A

Practice Questions For Chapters 2 - 4

1. Two cities A and B are connected to a water source W via pipes 1, 2, 3 and 4, see Figure A.1. The system works if both A and B receive water via at least one pipe. Let A_i be the event that the ith pipe is functioning, $i = 1, 2, 3, 4$. Let A be the event that the system is functioning. Express A in terms of $A_1, \ldots, A_4$, using complements, intersections and/or unions.

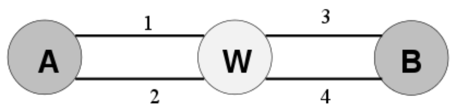


Figure A.1: A water distribution system

2. Thirty percent of people in Metropolis read the Daily Planet, 20% read the Inquirer, and 5% read both news papers. What is the probability that selected person uniformly at random from the population reads at least one newspaper?

3. Two fair dice are thrown and the sum of the face values, Z say, is noted. Give the pmf of Z in table form:

z	*	*	*	...
$\mathbb{P}(Z = z)$	*	*	*	...

4. A certain AIDS test has a 0.98 probability of giving a Positive result when the blood is infected, and a 0.07 probability of giving a Positive result when the blood is not infected (a so-called false positive). Suppose 1% of the population carries the HIV virus.

   Using Bayes' Rule, determine the probability that a person selected uniformly at random from the population is indeed infected, *given* that the test yields a Positive result.

5. We toss a fair coin three times. Give the sample space Ω for this experiment, if we observe the exact sequences of Heads and Tails, and assign the appropriate probabilities to the elementary events.

6. Let A, B and C be events such that: $\mathbb{P}(A) = 1/4$, $\mathbb{P}(B) = 1/3$, $\mathbb{P}(C) = 1/6$, A and B are independent, and $A \cup B$ and C are disjoint.

   Calculate the probability that at least one of the events A, B or C occurs.

7. We toss two fair dice and note the largest number M of the two dice. Derive the probability $\mathbb{P}(M = 4)$.

8. Consider two arrays (Array 1 and Array 2) of 10 bits. We combine Array 1 and Array 2, by using the AND operation (thus, 0 AND 0 = 0; 0 AND 1 = 0; 1 AND 0 = 0; 1 AND 1 = 1). An illustration of this procedure is given in Figure A.2.

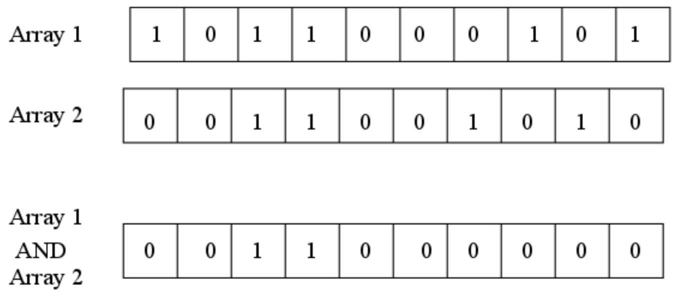


Figure A.2: The AND operation applied to two binary arrays.

Suppose that Array 1 and Array 2 are (independently of each other) filled according to 10 independent Bernoulli trials (coin flip experiments), with success (= 1) probability 1/3 for Array 1, and with success probability 3/4 for Array 2. What is the probability that the "combined" array has fewer than three 1-bits?

9. Consider the water cooling system schematically depicted in Figure A.3. The system has four unreliable components: two identical pumps (P1 and P2) and two valves (V1 and V2). The system works if, in the diagram, there is a path from left to right traversing only working components.

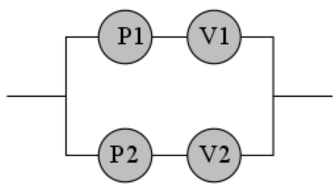


Figure A.3: The system works if there is a path from left to right with working components.

We assume that the pumps and valves fail independently of each other and that the reliability (=probability that it works) for each pump is 0.9 and for each valve is 0.8. Calculate the reliability of the system.

10. We throw with 2 fair dice. Find the probability that both dice show the same face, given that the sum of the dice is not greater than 4.

11. We drop a ball onto a quincunx, see Figure 11. This is a board in which pins are placed in a regular pattern, such that the ball bounces to the left or right of each pin with equal probability.

