Example 3.1A Show $\lim_{n\to\infty} \dfrac{n-1}{n+1} = 1$, directly from definition 3.1.

Solution. According to definition 3.1, we must show:

$$\text{(2)} \qquad \text{given } \epsilon > 0, \qquad \frac{n-1}{n+1} \underset{\epsilon}{\approx} 1 \quad \text{for } n \gg 1 .$$

We begin by examining the size of the difference, and simplifying it:

$$\left| \frac{n-1}{n+1} - 1 \right| = \left| \frac{-2}{n+1} \right| = \frac{2}{n+1} .$$

We want to show this difference is small if $n \gg 1$. Use the inequality laws:

$$\frac{2}{n+1} < \epsilon \quad \text{if} \quad n+1 > \frac{2}{\epsilon} , \quad \text{i.e., if} \quad n > N, \text{ where } N = \frac{2}{\epsilon} - 1 ;$$

this proves (2), in view of the definition (2.6) of "for $n \gg 1$". □

The argument can be written on one line (it's ungrammatical, but easier to write, print, and read this way):

Solution. Given $\epsilon > 0$, $\left| \dfrac{n-1}{n+1} - 1 \right| = \dfrac{2}{n+1} < \epsilon$, if $n > \dfrac{2}{\epsilon} - 1$. □

Example 3.1B Show $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=0$.

Solution. We use the identity $A-B=\dfrac{A^2-B^2}{A+B}$, which tells us that

$$(3) \qquad \left|(\sqrt{n+1}-\sqrt{n})\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}};$$

given $\epsilon>0$, $\dfrac{1}{2\sqrt{n}} < \epsilon$ if $\dfrac{1}{4n} < \epsilon^2$, i.e., if $n > \dfrac{1}{4\epsilon^2}$. $\square$

Home work questions

(a) $\dfrac{n}{n+1} \longrightarrow 1$

(c) $\dfrac{n^2+1}{n^2-1} \longrightarrow 1$

Prove that the sequence, sn = n + 1 /n + 2 does not converge to 0.

Note: The only limit point of a convergent sequence is its limit.

The converse does not hold: consider the sequence
{1, 1/2 , 2, 1/3 , 3, 1/4 , 4, 1/5……………….},
What is the limit point ?
Does limit point equals the limit?

$$\lim_{x \to \infty} \frac{2 - \cos x}{x + 3}$$

SOLUTION 2 : First note that

$$-1 \leq \cos x \leq +1$$

because of the well-known properties of the cosine function. Now multiply by -1, reversing the inequalities and getting

$$+1 \geq -\cos x \geq -1$$

or

$$-1 \leq -\cos x \leq +1 \ .$$

Next, add 2 to each component to get

$$1 \leq 2 - \cos x \leq 3 \ .$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $x + 3 > 0$. Thus,

$$\frac{1}{x+3} \leq \frac{2-\cos x}{x+3} \leq \frac{3}{x+3} \ .$$

Since

$$\lim_{x \to \infty} \frac{1}{x+3} = 0 = \lim_{x \to \infty} \frac{3}{x+3} \ ,$$

it follows from the Squeeze Principle that

$$\lim_{x \to \infty} \frac{2-\cos x}{x+3} = 0 \ .$$

$$\lim_{x \to \infty} \frac{x^2(2 + \sin^2 x)}{x + 100}$$

SOLUTION 5 : First note that

$$-1 \leq \sin x \leq +1 ,$$

so that

$$0 \leq \sin^2 x \leq 1$$

and

$$2 \leq 2 + \sin^2 x \leq 3 .$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $x+100 > 0$. Thus, dividing by $x+100$ and multiplying by x^2, we get

$$\frac{2}{x+100} \leq \frac{2+\sin^2 x}{x+100} \leq \frac{3}{x+100}$$

and

$$\frac{2x^2}{x+100} \leq \frac{x^2(2+\sin^2 x)}{x+100} \leq \frac{3x^2}{x+100} .$$

