6

Sequence and Series of Functions

6.1 Sequence of Functions

6.1.1 Pointwise Convergence and Uniform Convergence

Let J be an interval in $\mathbb{R}$.

Definition 6.1 For each $n \in \mathbb{N}$, suppose a function $f_n : J \to \mathbb{R}$ is given. Then we say that a a **sequence** (f_n) **of functions** on J is given.

More precisely, a sequence of functions on J is a map $F : \mathbb{N} \to \mathcal{F}(J)$, where $\mathcal{F}(J)$ is the set of all real valued functions defined on J. If $f_n := F(n)$ for $n \in \mathbb{N}$, then we denote F by (f_n), and call (f_n) as a sequence of functions. □

Definition 6.2 Let (f_n) be a sequence of functions on an interval J.

(a) We say that (f_n) **converges at a point** $x_0 \in J$ if the sequence $(f_n(x_0))$ of real numbers converges.

(b) We say that (f_n) **converges pointwise** on J if (f_n) converges at every point in J, i.e., for each $x \in J$, the sequence $(f_n(x))$ of real numbers converges. □

Definition 6.3 Let (f_n) be a sequence of functions on an interval J. If (f_n) converges pointwise on J, and if $f : J \to \mathbb{R}$ is defined by $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in J$, then we say that (f_n) **converges pointwise to** f on J, and f is the **pointwise limit** of (f_n), and in that case we write

$$f_n \to f \quad \text{pointwise on } J.$$

□

Thus, (f_n) converges to f pointwise on J if and only if for every $\varepsilon > 0$ and for each $x \in J$, there exists $N \in \mathbb{N}$ (depending, in general, on both ε and x) such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$.

***Exercise* 6.1** Pointwise limit of a sequence of functions is unique.

Example 6.1 Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{\sin(nx)}{n}, \qquad x \in \mathbb{R}$$

and for $n \in \mathbb{N}$. Then we see that for each $x \in \mathbb{R}$,

$$|f_n(x)| \le \frac{1}{n} \qquad \forall\, n \in \mathbb{N}.$$

Thus, (f_n) converges pointwise to f on $\mathbb{R}$, where f is the zero function on $\mathbb{R}$, i.e., $f(x) = 0$ for every $x \in \mathbb{R}$. $\square$

Suppose (f_n) converges to f pontwise on J. As we have mentioned, it can happen that for $\varepsilon > 0$, and for each $x \in J$, the number $N \in \mathbb{N}$ satisfying $|f_n(x) - f(x)| < \varepsilon$ $\forall\, n \ge N$ depends not only on ε but also on the point x. For instance, consider the following example.

Example 6.2 Let $f_n(x) = x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then we see that for $0 \le x < 1$, $f_n(x) \to 0$, and $f_n(1) \to 1$ as $n \to \infty$. Thus, (f_n) converges pointwise to a function f defined by

$$f(x) = \begin{cases} 0, & x \ne 1, \\ 1, & x = 1. \end{cases}$$

In particular, (f_n) converges pointwise to the zero function on $[0,1)$.

Note that if there exists $N \in \mathbb{N}$ such that $|x^n| < \varepsilon$ for all $n \ge N$ and for all $x \in [0,1)$, then, letting $x \to 1$, we would get $1 < \varepsilon$, which is not possible, had we chosen $\varepsilon < 1$. $\square$

For $\varepsilon > 0$, if we are able to find an $N \in \mathbb{N}$ which does not vary as x varies over J such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge N$, then we say that (f_n) *converges uniformly* to f on J. Following is the precise definition of uniform convergence of (f_n) to f on J.

Definition 6.4 Suppose (f_n) is a sequence of functions defined on an interval J. We say that (f_n) **converges to a function f uniformly on J** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (depending only on ε) such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \ge N \quad \text{and} \quad \forall\, x \in J,$$

and in that case we write

$$f_n \to f \quad \text{uniformly on} \quad J.$$

$\square$

We observe the following:

- If (f_n) converges uniformly to f, then it converges to f pointwise as well. Thus, if a sequence does not converge pointwise to any function, then it can not converge uniformly.

• If (f_n) converges uniformly to f on J, then (f_n) converges uniformly to f on every subinterval $J_0 \subseteq J$.

