrow \quad v\left[\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right]+u\left[\frac{d^2v}{dx^2}+P\frac{dv}{dx}\right]+2\frac{du}{dx}\cdot\frac{dv}{dx}=R$$

The first bracket is zero by virtue of relation (2), and the remaining is divided by u.

$$\frac{d^2v}{dx^2}+P\frac{dv}{dx}+\frac{2}{u}\frac{du}{dx}\frac{dv}{dx}=\frac{R}{u}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=\frac{R}{u} \qquad ...(3)$$

Let $\quad \frac{dv}{dx}=z$, so that $\frac{d^2v}{dx^2}=\frac{dz}{dx}$

Equation (3) becomes

$$\boxed{\frac{dz}{dx}+\left[P+\frac{2}{u}\frac{du}{dx}\right]z=\frac{R}{u}}$$

This is the linear differential equation of first order and can be solved (z can be found), which will contain one constant.

On integration $z=\frac{dv}{dx}$, we can get v.

Having found v, the solution is $y = uv$.

Example 9. *Solve $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is an integral included in the complementary function.*

Solution. Here, we have $y'' - 4xy' + (4x^2 - 2)y = 0 \qquad ...(1)$

On putting $y = v.e^{x^2}$ in (1), the reduced equation as in the article 4.7.

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=0 \qquad [P=-4x,\ Q=4x^2-2,\ R=0]$$

$$\Rightarrow \quad \frac{d^2v}{dx^2}+\left[-4x+\frac{2}{e^{x^2}}(2x\,e^{x^2})\right]\frac{dv}{dx}=0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2}+[-4x+4x]\frac{dv}{dx}=0 \Rightarrow \frac{d^2v}{dx^2}=0 \Rightarrow \frac{dv}{dx}=c_1 \Rightarrow v=c_1x+c_2$$

$$\therefore \quad y=uv \quad [u=e^{x^2}]$$

$$y=e^{x^2}(c_1x+c_2)$$ **Ans.**

Example 10. *Solve* $x^2 y'' - (x^2 + 2x)y' + (x + 2)y = x^3 e^x$ *given that* $y = x$ *is a solution.*

Solution. Here, we have $x^2 y'' - (x^2 + 2x)y' + (x + 2)y = x^3 e^x$

$$\Rightarrow \quad y'' - \frac{x^2 + 2x}{x^2} y' + \frac{x + 2}{x^2} y = x e^x \qquad \ldots(1)$$

On putting $y = vx$ in (1), the reduced equation as in the article 8.7.

$$\frac{d^2 v}{dx^2} + \left\{P + \frac{2}{u}\frac{du}{dx}\right\}\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2 v}{dx^2} + \left[-\frac{x^2 + 2x}{x^2} + \frac{2}{x}(1)\right]\frac{dv}{dx} = \frac{x e^x}{x}$$

$$\Rightarrow \quad \frac{d^2 v}{dx^2} - \frac{dv}{dx} = e^x \Rightarrow \frac{dz}{dx} - z = e^x, \qquad \left(z = \frac{dv}{dx}\right)$$

which is a linear differential equation

$$I.F. = e^{-\int dx} = e^{-x}$$

Its solution is $\quad z\, e^{-x} = \int e^x . e^{-x}\, dx + c$

$$\Rightarrow \quad z\, e^{-x} = x + c \quad \Rightarrow \quad z = e^x . x + c\, e^x$$

$$\Rightarrow \quad \frac{dv}{dx} = e^x . x + c\, e^x$$

$$\Rightarrow \quad v = x.e^x - e^x + c\, e^x + c_1$$

$$\Rightarrow \quad v = (x - 1 + c)\, e^x + c_1$$

$$y = vx = (x^2 - x + cx)\, e^x + c_1 x \qquad \textbf{Ans.}$$

Example 11. *Solve* $x\dfrac{d^2 y}{dx^2} - (2x - 1)\dfrac{dy}{dx} + (x - 1)y = 0$

given that $y = e^x$ *is an integral included in the complementary function.*

Solution. Here, we have $x\dfrac{d^2 y}{dx^2} - (2x - 1)\dfrac{dy}{dx} + (x - 1)\, y = 0$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} - \frac{2x - 1}{x}\frac{dy}{dx} + \frac{x - 1}{x} y = 0 \qquad \ldots(1)$$

