C

Practice Questions for Chapter 5

1. A continuous random variable X has pdf f given by,

$$f(x) = 3\,x^2, \quad x \in [0,1]\,.$$

   (a) Determine the cdf of X.
   (b) Calculate $\mathbb{P}(1/2 < X < 3/4)$.

2. An electrical component has a lifetime X (in years) with pdf

$$f(x) = c\,\mathrm{e}^{-x/2}, \quad x \geqslant 0, \qquad \text{and 0 otherwise.}$$

   (a) Calculate c.
   (b) What is the probability that the component is still functioning after 2 years?
   (c) What is the probability that the component is still functioning after 10 years, given it is still functioning after 7 years?

3. X has pdf

$$f(x) = 2\,x, \quad 0 \leqslant x \leqslant 1 \qquad \text{(and 0 otherwise).}$$

   Calculate $\mathbb{P}(1/4 < X < 3/4)$.

4. An electrical component has a lifetime X that is exponentially distributed with parameter $\lambda = 1/10$ per year. What is the probability the component is still alive after 5 years?

5. Let $X \sim \mathsf{Uniform}[0,1]$. What is the pdf of $Y = X^2$?

6. Let $X \sim \mathsf{Normal}(0,1)$, and $Y = 1 + 2X$. What is the pdf of Y?

7. Let $X \sim \mathsf{Normal}(0,1)$. Find $\mathbb{P}(X \leqslant 1.4)$ from the table of the $\mathsf{Normal}(0,1)$ distribution. Also find $\mathbb{P}(X > -1.3)$.

8. Let $Y \sim \mathsf{Normal}(1,4)$. Find $\mathbb{P}(Y \leqslant 3)$, and $\mathbb{P}(-1 \leqslant Y \leqslant 2)$.

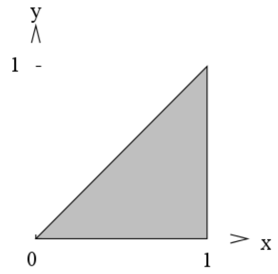
9. If $X \sim \mathsf{Exp}(1/2)$ what is the pdf of $Y = 1 + 2X$?

10. A continuous random variable X has probability density function

$$f(x) = \alpha\,x(1 - x^2)\,, \quad x \in [0,1]\,.$$

   Determine: (a) The value of α, (b) $\mathbb{P}(1/3 < X \leqslant 2/3)$, and (c) $\mathbb{P}(X \leqslant 1/3)$.

11. We select at random a point from the triangle $(0, 0) - (1, 0) - (1, 1)$; each point is equally likely. Let X be the x-coordinate and Y the y-coordinate of the point.



   (a) Determine the joint pdf of X and Y.
   (b) Determine the (marginal) pdf of Y.
   (c) Determine the expectation of Y.
   (d) Determine the conditional pdf of Y given $X = 1/2$.
   (e) Calculate $\mathbb{E}\,(Y \mid X = x)$ for all $x \in (0, 1)$.

12. The random variables X and Y have a joint probability density $f_{X,Y}$ given by

$$f_{X,Y}(x, y) = \begin{cases} c\,x\,y, & 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the constant c.
   (b) Are X and Y independent? Prove or disprove this.

13. Random variables $X_1, X_2, \ldots,$ are independent and have a standard normal distribution. Let $Y_n = X_1 + \cdots + X_n$, $n = 1, 2 \ldots$.

   (a) Determine $\mathbb{E}\,Y_5$ and $\mathrm{Var}(Y_{10})$.
   (b) Determine $\mathrm{Cov}(Y_5, Y_{10})$.
   (c) Give the distribution of $2X_1 + 6X_3 - 3Y_2$.
   (d) Give the joint distribution of $2X_2$ and $3X_1 + X_2$.

14. We draw a point (X, Y) uniformly at random from the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 4\}$.

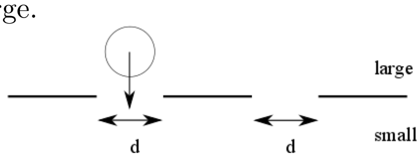
   (a) Determine the joint pdf of X and Y.
   (b) Determine the conditional pdf of Y given $X = 1$.

15. Let X have a uniform distribution on the interval $[-1, 2]$. Define $Y = 2X^2 - 5$. Derive the pdf of Y.

16. We draw at random a number in the interval [0,1] such that each number is "equally likely". Think of the *random generator* on you calculator.

   (a) Determine the probability that we draw a number less than 1/2.
   (b) What is the probability that we draw a number between 1/3 and 3/4?
   (c) Suppose we do the experiment two times (independently), giving us two numbers in [0,1]. What is the probability that the sum of these numbers is greater than 1/2? Explain your reasoning.

17. A sieve with diameter d is used to separate a large number of blueberries into two classes: small and large.



Suppose the diameters of the blueberries are normally distributed with an expectation $\mu = 1$ (cm) and a standard deviation $\sigma = 0.1$ (cm).

  (a) How large should the diameter of the sieve be, so that the proportion of large blueberries is 30%?

  (b) Suppose that the diameter is chosen such as in (a). What is the probability that out of 1000 blueberries, fewer than 280 end up in the "large" class?

18. Let $Y = \mathrm{e}^X$, where $X \sim \mathsf{Normal}(0, 1)$. Determine and sketch the pdf of Y.

19. Let $X \sim \mathsf{Normal}(1, 2)$, and $Y = 3 + 4X$. What is the pdf of Y?

20. Let $Y \sim \mathsf{Normal}(-1, 3)$. Find $\mathbb{P}(-2 \leqslant Y \leqslant 3)$.

21. If $X \sim \mathsf{Exp}(2)$ what is the pdf of $Y = 1 - 2X$? Sketch the graph.

22. A certain electronic system, that has to work under extreme environmental conditions, relies on the correct functioning of a certain silicon chip. A typical chip has an exponential lifetime with a mean of only 10 days.

  (a) What is the probability that a typical chip will still work after 20 days?

  (b) Calculate the probability that the system still works after 20 days.

23. Let $X \sim \mathsf{Exp}(1)$. Use the Moment Generating Functions to show that $\mathbb{E}[X^n] = n!$.

24. Using a $X \sim \mathsf{Uniform}[0, 1]$ random number, explain how to generate random numbers from the (a) $\mathsf{Uniform}[10, 15]$ distribution, (b) $\mathsf{Exp}(10)$ distribution.

25. If $X \sim \mathsf{Uniform}[0, 1]$, what is the distribution of $Y = 10 + 2X$?

26. Random variables X and Y have a joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} x \cos y & 0 < x < \pi/2, \quad 0 < y < x\,, \\ 0 & \text{otherwise.} \end{cases}$$

  (a) Determine the marginal probability density function of X.

  (b) Determine the conditional probability density function of Y given $X = x$.

  (c) Evaluate $\mathbb{P}(X + Y \geqslant \pi/2)$.

27. Consider the random experiment where we draw independently n numbers from the interval [0,1]; each number in [0,1] being equally likely to be drawn. Let the independent and $\mathsf{Uniform}[0, 1]$-distributed random variables $X_1, \ldots, X_n$ represent the numbers that are to be drawn.

  (a) Let M be the smallest of the n numbers, and $\bar{X}$ the average of the n numbers. Express M and $\bar{X}$ in terms of $X_1, \ldots, X_n$.

  (b) Determine the pdf of M.

(c) Give the expectation and variance of $\bar{X}$.

28. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables, with cdfs $F_1, F_2, \ldots, F_n$, respectively. Express the cdf of $M = \max(X_1, \ldots, X_n)$ in terms of the $\{F_i\}$.

29. A random vector (X, Y) has joint pdf f, given by

$$f(x, y) = 2\,\mathrm{e}^{-x-2y}, \quad x > 0, y > 0 \,.$$

(a) Calculate $\mathbb{E}[XY]$.

(b) Calculate the covariance of $X + Y$ and $X - Y$.

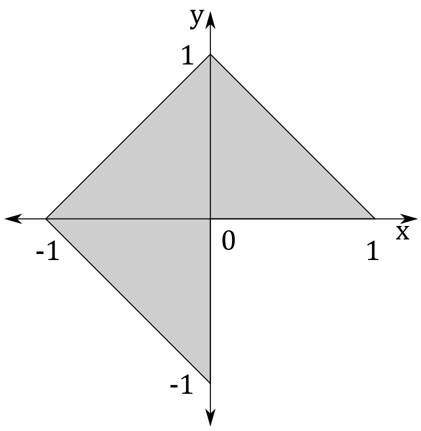
30. Let X have a uniform distribution on the interval $[1, 3]$. Define $Y = X^2 - 4$. Derive the probability density function (pdf) of Y. Make sure you also specify where this pdf is zero.