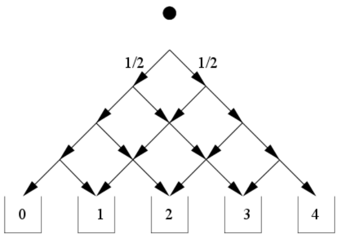


Figure A.4: A quincunx

(a) What is probability that the ball will end up in box 2?

(b) Let X be the number of the box in which the ball falls. What is the pmf of X?

12. Let X be the lifetime of a certain component in years. Suppose X has pmf

x	0	1	2	3	4	5	6
$\mathbb{P}(X = x)$	0.05	0.25	0.30	0.20	0.10	0.05	0.05

(a) Calculate the probability that a component has a lifetime of at least 5 years.

(b) Calculate the conditional probability that a component has a lifetime of at least 5 years, given that it is still alive after 2 years.

(c) Calculate the expected lifetime of a component.

13. A random variable X has expectation 2 and variance 1. Calculate $\mathbb{E}[X^2]$.

14. We flip a fair coin 10 times. How likely is it that fewer than 3 Heads appear?

15. We repeatedly throw two fair dice until two sixes are thrown. What is the probability that more than 20 throws are required?

16. A random variable X takes the values $0, 2, 5$ with probabilities $1/2, 1/3, 1/6$, respectively. What is the expectation of X?

17. We randomly select 3 balls from an urn with 365 balls, numbered 1, . . . ,365.

(a) How many possible outcomes of the experiment are there, if we put each ball back into the urn before we draw the next?

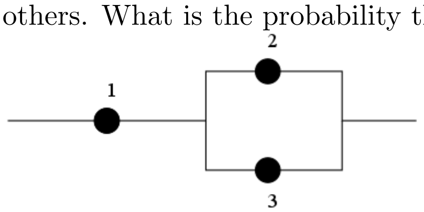
(b) Answer the same question as above, but now if we *don't* put the balls back.

(c) Calculate the probability that in case (a) we draw 3 times the same ball.

18. Prove, using the 3 axioms of probability, that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

19. Let $\mathbb{P}(A) = 0.9$ and $\mathbb{P}(B) = 0.8$. Show that $\mathbb{P}(A \cap B) \geqslant 0.7$.

20. Consider the following system. Each component has a probability 0.1 of failing, independently of the others. What is the probability that the system works?



21. What is the probability that none of 54 people in a room share the same birthday?

22. Answer whether the following statements are True or False. Explain your answer, e.g., by making Venn-diagrams.

    (a) $(A \cup B)^c = A^c \cup B^c$.
    (b) $(A \cap B)^c = A^c \cup B^c$.
    (c) $(A \cap \Omega) \cup (A^c \cap \Omega) = \Omega$.
    (d) $\mathbb{P}(A \cap \Omega) + \mathbb{P}(A^c \cap \Omega) = 1$.

23. We draw 3 cards from a full deck of cards, noting the order. Number the cards from 1 to 52.

    (a) Give the sample space. Is each elementary event equally likely?
    (b) What is the probability that we draw 3 Aces?
    (c) What is the probability that we draw 1 Ace, 1 King and 1 Queen?
    (d) What is the probability that we draw no pictures (no A,K,Q,J)?

24. In a binary communication channel, 0s and 1s are transmitted with equal probability. The probability that a 0 is correctly received (as a 0) is 0.95. The probability that a 1 is correctly received (as a 1) is 0.99. Suppose we receive a 0, what is the probability that, in fact, a 1 was sent?

25. Throw two fair dice one after the other.

    (a) What is the probability that the second die is 3, given that the sum of the dice is 6?
    (b) What is the probability that the first die is 3 and the second not 3?

26. We independently throw 10 balls into one of 3 boxes, numbered 1,2 and 3, with probabilities 1/4, 1/2 and 1/4, respectively.

    (a) What is the probability that box 1 has 2, box 2 has 5 and box 3 has 3 balls?
    (b) What is the probability that box 1 remains empty.

27. We draw at random 5 numbers from 1,... 100, *with replacement* (for example, drawing number 9 twice is possible). What is the probability that exactly 3 numbers are even?

28. We draw at random 5 numbers from 1,... 100, *without replacement.* What is the probability that exactly 3 numbers are even?

29. A radioactive source of material emits a radioactive particle with probability 1/1000 in each second. Let X be the number of particles emitted in one hour. X has approximately a Poisson distribution with what parameter?