By putting $y = ve^x$ in (1),we get the reduced equation as in the article 8.8.

$$\frac{d^2 v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = 0 \qquad \ldots(2)$$

Putting $u = e^x$ and $\dfrac{dv}{dx} = z$ in (2), we get

$$\frac{dz}{dx} + \left[-\frac{2x - 1}{x} + \frac{2}{e^x} e^x\right] z = 0$$

$$\Rightarrow \quad \left[\frac{dz}{dx} + \frac{-2x + 1 + 2x}{x} z\right] = 0 \qquad \Rightarrow \quad \frac{dz}{dx} + \frac{z}{x} = 0$$

$$\Rightarrow \quad \frac{dz}{z} = -\frac{dx}{x} \Rightarrow \log z = -\log x + \log c_1$$

$$\Rightarrow \quad z = \frac{c_1}{x} \Rightarrow \frac{dv}{dx} = \frac{c_1}{x} \Rightarrow dv = c_1 \frac{dx}{x} \Rightarrow v = c_1 \log x + c_2$$

$$y = u.\, v = e^x (c_1 \log x + c_2) \qquad \textbf{Ans.}$$

8.8 RULE TO FIND OUT PART OF THE COMPLEMENTARY FUNCTION

Rule	*Condition*	*Part of Complementary Function* $= u$
1	$1 + P + Q = 0$	e^x
2	$1 - P + Q = 0$	e^{-x}
3	$1 + \frac{P}{a} + \frac{Q}{a^2} = 0$	e^{ax}
4	$P + Qx = 0$	x
5	$2 + 2Px + Qx^2 = 0$	x^2
6	$n(n-1) + Pnx + Qx^2 = 0$	x^n

Example 12. *Solve* $x^2 \frac{d^2y}{dx^2} - 2x[1+x]\frac{dy}{dx} + 2(1+x)y = x^3$

Solution. $$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{2x(1+x)}{x^2}\frac{dy}{dx} + \frac{2(1+x)y}{x^2} = x \qquad \text{...(1)}$$

Here $$P + Qx = -\frac{2x(1+x)}{x^2} + \frac{2(1+x)}{x^2}x = 0$$

Hence $y = x$ is a solution of the C.F. and the other solution is v.
Putting $y = vx$ in (1), we get the reduced equation as in article 8.7

$$\frac{d^2v}{dx^2} + \left\{P + \frac{2}{u}\frac{du}{dx}\right\}\frac{dv}{dx} = \frac{x}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{-2x(1+x)}{x^2} + \frac{2}{x}(1)\right]\frac{dv}{dx} = \frac{x}{x}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \Rightarrow \frac{dz}{dx} - 2z = 1 \qquad \left[\frac{dv}{dx} = z\right]$$

which is a linear differential equation of first order and $I.F. = e^{\int -2\,dx} = e^{-2x}$

Its solution is $$z\, e^{-2x} = \int e^{-2x}\, dx + c_1$$

$$\Rightarrow \quad z\, e^{-2x} = \frac{e^{-2x}}{-2} + c_1 \Rightarrow z = \frac{-1}{2} + c_1 e^{2x}$$

$$\Rightarrow \quad \frac{dv}{dx} = -\frac{1}{2} + c_1 e^{2x} \quad \Rightarrow \quad dv = \left(-\frac{1}{2} + c_1 e^{2x}\right) dx$$

$$\Rightarrow \quad v = \frac{-x}{2} + \frac{c_1}{2} e^{2x} + c_2$$

$$y = uv = x\left(\frac{-x}{2} + \frac{c_1}{2} e^{2x} + c_2\right)$$ **Ans.**

Example 13. *Solve* $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$

Solution. Here, we have $\dfrac{d^2y}{dx^2} - \dfrac{2x+5}{x+2}\dfrac{dy}{dx} + \dfrac{2y}{x+2} = \dfrac{(x+1)e^x}{x+2}$ ...(1)

Here $\quad P = -\dfrac{2x+5}{x+2},\ Q = \dfrac{2}{x+2},\ R = \dfrac{(x+1)e^x}{x+2}$

$$1 + \frac{P}{a} + \frac{Q}{a^2} = 0, \text{ Choosing } a = 2$$

$$1 + \frac{P}{2} + \frac{Q}{4} = 1 - \frac{2x+5}{2x+4} + \frac{2}{4x+8} = 0$$

Hence $y = e^{2x}$ is a part of C.F.