31. The thickness of a printed circuit board is required to lie between the specification limits of 0.150 - 0.004 and 0.150 + 0.004 cm. A machine produces circuit boards with a thickness that is normally distributed with mean 0.151 cm and standard deviation 0.003 cm.

(a) What is the probability that the thickness X of a circuit board which is produced by this machine falls within the specification limits?

(b) Now consider the mean thickness $\bar{X} = (X_1 + \cdots + X_{25})/25$ for a batch of 25 circuit boards. What is the probability that this batch mean will fall within the specification limits? Assume that $X_1, \ldots, X_{25}$ are independent random variables with the same distribution as X above.

32. We draw a random vector (X, Y) uniformly from the diamond $(-1, 0)$–$(0, 1)$–$(1, 0)$–$(0, -1)$ *without* its lower-right corner (see figure).



(a) Write down the joint pdf of X and Y, clearly specifying where it is zero.

(b) Determine the marginal pdf of Y.

(c) Determine the conditional pdf of X given $Y = -1/4$.

D

Practice Questions For Chapter 5 (Solutions)

1. A continuous random variable X has pdf f given by,

$$f(x) = 3\,x^2, \quad x \in [0,1]\ .$$

(a) Determine the cdf of X.

(b) Calculate $\mathbb{P}(1/2 < X < 3/4)$.

Solution:

(a)

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,\mathrm{d}u = \begin{cases} 0, & x < 0, \\ x^3, & x \in [0,1], \\ 1, & x > 1. \end{cases}$$

(b)

$$\begin{aligned} \mathbb{P}(1/2 < X < 3/4) &= \mathbb{P}(1/2 < X \leqslant 3/4) \\ &= \mathbb{P}(X \leqslant 3/4) - \mathbb{P}(X \leqslant 1/2) \\ &= F_X(3/4) - F_X(1/2) \\ &= \left(\frac{3}{4}\right)^3 - \left(\frac{1}{2}\right)^3 = \frac{3^3 - 2^3}{4^3} = \frac{27-8}{64} \\ &= \frac{19}{64} = 0.296875. \end{aligned}$$

2. An electrical component has a lifetime X (in years) with pdf

$$f(x) = c\,\mathrm{e}^{-x/2}, \quad x \geqslant 0, \qquad \text{and 0 otherwise.}$$

(a) Calculate c.

(b) What is the probability that the component is still functioning after 2 years?

(c) What is the probability that the component is still functioning after 10 years, given it is still functioning after 7 years?

Solution:

(a)

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^{\infty} c \cdot \mathrm{e}^{-x/2}\,\mathrm{d}x = 2c \Rightarrow c = \frac{1}{2}.$$

(b)

$$\mathbb{P}(X > 2) = \int_2^\infty \frac{1}{2}\mathrm{e}^{-x/2}\,\mathrm{d}x = \mathrm{e}^{-(2/2)} = \mathrm{e}^{-1}.$$

(c)

$$\begin{aligned}\mathbb{P}(X > 10|X > 7) &= \frac{\mathbb{P}(\{X > 10\} \cap \{X > 7\})}{\mathbb{P}(X > 7)} = \frac{\mathbb{P}(X > 10)}{\mathbb{P}(X > 7)} \\ &= \frac{\mathrm{e}^{-10/2}}{\mathrm{e}^{-7/2}} = \mathrm{e}^{-3/2}.\end{aligned}$$

Alternatively, we may recognise that X is an $\mathsf{Exp}(1/2)$ *random variable, so the Memoryless Property says*

$$\mathbb{P}(X > 10|X > 7) = \mathbb{P}(X > 3),$$

and the answer follows immediately.

3. X has pdf
$$f(x) = 2\,x, \quad 0 \leqslant x \leqslant 1 \qquad \text{(and 0 otherwise)}.$$
Calculate $\mathbb{P}(1/4 < X < 3/4)$.

Solution: Firstly,

$$F(x) = \mathbb{P}(X \leqslant x) = \int_{-\infty}^x f(u)\,\mathrm{d}u = \begin{cases} 0, & x < 0, \\ x^2, & 0 \leqslant x \leqslant 1, \\ 1, & x > 1. \end{cases}$$

Then

$$\begin{aligned}\mathbb{P}\left(\frac{1}{4} < X < \frac{3}{4}\right) &= \mathbb{P}\left(\frac{1}{4} < X \leqslant \frac{3}{4}\right) \\ &= \mathbb{P}\left(X \leqslant \frac{3}{4}\right) - \mathbb{P}\left(X \leqslant \frac{1}{4}\right) = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2 \\ &= \frac{3^2 - 1^2}{4^2} = \frac{9-1}{16} = \frac{8}{16} = \frac{1}{2}.\end{aligned}$$

4. An electrical component has a lifetime X that is exponentially distributed with parameter $\lambda = 1/10$ per year. What is the probability the component is still alive after 5 years?

Solution:

$$\mathbb{P}(X > 5) = \mathrm{e}^{-5/10} = \mathrm{e}^{-1/2} \approx 0.6065.$$

5. Let $X \sim \mathsf{Uniform}[0,1]$. What is the pdf of $Y = X^2$? *Solution:*

$$f_X(x) = \begin{cases} 1, & x \in [0,1], \\ 0, & x \notin [0,1], \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & x \in [0,1], \\ 1, & x > 1. \end{cases}$$

$X \in [0,1] \Rightarrow Y = X^2 \in [0,1]$.

$$\begin{aligned} F_Y(Y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(X^2 \leqslant y) \quad \textit{(since } X \geqslant 0\textit{)} \\ &= \mathbb{P}(X \leqslant \sqrt{y}) = \begin{cases} 0, & y < 0, \\ \sqrt{y}, & y \in [0,1], \\ 1, & y > 1, \end{cases} \\ \Rightarrow f_Y(y) &= \begin{cases} \frac{1}{2} y^{-1/2}, & y \in [0,1], \\ 0, & y \notin [0,1]. \end{cases} \end{aligned}$$

6. Let $X \sim \mathsf{Normal}(0,1)$, and $Y = 1 + 2X$. What is the pdf of Y? *Solution:* $X \sim \mathsf{N}(0,1)$, $Y = 1 + 2X \Rightarrow Y \sim \mathsf{N}(1,4)$

$$f_Y(y) = \frac{1}{2 \cdot \sqrt{2\pi}} \exp\left(\frac{-1}{8}(y-1)^2\right), \quad y \in \mathbb{R}.$$

7. Let $X \sim \mathsf{Normal}(0,1)$. Find $\mathbb{P}(X \leqslant 1.4)$ from the table of the $\mathsf{Normal}(0,1)$ distribution. Also find $\mathbb{P}(X > -1.3)$. *Solution:* $\mathbb{P}(X \leqslant 1.4) \approx 0.9192$

$$\begin{aligned} \mathbb{P}(X > -1.3) &= \mathbb{P}(X \leqslant 1.3) \qquad \textit{(symmetry)} \\ &\approx 0.9032. \end{aligned}$$

8. Let $Y \sim \mathsf{Normal}(1,4)$. Find $\mathbb{P}(Y \leqslant 3)$, and $\mathbb{P}(-1 \leqslant Y \leqslant 2)$. *Solution:* $Y \sim \mathsf{Normal}(1,4)$,

$$\begin{aligned} &\mathbb{P}(Y \leqslant 3) = \mathbb{P}\left(Z \leqslant \frac{3-1}{2}\right) = \mathbb{P}(Z \leqslant 1) \qquad \textit{(where } Z \sim \mathsf{Normal}(0,1)\textit{)} \\ &\Rightarrow \mathbb{P}(Y \leqslant 3) \approx 0.8413. \end{aligned}$$

$$\begin{aligned} \mathbb{P}(-1 \leqslant Y \leqslant 2) &= \mathbb{P}(-1 < Y \leqslant 2) = \mathbb{P}(Y \leqslant 2) - \mathbb{P}(Y \leqslant -1) \\ &= \mathbb{P}\left(Z \leqslant \frac{2-1}{2}\right) - \mathbb{P}\left(Z \leqslant \frac{-1-1}{2}\right) \\ &= \mathbb{P}(Z \leqslant 0.5) - \mathbb{P}(Z \leqslant -1) \\ &= \mathbb{P}(Z \leqslant 0.5) - \mathbb{P}(Z \geqslant +1) \quad \textit{(symmetry)} \\ &= \mathbb{P}(Z \leqslant 0.5) - (1 - \mathbb{P}(Z \leqslant 1)) \\ &\approx 0.6915 - (1 - 0.8413) \\ &= 0.6915 - 0.1587 \\ &= 0.5328. \end{aligned}$$

9. If $X \sim \mathsf{Exp}(1/2)$ what is the pdf of $Y = 1 + 2X$? Sketch the graph.

Solution:

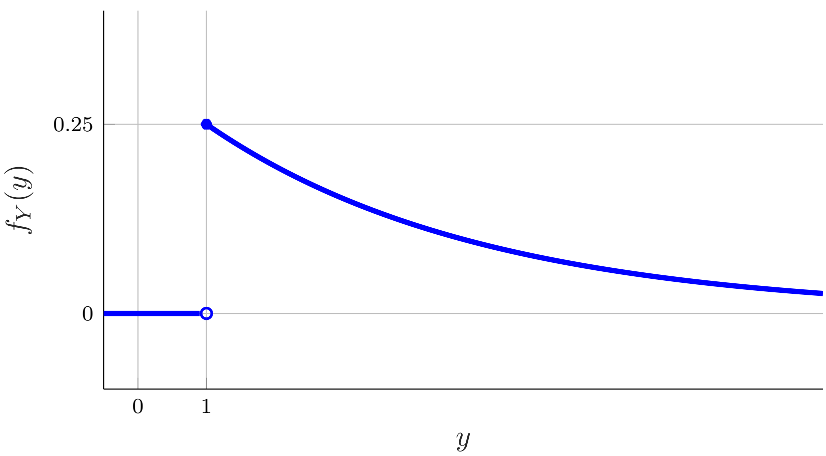
$$\mathbb{P}(X \leqslant x) = \begin{cases} 0, & x < 0, \\ 1 - \mathrm{e}^{-x/2}, & x \geqslant 0, \end{cases}$$

$$\mathbb{P}(Y \leqslant y) = \mathbb{P}(1 + 2X \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-1}{2}\right)$$

$$= \begin{cases} 0, & y > 1, \\ 1 - \mathrm{e}^{-\left(\frac{y-1}{4}\right)}, & y \geqslant 1, \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{4}\mathrm{e}^{-\left(\frac{y-1}{4}\right)}, & y \geqslant 1, \\ 0, & \textit{otherwise}. \end{cases}$$

PDF f_Y



10. A continuous random variable X has probability density function

$$f(x) = \alpha\, x(1 - x^2)\,, \quad x \in [0, 1]\,.$$

Determine: (a) The value of α, (b) $\mathbb{P}(1/3 < X \leqslant 2/3)$, and (c) $\mathbb{P}(X \leqslant 1/3)$.

Solution:

(a)

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^1 \alpha x(1 - x^2)\,\mathrm{d}x$$

$$= \alpha \cdot \left\{ \left[\frac{x^2}{2}\right]_{x=0}^{x=1} - \left[\frac{x^4}{4}\right]_{x=0}^{x=1} \right\}$$

$$= \alpha \cdot \left\{ \frac{1}{2} - \frac{1}{4} \right\} = \frac{\alpha}{4}$$

$$\Rightarrow \alpha = 4.$$

(b)

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u = \begin{cases} 0, & x < 0, \\ \int_0^x 4u(1-u^2)\,\mathrm{d}u, & x \in [0,1], \\ 1, & x > 1, \end{cases}$$
$$= \begin{cases} 0, & x < 0, \\ 4\left\{\frac{x^2}{2} - \frac{x^4}{4}\right\}, & x \in [0,1], \\ 1, & x > 1, \end{cases}$$
$$= \begin{cases} 0, & x < 0, \\ x^2(2-x^2), & x \in [0,1], \\ 1, & x > 1, \end{cases}$$

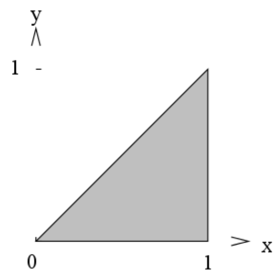
so

$$\begin{aligned} \mathbb{P}(1/3 < X \leqslant 2/3) &= F(2/3) - F(1/3) \\ &= \left(\frac{2}{3}\right)^2 \left(2 - \left(\frac{2}{3}\right)^2\right) - \left(\frac{1}{3}\right)^2 \left(2 - \left(\frac{1}{3}\right)^2\right) \\ &= \frac{13}{27} \approx 0.481481. \end{aligned}$$

(c)

$$\mathbb{P}(X \leqslant 1/3) = F(1/3) = \left(\frac{1}{3}\right)^2 \left(2 - \left(\frac{1}{3}\right)^2\right) = \frac{17}{81} \approx 0.209877.$$

11. We select at random a point from the triangle $(0,0) - (1,0) - (1,1)$; each point is equally likely. Let X be the x-coordinate and Y the y-coordinate of the point.



(a) Determine the joint pdf of X and Y.
(b) Determine the (marginal) pdf of Y.
(c) Determine the expectation of Y.
(d) Determine the conditional pdf of Y given $X = 1/2$.
(e) Calculate $\mathbb{E}(Y \mid X = x)$ for all $x \in (0,1)$.

Solution:

(a) Area of triangle is $1/2$*, so the joint pdf is*

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x, \\ 0, & \textit{otherwise}, \end{cases}$$

or equivalently,

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \leqslant y \leqslant 1,\ y \leqslant x \leqslant 1, \\ 0, & \textit{otherwise.} \end{cases}$$

(b)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \begin{cases} \int_y^1 2\,\mathrm{d}x = 2(1-y), & y \in [0,1], \\ 0, & y \notin [0,1]. \end{cases}$$

That is,

$$f_Y(y) = \begin{cases} 2(1-y), & 0 \leqslant y \leqslant 1, \\ 0, & \textit{otherwise.} \end{cases}$$

(c)

$$\begin{aligned} \mathbb{E}Y &= \int_0^1 y \cdot 2(1-y)\,\mathrm{d}y = \int_0^1 2y\,\mathrm{d}y - \int_0^1 2y^2\,\mathrm{d}y \\ &= \left[y^2\right]_{y=0}^{y=1} - \left[\frac{2}{3}y^3\right]_{y=0}^{y=1} \\ &= 1 - \frac{2}{3} = \frac{1}{3}. \end{aligned}$$

(d) Marginal of X:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \begin{cases} \int_0^x 2\,\mathrm{d}y = 2x, & x \in [0,1], \\ 0, & \textit{otherwise,} \end{cases}$$

i.e.

$$f_X(x) = \begin{cases} 2x, & x \in [0,1], \\ 0, & \textit{otherwise,} \end{cases}$$

so the conditional pmf of Y given $X = 1/2$ is

$$\frac{f(1/2,y)}{f_X(1/2)} = \begin{cases} \frac{2}{2\cdot(1/2)} = 2, & 0 \leqslant y \leqslant \frac{1}{2}, \\ 0, & \textit{otherwise.} \end{cases}$$

That is,

$$f_{Y|X=1/2}(y|1/2) = \begin{cases} 2, & 0 \leqslant y \leqslant \frac{1}{2}, \\ 0, & \textit{otherwise.} \end{cases}$$

*[Alternatively, **directly**, conditional is **still** uniform on the intersection of the triangle and the line $X = 1/2$.]*

(e)

$$f_{Y|X=x}(y|x) = \begin{cases} \frac{1}{x}, & 0 \leqslant y \leqslant x, \\ 0, & \textit{otherwise,} \end{cases} \quad \textit{for } x \in (0,1).$$

Therefore

$$\mathbb{E}[Y|X=x] = \int_0^x y \cdot \frac{1}{2}\,\mathrm{d}y = \frac{1}{x} \cdot \left[\frac{y^2}{2}\right]_{y=0}^{Y=x} = \frac{1}{x} \cdot \left[\frac{x^2}{2}\right] = \frac{x}{2}.$$

[Alternatively, directly, observe that $(Y|X=x)$ is $\mathsf{U}(0,x) \Rightarrow \mathbb{E}[Y|X=x] = x/2$.]

12. The random variables X and Y have a joint probability density $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} c\,x\,y, & 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Are X and Y independent? Prove or disprove this.

(c) Calculate $\mathbb{P}(2X > Y)$.

