Law of Total Probability

Recall the B_i are disjoint
$B_i \cap B_j = \emptyset$ for $i \neq j$
and $\bigcup_{i=1}^{n} B_i = \Omega$

Suppose $B_1, B_2, \ldots, B_n$ is a *partition* of Ω. Then, by the *sum rule*,

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) \, .$$

Note $A = \bigcup_{i=1}^{n} (B_i \cap A)$

$(B_i \cap A) \cap (B_j \cap A) = \emptyset$

$\Rightarrow$ Use sum rule (axiom 3)

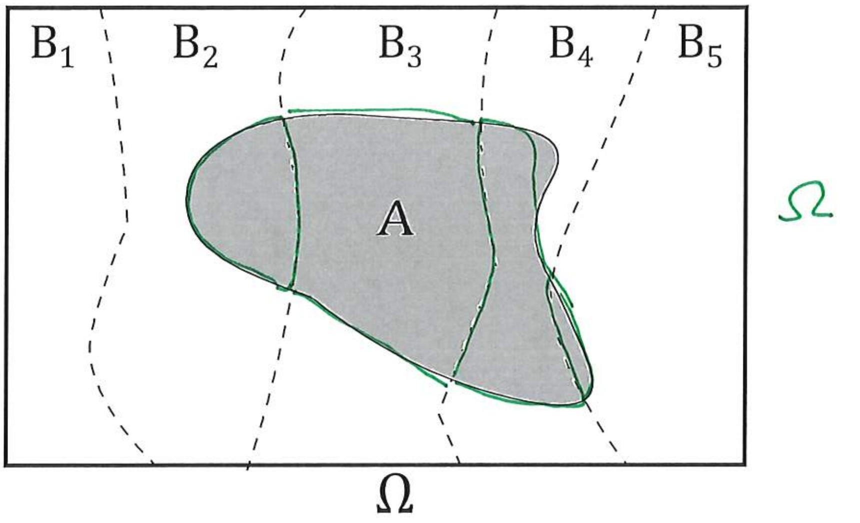
Hence,

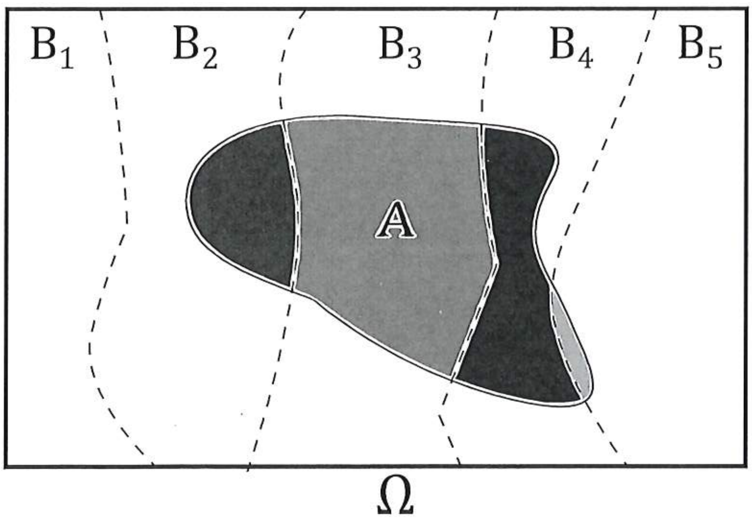
$\mathbb{P}(A \cap B_i) = \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$

$$\boxed{\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)}$$

(as long as $\mathbb{P}(B_1) > 0$, $\mathbb{P}(B_2) > 0$, $\ldots$, $\mathbb{P}(B_n) > 0$).

This is called the **Law of Total Probability**.





Trivial

Example. Draw a card from a full deck of 52 cards. What is the probability it is an Ace? $4/52 = 1/13$

Let A be the event that the card is an ace. Also, let B_1 be the event that the card is red and B_2 be the event that the card is black. Note that $\Omega = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$ so that $\{B_1, B_2\}$ is a valid partition. Always check this!

$$\mathbb{P}(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$
$$= 2/26 \times 1/2 + 2/26 \times 1/2 = 1/13$$

Bayes' Rule

Combining the definition of conditional probability with the Law of Total Probability gives the famous rule

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\,\mathbb{P}(B_j)}{\sum_{i=1}^{n}\mathbb{P}(A|B_i)\mathbb{P}(B_i)} = \frac{\mathbb{P}(A|B_j)\,\mathbb{P}(B_j)}{\mathbb{P}(A)}. = \frac{P(A \cap B_j)}{P(A)}$$

$B_1, \ldots, B_n$ form a partition of Ω

Example. Email arriving in your inbox is spam 30% of the time. Each of your emails is flagged *correctly* as spam with probability 0.99. However, each of your emails is flagged *incorrectly* as spam with probability 0.005.

