Chapter 6

Sequences and Series of Real Numbers

We often use sequences and series of numbers without thinking about it. A decimal representation of a number is an example of a series, the bracketing of a real number by closer and closer rational numbers gives us an example of a sequence. We want to study these objects more closely because this conceptual framework will be used later when we look at functions and sequences and series of functions. First, we will take on numbers.

Sequences have an ancient history dating back at least as far as Archimedes who used sequences and series in his "Method of Exhaustion" to compute better values of π and areas of geometric figures.

6.1 The Symbols $+\infty$ and $-\infty$

We often use the symbols $+\infty$ and $-\infty$ in mathematics, including courses in high school. We need to come to some agreement about these symbols. We will often write ∞ for $+\infty$ when it should not be confusing.

First of all, they are ***not*** real numbers and do ***not*** necessarily adhere to the rules of arithmetic for real numbers. There are times that we "act" as if they do, so we need to be careful.

We adjoin $+\infty$ and $-\infty$ to $\mathbb{R}$ and extend the usual ordering to the set $\mathbb{R} \cup \{+\infty, -\infty\}$. Explicitly, we will agree that $-\infty < a < +\infty$ for every real number $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. This gives the extended set with an ordering that satisfies our usual properties:

1) If $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$, then $a \leq b$ or $b \leq a$.
2) If $a \leq b$ and $b \leq a$, then $a = b$.
3) If $a \leq b$ and $b \leq c$, then $a \leq c$.

We will not extend the usual algebraic structure of the reals to $\mathbb{R} \cup \{+\infty, -\infty\}$.

Instead, when we have to we will discuss the algebra that might involve $+\infty$ and/or $-\infty$. Do not apply any theorem that is stated for the real numbers to the symbols $+\infty$ or $-\infty$.

The symbols make it convenient to extend our notation about intervals to the real line.

$$\begin{aligned} [a,\infty) &= \{x \in \mathbb{R} \mid a \le x\} & \qquad (a,\infty) &= \{x \in \mathbb{R} \mid a < x\} \\ (-\infty,b] &= \{x \in \mathbb{R} \mid x \le b\} & \qquad (-\infty,b) &= \{x \in \mathbb{R} \mid x < b\}. \end{aligned}$$

Occasionally, you will see $\mathbb{R} = (-\infty,\infty)$.

6.2 Sequences

Sequences are, basically, countably many numbers arranged in an ordered set that may or may not exhibit certain patterns.

Definition 6.1 *A sequence of real numbers is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \ge m\}$ where m is usually 0 or 1. Thus, a sequence is a function $f\colon \mathbb{N} \to \mathbb{R}$. Thus a sequence can be denoted by $f(m)$, $f(m+1)$, $f(m+2)$, $\ldots$. Usually, we will denote such a sequence by $\{a_i\}_{i=m}^{\infty}$ or $\{a_m, a_{m+1}, a_{m+2}, \ldots\}$, where $a_i = f(i)$. If $m = 1$, we may use the notation $\{a_n\}_{n\in\mathbb{N}}$.*

Example 6.1 The sequence $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$ is written as $\{1/i\}_{i=1}^{\infty}$. Keep in mind that this sequence can be thought of as an ordinary function. In this case $f(n) = 1/n$.

Example 6.2 Consider the sequence given by $a_n = (-1)^n$ for $n \ge 0$. This time we have started the sequence with 1 and the terms look like, $\{1, -1, 1, -1, 1, -1, \ldots\}$. Note that this time the function has domain $\mathbb{N}$ but the range is $\{-1, 1\}$.

Example 6.3 Consider the sequence $a_n = \cos\left(\frac{\pi n}{3}\right)$, $n \in \mathbb{N}$. The first term in the sequence is $\cos\frac{\pi}{3} = \cos 60^\circ = \frac{1}{2}$ and the sequence looks like

$$\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, \ldots\}.$$

Note that likes its predecessor, the function takes on only a finite number of values, but the sequence has an infinite number of elements.

Example 6.4 If $a_n = n^{1/n}$, $n \in \mathbb{N}$, the sequence is

$$1, \sqrt{2}, 3^{1/3}, 4^{1/4}, \ldots$$

We might use an approximation for each of these and, arbitrarily choosing 5 decimal places, the sequence would look like

$$1, 1.41421, 1.44225, 1.41421, 1.37973, 1.34801, 1.32047, 1.29684, 1.27652, 1.25893, \ldots$$

We would find that $a_{100} = 1.04713$ and $a_{10000} = 1.00092$.

Example 6.5 Consider the sequence $b_n = (1 + \frac{1}{n})^n$, $n \in \mathbb{N}$. This is the sequence

$$2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \ldots$$

or by approximation

$$2, 2.25, 2.37037, 2.44141, 2.48832, 2.52163, 2.54650, 2.56578, 2.58117, 2.59374 \ldots.$$

Again $a_{100} = 2.74081$ and $a_{10000} = 2.71815$.

In looking at these examples we might think that some of them are giving us a pattern of numbers that are "getting close" to some other real number. Others may not give us that indication. We are interested in what the long-term behavior of the sequence is. What happens for larger and larger values of n? Does the sequence approach a real number? Could it approach more than one real number?

Definition 6.2 *A sequence of real numbers is said to* ***converge*** *to a real number L if for every $\epsilon > 0$ there is an integer $N > 0$ such that if $k > N$ then $|a_k - L| < \epsilon$. The number L is called the* ***limit*** *of the sequence.*

If $\{a_k\}$ converges to L we will write $\lim\limits_{k\to\infty} a_k = L$ or simply $a_k \to L$. If a sequence does not converge, then we say that it **diverges**.

Note that the N in the definition depends on the ϵ that we were given. If you change the value of ϵ then you may have to "recalculate" N.

Consider the sequence $a_n = \dfrac{n}{2^n}$, $n \in \mathbb{N}$. Now, if we look at the values that the sequence takes

$$\frac{1}{2}, \frac{2}{2^2}, \frac{3}{2^3}, \frac{4}{2^4}, \ldots$$

n	$\approx n/2^n$
1	0.5
2	0.5
3	0.375
4	0.25
5	0.15625
6	0.09375
7	0.0546875
8	0.03125
9	0.0175781
10	0.00976562

we might think that the terms are getting smaller and smaller so maybe the limit of this sequence would be 0. Let's take a look and compare how N would vary as ϵ varies. Let's start with some simple small numbers and let ϵ be 0.1, 0.01, 0.001, and 0.0001, and 0.00001.

For $\epsilon = 0.1$, we need to find an integer N so that

$$\left|\frac{N}{2^N} - 0\right| < 0.1$$

Look in the table of values here and we see that for $N = 6$ we have satisfied the above condition. Following this we get the following by using a calculator or a computer algebra system:

$$\begin{aligned}
N > 0 &\quad \text{implies} \quad \left|\frac{N}{2^N} - 0\right| < 1 \\
N > 5 &\quad \text{implies} \quad \left|\frac{N}{2^N} - 0\right| < 0.1 \\
N > 9 &\quad \text{implies} \quad \left|\frac{N}{2^N} - 0\right| < 0.01 \\
N > 14 &\quad \text{implies} \quad \left|\frac{N}{2^N} - 0\right| < 0.001 \\
N > 18 &\quad \text{implies} \quad \left|\frac{N}{2^N} - 0\right| < 0.0001 \\
N > 22 &\quad \text{implies} \quad \left|\frac{N}{2^N} - 0\right| < 0.00001
\end{aligned}$$

We are going to establish several properties of convergent sequences. Many proofs will use a proof much like this next result. While this type of argument may not easy to get used to, it will appear again and again, so you should try to get as familiar with it as you can.