Solution:

(a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = c\int_0^1\int_0^1 xy\,\mathrm{d}x\,\mathrm{d}y \\ &= c\cdot\int_0^1 x\,\mathrm{d}x\int_0^1 y\,\mathrm{d}y = c\cdot\frac{1}{2}\cdot\frac{1}{2} = c\cdot\frac{1}{4} \\ \Rightarrow c &= 4. \end{aligned}$$

(b) Marginals:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \begin{cases} \int_0^1 4xy\,\mathrm{d}y = 2x, & x \in [0,1], \\ 0, & \textit{otherwise,} \end{cases} \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \begin{cases} \int_0^1 4xy\,\mathrm{d}x = 2y, & y \in [0,1], \\ 0, & \textit{otherwise.} \end{cases} \end{aligned}$$

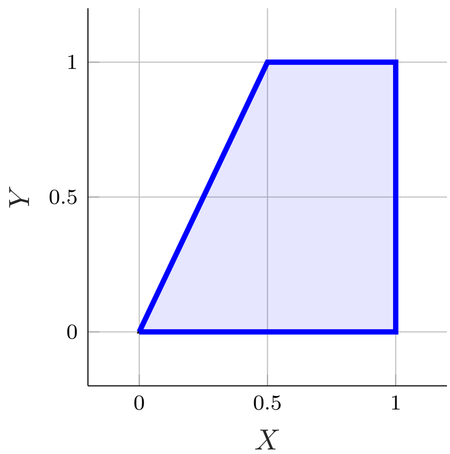
Since $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ for all $(x,y) \in \mathbb{R}^2$, X and Y are ***independent****.*

(c)

$$\mathbb{P}(2X > Y) = \mathbb{P}(2X - Y > 0).$$

Draw a picture of the region of integration:

The Event $\{2X > Y\}$



$$\begin{aligned}
\mathbb{P}(2X > Y) &= 1 - \mathbb{P}(2X \leqslant Y) \\
&= 1 - \int_0^1 \int_0^{y/2} 4xy \, \mathrm{d}x \, \mathrm{d}y \\
&= 1 - \int_0^1 4y \left[\int_0^{y/2} x \, \mathrm{d}x \right] \mathrm{d}y \\
&= 1 - \int_0^1 4y \cdot \frac{(y/2)^2}{2} \, \mathrm{d}y \\
&= 1 - \int_0^1 \frac{y^3}{2} \, \mathrm{d}y \\
&= 1 - \left[\frac{y^4}{8} \right]_{y=0}^{y=1} \\
&= 1 - \frac{1}{8} = \frac{7}{8}.
\end{aligned}$$

13. Random variables $X_1, X_2, \ldots,$ are independent and have a standard normal distribution. Let $Y_n = X_1 + \cdots + X_n$, $n = 1, 2 \ldots$.

   (a) Determine $\mathbb{E}\, Y_5$ and $\mathrm{Var}(Y_{10})$.

   (b) Determine $\mathrm{Cov}(Y_5, Y_{10})$.

   (c) Give the distribution of $2X_1 + 6X_3 - 3Y_2$.

   (d) Give the joint distribution of $2X_2$ and $3X_1 + X_2$.

   Solution:

   (a)

$$\begin{aligned}
\mathbb{E}[Y_5] &= \mathbb{E}[X_1 + \cdots + X_5] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_5] \\
&= 0 + \cdots + 0 = 0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(Y_{10}) &= \mathrm{Var}(X_1 + \cdots + X_{10}) \\
&= \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_{10}) && \textit{(from independence)} \\
&= 1 + \cdots + 1 = 10.
\end{aligned}$$

   (b)

$$\begin{aligned}
\mathrm{Cov}(Y_5, Y_{10}) &= \mathrm{Cov}(Y_5, Y_5 + X_6 + \cdots + X_{10}) \\
&= \mathrm{Cov}(Y_5, Y_5) + \mathrm{Cov}(Y_5, X_6 + \cdots + X_{10}) \\
&= \mathrm{Var}(Y_5) + 0 && \textit{(by independence)} \\
&= 5.
\end{aligned}$$

   (c)

$$\begin{aligned}
2X_1 + 6X_3 - 3Y_2 &= 2X_1 + 6X_3 - 3(X_1 + X_2) \\
&= -X_1 - 3X_2 + 6X_3 \sim \mathsf{N}\left(0, (-1)^2 + (-3)^2 + 6^2\right) \equiv \mathsf{N}(0, 46).
\end{aligned}$$

(d) Let $U = 2X_2$ and $V = 3X_1 + X_2$. Then

$$\begin{pmatrix} U \\ V \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathsf{N}(\mu, \Sigma),$$

where

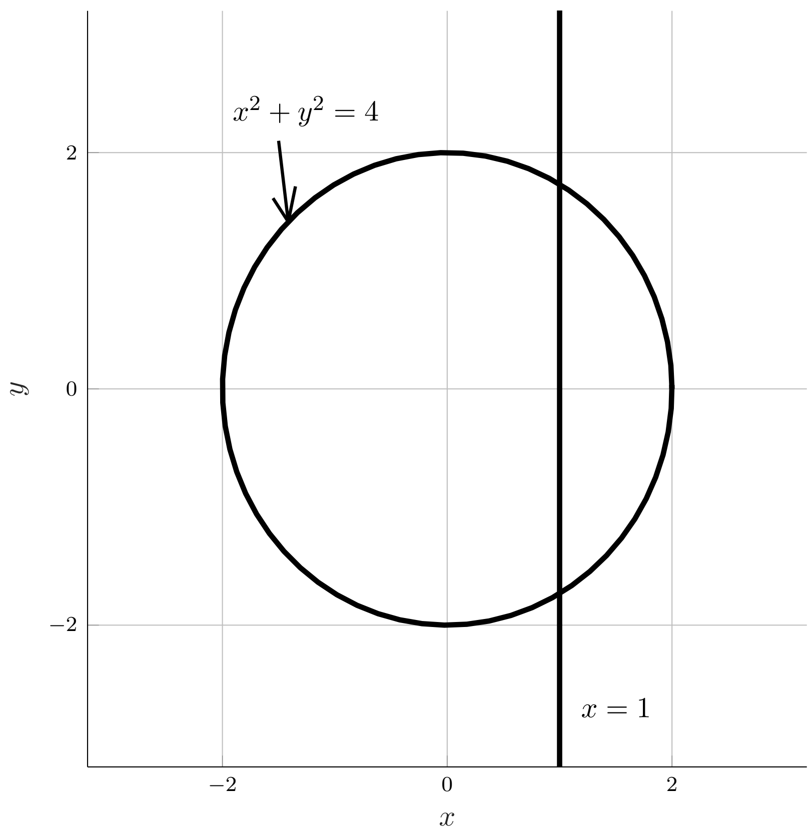
$$\begin{aligned} \mu &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{and} \\ \Sigma &= AA^T \\ &= \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 2 \\ 2 & 10 \end{pmatrix}, \\ \Rightarrow \begin{pmatrix} U \\ V \end{pmatrix} &\sim \mathsf{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 & 2 \\ 2 & 10 \end{pmatrix} \right). \end{aligned}$$

14. We draw at random a point (X, Y) from the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 4\}$.
    (a) Determine the joint pdf of X and Y.
    (b) Determine the conditional pdf of Y given $X = 1$.

Solution:

Visually:

The Region $\{x^2 + y^2 \leqslant 4\}$

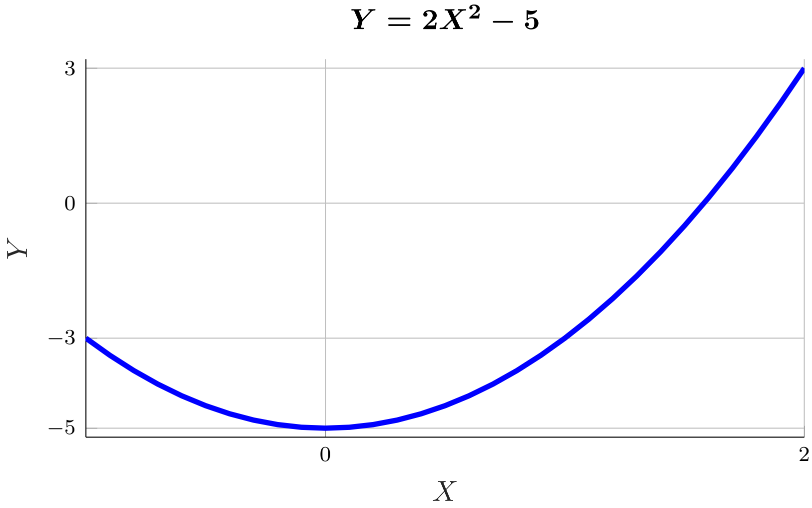


Given $X = 1$, clearly uniform. Intersection of line segment with disc has $y \in (-\sqrt{3}, \sqrt{3})$, so

$$f_{Y|X=1}(y|1) = \begin{cases} \frac{1}{2\sqrt{3}}, & y \in (-\sqrt{3}, \sqrt{3}), \\ 0, & \textit{otherwise.} \end{cases}$$

15. Let X have a uniform distribution on the interval $[-1, 2]$. Define $Y = 2X^2 - 5$. Derive the pdf of Y.

Solution: $X \sim \mathsf{U}[-1, 2]$ and $Y = 2X^2 - 5$. Find cdf of Y, beginning with X:



$$F_X(x) = \begin{cases} 0, & x < -1, \\ \frac{(x+1)}{3}, & -1 \leqslant x \leqslant 2, \\ 1, & x > 2. \end{cases}$$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(2X^2 - 5 \leqslant y) \\ &= \mathbb{P}\left(X^2 \leqslant \frac{y+5}{2}\right) = \mathbb{P}\left(-\sqrt{\frac{y+5}{2}} \leqslant X \leqslant \sqrt{\frac{y+5}{2}}\right) \\ &= \mathbb{P}\left(X \leqslant \sqrt{\frac{y+5}{2}}\right) - \mathbb{P}\left(X \leqslant -\sqrt{\frac{y+5}{2}}\right) \\ &= \begin{cases} 0, & y < -5, \\ \frac{1}{3}\left(\sqrt{\frac{y+5}{2}} + 1\right) - \frac{1}{3}\left(-\sqrt{\frac{y+5}{2}} + 1\right), & -5 \leqslant y \leqslant -3, \\ \frac{1}{3}\left(\sqrt{\frac{y+5}{2}} + 1\right), & -3 \leqslant y \leqslant 3, \\ 1 & y > 3, \end{cases} \end{aligned}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{3\sqrt{2(y+5)}}, & -5 \leqslant y \leqslant -3, \\ \frac{1}{6\sqrt{2(y+5)}}, & -3 \leqslant y \leqslant 3, \\ 0, & \textit{otherwise.} \end{cases}$$

16. We draw at random a number in the interval [0,1] such that each number is "equally likely". Think of the *random generator* on you calculator.

