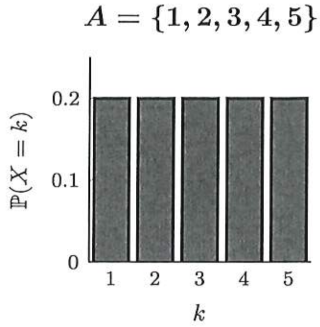

Figure 4.2: Probability mass function for $X \sim \mathsf{Uniform}(\{1,2,3,4,5\})$.

"is distributed as"

We write $X \sim \mathsf{DU}(A)$ or simply $X \sim \mathsf{Uniform}(A)$.

Examples.

mapping $\{H, T\} \to \{0, 1\}$

- fair coin toss where 1 if Heads and 0 if tails
- first number drawn in a lotto draw
- birthday of a randomly selected person $\{1, 2, \ldots, 365\}$ days.
- randi (MATLAB)

Bernoulli Distribution

We say that a random variable X has a **Bernoulli distribution** with success parameter p if $\Omega_X = \{0, 1\}$, and

$$\mathbb{P}(X = 1) = p \quad \text{and} \quad \mathbb{P}(X = 0) = 1 - p.$$

We write $X \sim \mathsf{Bernoulli}(p)$.

A Bernoulli random variable describes the outcome of a *Bernoulli trial.*

Example: Flip a biased coin with probability heads p. The sample space Ω is {Heads, Tails}, we can define a random variable as:

ω	$X(\omega)$	
Heads	1	with probability p
Tails	0	with probability $1-p$

This is a $\mathsf{Bernoulli}(p)$ random variable. If $p = 1/2$ it is *also* a discrete uniform random variable.

We can also define the random variable:

ω	$Y(\omega)$
Heads	0
Tails	1

$P_Y = 1 - P_X$

This is also a $\mathsf{Bernoulli}(p)$ random variable.

A Bernoulli trial is a fancy way of talking about an experiment that either succeeds or fails, with only those outcomes being possible.

Suppose we define a new random variable as:

ω	$Z(\omega)$
Heads	10
Tails	0

This is NOT a $\mathsf{Bernoulli}(p)$ random variable. However $Z = 10\,X$ so it is a *function* of a Bernoulli random variable. Notice that we can easily find the pmf of Z from the pmf of X.

Binomial Bistribution

We say that a random variable X has a **Binomial distribution** with parameters n and p if

number of trials

k successes

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n\,.$$

success probability of a single trial

n-k failures

We write $X \sim \mathsf{Binomial}(n, p)$. We have encountered this distribution several times already.

A Binomial random variable is used to describe the *total number of successes* in a sequence of n independent Bernoulli trials with success probability p.

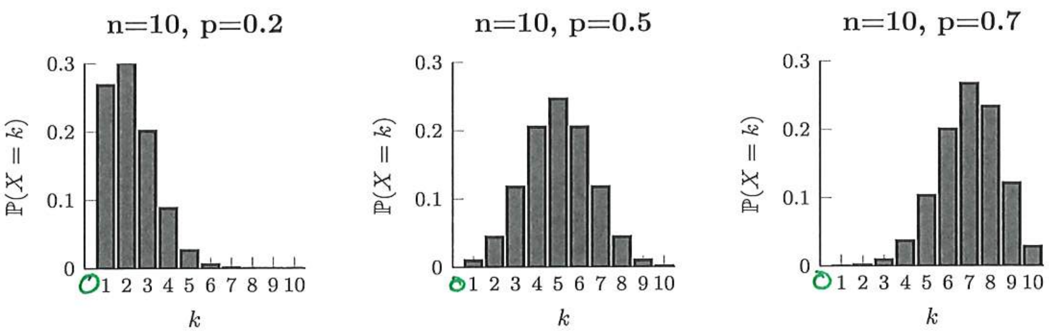

Figure 4.3: Probability mass function for $X \sim \mathsf{Binomial}(10, p)$ for $p = 0.2$, $p = 0.5$, and $p = 0.7$.

Example: A collection of 12 items contains 5 defectives. If you select one item from this collection (uniformly), the probability of the selected item being defective is $\boxed{5/12}$.

A sample of size 4 is taken from this collection *with replacement*. Let X be the number of defectives in the sample. Then $\boxed{X \sim \text{Binomial}(4, 5/12)}$.

$$\binom{4}{k} = {}^4C_k = \frac{4!}{k!(4-k)!}$$

$$\boxed{\mathbb{P}(X=k) = \binom{4}{k}\left(\tfrac{5}{12}\right)^k\left(\tfrac{7}{12}\right)^{4-k} \quad k = 0, 1, \ldots, 4.}$$

k	0	1	2	3	4	$\sum$
$\mathbb{P}(X=k)$	.116	.331	.354	.169	.030	1

Question. Can you give another example of a Binomial random variable?

opinion polls: sample of n individuals from the population + proportion p "agree" with some statement being tested.

We can write a function in MATLAB that generates a realisation from a Binomial(n,p) random variable using only the built in `rand` function as follows.

0 — p — 1

```
function output = Binomial(n,p)
    output = sum(rand(1,n) < p);
end
```

$\underbrace{\texttt{rand(1,n) < p}}_{}$ n. Bernoulli r.v.s with success prob p.

After saving this as 'Binomial.m' to our current working directory we can call this function as follows.

```
Binomial(100, 0.4)
ans =
    46

Binomial(100, 0.4)
ans =
    39
```

The function binornd built into MATLAB is a more sophisticated version of the function we just wrote. For more information on this function type "help binornd" into the MATLAB command line.

rbinom ?rbinom (R).

Geometric Distribution

Suppose that you have set up a printer on the local area network of your office. Suppose also that time is slotted into discrete units. In each time slot a single print job arrives with probability p indendently of the arrival of jobs at other time slots. We now have a sequence of independent Bernoulli random variables.

