Sequences

Def A sequence of real numbers is a function whose domain is a set of natural numbers and range is a set of real numbers. $f: N \to R$. The sequence is denoted by $\{a_n\}_{n=1}^{\infty}$ or $a_1, a_2, a_3 \ldots a_n \ldots$

ex

① $1, 3, 5, 7 \ldots\ldots$

② $1, \frac{1}{2}, \frac{1}{3} + \ldots\ldots$

1st term, 2nd term, 3rd term

③ $S_n = \{(-1)^n\}, n \in N.$

④ $S_n = \left(1+\frac{1}{n}\right)^n, \forall n \in N.$

Bounds of a sequence :-

A sequence is said to be bounded above if there exists a real number K such that $S_n \leq K, \forall n \in N$

A sequence is bounded below if there exists a real number k such that

$S_n \geq k$, $\forall n \in \mathbb{N}$

A sequence is said to be bounded if it is both bounded above and below.

<u>Convergence of a sequence :-</u>

A sequence $\{S_n\}$ is said to be converge to a real no. l, if for each $\epsilon > 0$, there exists a positive integer m such that $|S_n - l| < \epsilon$, for all $n \geq m$.

$\Rightarrow$ here m depends on ϵ.

$S_n \to l$ as $n \to \infty$. or $\lim_{n \to \infty} S_n = l$.

So here $l - \epsilon < S_n < l + \epsilon$, $\forall n \geq m$.

ex $a_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$. P.T 0 is the limit

Consider $|a_n - 0| < \epsilon$ $\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon$.

$\frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon} = m$.

for each ϵ, $\exists m$ such that $|a_n - 0| < \epsilon$, $m = \frac{1}{\epsilon}$

Theorems:

Thm:- Every convergent sequence is bounded.

proof: let $\{s_n\}$ converge to limit l.

then by the definition of convergent sequence, for each $\epsilon > 0$, $\exists m$ such that

$$|s_n - l| < \epsilon, \quad \forall n \geq m.$$

$$\Rightarrow \quad l - \epsilon < s_n < l + \epsilon, \quad \forall n \geq m.$$

here we see that all the sequence terms after s_m are bounded above and bounded below by $l+\epsilon$ & $l-\epsilon$ respectively.

let $s_1, s_2 \ldots s_{m-1}$ be the remaining terms

now let $g = \min\{l-\epsilon, s_1, s_2 \ldots s_{m-1}\}$

$G = \max\{l+\epsilon, s_1, s_2 \ldots s_{m-1}\}$

now we have $g \leq s_n \leq G, \; \forall n$

hence the sequence $\{s_n\}$ is bounded

$\therefore$ hence proved

thm A sequence converges to a unique limit

proof Let us suppose that a sequence $\{s_n\}$ converges to two different limits. Say l_1 & l_2

by definition of Convergence,

$\Rightarrow \{s_n\}$ converges to $l_1 \Rightarrow$ for each $\epsilon > 0$, $\exists m_1$

such that $|s_n - l_1| < \frac{\epsilon}{2}$, $\forall n \geq m_1$

$\Rightarrow \{s_n\}$ converges to $l_2 \Rightarrow$ for each $\epsilon > 0$, $\exists m_2$

such that $|s_n - l_2| < \frac{\epsilon}{2}$, $\forall n \geq m_2$.

($\because$ Since it is true for each ϵ, it must be true for $\frac{\epsilon}{2}$).

Now let $m = \max\{m_1, m_2\}$.

. now for each $\epsilon > 0$, $\exists m$ such that $\forall n \geq m$

. $|s_n - l_1| < \epsilon/2$ & $|s_n - l_2| < \frac{\epsilon}{2}$.

now consider $|l_1 - l_2| = |s_n - s_n + l_1 - l_2|$

$= |s_n - l_2 + l_1 - s_n|$ ($\because$ triangle inequality $|a+b| \leq |a| + |b|$)

$\leq |s_n - l_2| + |l_1 - s_n|$

$\leq |s_n - l_1| + |s_n - l_2|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\Rightarrow |l_1 - l_2| < \epsilon$ is true only if $l_1 = l_2$

Algebra of Convergent Sequences.

1) If Sequence $\{s_n\}$ converges to l & $c \in R$ then the sequence $\{ca_n\}$ converges to cl

i.e. $\lim_{n\to\infty} ca_n = c.\lim_{n\to\infty} a_n$

2) If sequence $\{a_n\}$ converges to l and $\{b_n\}$ converges to m then the sequence $\{a_n+b_n\}$ converges to $l+m$. i.e., $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

3) If the sequence $\{a_n\}$ converges to l and $\{b_n\}$ converges to m, then the sequence $\{a_n.b_n\}$ converges to $l.m$. i.e., $\lim_{n\to\infty}\{a_n.b_n\} = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$

4) If the sequence $\{a_n\}$ converges to l, if $a_n \neq 0$ for all $n \in N$, and $l \neq 0$, then the sequence $\{1/a_n\}$ converges to $1/l$. i.e., $\lim_{n\to\infty} 1/a_n = 1/\lim_{n\to\infty} a_n$

5). Suppose that the sequence $\{a_n\}$ converges to l and $\{b_n\}$ converges to m. If $b_n \neq 0$, $\forall n \in N$ and $m \neq 0$, then the sequence $\{a_n/b_n\}$ converges to l/m i.e $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

6) Let $\{a_n\}$ and $\{b_n\}$ be ~~a~~ the sequences such that $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n > 0$, then

$$\lim_{n\to\infty} a_n b_n = +\infty$$

7) For a sequence $\{a_n\}$ of positive real numbers $\lim_{n\to\infty} a_n = -\infty$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$

Limit Point of a Sequence.

