$P(A \mid B) = P(A \cap B) / P(B)$

Question. Returning to the printer example, suppose that no print jobs have arrived after k time slots, what is the probability that the next job arrives in the $m+k$ time slot ($m>0$)?

$P(X>k) = (1-p)^k$

$$
\begin{aligned}
\mathbb{P}(X = m+k \mid X > k) &= P(X=m+k)/P(X>k) \\
&= (1-p)^{m+k-1}\, p \,/\, (1-p)^k \\
&= (1-p)^{m-1}\, p \\
&=
\end{aligned}
$$

We just saw that

$$\mathbb{P}(X = m+k \mid X > k) = \mathbb{P}(X = m)\,.$$

This means that the distribution of the time we must wait for the next print job to arrive is independent of the amount of time we have already waited.

The Geometric distribution is the only discrete distribution with this extremely useful property.

Poisson Distribution

Consider a $\mathsf{Binomial}(10000, 0.005)$ random variable. This corresponds to a situation where we have 10000 independent Bernoulli trials with success probability 0.005.

$$\mathbb{P}(X = k) = \binom{10000}{k} 0.005^k\, 0.995^{10000-k}, \quad k = 0, 1, \ldots, 10000\,.$$

Question. Can any of you calculate $\mathbb{P}(X = 50)$ on your laptop? (Try to do this now!)

```
nchoosek(10000, 50)*0.005^50*0.995^9950
Warning: Result may not be exact. Coefficient is greater than
9.007199e+15 and is only accurate to 15 digits
> In nchoosek at 92
ans =
     0.0565
```

A Poisson distribution is the *limit of Binomial* distributions in the following sense:

Let $X_n \sim \mathsf{Binomial}(n, \lambda/n)$ with $\lambda > 0$ and X is such that

$\frac{\lambda}{n} < 1$

$$\mathbb{P}(X = k) = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2\ldots$$ pmf of Poisson

then

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k)\,,$$

for all k. We will justify this limiting result later when we look at ~~probability~~ Moment generating functions.

$$\sum_{k=0}^{\infty} \mathbb{P}(X=k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\boxed{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}} = e^{-\lambda}\cdot e^{\lambda} = 1$$

We say that the random variable X has a **Poisson distribution** with parameter $\lambda > 0$. When $\lambda = 0$, the corresponding pmf is just

$$\mathbb{P}(X = 0) = 1, \quad \text{and} \quad \mathbb{P}(X = k) = 0 \quad \text{for } k \geqslant 1.$$

We write $X \sim \mathsf{Poisson}(\lambda)$.

Question. For $X \sim \mathsf{Poisson}(50)$ can any of you calculate $\mathbb{P}(X = 50)$ on your laptop? (Try to do this now!)

```
exp(-50)*50^50/factorial(50)
ans =
      0.0563
```

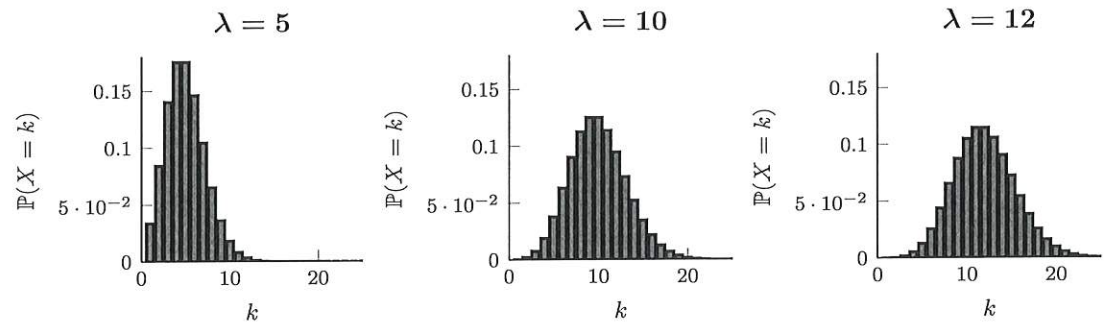

Figure 4.5: Probability mass function for $X \sim \mathsf{Poisson}(\lambda)$ for $\lambda = 5$, 10, and 12.

