D_c = the 2-out-of-3 system is functioning

$= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3)$

Axioms and Implications

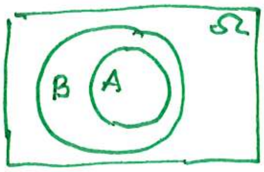
$[0,1]$

Definition. A **probability** $\mathbb{P}$ is a rule (or function) which assigns a number to each event, and which satisfies the following **axioms** (or properties):

- Axiom 1: $\mathbb{P}(A) \geqslant 0$.
- Axiom 2: $\mathbb{P}(\Omega) = 1$. one of the possible outcomes occur
- Axiom 3: **Sum Rule**: For any disjoint $A_1, A_2, \ldots$

$$\boxed{\mathbb{P}\left(\bigcup_j A_j\right) = \sum_j \mathbb{P}(A_j).}$$

Some **consequences** of the axioms:



A

$(B \cap A^c)$

- Consequence 1: If $A \subseteq B$ then $\mathbb{P}(A) \leqslant \mathbb{P}(B)$
- Consequence 2: $\mathbb{P}(\emptyset) = 0$

  like 3: Take $A = \Omega$, $\Omega^c = \emptyset$
- Consequence 3: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

  see below
- Consequence 4: $0 \leqslant \mathbb{P}(A) \leqslant 1$.

  like 1: $\emptyset \subseteq A \subseteq \Omega$
- Consequence 5: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. *

  If A and B are disjoint, this reduces to Axiom 3.

Proving these involves cleverly using the axioms, above.

Example: Prove Consequence 3.

$$\Omega = A \cup A^c$$

$$
\begin{aligned}
1 &= \mathbb{P}(\Omega) && \text{(Axiom 2)} \\
&= \mathbb{P}(A \cup A^c) \\
&= \mathbb{P}(A) + \mathbb{P}(A^c) && \text{(Axiom 3).}
\end{aligned}
$$

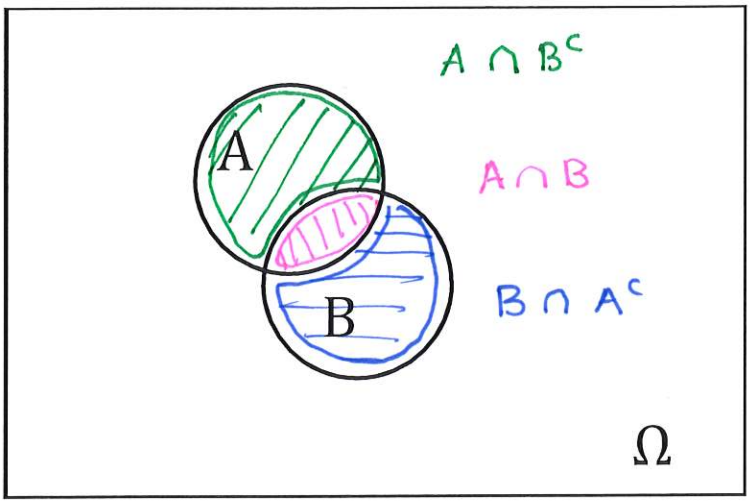
Now rearrange this equation. $\rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Question: Visually, what does Consequence 5 mean?

$\mathbb{P}(A) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B)$

$\mathbb{P}(B) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A \cap B)$



Note that these simple rules of probability are highly similar to those one would use to measure length, area, volume, weight, etc.

Example: Consider a fair, two-sided coin, with sample space $\Omega = \{H, T\}$ (here, the word *fair* means $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 1/2$). Write out all possible events and verify that the third probability axiom is satisfied in the situation $A_1 = \{H\}$, $A_2 = \{T\}$.

The events are $\emptyset, \{H\}, \{T\}, \{H, T\}$

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(\Omega) = 1$$

and since A_1 and A_2 are disjoint

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

Question: In the above example, $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(A_1) = 1/2$. Is it true that

$$\mathbb{P}(\Omega \cup A_1) = \mathbb{P}(\Omega) + \mathbb{P}(A_1) = 1 + \frac{1}{2}?$$

Why/why not?

This is FALSE because A_1 and Ω are not disjoint

In fact,

$\Omega \cup A_1 = \Omega$ $\mathbb{P}(\Omega \cup A_1) = \mathbb{P}(\Omega) = 1.$

Discrete Sample Spaces

If Ω is *countable* it is called a **discrete** sample space; otherwise it is called a **continuous** sample space.

Let Ω be a discrete sample space, e.g. $\Omega = \{a_1, a_2, \ldots, a_n\}$.