30. The sample space Ω of an experiment consists of the non-negative integers up to 10. Let A, B and C be the events defined by

    $$A = \{0, 1, 2, 4, 7, 8, 9\}, \quad B = \{0, 3, 4, 5, 9, 10\}, \quad \text{and} \quad C = \{0, 2, 4, 6, 8, 10\}.$$

    List the outcomes corresponding to each of the following events: (a) $A \cup B$, (b) $B \cup C$, (c) $A \cap B^c$. Verify that the distributive law holds for the events A, B and C listed above: that is, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

31. A discrete random variable X has probability mass function

$$\mathbb{P}(X = n) = \frac{6}{\pi^2 n^2}\,, \quad n = 1, 2, \ldots .$$

Determine the following probabilities: (a) $\mathbb{P}(X \leqslant 2)$, (b) $\mathbb{P}(X > 3)$, (c) $\mathbb{P}(1 < X \leqslant 3)$, and (d) $\mathbb{P}(1 \leqslant X \leqslant 3)$.

32. Consider the following reliability system. The system works if there is a path with working components (1, 2, 3 and 4) connecting the ends of the diagram. Let A_i be the event that the ith component is functioning, $i = 1, 2, 3, 4$. Let A be the event that the system is functioning. Express A in terms of $A_1, \ldots, A_4$, using complements, intersections and/or unions.

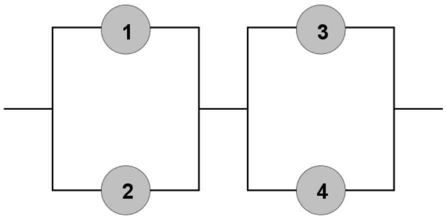


Figure A.5: Reliability system 1

33. Consider the following reliability system. The system works if there is a path with working components (1, 2, 3 and 4) connecting the ends of the diagram. Let A_i be the event that the ith component is functioning, $i = 1, 2, 3, 4$. Let A be the event that the system is functioning. Express A in terms of $A_1, \ldots, A_4$, using complements, intersections and/or unions.

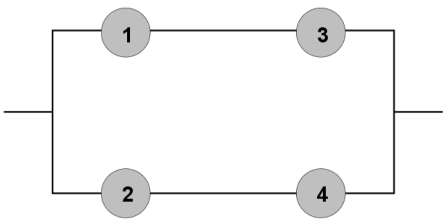


Figure A.6: Reliability system 2

34. A coding scheme translates the characters $\{a, b, c, d, e\}$ into binary codes $\{1, 01, 001, 0001, 0000\}$. The efficiency of this scheme depends on the "frequency" at which the characters appear in the message that is to be transmitted. Suppose these frequencies are $\{1/2, 1/4, 1/8, 1/16, 1/16\}$. Let X be the length (in bits) of an arbitrary character (chosen in accordance with the specified frequencies).

    (a) Determine the range and pmf of X.

    (b) Draw the graph of the cdf of X.

35. Consider the following model for the arrival of calls at a telephone exchange. During each of the 1-second time slots $[0, 1)$, $[1, 2)$, $[2, 3)$, $\ldots$, either 1 call arrives (with probability 0.005) or no calls arrive. Assume that the number of calls in the time slots are independent of each other.

(a) Let X be the number of calls in the first minute. Give the pmf of X.

(b) What is the probability that we have to wait more than 5 minutes before the first call arrives?

36. A discrete random variable X has probability mass function

$$\mathbb{P}(X = n) = \frac{1}{\ln(4)\,(2(n+1)^2 - n - 1)}\,, \qquad n = 0, 1, \ldots .$$

Determine the following probabilities: (a) $\mathbb{P}(X \geqslant 2)$, (b) $\mathbb{P}(X \leqslant 3)$, (c) $\mathbb{P}(X = 3)$, and (d) $\mathbb{P}(0 < X \leqslant 3)$.

37. A positive integer (natural number) is chosen. Use a Venn diagram to illustrate the relationship between the following events: E, the event that the chosen integer is even, P, the event that the chosen integer is prime, and D, the event that the chosen integer is divisible by 5.

Your Venn diagram should delineate seven regions corresponding to disjoint events whose union is the sample space $\Omega = \mathbb{N}$ (the natural numbers). Show each region is non-empty by indicating the smallest number that lies in each; for example, 10 is both divisible by 5 and even, and so it lies in the region represented by $E \cap D$, and 10 is clearly the smallest such number.