Note that unlike a Binomial random variable, which is restricted to take on values in $\{0, 1, \ldots, n\}$, a Poisson random variable can take on any non-negative integer value.

Examples.

- number of people in a queue
- number of trees in a particular area.
- number of calls arriving in given period.
-

We can generate a list of realisations from a Poisson(10) random variable in MATLAB as follows:

```
>> poissrnd(10,1,5)                    rpois    (R)
ans =
      10      8      10      9      11
```

A Printer Example

Suppose that print jobs during the first hour of each day last week for the printer in Priestley arrived in slotted time according to Figure 4.

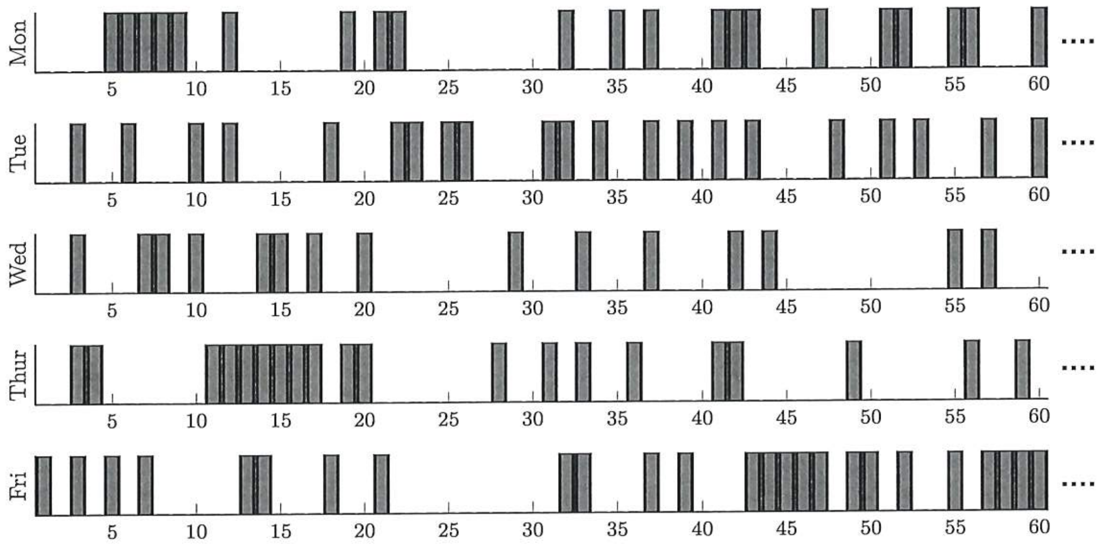

Figure 4.6: Print jobs in slotted time for first hour day during a week.

Question. How do we model the arrival of jobs in a given timeslot? (Assuming all time slots are independent.)

In this case the sample space for any given day contains all of the potential arrival sequences. For example

$$\omega = (0, 0, 1, 1, 0, \dots)\,.$$

Match the description on the left with a choice of states and then with a distribution.

Number of print jobs in any chosen n slots.	$\{1, 2, \dots\}$	Geometric$(1-p)$
Number of print jobs in any chosen n slots. (Large n and small p.)	$\{\dots, -2, -1, 0, 1, \dots\}$	Poisson(λ)
Indicator of print job arriving in a time slot.	$\{0, 1, \dots, n\}$	Bernoulli(p)
Time ~~between~~ to next print jobs.	$\{0, 1\}$	Geometric(p)
	$\{0, 1, 2, \dots\}$	Binomial(n, p)

in the limit $n \to \infty$

$np \to$ constant $n \to \infty$

Binomial$(1, p)$

Expected Value

Definition. Let X be a *discrete* random variable. The **expected value** (or **mean value** of X), denoted by $\mathbb{E}\,X$, is defined by

$$\mathbb{E}\,X = \sum_x x\,\mathbb{P}(X = x)\,.$$

mu

This number, sometimes written as μ_X, is a measure of location for the distribution.

