Let X be the lifetime of a certain component in years. Suppose X has pmf

x	0	1	2	3	4	5	6
$\mathbb{P}(X=x)$	0.05	0.25	0.30	0.20	0.10	0.05	0.05

(a) Calculate the probability that a component has a lifetime of at least 3 years.

$$\mathbb{P}(X \geqslant 3) = \mathbb{P}(X=3) + \mathbb{P}(X=4) + \mathbb{P}(X=5) + \mathbb{P}(X=6)$$

$$= 0\cdot2 + 0\cdot1 + 0\cdot05 + 0\cdot05 = 0\cdot4$$

(b) Calculate the conditional probability mass function of the lifetime of a component given that it is still working after 3 years.

$$\mathbb{P}(X=k \mid X \geqslant 3) \qquad k=3,4,5,6$$

$$= \frac{\mathbb{P}(X=k, X \geqslant 3)}{\mathbb{P}(X \geqslant 3)} = \frac{\mathbb{P}(X=k)}{\mathbb{P}(X \geqslant 3)}$$

k	3	4	5	6
$\mathbb{P}(X=k)$	$\frac{0\cdot2}{0\cdot4} = \frac{1}{2}$	$\frac{0\cdot1}{0\cdot4} = \frac{1}{4}$	$\frac{0\cdot05}{0\cdot4} = \frac{1}{8}$	$\frac{0\cdot05}{0\cdot4} = \frac{1}{8}$

(c) Calculate the expected lifetime of a component given it is still working after 3 years.

$$\mathbb{E}[X \mid X \geqslant 3] = \sum_{k=3}^{6} k \cdot \mathbb{P}(X=k \mid X \geqslant 3)$$

$$= 3 \times \frac{1}{2} + 4 \times \frac{1}{4} + 5 \times \frac{1}{8} + 6 \times \frac{1}{8} = 3\cdot875$$

From "Introduction to Probability" Blitzstein and Hwang – Chapter 9.

11. A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

X first roll

Y number of additional rolls needed to obtain a value of at least X.

We want $\mathbb{E}[Y]$

What is the conditional pmf of Y given $X=x$?

*

pmf $Y \mid X=6$ is geometric $(1/6)$

pmf $Y \mid X=5$ is geometric $(2/6)$

pmf $Y \mid X=k$ is geometric $\left(\frac{7-k}{6}\right)$ $k=1,2,\ldots,6$.

What is the conditional expectation of Y given $X=x$?

$$\mathbb{E}[Y \mid X=k] = \frac{6}{7-k} \qquad k=1,2,\ldots,6.$$

What is the expected value of Y?

$$\mathbb{E}[Y] = \cancel{\Sigma}\ \mathbb{E}[\mathbb{E}[Y \mid X]]$$

$$= \sum_{k=1}^{6} \mathbb{E}[Y \mid X=k] \times \mathbb{P}(X=k)$$

$$= \sum_{k=1}^{6} \frac{6}{7-k} \times \frac{1}{6} = 2.45.$$

Recall that if $X_i \sim \mathsf{Bernoulli}(p)$, then $\mathbb{P}(X_i = x) = p^x(1-p)^{1-x}$ for $x = 0, 1$.

$$M_{X_i}(s) = \mathbb{E}(e^{sX_i})$$
$$= (1-p)e^{s\times 0} + pe^{s\times 1} = 1-p+pe^s$$

Therefore,

$X \sim \text{Binomial}(n,p)$

$$M_X(s) = \prod_{i=1}^{n} M_{X_i}(s) \left\{= M_{X_1}(s) \times \cdots \times M_{X_n}(s)\right\}$$
$$= (1-p+pe^s)^n$$

Example. In the summary of formulas at the start of the workbook, the ~~PGF~~ MGF of $X \sim \mathsf{Poisson}(\lambda)$ is given as:

$$M_X(s) = \exp(\lambda(e^s - 1))$$

Hence, for $X \sim \mathsf{Poisson}(\lambda)$ and $Y \sim \mathsf{Poisson}(\mu)$ (indep.), the MGF of $Z = X + Y$ is:

$$M_Z(s) = M_X(s)\,M_Y(s) = \exp(\lambda(e^s-1)) \times \exp(\mu(e^s-1)) = \exp((\lambda+\mu)(e^s-1))$$

Which shows that $\boxed{Z \sim \text{Poisson}(\lambda+\mu)}$

A useful property of the MGF is that we can obtain the *moments* of X by *differentiating* M and evaluating it at $s = 0$.

$\mathbb{E}[X]$ 1st moment

$\mathbb{E}[X^2]$ 2nd moment.

