Sequences of functions.

Let $\{f_n(x)\}$ be a sequence of functions defined on an interval I.

ex:- $f_n(x) = \frac{x}{n}$, $\forall n \in \mathbb{N}$, $x \in [0,1]$.

clearly $f_n(0) = 0$, $\forall n \in \mathbb{N}$.

$f_1(x) = x$

$f_2(x) = \frac{x}{2}$

$\vdots$

→ for each point $x \in I$, There corresponds a sequence of numbers.

ex at $x = 1/2 \in [0,1]$.

$f_n(x) = f_n(1/2) = \frac{\frac{1}{2}}{n} = \frac{1}{2n}$.

now the Sequence becomes

$$\underset{f_1(1/2)}{\frac{1}{2}}, \underset{f_2(1/2)}{\frac{1}{4}}, \underset{f_3(1/2)}{\frac{1}{6}}, \underset{f_4(1/2)}{\frac{1}{8}} \ldots\ldots$$

→ So when a particular x value is considered it results into a new sequence for which Convergence can be checked.

Let $f_n(x)$ be a Sequence of functions on a given interval then

a) we say that the sequence of functions is convergent at a point $x = x_0 \in I$ if the sequence $f_n(x_0)$ is convergent.

ex the sequence $f_n(x) = \frac{x}{n}$.

at $x = 0.4$.

$$f_n(0.4) = \frac{0.4}{n}$$

$f_1(0.4) = 0.4$

$f_2(0.4) = \frac{0.4}{2}$.

⋮

the resulting sequence $f_n(0.4) = \frac{0.4}{4}$ is convergent

as $\lim_{n \to \infty} f_n(0.4) = \lim_{n \to \infty} \frac{0.4}{n} = 0$.

So the sequence of functions $\frac{x}{n}$, $x \in (0,1)$

is convergent at $x = 0.4$.

b) we say that it is point wise convergent if for each $x \in I$, the sequence $f_n(x)$ is convergent i.e.,

$$\lim_{n \to \infty} f_n(x) = f(x)$$

where $f(x)$ is a function of x.

by using defination of convergence

(2) Consider $f_n(x) = \frac{n}{1+xn}$, $x \in (0,1)$.

Consider $|f_n(x) - f| = \left|\frac{n}{1+xn} - \frac{1}{x}\right|$

Since $\lim_{n\to\infty} (f_n(x)) = \lim_{n\to\infty} \frac{n}{1+xn}$

$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(x+1/n)}$

$= \frac{1}{x}$.

now $\left|\frac{n}{1+xn} - \frac{1}{x}\right| = \left|\frac{\cancel{nx} - \cancel{nx}-1}{x(1+xn)}\right| < \epsilon$

$\frac{1}{x(1+xn)} < \epsilon \Rightarrow x(1+xn) > \epsilon$

$1+xn > \frac{\epsilon}{x}$

$nx > \frac{\epsilon}{x} - 1$

$n > \frac{\epsilon}{x^2} - \frac{1}{x} = m$.

for each $\epsilon > 0$, $\exists m$ for each $x \in (0,1)$

such that $|f_n(x) - f(x)| < \epsilon$.

so $\frac{n}{1+xn}$ converges point wise on $(0,1)$.

3) $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$, ~~$0 \le x \le$~~

for $0 \le x < 1$, $\lim_{n\to\infty} x^n = 0$.

i.e., for $[0,1)$ the function converges to zero function but at $x = 1$,

$f_n(1) = 1$,

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

so $f_n(x)$ is point wise convergent to zero function on $[0,1)$.

c) Uniform Convergence:-

we say $\{f_n(x)\}$ is uniformly convergent to a function on I, if for each $\epsilon > 0$, there exists $m \in \mathbb{N}$. such that

$$|f_n(x) - f(x)| < \epsilon, \quad \forall n \ge m, \forall x \in I$$

in this case we say $f_n \to f$ uniformly on I.

* here m depends only on ϵ. & not x.

Note:- If $\{f_n\}$ converges uniformly to f, then it converges point wise also. but if a sequence does not converge point wise to any function, it can not converge uniformly uniform convergence examples.

Ex ① $f_n(x) = \frac{1}{n+x}$.

$\lim_{n \to \infty} f(x) = 0, \quad \forall x \in [0, b]$.

for any $\epsilon > 0$, $|f_n(x) - f(x)|$

$$= \left|\frac{1}{n+x} - 0\right| < \epsilon$$

$$\frac{1}{n+x} < \epsilon$$

$$n+x > \frac{1}{\epsilon}$$

$n > \frac{1}{\epsilon} - x$, this decreases as x increases.

with max value $\frac{1}{\epsilon}$.