You received one email, today. What is the probability that it really is spam given that it is flagged as spam?

Define the events

$A =$ email is flagged as spam

and

$B =$ email <u>is</u> spam

We know

Note B and B^c form a partition of Ω

$$\mathbb{P}(B) = 0.3$$
$$\mathbb{P}(B^c) = 0.7$$
$$\mathbb{P}(A\,|\,B) = 0.99, \text{ and}$$
$$\mathbb{P}(A\,|\,B^c) = 0.005$$

so Bayes' Rule gives

$$\mathbb{P}(B\,|\,A) = \frac{\mathbb{P}(A|B)\,\mathbb{P}(B)}{\mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c)}$$
$$= 0.99 \times 0.3 / (0.99 \times 0.3 + 0.005 \times 0.7)$$
$$\approx 0.988$$

Question 6

Total alpha-fetoprotein is a serological marker for hepatocellular carcinoma (HCC). A test using this marker correctly diagnoses a patient with HCC with 0.60 probability and it correctly diagnoses a patient without HCC with probability 0.90. In a certain population the prevalence of HCC is 4.1 per 100,000. What is the probability that someone has HCC given the total alpha-fetoprotein marker is positive.

a) 0.000025
b) 0.00083
→ c) 0.00025
d) 0.0010
e) 0.60

$$P(HCC) = \frac{4.1}{100000}$$

$$P(\text{test +ve} \mid HCC) = 0.6$$

$$P(\text{test -ve} \mid \text{not } HCC) = 0.9$$

We want $P(HCC \mid \text{test +ve})$

$$= \frac{P(\text{test +ve} \mid HCC)\, P(HCC)}{P(\text{test +ve})}$$

$$P(\text{test +ve}) = P(\text{test +ve} \mid HCC)P(HCC) + P(\text{test +ve} \mid \text{not } HCC)\, P(\text{not } HCC)$$

$$= 0.6 \times \frac{4.1}{100000} + 0.1 \times \left(1 - \frac{4.1}{100000}\right)$$

$$\simeq 0.1$$

$$P(HCC \mid \text{test +ve}) = \frac{0.6 \times \frac{4.1}{100000}}{0.1} \simeq 0.0002459.$$

Independent events

We say A and B are *independent* if the knowledge that ~~A~~ B has occurred does not change the *probability* that ~~B~~ A occurs. Mathematically, this is written

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)\,.$$

If $A \cap B = \emptyset$

$P(A \cap B) = 0 = P(A)P(B)$

Since $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$ an alternative definition is: they cannot be independent unless $P(A)=0$ or $P(B)=0$.

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

This definition covers the case $B = \emptyset$, which is always independent of every event.

Mutually Independent Events

We can extend this to *arbitrary* many events:

The events $A_1, A_2, \ldots,$ are said to be (mutually) independent if for any k and any choice of distinct indices $i_1, \ldots, i_k$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})\,.$$

Example: Suppose that a couple have four children, where each child is either a boy or a girl, each with probability 1/2, independent of the outcome for any other child. What is the probability of the children being born in the order (boy, girl, girl, girl)?

In our heads, we are thinking

$B \cap G \cap G \cap G$

$= B \cap G$

Bad notation!

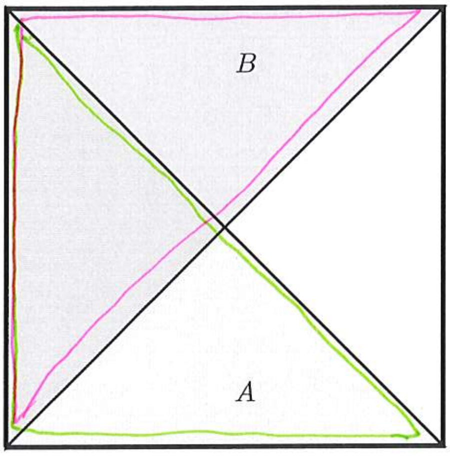
$$\begin{aligned}\mathbb{P}(BGGG) &= P(B) \times P(G) \times P(G) \times P(G)\\ &= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\\ &= \tfrac{1}{16}\\ &= .\end{aligned}$$

Mathematically, what's happening is we are treating the outcomes as events: Let B_1 be the event that the first child is a boy and let each G_i be the event that the ith child is a girl.