Theorem 6.1 (Convergent sequences are bounded) *Let $\{a_n\}$, $n \in \mathbb{N}$ be a convergent sequence. Then the sequence is bounded, and the limit is unique.*

PROOF: The easier property to show is that the limit is unique, so let's do that first. Suppose the sequence has two limits, L and K. Take any $\epsilon > 0$. Then there is an integer N such that

$$|a_k - L| < \frac{\epsilon}{2} \text{ if } k > N.$$

Also, there is another integer N' such that

$$|a_k - K| < \frac{\epsilon}{2} \text{ if } k > N'.$$

Then, by the triangle inequality:

$$|L - K| < |a_k - L| + |a_k - K| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ if } k > \max\{N, N'\}.$$

Therefore $|L - K| < \epsilon$ for any $\epsilon > 0$. But the only way that that can happen is for $L = K$, so that the limit is indeed unique.

Next, we need to prove boundedness. Since the sequence converges, we can take any ϵ we wish, and tradition shows us to take $\epsilon = 1$. Then there is an integer N so that

$$|a_k - L| < 1 \text{ if } k > N.$$

Fix that integer N. Then we have that

$$|a_n| \leq |a_n - L| + |L| < 1 + |L| = P \text{ for all } n > N.$$

Now, define $M = \max\{\{|a_k|, k = 1, \ldots, N\}, P\}$. Then $|a_n| < M$ for all n, which makes the sequence bounded. ∎

6.3 The Algebra of Convergent Sequences

This section proves some basic results that do not come as a surprise to the student.

Theorem 6.2 *If the sequence $\{a_n\}$ converges to L and $c \in \mathbb{R}$, then the sequence $\{ca_n\}$ converges to cL; i.e., $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$.*

Proof: Let's assume that $c \neq 0$, since the result is trivial if $c = 0$. Let $\epsilon > 0$. Since $\{a_n\}$ converges to L, we know that there is an $N \in \mathbb{N}$ so that if $n > N$

$$|a_n - L| < \frac{\epsilon}{|c|}.$$

Thus, for $n > N$ we then have that

$$|ca_n - cL| = |c||a_n - L| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

which is what we needed to prove. ∎

Theorem 6.3 *If the sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M, then the sequence $\{a_n + b_n\}$ converges to $L + M$; i.e., $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.*

Proof: Let $\epsilon > 0$. We need to find an $N \in \mathbb{N}$ so that if $n > N$

$$|(a_n + b_n) - (L + M)| < \epsilon.$$

Since $\{a_n\}$ and $\{b_n\}$ are convergent, for the given ϵ there are integers $N_1, N_2 \in \mathbb{N}$ so that

$$\text{If } n > N_1 \text{ then } |a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad \text{if } n > N_2 \text{ then } |b_n - M| < \frac{\epsilon}{2}$$

Thus, if $n > \max\{N_1, N_2\}$ then

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

∎

Theorem 6.4 *If the sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M, then the sequence $\{a_n \cdot b_n\}$ converges to $L \cdot M$; i.e., $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.*

The trick with the inequalities here is to look at the inequality

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - a_n M + a_n M - LM| \\ &\leq |a_n b_n - a_n M| + |a_n M - LM| \\ &= |a_n||b_n - M| + |M||a_n - L|. \end{aligned}$$

So for large values of n, $|b_n - M|$ and $|a_n - L|$ are small and $|M|$ is constant. Now, by Theorem 6.1 shows that $|a_n|$ is bounded, so that we will be able to show that $|a_n b_n - LM|$ is small.

Proof: Let $\epsilon > 0$. By Theorem 6.1 there is a constant $K > 0$ such that $|a_n| \leq K$ for all n. Since $\{b_n\}$ is convergent, for the given ϵ there is an integer $N_1 \in \mathbb{N}$ so that

$$\text{If } n > N_1 \text{ then } |b_n - M| < \frac{\epsilon}{2K}.$$

Also, since $\{a_n\}$ is convergent there is an integer $N_2 \in \mathbb{N}$ so that

$$\text{If } n > N_2 \text{ then } |a_n - L| < \frac{\epsilon}{2(|M| + 1)}.$$

Thus if $N = \max\{N_1, N_2\}$ then if $n > N$

$$\begin{aligned} |a_n b_n - LM| &\leq |a_n||b_n - M| + |M||a_n - L| \\ &\leq K \cdot \frac{\epsilon}{2K} + |M|\frac{\epsilon}{2(|M|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

∎

Lemma 6.1 *If the sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M and if $a_n \leq b_n$ for all $n \geq m$, then $L \leq M$.*

Lemma 6.2 *If the sequence $\{a_n\}$ converges to L, if $a_n \neq 0$ for all $n \in \mathbb{N}$, and if $L \neq 0$, then* $\text{glb}\{|a_n| \mid n \in \mathbb{N}\} > 0$.

Proof: Let $\epsilon = \frac{1}{2}|L| > 0$. Since $\{a_n\}$ converges to L, there is an $N \in \mathbb{N}$ so that if $n > N$ then $|a_n - L| < |L|/2$.

Now if $n > N$ we must have that $|a_n| \geq |L|/2$. If not then the triangle inequality would imply

$$|L| = |L - a_n + a_n| \leq |L - a_n| + |a_n| < \frac{|M|}{2} + \frac{|M|}{2} = |M|.$$

Now we set

$$m = \min\left\{\frac{|L|}{2}, |a_1|, |a_2|, \ldots, |a_N|\right\}.$$

Then clearly $m > 0$ and $|a_n| \geq m$ for all $n \in \mathbb{N}$. ∎

Theorem 6.5 *If the sequence $\{a_n\}$ converges to L, if $a_n \neq 0$ for all $n \in \mathbb{N}$, and if $L \neq 0$, then the sequence $\{1/a_n\}$ converges to $1/L$; i.e., $\lim_{n\to\infty} 1/a_n = 1/\lim_{n\to\infty} a_n$.*

PROOF: Let $\epsilon > 0$. By Lemma 6.2 there is an $m > 0$ such that $|a_n| \geq m$ for all n. Since $\{a_n\}$ is convergent there is an integer $N \in \mathbb{N}$ so that if $n > N$ $|L - a_n| < \epsilon \cdot m|L|$. Then for $n > N$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n L|} \leq \frac{|a_n - L|}{m|L|} < \epsilon.$$

∎

Theorem 6.6 *Suppose that the sequence $\{b_n\}$ converges to M and if $\{a_n\}$ converges to L. If $b_n \neq 0$ for all $n \in \mathbb{N}$, and if $M \neq 0$, then the sequence $\{a_n/b_n\}$ converges to L/M; i.e., $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.*

PROOF: We use two of the previous theorems to prove this. By Theorem 6.5 $\{1/b_n\}$ converges to $1/M$, so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = L \cdot \frac{1}{M} = \frac{L}{M}.$$

∎

Example 6.6 Let $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Let $\epsilon > 0$ and let $N = (\frac{1}{\epsilon})^{1/p}$. Then $n > N$ implies that $n^p > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^p}$. Since $\frac{1}{n^p} > 0$, this shows that $n > N$ implies $|\frac{1}{n^p} - 0| < \epsilon$.

Example 6.7 Let $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$.

Suppose $a \neq 0$, because $\lim a^n = 0$ is clear for $a = 0$. Since $|a| < 1$, we can write $|a| = \dfrac{1}{1+|b|}$ where $b > 0$. By the binomial theorem

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2}b^2 + \cdots + b^n \geq 1 + nb > nb,$$

so

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Now consider $\epsilon > 0$ and let $N = \frac{1}{\epsilon b}$. Then if $n > N$, we have $n > \frac{1}{\epsilon b}$ and hence $|a^n - 0| < \frac{1}{nb} < \epsilon$.

Example 6.8 $\lim_{n\to\infty}(n^{1/n}) = 1$.

Let $a_n = (n^{1/n}) - 1$ and note that $a_n \geq 0$ for all n. By Theorem 6.3 it is sufficient for us to show that $\lim_{n\to\infty} a_n = 0$.

