Outline

Text Book(s):

1. Erwin Kreyszig, Advanced Engineering Mathematics, 10^{th} Edition, John Wiley & Sons (Wiley student Edison) (2015).

Reference Books:

1. B. S. Grewal, Higher Engineering Mathematics, 42^{nd} Edition (2013), Khanna Publishers, New Delhi.
2. G. DennisZill, Patrick D. Shanahan, A first course in complex analysis with applications, 3^{rd} Edition, 2013, Jones and Bartlett Publishers Series in Mathematics.
3. Michael, D. Greenberg, Advanced Engineering Mathematics, 2^{nd} Edition, Pearson Education (2002).
4. Peter V. O Neil, Advanced Engineering Mathematics, 7^{th} Edition, Cengage Learning (2011).
5. J. H. Mathews, R. W. Howell, Complex Analysis for Mathematics and Engineers, Fifth Edition (2013), Narosa Publishers.

- Complex numbers.
- $z = x + iy$ form of complex numbers.
- Algebraic Operations: Addition, Subtraction, Multiplication, Division.
- Conjugate, Modulus of a complex number.
- Properties of the complex numbers.
- Basic identities and inequalities.
- Non-zero complex numbers: Polar form, Trigonometric form, Exponential form, Argument function.
- Powers and Roots of complex numbers.
- Analytic function, Harmonic function.
- Properties of the analytic functions.
- Construction of analytic functions.
- Application of analytic functions.

Complex Numbers

Definition

A complex number z is defined to be an ordered pair of real numbers x and y as $z = (x, y)$. That is, the set of complex numbers is denoted by $\mathbb{C}$ and is given by

$$\mathbb{C} \equiv \{z = (x, y) : x \text{ and } y \text{ are real numbers}\}.$$

The ordered pair here means the order in which we write x and y in defining the complex number $z = (x, y)$. For example, the number $(1, 2)$ is not the same as $(2, 1)$.

In the complex number $z = (x, y)$,

- x is called the real part of z and is denoted by $Re(z)$ or $\Re(z)$.
- y is called the imaginary part of z and is denoted by $Im(z)$ or $\Im(z)$.

Cont.

- The numbers of the form $(0, y)$ are called pure imaginary numbers.
- The numbers of the form $(x, 0)$ are called real numbers.
- The set of real numbers can be identified as a subset

$$\mathbb{R} \equiv \{z = (x, y) \in \mathbb{C} : x \in \mathbb{R} \text{ and } y = 0\} \in \mathbb{C}. \text{ That is, } \mathbb{R} \subset \mathbb{C}.$$

- Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.
- i is used for the symbol of $\sqrt{-1}$ which is an imaginary unit.
- A real constant multiple of imaginary unit is pure imaginary number. Example: $z = 6i$.
- Re(z) = Im(iz) and Im(z) = -Re(iz).

Algebraic form of Complex Number (x + i y notation)

Set, $i = (0, 1)$. It is called iota. Such that,

$$(x, y) = (x, 0)(1, 0) + (0, 1)(y, 0) = x.1 + i.y = x + iy$$
$$\Rightarrow (x, y) = x + iy,$$
$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$
$$i^3 =?,\ i^4 =?, i^5 =? \texttt{ and so on.}$$

The form $z = x + iy$ is called the algebraic form of a complex number.

Hereafter, we prefer to use $x + iy$ form instead of ordered pair (x, y) form to write complex numbers.

Note: Electrical engineers use the letter j instead of i.

History: The representation of complex numbers in the plane was proposed independently by Casper Wessel (1797), K. F. Gauss (1799) and Jean Robert Argand (1806).