Example. Find $\mathbb{E}\,X$ where X is the outcome of a roll of a fair die.

Since $\mathbb{P}(X = 1) = \ldots = \mathbb{P}(X = 6) = 1/6$

$$\mathbb{E}[X] = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6} = 3.5$$

Note: $\mathbb{E}[X]$ is not necessarily a possible outcome of the random experiment as in the previous example.

Example. Find $\mathbb{E}[X]$ for $X \sim \mathsf{Binomial}(n, p)$.

$n - x = (n-1) - (x-1)$
$= n - 1 - x + 1$

$$\mathbb{E}[X] = \sum_{x=0}^{n} x \cdot \mathbb{P}(X = x)$$

$$= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

Recall
$x! = x \times (x-1) \times (x-2) \cdots 1$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n \times (n-1)!}{(x-1)!\,((n-1)-(x-1))!}\, p \times p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \left[\frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!}\right] p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$k = x - 1$

$$= np \boxed{\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}} = np$$

← pmf Binomial$(n-1, p)$

Let X be a discrete random random variable and let g be a real-valued function defined on the support of X. We can define a new random variable Y as $Y := g(X)$. Note that Y is also a discrete random variable. The probability mass function of Y is

$$\mathbb{P}(Y = y) = \sum_{x:g(x)=y} \mathbb{P}(X = x).$$

Suppose X has discrete uniform distribution $\{-1, 0, 1\}$
Let $Y := X^2$. What is the pmf of Y?

$$\mathbb{P}(Y = y) = \begin{cases} 1/3, & y = 0 \\ 2/3 & y = 1 \end{cases}$$

$Y = g(X)$

The expected value of Y is then

$$\mathbb{E}Y = \sum_y y\mathbb{P}(Y=y) = \sum_y y\left[\sum_{x:g(x)=y} \mathbb{P}(X=x)\right] = \sum_x g(x)\mathbb{P}(X=x).$$

This leads to the following natural defintion for the expectation of $g(X)$ for any real valued function g.

Definition. If X is a *discrete* random variable, then for any real-valued function g

$$\mathbb{E}\left[g(X)\right] = \sum_x g(x)\,\mathbb{P}(X=x)\ .$$

LOTUS: Law of the unconscius statistician.

Example. Find $\mathbb{E}\left[X/n\right]$ where $X \sim \mathsf{Binomial}(n,p)$. We have

$$\boxed{\mathbb{E}\left[X/n\right] = \sum_{x=0}^{n} \frac{x}{n}\cdot\binom{n}{x}p^x(1-p)^{n-x} = \frac{1}{n}\sum_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x}} = \frac{1}{n}\times np$$

$= p$

In general, for any random variable X and real constants a and b

$$\boxed{\mathbb{E}\left[aX+b\right] = a\times\mathbb{E}(X) + b}$$

Example. Find $\mathbb{E}\left[X\right]$ for $X \sim \mathsf{Poisson}(\lambda)$. $\mathbb{E}(X) = \lambda$

$$
\begin{aligned}
\mathbb{E}\left[X\right] &= \sum_{x=0}^{\infty} x\cdot\mathbb{P}(X=x) \\
&= \sum_{x=0}^{\infty} x\cdot\frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} x\cdot\frac{e^{-\lambda}\lambda^x}{x\times(x-1)!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}\lambda}{(x-1)!} \\
&= \lambda\boxed{\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}} \\
&= \lambda\ .
\end{aligned}
$$

$k = x-1$

$\mathbb{E}\left[X(X-1)\right]$

pmf of poisson