Linear Algebra

Eigen Values and Eigen Vectors

Source: Introduction to linear algrebra by Gilbert Strang

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors*** $\boldsymbol{x}$ ***are in the same direction as*** $A\boldsymbol{x}$. ***Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector $A\boldsymbol{x}$ is a number λ times the original $\boldsymbol{x}$.

The basic equation is $A\boldsymbol{x} = \lambda\boldsymbol{x}$. The number λ is the ***"eigenvalue"***. It tells whether the special vector $\boldsymbol{x}$ is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $A\boldsymbol{x} = 0\boldsymbol{x}$ means that this eigenvector $\boldsymbol{x}$ is in the nullspace.

If A is the identity matrix, every vector has $A\boldsymbol{x} = \boldsymbol{x}$. All vectors are eigenvectors. The eigenvalue (the number lambda) is $\lambda = 1$. This is unusual to say the least.

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}$$

$$A \qquad\qquad A^2 \qquad\qquad A^3 \qquad\qquad\qquad A^{100}$$

$$\boldsymbol{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \boldsymbol{x}_1 \quad (A\boldsymbol{x} = \boldsymbol{x} \text{ means that } \lambda_1 = 1)$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}\boldsymbol{x}_2 \text{ so } \lambda_2 = \tfrac{1}{2})$$

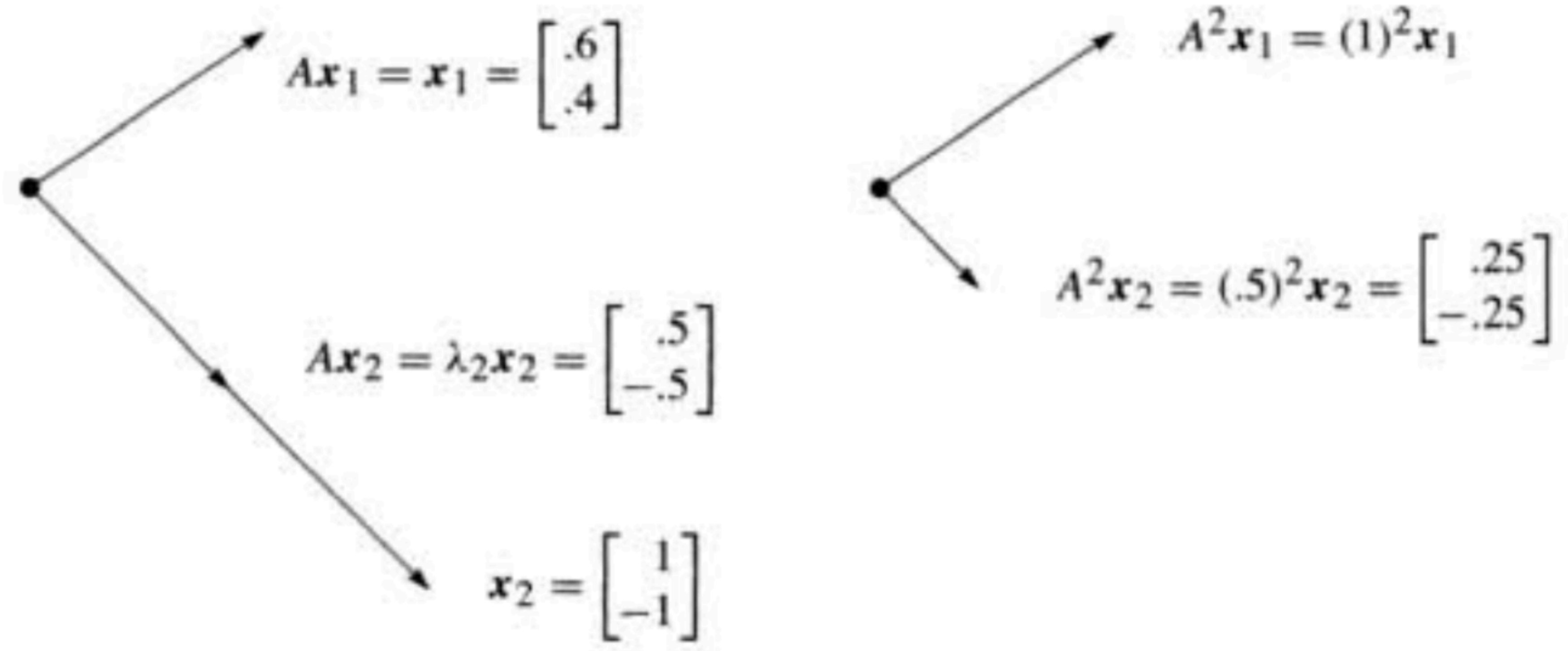

Figure 6.1 The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

If we again multiply $\boldsymbol{x}_1$ by A, we still get $\boldsymbol{x}_1$. Every power of A will give $A^n\boldsymbol{x}_1 = \boldsymbol{x}_1$. Multiplying $\boldsymbol{x}_2$ by A gave $\frac{1}{2}\boldsymbol{x}_2$, and if we multiply again we get $(\frac{1}{2})^2\boldsymbol{x}_2$. ***When A is squared, the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ stay the same.*** The λ's are now 1^2 and $(\frac{1}{2})^2$. ***The eigenvalues are squared!*** This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100} =$ very small number.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\boldsymbol{x}_1 + (.2)\boldsymbol{x}_2$:

$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + (.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \tag{1}$$

Multiplying by A gives the first column of A^2. Do it separately for $\boldsymbol{x}_1$ and $(.2)\boldsymbol{x}_2$. Of course $A\boldsymbol{x}_1 = \boldsymbol{x}_1$. And A multiplies $\boldsymbol{x}_2$ by its eigenvalue $\frac{1}{2}$:

$$A\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + \frac{1}{2}(.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need these eigenvectors to find A^2. But it is the good way to do 99 multiplications.