38. The joint pmf of X and Y is given by the table

	y			
x	1	2	3	4
1	0.1	0.2	0.1	0
2	0.1	0.1	0	0
3	0	0.1	0.2	0.1

(a) Determine the (marginal) pmf of X and of Y.

(b) Calculate $\mathrm{Cov}(X, Y)$.

39. Let U and V be independent random variables, with $\mathbb{P}(U = 1) = \mathbb{P}(V = 1) = 1/4$ and $\mathbb{P}(U = -1) = \mathbb{P}(V = -1) = 3/4$. Define $X = U/V$ and $Y = U + V$.

(a) Give a table for the joint pmf of X and Y.

(b) Derive the expectation and variance of X.

(c) Calculate $\mathrm{Cov}(X, Y)$.

40. A certain multiple choice exams has 30 questions, each providing 3 choices. To pass the exam one needs at least 20 out of 30 correct answers. Suppose a student knows the answers to 27 questions for certain, and fills in the remaining three questions “at random”. What is the probability that the student will get full marks?

41. We draw at random 10 balls from an urn with 25 red and 75 white balls. What is the expected number of red balls amongst the 10 balls drawn? Does it matter if we draw the balls with or without replacement?

42. We repeatedly throw two fair dice until two sixes are thrown. What is the expected number of throws required?

43. Suppose we divide the population of Brisbane (say 2,000,001 people) randomly in groups of 3.

    (a) How many groups would you expect there to be in which all persons have the same birthday?

    (b) What is the probability that there is at least one group in which all persons have the same birthday?

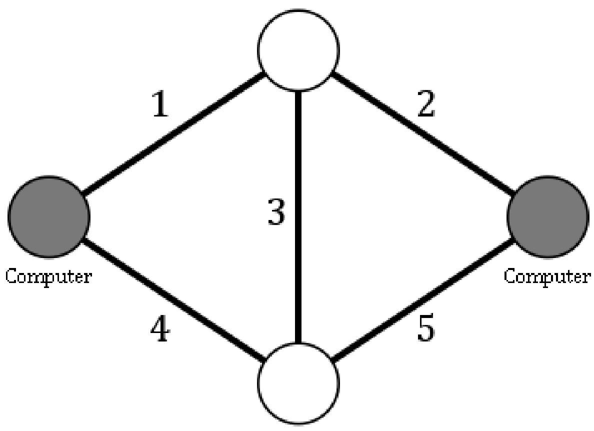
44. Let X, Y be independent $\mathsf{Bernoulli}(p)$ distributed random variables. Let $U = X+Y$ and $V = |X - Y|$. Determine the joint pmf of U and V. Are they independent? What is the covariance?

45. A janitor with n keys wants to open a door and tries the keys independently and at random. Consider the following two selection strategies: (A) unsuccessful keys are not eliminated from further selection and (B) unsuccessful keys *are* eliminated. Let N be the number of keys tried before the correct one is found.

    (a) Give the expectation and variance of N for strategy (A).

    (b) Give the probability mass function and expectation of N for strategy (B).

46. Two computers are connected by five links (see Figure). Each link *works* with probability p, independently of all other links. Denote by A_i the event that link i works, for $i = 1, 2, \ldots, 5$.



    (a) Write down the event B that the two computers are connected by working links in terms of the individual events $A_1, \ldots, A_5$.

    (b) By using the partition $\{A_3, A_3^c\}$ and the law of total probability, determine $\mathbb{P}(B)$, the probability that the two computers are connected by working links.

    (c) If $p = 0.5$, what is the probability that link 3 works, given that the two computers are connected by working links?

47. Suppose that a widget manufacturing plant has two production systems working in parallel to produce widgets. The first widget system is older, and produces 10000 widgets a day, with each widget being defective with probability 0.005, independently. The second widget system is brand new, and produces 25000 widgets a day,

with each widget being defective with probability 0.002, independently. Moreover, the systems operate independently. At the end of each day, all of the day's widgets are collected together in a box.

Let X and Y be the number of widgets the first and second system produce per day, respectively.

(a) What are the expectations and variances of X and Y?

(b) Determine the probability that a randomly chosen defective widget from the box was produced by the second (brand new) system.

48. Let U and V be independent and identically distributed $\mathsf{Bernoulli}(p)$ random variables. Define $X = \min\{U, V\}$ and $Y = \max\{U, V\}$

    (a) Write down the joint pmf (as a table) of X and Y.

    (b) Determine the marginal pmf of X.

    (c) Give the conditional pmf of Y given $X = 1$.