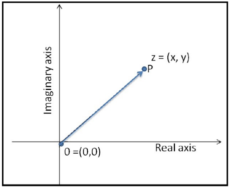
Basic algebraic properties

Let $z_1, z_2, z_3 \in \mathbb{C}$:

- **Commutative and associative law for addition:** $z_1 + z_2 = z_2 + z_1$ and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
- **Additive identity:** $z + 0 = 0 + z = z \ \forall \ z \in \mathbb{C}$.
- **Additive inverse:** For every $z \in \mathbb{C}$ there exists $-z \in \mathbb{C}$ such that $z + (-z) = 0 = (-z) + z$.
- **Commutative and associative law for multiplication:** $z_1 z_2 = z_2 z_1$ and $z_1(z_2 z_3) = (z_1 z_2) z_3$.
- **Multiplicative identity:** $z.1 = z = 1.z \ \forall \ z \in \mathbb{C}$.
- **Multiplicative inverse:** For every non-zero $z \in \mathbb{C}$ there exists $w(= \frac{1}{z}) \in \mathbb{C}$ such that $zw = 1 = wz$.
- **Distributive law:** $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Note: $\mathbb{C}$ is a field.

Complex Plane



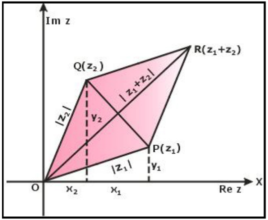
- The complex number $z = x + iy$ can be viewed as a point P having cartesian coordinates (x, y) in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
- The x-axis and y-axis are called the real axis and the imaginary axis respectively.
- The complex number $z = (x, y)$ can also be represented by a vector connecting the origin $0 = (0, 0)$ to the point P.
- This plane is called the complex plane or z-plane. It is also known as the Gauss plane or the Argand Plane.
- Sum, difference, product and division of any two complex numbers is itself a complex number .

Addition Operation

For any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the addition of z_1 and z_2 is defined

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Geometric Interpretation of Addition of two complex numbers:



If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are not collinear, then $\overrightarrow{OR}$ is the diagonal of the parallelogram with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as adjacent sides.

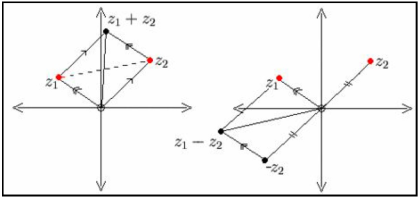
Subtraction Operation

For any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the difference of z_1 and z_2 is defined

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

The difference $z_1 - z_2$ can be viewed as the sum of the complex numbers z_1 and $-z_2$(additive inverse of z_2).

Geometric Interpretation of Subtraction of two complex numbers:



Multiplication and Division

Multiplication Operation

For any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the product of z_1 and z_2 is defined

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Division Operation

For any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the division of z_1 and z_2 is defined

$$\frac{z_1}{z_2} = \left(\frac{1}{x_1^2 + x_2^2}\right)((x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)).$$

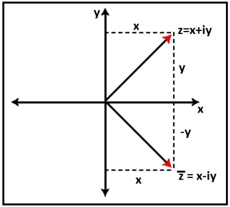
Note: The set of complex numbers $\mathbb{C}$ with these operations addition and multiplication forms a field. The identity element of addition is $(0, 0)$ and the identity element of multiplication is $(1, 0)$. $\mathbb{R}$ is a sub-field of $\mathbb{C}$.

Conjugate of a Complex Number

The complex conjugate, or simply, the conjugate of a complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined by

$$\bar{z} = x - iy$$

Geometrically, the point $\bar{z} = x - iy$ is the reflection (mirror image) of the point $z = x + iy$ on the real axis.



Examples: If $z = 3 + 4i$ then $\bar{z} = 3 - 4i$. If $z = -5$ then $\bar{z} = -5$.

Complex conjugate properties

If $z = x + iy$ is a complex number then its conjugate is denoted by $\bar{z} = x - iy$. Let $z_1, z_2 \in \mathbb{C}$ then,

- $z_1 = z_2$ if and only if $\bar{z}_1 = \bar{z}_2$.
- $Re(z) = Re(\bar{z})$ and $Im(z) = -Im(\bar{z})$.
- $\bar{z} = z$ if and only if z is a real number.
- $\bar{\bar{z}} = z$.
- $z + \bar{z} = 2Re(z) = 2x$ if $z = x + iy$.
- $z - \bar{z} = 2iIm(z) = 2iy$ if $z = x + iy$.
- $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$ and $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.
- $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$ provided $\bar{z}_2 = 0$.

