# Lecture 33: Real Numbers and the Completeness Axiom

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**Definitions:**
Let $S$ be a nonempty subset of $\mathbb{R}$, i.e. $\phi \neq S \subseteq \mathbb{R}$

(1) If $x_0 \in S$ and $x \leq x_0$ for all $x \in S$,
then $x_0$ is called the maximum of $S$. ($x_0 = \max S$.)

(2) If $x_0 \in S$ and $x_0 \leq x$ for all $x \in S$,
then $x_0$ is called the minimum of $S$. ($x_0 = \min S$.)

(3) If $\exists M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$,
then $M$ is called an **upper bound** of $S$ and the set $S$ is **bounded above**.

(4) If $\exists m \in \mathbb{R}$ such that $m \leq x$ for all $x \in S$,
then $m$ is called a **lower bound** of $S$ and the set $S$ is **bounded below**.

(5) If $\exists m, M \in \mathbb{R}$ such that $m \leq x \leq M$ $\forall x \in S$, then $S$ is **bounded**.

(6) If $S$ is bounded above and $S$ has a least upper bound $M_0$, then $M_0$ is called the supremum of $S$ and denoted by $\sup S$.

(7) If $S$ is bounded below and $S$ has a greatest lower bound $m_0$, then $m_0$ is called the infimum of $S$ and denoted by $\inf S$.

**The Completeness Axiom**

A fundamental property of the set $\mathbb{R}$ of real numbers :

Completeness Axiom : $\mathbb{R}$ has "no gaps".

$\forall S \subseteq \mathbb{R}$ and $S \neq \emptyset$,

$$\boxed{\text{If } S \text{ is bounded above, then } \sup S \text{ exists and } \sup S \in \mathbb{R}.}$$

(that is, the set $S$ has a least upper bound which is a real number).

Note : "The Completeness Axiom" distinguishes the set of real numbers $\mathbb{R}$ from other sets such as the set $\mathbb{Q}$ of rational numbers.

**Example:** Let $A := \{r \in \mathbb{Q} | 0 \leq r \leq \sqrt{2}\} \subseteq \mathbb{Q}$.

(1) Is the set $A$ bounded above?

(2) Does it has a least upper bound in $A$?

**Examples:** Find the inf and sup of the following sets, if possible. State whether or not these numbers are in S.

1. $S = \{x \,|\, 0 < x \leq 3\}$

2. $S = \{x \,|\, x^2 - 2x - 3 < 0\}$

3. $S = \{x \,|\, 0 < x < 5, \cos(x) = 0\}$

4. $S = \{x \mid x = \frac{1}{n}, n \in \mathbb{N}\}$

**Some properties of sup and inf Theorem.** If $x_1$ and $x_2$ are least upper bounds for the set $A$, then $x_1 = x_2$.

**Theorem.** If the sets $A$ and $B$ are bounded above and $A \subseteq B$, then $\sup(A) \leq \sup(B)$.