    (a) Determine the probability that we draw a number less than 1/2.

    (b) What is the probability that we draw a number between 1/3 and 3/4?

    (c) Suppose we do the experiment two times (independently), giving us two numbers in [0,1]. What is the probability that the sum of these numbers is greater than 1/2? Explain your reasoning.

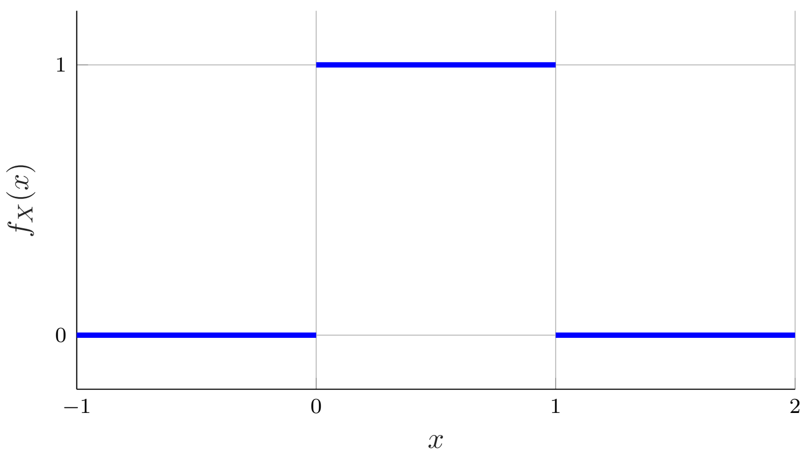
Solution:

(a) 1/2.

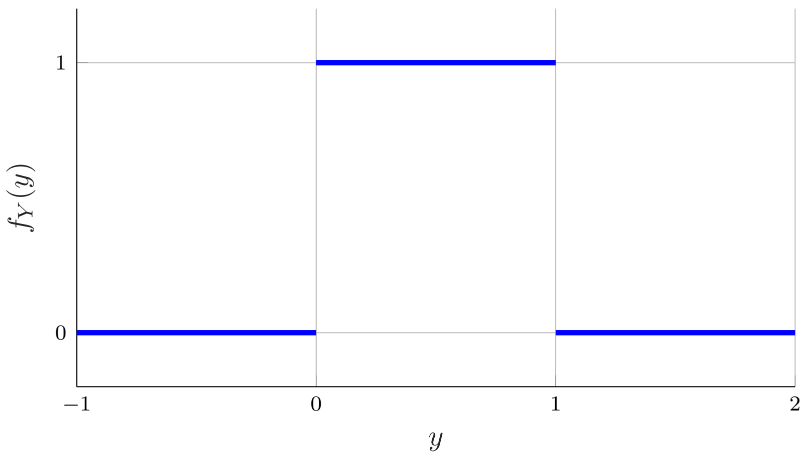
(b) $\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12} = \frac{5}{12}$.

(c) Let $X, Y \sim \mathsf{U}(0,1)$ *be independent, and let* $Z = X + Y$.

PDF f_X



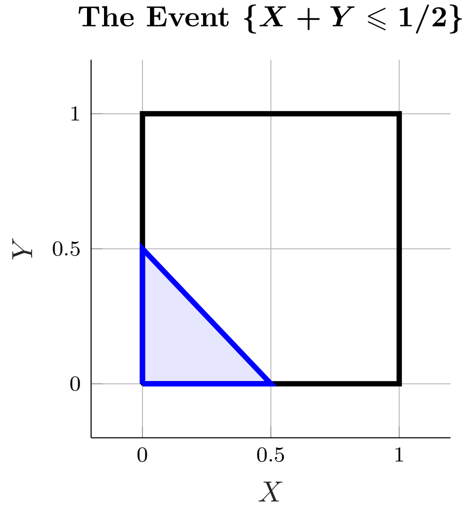
PDF f_Y



The joint pdf is $f_{X,Y} = f_X f_Y$*, by independence, so*

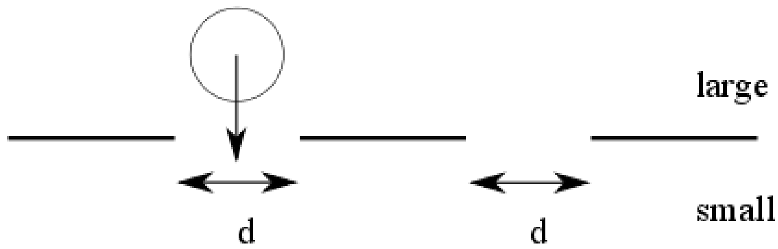
$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 1, & (x,y) \in [0,1]^2, \\ 0, & \textit{otherwise.} \end{cases}$$

The region when $X + Y < 1/2$:



$$\begin{aligned}\mathbb{P}(Z > 1/2) &= 1 - \mathbb{P}(Z \leqslant 1/2) \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{7}{8}.\end{aligned}$$

17. A sieve with diameter d is used to separate a large number of blueberries into two classes: small and large.



Suppose the diameters of the blueberries are normally distributed with an expectation $\mu = 1$ (cm) and a standard deviation $\sigma = 0.1$ (cm).

(a) How large should the diameter of the sieve be, so that the proportion of large blueberries is 30%?

(b) Suppose that the diameter is chosen such as in (a). What is the probability that out of 1000 blueberries, fewer than 280 end up in the "large" class?

Solution: Each blueberry has diameter $D \sim \mathsf{N}(1, 0.1^2)$.

(a) Want to find the diameter d such that

$$\begin{aligned} &\mathbb{P}(D > d) = 0.3 \\ \iff &\mathbb{P}(D \leqslant d) = 0.7 \\ \iff &\mathbb{P}\left(Z \leqslant \frac{d-1}{0.1}\right) = 0.7.\end{aligned}$$

Now $\mathbb{P}(Z \leqslant 0.52) \approx 0.6985$ *from table, and* $\mathbb{P}(Z \leqslant 0.53) \approx 0.7019$. *From these,*

$$\frac{d-1}{0.1} = 0.52 \Rightarrow d = 1.052,$$
$$\frac{d-1}{0.1} = 0.53 \Rightarrow d = 1.053,$$

so d *should be between* 1.052 *to* $1.053cm$ *to ensure about* 30% *of blueberries fall in the "large" class.*

(b) *Each blueberry falling in large class with probability* 0.3 *independent of all others. Number in large class is*

$$\begin{aligned} X &\sim \mathsf{Bin}(1000, 0.3) \sim_{approx} \mathsf{N}(300, 210) \\ \Rightarrow \mathbb{P}(X < 280) &\approx \mathbb{P}\left(Z < \frac{280-300}{\sqrt{210}}\right) \\ &= \mathbb{P}\left(Z < -2 \times \sqrt{\frac{10}{21}}\right) \approx \mathbb{P}(Z < -1.38013) \\ &\approx \mathbb{P}(Z > 1.38) \qquad \textit{(symmetry)} \\ &= 1 - \mathbb{P}(Z \leqslant 1.38) \\ &= 1 - 0.9162 = 0.0838. \end{aligned}$$

18. Let $Y = \mathrm{e}^X$, where $X \sim \mathsf{Normal}(0,1)$. Determine and sketch the pdf of Y.

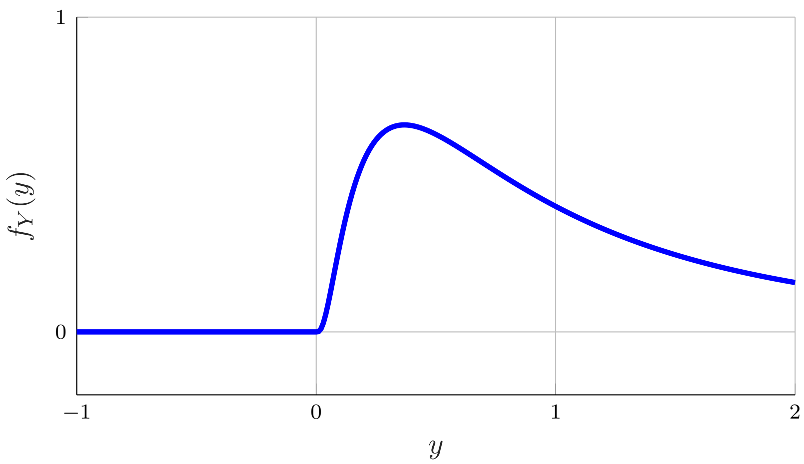
Solution: If $x \in \mathbb{R}$ *then* $y = \mathrm{e}^x \in (0, \infty)$.

