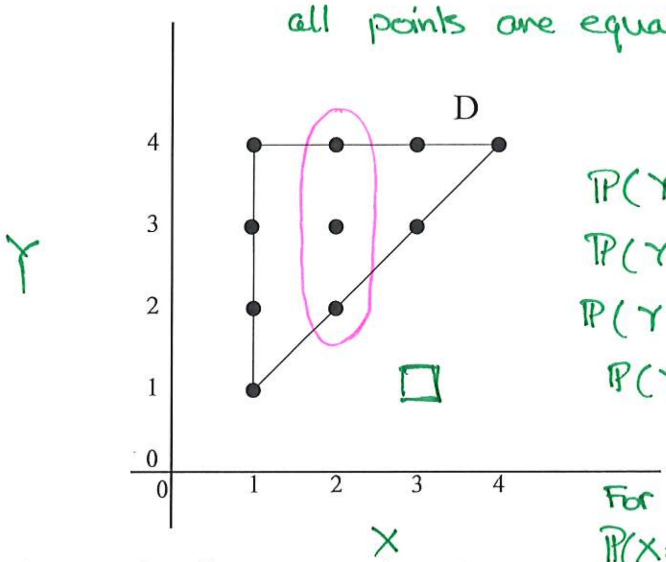
$$\begin{aligned}\mathrm{Var}(X_1 + X_2 + \cdots + X_n) &= \cancel{p(1-p)}\, Var(X_1) + \cdots + Var(X_n) \\ &= p(1-p) + \cdots + p(1-p) = np(1-p)\end{aligned}$$

The variance of a Binomial(n,p) r.v. is $np(1-p)$.

Conditional probability mass function

Example. We draw at random a point (X, Y) from the 10 points on the triangle D below.

all points are equally likely to be selected.



$\mathbb{P}(X=3, Y=1) = 0$

$\neq \mathbb{P}(X=3) \times \mathbb{P}(Y=1)$

$\mathbb{P}(Y=4) = 4/10$

$\mathbb{P}(Y=3) = 3/10$

$\mathbb{P}(Y=2) = 2/10$

$\mathbb{P}(Y=1) = 1/10$

For independence we need $\mathbb{P}(X=i, X=j) = \mathbb{P}(X=i)\mathbb{P}(Y=j)$ $\forall i,j$

The joint and marginal pmf's are easy to determine:

$$\begin{aligned} \mathbb{P}(X = i, Y = j) &= 1/10 && (i,j) \in D\,, \\ \mathbb{P}(X = i) &= \frac{5-i}{10}\,, && i \in \{1,2,3,4\}\,, \\ \mathbb{P}(Y = j) &= j/10 && j \in \{1,2,3,4\} \end{aligned}$$

Clearly X and Y are *not independent.* In fact, if we know that $X = 2$, then Y can only take the values $j = 2, 3$ or 4.

The corresponding probabilities are

$$f_{Y\,|\,X}(j,2) = \begin{cases} \mathbb{P}(Y = j \,|\, X = 2) = \dfrac{\mathbb{P}(Y = j, X = 2)}{\mathbb{P}(X = 2)} = \dfrac{1/10}{3/10} = \dfrac{1}{3} & \text{if } j \in \{2,3,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

We thus have determined the **conditional pmf** of Y given $X = 2$.

Definition. If X and Y are *discrete* and $\mathbb{P}(X = x) > 0$, then the probabilities

$$f_{Y\,|\,X}(y\,|\,x) = \mathbb{P}(Y = y \,|\, X = x) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)} = \frac{f_{X,Y}(x,y)}{f_X(x)}\,,$$

conditional pmf of Y given $X = x$.

$f_X(x) f_{Y|X}(y|x) = f_{X,Y}(x,y).$

for all y, give the **conditional pmf** of Y given $X = x$.

For a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ we also have the **chain rule**

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1)\, f_{X_2\,|\,X_1}(x_2\,|\,x_1) \ldots f_{X_n\,|\,X_1,\ldots,X_{n-1}}(x_n\,|\,x_1, \ldots, x_{n-1}).$$

$$\mathbb{P}(X_1=x_1, X_2=x_2, \ldots, X_n=x_n) = \mathbb{P}(X_1=x_1)\mathbb{P}(X_2=x_2|X_1=x_1)\cdots\mathbb{P}(X_n=x_n|X_{n-1}=x_{n-1}, \ldots X_1=x_1)$$

This is also known as *factorising* the joint distribution.

