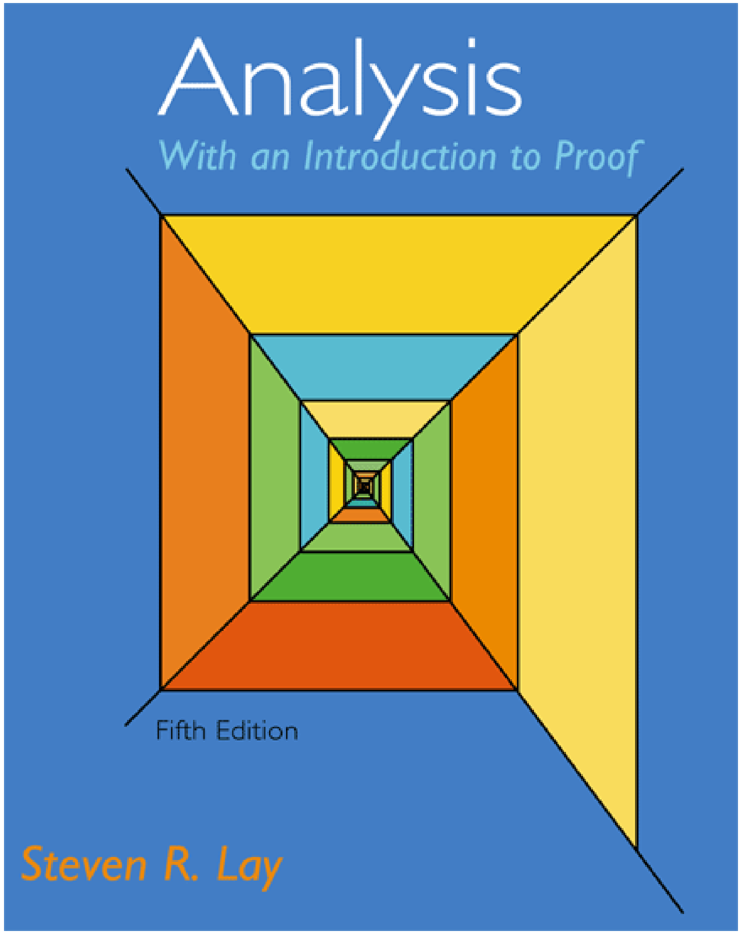
Chapter 3

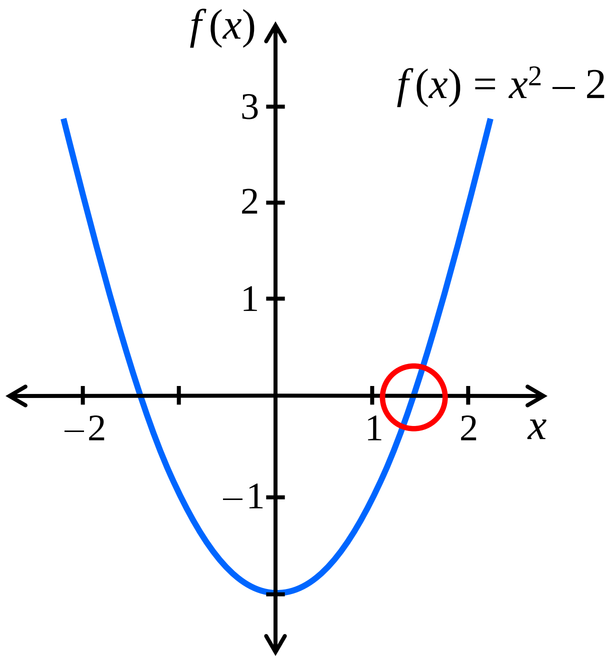
The Real Numbers



1) Bounded and Unbounded sets
2) Lower and Upper Bounds
3) Supremum, Infimum
4) The Completeness / LUB Axiom

There is one additional axiom that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. It is called the **completeness axiom**. Before presenting this axiom, let's look at why it's needed.

When we graph the function $f(x) = x^2 - 2$, it appears to cross the horizontal axis at a point between 1 and 2. But does it really?