Since $1 + a_n = n^{1/n}$, we have $n = (1+a_n)^n$. For $n \geq 2$ the binomial theorem gives us

$$n = (1+a_n)^n \geq 1 + na_n + \frac{1}{2}n(n-1)a_n^2 > \frac{1}{2}n(n-1)a_n^2.$$

Thus, $n > \frac{1}{2}n(n-1)a_n^2$, so $a_n^2 < \frac{2}{n-1}$. Thus, we have shown that $a_n < \sqrt{\frac{2}{n+1}}$ for $n \geq 2$. Thus, $\lim a_n = 0$.

Example 6.9 $\lim_{n\to\infty}(a^{1/n}) = 1$ for $a > 0$.

Suppose $a \geq 1$. Then for $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$, it easily follows that $\lim a^{1/n} = 1$. Suppose that $0 < a < 1$. Then $\frac{1}{a} > 1$, so that $\lim(\frac{1}{a})^{1/n} = 1$. Thus,

$$\begin{aligned}
\lim\left(\frac{1}{a}\right)^{1/n} &= 1\\
\lim\frac{1^{1/n}}{a^{1/n}} &= 1\\
\frac{1}{\lim a^{1/n}} &= 1\\
\lim a^{1/n} &= 1
\end{aligned}$$

Definition 6.3 *For a sequence $\{a_n\}$ we write $\lim a_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies that $a_n > M$.*

In this case we will say that $\{a_n\}$ **diverges** to $+\infty$.

We can make a similar definition for $\lim a_n = -\infty$.

Of course, we cannot use the previous theorems when dealing with infinite limits.

Theorem 6.7 *Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim a_n = +\infty$ and $\lim b_n > 0$. Then $\lim a_n b_n = +\infty$.*

PROOF: Let $M > 0$. Choose a real number m so that $0 < m < \lim b_n$. Whether $\lim b_n = +\infty$ or not, there exists N_1 so that if $n > N_1$ then $b_n > m$. Since $\lim a_n = +\infty$ there is an N_2 so that if $n > N_2$ then

$$a_n > \frac{M}{m}.$$

Setting $N = \max\{N_1, N_2\}$ means that for $n > N$ $a_n b_n > \frac{M}{m} \cdot m = M$. ∎

Theorem 6.8 *For a sequence $\{a_n\}$ of positive real numbers $\lim a_n = +\infty$ if and only if $\lim \frac{1}{a_n} = 0$.*

Proof: Let $\{a_n\}$ be a sequence of positive numbers. We need to show

$$\text{If } \lim a_n = +\infty \text{ then } \lim \frac{1}{a_n} = 0 \tag{6.1}$$

and

$$\text{If } \lim \frac{1}{a_n} = 0 \text{ then } \lim a_n = +\infty. \tag{6.2}$$

To prove 6.1 we will suppose that $\lim a_n = +\infty$. Let $\epsilon > 0$ and let $M = 1/\epsilon$. Since $\{a_n\}$ diverges to $+\infty$, there is an N so that if $n > N$ then $a_n > M = 1/\epsilon$. Therefore, if $N > n$ then $\epsilon > 1/a_n > 0$, so

$$\text{if } n > N \text{ then } \left|\frac{1}{a_n} - 0\right| < \epsilon.$$

Thus, $\lim 1/a_n = 0$. This proves 6.1.

To prove 6.2 suppose that $\lim 1/a_n = 0$ and let $M > 0$. Let $\epsilon = 1/M$. Then since $\epsilon > 0$ there is an N so that if $n > N$ then $\left|\frac{1}{a_n} - 0\right| < \epsilon = \frac{1}{M}$. Since $a_n > 0$ we then know that

$$\text{if } n > N \text{ then } 0 < \frac{1}{a_n} < \frac{1}{M}$$

and hence

$$\text{if } n > N \text{ then } a_n > M.$$

This means that $\lim a_n = +\infty$ and 6.2 holds. ∎

6.4 Monotonicity and Cauchy Sequences

The previous section showed us how to work with convergent sequences, but does not tell us how to determine (quickly) if a sequence does converge. We need such a tool.

Definition 6.4 *A sequence $\{a_n\}$ of real numbers is called* ***monotonically increasing***[1] *if $a_n \leq a_{n+1}$ for all n, and $\{a_n\}$ is called a* ***monotonically decreasing***[2] *if $a_n \geq a_{n+1}$ for all n.*

If a sequence is monotonically increasing or monotonically decreasing, we will call it a *monotonic sequence* or a *monotone sequence.*

Theorem 6.9 *All bounded monotone sequences converge.*

[1]Sometimes called **nondecreasing**
[2]Sometimes called **nonincreasing**.

PROOF: Let $\{a_n\}$ be a bounded monotonically increasing sequence and let $S = \{a_n \bmod n \in \mathbb{N}\}$. Since the sequence is bounded, $a_n < M$ for some real number M and for all $n \in \mathbb{N}$. This means that the set S is bounded, and thus it has a least upper bound. Let $u = \operatorname{lub} S$. Let $\epsilon > 0$. Since $u = \operatorname{lub} S$ and $\epsilon > 0$, $u - \epsilon$ is not an upper bound for S. This means that there must be some N so that $a_N > u - \epsilon$. Since $\{a_n\}$ is monotonically increasing we have that for all $n > N$ $a_n \geq a_N$ and hence for all $n > N$ it follows that $u - \epsilon < a_n \leq u$. Thus, $|a_n - u| < \epsilon$ for all $n > N$. Thus, $\lim a_n = u = \operatorname{lub} S$.

The proof for bounded monotonically decreasing sequences is the same with the greatest lower bound playing the role of the least upper bound. ∎

Now, we can also handle unbounded monotone sequences.

Theorem 6.10 *Let $\{a_n\}$ be a sequence of real numbers.*

(i) *If $\{a_n\}$ is an unbounded monotonically increasing sequence, then $\lim a_n = +\infty$.*

(ii) *If $\{a_n\}$ is an unbounded monotonically decreasing sequence, then $\lim a_n = -\infty$.*

Let $\{a_n\}$ be a bounded sequence of real numbers. While it may converge or may not converge, the limiting behavior of $\{a_n\}$ depends only on the "tails" of the sequence, or sets of the form $\{a_n \mid n > N\}$. This leads us to a concept that we can discuss without knowing *a priori* if a given sequence converges or diverges.

Let $u_N = \operatorname{glb}\{a_n \mid n > N\} = \inf\{a_n \mid n > N\}$ and let $v_n = \operatorname{lub}\{a_n \mid n > N\} = \sup\{a_n \mid n > N\}$. We have seen that if $\lim a_n$ exists, then it must lie in the interval $[u_N, v_N]$. As N increases, the sets $\{a_n \mid n > N\}$ get smaller, so we have

$$u_1 \leq u_2 \leq u_3 \leq \ldots \text{ and } v_1 \geq v_2 \geq v_3 \geq \ldots.$$

By the above theorem the limits $u = \lim_{N\to\infty} u_N$ and $v = \lim_{N\to\infty} v_N$ both exist and $u \leq v$ since $u_N \leq v_N$ for all N. If the limit exists, then $u_N \leq \lim a_n \leq v_N$ so $u \leq \lim a_n \leq v$. These numbers u and v turn out to be useful whether $\lim a_n$ exists or not.

Definition 6.5 *Let $\{a_n\}$ be a sequence of real numbers. We define*

$$\limsup a_n = \lim_{N\to\infty} \operatorname{lub}\{s_n \mid n > N\}$$

and

$$\liminf a_n = \lim_{N\to\infty} \operatorname{glb}\{s_n \mid n > N\}.$$

Note that we do not require that $\{a_n\}$ be bounded. We will take some precautions and adopt the following conventions. If $\{a_n\}$ is not bounded above, $\operatorname{lub}\{s_n \mid n > N\} = +\infty$ for all N and we define $\limsup a_n = +\infty$. Likewise, if $\{a_n\}$ is not bounded below, $\operatorname{glb}\{s_n \mid n > N\} = -\infty$ and we define $\liminf a_n = -\infty$ in this case.

