STAT2203: Probability Models and Data Analysis for Engineering
Assignment 2

Due by 11:00 am on Tuesday the 24th of September, 2019
via the Electronic Assignment Submission System (62-225)

The marks for each question is indicate by the number in square brackets. There are a total of 12 marks for this assignment.

1. A continuous random variable X has probability density function

$$f_X(x) = c\exp(-2|x| + x) = \begin{cases} ce^{-x}, & x \geqslant 0 \\ ce^{3x}, & x \leqslant 0 \end{cases}$$

   (a) Determine the value of c. [1]
   Solution:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx = c\int_0^{\infty} e^{-x}dx + c\int_{-\infty}^{0} e^{3x}dx \\ &= c + \frac{c}{3} = \frac{4c}{3}. \end{aligned}$$

   So $c = 3/4$.

   (b) Determine the moment generating function $M_X(t)$ of X, remembering to state the valid range for t. [1]
   Solution:

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\ &= \frac{3}{4}\int_0^{\infty} e^{x(t-1)}dx + \frac{3}{4}\int_{-\infty}^{0} e^{x(t+3)}dx \\ &= \frac{3}{4}\left(\frac{1}{1-t} + \frac{1}{t+3}\right) = \frac{3}{3-2t-t^2} \qquad -3 < t < 1. \end{aligned}$$

   (c) Hence, or otherwise, determine the mean and variance of X. [2]
   Solution:

$$\begin{aligned} M'(t) &= 3(2+2t)(3-2t-t^2)^{-2} \\ \mathbb{E}[X] &= M'(0) = \frac{2}{3} \\ M''(t) &= 6(3-2t-t^2)^{-2} + 6(2+2t)^2(3-2t-t^2)^{-3} \\ \mathbb{E}[X^2] &= M''(0) = \frac{2}{3} + \frac{8}{9} = \frac{14}{9} \\ \mathrm{Var}(X) &= [X^2] - (\mathbb{E}[X])^2 = \frac{14}{9} - \frac{4}{9} = \frac{10}{9} \end{aligned}$$

   (d) Define the random variable $Y := X^4$. Give the probability density function for Y. [2]

Solution: $Y = X^4$ so for $y \geqslant 0$,

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(X^4 \leqslant y) = \mathbb{P}(-y^{1/4} \leqslant X \leqslant y^{1/4}) \\
&= F_X(y^1/4) - F_X(-y^{1/4}). \\
f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{1}{4} y^{-3/4} f_X(y^{1/4}) + \frac{1}{4} y^{-3/4} f_X(-y^{1/4}) \\
&= \frac{1}{4} y^{-3/4} \times \frac{3}{4} \exp(-y^{1/4}) + \frac{1}{4} y^{-3/4} \times \frac{3}{4} \exp(-3y^{1/4}) \\
&= \frac{3}{16} y^{-3/4} \left(\exp(-y^{1/4}) + \exp(-3y^{1/4}) \right).
\end{aligned}$$

2. Let $X_1, X_2, \ldots$ be a sequence of independent random variables, each with a $\mathsf{Geometric}(1/2)$ distribution. Let N be a random variable with a $\mathsf{Geometric}(1/3)$ distribution, independent of $X_1, X_2, \ldots$. Define the random variable

$$Y = \sum_{i=1}^{N} X_i,$$

where $Y = 0$ if $N = 0$. Determine the probability generating function of Y and identify its distribution. [2]

Solution: This is the type of problem study on page 62 of notes and in quiz 4. In general,

$$\begin{aligned}
G_Y(s) &= \mathbb{E}[s^Y] = \mathbb{E}[\mathbb{E}[s^Y | N]] = \mathbb{E}[\mathbb{E}[s^{\sum_{i=1}^N X_i} | N]] \\
\mathbb{E}[s^Y | N = n] &= \mathbb{E}[s^{\sum_{i=1}^N X_i} | N = n] = \prod_{i=1}^{n} \mathbb{E}[s^{X_i}] = G_X(s)^n \\
G_Y(s) &\mathbb{E}[G_X(s)^N] = G_N(G_X(s)).
\end{aligned}$$

As $X \sim \mathsf{Geometric}(1/2)$, $G_X(s) = \frac{\frac{1}{2}s}{1-\frac{1}{2}s}$ and $N \sim \mathsf{Geometric}(1/3)$ so $G_N(s) = \frac{\frac{1}{3}s}{1-\frac{1}{3}s}$. Therefore,

$$G_Y(s) = \frac{\frac{1}{3} \frac{\frac{1}{2}s}{1-\frac{1}{2}s}}{1 - \frac{2}{3} \frac{\frac{1}{2}s}{1-\frac{1}{2}s}} = \frac{\frac{1}{6}s}{1 - \frac{5}{6}s}.$$

So Y has a $\mathsf{Geometric}(1/6)$ distribution.