    (d) Calculate the covariance of X and Y when $p = 1/2$. Are X and Y independent?

49. Let U and V be independent and identically distributed $\mathsf{Bernoulli}(p)$ random variables. Define $X = 2|U - V| - 1$ and $Y = U + V - 1$

    (a) Write down the joint pmf (as a table) of X and Y.

    (b) Determine the marginal pmf of X.

    (c) Give the conditional pmf of Y given $X = 1$.

    (d) Calculate the covariance of X and Y when $p = 1/2$. Are X and Y independent?

50. A space shuttle has two booster rockets. Each booster rocket consists of four segments. The joint between adjoining segments is sealed by a pair of O-rings. The joint is properly (safely) sealed if at least one of the two O-rings works. Let A_i be the event that the ith joint is properly sealed, $i = 1, \ldots, 6$. The system fails if at least one of the joints is not properly sealed. If the 12 O-rings fail independently of each other, each with failure probability 0.1, what is the probability of system failure?

51. A system is comprised of two components connected in series so the system is working if and only if both components are working. The two components have independent $\mathsf{Geometric}(p)$ lifetimes. Find the probability mass function of the lifetime of the system.

B

Practice Questions For Chapters 2 - 4 (Solutions)

1. Two cities A and B are connected to a water source W via pipes 1, 2, 3 and 4, see Figure B.1. The system works if both A and B receive water via at least one pipe. Let A_i be the event that the ith pipe is functioning, $i = 1, 2, 3, 4$. Let A be the event that the system is functioning. Express A in terms of $A_1, \ldots, A_4$, using complements, intersections and/or unions.

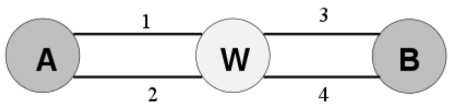


Figure B.1: A water distribution system

Solution:

$$A = (A_1 \cup A_2) \cap (A_3 \cup A_4).$$

2. Thirty percent of people in Metropolis read the Daily Planet, 20% read the Inquirer, and 5% read both news papers. What is the probability that an arbitrarily selected person reads at least one newspaper?

Solution: Let A be the event that a person reads the Daily Planet, B be the event that a person reads the Inquirer. Then $A \cap B$ is the event that they read both. We seek

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= 0.3 + 0.2 - 0.05 \\ &= 0.45.\end{aligned}$$

3. Two fair dice are thrown and the sum of the face values, Z say, is noted. Give the pmf of Z in table form:

z	$*$	$*$	$*$	$\ldots$
$\mathbb{P}(Z = z)$	$*$	$*$	$*$	$\ldots$

Solution:[Corrected by Ron.]

z	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
$\mathbb{P}(Z=z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The numerator is the number of distinct ways in which the sum can be thrown on a pair of two fair six-sided dice, taking order into account.

4. A certain AIDS test has a 0.98 probability of giving a Positive result when the blood is infected, and a 0.07 probability of giving a Positive result when the blood is not infected (a so-called false positive). Suppose 1% of the population carries the HIV virus.

   Using Bayes' Rule, determine the probability that a randomly selected person is indeed infected, *given* that the test yields a Positive result.

   Solution: Let I be the event that a randomly selected individual is infected. Let P be the event that the blood test is positive. Then we seek

$$\mathbb{P}(I|P) = \frac{\mathbb{P}(P|I)\mathbb{P}(I)}{\mathbb{P}(P|I)\mathbb{P}(I) + \mathbb{P}(P|I^c)\mathbb{P}(I^c)}$$

   (from Bayes' Rule). We know:

$$\begin{array}{ll} \mathbb{P}(P|I) = 0.98, & \mathbb{P}(P|I^c) = 0.07, \\ \mathbb{P}(I) = 0.01, & \mathbb{P}(I^c) = 0.99. \end{array}$$

   Therefore

$$\mathbb{P}(I|P) = \frac{0.98 \times 0.01}{0.98 \times 0.01 + 0.07 \times 0.99} = \frac{98}{98 + 693} = \frac{98}{791} \approx 0.12389.$$

5. We toss a fair coin three times. Give the sample space Ω for this experiment, if we observe the exact sequences of Heads (= 1) and Tails (= 0), and assign the appropriate probabilities to the elementary events.

   Solution:

$$\Omega = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}.$$

   Each elementary event is equally likely, so has probability $\frac{1}{|\Omega|} = \frac{1}{8}$.

