Question. If $X \sim \text{Normal}(\mu, \sigma^2)$, what is the distribution of $Z = (X - \mu)/\sigma$?

$$
\begin{aligned}
F_Z(z) &= \mathbb{P}\left(\frac{X-\mu}{\sigma} \leqslant z\right) \\
&= \mathbb{P}(X \leqslant \mu + \sigma z) \\
&= F_X(\mu + \sigma z) \\
f_Z(z) = \frac{d}{dz} F_Z(z) &= \frac{d}{dz} F_X(\mu + \sigma z) \\
&= \sigma \cdot f_X(\mu + \sigma z) = \cancel{\sigma} \cdot \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \exp\left(-\frac{(\cancel{\mu} + \sigma z - \cancel{\mu})^2}{2\sigma^2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)
\end{aligned}
$$

So the distribution of Z is $\boxed{N(0,1)}$.

This is an important property that enables us to use the tabulated values of the standard normal cdf to determine the cdf of a normal distribution in general.

Question. Use the tables of the standard normal cdf to determine the following probabilities:

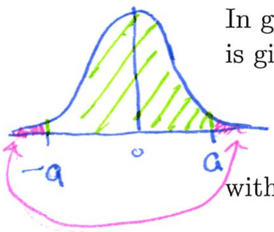
Let $Z = \frac{X-1}{1} \to Z \sim N(0,1)$

Let $X \sim \text{Normal}(1, 1)$. Find $\mathbb{P}(X \leqslant 1)$.

$\mathbb{P}(X \leqslant 1) = \mathbb{P}(Z \leqslant \frac{1-1}{1}) = \mathbb{P}(Z \leqslant 0) = 1/2$.

Let $X \sim \text{Normal}(0, 4)$. Find $\mathbb{P}(X \leqslant 3.92)$.

$\mathbb{P}(X \leqslant 3.92) = \mathbb{P}(Z \leqslant \frac{3.92-0}{2}) = \mathbb{P}(Z \leqslant 1.96) = 1 - \mathbb{P}(Z \geqslant 1.96) = 1 - 0.025$
$= 0.975$



$\mathbb{P}(-a \leqslant X \leqslant a)$
$= 1 - 2\mathbb{P}(X \geqslant a)$
$X \sim N(0,1)$.

In general, for a continuous random variable X with pdf $f_X(x)$, the pdf of $Y = aX + b$ is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right),$$

with support $\{y : y = ax + b,\ x \in \text{supp}(X)\}$.

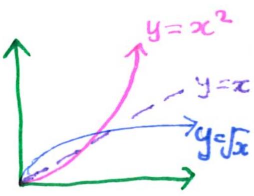
$$f_X(x) = \begin{cases} e^{-x}, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$

Question. If $X \sim \text{Exponential}(1)$, what is the distribution of $Y = 5X$?

exponential $(\frac{1}{5})$ distribution

$$f_Y(y) = \begin{cases} \frac{1}{5}\exp(-y/5), & y \geqslant 0 \\ 0, & y < 0 \end{cases}$$

Monotone transformations



If X is a continuous random variable and $Y = g(X)$, where $g : \mathbb{R} \to \mathbb{R}$ is monotonic, then we can easily obtain the distribution of Y from that of X. If g is increasing, then, for all $y \in \mathbb{R}$,

$$F_Y(y) = \mathbb{P}(g(X) \leqslant y) = \mathbb{P}(X \leqslant g^{-1}(y)) = F_X(g^{-1}(y)).$$

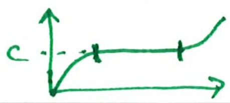
where g^{-1} is the inverse of g. The pdf of Y is then given by

\# $g^{-1}(x)$ is not $\frac{1}{g(x)}$

\# If g is increasing, then g^{-1} is also increasing.

strictly increasing

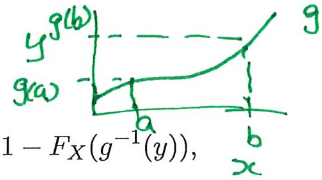
$\frac{d}{dy} g^{-1}(y) = \frac{1}{g'(y)}$



$Y = g(X)$ $\quad \mathbb{P}(Y = c) > 0$

$$\boxed{\frac{d}{dy}F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \frac{1}{g'(y)} f_X(g^{-1}(y))}$$

If X has support $[a, b]$, then the support of Y is $\boxed{[g(a), g(b)]}$.