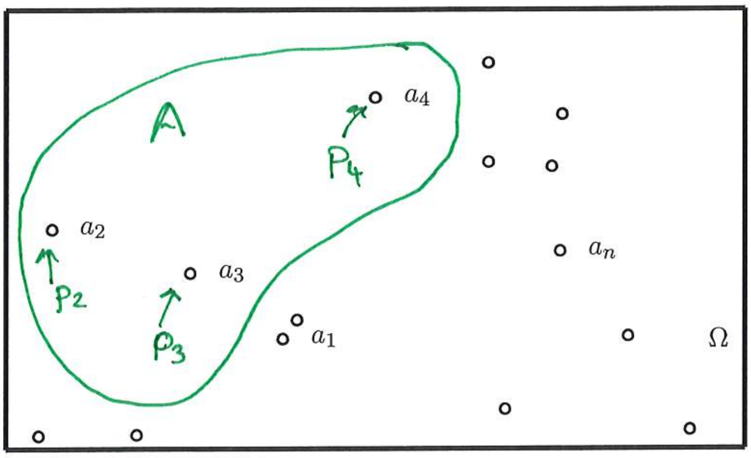


Figure 2.1: A finite sample space $\Omega = \{a_1, a_2, \dots, a_n\}$.

Let $\mathbb{P}(\{a_i\}) = p_i$, for $i = 1, \dots, n$, and define

$$\mathbb{P}(A) = \sum_{i:a_i \in A} p_i\,, \quad \text{for all} \quad A \subset \Omega.$$

Then $\mathbb{P}$ is a probability measure.

Thus we can *specify* $\mathbb{P}$ by specifying only the probabilities of the **elementary events** $\{a_i\}$.

Example. Experiment: throw a fair die.

Sample space:

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Define $\mathbb{P}$ by:

for each $i = 1, \dots, 6$
$p_i = 1/6$

$$\mathbb{P}(A) = \frac{\#\text{ elements in } A}{6} = \frac{|A|}{6}$$

cardinality of A.

This completely specifies/models the experiment. For example, the probability of getting an even number is

$$\mathbb{P}(\{2, 4, 6\}) = \frac{3}{6} = 1/2$$

Example. We draw two cards from a full deck of 52 cards. What is the probability of drawing at least one Ace?

Give all the cards a number from 1 to 52. Draw the cards one-by-one. Write a possible outcome as (x, y), $x \neq y$.

four suits, each suit has 13 cards
2, ..., 10, J, Q, K, A

Each elementary event $\{(x, y)\}$ has the same probability

1st card is any one of 52	$\dfrac{1}{52 \times 51}$	2nd card is any one of remaing 51

Let A be the event: "at least one Ace". Then,

$$\mathbb{P}(A) = \sum_{(x,y) \in A} \frac{1}{52 \times 51} = \frac{|A|}{52 \times 51}$$

We need to *count* how many elements are in A. Easier: $|A^c| = 48 \times 47$. Hence,

A^c neither card is an ace

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{|A^c|}{52 \times 51} = 1 - \frac{48 \times 47}{52 \times 51} \simeq 0.15$$

Remark. In many cases, as above, we can choose Ω such that each elementary event is equally likely, i.e. $\mathbb{P}(\{a_i\}) = 1/n$.

This is sometimes known as the *Equally-Likely Principle*, or the *Equilikely Principle.*

Question: What if we choose the two cards *at the same time* (no order). Does that change the model? Does it change the probability?

sample space changes

Example: Consider tossing a fair coin twice (so that you sample from $\{H, T\}$ *with replacement*). What is the probability of getting both one heads and one tails?

- **Model I:** Order recorded:

(H, H)

$$\Omega = \{HH, HT, TH, TT\}$$
$$A = \{HT, TH\}$$
$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{2^2} = \frac{2}{4} = \frac{1}{2}$$

- **Model II:** Order ignored:

$$\widetilde{\Omega} = \{\overset{1/4}{\{H, H\}}, \overset{1/2}{\{H, T\}}, \overset{1/4}{\{T, T\}}\}$$
$$\widetilde{A} = \{\{H, T\}\}$$
$$\widetilde{\mathbb{P}}(\widetilde{A}) = \frac{1}{2}$$

is associated with HT and TH

Question: Did we apply the Equally-Likely Principle in Model I of the above example? YES
Did we apply it in Model II? NO

Example: Suppose that we have three balls (labelled 1, 2, and 3) in an urn, and select (*without replacement*) two of the balls. Consider the event of 1 being chosen as one of the two balls.