6. Let A, B and C be events such that: $\mathbb{P}(A) = 1/4$, $\mathbb{P}(B) = 1/3$, $\mathbb{P}(C) = 1/6$, A and B are independent, and $A \cup B$ and C are disjoint.

   Calculate the probability that at least one of the events A, B or C occurs.

   Solution: We seek $\mathbb{P}(A \cup B \cup C)$. Using the fact that $(A \cup B)$ and C are disjoint, we may write (by Axiom 2)

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C).$$

   We further have in general that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

   Since A and B are independent,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

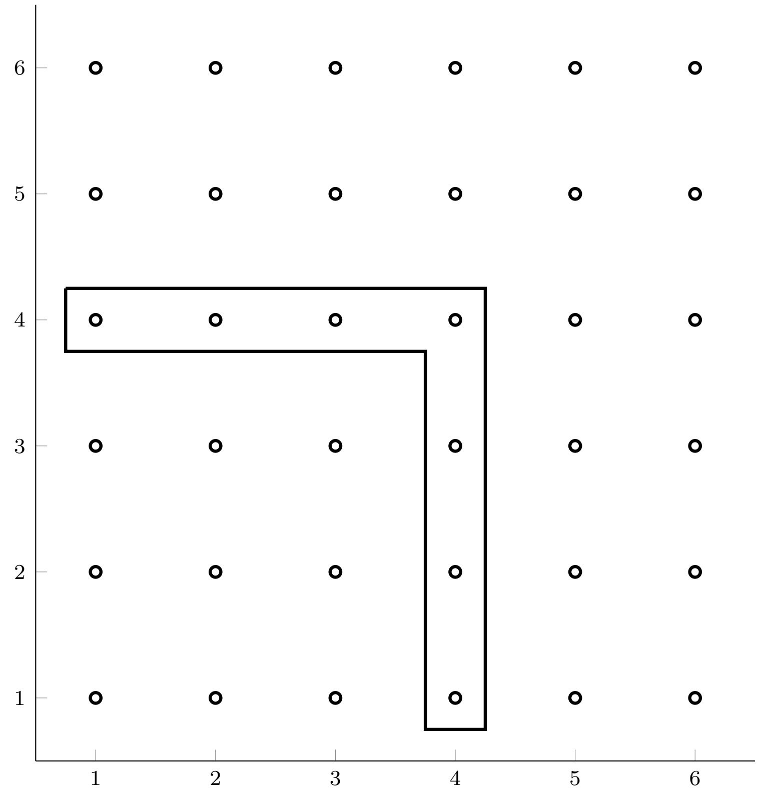
Putting this all together, we have

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(C) \\ &= \frac{1}{4} + \frac{1}{3} - \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{6} \\ &= \frac{3}{12} + \frac{4}{12} - \frac{1}{12} + \frac{2}{12} \\ &= \frac{8}{12} \\ &= \frac{2}{3}.\end{aligned}$$

7. We toss two fair dice and note the largest number M of the two dice. Derive the probability $\mathbb{P}(M = 4)$.

   Solution: Consider the following diagram, representing the pairs of outcomes.

Possible Dice Outcomes



The enclosed points correspond to the event $\{M = 4\}$*, so* $\mathbb{P}(M = 4) = 7/36$.

8. Consider two arrays (Array 1 and Array 2) of 10 bits. We combine Array 1 and Array 2, by using the AND operation (thus, 0 AND 0 = 0; 0 AND 1 = 0; 1 AND 0 = 0; 1 AND 1 = 1). An illustration of this procedure is given in Figure B.2.

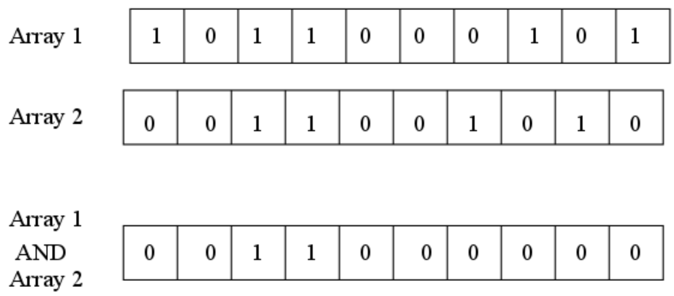


Figure B.2: The AND operation applied to two binary arrays.