On the other hand, if g is decreasing, then

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(X \geqslant g^{-1}(y)) = 1 - \lim_{z \uparrow y} F_X(g^{-1}(z)) = 1 - F_X(g^{-1}(y)),$$

$\mathbb{P}(g(X) \leqslant y)$ because g^{-1} is decreasing.

The pdf of Y is then given by

$$\frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right) = \frac{1}{|g'(y)|} f_X(g^{-1}(y)).$$

If X has support $[a, b]$, then the support of Y is $\boxed{[g(b), g(a)]}$.

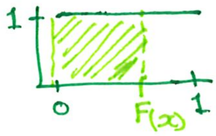
An important type of monotone transformation is given by the *inverse cdf* or *quantile function.*

Defintion. Let X be a continuous random variable. The function $q_X : \mathbb{R} \to \mathbb{R}$ such that

$$F_X(q_X(x)) = x,$$

is called the **quantile function** of X.

Note that the quantile function is an increasing function.



Suppose F_X is the cdf of a continuous random variable X and q_X is its quantile function. If $U \sim \mathsf{Uniform}(0, 1)$, then the cdf of $q_X(U)$ is

$$\boxed{\begin{aligned}\mathbb{P}(q_X(U) \leqslant x) &= \mathbb{P}(F_X(q_X(U)) \leqslant F_X(x)) \\ &= \mathbb{P}(U \leqslant F_X(x)) = F_X(x)\end{aligned}}$$

Example. For $X \sim \mathsf{Exp}(\lambda)$ (consider Figure 5.4) we have:

$$\begin{aligned} x &= F_X(q) = 1 - e^{-\lambda q} \\ 1 - x &= e^{-\lambda q} \\ \log(1-x) &= -\lambda q \\ -\frac{\log(1-x)}{\lambda} &= q \end{aligned}$$

$$\Rightarrow \quad q_X(x) = -\frac{\ln(1-x)}{\lambda}$$

when $1 > x \geqslant 0$. Note that if $U \sim \mathsf{Uniform}(0, 1)$, then $V = 1 - U$ has a $\boxed{\mathsf{Uniform}(0,1)}$ distribution.

As a result we can define a MATLAB function to generate samples from the Exponential distribution as follows.

$v \in (0,1)$

$$\begin{aligned} F_V(v) &= \mathbb{P}(V \leqslant v) \\ &= \mathbb{P}(1 - U \leqslant v) \\ &= \mathbb{P}(U \geqslant 1 - v) \\ &= 1 - \mathbb{P}(U < 1 - v) = 1 - (1 - v) = v \end{aligned}$$

so ...