Now, is it true that $\limsup a_n = \text{lub}\{a_n \mid n > N\}$? Not necessarily, because while it is true that $\limsup a_n \leq \text{lub}\{a_n \mid n > N\}$, some of the values a_n may be much larger than $\limsup a_n$. Note that $\limsup a_n$ is the largest value that *infinitely many* a_n's can get close to.

Theorem 6.11 *Let $\{a_n\}$ be a sequence of real numbers.*

(i) *If $\lim a_n$ is defined [as a real number, $+\infty$ or $-\infty$], then $\liminf a_n = \lim a_n = \limsup a_n$.*

(ii) *If $\liminf a_n = \limsup a_n$, then $\lim a_n$ is defined and $\lim a_n = \liminf a_n = \limsup a_n$.*

PROOF: Let us use the notation from above; *i.e.*,Let $u_N = \text{glb}\{a_n \mid n > N\}$, $v_n = \text{lub}\{a_n \mid n > N\}$, $u = \lim u_N = \liminf a_n$ and $v = \lim v_N = \limsup a_n$.

(i) Suppose $\lim a_n = +\infty$. Let $M > 0$ be a positive number. Then there is $N \in \mathbb{N}$ so that if $n > N$ then $a_n > M$. Then $u_N = \text{glb}\{a_n \mid n > N\} \geq M$. It follows that $m > N$ implies that $u_m \geq M$. Thus, the sequence $\{u_N\}$ satisfies the condition that $\lim u_N = +\infty$ or $\liminf a_n = +\infty$. Likewise, we can show that $\limsup a_n = +\infty$. We do the case that $\lim a_n = -\infty$ similarly.

Suppose that $\lim a_n = L \in \mathbb{R}$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ so that $|a_n - L| < \epsilon$ for $n > N$. Thus $a_n < L + \epsilon$ for $n > N$. This means that

$$v_N = \text{lub}\{a_n \mid n > N\} \leq L + \epsilon.$$

Also, if $m > N$ then $v_m \leq L + \epsilon$ for all $\epsilon > 0$, no matter how small. This means that $\limsup a_n \leq L = \lim a_n$. Similarly, we can show that $\lim a_n \leq \liminf a_n$. Since $\liminf a_n \leq \limsup a_n$, these inequalities give us that

$$\liminf a_n = \lim a_n = \limsup a_n.$$

(ii) If $\liminf a_n = \limsup a_n = \pm\infty$ it is easy to show that $\lim a_n = \pm\infty$. Suppose that $\liminf a_n = \limsup a_n = L \in \mathbb{R}$. We need to show that $\lim a_n = L$. Let $\epsilon > 0$. Since $L = \lim v_N$ there is an $N_0 \in \mathbb{N}$ so that

$$|L - \text{lub}\{a_n \mid n > N_0\}| < \epsilon.$$

Thus, $\text{lub}\{a_n \mid n > N_0\} < L + \epsilon$, so

$$a_n < L + \epsilon \text{ for all } n > N_0.$$

Similarly, since $L = \lim u_N$ there is an $N_1 \in \mathbb{N}$ so that

$$|L - \text{glb}\{a_n \mid n > N_1\}| < \epsilon.$$

Thus, $\text{glb}\{a_n \mid n > N_1\} > L - \epsilon$, so

$$a_n > L - \epsilon \text{ for all } n > N_1.$$

These two conditions tell us that

$$L - \epsilon < a_n < L + \epsilon \text{ for } n > \max\{N_0, N_1\},$$

or, equivalently,

$$|a_n - L| < \epsilon \text{ for } n > \max\{N_0, N_1\}.$$

This proves that $\lim a_n = L$ as needed.

∎

This tells us that if $\{a_n\}$ converges, then $\liminf a_n = \limsup a_n$, so for large N the numbers $\text{lub}\{a_n \mid n > N\}$ and $\text{glb}\{a_n \mid n > N\}$ must be close together. This means that all of the numbers in the set $\{a_n \mid n > N\}$ must be close together. This leads to the following definition.

Definition 6.6 *A sequence $\{a_n\}$ of real numbers is called a* ***Cauchy sequence*** *if for each $\epsilon > 0$ there is a number $N \in \mathbb{N}$ so that if $m, n > N$ then $|a_n - a_m| < \epsilon$.*

Lemma 6.3 *Convergent sequences are Cauchy sequences.*

PROOF: Suppose that $\lim a_n = L$. Note that

$$|a_n - a_m| = |a_n - L + L - a_M| \leq |a_n - L| + |a_m - L|.$$

Thus, given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that if $k > N$ then $|a_k - L| < \frac{\epsilon}{2}$. Thus, if $m, n > N$ we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\{a_n\}$ is a Cauchy sequence. ∎

Lemma 6.4 *Cauchy sequences are bounded.*

This leads us to the following theorem.

Theorem 6.12 *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Proof: We have just proven half of this above. That means that we are left to show that any Cauchy sequence must converge. To see this let $\{a_n\}$ be a Cauchy sequence. From the above lemma we know that it is bounded. That means then that we only need to show that $\liminf a_n = \limsup a_n$.

Let $\epsilon > 0$. Since $\{a_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ so that if $m, n > N$ then $|a_n - a_m| < \epsilon$. In particular, $a_n < a_m + \epsilon$ for all $m, n > N$. This shows that $a_m + \epsilon$ is an upper bound for $\{a_n \mid n > N\}$. Thus $v_N = \text{lub}\{a_n \mid n > N\} \leq a_m + \epsilon$ for $m > N$. Now, this shows that $v_N - \epsilon$ is a lower bound for $\{a_m \mid m > N\}$, so that $v_N - \epsilon \leq \text{glb}\{a_m \mid m > N\} = u_N$. Therefore

$$\limsup a_n \leq v_N \leq u_N + \epsilon \leq \liminf a_n + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have that $\limsup a_n \leq \liminf a_n$ and this is enough to give us that the two quantities are equal. ∎

6.5 Subsequences

So far we have learned the basic definitions of a sequence (a function from the natural numbers to the reals), the concept of convergence, and we have extended that concept to one which does not pre-suppose the unknown limit of a sequence (Cauchy sequence). Unfortunately, however, not all sequences converge. We will now introduce some techniques for dealing with those sequences. The first is to change the sequence into a convergent one (extract subsequences) and the second is to modify our concept of limit as we did with lim sup and lim inf.

Definition 6.7 *Let $\{a_n\}$ be a sequence. When we extract from this sequence only certain elements and drop the remaining ones we obtain a new sequences consisting of an infinite subset of the original sequence. That sequence is called a* ***subsequence*** *and denoted by $\{a_{n_k}\}$.*

One can extract infinitely many subsequences from any given sequence.

Example 6.10 Take the sequence $\{(-1)^n\}$, which we know does not converge. Extract every other member, starting with the first. Does this sequence converge? What if we extract every other member, starting with the second. What do you get in this case?

Example 6.11 Take the sequence $\{1/n\}$. Extract three different subsequences of your choice. Do these subsequences converge? Is so, to what limit?

The last example is an indication of a general result.

Theorem 6.13 **(i)** *If $\{a_n\}$ is a convergent sequence, then every subsequence of that sequence converges to the same limit.*

(ii) *If is a sequence such that every possible subsequence extracted from that sequences converges to the same limit, then the original sequence also converges to that limit.*

The next statement is probably one on the most fundamental results of basic real analysis, and generalizes the above proposition. It also explains why subsequences can be useful, even if the original sequence does not converge.

