STAT7203: Applied Probability and Statistics
Week 8 Exercises

1. A pair of random variables (X, Y) has a joint probability distribution in which $X \sim \mathsf{Uniform}(0, 1)$ and the conditional probability density function of Y given $\{X = x\}$ is
$$f_{Y|X}(y|x) = \begin{cases} 1, & x \leqslant y \leqslant 1 + x \\ 0, & \text{else.} \end{cases}$$

   (a) Determine the marginal probability density function for Y.

   (b) Using the fact that $\mathbb{E}[XY] = \mathbb{E}[X\mathbb{E}[Y|X]]$, compute the covariance between X and Y.

   (c) Suppose that $U \sim \mathsf{Uniform}(0, 1)$ and define the random variable $Z := g(U)$, where
$$g(u) = \begin{cases} \sqrt{2u}, & 0 \leqslant u \leqslant \frac{1}{2}, \\ 2 - \sqrt{2(1-u)}, & \frac{1}{2} < u \leqslant 1. \end{cases}$$
   Show the distribution of Z is the same as the marginal distribution of Y in part (a).

2. Suppose the random variable X has a $\mathsf{N}(2, 3)$ distribution. Conditional on $\{X = x\}$, the random variable Y has a $\mathsf{N}(1 + x, 2)$ distribution.

   (a) Detemine the probability that $X \geqslant 3$ using the normal distribution table and using the `R` function `pnorm`.

   (b) Determine the moment generating function of Y. Hence, identify the marginal distribution of Y .

   (c) Determine the correlation between X and Y.