In Example 6.2 we obtained a sequence of functions which converges pointwise but not uniformly. Here is another example of a sequence of functions which converges pointwise but not uniformly.

Example 6.3 For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{nx}{1+n^2x^2}, \qquad x \in [0,1].$$

Note that $f_n(0) = 0$, and for $x \neq 0$, $f_n(x) \to 0$ as $n \to \infty$. Hence, (f_n) converges poitwise to the zero function. We do not have uniform convergence, as $f_n(1/n) = 1/2$ for all n. Indeed, if (f_n) converges uniformly, then there exists $N \in \mathbb{N}$ such that

$$|f_N(x)| < \varepsilon \qquad \forall\, x \in [0,1].$$

In particular, we must have

$$\frac{1}{2} = |f_N(1/N)| < \varepsilon \qquad \forall\, x \in [0,1].$$

This is not possible if we had chosen $\varepsilon < 1/2$. □

Example 6.4 Consider the sequence (f_n) defined by

$$f_n(x) = \tan^{-1}(nx), \qquad x \in \mathbb{R}.$$

Note that $f_n(0) = 0$, and for $x \neq 0$, $f_n(x) \to \pi/2$ as $n \to \infty$. Hence, the given sequence (f_n) converges pointwise to the function f defined by

$$f(x) = \begin{cases} 0, & x = 0, \\ \pi/2, & x \neq 0. \end{cases}$$

However, it does not converge uniformly to f on any interval containing 0. To see this, let J be an interval containing 0 and $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in J$. In particular, we have

$$|f_N(x) - \pi/2| < \varepsilon \qquad \forall\, x \in J \setminus \{0\}.$$

Letting $x \to 0$, we have $\pi/2 = |f_N(0) - \pi/2| < \varepsilon$ which is not possible if we had chooses $\varepsilon < \pi/2$. □

Now, we give a theorem which would help us to show non-uniform convergence of certain sequence of functions.

Theorem 6.1 *Suppose f_n and f are functions defined on an interval J. If there exists a sequence (x_n) in J such that $|f_n(x_n) - f(x_n)| \not\to 0$, then (f_n) does not converge uniformly to f on J.*

Proof. Suppose (f_n) converges uniformly to f on J. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, n \geq N, \quad \forall\, x \in J.$$

In particular,

$$|f_n(x_n) - f(x_n)| < \varepsilon \quad \forall\, n \geq N.$$

Hence, $|f_n(x_n) - f(x_n)| \to 0$ as $n \to \infty$. This is a contradiction to the hypothesis that $|f_n(x_n) - f(x_n)| \not\to 0$. Hence our assumption that (f_n) converges uniformly to f on J is wrong. ∎

In the case of Example 6.2, taking $x_n = n/(n+1)$, we see that

$$f_n(x_n) = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}.$$

Hence, by Theorem 6.1, (f_n) does not converge to $f \equiv 0$ uniformly on $[0, 1)$.

In Example 6.3, we may take $x_n = 1/n$, and in the case of Example 6.4, we may take $x_n = \pi/n$, and apply Theorem 6.1.

***Exercise* 6.2** Suppose f_n and f are functions defined on an interval J. If there exists a sequence (x_n) in J such that $[f_n(x_n) - f(x_n)] \not\to 0$, then (f_n) does not converge uniformly to f on J. Why?

[Suppose $a_n := [f_n(x_n) - f(x_n)] \not\to 0$. Then there exists $\delta > 0$ such that $|a_n| \geq \delta$ for infinitely many n. Now, if $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta/2$ for all $n \geq N$. In particular, $|a_n| < \delta/2$ for all $n \geq N$. Thus, we arrive at a contradiction.] ◀

Here is a sufficient condition for uniform convergence. Its proof is left as an exercise.

Theorem 6.2 *Suppose f_n for $n \in \mathbb{N}$ and f are functions on J. If there exists a sequence (α_n) of positive reals satisfying $\alpha_n \to 0$ as $n \to \infty$ and*

$$|f_n(x) - f(x)| \leq \alpha_n \quad \forall n \in \mathbb{N}, \quad \forall\, x \in J,$$

then (f_n) converges uniformly to f.

***Exercise* 6.3** Supply detailed proof for Theorem 6.2. ◀

Here are a few examples to illustrate the above theorem.