Putting $y = e^{2x}v$ in (1), the reduced equation as in the article 4.7.

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u}$$

On putting the values of P, Q, and R, we get

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left[-\frac{2x+5}{x+2} + \frac{2}{e^{2x}} 2e^{2x}\right]\frac{dv}{dx} = \frac{(x+1)e^x}{e^{2x}(x+2)}$$

$$\Rightarrow \text{ as`1`} \quad \frac{d^2v}{dx^2} + \left[-\frac{2x+5}{x+2} + 4\right]\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

$$\frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

$$\Rightarrow \quad \frac{dz}{dx} + \frac{2x+3}{x+2}z = \frac{x+1}{x+2}e^{-x} \qquad \left(\frac{dv}{dx} = z\right)$$

which is a linear differential equation,

$$I.F. = e^{\int\frac{2x+3}{x+2}dx} = e^{\int\left(2 - \frac{1}{x+2}\right)dx} = e^{2x - \log(x+2)} = \frac{e^{2x}}{x+2}$$

Its solution is

$$z.\frac{e^{2x}}{x+2} = \int\frac{e^{2x}}{x+2}\left[\frac{x+1}{x+2}\right]e^{-x}\,dx + c$$

$$= \int\frac{e^x(x+1)}{(x+2)^2}dx + c = \int e^x\left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right]dx + c$$

$$= \int \frac{e^x dx}{x+2} - \int \frac{e^x dx}{(x+2)^2} + c = \frac{e^x}{x+2} + \int \frac{e^x dx}{(x+2)^2} - \int \frac{e^x dx}{(x+2)^2} + c = \frac{e^x}{x+2} + c$$

$$\Rightarrow \quad z = e^{-x} + c\,(x+2)e^{-2x}$$

$$\Rightarrow \quad \frac{dv}{dx} = e^{-x} + c(x+2)e^{-2x}$$

$$v = \int e^{-x}\,dx + c\int (x+2)e^{-2x}\,dx + c_1$$

$$= -e^{-x} + c\left[\frac{(x+2)\,e^{-2x}}{-2} - \frac{e^{-2x}}{4}\right] + c_1$$

$$= -e^{-x} + \frac{ce^{-2x}}{4}[2x+5] + c_1$$

$$y = u.v$$

$$y = e^{2x}\left[-e^{-x} + \frac{ce^{-2x}}{4}(2x+5) + c_1\right]$$

$$y = -e^{x} - \frac{c}{4}(2x+5) + c_1\,e^{2x}$$

Ans.

EXERCISE 8.3

Solve the following differential equations :

1. $(3-x)\dfrac{d^2y}{dx^2} - (9-4x)\dfrac{dy}{dx} + (6-3x)\,y = 0$, given $y = e^x$ is a solution.

   Ans. $y = \dfrac{c_1}{8}e^{3x}(4x^3 - 42x^2 + 150x - 183) + c_2e^x$

2. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + (1-x)y = x^2\,e^{-x}$ given $y = e^x$ is an integral included in C.F.

   Ans. $y = c_2\,e^x + c_1\,(2x+1)\,e^{-x} - \dfrac{1}{4}(2x^2 + 2x + 1)e^{-x}$

3. $(1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x(1-x^2)^{3/2}$, given $y = x$ is part of C.F.

   Ans. $y = -\dfrac{x}{9}(1-x^2)^{3/2} - c_1\left[\sqrt{(1-x^2)} + x\sin^{-1}x\right] + c_2x$

4. $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution. **Ans.** $cy = 1 + (c_1 - x)\cot x$

5. $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = x$, given $y = x$ is a part of C.F. **Ans.** $y = 1 + c_1x\displaystyle\int \frac{1}{x^2}e^{\frac{x^3}{3}}dx + c_2x$

6. $(x\sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = 0$ given $y = x$ is solution.