How can we be sure that there is a "number" x on the axis such that $x^2 - 2 = 0$?

It turns out that if the x-axis consists only of rational numbers, then no such number exists.

That is, there is no rational number whose square is 2.

In fact, we can easily prove the more general result that $\sqrt{p}$ is irrational (not rational) for any prime number p.

Recall that an integer $p > 1$ is prime iff its only divisors are 1 and p.

In order to state the completeness axiom for $\mathbb{R}$, we need some preliminary definitions.

Upper Bounds and Suprema

Definition 3.3.2

- Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is called an **upper bound** of S, and we say that S is bounded above.

If $m \leq s$ for all $s \in S$, then m is a **lower bound** of S and S is bounded below.

- The set S is said to be **bounded** if it is bounded above and bounded below.

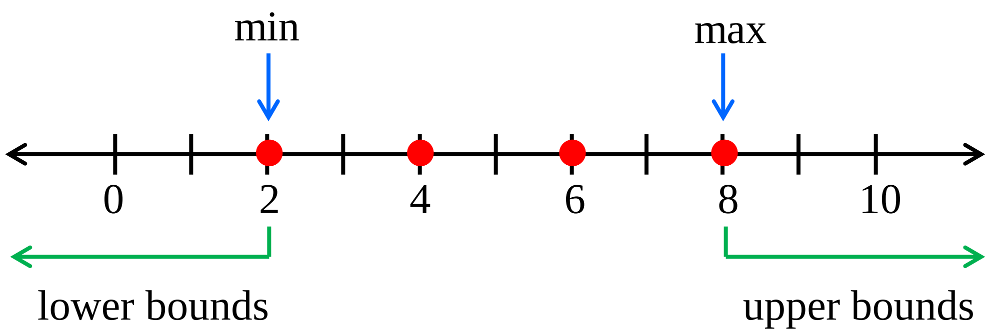
If an upper bound m of S is a member of S, then m is called the **maximum** (or largest element) of S, and we write $m = \max S$.

Similarly, if a lower bound of S is a member of S, then it is called the **minimum** (or least element) of S, denoted by $\min S$.

While a set may have many upper and lower bounds, if it has a maximum or a minimum, then those values are unique (Exercise 6).

Example 3.3.3

(a) Let $S = \{2, 4, 6, 8\}$.



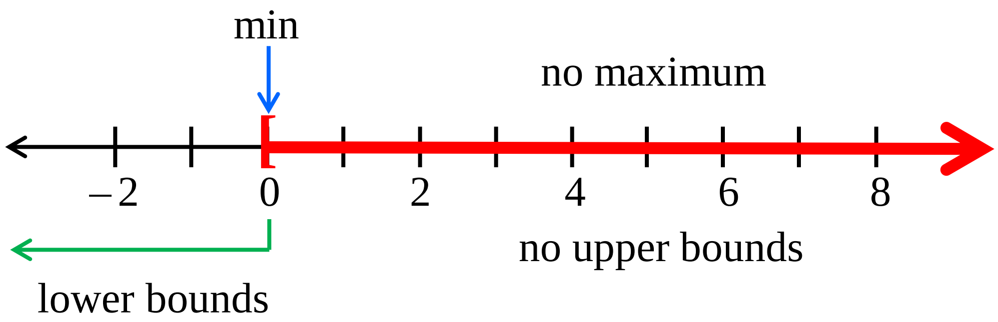
Then S is bounded above by 8, 9, 8.5, π^2, and any other real number greater than or equal to 8. Since $8 \in S$, we have $\max S = 8$.

Similarly, S has many lower bounds, including 2, which is the largest of the lower bounds and the minimum of S.

It is easy to see that any finite set is bounded and always has a maximum and a minimum.

