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## Neighbourhood of a point $z_0$

Let $z_0$ be a complex number and $\delta$ be a positive real number. Then the set of all points $z$ satisfying $|z - z_0| < \delta$ is called a Neighbourhood(nbd). Thus, $N_\delta(z_0) = \{z : |z - z_0| < \delta\}$. we observe that, $|z - z_0| < \delta$ represent the interior of the circle with center $Z_0$ and radius $\delta$.

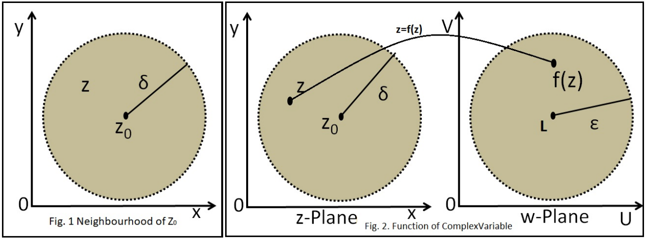



Fig. 1 Neighbourhood of $Z_0$ Fig. 2. Function of ComplexVariable

## Functions of Complex Variables

If for each value of the complex variable $z = x + iy$ in a given region $\mathbb{R}$, we have one or more values of $w = u + iv$, the $w$ is called function of the complex variable $z$ and we write $w = f(z) = u(x, y) + iv(x, y)$ where $u$ and $v$ are real functions of $x$ and $y$.

## Limit Point

A point $z_0$ is said to be a limit point of a set if every neighbourhood of $z_0$ contains infinitely many points.

## Limit Point of a Function

Let $f(z)$ is a single valued function in a region $\mathbb{R}$ and let $z_0$ be a limit point of $\mathbb{R}$. Then, $L$ is said to be the limit of $f(z)$ at $z_0$ if for every $\varepsilon > 0$, there exists a positive $\delta$ such that $|f(z) - L| < \varepsilon$ whenever $|z - z_0| < \delta$

$$i.e., \lim_{z \to z_0} f(z) = L.$$

## Continuity of a Function

Let $f(z)$ is a single valued function in a region $\mathbb{R}$ and let $z_0$ be a limit point of $\mathbb{R}$. If

1. $\lim_{z \to z_0} f(z)$ exists and
2. $\lim_{z \to z_0} f(z) = f(z_0)$, then $f(z)$ is said to be continuous at $z = z_0$.

**Example:** Show that $f(x,y)=\dfrac{2xy}{x^2+y^2}$ discontinuous at $(0,0)$ given that $f(0)=0$.

**Soln:** Given $f(x,y)=\dfrac{2xy}{x^2+y^2}$.

$$\lim_{x\to 0},\lim_{y\to 0} f(x,y)=\lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy}{x^2+y^2}\right)=0,$$

$$\lim_{x\to 0},\lim_{y\to 0} f(x,y)=\lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right)=0.$$

Along the path $y=mx$, let $x\to 0$, $y\to 0$ simultaneously,

$$\lim_{y=mx},\lim_{x\to 0} f(x,y)=\lim_{y=mx},\lim_{x\to 0}\frac{2xy}{x^2+y^2}$$

$$=\lim_{y\to 0},\lim_{x\to 0}\frac{2mx^2}{x^2+m^2x^2}=\lim_{y\to 0},\lim_{x\to 0}\frac{2m}{1+m^2}\neq 0.$$

Hence, the function is not continuous at orgin.

**Example:** Test continuity of $f(x,y)=\dfrac{2xy^2}{x^2+y^4}$ at the orgin given that f(0,0)=0.

**Soln:** Given $f(x,y)=\dfrac{2xy^2}{x^2+y^4}$.

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy^2}{x^2+y^2}\right)=0,\ \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy^2}{x^2+y^4}\right)=0.$$

Along the curve $x=y^2$, let $x\to 0$, $y\to 0$ simultaneously,

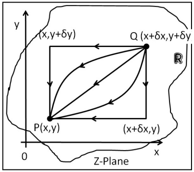
$$\lim_{x=y^2},\lim_{x\to 0}\frac{2xy^2}{x^2+y^4}=\lim_{y\to 0},\lim_{x\to 0}\frac{2y^4}{2y^4}=1\neq 0.$$

Hence, the function is not continuous at orgin.