   Ans. $y = c_2x - c_1\cos x$

7. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$, given that $y=x+\dfrac{1}{x}$ is one integral. **Ans.** $y=c_2\left(x+\dfrac{1}{x}\right)+\dfrac{c_1}{x}$

8. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-12y=x^3\log x$ *(U.P., II Semester 2004)*

[**Hint.** $n(n-1)+pnx+Qx^2=0$, $n=3$, satisfies this equation. Put $y=vx^3$, $\dfrac{dv}{dx}=z$]

Ans. $y=\left(c_1x^3+\dfrac{c_2}{x^4}\right)+\dfrac{x^3}{98}\log x\,(7\log x-2)$

TYPE (VI)

8.9 NORMAL FORM (REMOVAL OF FIRST DERIVATIVE)

When the part of complementary function cannot be found by previous method then we find the complete solution by removing $\dfrac{dy}{dx}$ from

$$y''+Py'+Qy=R \qquad \text{...(1)}$$

Method:

Let uv be the complete solution of the given differential equation.

$$y=u.v.$$

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

On putting the values of $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ in (1), we get

$$\left(u\frac{d^2v}{dx^2}+2\frac{dv}{dx}\frac{du}{dx}+v\frac{d^2u}{dx^2}\right)+P\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)+Q.uv=R$$

$$\Rightarrow \quad v\frac{d^2u}{dx^2}+\frac{du}{dx}\left(Pv+2\frac{dv}{dx}\right)+u\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Q.v\right)=R$$

$$\Rightarrow \quad \frac{d^2u}{dx^2}+\frac{du}{dx}\left(P+\frac{2}{v}\frac{dv}{dx}\right)+\frac{u}{v}\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Q.v\right)=\frac{R}{v} \qquad \text{...(2)}$$

Here in the last bracket on L.H.S. is not zero as $y=v$ is not a part of C.F.

Here we shall remove the first derivative.

$$P+\frac{2}{v}\frac{dv}{dx}=0 \Rightarrow \frac{dv}{v}=-\frac{1}{2}P\,dx \Rightarrow \log v=\frac{-1}{2}\int P\,dx$$

$$\Rightarrow \quad v=e^{-\frac{1}{2}\int P\,dx}$$

In (2) we have to find out the value of the last bracket *i.e.*, $\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv$

$$\frac{dv}{dx}=-\frac{P}{2}e^{-\frac{1}{2}\int P\,dx}=-\frac{1}{2}Pv \qquad \left[\because v=e^{-1/2\int P\,dx}\right]$$

$$\frac{d^2v}{dx^2}=-\frac{1}{2}\frac{dP}{dx}v-\frac{P}{2}\frac{dv}{dx}=-\frac{1}{2}\frac{dP}{dx}v-\frac{P}{2}\left(-\frac{1}{2}Pv\right)=-\frac{1}{2}\frac{dP}{dx}v+\frac{1}{4}P^2v$$

$$\therefore \quad \frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv=-\frac{1}{2}\frac{dP}{dx}v+\frac{1}{4}P^2v+P\left(-\frac{1}{2}Pv\right)+Qv$$

$$=v\left[Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2\right]$$

Equation (1) is transformed as

$$\frac{d^2u}{dx^2}+\frac{u}{v}v\left\{Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}\right\}=\frac{R}{v}$$

$$\Rightarrow \quad \frac{d^2u}{dx^2}+u\left\{Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}\right\}=Re^{\frac{1}{2}\int P\,dx}$$

$$\frac{d^2u}{dx^2}+Q_1u=R_1 \quad \ldots(3), \qquad \text{where } Q_1=\left[Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}\right]$$

$$R_1=Re^{\frac{1}{2}\int P\,dx} \quad \text{or} \quad \frac{R}{v}$$

On putting the values of Q_1 and R_1 in (3), we can find u by solving (3).
Hence the solution of (1) is

$$y=uv \qquad \textbf{Ans.}$$

Example 14. *Solve* $\frac{d}{dx}\left[\cos^2 x\frac{dy}{dx}\right]+\cos^2 x.\,y=0$ *by removing the first derivative.*

Solution. We have, $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right)+\cos^2 x.\,y=0$

$$\Rightarrow \quad \frac{d^2y}{dx^2}\cos^2 x-2\cos x\sin x\frac{dy}{dx}+(\cos^2 x)y=0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-2\tan x.\frac{dy}{dx}+y=0$$