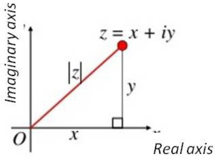
Note: The numbers z and $\bar{z}$ are called the complex conjugate coordinates, or simply the conjugate coordinates corresponding to the point $z = (x, y) = x+iy$. Also they have been called the isotropic coordinates and the minimal coordinates of the point.

Modulus of a Complex Number

The modulus or absolute value of a complex number $z = x + iy$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}.$$

Here, as usual, the radical stands for the principal (non-negative) square root of $x^2 + y^2$.



Example: The modulus of the complex number $4 + 3i$ is $|4 + 3i| = \sqrt{4^2 + 3^2} = 5$.

Complex conjugate and Modulus properties

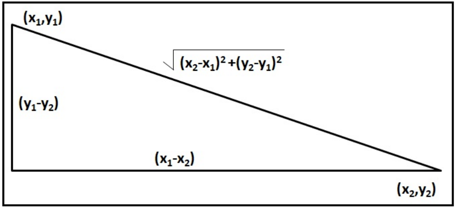
- $|z| \geq 0 \ \forall \ z \in \mathbb{C}$. $|z| = 0$ if and only if $z = 0$.
- $|\bar{z}| = |z| = |-z|$.
- $|z|^2 = z\bar{z}$.
- $|z_1 z_2| = |z_1||z_2|$.
- If $z = x + iy$, $|z| \leq |x| + |y|$.
- If $z = x + iy$, $|x| \leq |z|$ and $|y| \leq |z|$.
- Parallelogram Law: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.
- Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$.
- $|z_1 - z_2| \leq |z_1| + |z_2|$, $|z_1 - z_2| \geq ||z_1| - |z_2||$, $|z_1 + z_2| \geq ||z_1| - |z_2||$.
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$.
- If $n \in \mathbb{N}$, then $|z^n| = |z|^n$. If $-n \in \mathbb{N}$, then $|z^n| = |z|^n$ for $z \neq 0$.

Distance between Two Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers. Then the (Usual/Euclidean) distance between z_1 and z_2 is defined by

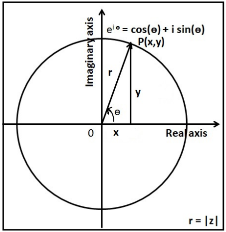
$$d(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$similarly, |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$



Example: If $z_1 = 1 + i$ and $z_2 = 1 - i$ then
$|z_1 - z_2| = \sqrt{(1-1)^2 + (1-(-1))^2} = 2.$
Note: $|z| = d(0, z)$. $(\mathbb{C}, d)$ is a metric space.

Polar representation of Complex Numbers



- In addition to the cartesian coordinates (x, y) in the complex plane, we also employ the usual polar coordinates (r, θ) defined by

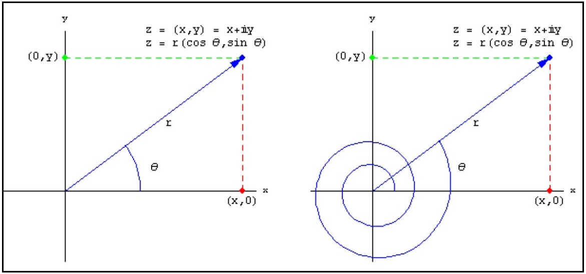
$$x = rcos(\theta), y = rsin(\theta).$$

- Then, $z = x + iy$ takes polar form

$$z = r(cos(\theta) + isin(\theta)) = re^{i\theta}$$

where, the Euler's formula $e^{i\theta} = cos(\theta) + isin(\theta)$.

Cont.



