(1) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$. Answer: $\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}$.

The matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

is the 4×4 HILBERT MATRIX. Use Gauss-Jordan elimination to compute $K = H^{-1}$. Then K_{44} is (exactly) ____________ . Now, create a new matrix H' by replacing each entry in H by its approximation to 3 decimal places. (For example, replace $\frac{1}{6}$ by 0.167.) Use Gauss-Jordan elimination again to find the inverse K' of H'. Then K'_{44} is ____________________ .

(2) Prove rank $A^T A$ = rank A for any $A \in M_{m \times n}$

(3) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a linearly independent set in a vector space V. Use the **definition** of *linear independence* to give a careful proof that the set $\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$ is linearly independent in V.

(4) The system of equations:

$$\begin{cases} \quad 2y+3z = 7 \\ x+\ y-\ z = -2 \\ -x+\ y-5z = 0 \end{cases}$$

is solved by applying Gauss-Jordan reduction to the augmented coefficient matrix $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & -1 & -2 \\ -1 & 1 & -5 & 0 \end{bmatrix}$. Give the names of the elementary 3×3 matrices $X_1, \ldots, X_8$ which implement the following reduction.

$$A \xrightarrow{X_1} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ -1 & 1 & -5 & 0 \end{bmatrix} \xrightarrow{X_2} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & -6 & -2 \end{bmatrix} \xrightarrow{X_3} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -9 & -9 \end{bmatrix}$$

$$\xrightarrow{X_4} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{X_5} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{X_6} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{X_7} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{X_8} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Answer: $X_1 =$ __________ , $X_2 =$__________ , $X_3 =$__________ , $X_4 =$__________ ,

$X_5 =$__________ , $X_6 =$__________ , $X_7 =$__________ , $X_8 =$__________ .

(5) Prove that the vectors $(1, 1, 0)$, $(1, 2, 3)$, and $(2, -1, 5)$ form a basis for $\mathbb{R}^3$.