## Differentiability of a Point

Let $f(z)$ is a single valued function in a region $\mathbb{R}$ and let $z_0$ be a limit point of $\mathbb{R}$. Then, the function $f(z)$ is differentiable at $z_0$, if the function continuous at $z_0$ and

$$f'(z) = \frac{dw}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} exists.$$




Here, $P(z)$ and $Q(z + \delta z)$ be two neighbouring points . The point $Q$ may approach $P$ along straight line or curved path in the given region.
**Note:** for each value of $z$ there corresponding one and only one value of $w$ is called single value function($z^2$, $1/z$) of $z$, otherwise it is called a multi valued function($\sqrt{z}$, $\sqrt[3]{z}$) of $z$.

### Analytic or Holomorphic or Regular Function

A single valued function $f(z)$ is defined in a region $\mathbb{R}$ is said to be analytic at the point $z_0$ if $f(z)$ is differentiable at every point of some neighbourhood of $z_0$.

### Entire or Integral Function

A single valued function $f(z)$ is defined in a region $\mathbb{R}$ is said to be entire at the point $z_0$ if $f(z)$ is analytic at every point of some neighbourhood of $z_0$. An entire function is analytic everywhere except at $z = \infty$.
Example: $z^2$, $e^z$, $cos(z)$, $sin(z)$, $cosh(z)$, $sinh(z)$.

Note: A point at which the function $w = f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Any functions : Continuous $\to$ Differentiable $\to$ Analytic $\to$ Entire

## Necessary condition for $f(z)$ to be analytic(Cauchy-Riemann Equations)

If $f(z) = u(x, y) + iv(x, y)$ is an analytic function in a region $\mathbb{R}$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, i.e., u_x = v_y \text{ and } v_x = -u_y.$$

## Sufficient condition for $f(z)$ to be analytic

A single valued function $f(z) = u(x, y) + iv(x, y)$ defined in a region $\mathbb{R}$, if

1. The first order partial derivatives with respect to $x$ and $y$ exists in $\mathbb{R}$

$$i.e., \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} \text{ exists and}$$

2. all these partial derivatives are continuous and
3. $u_x = v_y$ and $v_x = -u_y$ at every point in $\mathbb{R}$, then $f(z)$ is an analytic function in that region $\mathbb{R}$.

### Polar Form of C-R Equations

Let $(r, \theta)$ denote the polar co-ordinates of the point $(x, y)$. Then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}, i.e., u_r = \frac{1}{r}v_\theta \text{ and } v_r = -\frac{1}{r}u_\theta.$$

### Harmonic Functions

A function $f(z)$ is said to be a harmonic function if it satisfy the Laplace equations

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, i.e., \nabla^2 f = 0.$$

Note: Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function, then

- $u(x, y)$ is conjugate harmonic function of $v(x, y)$ and
- $v(x, y)$ is conjugate harmonic function of $u(x, y)$.

**Example:** Prove that the function $f(z) = z^2$ is analytic.

**Soln:** Let $f(z) = u + iv$, where $z = x + iy$

$\Rightarrow u + iv = z^2 = (x + iy)^2 = x^2 + i^2y^2 + i2xy = (x^2 - y^2) + i2xy.$

Equating real and imaginary parts, we get

$u = x^2 - y^2$ and $v = 2xy$

Diff. w. r. t. $x$ and $y$, we obtain

$u_x = 2x$ and $v_x = 2y$,

$u_y = -2y$ and $v_y = 2x$.

From above equations,

$u_x = v_y = 2x$ and $u_y = -v_x = -2y$.

Hence, Cauchy-Rieman equations satisfied.