Here, $P=-2\tan x,\ Q=1,\ R=0$

$$Q_1=Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}$$

$$=1-\frac{1}{2}(-2\sec^2 x)-\frac{4\tan^2 x}{4}$$

$$= 1 + \sec^2 x - \tan^2 x = 1 + 1 = 2$$

$$R_1 = R\, e^{\frac{1}{2}\int P\,dx} = 0$$

$$v = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-2\tan x)dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

Normal equation is

$$\frac{d^2u}{dx^2} + Q_1 u = R_1$$

$$\frac{d^2u}{dx^2} + 2u = 0 \quad \Rightarrow \quad (D^2 + 2)\, u = 0$$

A.E. is $m^2 + 2 = 0 \quad \Rightarrow \quad m = \pm i\sqrt{2}$

$$u = C.F. = c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x$$

$$y = u.v$$

Hence the complete solution is $y = [c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}x] \sec x$ **Ans.**

Example 15. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 2(x^2 + x)\dfrac{dy}{dx} + (x^2 + 2x + 2)y = 0$ *by removing the first derivative.*

Solution. We have, $\dfrac{d^2y}{dx^2} - \dfrac{2(x^2+x)}{x^2}\dfrac{dy}{dx} + \left(\dfrac{x^2+2x+2}{x^2}\right)y = 0$ ...(1)

Here, $P = -2\left(1 + \dfrac{1}{x}\right)$, $Q = \dfrac{x^2 + 2x + 2}{x^2}$, $R = 0$

In order to remove the first derivative, we put $y = u.v$ in (1) to get the normal equation

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \qquad ...(2)$$

where $v = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\left(1+\frac{1}{x}\right)dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^x . e^{\log x} = x\,e^x$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = \frac{x^2 + 2x + 2}{x^2} - \frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{4}{4}\left(1 + \frac{1}{x}\right)^2$$

$$= 1 + \frac{2}{x} + \frac{2}{x^2} - \frac{1}{x^2} - 1 - \frac{1}{x^2} - \frac{2}{x} = 0$$

$$R_1 = R e^{\frac{1}{2}\int p\,dx} = 0$$

On putting the values of Q_1 and R_1 in (2), we get

$$\frac{d^2u}{dx^2} + 0\,(u) = 0 \quad \Rightarrow \quad \frac{d^2u}{dx^2} = 0$$

$$\frac{du}{dx} = c_1 \quad \Rightarrow \quad u = c_1 x + c_2$$

Hence the complete solution is $y = u.v = (c_1 x + c_2)\, x\, e^x$ **Ans.**

Example 16. *Solve:* $\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$ *by removing the fir[st] derivative.*

Solution. We have, $\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(4x^2-1)\,y=-3e^{x^2}\sin 2x$ ...(1)

Here $P=-4x$, $\quad Q=4x^2-1$, $\quad R=-3e^{x^2}\sin 2x$

In order to remove the first derivative, $v=e^{-\frac{1}{2}\int P dx}=e^{-\frac{1}{2}\int -4x\,dx}=e^{2\int x\,dx}=e^{x^2}$

On putting $y=uv$, the normal equation is

$$\frac{d^2u}{dx^2}+Q_1\,u=R_1 \qquad ...(2)$$

where
$$Q_1=Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}=(4x^2-1)-\frac{1}{2}(-4)-\frac{16x^2}{4}$$
$$=4x^2-1+2-4x^2=1$$
$$R_1=\frac{R}{v}=\frac{-3e^{x^2}\sin 2x}{e^{x^2}}=-3\sin 2x$$

Equation (2) becomes $\dfrac{d^2u}{dx^2}+u=-3\sin 2x$

$$(D^2+1)u=-3\sin 2x$$

A.E. is $m^2+1=0\Rightarrow m=\pm i$

$\Rightarrow$ $\qquad C.F.=c_1\cos x+c_2\sin x$

$$P.I.=\frac{1}{D^2+1}(-3\sin 2x)=\frac{-3\sin 2x}{-4+1}=\sin 2x$$

Its solution is $\qquad u=C.F.+P.I.$

$$u=c_1\cos x+c_2\sin x+\sin 2x$$

Hence, complete solution is $\quad y=u.\,v=(c_1\cos x+c_2\sin x+\sin 2x)e^{x^2}$ **Ans.**