In the case $\mathbf{X} = (X_1, X_2)$ we have

$$\boxed{f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2|X_1}(x_2|x_1)}$$

The choice of how to factorise the distribution often depends on what we are modelling or what information is available. Another possible factorisation is given by

$$\boxed{f_{X_1,X_2}(x_1, x_2) = f_{X_2}(x_2)\, f_{X_1|X_2}(x_1|x_2)}$$

When X and Y are independent, this also gives us

$$\boxed{f_{X\,|\,Y}(x\,|\,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x).}$$

Example. Let $Y \sim \mathsf{Poisson}(\lambda)$. The conditional on Y, X has a $\mathsf{Binomial}(Y, p)$ distribution. Find the joint pmf of (X, Y) and the marginal pmf of X.

To find the joint pmf of (X, Y):

$$\boxed{f_{XY}(x,y) = f_{X\,|\,Y}(x\,|\,y) f_Y(y) = \binom{y}{x} p^x (1-p)^{y-x} \times \frac{e^{-\lambda}\lambda^y}{y!}}$$

$x, y \in \{0, 1, \ldots\}$ such that $x \leq y$.

To find the marginal pmf of X:

$$f_X(x) = \sum_y f_{X,Y}(x,y) = \sum_y \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^y}{y!}$$

$$= \sum_{y=x}^{\infty} \frac{\cancel{y!}}{x!(y-x)!} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^{y-x}\cdot\lambda^x}{\cancel{y!}}$$

$$= \frac{\lambda^x p^x}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x} e^{-\lambda(1-p)} e^{-\lambda p}$$

$$= \left[\frac{e^{-\lambda p}(\lambda p)^x}{x!}\right] \left[\sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x} e^{-\lambda(1-p)}\right]$$

$$= \frac{e^{-\lambda p}(\lambda p)^x}{x!} \left[\sum_{k=0}^{\infty} \frac{1}{k!} [(1-p)\lambda]^k e^{-\lambda(1-p)}\right] = 1$$

↑ pmf of Poisson $(\lambda(1-p))$

The marginal pmf of X is Poisson (λp)

U	V		X	Y
0	0	$\longrightarrow$	0	0
0	1		0	1
1	0		0	1
1	1		1	1

$\mathbb{P}(X=0, Y=0) = 1/4$

$\mathbb{P}(X=0, Y=1) = \mathbb{P}(U=0, V=1) + \mathbb{P}(U=1, V=0)$

Let U and V be independent Beroulli (1/2) random variables. Define $X = \min(U,V)$ and $Y = \max(U,V)$.

a) Write down the joint pmf (as a table) of X and Y.

V \ U	0	1
0	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

Y \ X	0	1
0	$\frac{1}{4}$	0
1	$\frac{1}{2}$	$\frac{1}{4}$

b) Determine the marginal pmf of X and marginal pmf of Y.

x	0	1
$\mathbb{P}(X=x)$	$3/4$	$1/4$

y	0	1
$\mathbb{P}(Y=y)$	$1/4$	$3/4$

c) Give the conditional pmf of Y given $X=1$.

$$f_{Y|X}(y \mid 1) = \begin{cases} \dfrac{\mathbb{P}(Y=0, X=1)}{\mathbb{P}(X=1)} = \dfrac{0}{1/4} = 0, & y=0 \\[2ex] \dfrac{\mathbb{P}(Y=1, X=1)}{\mathbb{P}(X=1)} = \dfrac{1/4}{1/4} = 1, & y=1 \end{cases}$$

d) Calculate the covariance of X and Y.

$$\mathbb{E}[X] = \sum_{x=0}^{1} x\, \mathbb{P}(X=x) = 0 \times 3/4 + 1 \times 1/4 = 1/4$$

$$\mathbb{E}[Y] = 0 \times 1/4 + 1 \times 3/4 = 3/4$$

$$\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

$$\mathbb{E}[XY] = \sum_{x=0}^{1} \sum_{y=0}^{1} xy\, \mathbb{P}(X=x, \cancel{y} Y=y)$$

$$= 1/4 - 1/4 \times 3/4 = 1/16$$

$$= 0 \times 0 \times 1/4 + 0 \times 1 \times 1/2 + 1 \times 0 \times 0 + 1 \times 1 \times 1/4 = 1/4$$

e) Are X and Y independent? No

- $\mathbb{P}(X=1, Y=0) = 0$ but $\mathbb{P}(X=\cancel{0}\,1)\,\mathbb{P}(Y=0) = 1/4 \times 1/4 \neq 0$

- $\operatorname{cov}(X,Y) = 1/16$ but if X and Y are independent, then $\operatorname{cov}(X,Y) = 0$.

(d continued)

$$\operatorname{cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$= \mathbb{E}\left[XY - Y\mathbb{E}X - X\mathbb{E}Y + \mathbb{E}[X]\mathbb{E}[Y]\right]$$

$$= \mathbb{E}[XY] - \mathbb{E}[Y]\mathbb{E}[X] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathbb{E}[XY] - \mathbb{E}[Y]\mathbb{E}[X]$$