Differentiating $M(s)$ with respect to s gives

$$M'(s) = \frac{\mathrm{d}\,\mathbb{E}\,\mathrm{e}^{sX}}{\mathrm{d}s} = \mathbb{E}\,X\mathrm{e}^{sX} \longrightarrow M'(0) = \mathbb{E}[X]$$
$$M''(s) = \frac{\mathrm{d}\,\mathbb{E}\,X\mathrm{e}^{sX}}{\mathrm{d}s} = \mathbb{E}\,X^2\mathrm{e}^{sX} \rightarrow M''(0) = \mathbb{E}[X^2]$$
$$M'''(s) = \mathbb{E}\,X^3\mathrm{e}^{sX}$$
$$\vdots$$
$$M^{(k)}(0) = \mathbb{E}\left[X^k\right]$$

In particular

$$\mathbb{E}[X] = M'(0)$$

and

$$\boxed{\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = M''(0) - (M'(0))^2}$$

Exercise: Prove this variance formula.

$$\mathrm{Var}(X) =$$

but

$$M'_X(0) =$$
$$M''_X(0) =$$
$$=$$
$$\Rightarrow \mathbb{E}\left[X^2\right] =$$

$$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$$

Exercise. Using the MGF find the variance of a Geometric(p) distribution.

$$M'_X(s) = \frac{pe^s(1-(1-p)e^s) + (1-p)e^s pe^s}{(1-(1-p)e^s)^2} = \frac{pe^s}{(1-(1-p)e^s)^2}$$

$$M''_X(s) = \frac{pe^s(1-(1-p)e^s)^2 + 2(1-p)e^s(1-(1-p)e^s)\cdot pe^s}{(1-(1-p)e^s)^4}$$

so

$$\mathbb{E}[X] = M_X'(0) = \frac{p}{p^2} = \frac{1}{p}$$

$$\mathbb{E}[X^2] = M''_X(0) = \frac{p^3 + 2(1-p)p^2}{p^4} = \frac{\cancel{p}(2-p)}{p^2}$$

$$\mathrm{Var}(X) = M_X''(0) - (M_X'(0))^2$$
$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}.$$
$$=$$

Finally, we stated earlier that the Poisson distribution can be viewed as the limit of a sequence of Binomial distributions with the number of trials increasing and the success probability decreasing at a particular rate. We can justify this using MGFs.

Let X_n, $n = 1, 2, \ldots$ be a sequence of non-negative integer valued random variables with MGFs $M_{X_n}(s)$ and let $M_X(s)$ be the MGF of X. If $M_{X_n}(s) \to M_X(s)$ for all s in some neighbourhood of 0, then

$$\lim_n \mathbb{P}(X_n = k) = \mathbb{P}(X = k), \quad \text{for all } k = 0, 1, 2 \ldots$$

We say that X_n *converges in distribution* to X.

MGF for Binomial (n,p) $\qquad (1-p+pe^s)^n$

Example. Consider the sequence of random variable $X_n \sim \text{Binomial(n}, \lambda/\text{n)}$. We can show that X_n converges in distribution to a $\text{Poisson}(\lambda)$ random variable.

$$M_{X_n}(s) = \left(1-\frac{\lambda}{n}+\frac{\lambda}{n}e^s\right)^n = \left(1+\boxed{\frac{\lambda}{n}(e^s-1)}\right)^n$$

It is known that $\lim_{n\to\infty}(1+x/n)^n = e^x$. So

$$\lim_{n\to\infty} M_{X_n}(s) = \exp(\lambda(e^s-1))$$

which we can identify as the MGF of a Poisson distribution with mean λ.

Application: Run time of quicksort

Consider a list of n distinct numbers which we want to sort into increasing order. The quicksort algorithm begins by choosing an element from the list called the pivot. The pivot is compared with all other elements in the list and the list is then divided into two sublists:

- one comprising those elements less than the pivot,
- the other comprising those elements greater than the pivot.

This procedure is then repeated on the sublists until the entire list is sorted.

(Initial list)	10	5	3	7	9	2	1
(With 7 as pivot)	5	3	2	1	**7**	10	9
(With 3 as pivot for left sub-list)	2	1	**3**	5	7	10	9
...							
(List sorted)	1	2	3	5	7	9	10

If the pivot is simply taken to be the first element of the list and the list is already sorted then all pairs of numbers will need to be compared. Therefore, the number of comparisons is $\boxed{}$.

On the other hand, if the pivot is selected uniformly at random from the list, then the number of comparisons tends to be much smaller.

Let X_n be the random variable giving the number of comparison required to sort a list of n items. By convention $\mathbb{P}(X_0 = 0) = 1$. To compute $\mathbb{E}\, X_n$ we will condition of the selection of the first pivot

Moment generating function of Poisson(λ).

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

$$\mathbb{E}[e^{sX}] = \sum_{k=0}^{\infty} e^{sk}\, \mathbb{P}(X = k)$$

$$= \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \boxed{\sum_{k=0}^{\infty} \frac{(e^s\lambda)^k}{k!}} = e^{-\lambda} \exp(\lambda e^s) = \exp(\lambda(e^s - 1)).$$