Theorem 6.14 (Bolzano-Weierstrass) *Let $\{a_n\}$ be a sequence of real numbers that is bounded. Then there exists a subsequence $\{a_{n_k}\}$ that converges.*

PROOF: Since the sequence is bounded, there exists a number M such that $|a_n| < M$ for all n. Then either $[-M, 0]$ or $[0, M]$ contains infinitely many elements of the sequence. Say that $[0, M]$ does. Choose one of them, and call it a_{n_1}. Now, either $[0, M/2]$ or $[M/2, M]$ contains infinitely many elements of the (original) sequence. Say it is $[0, M/2]$. Choose one of those elements, and call it a_{n_2}. Again, either $[0, M/4]$ or $[M/4, M/2]$ contains infinitely many elements of the (original) sequence. This time, say it is $[M/4, M/2]$. Pick one of those elements and call it a_{n_3}.

Keep on going in this way, halving each interval from the previous step at the next step, and choosing one element from that new interval. Here is what we get:

- $|a_{n_1} - a_{n_2}| < M$, because both are in $[0, M]$
- $|a_{n_2} - a_{n_3}| < M/2$, because both are in $[0, M/2]$
- $|a_{n_3} - a_{n_4}| < M/4$, because both are in $[M/2, M/4]$

and in general, we see that

$$|a_{n_k} - a_{n_{k+1}}| < M/2^{k-1},$$

because both are in an interval of length $M/2^{k-1}$.

So, this proves that consecutive elements of this subsequence are close together. That is not enough, however, to say that the sequence is Cauchy, since for that not only consecutive elements must be close together, but all elements must get close to each other eventually.

So take any $\epsilon > 0$, and pick an integer N such that for any $k, m > N$ (with $m > k$)

we have:

$$\begin{aligned}
|a_{n_k} - a_{n_m}| &= |(a_{n_k} - a_{n_{k+1}}) + (a_{n_{k+1}} - a_{n_{k+2}}) + \cdots + (a_{n_{m-1}} - a_{n_m})| \\
&\leq |a_{n_k} - a_{n_{k+1}}| + |a_{n_{k+1}} - a_{n_{k+2}}| + \cdots + |a_{n_{m-1}} - a_{n_m}| \\
&= M\left(\frac{1}{2^{k-1}} + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{m-2}}\right) \\
&= \frac{M}{2^{k-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-k-1}}\right) \\
&\leq \frac{M}{2^{k-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right) \\
&= \frac{M}{2^{k-1}}\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j = \frac{2M}{2^{k-1}} = \frac{M}{2^{k-2}}
\end{aligned}$$

In order for this to be less than ϵ for $m, k > N$ we would need to take N so that

$$\begin{aligned}
\frac{M}{2^{N-2}} &< \epsilon \\
2^{N-2} &> \frac{M}{\epsilon} \\
(N-2)\ln 2 &> \ln M - \ln \epsilon \\
N &> 2 + \frac{\ln M - \ln \epsilon}{\ln 2}.
\end{aligned}$$

This proves what we wanted. ∎

6.6 Series

Now we will investigate what may happen when we add all terms of a sequence together to form what will be called an infinite series. The old Greeks already wondered about this, and actually did not have the tools to quite understand it. This is illustrated by the old tale of Achilles and the Tortoise.

Zeno's Paradox (Achilles and the Tortoise)
Achilles, a fast runner, was asked to race against a tortoise. Achilles can run 10 yards per second, the tortoise only 5 yards per second. The track is 100 yards long. Achilles, being a fair sportsman, gives the tortoise 10 yard advantage. Who will win?

Both start running, with the tortoise being 10 yards ahead. After one second, Achilles has reached the spot where the tortoise started. The tortoise, in turn, has run 5 yards. Achilles runs again and reaches the spot the tortoise has just been. The tortoise, in turn, has run 2.5 yards. Achilles runs again to the spot where the tortoise has just been. The tortoise, in turn, has run another 1.25 yards ahead.

This continuous for a while, but whenever Achilles manages to reach the spot where the tortoise has just been a second ago, the tortoise has again covered a little bit of distance, and is still ahead of Achilles. Hence, as hard as he tries, Achilles only manages to cut the remaining distance in half each time, implying, of course, that Achilles can actually never reach the tortoise. So, the tortoise wins the race, which does not make Achilles very happy at all. What is wrong with this line of thinking?

Let us look at the difference between Achilles and the tortoise:

Time	Difference
$t = 0$	10 yards
$t = 1$	$5 = 10/2$ yards
$t = 1 + \frac{1}{2}$	$2.5 = 10/4$ yards
$t = 1 + \frac{1}{2} + \frac{1}{4}$	$1.25 = 10/8$ yards
$t = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$0.625 = 10/16$ yards

and so on. In general we have:

Time	Difference
$t = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$	$\frac{10}{2^n}$ yards

Now we want to take the limit as n goes to infinity to find out when the distance between Achilles and the tortoise is zero. But that involves adding infinitely many numbers in the above expression for the time, and we (the Greeks and Zeno) don't know how to do that. However, if we define

$$s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$$

then, dividing by 2 and subtracting the two expressions:

$$s_n - \frac{1}{2}s_n = 1 - \frac{1}{2^{n+1}}$$

or equivalently, solving for s_n:

$$s_n = 2\left(1 - \frac{1}{2^{n+1}}\right).$$

Now s_n is a simple sequence, for which we know how to take limits. In fact, from the last expression it is clear that $\lim s_n = 2$ as n approaches infinity.

Hence, we have - mathematically correctly - computed that Achilles reaches the tortoise after exactly 2 seconds, and then, of course passes it and wins the race. A much simpler calculation not involving infinitely many numbers gives the same result:

- Achilles runs 10 yards per second, so he covers 20 yards in 2 seconds.

- The tortoise runs 5 yards per second, and has an advantage of 10 yards. So, it also reaches the 20 yard mark after 2 seconds.

- Therefore, both are even after 2 seconds.

Of course, Achilles will finish the race after 10 seconds, while the tortoise needs 18 seconds to finish, and Achilles will clearly win.

The problem with Zeno's paradox is that Zeno was uncomfortable with adding infinitely many numbers. In fact, his basic argument was that if you add infinitely many numbers, then — no matter what those numbers are — you must get infinity. If that was true, it would take Achilles infinitely long to reach the tortoise, and he would loose the race. However, reducing the infinite addition to the limit of a sequence, we have seen that this argument is false.

One reason for looking so carefully at sequences is that it allows us to to quickly obtain the properties of infinite series.

We know (at least theoretically) how to deal with finite sums of real numbers.

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots a_n.$$

More interest in mathematics though tends to lie in the area of *infinite series*:

$$\sum_{k=m}^{\infty} a_k = a_m + a_{m+1} + a_{m+2} + \ldots.$$

What do we mean by this *infinite series*, $\sum_{k=m}^{\infty} a_k$? Define the n^{th} partial sum, S_n by

$$S_n = a_m + a_{m+1} + \ldots a_n = \sum_{k=m}^{n} a_k.$$

This now gives us a sequence, *the sequence of partial sums*, $\{S_k\}_{k=m}^{\infty}$. The infinite series $\sum_{k=m}^{\infty} a_k$ is said to converge provided the sequence of partial sums converges to a real number S. In this case we define $\sum_{n=m}^{\infty} a_n = S$. Thus

$$\sum_{n=m}^{\infty} a_n = S \text{ means } \lim_{n\to\infty} S_n = S \text{ or } \lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k\right)) = S.$$

If a series does not converge we say that it *diverges*. We can then say that a series *diverges to* $+\infty$ if $\lim s_n = +\infty$ or that it *diverges to* $-\infty$ if $\lim s_n = -\infty$. Some texts will indicate that the symbol $\sum_{n=m}^{\infty} a_n$ ***has no meaning*** unless the series converges or diverges to $+\infty$ or $-\infty$. Thus, $\sum_{n=0}^{\infty}(-1)^n$ will have no meaning.