Therefore, $f(z) = z^2$ is analytic.

**Example:** Show that the following

(i) $f(z) = e^z$ (ii) $f(z) = sin(z)$ are analytic.

**Hint:**

(i) $e^{i\theta} = cos\theta + i\sin\theta$,

(ii) $sin(A + B) = sin(A)cos(B) + cos(A)sin(B)$,

$sin(i\theta) = isinh(\theta)$,

$cos(i\theta) = cosh(\theta)$.

**Example:** Show that the function $f(z) = log(z)$ is analytic everywhere except at the orgin. Also, find its derivative.

**Soln:** Let $f(z) = u + iv$, where $z = re^{i\theta}$

$\Rightarrow u + iv = logz = log(re^{i\theta}) = log(r) + log(e^{i\theta}) = log(r) + i\theta. \quad \longrightarrow (1)$

At the orgin $r = 0$,

$\Rightarrow f(z) = log(0) + i\theta = -\infty + i\theta.$

So, $f(z)$ is not analytic at orgin. To show other than orgin, equating real and imaginary parts, we have

$u(r, \theta) = log(r)$, $v(r, \theta) = \theta.$

Diff. w. r. t. $r$ and $\theta$, we obtain

$u_r = 1/r$ and $v_r = 0$,

$u_\theta = 0$ and $v_\theta = 1$.

From above equations,

$u_r = (1/r)v_\theta = 1/r$ and $v_r = (-1/r)u_\theta = 0$.

Hence, Cauchy-Rieman equations satisfied.

So, $u_r$, $u_\theta$, $v_r$ and $v_\theta$ are continuous everywhere except at orgin. The function $f(z)$ satisfies all sufficient condition for existence of derivatives except at orgin.

Therefore, $f(z) = z^2$ is analytic everywhere except at orgin.

The derivatives is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{u_r + i(0)}{e^{i\theta}} = \frac{1/r}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

**Example:** Show that $f(z) = |xy|^{1/2}$ is not analytic at orgin eventhough C-R equations are satisfied at the point.

**Soln:** Let $f(z) = u + iv$, where $z = x + iy$

$\Rightarrow u + iv = |xy|^{1/2}$. $\longrightarrow (1)$

Equating real and imaginary parts, we get

$u = |xy|^{1/2}$ and $v = 0$.

Diff. w. r. t. $x$ and $y$ and substitute $x = 0$, $y = 0$ (at orgin), we get

$$\frac{\partial u}{\partial x}_{(0,0)} = \lim_{x\to 0}\left[\frac{u(x,0)-u(0,0)}{x}\right] = \lim_{x\to 0}\left[\frac{(0-0)}{x}\right] = 0,$$

$$\frac{\partial u}{\partial y}_{(0,0)} = \lim_{y\to 0}\left[\frac{u(0,y)-u(0,0)}{y}\right] = 0, \quad \frac{\partial v}{\partial x}_{(0,0)} = \lim_{x\to 0}\left[\frac{v(x,0)-u(0,0)}{x}\right] = 0 \text{ and}$$

$$\frac{\partial v}{\partial y}_{(0,0)} = \lim_{y\to 0}\left[\frac{v(0,y)-u(0,0)}{y}\right] = 0.$$

Clearly, C-R equations are satisfied. That is, $u_x = v_y$ and $u_y = -v_x$ at orgin.

$$\text{Now, } \lim_{z\to 0}\left[\frac{f(z)-f(0)}{z}\right] = \lim_{x\to 0, y\to 0}\left[\frac{\sqrt{|xy|-0}-0}{xy}\right]$$

$$= \lim_{y=mx, x\to 0}\left[\frac{\sqrt{m|x|^2}-0}{x(1+im)}\right] = \frac{\sqrt{m}}{(1+im)}. \quad [\because \text{ Along } y = mx]$$

The limit is not unique, since it depends on the value $m$. Therefore, $f'(z)$ does not exist.

Hence, $f(z)$ is not analytic at orgin.