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(\mathrm{e}^X \leqslant y) = \mathbb{P}(X \leqslant \ln(y)) = F_X(\ln(y)), \quad y \in (0,\infty),$$

so

$$f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln(y)), & y \in (0,\infty), \\ 0, & \textit{otherwise}, \end{cases}$$
$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{y\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}(\ln(y))^2}, & y \in (0,\infty), \\ 0, & \textit{otherwise}. \end{cases}$$

PDF f_Y



19. Let $X \sim \mathsf{Normal}(1, 2)$, and $Y = 3 + 4X$. What is the pdf of Y?

    Solution: $\mathbb{E}Y = 3 + 4 = 7$ *and* $\mathrm{Var}(Y) = 4^2 \times 2 = 32$*, so* $Y \sim \mathsf{N}(7, 32)$*. Therefore*

$$f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 32}} \mathrm{e}^{\frac{-1}{64}(y-7)^2}, \quad y \in \mathbb{R}.$$

20. Let $Y \sim \mathsf{Normal}(-1, 3)$. Find $\mathbb{P}(-2 \leqslant Y \leqslant 3)$.

    Solution:

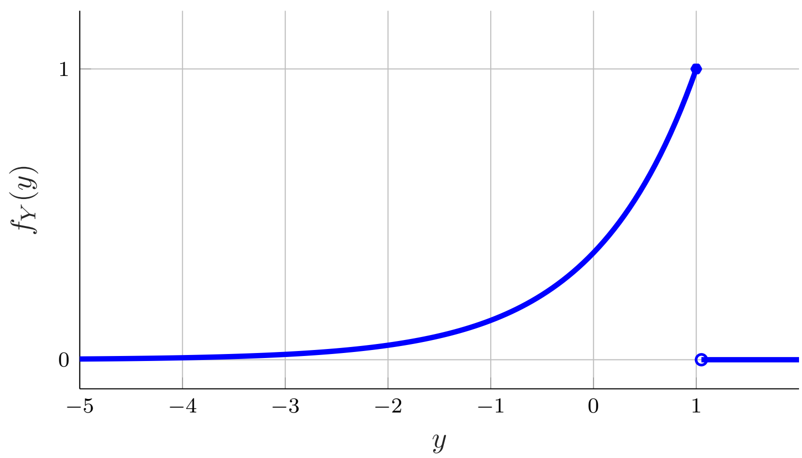
$$\begin{aligned}
\mathbb{P}(-2 \leqslant Y \leqslant 3) &= \mathbb{P}(Y \leqslant 3) - \mathbb{P}(Y \leqslant -2) \\
&= \mathbb{P}(Z \leqslant 4/\sqrt{3}) - \mathbb{P}(Z \leqslant -1/\sqrt{3}) \qquad \textit{(where } Z \sim \mathsf{N}(0,1)\textit{)} \\
&\approx \mathbb{P}(Z \leqslant 2.31) - (1 - \mathbb{P}(Z \leqslant 0.58)) \\
&\approx 0.9896 - (1 - 0.7190) = 0.7086.
\end{aligned}$$

21. If $X \sim \mathsf{Exp}(2)$ what is the pdf of $Y = 1 - 2X$? Sketch the graph.

    Solution: $X \leqslant 0 \Rightarrow Y \leqslant 1$*, and*

$$\begin{aligned}
F_X(x) &= \begin{cases} 0, & x < 0, \\ 1 - \mathrm{e}^{-2x}, & x \leqslant 0, \end{cases} \\
F_Y(y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(1 - 2X \leqslant y) \\
&= \mathbb{P}\left(-X \leqslant \frac{y-1}{2}\right) = \mathbb{P}\left(X \geqslant \frac{1-y}{2}\right) = \mathrm{e}^{-2(\frac{1-y}{2})} \\
&= \begin{cases} \mathrm{e}^{-(1-y)}, & y \leqslant 1, \\ 0, & y > 1, \end{cases} \\
\Rightarrow f_Y(y) &= \begin{cases} \mathrm{e}^{-(1-y)}, & y \leqslant 1, \\ 0, & \textit{otherwise.} \end{cases}
\end{aligned}$$

PDF f_Y



22. A certain electronic system, that has to work under extreme environmental conditions, relies on the correct functioning of a certain silicon chip. A typical chip has an exponential lifetime with a mean of only 10 days.

To increase the reliability of the system, 20 chips are put in parallel. As long as one of the chips works, the system functions.

(a) What is the probability that a typical chip will still work after 20 days?

(b) Calculate the probability that the system still works after 20 days.

Solution: Let X_i be the lifetime of the i^{th} chip. Then $X_i \sim \mathsf{Exp}(1/10)$, $i = 1, \ldots, 20$.

(a)

$$\mathbb{P}(X_1 > 20) = \mathrm{e}^{-20/10} = \mathrm{e}^{-2} \approx 0.1353.$$

(b) System works if at least one chip still alive.

$$\begin{aligned}
&\mathbb{P}(\{X_1 > 20\} \cup \cdots \cup \{X_{20} > 20\}) \\
&\qquad = 1 - \mathbb{P}(\{X_1 \leqslant 20\} \cap \cdots \cap \{X_{20} \leqslant 20\}) \\
&\qquad = 1 - \mathbb{P}(X_1 \leqslant 20) \times \cdots \times \mathbb{P}(X_{20} \leqslant 20) \qquad \textit{(independence)} \\
&\qquad = 1 - \mathbb{P}(X_1 \leqslant 20)^{20} \qquad \textit{(identical distributions)} \\
&\qquad = 1 - (1 - \mathrm{e}^{-2})^{20} \approx 0.9454.
\end{aligned}$$

23. Let $X \sim \mathsf{Exp}(1)$. Use the Moment Generating Functions to show that $\mathbb{E}[X^n] = n!$.

Solution:

$$X \sim \mathsf{Exp}(1) \Rightarrow M_X(t) = \frac{1}{1-t}, \ t < 1.$$

Note

$$\begin{aligned}
M_X(t) &= \sum_{k=0}^{\infty} t^k = 1 + t + \cdots + t^{n-1} + t^n + t^{n+1} + \cdots \qquad |t| < 1, \\
\Rightarrow M_X^{(n)} &= 0 + \cdots + 0 + n! + \frac{(n+1)!}{(n+1-n)!} t + \cdots \qquad t \in (-1, 1) \\
\Rightarrow M_X^{(n)}(0) &= n!.
\end{aligned}$$

24. Using a $X \sim \mathsf{Uniform}[0,1]$ random number, explain how to generate random numbers from the (a) $\mathsf{Uniform}[10,15]$ distribution, (b) $\mathsf{Exp}(10)$ distribution.

Solution: Inverse transform method: Solve $U = F(X)$, where F is the cdf, and U is $\mathsf{U}(0,1)$.

(a)

$$F(x) = \begin{cases} 0, & x < 10, \\ \left(\frac{x-10}{5}\right), & 10 \leqslant x \leqslant 15, \\ 1, & x > 15, \end{cases}$$

so rearranging $u = \left(\frac{x-10}{5}\right)$ gives $x = 10 + 5yu$. Thus we simulate $U \sim \mathsf{U}(0,1)$ and return $X = 10 + 5U$.

(b)

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \mathrm{e}^{-10x}, & x \geqslant 0, \end{cases}$$

so solving for x:

$$\begin{aligned} u &= 1 - \mathrm{e}^{-10x} \\ \mathrm{e}^{-10x} &= 1 - u \\ -10x &= \ln(1-u) \\ 10x &= -\ln(1-u) \\ x &= \frac{-1}{10}\ln(1-u). \end{aligned}$$

Now simulate $U \sim \mathsf{U}(0,1)$ and return $X = \frac{-1}{10}\ln(1-U)$.
*NB: For $U \sim \mathsf{U}(0,1)$, $1-U$ has the **same** distribution as U, so equivalently, we can:*
Simulate $U \sim \mathsf{U}(0,1)$ and return $X = \frac{-1}{10}\ln(U)$.