$m = \frac{1}{\epsilon}$

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Statement :- Suppose $f_n(x)$ for $n \in \mathbb{N}$ and f are functions on given interval.

if there exists a sequence a_n of positive real numbers such that $\lim_{n\to\infty} a_n = 0$,

and $|f_n(x) - f(x)| < \epsilon$, $\forall n \in \mathbb{N}$ & $\forall x \in J$

then $f_n(x)$ converges uniformly to $f(x)$.

ex $f_n(x) = \dfrac{2nx}{1+n^4x^2}$, $x \in [0,1]$.

we have $1+n^4x^2 \geq 2n^2x$ $(\because a^2+b^2 \geq 2ab)$

$$0 \leq f_n(x) \leq \frac{2nx}{2n^2x}$$

$$= \frac{1}{n}.$$

now we have $|f_n(x) - f(x)|$

$$= \left|\frac{2nx}{1+n^4x^2} - 0\right| \leq \frac{2nx}{1+n^4x^2} \quad \left(\because 1+n^4x^2 \geq 2n^2x\right)$$

$$\leq \frac{2nx}{2n^2x} = \frac{1}{n}$$

& $\lim_{n\to\infty} \frac{1}{n} = 0$.

hence by previous statement the given sequence uniformly converges to $f(x) = 0$.

Ex (i) $f_n(x) = \dfrac{nx}{1+n^2x^2}$

$f_n(0) = 0$ for $x = 0$.

& for $x \neq 0$ $\lim_{n\to\infty} f_n(x)$

$= \lim_{n\to\infty} \dfrac{n x}{n^2 (x^2 + \frac{1}{n^2})}$

$= \lim_{n\to\infty} \dfrac{x}{n (x^2 + \frac{1}{n^2})}$.

$= 0$

So f_n converges pointwise to zero function

when $x = 1/n$.

$f_n(x) = f_n\left(\frac{1}{n}\right) = \dfrac{n \cdot \frac{1}{n}}{1 + n^2 \frac{1}{n^2}}$

$= \frac{1}{2}$.

Then $\because$ $|f_n(x) - f| = |\frac{1}{2} - 0| = \frac{1}{2} < \epsilon \Rightarrow \epsilon > \frac{1}{2}$.

but if we choose $\epsilon < 1/2$ it is

not possible.

So it is not uniformly convergent.

ex $f_n(x) = \frac{1}{n^3} \log(1+n^4x^2)$, $x \in [0,1]$

$$0 \le f_n(x) - 0 \le \frac{1}{n^3} \log(1+n^4).$$

now consider $\frac{1}{n^3} \log(1+n^4)$

$$\lim_{n\to\infty} \frac{\log(1+n^4)}{n^3}.$$

$$\lim_{n\to\infty} \frac{4n^{\cancel{3}}}{(1+n^4)\cdot 3\cancel{n^2}}$$ using L hospital's rule

$$\lim_{n\to\infty} \frac{4n}{1+n^4}$$

$$\lim_{n\to\infty} \frac{4\cancel{n}}{n^{\cancel{4}3}\left(\frac{1}{n^4}+1\right)}.$$

$$\lim_{n\to\infty} \frac{4}{n^3\left(1+\frac{1}{n^4}\right)}$$

$$= 0$$

by previous theorem above sequence of functions converges to zero function.

Series of Functions :-

def Series of functions is denoted by $\sum_{n=1}^{\infty} f_n(x)$.

where $f_n(x)$ is sequence of functions.

def For any given series, $\sum_{n=1}^{\infty} f_n(x)$ on any interval I, let $S_n(x) = \sum_{n=1}^{\infty} f_i(x)$, $x \in I$

where $S_n(x)$ is called n^{th} partial sum of given series of functions.

Consider the series $\sum_{n=1}^{\infty} f_n(x)$ of functions on I. & $S_n(x)$ be its n^{th} partial sum. then $\sum_{n=1}^{\infty} f_n(x)$

(a) converges at a point $x_0 \in I$, if $S_n(x)$ converges at x_0.

(b) converges pointwise on I if $S_n(x)$ converges pointwise on I.

(c) Converges uniformly on I, if $S_n(x)$ converges uniformly on I.

we say that an series of functions $\sum f_n(x)$ is dominated series if there exists a sequence a_n of positive real numbers such that $|f_n(x)| \leq a_n$, $\forall x \in I$, and $\forall n \in \mathbb{N}$. and the sequence series $\sum_{n=1}^{\infty} a_n$ converges, then

<u>Statement</u> :- A dominated series converges uniformly

<u>ex</u> $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ & $\sum \frac{\sin nx}{n^2}$. are dominated series as $\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}$, $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$

& $\sum \frac{1}{n^2}$ is convergent as $p > 1$

So the series of functions are uniformly convergent.

2). $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ $x \in [c, \infty)$, $c > 0$.

$$\frac{x}{1+n^2x^2} \leq \frac{x}{2n^2x^2} \leq \frac{1}{n^2x} \leq \frac{1}{n^2c}$$

now $\sum \frac{1}{n^2c}$ is convergent , $p > 1$

So, given series is uniformly convergent