STAT2203: Probability Models and Data Analysis for Engineering
Quiz 7

1. In the Jelinski and Moranda model of software reliability, a program contains N bugs. Each of these bugs cause the program to fail independently. Once a bug is found, it is immediately fixed and no longer causes any issues for the program. The time T_i for bug i to cause the program to fail is assumed to have an exponential distribution with rate parameter λ.

   In this question we shall assumed that $N = 2$.

   (a) Find the cdf of the time for the program to fail for the first time, that is the cdf of $\min\{T_1, T_2\}$.

   Solution: Let $Y = \min\{T_1, T_2\}$. Then

$$\begin{aligned}
\mathbb{P}(Y \leqslant y) &= \mathbb{P}(\{T_1 \leqslant y\} \cup \{T_2 \leqslant y\}) \\
&= \mathbb{P}(T_1 \leqslant y) + \mathbb{P}(T_2 \leqslant y) - \mathbb{P}(T_1 \leqslant y, T_2 \leqslant y) \\
&= \mathbb{P}(T_1 \leqslant y) + \mathbb{P}(T_2 \leqslant y) - \mathbb{P}(T_1 \leqslant y)\mathbb{P}(T_2 \leqslant y) \qquad \text{[by independence]} \\
&= (1 - e^{-\lambda y}) + (1 - e^{-\lambda y}) - (1 - e^{-\lambda y})(1 - e^{-\lambda y}) \qquad \text{[using exponential cdf]} \\
&= 2 - e^{-\lambda y} - (1 - 2e^{-\lambda y} + e^{-2\lambda y}) \\
&= 1 - e^{-2\lambda y}.
\end{aligned}$$

   (b) Find the cdf of the time for both bugs to have occured, that is the cdf of $\max\{T_1, T_2\}$.

   Solution: Let $Y = \max\{T_1, T_2\}$. Then

$$\begin{aligned}
\mathbb{P}(Y \leqslant y) &= \mathbb{P}(T_1 \leqslant y, T_2 \leqslant y) \\
&= \mathbb{P}(T_1 \leqslant y)\mathbb{P}(T_2 \leqslant y) \qquad \text{[by independence]} \\
&= (1 - e^{-\lambda y})(1 - e^{-\lambda y}) \qquad \text{[using exponential cdf]} \\
&= 1 - 2e^{-\lambda y} + e^{-2\lambda y}.
\end{aligned}$$

   (c) What is the expected time for both bugs to have occured?

   Solution:
   First, we get the pdf of $Y = \max\{T_1, T_2\}$.

$$f_Y(y) = \frac{d}{dy}(1 - 2e^{-\lambda y} + e^{-2\lambda y}) = 2\lambda e^{-\lambda y} - 2\lambda e^{-2\lambda y},$$

   for $y \geqslant 0$. The expected value of Y is

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^\infty y(2\lambda e^{-\lambda y} - 2\lambda e^{-2\lambda y})dy \\
&= 2\int_0^\infty y\lambda e^{-\lambda y}dy - \int_0^\infty y2\lambda e^{-2\lambda y}dy \qquad \text{[using the mean of an exponential distribution]} \\
&= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}.
\end{aligned}$$

(d) What is the variance of the time for both bugs to have occured?

Solution: To compute the variance of Y we first compute $\mathbb{E}(Y^2)$.

$$\begin{aligned}
\mathbb{E}(Y^2) &= \int_0^\infty y^2(2\lambda e^{-\lambda y} - 2\lambda e^{-2\lambda y})dy \\
&= 2\int_0^\infty y^2\lambda e^{-\lambda y}dy - \int_0^\infty y^2 2\lambda e^{-2\lambda y}dy \qquad \text{[see page 72 of notes]} \\
&= \frac{4}{\lambda^2} - \frac{1}{2\lambda^2} = \frac{7}{2\lambda^2}.
\end{aligned}$$

So

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 = \frac{7}{2\lambda^2} - \frac{9}{4\lambda^2} = \frac{5}{4\lambda^2}$$

Can you generalise this to an arbitrary number of bugs in the program?

Just something to think about ...

2. Suppose the current price of a stock is \$ 1. A simple model of stock price fluctations is the following: Let X_n be the stock price after n days. Then $X_0 = 1$ and

$$X_n = \exp\left(\sum_{i=1}^n Z_i\right),$$

where the Z_i are independent standard normal random variables.

(a) What is the probability that the stock price has doubled after 10 days.