25. If $X \sim \mathsf{Uniform}[0,1]$, what is the distribution of $Y = 10 + 2X$?
Solution: $Y \sim \mathsf{Uniform}[10,12]$ since $X \sim \mathsf{Uniform}[0,1]$.

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & x \in [0,1], \\ 1, x > 1, & \end{cases}$$

and $Y = 10 + 2X$ so $10 \leqslant Y \leqslant 12$.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(10 + 2X \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-10}{2}\right) \\ &= F_X\left(\frac{y-10}{2}\right) = \begin{cases} 0, & y < 10, \\ \frac{y-10}{2}, & 10 \leqslant y \leqslant 12, \\ 1, & y > 12, \end{cases} \\ \Rightarrow f_Y(y) &= \begin{cases} \frac{1}{2}, & y \in [10,12], \\ 0, & \textit{otherwise}. \end{cases} \end{aligned}$$

26. Random variables X and Y have a joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x \cos y & 0 < x < \pi/2, \quad 0 < y < x\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the marginal probability density function of X.
(b) Determine the conditional probability density function of Y given $X = x$.
(c) Evaluate $\mathbb{P}(X + Y \geqslant \pi/2)$.

Solution:

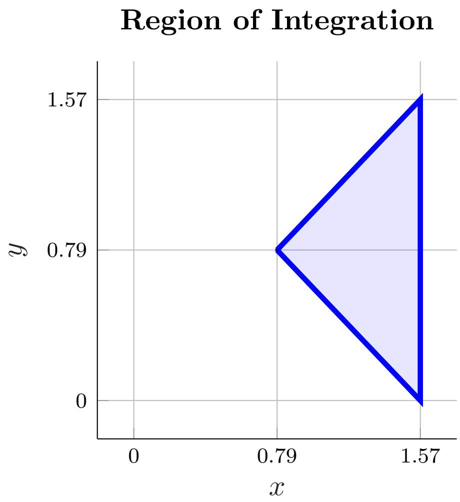
(a)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_0^x x\cos y\,\mathrm{d}y \\ &= \begin{cases} x \sin x, & 0 < x < \pi/2, \\ 0, & \textit{otherwise}. \end{cases} \end{aligned}$$

(b)

$$
\begin{aligned}
f_{Y|X=x}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \begin{cases} \frac{x\cos y}{x\sin x} \equiv \frac{\cos y}{\sin x}, & 0 < y < x, \\ 0, & \textit{otherwise.} \end{cases}
\end{aligned}
$$

(c) Region of interest:



$$
\begin{aligned}
\mathbb{P}(X+Y \geqslant \pi/2) &= \int_{\pi/4}^{\pi/2}\int_{\pi/2-x}^{x} x\cos y \,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\pi/4}^{\pi/2} x\left\{\sin(x) - \sin(\pi/2 - x)\right\}\,\mathrm{d}x \\
&= \int_{\pi/4}^{\pi/2} x\left\{\sin(x) - \cos(x)\right\}\,\mathrm{d}x \\
&= \left[\sin(x) - x\cos(x)\right]_{x=\pi/4}^{x=\pi/2} - \left[\cos(x) + x\sin(x)\right]_{x=\pi/4}^{x=\pi/2} \\
&= \left[1 - \frac{\pi}{2}\times 0 - \left(\frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\right)\right] - \left[0 + \frac{\pi}{2}\times 1 - \left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\right)\right] \\
&= 1 - \frac{\pi}{2} + \frac{\pi}{2\sqrt{2}} \approx 0.539924.
\end{aligned}
$$

27. Consider the random experiment where we draw independently n numbers from the interval [0,1]; each number in [0,1] being equally likely to be drawn. Let the independent and $\mathsf{Uniform}[0,1]$-distributed random variables $X_1, \dots, X_n$ represent the numbers that are to be drawn.

    (a) Let M be the smallest of the n numbers, and $\bar{X}$ the average of the n numbers. Express M and $\bar{X}$ in terms of $X_1, \dots, X_n$.

    (b) Determine the pdf of M.

    (c) Give the expectation and variance of $\bar{X}$.

    Solution: $X_1, \dots, X_n \sim_{\text{iid}} \mathsf{U}(0,1)$.

(a)

$$M = \min\{X_1, \ldots, X_n\}, \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(b)

$$\begin{aligned}
F_M(m) &= \mathbb{P}(M \leqslant m) = 1 - \mathbb{P}(M > m) \\
&= 1 - \mathbb{P}(\min\{X_1, \ldots, X_n\} > m) \\
&= 1 - \mathbb{P}(X_1 > m, \ldots, X_n > m) \\
&= 1 - \mathbb{P}(X_1 > m) \times \cdots \times \mathbb{P}(X_n > m) && \textit{(independence)} \\
&= 1 - \mathbb{P}(X_1 > m)^n && \textit{(identical distributions)} \\
&= 1 - (1 - \mathbb{P}(X_1 \leqslant m))^n \\
&= \begin{cases} 0, & m < 0, \\ 1 - (1-m)^n, & 0 \leqslant m \leqslant 1, \\ 1, & m > 1, \end{cases} \\
\Rightarrow f_M(m) &= \begin{cases} n(1-m)^{n-1}, & 0 \leqslant m \leqslant 1, \\ 0, & \textit{otherwise.} \end{cases}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{E}(\bar{X}) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}X_i \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\right) && \textit{(identical distributions)} \\
\operatorname{Var}(\bar{X}) &= \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{Var}(X_i) && \textit{(independence)} \\
&= \frac{1}{n}\operatorname{Var}(X_1) && \textit{(identical distributions)} \\
&= \frac{1}{12n}.
\end{aligned}$$

28. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables, with cdfs $F_1, F_2, \ldots, F_n$, respectively. Express the cdf of $M = \max(X_1, \ldots, X_n)$ in terms of the $\{F_i\}$.

    Solution:

$$\begin{aligned}
F_M(m) &= \mathbb{P}(\max\{X_1, \ldots, X_n\} \leqslant m) \\
&= \mathbb{P}(X_1 \leqslant m, \ldots, X_n \leqslant m) \\
&= \mathbb{P}(X_1 \leqslant m) \times \cdots \times \mathbb{P}(X_n \leqslant m) && \textit{(independence)} \\
&= F_1(m) \times \cdots \times F_n(m) \\
&\equiv \prod_{i=1}^{n} F_i(m).
\end{aligned}$$

29. A random vector (X, Y) has joint pdf f, given by

$$f(x, y) = 2\,\mathrm{e}^{-x-2y}, \quad x > 0, y > 0\ .$$

(a) Calculate $\mathbb{E}[XY]$.

(b) Calculate the covariance of $X + Y$ and $X - Y$.

Solution: Notice that $f_{X,Y}(x, y) = f_X(x) f_Y(y)$, *for all* $(x, y) \in \mathbb{R}^2$, *where*

$$f_X(x) = \begin{cases} \mathrm{e}^{-x}, & x > 0, \\ 0, & \textit{otherwise}, \end{cases} \qquad \textit{and} \qquad f_Y(y) = \begin{cases} 2\mathrm{e}^{-2y}, & y > 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Thus X *and* Y *are **independent** with* $X \sim \mathsf{Exp}(1)$ *and* $Y \sim \mathsf{Exp}(2)$.

(a) Hence,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 1 \times \frac{1}{2} = \frac{1}{2}.$$

***Alternatively,** this can be done via direct calculation, showing that*

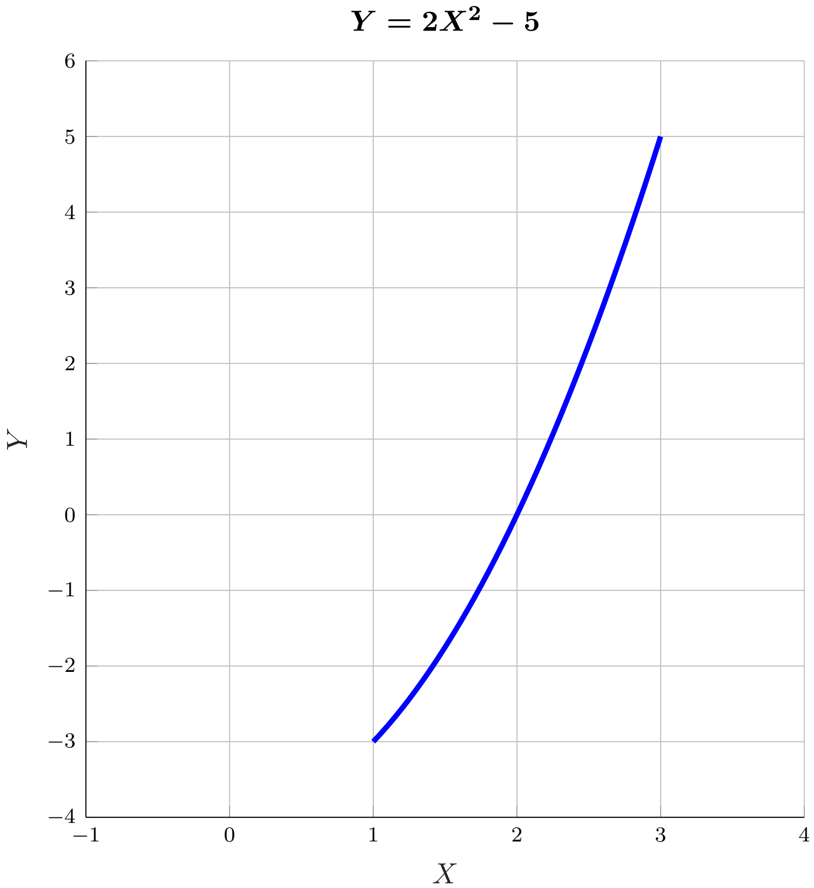
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}.$$

