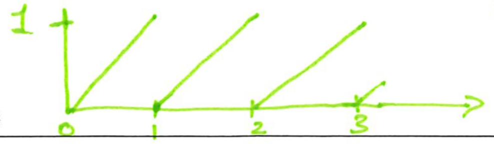

$y = x - \lfloor x \rfloor$

rounding down X

Example. Suppose $X \sim \mathsf{Exp}(\lambda)$ and $Y = X - \lfloor X \rfloor$. Find the pdf of Y. (*Note that* Y *is the fractional part of* X.)

Let $y \in (0,1)$. The event $\{Y \leqslant y\}$ can be written in terms of the random variable X as

$$\bigcup_{n=1}^{\infty} \{n-1 \leqslant X \leqslant n-1+y\}$$

$$\{0 \leqslant X \leqslant y\} \cup \{1 \leqslant X \leqslant 1+y\} \cup \{2 \leqslant X \leqslant 2+y\} \cdots$$

$$\mathbb{P}(Y \leqslant y) = \mathbb{P}\Big(\bigcup_{n=1}^{\infty} \{n-1 \leqslant X \leqslant n-1+y\}\Big) = \sum_{n=1}^{\infty} \mathbb{P}(n-1 \leqslant X \leqslant n-1+y) = \sum_{n=1}^{\infty} [F_X(n-1+y) - F_X(n-1)]$$

if A and B are disjoint $P(A \cup B) = P(A) + P(B)$

so that

$$f_Y(y) = \frac{d}{dy} \sum_{n=1}^{\infty} [F_X(n-1+y) - F_X(n-1)] = \sum_{n=1}^{\infty} f_X(n-1+y) = \sum_{n=1}^{\infty} \lambda e^{-\lambda(n-1+y)} = \lambda e^{-\lambda y} \sum_{n=1}^{\infty} e^{-\lambda(n-1)} = \lambda e^{-\lambda y}(1-e^{-\lambda})^{-1}$$

$y \in (0,1)$

Note. We have previously defined the expected value of a continuous random variable X and the expected value of a function g of X. Now that we have studied the effect of transformations on the distribution of a random variable we can see that these two definitions are consistent. That is, given a continuous random variable X and continouos function g, if we define the random variable $Y := g(X)$, then $\mathbb{E}\, Y = \mathbb{E}\,[g(X)]$.

Multiple continuous random variables

As was the case with discrete random variables, we will often have need to work with multiple random variables at once. Recall that the **joint distribution** of the random variables $X_1, \ldots, X_n$, defined in the same random experiment, can be specified through the **joint cumulative distribution function** F defined by

$$F(x_1, \ldots, x_n) = \mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n)\,.$$

This completely specifies the probability distribution of the vector $\boldsymbol{X} := (X_1, \ldots, X_n)$.

We will now just work with a pair of continuous random variables (X, Y) having joint cdf $F_{X,Y}$. The extension to more than two random variables is straightforward.

Lets first recall some basic notions that we saw previously in connection with the joint distribution of multiple discrete random variables.

It is clear from the law of total probability that

$\mathbb{P}(X \leqslant x) = \mathbb{P}(X \leqslant x, Y < \infty)$

$$F_X(x) = \mathbb{P}(X \leqslant x) = \lim_{y \to \infty} F_{X,Y}(x,y)$$

and similarly for F_Y. We refer to F_X and F_Y as **marginal cumulative distribution functions**. From the joint cumulative distribution $F_{X,Y}$ we can determine if X and Y are independent: X and Y are said to be **independent** if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y),$$

Recall X and Y are independent if and only if all pairs $\{X \leqslant x\}$ and $\{Y \leqslant y\}$ are independent.

for all $(x,y) \in \mathbb{R}^2$. Instead of using the cdf to describe the distribution of a single continuous random variable, we usually used its probability density function. Similarly, for multiple continuous random variables we usually use the joint probability density function.

Definition. If there exists a function $f_{X,Y}(x,y)$ such that for all $(x,y) \in \mathbb{R}^2$

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{x} f_{X,Y}(u,v)\, \mathrm{d}u \right\} \mathrm{d}v,$$

treat as two one-variable integrals.

we call $f_{X,Y}$ the **joint probability density function** of (X,Y).

Note that in the above double integral we integrate the variable u first, treating v as constant, and then integrate the variable v. In this setting, the order in which we perform this integration is not important since it can be shown that

$$\int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, \mathrm{d}u\, \mathrm{d}v = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, \mathrm{d}v\, \mathrm{d}u.$$

The joint pdf is not prescribed uniquely by this definition, but, if both of the *partial derivatives* of $F_{X,Y}$ exist at the point (x,y), then

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y).$$

differentiate wrt y, then differentiate wrt x.

The symbol $\frac{\partial^2}{\partial x \partial y}$ means to differentiate $F_{X,Y}(x,y)$ first with respect to y, treating x as constant and then differentiate with respect to x, treating y as constant. The order in which we perform this differentiation is not imporant since it can be shown that

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{X,Y}(x,y).$$

The joint pdf completely specifies the distribution of (X,Y), as does the joint cdf.

Basic properties of $f_{X,Y}$:

- $f_{X,Y}(x,y) \geqslant 0$ for all $(x,y) \in \mathbb{R}^2$;
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, \mathrm{d}x \mathrm{d}y = 1$. $\iff$ $\mathbb{P}(\Omega) = 1$
- $\mathbb{P}(a \leqslant X \leqslant b, c \leqslant Y \leqslant d) = \int_a^b \int_c^d f_{X,Y}(x,y)\, \mathrm{d}y\, \mathrm{d}x$, where a or c can be $-\infty$ and b or d can be ∞, and any of the inequalities can be replaced by strict ones.