Suppose that Array 1 and Array 2 are (independently of each other) filled according to 10 independent Bernoulli trials (coin flip experiments), with success ($= 1$) probability 1/3 for Array 1, and with success probability 3/4 for Array 2. What is the probability that the "combined" array has fewer than three 1-bits?

*Solution: Each entry in the combined array is "1" only if **both** corresponding entries in Array 1 and Array 2 are "1".*

Since they are filled independently, the probability of an entry in the combined arrow being "1" is $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$ and the entry itself can be viewed as the outcome of a Bernoulli trial with success parameter $\frac{1}{4}$.

The total number of "1"s in the combined array is the sum of 10 independent and identically distributed $\mathsf{Ber}(1/4)$ *random variables — denoting this number by N, we therefore have $N \sim \mathsf{Bin}(10, 1/4)$, so*

$$
\begin{aligned}
\mathbb{P}(N < 3) &= \mathbb{P}(N = 0) + \mathbb{P}(N = 1) + \mathbb{P}(N = 2) \\
&= \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} + \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{9} + \binom{10}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{8} \\
&= \left(\frac{3}{4}\right)^{10}\left\{1 + \frac{10}{3} + \frac{4\cdot 5}{9}\right\} \\
&= \left(\frac{3}{4}\right)^{10} \times \frac{28}{3} = 7 \times \left(\frac{3}{4}\right)^{9} \\
&\approx 0.52559.
\end{aligned}
$$

9. Consider the water cooling system schematically depicted in Figure B.3. The system has four unreliable components: two identical pumps (P1 and P2) and two valves (V1 and V2). The system works if, in the diagram, there is a path from left to right traversing only working components.

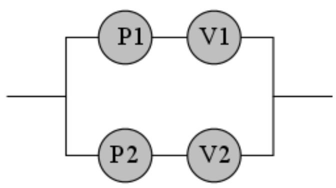


Figure B.3: The system works if there is a path from left to right with working components.

We assume that the pumps and valves fail independently of each other and that the reliability (=probability that it works) for each pump is 0.9 and for each valve is 0.8. Calculate the reliability of the system.

Solution: Let P_i denote the event that Pump i works and let V_i denote the event that Valve i works, for $i = 1, 2$. Denoting by S the event that the system works, we may write

$$S = (P_1 \cap V_1) \cup (P_2 \cap V_2),$$

which gives

$$\begin{aligned}
\mathbb{P}(S) &= \mathbb{P}((P_1 \cap V_2) \cup (P_1 \cap V_2)) = \mathbb{P}(P_1 \cap V_1) + \mathbb{P}(P_2 \cap V_2) - \mathbb{P}(P_1 \cap V_1 \cap P_2 \cap V_2) \\
&= \mathbb{P}(P_1)\mathbb{P}(V_1) + \mathbb{P}(P_2)\mathbb{P}(V_2) - \mathbb{P}(P_1)\mathbb{P}(V_1)\mathbb{P}(P_2)\mathbb{P}(V_2) \quad \textit{(independence)} \\
&= 0.9 \times 0.8 + 0.9 \times 0.8 - 0.9 \times 0.8 \times 0.9 \times 0.8 \\
&= \frac{7200 + 7200 - 5184}{10000} = \frac{9216}{10000} = 0.9216.
\end{aligned}$$

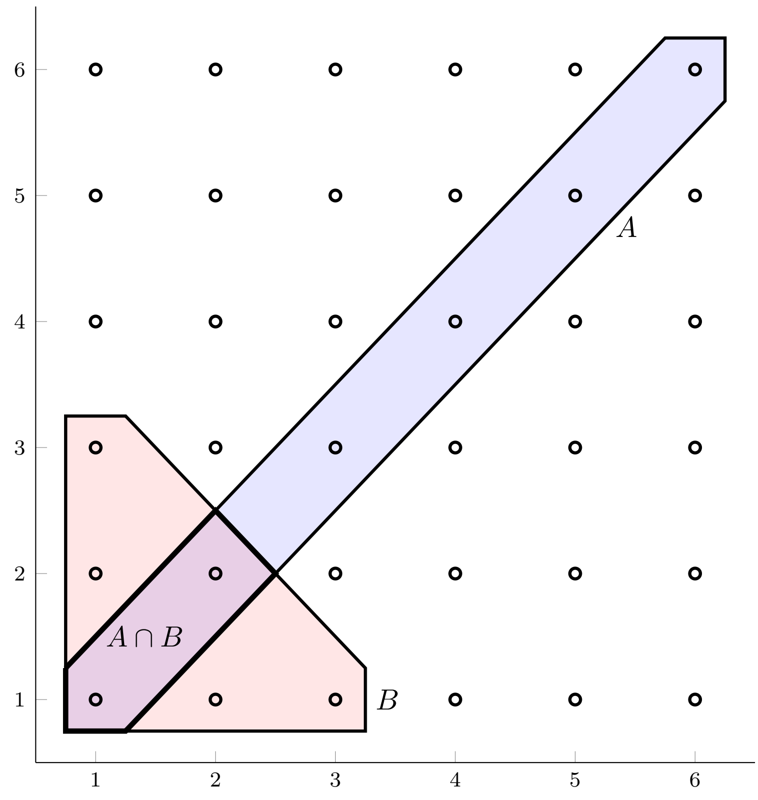
10. We throw with 2 fair dice. Find the probability that both dice show the same face, given that the sum of the dice is not greater than 4.