Good notation

$$\begin{aligned}\mathbb{P}(B_1G_2G_3G_4) &= P(B_1) \times P(G_2) \times P(G_3) \times P(G_4)\\ &= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\\ &= \tfrac{1}{16}\end{aligned}$$

Exercise. We "uniformly" select a point in the unit square. Show that the events $\{(x, y) : x + y \leqslant 1\}$ and $\{(x, y) : x - y \leqslant 0\}$ are independent.



"uniformly"

$\mathbb{P}(\{(x,y) \in C\}) = \text{area of } C$

$\Omega = [0, 1] \times [0, 1]$

Figure 3.2: $A = \{x + y \leqslant 1\}$ and $B = \{x - y \leqslant 0\}$.

We need to check if

$$\mathbb{P}(A \cap B) = 1/4$$

$$\mathbb{P}(A) = 1/2, \text{ and}$$

$$\mathbb{P}(B) = 1/2$$

$$\Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Therefore A and B are independent events.

Remark. In most cases independence of events is a *model assumption.* That is, we assume that there exists a $\mathbb{P}$ such that certain events are independent.

Figure 3.3: The system fails if either component fails.

Example: Consider the system of Figure 3.3, with two components (1 and 2). The system fails if one (or both) of the components fails. Each component fails independently of the other with probability 0.01. What is the probability of a system failure?

If A_1 and A_2 are independent, are A_1^c and A_2^c also independent?

Let A_i be the event that Component i fails. Let A be the event that the system fails.
Note that $A = A_1 \cup A_2$
To apply independence, we need an intersection, so instead can look at $A^c = (A_1 \cup A_2)^c$
$= A_1^c \cap A_2^c$

$P(A) = P(A_1 \cup A_2)$

$$\begin{aligned}
\mathbb{P}(A^c) &= P(A_1^c \cap A_2^c) = P(A_1^c)\,P(A_2^c) \\
&= (1 - P(A_1))(1 - P(A_2)) \\
\mathbb{P}(A) &= 1 - P(A^c) \\
&= 1 - (1 - 0.01)(1 - 0.01) \\
&\approx 0.0199
\end{aligned}$$

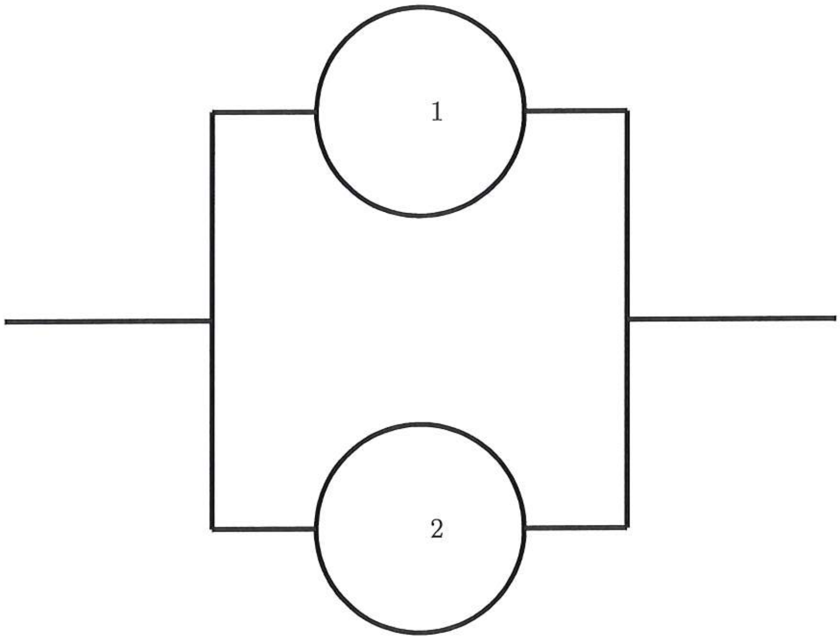


Figure 3.4: The system fails if both components fail.

Example: Consider the system of Figure 3.4, with two components (1 and 2), but this time, the system fails only if *both* of the components fails. Again, each component fails independently of the other with probability 0.01. What is the probability of a system failure?

Let A_i be the event that Component i fails. Let A be the event that the system fails.
Note that $A =$, this time.

$$\begin{aligned}
\mathbb{P}(A) &= \\
&= \\
&= \\
&= .
\end{aligned}$$