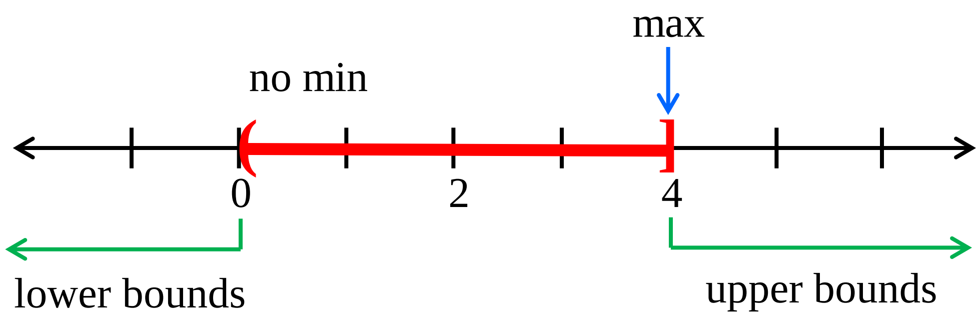
Example 3.3.3

(b) The interval $[0, \infty)$ is not bounded above.



It is bounded below by any nonpositive number, and of these lower bounds, 0 is the largest. Since $0 \in [0, \infty)$, 0 is the minimum of $[0, \infty)$.

(c) The interval $(0, 4]$ has a maximum of 4, and this is the smallest of the upper bounds.



It is bounded below by any nonpositive number, and of these lower bounds, 0 is the largest. Since $0 \notin (0,1]$, the set has no minimum.

Example 3.3.3

(d) The empty set $\varnothing$ is bounded above by any $m \in \mathbb{R}$. Note that the condition $m \geq s$ for all $s \in \varnothing$ is equivalent to the implication "if $s \in \varnothing$, then $m \geq s$."

This implication is true since the antecedent is false.

Likewise, $\varnothing$ is bounded below by any real m.

Definition 3.3.5

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called its **supremum** and is denoted by sup S. Thus $m = \sup S$ iff

(a) $m \geq s$, for all $s \in S$,

That is, m is an upper bound of S.

(b) if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

Nothing smaller than m is an upper bound of S.

If S is bounded below, then the greatest lower bound of S is called its **infimum** and is denoted by inf S.

Definition 3.3.6

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded below, then the greatest lower bound of S is called its **infimum** and is denoted by inf S. Thus $m = \inf S$ iff

(a) $m <= s$, for all $s \in S$, That is, m is a lower bound of S.

(b) if $m' > m$, then there exists $s' \in S$ such that $s' < m'$.

Nothing greater than m is an lower bound of S.

Let $T = \left\{q \in \mathbb{Q} : 0 \le q \le \sqrt{2}\right\}$. Does T have a supremum? It depends on the context.

If we think of T as a subset of the real numbers, then $\sup T = \sqrt{2}$.
But $\sqrt{2}$ is not a rational number. So T does not have a supremum in $\mathbb{Q}$.

When considering subsets of $\mathbb{R}$, it has been true that each set bounded above has had a least upper bound. This supremum may be a member of the set, as in the interval [0,1], or it may be outside the set, as in the interval [0,1), but in both cases the supremum *exists* as a real number. This fundamental difference between $\mathbb{Q}$ and $\mathbb{R}$ is the basis for our final axiom of the real numbers, the **completeness axiom**:

> Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound.
> That is, sup S exists and is a real number.

It follows readily from this that every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound. So, inf S exists and is a real number.

ILLUSTRATIONS

1. The set **N** of natural numbers is bounded below but not bounded above. 1 is a lower bound.

2. The sets **I**, **Q** and **R** are not bounded.

3. Every finite set of numbers is bounded.

4. The set S_1 of all positive real numbers $S_1 = \{x : x > 0, x \in \mathbf{R}\}$ is not bounded above, but is bounded below. The infimum zero is not a member of the set S_1.

5. The infinite set $S_2 = \{x : 0 < x < 1, x \in \mathbf{R}\}$ is bounded with supremum 1 and infimum zero, both of which do not belong to S_2.

6. The infinite set $S_3 = \{x : 0 \leqslant x \leqslant 1, x \in \mathbf{Q}\}$ is bounded, with supremum 1 and infimum 0 both of which are members of S_3.

7. The set $S_4 = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is bounded. The supremum 1 belongs to S_4 while infimum 0 does not.

8. Each of the following intervals is bounded:

$$[a, b], \quad]a, b], \quad [a, b[, \quad]a, b[.$$