Infinite Series.

Def :- Let $u_1, u_2 \ldots \ldots u_n \ldots$ be an infinite sequence of real numbers, then

$$u_1 + u_2 + \ldots \ldots + u_n + \ldots \infty$$

is called an infinite series denoted by $\sum u_n$. and sum of its first n terms is denoted by $S_n = u_1 + u_2 \ldots + u_n$.

Ex. Geometric series, Harmonic series

→ Convergence :- Let $\sum u_n = u_1 + u_2 \ldots + u_n + \ldots \infty$ be an infinite series & S_n be the sum of first n terms. i.e., $S_n = u_1 + u_2 \ldots + u_n$.

def :- The series is said to be convergent if the n^{th} partial sum S_n tends to a finite limit as $n \to \infty$.

Ex :- $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$

S_1 = first partial sum $= \frac{1}{2} = \frac{2^1 - 1}{2}$

S_2 = Second partial sum $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \frac{2^2 - 1}{2^2}$

$\vdots$

$S_n = \frac{2^n - 1}{2^n}$ here $\lim_{n \to \infty} \sum u_n = \lim_{n \to \infty} S_n$

$= \lim_{n \to \infty} \frac{2^n - 1}{2^n} = \lim_{n \to \infty} \frac{1 - 1/2^n}{1} = \frac{1-0}{1} = 1$

Divergent Series :- The series $\sum u_n$ is said to be divergent if the n^{th} partial sum S_n tends to $\pm\infty$ as $n \to \infty$

ex $\sum u_n = 1+2+3+ \cdots\cdots$

here $S_n = \frac{n(n+1)}{2}$ (sum of first n terms)

$$\lim_{n\to\infty} \sum u_n = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2}$$

$$= \infty.$$

$\therefore$ The given series is divergent

Oscillation of a Series :- Series $\sum u_n$ is said to be oscillatory or non-convergent if it's n^{th} partial sum S_n does not tend to a unique limit as $n \to \infty$.

ex $\sum u_n = 5-4-1+5-4-1+5-4-1+\cdots\cdots$

here possible values of S_n are $5, 0, 1$ and S_n does not tend to a unique limit. so $\sum u_n$ is oscillatory

Fundamental necessary condition for convergence

Statement:- If a positive term series $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

def:- An infinite series of which all the terms after some perticular terms are positive is a positive term series.

Proof:- let $S_n = u_1 + u_2 + u_3 \dots + u_n$.

Since $\sum u_n$ is convergent

$\lim_{n\to\infty} S_n =$ a finite real no. (say k)

& $\lim_{n\to\infty} S_{n-1} = k$.

now consider $u_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= k - k$$

$$\lim_{n\to\infty} u_n = 0$$

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Note :- Converse of the above theorem need not be true. i.e.,

For any infinite series if $\lim_{n\to\infty} u_n = 0$ then the series $\sum u_n$ need not be convergent.

Ex. $\sum u_n = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

here. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

but the series $\sum \frac{1}{n}$ is divergent

Contrapositive :- if $\lim_{n\to\infty} u_n \neq 0$ then the infinite series is divergent. is also called as divergent test

ex $\sum \frac{n}{n+1}$

here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$

$= \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$

$\therefore \sum \frac{n}{n+1}$ is divergent.

Comparison Tests.

1. If two positive term series Σu_n & Σv_n be such that

(i) Σv_n converges

(ii) $u_n \leq v_n$ for all n. then

Σu_n also converges.

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(ii) $u_n \geq v_n$ for all n, then

Σu_n also diverges.

① $\sum \frac{1}{2^n+n}$.

let $\sum u_n = \sum \frac{1}{2^n+n}$

now consider $2^n+n > 2^n$ $\forall n$.

$$\frac{1}{2^n+n} < \frac{1}{2^n}. \quad \forall n.$$

here $\sum \frac{1}{2^n}$ is convergent

Since $\sum \frac{1}{2^n}$ is convergent & $\frac{1}{2^n+n} < \frac{1}{2^n}$, $\forall n$

the series $\sum \frac{1}{2^n+n}$ is convergent

② Consider the series $\sum \frac{1}{n+1}$

now $n+1 < n+n$ $\forall n$

$$\frac{1}{n+1} > \frac{1}{2n}.$$

$$\frac{1}{n+1} > \frac{1}{2n}.$$

here $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent $\left(\because \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\right)$

& $\frac{1}{n+1} > \frac{1}{2n}$, $\forall n$ the given series

diverges.

Limit form of Comparison Test :-

If two positive terms series $\sum u_n$ & $\sum v_n$ be such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite value $(\neq 0)$ then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

ex
$$\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$$

now
$$\frac{4n^2}{n^{7/3}} = n^{2-\frac{7}{3}} = n^{-1/3} = \frac{1}{n^{1/3}}$$

let
$$v_n = \frac{1}{n^{1/3}}$$

$$\lim_{n\to\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \times \frac{n^{1/3}}{1} =$$

$$\lim_{n\to\infty} \frac{\cancel{n^{7/3}}\left(4+n^{4/3-7/3}\right)}{\cancel{n^{7/3}}\sqrt[3]{1+\frac{1}{n^4}}}$$

$$= \lim_{n\to\infty} \frac{4+n^{-3/3}}{\sqrt[3]{1+\frac{1}{n^4}}} = \lim_{n\to\infty} \frac{4+\frac{1}{n}}{\sqrt[3]{1+1/n^4}}$$

$$= \frac{4}{1} = 4 \neq 0.$$

Since $\sum \frac{1}{n^{1/3}}$ diverges, given series also diverges

D' ALEMBERT'S Ratio Test :

In a positive series $\sum u_n$, if $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$ then the series converges if $\lambda < 1$ and diverges if $\lambda > 1$. and test fails if $\lambda = 1$

Reciprocal of Ratio test :-

In the positive terms series $\sum u_n$

if $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$ and diverges if $k < 1$ but fails for $\lambda = 1$.

Ex:- $1 + \frac{a+1}{b+1} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)} \cdots$

$$u_{n+1} = u_n \cdot \frac{na+1}{nb+1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{nb+1}{na+1}.$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(b+1/n)}{\cancel{n}(a+1/n)}$$

$$= \frac{b}{a}$$

So Given Series converges if $\frac{b}{a} > 1$ or $b > a$
diverges if $\frac{b}{a} < 1$ or $b < a$