Example 6.12 $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is an infinite series. The sequence of partial sums looks like:

$$S_0 = 1,\ S_1 = \frac{3}{2},\ S_2 = \frac{7}{4},\ S_3 = \frac{15}{8}, \ldots$$

We saw above that this sequence converges to 2, so

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \lim S_n = 2.$$

Example 6.13 The ***harmonic series*** is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

The first few terms in the sequence of partial sums are:

$$S_1 = 1,\ S_2 = \frac{3}{2},\ S_3 = \frac{11}{6},\ S_4 = \frac{25}{12},\ S_5 = \frac{137}{60},$$
$$S_6 = \frac{49}{20},\ S_7 = \frac{363}{140},\ S_8 = \frac{761}{280},\ S_9 = \frac{7129}{2520}, S_{10} = \frac{7381}{2520}$$

This series diverges to $+\infty$. To prove this we need to estimate the n^{th} term in the sequence of partial sums. The n^{th} partial sum for this series is

$$S_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}.$$

Now consider the following subsequence extracted from the sequence of partial sums:

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} \\
S_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} \\
S_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}
\end{aligned}$$

In general, by induction we have that that

$$S_{2^k} \geq 1 + \frac{k}{2}$$

for all k. Hence, the subsequence $\{S_{2^k}\}$ extracted from the sequence of partial sums $\{S_N\}$ is unbounded. But then the sequence $\{S_N\}$ cannot converge either, and must, in fact, diverge to infinity.

If the terms a_n of an infinite series $\sum a_n$ are all nonnegative, then the partial sums $\{S_n\}$ form a nondecreasing sequence, so by Theorems 6.9 and 6.10 $\sum a_n$ either converges or diverges to $+\infty$. In particular, $\sum |a_n|$ is meaningful for any sequences $\{a_n\}$ whatsoever. The series $\sum a_n$ is said to *converge absolutely* if $\sum |a_n|$ converges.

Example 6.14 A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants a and r is called a *geometric series.* For $r \neq 1$ the partial sums are given by

$$S_N = \sum_{k=0}^{N} ar^k = a\frac{1 - r^{n+1}}{1 - r}.$$

Taking the limit as N goes to infinity, gives us that

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ +\infty & \text{if } a \neq 0 \text{ and } |r >\geq 1 \end{cases}$$

Example 6.15 [p-Series]For a positive number p

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

The exact value of this series for $p > 1$ is extremely difficult to determine. A few are known. The first of these below is due to Euler.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

If $p > 1$ then the sum of the series is $\zeta(p)$, *i.e.*, the Riemann zeta function evaluated at p. There If p is an even integer then there are formulas like the above, but there are no elegant formulas for p an odd integer.

A series converges conditionally, if it converges, but not absolutely.

Example 6.16 1. Does the series $\sum_{n=0}^{\infty}(-1)^n$ converge absolutely, conditionally, or not at all?

2. Does the series $\sum_{n=0}^{\infty}(\frac{1}{2})^n$ converge absolutely, conditionally, or not at all?

3. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge absolutely, conditionally, or not at all (this series is called alternating harmonic series)?

Conditionally convergent sequences are rather difficult with which to work. Several operations that one would expect to be true do not hold for such series. The perhaps most striking example is the associative law. Since a + b = b + a for any two real numbers a and b, positive or negative, one would expect also that changing the order of summation in a series should have little effect on the outcome. Not true.

Theorem 6.15 (Order of Summation) **(i)** *Let $\sum a_n$ be an absolutely convergent series. Then any rearrangement of terms in that series results in a new series that is also absolutely convergent to the same limit.*

(ii) *Let be a conditionally convergent series. Then, for any real number c there is a rearrangement of the series such that the new resulting series will converge to c.*

This will be proved later. One sees, however, that conditionally convergent series probably contain a few surprises. Absolutely convergent series, however, behave just as one would expect.

Theorem 6.16 (Algebra on Series) *Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. Then*

(i) *The sum of the two series is again absolutely convergent. Its limit is the sum of the limit of the two series.*

(ii) *The difference of the two series is again absolutely convergent. Its limit is the difference of the limit of the two series.*

(iii) *The product of the two series is again absolutely convergent. Its limit is the product of the limit of the two series.*

The Cauchy product of two series $\sum a_n$ and $\sum b_n$ of real is defined as follows. The Cauchy product is

$$\left(\sum_{n=m}^{\infty} a_n\right) \cdot \left(\sum_{n=m}^{\infty} b_n\right) = \left(\sum_{n=m}^{\infty} c_n\right) \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for $n = 0, 1, 2, \ldots$.

6.7 Convergence Tests

Definition 6.8 *We say that a series $\sum a_n$ satisfies the Cauchy criterion if its sequence of partial sums is a Cauchy sequence.*

This means that for each $\epsilon > 0$ there exists a number N such that if $m, n > N$ then $|S_n - S_m| < \epsilon$. Nothing is lost in this definition if we impose the restriction $n \geq m$. Moreover, it is only a notational matter to work with $m-1$ where $m \leq n$ instead of m where $m < n$. That means that the definition is equivalent to for each $\epsilon > 0$ there exists a number N such that if $n \geq m > N$ then $|S_n - S_{m-1}| < \epsilon$. The reason for doing this is that $S_n - S_{m-1} = \sum_{k=m}^{n} a_k$. Then this condition can be rewritten as for each $\epsilon > 0$ there exists a number N such that if $n \geq m > N$ then $|\sum_{k=m}^{n}| < \epsilon$.

Theorem 6.17 *A series converges if and only if it satisfies the Cauchy criterion.*

Corollary 6.1 *If a series $\sum a_n$ converges, then $\lim a_n = 0$.*

It is often easier to prove that a limit exists or that a series converges than it is to determine its exact value. As an example consider the following.

Theorem 6.18 (Comparison Test) *Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.*

(i) *If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.*

(ii) *If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$.*

Proof: **(i)** For $n \geq m$ we have

$$\left|\sum_{k=m}^{n} b_k\right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k$$

where the first inequality follows from the Triangle Inequality. Since $\sum a_n$ converges, it satisfies the Cauchy criterion. It then follows from the above that $\sum b_n$ also satisfies the Cauchy criterion and hence it also converges.

(ii) Let $\{S_n\}$ and $\{T_n\}$ be the sequences of partial sums for $\sum a_n$ and $\sum b_n$, respectively. Since $b_n \geq a_n$ for all n we then clearly have that $T_n \geq S_n$ for all n. Since $\lim S_n = +\infty$, we conclude that $\lim T_n = +\infty$, and $\sum b_n = +\infty$.

■

Corollary 6.2 *Absolutely convergent series are convergent.*

Proof: Suppose that $\sum a_n$ is absolutely convergent. This means that $\sum b_n$ converges where $b_n = |a_n|$ for all n. Then $|a_n| \leq b_n$, so that $\sum a_n$ converges by the Comparison Test. ■

Theorem 6.19 (Limit Comparison Test) *Suppose* $\sum a_n$ *and* $\sum b_n$ *are two infinite series. Suppose also that* $r = \lim |a_n/b_n|$ *exists, and* $0 < r < +\infty$*. Then* $\sum a_n$ *converges absolutely if and only if* $\sum b_n$ *converges absolutely.*

Proof: Since $r = \lim |a_n/b_n|$ exists, and r is between 0 and $+\infty$, there exist constants c and C, $0 < c < C < +\infty$ such that for some $N > 1$ we have that if $n > N$

$$c < \left|\frac{a_n}{b_n}\right| < C.$$

Assume that $\sum a_n$ converges absolutely. For $n > N$ we have that $c|b_n| < |a_n|$. Therefore, $\sum b_n$ converges absolutely by the Comparison Test.