Example 6.5 For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{2nx}{1 + n^4 x^2}, \qquad x \in [0, 1].$$

Since $1 + n^4x^2 \geq 2n^2x$ (using the relation $a^2 + b^2 \geq 2ab$), we have

$$0 \leq f_n(x) \leq \frac{2nx}{2n^2x} = \frac{1}{n}.$$

Thus, by Theorem 6.2, (f_n) converges uniformly to the zero function. □

Example 6.6 For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{1}{n^3} \log(1 + n^4x^2), \qquad x \in [0, 1].$$

Then we have

$$0 \leq f_n(x) \leq \frac{1}{n^3} \log(1 + n^4) =: \alpha_n \quad \forall\, n \in \mathbb{N}.$$

Taking $g(t) := \frac{1}{t^3} \log(1 + t^4)$ for $t > 0$, we see, using L'Hospital's rule that

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{4t^3}{3t^2(1 + t^4)} = 0.$$

In particular,

$$\lim_{n\to\infty} \frac{1}{n^3} \log(1 + n^4) = 0.$$

Thus, by Theorem 6.2, (f_n) converges uniformly to the zero function. □

6.2 Series of Functions

Definition 6.5 By a **series of functions** on a interval J, we mean an expression of the form

$$\sum_{n=1}^{\infty} f_n \quad \text{or} \quad \sum_{n=1}^{\infty} f_n(x),$$

where (f_n) is a sequence of functions defined on J. □

Definition 6.6 Given a series $\sum_{n=1}^{\infty} f_n(x)$ of functions on an interval J, let

$$s_n(x) := \sum_{i=1}^{n} f_i(x), \quad x \in J.$$

Then s_n is called the **n-th partial sum** of the series $\sum_{n=1}^{\infty} f_n$. □

Definition 6.7 Consider a series $\sum_{n=1}^{\infty} f_n(x)$ of functions on an interval J, and let $s_n(x)$ be its n-th partial sum. Then we say that the series $\sum_{n=1}^{\infty} f_n(x)$

(a) **converges at a point** $x_0 \in J$ if (s_n) converges at x_0,

(b) **converges pointwise on** J if (s_n) converges pointwise on J, and

(c) **converges uniformly on** J if (s_n) converges uniformly on J. □

The proof of the following two theorems are obvious from the statements of Theorems 6.4 and 6.5 respectively.

Theorem 6.6 *Suppose (f_n) is a sequence of continuous functions on J. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on J, say to $f(x)$, then f is continuous on J, and for $[a,b] \subseteq J$,*

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

Theorem 6.7 *Suppose (f_n) is a sequence of continuously differentiable functions on J. If $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on J, and if $\sum_{n=1}^{\infty} f_n(x)$ converges at some point $x_0 \in J$, then $\sum_{n=1}^{\infty} f_n(x)$ converges to a differentiable function on J, and*

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} f_n'(x).$$

Next we consider a useful sufficient condition to check uniform convergence. First a definition.

Definition 6.8 We say that $\sum_{n=1}^{\infty} f_n$ is a **dominated series** if there exists a sequence (α_n) of positive real numbers such that $|f_n(x)| \leq \alpha_n$ for all $x \in J$ and for all $n \in \mathbb{N}$, and the series $\sum_{n=1}^{\infty} \alpha_n$ converges. □

Theorem 6.8 *A dominated series converges uniformly.*

Proof. Let $\sum_{n=1}^{\infty} f_n$ be a dominated series defined on an interval J, and let (α_n) be a sequence of positive reals such that

(i) $|f_n(x)| \le \alpha_n$ for all $n \in \mathbb{N}$ and for all $x \in J$, and

(ii) $\sum_{n=1}^{\infty} \alpha_n$ converges.