**Example:** Prove that $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, (z \neq 0), f(0) = 0$ is continuous and that C-R equations are satisfied at orgin, yet $f'(z)$ does not exist there.

**Soln:** Let $f(z) = u + iv = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, where $z = x + iy$

$$\Rightarrow u + iv = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2+y^2}. \qquad \longrightarrow (1)$$

Equating real and imaginary parts, we get

$$u = \frac{(x^3 - y^3)}{(x^2+y^2)} \text{ and } v = \frac{(x^3 + y^3)}{(x^2+y^2)}.$$

Diff. w. r. t. $x$ and $y$ and substitute $x = 0$, $y = 0$ (at orgin), we get

$$\frac{\partial u}{\partial x}_{(0,0)} = \lim_{x\to 0}\left[\frac{u(x,0) - u(0,0)}{x}\right] = \lim_{x\to 0}\left[\frac{(x-0)}{x}\right] = 1,$$
$$\frac{\partial u}{\partial y}_{(0,0)} = \lim_{y\to 0}\left[\frac{u(0,y) - u(0,0)}{y}\right] = \lim_{y\to 0}\left[\frac{(-y-0)}{y}\right] = -1,$$
$$\frac{\partial v}{\partial x}_{(0,0)} = \lim_{x\to 0}\left[\frac{v(x,0) - u(0,0)}{x}\right] = \lim_{x\to 0}\left[\frac{(x-0)}{x}\right] = 1 \text{ and}$$
$$\frac{\partial v}{\partial y}_{(0,0)} = \lim_{y\to 0}\left[\frac{v(0,y) - u(0,0)}{y}\right] = \lim_{y\to 0}\left[\frac{(y-0)}{y}\right] = 1.$$

Clearly, C-R equations are satisfied.

That is, $u_x = v_y = 1$ and $u_y = -v_x = -1$ at orgin.

Now, $\displaystyle\lim_{z\to 0}\left[\frac{f(z) - f(0)}{z}\right] = \lim_{x\to 0, y\to 0}\left[\frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2+y^2)(x+iy)}\right]$

# Cont.

.Along $y = 0$,

$$\Rightarrow \lim_{x \to 0} \frac{(x^3 + ix^3)}{(x^2.x)} = (1 + i).$$

Along $y = x$,

$$\Rightarrow \lim_{x \to 0} \frac{2ix^3}{2x^3(1 + i)} = \frac{i}{(1 + i)}.$$

Here, $f'(0)$ have different values for different curves. So, the limit is not unique and not exist.

Hence, $f'(0)$ is not a regular(analytic) function at the orgin.

## Examples:

1. Verify whether $\dfrac{x - iy}{x^2 + y^2}$ is an analytic function.
2. Determine whether the function with $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2 + y^2}$ is an analytic function or not.
3. Prove that $z^3 + z$ is an analytic function.
4. Show that $f(z) = |z|^2$ is differentiable at orgin but it not analytic.

**Example:** Prove that $u = e^x cos(y)$ is harmonic.

**Soln:** Given $u = e^x cos(y)$. $\rightarrow (1)$

We know that $\nabla^2 u = 0$, if $u$ is harmonic.

Diff. (1) partially w.r.t, we have

$$\frac{\partial u}{\partial x} = e^x cos(y) \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x cos(y) \quad \rightarrow (2)$$

$$\frac{\partial u}{\partial y} = e^x(-sin(y)) \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x cos(y) \quad \rightarrow (3).$$

$(2) + (3) \Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x(cos(y) - cos(y)) = 0.$$

Hence, the given function $u$ is harmonic.

**Examples:** Show that, the following functions are harmonic

1. $u(x, y) = \frac{1}{2} log(x^2 + y^2)$,
2. $u(x, y) = e^x(xcos(y) - ysin(y))$,
3. $v(x, y) = 3x^2 y - y^3$,
4. $v(x, y) = e^{-x}(2xycos(y) + (y^2 - x^2)sin(y))$.