Def :- A real no. l is said to be a limit point of the sequence $\{s_n\}$ if every neighbourhood of l contains an infinite no of members of sequence.

$l-\epsilon \quad l \quad l+\epsilon$

(or) for each $\epsilon > 0$, if the ϵ-neighbourhood of l has an infinite. no. of members of sequence for an infinite values of n.

i.e $s_n \in (l-\epsilon, l+\epsilon)$ for an infinite values of n.

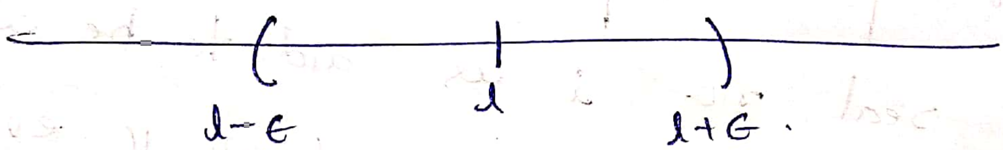
i.e $|s_n - l| < \epsilon$.

Note :- limit point of a range set of a sequence is also a limit point of the sequence but the converse need not be true always.

Ex ① $S_n = 1, \ \forall n$, has only one limit point that is 1

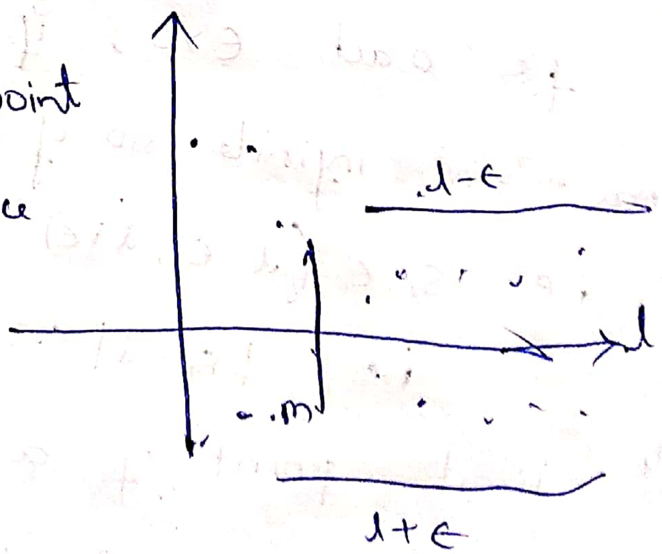
② $S_n = \frac{1}{n}, \ \forall n \in \mathbb{N}$, 0 is the limit point and 0 is the limit point of range $\{1, \frac{1}{2}, \frac{1}{3} \ldots\}$

③ $S_n = 1 + (-1)^n$ has 0 & 2 as limit points



limit point $\Rightarrow$ there are infinite terms of the sequence in the open interval $(l-\epsilon, l+\epsilon)$.

limit $\Rightarrow$ after certain point say m, all the sequence terms are inside the interval $(l-\epsilon, l+\epsilon)$.



theorem Bolzano Weistrass theorem

Statement :- Every bounded Sequence has a limit point

Converse of above theorem need not be true.

i.e if a sequence has a limit point, it need not be bounded

ex $\{1, 2, 1, 4, 1, 6 \ldots\}$ is unbounded and 1 is the limit point.

Convergent Sequences :-

theorem :- Every bounded Sequence with a unique limit point is convergent.

theorem :- A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.

Def :- A Sequence is Said to be convergent if it is bounded and has a unique limit point.

theorem :- A necessary and sufficient condition for a sequence $\{S_n\}$ to converge is that for each $\epsilon > 0$, $\exists m$ such that $|S_n - l| < \epsilon$, $\forall n \geq m$

Monotonic Sequences.

<u>def</u> A sequence is said to be monotonic increasing if $S_{n+1} \geq S_n$ $\forall n$, and monotonic decreasing if $S_{n+1} \leq S_n$ $\forall n$.

Ex monotonic increasing $S_n = n^2$.

motonic decreasing $S_n = \frac{1}{n}$.

<u>def</u> A sequence is said to be monotonic increasing strictly if $S_{n+1} > S_n$ and strictly decreasing if $S_{n+1} < S_n$.

<u>theorm</u> A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

<u>theorm</u> :- Let $\{a_n\}$ be a sequence of real no's

i) If $\{a_n\}$ is an unbounded monotonically increasing sequence then $\lim_{n\to\infty} a_n = +\infty$

ii) If $\{a_n\}$ is an unbounded monotonically decreasing sequence then $\lim_{n\to\infty} a_n = -\infty$

Sub Sequences :-

def If $\{s_n\} = \{s_1, s_2, \dots\}$ be a sequence, then any infinite succession of its terms, picked in such a way that the original order is preserved is called subsequence

ex ① $\{s_2, s_4, s_6 \dots\}$ is a subsequence of $\{s_n\}$ even terms

② for $s_n = 1/n$, the subsequences are

$s_n = \frac{1}{2n}$, $s_n = \frac{1}{3n}$ ~~are~~

theorem :- A sequence $\{s_n\}$ converges to s iff it's every subsequence converges to s.

similarly $\lim_{n\to\infty} s_n = \infty$ or $-\infty$ iff every subsequence of $\{s_n\}$ tends to ∞ or $-\infty$.

theorem :- If l is a limit point of a sequence $\{s_n\}$ then there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ which converges to l. i.e., $\lim_{n\to\infty} s_{n_k} = \xi$