Let $s_n(x) = \sum_{i=1}^{n} f_i(x)$, $n \in \mathbb{N}$. Then for $n > m$,

$$|s_n(x) - s_m(x)| = \left|\sum_{i=m+1}^{n} f_i(x)\right| \le \sum_{i=m+1}^{n} |f_i(x)| \le \sum_{i=m+1}^{n} \alpha_i = \sigma_n - \sigma_m,$$

where $\sigma_n = \sum_{k=1}^{n} \alpha_k$. Since $\sum_{n=1}^{\infty} \alpha_n$ converges, the sequence (σ_n) is a Cauchy sequence. Now, let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ be such that

$$|\sigma_n - \sigma_m| < \varepsilon \quad \forall n, m \ge N.$$

Hence, from the relation: $|s_n(x) - s_m(x)| \le \sigma_n - \sigma_m$, we have

$$|s_n(x) - s_m(x)| < \varepsilon \quad \forall n, m \ge N, \forall x \in J.$$

This, in particular implies that $\{s_n(x)\}$ is also a Cauchy sequence at each $x \in J$. Hence, $\{s_n(x)\}$ converges for each $x \in J$. Let $f(x) = \lim_{n\to\infty} s_n(x)$, $x \in J$. Then, we have

$$|f(x) - s_m(x)| = \lim_{n\to\infty} |s_n(x) - s_m(x)| < \varepsilon \quad \forall m \ge N, \forall x \in J.$$

Thus, the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on J. ∎

Example 6.9 The series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ and $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ are dominated series, since

$$\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}, \qquad \left|\frac{\sin nx}{n^2}\right| \le \frac{1}{n^2} \quad \forall n \in \mathbb{N}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. □

Example 6.10 The series $\sum_{n=0}^{\infty} x^n$ is a dominated series on $[-\rho, \rho]$ for $0 < \rho < 1$, since $|x^n| \le \rho^n$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \rho^n$ is convergent. Thus, the given series is a dominated series, and hence, it is uniformly convergent. □

Example 6.11 Consider the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ on $\mathbb{R}$. Note that

$$\frac{x}{n(1+nx^2)} \le \frac{1}{n}\left(\frac{1}{2\sqrt{n}}\right),$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. Thus, the given series is dominated series, and hence it converges uniformly on $\mathbb{R}$. □

Example 6.12 Consider the series $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ for $x \in [c, \infty)$, $c > 0$. Note that

$$\frac{x}{1+n^2x^2} \leq \frac{x}{n^2x^2} =\leq \frac{1}{n^2x} \leq \frac{1}{n^2c}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus, the given series is dominated series, and hence it converges uniformly on $[c, \infty)$. □

Example 6.13 The series $\sum_{n=1}^{\infty} \left(xe^{-x}\right)^n$ is dominated on $[0, \infty)$: To see this, note that

$$\left(xe^{-x}\right)^n = \frac{x^n}{e^{nx}} \leq \frac{x^n}{(nx)^n/n!} = \frac{n!}{n^n}$$

and the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

It can also be seen that $|xe^{-x}| \leq 1/2$ for all $x \in [0, \infty)$. □

Example 6.14 The series $\sum_{n=1}^{\infty} x^{n-1}$ is not uniformly convergent on $(0, 1)$; in particular, not dominated on $(0, 1)$. This is seen as follows: Note that

$$s_n(x) := \sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x} \to f(x) := \frac{1}{1-x} \quad \text{as} \quad n \to \infty.$$

Hence, for $\varepsilon > 0$,

$$|f(x) - s_n(x)| < \varepsilon \quad \Longleftrightarrow \quad \left|\frac{x^n}{1-x}\right| < \varepsilon.$$

Hence, if there exists $N \in \mathbb{N}$ such that $|f(x) - s_n(x)| < \varepsilon$ for all $n \geq N$ for all $x \in (0, 1)$, then we would get

$$\frac{|x|^N}{|1-x|} < \varepsilon \quad \forall x \in (0, 1).$$

This is not possible, as $|x|^N/|1-x| \to \infty$ as $x \to 1$.

However, we have seen that the above series is dominated on $[-a, a]$ for $0 < a < 1$.

□

Example 6.15 The series $\sum_{n=1}^{\infty}(1-x)x^{n-1}$ is not uniformly convergent on $[0, 1]$; in particular, not dominated on $[0, 1]$. This is seen as follows: Note that

$$s_n(x) := \sum_{k=1}^{n}(1-x)x^{k-1} = \begin{cases} 1-x^n & \text{if} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1. \end{cases}$$

In particular, $s_n(x) = 1 - x^n$ for all $x \in [0, 1)$ and $n \in \mathbb{N}$. By Example 6.2, we know that $(s_n(x))$ converges to $f(x) \equiv 1$ pointwise, but not uniformly. □