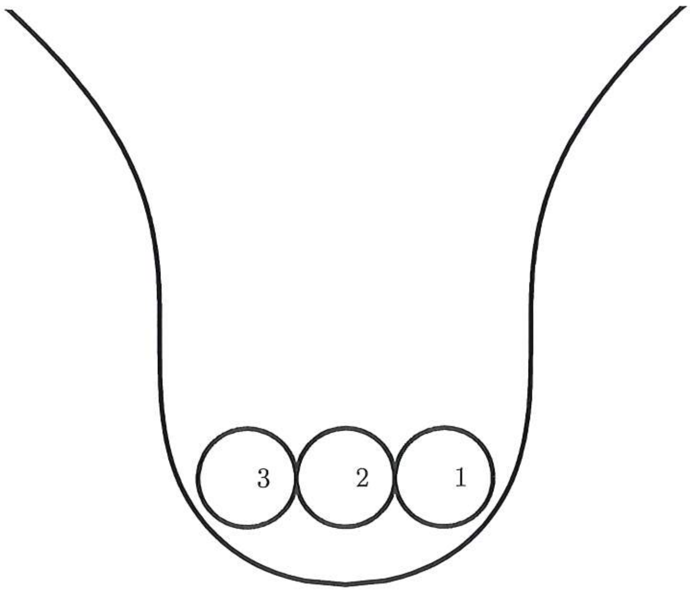


Figure 2.2: Balls 1, 2, and 3 in an urn.

- **Model I:** Order recorded:

$$\Omega = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

Ball 1 is chosen $A = \{(1,2), (1,3), (2,1), (3,1)\}$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{6} = \frac{4}{6} = \frac{2}{3}$$

- **Model II:** Order ignored:

$$\widetilde{\Omega} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Ball 1 chosen $\widetilde{A} = \{\{1,2\}, \{1,3\}\}$

$$\widetilde{\mathbb{P}}(\widetilde{A}) = \frac{|\widetilde{A}|}{|\widetilde{\Omega}|} = \frac{|\widetilde{A}|}{3} = \frac{2}{3}.$$

- each outcome in $\widetilde{\Omega}$ is associated with two outcomes in Ω
- Equally likely principle still holds.

Counting

In the previous examples we drew two cards from a full deck, and balls from an urn, *without replacement*. In general, if choosing r objects from a collection of n objects, without replacement, then there are

$$n \times (n-1) \times \cdots \times (n-r+1)$$

ways of doing this. We write ${}^nP_r$ for this quantity. (Check your calculator!) Another way to write this (n permutation r) is

$${}^nP_r = \frac{n!}{(n-r)!},$$

where $n!$ ("n factorial") is given by

$$n! = n \times (n-1) \times \cdots \times 2 \times 1.$$

Notice that the *order* of the letters matters.

Example. Two letters are chosen *with replacement* from the word *PING*. What is the sample space? How many outcomes are there?

$$\Omega = \{PP, PI, PN, PG, IP, II, IN, IG, NP, NI, NN, NG, GP, GI, GN, GG\}$$
$$|\Omega| = 4 \times 4 = 16$$

Notice that the *order* of the letters matters.

Example. How many three-letter words can be obtained from the letters in the word *SURFING*? This is sampling without replacement. Does the order matter?

$${}^7P_3$$

$$|\Omega| = 7 \times 6 \times 5 = \frac{7!}{(7-3)!} = 210$$

Example. Three people from the set {Joffrey, Balon, Robb, Stannis, Renly} must be chosen to go into a fighting pit. How many different combinations of combatants are there?

Suppose the order is important

$$|\Omega| = {}^5P_3 = 5 \times 4 \times 3 = 60$$

Some outcomes are equivalent though, so this is too many! Our event is a union of several outcomes.
Considering the opponents {Robb, Balon, Stannis}. This combination has been counted ${}^3P_3 = 6$ times:

$$RBS, \quad RSB, \quad BRS, \quad BSR, \quad SRB, \quad SBR$$

If N is the number of ways of choosing opponents, then $N \times {}^3P_3 = {}^5P_3$, so

$$N = {}^5P_3/{}^3P_3 = \frac{5!}{(5-3)!3!} = 10 = {}^5C_3$$

In general, if choosing r objects from a collection of n objects, without replacement, then the number of combinations is:

$${}^nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r} \leftarrow \text{notation we will use.}$$

Summary.

	Order matters	Order does not matter
With replacement	n^r	$\binom{n+r-1}{r}$
Without replacement	${}^nP_r = \frac{n!}{(n-r)!}$	${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$

Matlab code.

```
n = 6;
r = 2;
factorial(n)
ans =
    720
nchoosek(n,r)
ans =
     15
nPr = nchoosek(n,r)*factorial(r)
nPr =
     30
```

Example. How many ways are there to order the letters in the word *INDOOROOPILLY*?

Notice that this situation does not fall into any of the above categories. In general, the number of permutations of n objects with n_1 of type 1, n_2 of type 2, et cetera, is given by

$$\frac{n!}{n_1!n_2!\dots n_k!},$$

where k is the number of types.

Hence, the number of ways to order the letters in *INDOOROOPILLY* is

13P13/(4P4*2P2*2P2) =21621600

Example. Suppose that you have two red balls and three blue balls. How many *distinct* orderings of all balls are there?

4C1+4C2=10