Now assume that $\sum b_n$ converges absolutely. From the above inequality we have that $|a_n| < C|b_n|$ for $n > N$. But since the series $C \sum b_n$ also converges absolutely, we can use again the Comparison Test to see that $\sum a_n$ must converge absolutely. ∎

Theorem 6.20 (Cauchy Condensation Test) *Suppose* $\{a_n\}$ *is a decreasing sequence of positive terms. Then the series* $\sum a_n$ *converges if and only if the series* $\sum 2^k a_{2^k}$ *converges.*

Proof: Assume that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Since $\{a_n\}$ is a decreasing sequence, we have that

$$\begin{aligned} 2^{k-1} a_{2^k} &= a_{2^k} + a_{2^k} + a_{2^k} + \ldots + a_{2^k} \\ &\leq a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^{k-1}+2^{k-2}} + a_{2^k} = \sum_{m=2^{k-1}+1}^{2^k} a_m. \end{aligned}$$

Therefore, we have that

$$\sum_{k=1}^{N} 2^{k-1} a_{2^k} \leq \sum_{k=1}^{N} \left(\sum_{m=2^{k-1}+1}^{2^k} a_m \right) = \sum_{m=2}^{2^N} a_m.$$

Now the partial sums on the right are bounded, by assumption. Hence the partial sums on the left are also bounded. Since all terms are positive, the partial sums now form an increasing sequence that is bounded above, hence it must converge. Multiplying the left sequence by 2 will not change convergence, and hence the series $\sum_{k=1}^{N} 2^k a_{2^k}$ converges.

Now, assume that $\sum_{k=1}^{N} 2^k a_{2^k}$ converges: We have

$$\begin{aligned} \sum_{m=2^{k-1}+1}^{2^N} a_m &= a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^{k-1}+2^{k-2}} + a_{2^k} \\ &\leq a_{2^{k-1}} + a_{2^{k-1}} + a_{2^{k-1}} + \ldots + a_{2^{k-1}} = 2^{k-1} a_{2^{k-1}} \end{aligned}$$

Therefore, similar to above, we get:

$$\sum_{m=2}^{2^N} a_m = \sum_{k=1}^{N}\left(\sum_{m=2^{k-1}+1}^{2^k} a_m\right) \leq \sum_{k=1}^{N} 2^{k-1} a_{2^k}.$$

Now the sequence of partial sums on the right is bounded, by assumption. Therefore, the left side forms an increasing sequence that is bounded above, and therefore must converge. ∎

Corollary 6.3 *For a positive number p*

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converges if and only if } p > 1.$$

PROOF: If $p < 0$ then the sequence $\left\{\frac{1}{n^p}\right\}$ diverges to infinity. Hence, the series diverges by the Divergence Test.

If $p > 0$ then consider the series

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} (2^{1-p})^n.$$

This right-hand side is a geometric series. Thus, we know that

- if $0 < p \leq 1$ then $2^{1-p} \geq 1$, hence the right-hand series diverges;
- if $p > 1$ then $2^{1-p} < 1$ and the right-hand series converges,

Now the result follows from the Cauchy Condensation Test . ∎

Theorem 6.21 (Root Test) *Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$*

(i) *converges absolutely if $\alpha < 1$,*

(ii) *diverges if $\alpha > 1$.*

(iii) *Otherwise $\alpha = 1$ and the test gives no information.*

Although this Root Test is more difficult to apply, it is better than the Ratio Test in the following sense. There are series for which the Ratio Test give no information, yet the Root Test will be conclusive. We will use the Root Test to prove the Ratio Test, but you cannot use the Ratio Test to prove the Root Test. It is important to remember that when the Root Test gives 1 as the answer for the lim sup, then no conclusion at all is possible.

The use of the lim sup rather than the regular limit has the advantage that we do not have to be concerned with the existence of a limit. On the other hand, if the regular limit exists, it is the same as the lim sup, so that we are not giving up anything using the lim sup.

PROOF:

(i) Suppose that $\alpha < 1$. Then choose an $\epsilon > 0$ so that $\alpha + \epsilon < 1$. Then by the definition of the limit superior there is a natural number N such that

$$\alpha - \epsilon < \text{lub}\{|a_n|^{1/n} \mid n > N\}, \alpha + \epsilon.$$

In particular, we have $|a_n|^{1/n} < \alpha + \epsilon$ for $n > N$, so

$$|a_n| < (\alpha + \epsilon)^n \text{ for } n > N.$$

Since $0 < \alpha + \epsilon < 1$, the geometric series $\sum_{n=N+1}^{\infty} (\alpha + \epsilon)^n$ converges. Thus, the Comparison Test shows that $\sum_{n=N+1}^{\infty} a_n$ converges. This means that $\sum a_n$ also converges.

(ii) If $\alpha > 1$, then there is a subsequence of $|a_n|^{1/n}$ that has limit $\alpha > 1$. That means that $|a_n| > 1$ for infinitely many choices of n. In particular, the sequence $\{a_n\}$ cannot converge to 0, so the series $\sum a_n$ cannot converge.

(iii) For the series $\sum \frac{1}{n}$ and for the series $\sum \frac{1}{n^2}$, α turns out to be 1. Since the harmonic series diverges and the series $\sum \frac{1}{n^2}$ converges, the equality $\alpha = 1$ cannot guarantee either convergence or divergence of the series.

∎

Theorem 6.22 (Ratio Test) *A series $\sum a_n$ of nonzero series*

(i) *converges absolutely if* $\limsup |a_{n+1}/a_n| < 1$*,*

(ii) *diverges is* $\liminf |a_{n+1}/a_n| > 1$*.*

(iii) *Otherwise* $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ *and the test gives no information.*

PROOF:
(i) Suppose that $\limsup |a_{n+1}/a_n| < 1$. Then choose an $\epsilon > 0$ so that $|a_{n+1}/a_n| + \epsilon < 1$. Then by the definition of the limit superior there is a natural number N such that for $n > N$

$$\left| \frac{a_{n+1}}{a_n} \right| < 1 - \epsilon.$$

Multiplying both sides by $|a_n|$ we get

$$|a_{n+1}| < (1 - \epsilon)|a_n| \text{ for } n > N.$$

Therefore, we also have

$$|a_{n+2}| < (1 - \epsilon)|a_{n+1}| < (1 - \epsilon)^2 |a_n| \text{ for } n > N.$$

Repeating this procedure, we get that

$$|a_k| < (1-\epsilon)^{k-N}|a_N| \text{ for } k > N.$$

These terms form a convergent geometric series. Thus, the Comparison Test shows that $\sum a_n$ also converges.

The other two parts are proven in much the same fashion as the previous theorem. ∎

The next lemma is used to prove Abel's Convergence Test. It is computational in nature.

Lemma 6.5 (Summation by Parts) *Consider the two sequences $\{a_n\}$ and $\{b_n\}$. Let $S_n = \sum_{k=1}^{n} a_n$ be the n-th partial sum. Then for any $0 \le m \le n$ we have*

$$\sum_{j=m}^{n} a_j b_j = [S_n b_n - S_{m-1} b_m] + \sum_{j=m}^{n+1} S_j (b_j - b_{j+1}).$$

Proof: Just be careful with the subscripts:

$$\begin{aligned}
\sum_{j=m}^{n} a_j b_j &= \sum_{j=m}^{n} (S_j - S_{j-1}) b_j \\
&= \sum_{j=m}^{n} S_j b_j - \sum_{j=m}^{n} S_{j-1} b_j \\
&= \sum_{j=m}^{n} S_j b_j - \sum_{j=m-1}^{n-1} S_j b_{j+1} \\
&= \sum_{j=m}^{n-1} S_j (b_j - b_{j+1} + (S_n b_n - S_{m-1} b_m)
\end{aligned}$$

∎

Theorem 6.23 (Abel's Test) *Consider the series $\sum a_n b_n$. Suppose that*

(i) *the partial sums $S_N = \sum_{n=1}^{N} a_n$ form a bounded sequence,*

(ii) *the sequence $\{b_n\}$ is decreasing,*

(iii) $\lim b_n = 0$.