Solution: Since $X_0 = 1$, the stock will have doubled in 10 days if $X_{10} = \exp(\sum_{i=1}^{10} Z_i) \geqslant 2$. As the Z_i are independent $\mathsf{N}(0,1)$ random variables, $\sum_{i=1}^{10} Z_i \sim \mathsf{N}(0,10)$.

$$\begin{aligned}
\mathbb{P}(\log(X_{10}) \geqslant \log(2)) &= \mathbb{P}\left(\frac{\log(X_{10}) - 0}{\sqrt{10}} \geqslant \frac{\log(2) - 0}{\sqrt{10}}\right) \\
&= \mathbb{P}(Z \geqslant 0.2191924) \\
&= 1 - \mathbb{P}(Z < 0.2191924) \\
&= 1 - 0.5871 = 0.4129 \qquad \text{[using tables of standard normal cdf].}
\end{aligned}$$

(b) What is the expected value and variance of the stock price after 10 days.

Solution: Let $Y = \sum_{i=1}^{10} Z_i$. As the Z_i are independent $\mathsf{N}(0,1)$ random variables, Y has a $\mathsf{N}(0,10)$ distribution. Therefore,

$$\begin{aligned}
\mathbb{E}(X_{10}) &= \mathbb{E}(\exp(Y)) \\
&= \int_{-\infty}^\infty e^y \frac{1}{\sqrt{10}\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\times 10}\right) dy \\
&= \int_{-\infty}^\infty \frac{1}{\sqrt{20\pi}} \exp\left(-\frac{(y-10)^2}{20} + 5\right) dy \\
&= \exp(5)\left[\int_{-\infty}^\infty \frac{1}{\sqrt{20\pi}} \exp\left(-\frac{(y-10)^2}{20}\right) dy\right] \qquad \text{[integral is of pdf of a } \mathsf{N}(10,10)\text{]} \\
&= \exp(5).
\end{aligned}$$

To compute the variance

$$
\begin{aligned}
\mathbb{E}(X_{10}^2) &= \mathbb{E}(\exp(2Y)) \\
&= \int_{-\infty}^{\infty} e^{2y} \frac{1}{\sqrt{10}\sqrt{2\pi}} \exp\left(-\frac{y^2}{2 \times 10}\right) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{20\pi}} \exp\left(-\frac{(y-20)^2}{20} + 20\right) dy \\
&= \exp(20) \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{20\pi}} \exp\left(-\frac{(y-20)^2}{20}\right) dy\right] \quad \text{[integral is of pdf of a } \mathsf{N}(20,10)] \\
&= \exp(20).
\end{aligned}
$$

So

$$
\operatorname{Var}(X_{10}) = \exp(20) - (\exp(5))^2 = (\exp(20) - \exp(10)).
$$

3. Let $X_1, X_2, \ldots, X_n$ be a collection of n independent random variables with $\mathbb{E}X_i = \mu$ and $\operatorname{Var}(X_i) = \sigma^2$. Define the random variables

$$
\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.
$$

(a) What is the expected value and variance of $\bar{X}$?

Solution:

$$
\mathbb{E}[\bar{X})] = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu = \mu.
$$

$$
\operatorname{Var}(\bar{X}) = \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}.
$$

(b) Compute the covariance of X_1 and $\bar{X}$.

Solution:

$$
\begin{aligned}
\operatorname{Cov}(X_1, \bar{X}) &= \operatorname{Cov}\left(X_1, \frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n}\operatorname{Cov}\left(X_1, \frac{1}{n}X_i\right) \\
&= \frac{1}{n}\operatorname{Cov}(X_1, X_1) = \frac{1}{n}\operatorname{Var}(X_1) = \frac{\sigma^2}{n}.
\end{aligned}
$$

(c) What is the expected value of S^2?

Solution:

$$\begin{aligned}
\mathbb{E}\left[S^2\right] &= \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}\left[\left(X_i-\bar{X}\right)^2\right] \\
&= \frac{n}{n-1}\mathbb{E}\left[\left(X_1-\mu+\mu-\bar{X}\right)^2\right] \\
&= \frac{n}{n-1}\left(\mathbb{E}\left[(X_1-\mu)^2\right]-2\mathbb{E}\left[(X_1-\mu)(\bar{X}-\mu)\right]+\mathbb{E}\left[(\bar{X}-\mu)^2\right]\right) \\
&= \frac{n}{n-1}\left(\mathrm{Var}(X_1)-2\mathrm{Cov}(X_1,\bar{X})+\mathrm{Var}(\bar{X})\right) \\
&= \frac{n}{n-1}\left(\sigma^2-\frac{2}{n}\sigma^2+\frac{1}{n}\sigma^2\right) \\
&= \sigma^2.
\end{aligned}$$