(b)

$$\begin{aligned} \mathrm{Cov}(X+Y, X-Y) &= \mathrm{Cov}(X, X) - \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) - \mathrm{Cov}(Y, Y) \\ &= \mathrm{Var}(X) - \mathrm{Var}(Y) \\ &= \frac{1}{1^2} - \frac{1}{2^2} \quad \textit{(since Var of } \mathsf{Exp}(\lambda) \textit{ is } \frac{1}{\lambda^2}\textit{)} \\ &= 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

30. Let X have a uniform distribution on the interval $[1, 3]$. Define $Y = X^2 - 4$. Derive the probability density function (pdf) of Y. Make sure you also specify where this pdf is zero.

Solution: $X \sim \mathsf{U}[1, 3]$, $Y = X^2 - 4$, *so* $Y \in [-3, 5]$.



$$F_X(x) = \begin{cases} 0, & x < 1, \\ \frac{(x-1)}{2}, & 1 \leqslant x \leqslant 3, \\ 1, & x > 1, \end{cases}$$

$$\begin{aligned} \Rightarrow F_Y(y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(X^2 - 4 \leqslant y) = \mathbb{P}(X^2 \leqslant y + 4) \\ &= \mathbb{P}\left(X \leqslant \sqrt{y+4}\right) \\ &= \begin{cases} 0, & y < -3, \\ \frac{(\sqrt{y+4}-1)}{2}, & -3 \leqslant y \leqslant 5, \\ 1, & y > 5, \end{cases} \end{aligned}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y+4}}, & -3 \leqslant y \leqslant 5, \\ 0, & \textit{otherwise.} \end{cases}$$

31. The thickness of a printed circuit board is required to lie between the specification limits of 0.150 - 0.004 and 0.150 + 0.004 cm. A machine produces circuit boards with a thickness that is normally distributed with mean 0.151 cm and standard deviation 0.003 cm.

    (a) What is the probability that the thickness X of a circuit board which is produced by this machine falls within the specification limits?

(b) Now consider the mean thickness $\bar{X} = (X_1 + \cdots + X_{25})/25$ for a batch of 25 circuit boards. What is the probability that this batch mean will fall within the specification limits? Assume that $X_1, \ldots, X_{25}$ are independent random variables with the same distribution as X above.

Solution: Let X denote the thickness of a typical board. Then $X \sim \mathsf{N}(0.151, 0.003^2)$.

(a)

$$\begin{aligned}
&\mathbb{P}(0.150 - 0.004 \leqslant X \leqslant 0.150 + 0.004)\\
&\quad = \mathbb{P}(0.146 \leqslant X \leqslant 0.154)\\
&\quad = \mathbb{P}\left(\frac{0.146 - 0.151}{0.003} \leqslant Z \leqslant \frac{0.154 - 0.151}{0.003}\right) \qquad (Z \sim \mathsf{N}(0,1))\\
&\quad = \mathbb{P}\left(\frac{-5}{3} \leqslant Z \leqslant 1\right)\\
&\quad = \mathbb{P}(Z \leqslant 1) - \mathbb{P}(Z \leqslant -5/3)\\
&\quad = \mathbb{P}(Z \leqslant 1) - (1 - \mathbb{P}(Z \leqslant 5/3))\\
&\quad = \mathbb{P}(Z \leqslant 1) - (1 - \mathbb{P}(Z \leqslant 1.666\ldots)),
\end{aligned}$$

between

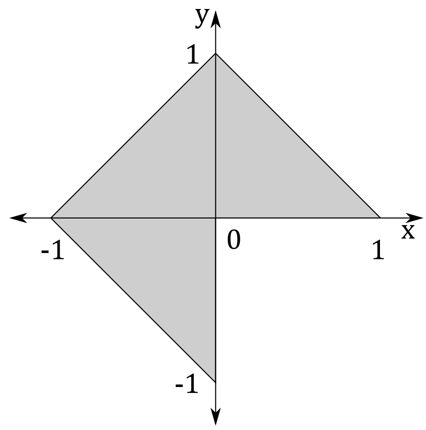
$$\begin{aligned}
&0.8413 - (1 - 0.9515) = 0.7928\\
\textit{and}\quad &0.8413 - (1 - 0.9525) = 0.7938.
\end{aligned}$$

(b)

$$\begin{aligned}
&\bar{X} \sim \mathsf{N}\left(0.151, \frac{0.003^2}{25}\right)\\
&\Rightarrow \mathbb{P}(0.146 \leqslant \bar{X} \leqslant 0.154)\\
&\quad = \mathbb{P}\left(\frac{0.146 - 0.151}{0.003/\sqrt{25}} \leqslant Z \leqslant \frac{0.154 - 0.151}{0.003/\sqrt{25}}\right) \qquad (Z \sim \mathsf{N}(0,1))\\
&\quad \approx \mathbb{P}(-8.333 \leqslant Z \leqslant 5)\\
&\quad = \underbrace{\mathbb{P}(Z \leqslant 5)}_{>0.9999} - \underbrace{(1 - \mathbb{P}(Z \leqslant 8.333))}_{<0.0001},
\end{aligned}$$

which is definitely greater than 0.9998.

32. We draw a random vector (X, Y) uniformly from the diamond $(-1,0)$–$(0,1)$–$(1,0)$–$(0,-1)$ *without* its lower-right corner (see figure).



(a) Write down the joint pdf of X and Y, clearly specifying where it is zero.
(b) Determine the marginal pdf of Y.
(c) Determine the conditional pdf of X given $Y = -1/4$.

Solution:

(a) Area of shape is $3/2$, *so*

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3}, & -1 \leqslant x \leqslant 0,\ -1-x \leqslant y \leqslant 1+x \\ & \text{or } 0 < x < 1,\ 0 \leqslant y \leqslant 1-x, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{2}{3}, & -1 \leqslant y \leqslant 0,\ -1-y \leqslant x \leqslant 0 \\ & \text{or } 0 < y \leqslant 1,\ y-1 \leqslant x \leqslant 1-y, \\ 0, & \text{otherwise}. \end{cases}$$

(b)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \begin{cases} \int_{-1-y}^{0} \frac{2}{3}\,\mathrm{d}x, & -1 \leqslant y \leqslant 0, \\ \int_{y-1}^{1-y} \frac{2}{3}\,\mathrm{d}x, & 0 < y \leqslant 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{2}{3}(1+y), & -1 \leqslant y \leqslant 0, \\ \frac{4}{3}(1-y), & 0 < y \leqslant 1, \\ 0, & \text{otherwise}. \end{cases}$$

(c)

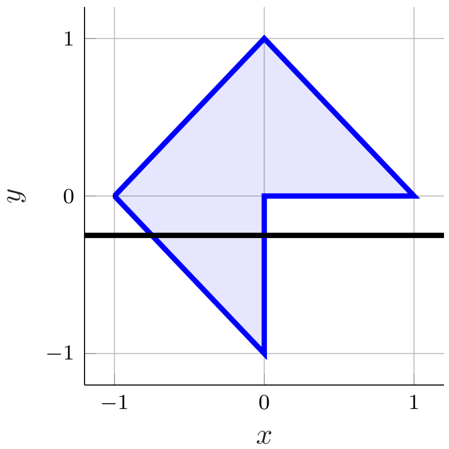
$$f_{X|Y=-1/4}(x|-1/4) = \frac{f_{X,Y}(x,-1/4)}{f_Y(-1/4)}$$

$$= \begin{cases} \frac{2/3}{2(1-1/4)/3} = \frac{4}{3}, & \frac{-3}{4} \leqslant x \leqslant 0, \\ 0, & \text{otherwise}, \end{cases}$$

so $(X|Y=-1/4) \sim \mathsf{U}(-3/4, 0)$.

Alternatively, *directly look at the intersection of* $Y = -1/4$ *with the shape:*

The Conditional Region



Uniform on intersection.