Then the series $\sum a_n b_n$ converges.

This test is rather sophisticated. Its main application is to prove the Alternating Series test, but one can sometimes use it for other series as well, if the more obvious tests do not work.

PROOF: First, let's assume that the partial sums S_N are bounded by K. Next, since the sequence $\{b_n\}$ converges to zero, we can choose an integer N such that $|b_n| < \epsilon/2K$. Using the Summation by Parts lemma, we then have:

$$\begin{aligned}\left|\sum_{j=m}^{n} a_j b_j\right| &= \left|[S_n b_n - S_{m-1} b_m] + \sum_{j=m}^{n+1} S_j (b_j - b_{j+1})\right| \\ &\leq K|b_n| + K|b_m| + K\sum_{j=m}^{n-1} |b_j - b_{j+1}|.\end{aligned}$$

But the sequence $\{b_n\}$ is decreasing to zero, so in particular, all terms must be positive, and all absolute values inside the summation above are superfluous. But then the sum is a telescoping sum. Therefore, all that remains is the first and last term, and we have:

$$\left|\sum_{j=m}^{n} a_j b_j\right| \leq K(b_m + b_n + b_m - b_n) = 2K b_m.$$

But by our choice of N, this is less than ϵ if we choose n and m larger than the predetermined N. ∎

Theorem 6.24 (Alternating Series Test) *If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \ldots 0$ and $\{a_n\}$ converges to zero, then the alternating series $\sum (-1)^n a_n$ converges.*

This test does not prove absolute convergence. In fact, when checking for absolute convergence the term 'alternating series' is meaningless. It is important that the series truly alternates, that is each positive term is followed by a negative one, and visa versa. If that is not the case, the alternating series test does not apply (while Abel's Test may still work).

PROOF: Let $a_n = (-1)^n$. Then the formal sum $\sum_{n=1}^{\infty} a_n$ has bounded partial sums although the sum does not converge. Then, with the given choice of $\{b_n\}$ Abel's test applies directly, showing that the series converges. ∎

6.7.1 Riemann Series Theorem

This is Theorem 6.15. We will prove it here. Let me restate it in two parts.

We are used to using the law of commutativity when adding real numbers. It does not matter in which order we calculate a sum

$$\sum_{k=1}^{n} a_k$$

we always end up with the same number. We can express this in a more formal way by introducing the notion of **rearrangement**. A rearrangement is a one-to-one, onto function

$$\pi\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$$

As a consequence every number $k \in \{1,\ldots,n\}$ can be written as $k = \pi(\ell)$ for some $\ell \in \{1,\ldots,n\}$. Often, such a function is called a permutation of the numbers $\{1,\ldots,n\}$. For example take the set $\{1,2,3\}$ and consider the map

$$\pi_1(1) = 2, \quad \pi_1(2) = 3, \quad \pi_1(3) = 1.$$

This is a rearrangement or permutation of the the three numbers.

Question: In how many ways can we rearrange the numbers $\{1,\ldots,n\}$?

Returning to the series we can express the fact that it does not matter in which order we add the numbers by the equation

$$\sum_{k=1}^{n} a_{\pi(k)} = \sum_{k=1}^{n} a_k$$

What do we mean by a rearrangement of a series? If we denote the natural numbers by $\mathbb{N}$, we have to be precise what we mean by a rearrangement of $\mathbb{N}$. We have to specify the rearrangement π for infinitely many numbers. Again, we must have that π is one-to-one and onto. This means that every element $k \in \mathbb{N}$ can be written as $\pi(\ell)$ for some $\ell \in \mathbb{N}$.

With the help of such a map π we can rearrange series simply by writing

$$\sum_{k} a_{\pi(k)}.$$

It is natural to expect, that for any convergent series it doe not really matter how we sum up this series, *i.e.*,

$$\sum_{k} a_{\pi(k)} = \sum_{k} a_k.$$

The Reimann Series Theorem tells us first that it is not true in general, but it is true under certain reasonable conditions.

Theorem 6.25 (Riemann Series Theorem) *Let $\sum_k a_k$ be an absolutely convergent series. then for any rearrangement π we have that $\sum_k a_{\pi(k)} = \sum_k a_k$.*

Thus, absolutely convergent series really do behave like finite sums when it comes to changing the order of summation.

PROOF: We will do this by looking at the partial sums. The partial sum for the rearranged series we will call

$$R_n = \sum_{k=1}^{n} a_{\pi(k)}$$

whereas

$$S_n = \sum_{k=1}^{n} a_k.$$

Since we are only interested in absolute convergence, we may assume that the numbers $a_k > 0$.

Define the number

$$D(n) = \max_{1 \le k \le n} |\pi(k) - k|.$$

This number describes the range of the rearrangement, in particular we have that

$$\pi(k) \le k + D(n)$$

for all $1 \le k \le n$. Now we have that

$$R_n = \sum_{k=1}^{n} a_{\pi(k)} \le \sum_{k=1}^{n+D(n)} a_k.$$

In the first expression the n indices $\pi(k)$ for $k = 1, \ldots, n$ are distinct numbers, each somewhere between 1 and $n + D(n)$. Here is one place that we use the fact that the function π is one-to-one. These indices are also included on the right side of the equation but there may be more and hence the inequality sign.

In particular we realize that

$$R_n \le S_n + \sum_{k=n+1}^{n+D(n)} a_k.$$

Since the series $\sum_k a_k$ is absolutely convergent we have that

$$R_n \le S_n + \sum_{k=n+1}^{\infty} a_k,$$

and as $n \to \infty$ S_n has a limit L and $\sum_{n+1}^{\infty} a_k \to 0$. Hence R_n is a bounded sequence and since it is increasing it has a limit R which satisfies $R \le L$.

Now we have to show that the converse is true. Notice that we have not used the assumption that π is onto, which means that we do not miss any terms. Pick any n

and notice that there exists a number $E(n)$ so that the numbers $a_1, \ldots, a_n$ are among the numbers $a_{\pi(1)}, \ldots, a_{\pi(E(n))}$. Thus

$$S_n \leq \sum_{k=1}^{E(n)} a_{\pi(k)}.$$

As $n \to \infty$ $E(n) \to \infty$ and since both sides converge, we have $L \leq R$ and our theorem is proven. ∎

The following is one of the most scandalous results in series.

Theorem 6.26 *Let $\sum_k a_k$ be a conditionally convergent series. For any given number L there exists a rearrangement π of $\mathbb{N}$, so that the series $\sum_k a_{\pi(k)}$ converges to L.*

Thus we see that our expectations fail in a spectacular way and conditionally convergent sequences continue to be promiscuous.

PROOF: We assume that L is positive. We have to find the desired rearrangement. The procedure goes as follows. Make two boxes, in one we put all the positive numbers a_k in an ordered fashion and in the other we put all the negative numbers a_k also in an ordered fashion. Since the series is conditionally convergent we have that the series consisting of the numbers in the positive box diverges to $+\infty$ whereas the series consisting of the numbers in the negative box has to diverge towards $-\infty$.

Now start picking numbers from the positive box until the sum overshoots L for the first time. Notice that this is possible since the series in the positive box diverges. Then we start adding the numbers from the negative box until we undershoot the number L for the first time. Notice again, that this must happen since the series from the negative box diverges. Now we keep doing this, and produce in this fashion a sequence of partial sums of a rearranged series that oscillates about the value L. it remains to see that this sequence converges to L, but this follows immediately from the fact that the numbers $a_k \to 0$ as $k \to \infty$. ∎