At every step $\boldsymbol{x}_1$ is unchanged and $\boldsymbol{x}_2$ is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99}\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad \boldsymbol{x}_1 + (.2)(\frac{1}{2})^{99}\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100}. The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

Example 1 **The projection matrix $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has eigenvalues 1 and 0.**

1. Each column of P adds to 1, so $\lambda = 1$ is an eigenvalue.

2. P is singular, so $\lambda = 0$ is an eigenvalue.

3. P is symmetric, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only possible eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\boldsymbol{x} = 0\boldsymbol{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\boldsymbol{x} = \boldsymbol{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself.

First move $\lambda \boldsymbol{x}$ to the left side. Write the equation $A\boldsymbol{x} = \lambda \boldsymbol{x}$ as $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector $\boldsymbol{x}$ is the zero vector. ***The eigenvectors make up the nullspace of*** $A - \lambda I$! When we know an eigenvalue λ, we find an eigenvector by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of*** $\boldsymbol{A - \lambda I}$ ***must be zero.*** This is how to recognize an eigenvalue λ:

6A The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0. \tag{3}$$

This "characteristic equation" involves only λ, not $\boldsymbol{x}$. When A is n by n, $\det(A - \lambda I) = 0$ is an equation of degree n. Then A has n eigenvalues and each λ leads to $\boldsymbol{x}$:

$$\textbf{For each } \lambda \textbf{ solve} \quad (A - \lambda I)\boldsymbol{x} = \mathbf{0} \quad \text{or} \quad A\boldsymbol{x} = \lambda \boldsymbol{x} \quad \textbf{to find an eigenvector } \boldsymbol{x}. \tag{4}$$

Example 3 $A = \left[\begin{smallmatrix}1 & 2\\ 2 & 4\end{smallmatrix}\right]$ is already singular (zero determinant). Find its λ's and $\boldsymbol{x}$'s.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $A\boldsymbol{x} = 0\boldsymbol{x}$ has solutions. They are the eigenvectors for $\lambda = 0$. But here is the way to find *all* λ's and $\boldsymbol{x}$'s! Always subtract λI from A:

$$\textbf{\textit{Subtract } } \lambda \textbf{\textit{ from the diagonal to find}} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}.$$

Take the determinant "ad − bc" of this 2 by 2 matrix. From $1-\lambda$ times $4-\lambda$, the "*ad*" part is $\lambda^2 - 5\lambda + 4$. The "*bc*" part, not containing λ, is 2 times 2.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda. \tag{5}$$

Set this determinant $\boldsymbol{\lambda^2 - 5\lambda}$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues} \quad \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5\,.$$

Now find the eigenvectors. Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ is $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

We need to emphasize: *There is nothing exceptional about* $\lambda = 0$. Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $A\boldsymbol{x} = 0\boldsymbol{x} = \boldsymbol{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*. In the example, the shifted matrix $A - 5I$ was singular and 5 was the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.

2. ***Find the roots of this polynomial***, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.

3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ ***to find an eigenvector*** $\boldsymbol{x}$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1 \quad Ax = x$$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda = -1 \quad Ax = -x$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$$

st.

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_1 = 4 \qquad \lambda_2 = 2 \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $Ax = \lambda x$

n $(A + 3I)x = \lambda x + 3x = (\lambda + 3)x.$

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the* λ's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 2.$$

Good news second: The *product* λ_1 *times* λ_2 *and the sum* $\lambda_1 + \lambda_2$ *can be found quickly from the matrix*. For this A, the product is 0 times 2. That agrees with the determinant (which is 0). The sum of eigenvalues is $0 + 2$. That agrees with the sum down the main diagonal (which is $1 + 1$). These quick checks always work:

6B *The product of the n eigenvalues equals the determinant of A.*

6C *The sum of the n eigenvalues equals the sum of the n diagonal entries of A.* This sum along the main diagonal is called the ***trace*** of A:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \textit{trace} = a_{11} + a_{22} + \cdots + a_{nn}. \tag{6}$$

Example 4 ***The 90° rotation*** $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ***has no real eigenvectors or eigenvalues.*** *No vector* $Q\boldsymbol{x}$ *stays in the same direction as* $\boldsymbol{x}$ (except the zero vector which is useless). There cannot be an eigenvector, unless we go to ***imaginary numbers***. Which we do.

If $Ax = \lambda x$, B has eigenvalues $\alpha_1, \ldots$

$Bx = \alpha x$

$(A+B)x = (\lambda + \alpha)x$