Then

$$\lim_{x\to\infty} \frac{2x^2}{x+100} = \lim_{x\to\infty} \frac{2x^2}{x+100} \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{2x}{1+\frac{100}{x}}$$

$$= \frac{\infty}{1+0}$$

$$= \infty .$$

Similarly,

$$\lim_{x\to\infty} \frac{3x^2}{x+100} = \infty .$$

Thus, it follows from the Squeeze Principle that

$$\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100} = \infty \text{ (does not exist).}$$

Problems on sandwich theorem

$$\lim_{x \to -\infty} \frac{5x^2 - \sin(3x)}{x^2 + 10}$$

Ans: 5

$$\lim_{x \to \infty} \frac{\cos^2(2x)}{3 - 2x}$$

Ans: 0

For problems 3 and 4, determine if the sequence is increasing or decreasing by calculating $a_{n+1} - a_n$.

3. $\left\{\dfrac{1}{4^n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) decreasing.}}$

4. $\left\{\dfrac{2n-3}{3n-2}\right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) increasing.}}$

1. Determine whether the sequences are increasing or decreasing:

(i) $\left\{\dfrac{n}{n^2+1}\right\}_{n\geq 1}$ (ii) $\left\{\dfrac{2^n 3^n}{5^{n+1}}\right\}$ (iii) $\left\{\dfrac{1-n}{n^2}\right\}_{n\geq 1}$.

18. In the previous set of assigned problems it was shown that **if** the sequence

$$\sqrt{30}, \sqrt{30+\sqrt{30}}, \sqrt{30+\sqrt{30+\sqrt{30}}}, \ldots$$

converged to a limit, that limit was 6. Now we will show that the sequence is bounded above and increasing; thus, it must converge.

(a) Define the sequence recursively.

$a_1 = \sqrt{30}, a_{n+1} = \sqrt{30 + a_n}$ for integers $n \geq 1$.

(b) Show that the sequence has an upper bound of 6.

$a_1 = \sqrt{30} < \sqrt{36} = 6$, so $a_1 < 6$.
$a_2 = \sqrt{30 + a_1} < \sqrt{30 + 6} = 6$, so $a_2 < 6$.
$a_3 = \sqrt{30 + a_2} < \sqrt{30 + 6} = 6$, so $a_3 < 6$.
This continues indefinitely, so $a_n < 6$ for all integers $n \geq 1$, i.e. the sequence is bounded from above by 6. (It is also bounded from below by 0).

(c) Show that the sequence is increasing by computing $a_{n+1}^2 - a_n^2$.

$a_{n+1}^2 - a_n^2 = 30 + a_n - a_n^2 = (5 + a_n)(6 - a_n)$.
Now from part (b) $0 < a_n < 6$, so $5 + a_n > 0$ and $6 - a_n > 0$, so $a_{n+1}^2 - a_n^2 > 0$.
Also, $a_{n+1}^2 - a_n^2 = (a_{n+1} - a_n)(a_{n+1} + a_n)$, so $(a_{n+1} - a_n)(a_{n+1} + a_n) > 0$.
Since every term in the sequence is positive, we now have $(a_{n+1} - a_n) > 0$, or $a_{n+1} > a_n$, i.e. the sequence is (strictly) increasing.

Examples: 1. Let $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}}$ for $n > 1$. Then use induction to see that $0 \leq x_n \leq 2$ and (x_n) is increasing. Therefore, by previous result (x_n) converges. Suppose $x_n \to \lambda$. Then $\lambda = \sqrt{2 + \lambda}$. This implies that $\lambda = 2$.

2. Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$. Note that $\frac{x_{n+1}}{x_n} < 1$. Hence the sequence is decreasing. Since $x_n > 0$, the sequence is bounded below. Therefore (x_n) converges. Suppose $x_n \to \lambda$. Then $\lambda = \frac{\lambda}{2} + 2$. Therefore, $\lambda = 2$.