Question. What is the pmf of the time until the first job arrives?

$X :=$ time to first job arriving.

$$\mathbb{P}(X=5) = (1-p)\times(1-p)\times(1-p)\times(1-p)\times p$$
$$= (1-p)^4 p$$
$$\mathbb{P}(X=k) = (1-p)^{k-1} p \qquad k = 1, 2, 3, \ldots$$

We say that the random variable X has a **Geometric distribution** with parameter p and we write $X \sim \mathsf{Geometric}(p)$.

Notice that after each arrival the time until the next arrival is again $\mathsf{Geometric}(p)$.

A Geometric random variable is used to describe the *time of first success* in a sequence of independent Bernoulli trials with success probability p.

when p is small, many trials are needed to observe success

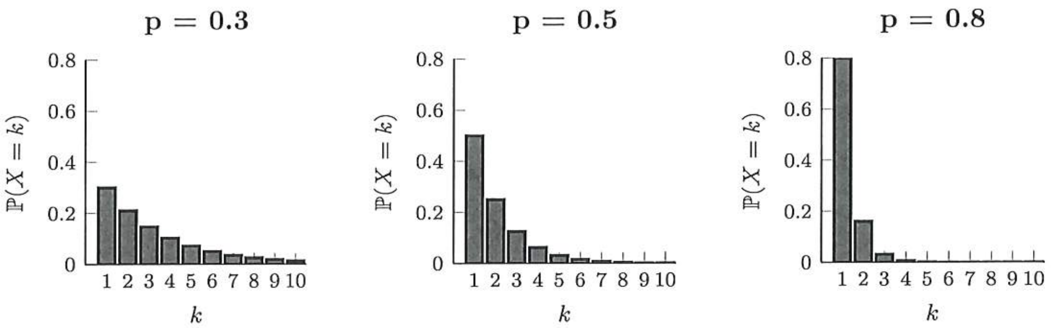

Figure 4.4: Probability mass function for $X \sim \mathsf{Geometric}(p)$ for $p = 0.3$, 0.5, and 0.8.

Question. Given an outcome $\omega \in \Omega$, what are the possible range of outcomes for $X(\omega) \sim \mathsf{Geometric}(p)$?

$$P(X=k) = (1-p)^{k-1}p, \quad k \in \{1, 2, 3, \ldots\}.$$ positive integers.

Remark. Be careful as other sources may define a Geometric random variable as the number of trials *before the first success.*

Example: Let X be the number of throws with a (fair) die, needed to get a six.

Then $\boxed{X \sim \mathsf{Geometric}(1/6)}$.

Examples.

- number of people need to ask before finding
- someone with the same birthday as me.

- number of visitors to an e-commerce website
- before first purchase is made.

Question. How would you write a function in MATLAB that generates a realisation from a $\mathsf{Geometric}(p)$ random variable using only the built in rand function?

geornd (MATLAB)

$P(A|B) = P(A \cap B)/P(B)$

Question. Returning to the printer example, suppose that no print jobs have arrived after k time slots, what is the probability that the next job arrives in the $m+k$ time slot ($m>0$)?

$P(X>k) = (1-p)^k$

$$\begin{aligned}\mathbb{P}(X=m+k\,|\,X>k) &= P(X=m+k)/P(X>k)\\ &= (1-p)^{m+k-1}\,p\,/\,(1-p)^k\\ &= (1-p)^{m-1}\,p\\ &=\end{aligned}$$

We just saw that

$$\mathbb{P}(X=m+k\,|\,X>k)=\mathbb{P}(X=m)\,.$$

This means that the distribution of the time we must wait for the next print job to arrive is independent of the amount of time we have already waited.

The Geometric distribution is the only discrete distribution with this extremely useful property.

Poisson Distribution

Consider a $\mathsf{Binomial}(10000, 0.005)$ random variable. This corresponds to a situation where we have 10000 independent Bernoulli trials with success probability 0.005.

$$\mathbb{P}(X=k)=\binom{10000}{k}0.005^k\,0.995^{10000-k}\,,\quad k=0,1,\ldots,10000\,.$$

Question. Can any of you calculate $\mathbb{P}(X=50)$ on your laptop? (Try to do this now!)

```
nchoosek(10000, 50)*0.005^50*0.995^9950
Warning: Result may not be exact. Coefficient is greater than
9.007199e+15 and is only accurate to 15 digits
> In nchoosek at 92
ans =
      0.0565
```

A Poisson distribution is the *limit of Binomial* distributions in the following sense:

Let $X_n \sim \mathsf{Binomial}(n, \lambda/n)$ with $\lambda > 0$ and X is such that

$$\mathbb{P}(X=k)=\mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}\,,\quad k=0,1,2\ldots$$

then

$$\lim_{n\to\infty}\mathbb{P}(X_n=k)=\mathbb{P}(X=k)\,,$$

for all k. We will justify this limiting result later when we look at ~~probability~~ Moment generating functions.

Alternate derivation for $X \sim \text{Geometric}(p)$

$$P(X > k) = \sum_{j=k+1}^{\infty} P(X=j)$$

$$= \sum_{j=k+1}^{\infty} (1-p)^{j-1} p$$

$$= \sum_{j=k+1}^{\infty} (1-p)^{j-k+k-1} p$$

$$= (1-p)^{k} \sum_{j=k+1}^{\infty} (1-p)^{j-k-1} p$$

$$= (1-p)^{k} \sum_{j=0}^{\infty} (1-p)^{j} p$$

$$= (1-p)^{k}$$

using the geometric series

$$\sum_{j=0}^{\infty} (1-p)^{j} = \frac{1}{1-(1-p)} = 1/p$$