EXERCISE 8.4

Solve the following differential equations by removing the first derivative:

1. $\dfrac{d^2y}{dx^2}-2\tan x.\dfrac{dy}{dx}-5y=0$ **Ans.** $y=(a\,e^{2x}+b\,e^{-3x})\sec x$

2. $\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(4x^2-3)\,y=e^{x^2}$ **Ans.** $y=(c_1e^{x}+c_2\,e^{-x}-1)$

3. $\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+(x^2+2)y=e^{\frac{1}{2}(x^2+2x)}$ **Ans.** $y=(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}\,x)\,e^{\frac{x^2}{2}+\frac{1}{4}e^{x}.e^{\frac{x^2}{2}}}$

4. $\dfrac{d^2y}{dx^2}-\dfrac{2}{x}\dfrac{dy}{dx}+\left(n^2+\dfrac{2}{x^2}\right)y=0$ **Ans.** $y=(c_1\cos nx+c_2\sin nx)x$

5. $\frac{d^2 y}{d x^2}+\frac{2}{x}\frac{d y}{d x}-n^2 y=0$ **Ans.** $y=(c_1 e^{nx}+c_2 e^{-nx})\frac{1}{x}$

6. $\frac{d^2 y}{d x^2}+\frac{1}{x^{\frac{1}{3}}}\frac{d y}{d x}+\left(\frac{1}{4x^{2/3}}-\frac{1}{x^{4/3}}-\frac{6}{x^2}\right)y=0$ **Ans.** $y=(c_1 x^3+c_2 x^{-2})e^{-\frac{3}{4}x^{\frac{2}{3}}}$

7. $\frac{d^2 y}{d x^2}-\frac{1}{\sqrt{x}}\frac{d y}{d x}+\frac{y}{4x^2}(-8+\sqrt{x}+x)=0$ **Ans.** $y=(c_1 x^2+c_2 x^{-1})\,e^{\sqrt{x}}$

TYPE (VII)

8.10 METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS BY CHANGING THE INDEPENDENT VARIABLE

Consider, $\frac{d^2 y}{d x^2}+P\frac{d y}{d x}+Qy=R$ ...(1)

Let us change the independent variable x to z and $z=f(x)$

$$\frac{d y}{d x}=\frac{d y}{d z}\frac{d z}{d x}$$

$$\frac{d^2 y}{d x^2}=\frac{d^2 y}{d z^2}\left(\frac{d z}{d x}\right)^2+\frac{d y}{d z}\frac{d^2 z}{d x^2}$$

Putting the values of $\frac{d y}{d x}$ and $\frac{d^2 y}{d x^2}$ in (1), we get

$$\left(\frac{d^2 y}{d z^2}\left(\frac{d z}{d x}\right)^2+\frac{d y}{d z}\frac{d^2 z}{d x^2}\right)+P\left(\frac{d y}{d z}\frac{d z}{d x}\right)+Qy=R$$

$$\Rightarrow \quad \frac{d^2 y}{d z^2}\left(\frac{d z}{d x}\right)^2+\left(P\frac{d z}{d x}+\frac{d^2 z}{d x^2}\right)\frac{d y}{d z}+Qy=R$$

$$\Rightarrow \quad \frac{d^2 y}{dz^2}+\frac{\left(P\frac{d z}{d x}+\frac{d^2 z}{d x^2}\right)}{\left(\frac{d z}{d x}\right)^2}\frac{d y}{d z}+\frac{Qy}{\left(\frac{d z}{d x}\right)^2}=\frac{R}{\left(\frac{d z}{d x}\right)^2} \qquad ...(2)$$

On putting $\frac{\left(P\frac{d z}{d x}+\frac{d^2 z}{d x^2}\right)}{\left(\frac{d z}{d x}\right)^2}$ P_1, $\frac{Q}{\left(\frac{d z}{d x}\right)^2}=Q_1$ and $\frac{R}{\left(\frac{d z}{d x}\right)^2}$ R_1 in (2), we get

$$\Rightarrow \quad \frac{d^2 y}{d z^2}+P_1\frac{d y}{d z}+Q_1 y=R_1 \qquad ...(3)$$

Equation (3) is solved either by taking $P_1=0$ (first method) or $Q_1=a$ constant (second method).