3. Suppose the random variable X has a $\mathsf{N}(2,3)$ distribution. Conditional on $\{X=x\}$, the random variable Y has a $\mathsf{N}(1+x,2)$ distribution.

(a) Detemine the probability that $X \geqslant 3$. [1]

Solution: $\mathbb{P}(X \geqslant 3) = \mathbb{P}\left(\frac{X-2}{\sqrt{3}} \geqslant \frac{3-2}{\sqrt{3}}\right) = \mathbb{P}(Z \geqslant \frac{1}{\sqrt{3}}) = 0.2818514.$

(b) Determine the marginal distribution of Y. [2]

Solution: Several approaches possible.

The most straightforward approach is with moment generating functions:

$$\begin{aligned}
\mathbb{E}[e^{tY}] &= \mathbb{E}[\mathbb{E}[e^{tY}|X]] \\
\mathbb{E}[e^{tY}|X=x] &= \exp((1+x)t+t^2) \\
\mathbb{E}[e^{tY}] &= \mathbb{E}[\exp(t+t^2 tX)] = \exp(t+t^2)\mathbb{E}[e^{tX}] \\
&= \exp(t+t^2)\exp(2t+3t^2/2) \\
&= \exp(3t+5t^2/2).
\end{aligned}$$

This is the moment generating function of a $\mathsf{N}(3,5)$ distribution.

Alternatively, one could write down the joint probability density function from which the marginal distribution of Y can be identified. The joint pdf of a bivariate normal is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\varrho^2}} \times$$
$$\exp\left(-\frac{1}{2(1-\varrho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\varrho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right).$$

In this form the marginal of Y is $\mathsf{N}(\mu_Y, \sigma_Y^2)$. So

$$\begin{aligned}
f_{X,Y}(x,y) &= f_X(x) f_{Y|X}(y|x) \\
&= \frac{1}{\sqrt{6\pi}}\exp\left(-\frac{1}{2}\frac{(x-2)^2}{3}\right) \times \frac{1}{\sqrt{4\pi}}\exp\left(-\frac{1}{2}\frac{(y-(1+x))^2}{2}\right) \\
&= \frac{1}{\pi\sqrt{24}}\exp\left(-\frac{1}{2}\left(\frac{(x-2)^2}{3} + \frac{(y-(1+x))^2}{2}\right)\right) \\
&= \frac{1}{\pi\sqrt{24}}\exp\left(-\frac{1}{2}\left(\frac{(x-2)^2}{3} + \frac{((y-3)-(x-2))^2}{2}\right)\right) \\
&= \frac{1}{\pi\sqrt{24}}\exp\left(-\frac{1}{2}\left(\frac{(x-2)^2}{(6/5)} - (x-2)(y-3) + \frac{(y-3)^2}{2}\right)\right)
\end{aligned}$$

Equating terms we see that $(1-\varrho^2)\sigma_X^2 = 6/5$ and $(1-\varrho^2)\sigma_X\sigma_Y = 2\varrho$. We already know that $\mu_X = 2$ and $\sigma_X^2 = 3$ so $\varrho = \sqrt{3/5}$, $\mu_Y = 3$ and $\sigma_Y^2 = 5$.

A third alternative is to recognise/state that Y must be marginally normal and so they only need to determine the mean and variance of Y.

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[1+X] = 1+\mathbb{E}[X] = 1+2 = 3\\
\mathbb{E}[Y^2] &= \mathbb{E}[\mathbb{E}[Y^2|X]] = \mathbb{E}[2+(1+X)^2] = 3+2\mathbb{E}[X]+\mathbb{E}[X^2]\\
&= 3+2\times 2+(3+2^2) = 14\\
\mathrm{Var}(Y) &= \mathbb{E}[Y^2]-(\mathbb{E}[Y])^2 = 14-3^2 = 5.
\end{aligned}$$

So $Y \sim \mathsf{N}(3,5)$.

(c) Determine the correlation between X and Y. [1]

Solution:

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]\\
\mathbb{E}[XY] &= \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = \mathbb{E}[X(1+X)]\\
&= \mathbb{E}[X]+\mathbb{E}[X^2] = 2+(3+2^2) = 9\\
\mathrm{Cov}(X,Y) &= 9-2\times 3 = 3\\
\mathrm{corr}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{3}{\sqrt{3\times 5}} = \sqrt{\frac{3}{5}}.
\end{aligned}$$

Total [12]