Probabilities now given by volumes under the joint pdf.

Example. Let (X, Y) be a pair of random variables having joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c(x+y+xy), & \text{if } (x,y) = [0,1]^2 \\ 0, & \text{else.} \end{cases}$$

What value must c take for $f_{X,Y}$ to be a valid joint pdf?

Recall that the second axiom of probability states that $\mathbb{P}(\Omega) = 1$, meaning that *something* must happen with probability 1. Here that implies

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,\mathrm{d}u\mathrm{d}v = \int_0^1\int_0^1 c(u+v+uv)\,du\,dv$$
$$= c\int_0^1 \left\{ \left[\tfrac{1}{2}u^2 + vu + \tfrac{1}{2}u^2v \right]_{u=0}^{1} \right\} dv$$
$$= c\int_0^1 \left(\tfrac{1}{2} + v + \tfrac{1}{2}v \right) dv = c\left[\tfrac{1}{2}v + \tfrac{3}{4}v^2 \right]_{v=0}^{1}$$
$$= c\left(\tfrac{1}{2} + \tfrac{3}{4} \right) = c \cdot 5/4$$

Therefore, $c =$ $4/5$.

The marginal distribution functions F_X and F_Y can be expressed in terms of $f_{X,Y}$; for example,

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,\mathrm{d}v\mathrm{d}u$$

and this leads to the **marginal probability density function** $f_X(x)$ given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,\mathrm{d}v.$$

The marginal pdf f_Y can be similarly determined.

If X and Y are two independent random variables, then their joint pdf can be factorised

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

for all $(x,y) \in \mathbb{R}^2$. The converse is also true so if a joint pdf can be factorised in this way, then the random variables are independent.

Example. Consider again the pair of random variables (X, Y) having joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 4/5(x+y+xy), & \text{if } (x,y) = [0,1]^2 \\ 0, & \text{else.} \end{cases}$$

What is the marginal pdf X? Are X and Y independent?

For $x \in [0,1]$, $\quad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,v)\,\mathrm{d}v = \int_0^1 4/5(x+y+xy)\,dy$

$$= \frac{4}{5}\left[xy + \frac{1}{2}y^2 + \frac{x}{2}y^2\right]_{y=0}^{1}$$

$$= 4/5\,(1/2 + 3/2\,x) = 2/5 + 6/5\,x.$$

Similarly, $\boxed{f_Y(y) = \frac{2}{5} + \frac{6}{5}y}$ for $y \in [0,1]$. To check independence, for $(x,y) \in [0,1]^2$

$$f_X(x)f_Y(y) = \left(\frac{2}{5} + \frac{6}{5}x\right)\left(\frac{2}{5} + \frac{6}{5}y\right)$$

$$\neq \frac{4}{5}(x+y+xy)$$

so X and Y are not independent.

Similar to the case of a single continuous variable, the expected value of $Z = g(X,Y)$ can be evaluated as

$$\mathbb{E}g(X,Y) = \mathbb{E}Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,\mathrm{d}x\mathrm{d}y$$

and in general, the expected value of $Z = g(X_1,\ldots,X_n)$ can be evaluated as

$$\mathbb{E}Z = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_n) f_{\mathbf{X}}(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where $f_{\mathbf{X}}$ is the joint pdf of $(X_1,\ldots,X_n)$.

The properties of expectation that we determined for discrete random variables also hold for continuous random variables. In particular, if X and Y are two continuous random variables measured on the same random experiment, then

$\mathbb{E}[aX+bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$ [for any constants $a,b \in \mathbb{R}$]

$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ [if X and Y are independent]

$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$ [for any constants $a,b \in \mathbb{R}$]

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$ (see page 54)

$\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]$ [by definition]

$\quad = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$

$\mathrm{Cov}(X,Y) = 0$ [if X and Y are independent]

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$ [if X and Y are independent]

Example. Consider again the pair of random variables (X, Y) having joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{4}{5}(x+y+xy), & \text{if } (x,y) = [0,1]^2 \\ 0, & \text{else.} \end{cases}$$

Compute the covariance of X and Y.

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[XY] = \int_0^1 \int_0^1 xy \cdot \frac{4}{5}(x+y+xy)\, dx\, dy$$

$$= \frac{4}{5} \int_0^1 \int_0^1 (x^2y + xy^2 + x^2y^2)\, dx\, dy$$

$$= \frac{4}{5} \int_0^1 \left\{ \left[\frac{1}{3}x^3y + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3y^2 \right]_{x=0}^{1} \right\} dy$$

$$= \frac{4}{5} \int_0^1 \left(\frac{1}{3}y + \frac{1}{2}y^2 + \frac{1}{3}y^2 \right) dy$$

$$= \frac{4}{5} \left[\frac{1}{6}y^2 + \frac{1}{6}y^3 + \frac{1}{9}y^3 \right]_{y=0}^{1}$$

$$= \frac{4}{5}\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{9} \right) = 16/45$$

$$\mathbb{E}X = \int_0^1 x \cdot \underbrace{\left(\frac{2}{5} + \frac{6}{5}x \right)}_{f_X(x)} dx = \int_0^1 \left(\frac{2}{5}x + \frac{6}{5}x^2 \right) dx$$

$$= \left[\frac{1}{5}x^2 + \frac{2}{5}x^3 \right]_{x=0}^{1}$$

$$= 3/5$$

$\mathbb{E}[Y] = 3/5$ (as Y had same marginal pdf as X)

$$\mathrm{Cov}(X,Y) = \frac{16}{45} - \frac{3}{5} \times \frac{3}{5} = \frac{-1}{225}.$$