- Any non-zero complex number $z = x + iy$ can be uniquely specified by its magnitude(length from origin) and direction(the angle it makes with positive x-axis).
- Here, r is called the absolute value or modulus of z and is denoted by $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$.
- and θ is called the argument of z and is denoted by $arg(z) = \theta = tan^{-1}\left(\frac{y}{x}\right)$.

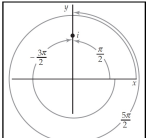
Cont.

- An argument of a complex number z is not unique since $cos(\theta)$ and $sin(\theta)$ are 2π-periodic; in other words, if θ_0 is an argument of z, then necessarily the angles $\theta_0 \pm 2\pi$, $\theta_0 \pm 2\pi$, ... are also arguments of z.
- So that $arg(z)$ is a multi-valued function.

$$arg(z) = Arg(z) + 2k\pi, k \in \mathbb{Z}.$$

- $Arg(z)$ is a principal value of $arg(z)$, the principal argument of z chosen in $(-\pi, \pi]$.

  Examples: 1. Let $z = 0 + i$, $arg(i) = \pi/2 + 2k\pi, n \in \mathbb{Z}$, where $Arg(i) = \pi/2$ since i lies in first quadrant. 2. $arg(5) = \{2k\pi, k \in \mathbb{Z}\}$; 3. $arg(-3) = \{(2k+1)\pi, k \in \mathbb{Z}\}$. 4. Find $arg(1-i)$. 5. Find $arg(-i)$.



Properties

- Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 [cos(\theta_1 + \theta 2) + isin(\theta 1 + \theta 2)] = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

- If $z_1 \neq 0$ and $z_2 \neq 0$,

$$arg(z_1 z_2) = arg(z_1) + arg(z_2),$$

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2).$$

- As $|e^{i\theta}| = 1, \ \forall \ \theta \in \mathbb{R}$, it follows that $|z_1 z_2| = |z_1||z_2|$.

Powers of Complex Numbers

De Moiver's formula:

$$z^n = [r(cos(\theta) + isin(\theta))]^n = r^n(cos(n\theta) + isin(n\theta)) = r^n e^{in\theta}.$$

- Problem: Given a non-zero complex number z_0 and a natural number $n \in N$. Find all distinct complex numbers w such that $z_0 = w^n$.
- If w satisfies the above then $|w| = |z_0|^{1/n}$. So, if $z_0 = |z_0|(cos(\theta) + isin(\theta))$, we try to find α such that

$$|z_0|(cos(\theta) + isin(\theta)) = [|z_0|^{1/n}(cos(\alpha) + isin(\alpha))]^n.$$

- By De Moiver's formula the absolute values $cos(\theta) = cos(n\alpha)$ and $sin(\theta) = sin(n\alpha)$ that is, $n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}$. Then $w = z_0^{1/n} = |z_0|^{1/n}\left(cos\left(\frac{\theta + 2k\pi}{n}\right) + isin\left(\frac{\theta + 2k\pi}{n}\right)\right)$ for $k = 0, 1, 2, ..., n-1$.

Cont.

Example 1. Simplify $(\sqrt{3}+i)^7$. We know that $z^n = r^n e^{in\theta}$,

$$(\sqrt{3}+i)^7 = \left(2e^{i\pi/6}\right)^7 = \left(2^6 e^{i\pi}(2e^{i\pi/6})\right) = -64\left(2e^{i\pi/6}\right) = -64(\sqrt{3}+i).$$

Example 2. Find cube roots of $z = i$.
WKT, we are basically solving the equation $w^3 = i$. Now $r = 1$, $\theta = arg(i) = \pi/2$. Then, the polar form of the given number is given by

$$z = 1(cos(\pi/2) + isin(\pi/2))$$

using De Moiver's formula, we get

$$w_k = \sqrt[3]{1}\left[cos\left(\frac{\pi/2+2k\pi}{3}\right) + isin\left(\frac{\pi/2+2k\pi}{3}\right)\right],\ k = 0, 1, 2.$$