$\mathbb{P}(U \leqslant x) = x$

$x \in (0,1)$

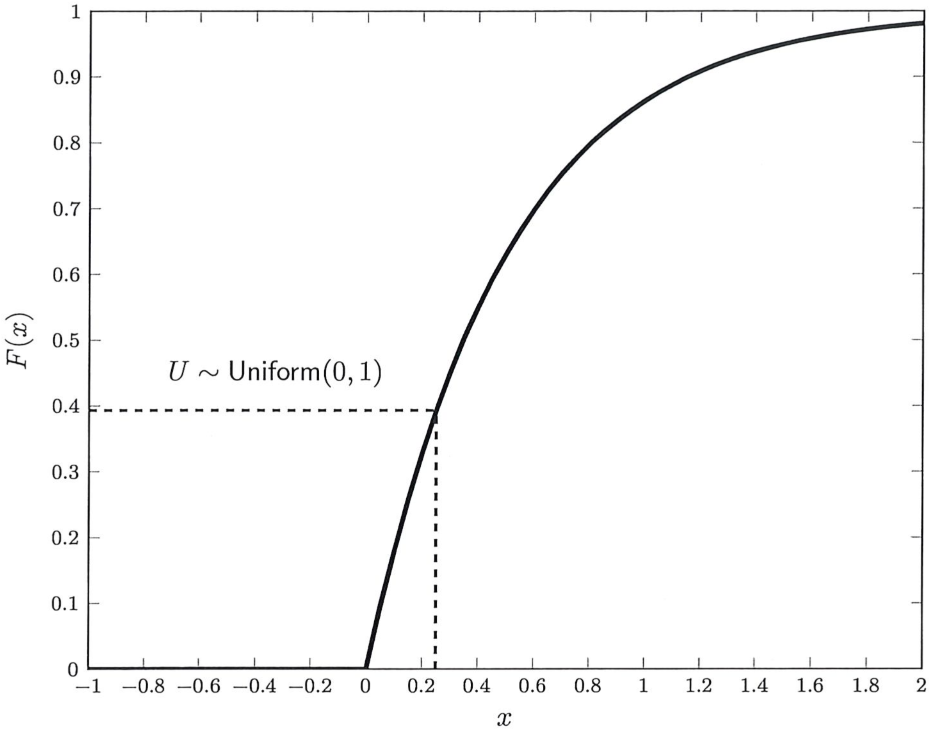


Figure 5.4: The cdf of $X \sim \mathsf{Exp}(2)$.

rand MATLAB function generating Uniform(0,1)

```
function output = Exponential(lambda)
    output = -log(rand)/lambda;
end
```

Upon saving this as 'Exponential.m' to our working directory we can then use this function as follows:

```
>> Exponential(2)
ans =
      0.0453
>> Exponential(2)
ans =
      0.2291
>> Exponential(2)
ans =
      1.1637
```

Be careful, as the built in MATLAB function `exprnd` generates samples from the $\mathsf{Exp}(\lambda^{-1})$ distribution.

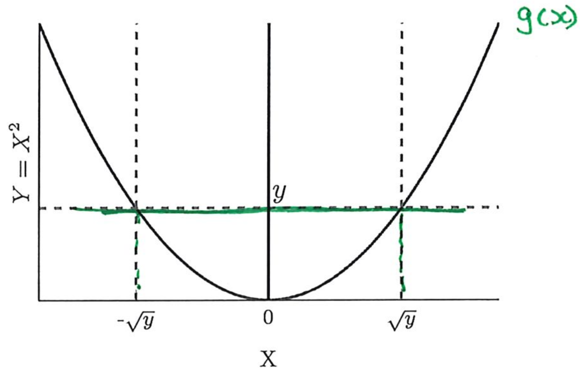
Non-monotone transformations

If the function g is not monotone, then we can still make progress by considering separately those intervals over which it is monotone. The general procedure to follow is given below:

$Y = g(X)$

1. Determine the support of the Y.
2. Determine the event in terms of the random variable X that maps to the event $\{Y \leqslant y\}$. Typically this will be in the form of a union of disjoint events of the form $\{a \leqslant X \leqslant b\}$.
3. Find the probability $F_Y(y)$ of the event $\{Y \leqslant y\}$ in terms of F_X, the cumulative distribution function of X.
4. Differentiate the result to find the probabilty density function of Y.

Example. Suppose $X \sim \mathsf{Normal}(0,1)$ and $Y = X^2$. Find the pdf of Y.



Step 1: The function $g(x) = x^2$ maps $\mathbb{R}$ to $[0,\infty)$. So the support of Y is $[0,\infty)$.

Step 2: From the figure it is clear that $\{Y \leqslant y\}$, where $y \geqslant 0$, corresponds to $\{-\sqrt{y} \leqslant X \leqslant \sqrt{y}\}$.

Steps 3 and 4:

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(-\sqrt{y} \leqslant X \leqslant \sqrt{y})$$
$$= \mathbb{P}(X \leqslant \sqrt{y}) - \mathbb{P}(X \leqslant -\sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

so that $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}, \quad y \geqslant 0.$$

This distribution is called the χ^2_1-distribution.

$$\frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} + \frac{1}{2\sqrt{y}\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2}$$