Real numbers and completeness axiom.

Compositions.

def: An addition Composition is defined in a set S, if for any $a, b \in S$, there corresponds a member $a+b$ of S. i.e., $a+b \in S$.

def A multiplicative composition is defined in a set S, if for any $a, b \in S$, there corresponds a member $a.b$ of S, i.e., $ab \in S$.

def:- A Set is said to be algebraic Structure if two compositions addition & multiplication are defined.

Field Structure :- A set is said to field if the following properties are salisfied, $\forall a, b, c \in S$.

A1 :- closed w.r.t addition, $a, b \in S \Rightarrow a+b \in S$.

A2 :- Commutative w.r.t addition $a+b = b+a$, $a, b \in S$.

A3 :- associative w.r.t addition $c+(a+b) = b+(a+c)$

A4 :. additive identity exists, for each $a \in S$, $\exists\, 0 \in S$ such that $a+0 = a = 0+a$

A5 :. additive inverse exists, for each $a \in S$, $\exists -a \in S$ such that $a+(-a) = 0 = (-a)+a$.

M1 :: S is closed under multiplication.

$a, b \in S. \Rightarrow ab \in S$ ::

M-2 :: Commutative under multiplication

$a, b \in S \Rightarrow a \cdot b = b \cdot a$

M3 :: associative under multiplication

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $a, b, c \in S$.

M4. Multiplicative identity exists. $\exists$ a member $1 \in S$, such that $a \cdot 1 = 1 \cdot a = a$.

M5 Multiplicative inverse exists $\exists$ a member $\frac{1}{a} \in S$, Such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

A-M Multiplication is distributive with respect to addition i.e., $a(b+c) = ab + ac$.

a set S, if it possesses two compositions of addition and multiplication and satisfies above properties is called field structure.

Ex:- $\mathbb{Q}$ & $\mathbb{R}$ are fields

$\mathbb{N}$ & W, $\mathbb{Z}$ are not fields.

Order structure:-

A field is an ordered field if it satisfies the following properties.

1) Law of Trichotomy :- for any two elements $a, b \in S$, ~~eith~~ one of the following must be true. $a > b$, $a = b$ or $b > a$

2) Transitivity : $\forall a, b, c \in S$

$$a > b \wedge b > c \Rightarrow a > c$$

3) Compatability w.r.t addition composition and multiplication composition

$\forall a, b, c \in S$, $a > b \Rightarrow a + c > b + c$

$$a > b \; ; \wedge c > 0 \Rightarrow ac > bc.$$

here $\mathbb{Q}$ & $\mathbb{R}$ are ordered fields.

<u>Problem</u> :- Prove that $\sqrt{2}$ is irrational.

If possibel let us suppose that $\sqrt{2}$ is rational i.e., $\sqrt{2} = \frac{p}{q}$, $q \neq 0$, $p, q \in \mathbb{Z}$

$(\sqrt{2})^2 = \left(\frac{p}{q}\right)^2$. p, q have no common factor.

$p^2 = 2q^2$.

Since q is an integer $2q^2$ is integer

p^2 is integer divisible by 2.

So p must be divisible by 2

Let $p = 2m$, where m is integer.

$(2m)^2 = p^2$

$(2m)^2 = (2q^2) = 2q^2$.

$2^2 m^2 = 2q^2$

$2m^2 = q^2$.

$\Rightarrow$ q is also divisible by 2.

hence p & q are divisible by 2 which contradicts the statement that p & q have no common factor.

Open interval :- If a & b be two real numbers such that $a<b$, then the set $\{x : a<x<b\}$. consists of real numbers between a & b. and denoted by (a,b) called as open interval

closed interval :- The set $\{x : a \le x \le b\}$. which also includes a & b and all real numbers between a & b is called closed interval & denoted by $[a,b]$.

Semi closed or Semi open intervals

$(a, b]$ or $[a, b)$ are called Semi open or semi closed intervals.

Bounded and unbounded sets.

⇒ A subset S of Real numbers is said to be bounded above if ∃ a real number k such that every member of set S is less than or equal to k i.e., $x \leq k$, $\forall x \in S$.

⇒ The number k is called upperbound

there can be infinite no. of upper bounds for any subset.

⇒ A subset S of Real numbers is said to be bounded below if ∃ a real number k such that every member of S is greater than or equal to k i.e., $x \geq k$, $\forall x \in S$

⇒ the number k is called lower bound.

there can be infinite no. of lower bounds for any set ex ① Sets I, Q & $\mathbb{R}$ are not bounded.

② $S = \{x : x > 0,\ x \in \mathbb{R}\}$ is bounded below but not above

③ $\mathbb{N}$ is bounded below but not above.

def If an upperbound m of S is a member of S or belongs to S, then we say m is called largest / greatest / maximum of S.

def If a lower bound m of S is a member of S or belongs to S, then we say m is called smallest / least / minimum of S.

ex $S = \{2, 4, 6, 8\}$

$\downarrow$ min (2), $\downarrow$ max. (8)

→ all the numbers greater than 8 & including 8 are upper bounds

→ all the numbers lower than 2 & including 2 are lower bounds.

→ S is bounded both above & below.

(2) $[0, \infty)$ is not bounded above and bounded below by any non +ve no.

0 is also minimum as $0 \in S$.

(3) $(0, 4]$ is bounded below by any non +ve no. and bound above by 4 & any no > 4. 4 is also maximum as $4 \in S$

def Supremum:- let S be any non empty subset of R. If S is bounded above, then the least upper bound of S is also called its Supremum is denoted by sup S. Then $m = \sup S$ if

i) $m \geq s, \ \forall s \in S$.

ii) if $m' < m$, there exists $s' \in S$, such that $s' > m'$

i.e., no number smaller than m is an upper bound of S.

ex $S = \{x : 0 < x < 1, x \in R\}$, is bounded with supremum 1, as it is an upper bound as nothing less than 1 is an upper bound def. Let S be any non empty subset of $\mathbb{R}$ if S is bounded below, then the greatest lower bound of S is called infimum and denoted by $\inf S$. $m = \inf S$ iff

i) $m <= s, \forall s \in S$. &

ii) if $m' > m$, then there exists $s' \in S$, such that $s' < m'$.

i.e., no number greater than m is a lower bound of S.

ex $S = \{x : 0 \leq x \leq 1\}$. is bounded

$1 \rightarrow$ Supremum, upper bound & maximum

$0 \rightarrow$ Infimum, lower bound & minimum

Ex $S = \{q \in Q : 0 \leq q \leq \sqrt{2}\}$

if S is a subset of Real numbers, supremum of S is $\sqrt{2}$. but if S is set of rational numbers it does not have supremum in Q.

Completeness axiom: Every non empty set S of $\mathbb{R}$ which is bounded above (below) has a (supremum (infimum)).