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1. $\lim_{x\to\infty} \frac{2 - \cos x}{x+3}$

$-1 \leq \cos x \leq +1$

$= \lim_{x\to\infty} \frac{2}{x+3} - \lim_{x\to\infty} \frac{\cos x}{x+3}$

$= 0 - \lim_{x\to\infty} \frac{\cos x}{x+3}$

$\lim_{x\to\infty} \frac{-1}{x+3} \leq \lim_{x\to\infty}\left(\frac{\cos x}{x+3}\right) \leq \lim_{x\to\infty} \frac{1}{x+3}$ (using sandwich theorem)

$0 \leq \lim_{x\to\infty}\left(\frac{\cos x}{x+3}\right) \leq 0$

$\boxed{\cancel{= \lim_{x\to\infty} \frac{\cos x}{x+3} (-\infty, \infty)}}$

So,

$= 0 \leq -\lim_{x\to\infty} \frac{\cos x}{x+3} \leq 0$

$= \underline{\underline{0}}$

2. [a] $\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2(n)}$, using comparison test

$$\frac{U_n}{V_n} = \frac{\lim_{n\to\infty} \frac{(2n^3+7)}{n^4 \sin^2(n)}}{\lim_{n\to\infty} \frac{1}{n}} = \lim_{n\to\infty} \frac{(2n^4+7n)n}{n^4 \times \sin^2 n} \approx \cancel{\lim_{n\to\infty} \frac{n}{\sin^2 n}}$$

$= 1 =$ constant

Now by comparing U_n with $\Sigma(\frac{1}{n})$

we know that $\Sigma(\frac{1}{n})$ & U_n either converges or diverges together for being constant.

$\Sigma(\frac{1}{n})$ diverges using p' test.

So, ~~is~~ U_n is also divergent.

So, given seris diverges.

(b) $\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$

$U_n = \frac{e^{4n}}{(n-2)!}$

using couchy's root test :-

$\lim_{n\to\infty} U_n^{1/n} = K$

if $K < 1$, the series converges

$K = 1$ fails the test.

$\lim_{n\to\infty} U_n^{1/n} = \left[\frac{e^{4n}}{(n-2)!}\right]^{1/n} = \lim_{n\to\infty} \frac{e^4}{[(n-2)!]^{1/n}}$

$= \cancel{\lim e^4}$

$= \lim_{n\to\infty} \frac{e^4}{(n-2)!^{1/n}} < \lim_{n\to\infty} \frac{e^4}{n^{1/n}}$

So, $e^4 \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n}$

$= \lim_{x\to 0} (x)^x = e^{(\ln \lim_{x\to 0} (x)^x)}$

$= e^{(\lim_{x\to 0} \ln x^x)} = e^{(\lim_{x\to 0} x \ln x)}$

$= e^{(\lim_{x\to 0} x \ln x)} = e^{\left(\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}\right)}$

$= e^{\left(\lim_{x\to 0} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\left(\frac{1}{x}\right)}\right)}$ [using L'Hospital's rule]

$= e^{(\lim_{x\to 0}(-x))} = e^0 = 1$

Since, $\lim_{n\to\infty} \frac{e^4}{(n-2)!^{1/n}} < \lim_{n\to\infty} \frac{e^4}{n^{1/n}} = 1$

So, $\lim_{n\to\infty} \frac{e^4}{(n-2)!^{1/n}} < 1$ So, the series converges using Couchy's test.

③ $f_n(x) = \dfrac{1}{(1+x^n)}$ for $x \in [0,1]$

[a] $\displaystyle\lim_{n\to\infty} \frac{1}{(1+x^n)} = f(x)$

$\displaystyle\lim_{n\to\infty} x^n = 0$ for $0 < x < 1$

$$= \lim_{n\to\infty} \frac{1}{1+x^n} = 1 = f(x)$$

[b] $0 < a < 1$, $\{f_n\}$, on $[0,a]$

$0 < a < 1$, so,

$|f_n(x) - f(x)| < \epsilon$

$\left|\dfrac{1}{1+x^n} - 1\right| < \epsilon \quad = \quad \dfrac{1}{1+x^n} < \epsilon + 1$

$1 + x^n > \dfrac{1}{\epsilon+1} \Rightarrow x^n > \dfrac{-\epsilon}{\epsilon+1}$

$n \log x > \log\left(\dfrac{\epsilon+1}{\epsilon}\right)$

$n > \dfrac{\log(\epsilon+1) - \log\epsilon}{\log(x)}$

[C] f_n does not converge uniformly on $[0,\infty)$

$|f_n(x) - f(x)| < \epsilon$

$\left|\dfrac{1}{1+x^n} - 1\right| < \epsilon \quad = \quad \dfrac{1}{1+x^n} < \epsilon + 1$

$1 + x^n = \dfrac{1}{\epsilon+1} \Rightarrow x^n > \dfrac{-\epsilon}{\epsilon+1}$

$n \log x > \log\left(\dfrac{\epsilon+1}{\epsilon}\right)$

$n > \dfrac{\log(\epsilon+1) - \log\epsilon}{\log(x)} = m$

Since m depends on the value of ϵ and x, so it is not uniformly convergent, rather it is point-wise convergent.

[b]

$$f_n(x) = \frac{1}{1+x^n}$$

for $0 < x < a < 1$

$$\lim_{n\to\infty} a^n = 0 \quad , b<2, 0<a<1$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+x^n} = \lim_{n\to\infty} \frac{1}{1+a^n} = 1 \quad \cancel{\text{for an}}$$

So, $f_n(x)$ converges uniformly to $f(x)$ for any value of x in $[0,a]$, so, it is uniformly convergent.

4.[1] $y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx$

order = 2nd order

degree = 2

~~Homo~~ Linearly → non-linear

Q. 2. $x \sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y \sin\left(\frac{y}{x}\right) + x$

$y = xv$

$\frac{dy}{dx} = v + \cancel{dy}\,\frac{dv}{dx}x$

$\frac{dy}{dx} = \frac{y}{x}\cancel{\sin\left(\frac{y}{x}\right)} + \frac{\cancel{x}\,1}{\sin\left(\frac{y}{x}\right)}$

$v + \frac{dv}{dx}x = v + \frac{1}{\sin v}$

$\frac{dv}{dx}x = \frac{1}{\sin v}$

$(\sin v)\,dv = \frac{1}{x}dx$

$\cancel{\cos}\ \cos v = \log x + c$

Substituting $\cancel{\cos v}$ $v = \frac{y}{x}$

$\cos\frac{y}{x} = \log x + c$