Power Series

A Series of the form $a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n + \ldots$ where the a's are independent of x, is called a power Series in x.

Such Series may converge for Some or all the Values of x.

Interval of Convergence :-

In the power series, (i.e.)

$$u_n = a_n x^n.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) \cdot x$$

if $\lim \frac{a_{n+1}}{a_n} = l, \Rightarrow \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) x = l x$

by ratio test, given series converges if

$$\lim_{n\to\infty} \left| \left(\frac{a_{n+1}}{a_n}\right) \cdot x \right| = |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| \cdot l < 1$$

$$|x| < \frac{1}{l}.$$ (∵ absolute Convergence ⇒ Convergence)

$-\frac{1}{l} < x < \frac{1}{l}$ is called as interval of convergence. and for all other values of x the Series diverges

Ex $\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \dots \infty$

let $u_n = \frac{1}{n(1-x)^n}$, $u_{n+1} = \frac{1}{(n+1)(1-x)^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)(1-x)^{\cancel{n}+1}} \, n\,(1-\cancel{x})^{\cancel{n}} \right|$$

$$= \left|\frac{1}{1-x}\right| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \left|\frac{1}{1-x}\right| \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+1/n)}$$

$$= \left|\frac{1}{1-x}\right| \cdot 1$$

by ratio test it converges if $\left|\frac{1}{1-x}\right| < 1$

$$\left|\frac{1}{1-x}\right| < 1 \Rightarrow |1-x| > 1$$

$1-x < -1$ & $1-x > 1$

or for $x < 0$ & $x > 2$ the Series is Convergent

$x = 0 \Rightarrow \cdot 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots$ is divergent $\cdot \sum \frac{1}{n}$.

for $x = 2$ $\cdot -1 + \frac{1}{2} + \frac{-1}{3} + \dots + \frac{(-1)^n}{n} + \dots = \sum \frac{(-1)^n}{n}$

by Leibnitz rule, the above Series convergent

& the given Series diverges for other values

& interval of Convergence is $x < 0$ & $x \geq 2$

Convergence of Exponential Series

The Series $1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \infty$.

Th. let $u_n = \frac{x^n}{n!}$, $u_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}$$

$$= \lim_{n\to\infty} \frac{x \cdot \cdot \cancel{n!}}{(n+1)\cancel{n!}}$$

$$= x \cdot \lim_{n\to\infty} \frac{1}{n+1} = x \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

above Series converges for any value of x.

2). Convergence of Lagorithemic Series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \cdot \frac{x^n}{n} + \cdots \infty$$

$$u_{n+1} = \frac{(-1)^{n+1} \cdot x^{n+1}}{(n+1)}, \quad u_n = \frac{(-1)^n \cdot x^n}{n}.$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot x^{n+1}}{(n+1)} \cdot \frac{(n)}{(-1)^n \cdot x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\, x}{(n+1)} \right|$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |x|$$

So lagorithemic Series converges if $|x| < 1$ & diverges if $|x| > 1$

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If $x=1$, the series becomes.

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots = \sum\frac{(-1)^n}{n}$ is convergent

by Leibinitz's test.

$x=-1$, the series becomes

$-1-\frac{1}{2}+\frac{-1}{3}-\frac{1}{4}+\cdots$

$= -\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\right)$

$= -\sum\frac{1}{n}$ is divergent $\therefore$ converges for $-1<x\leq 1$.

3) Convergence of binomial series.

$1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{r!}.x^r+\cdots\infty$

$u_r = \frac{n(n-1)(n-2)\cdots(n-r)}{(r-1)!}x^{r-1}$, $u_{r+1} = \frac{n(n-1)\cdots(n-r+1)}{r!}.x^r$

$\lim\limits_{n\to\infty}\frac{u_{r+1}}{u_r} = \lim\limits_{r\to\infty}\frac{n\cancel{(n-1)}\cdots\cancel{(n-r)}(n-r+1)\,x^{\cancel{r}}\,(r-1)!}{r!\,\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(n-r)}.x^{\cancel{r}-1}}$

$\lim\limits_{r\to\infty}\frac{\cancel{(r-1)!}(n-r+1)}{r\cancel{(r-1)!}\,x^{-1}}$

$\lim\limits_{r\to\infty}\frac{x\,(n-r+1)}{r}$

$\lim\limits_{r\to\infty} x\left(\frac{n+1}{r}-\frac{\cancel{r}}{\cancel{r}}\right)$

$x.\lim\limits_{r\to\infty}\left(\frac{n+1}{r}-1\right)$

$x \cdot (0-1)$ (when $r > n+1$)

$= -x.$

$\therefore$ the Series Converges for $|-x| = |x| < 1$

and diverges for $|x| > 1$. or $|x| > 1$

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