Solution: Let A denote the event that the dice show the same face, and B denote the event that the sum of the dice is not greater than A. Then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We can find the numerator and denominator probabilities easily from the following diagram, which represents the pair of dice throws:

Possible Dice Outcomes



This gives

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}.$$

11. We drop a ball onto a quincunx, see Figure 1. This is a board in which pins are placed in a regular pattern, such that the ball bounces to the left or right of each pin with equal probability.

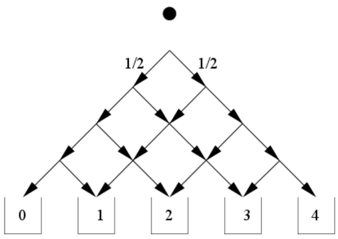


Figure B.4: A quincunx

(a) What is probability that the ball will end up in box 2?

(b) Let X be the number of the box in which the ball falls. What is the pmf of X?

Solution: We can relate to the box number, X, the number of "right" bounces; $X = R_1 + R_2 + R_3 + R_4$, where each

$$R_i = \begin{cases} 1, & i\text{th bounce is "right"}, \\ 0, & i\text{th bounce is "left"}, \end{cases}$$

is independent of all others. Now each R_i is $\mathsf{Ber}(1/2)$ and so $X \sim \mathsf{Bin}(4, 1/2)$.

(a)

$$\mathbb{P}(X = 2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{6}{16} = \frac{3}{8} = 0.375.$$

(b)

$$\mathbb{P}(X = x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x} = \frac{1}{16}\binom{4}{x}, \quad x = 0, 1, \ldots, 4.$$

12. Let X be the lifetime of a certain component in years. Suppose X has pmf

x	0	1	2	3	4	5	6
$\mathbb{P}(X = x)$	0.05	0.25	0.30	0.20	0.10	0.05	0.05

(a) Calculate the probability that a component has a lifetime of at least 5 years.

(b) Calculate the conditional probability that a component has a lifetime of at least 5 years, given that it is still alive after 2 years.

(c) Calculate the expected lifetime of a component.

Solution:

(a)

$$\mathbb{P}(X \geqslant 5) = \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = 0.05 + 0.05 = 0.1.$$

(b)

$$\begin{aligned}\mathbb{P}(X \geqslant 5 | X \geqslant 2) &= \frac{\mathbb{P}(\{X \geqslant 5\} \cap \{X \geqslant 2\})}{\mathbb{P}(X \geqslant 2)} = \frac{\mathbb{P}(X \geqslant 5)}{\mathbb{P}(X \geqslant 2)} \\ &= \frac{0.1}{1 - \mathbb{P}(X \leqslant 1)} = \frac{0.1}{1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1)} \\ &= \frac{0.1}{1 - 0.05 - 0.25} = \frac{0.1}{1 - 0.3} = \frac{0.1}{0.7} = \frac{1}{7}.\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{E}X &= \sum_{x=0}^{6} x \cdot \mathbb{P}(X = x) \\ &= 0 \times 0.05 + 1 \times 0.25 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 + 5 \times 0.05 + 6 \times 0.05 \\ &= 0.25 + 0.6 + 0.6 + 0.4 + 0.25 + 0.3 \\ &= 2.4.\end{aligned}$$

13. A random variable X has expectation 2 and variance 1. Calculate $\mathbb{E}[X^2]$.

    Solution:

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 \\ \Rightarrow 1 &= \mathbb{E}\left[X^2\right] - (2)^2 \\ \Rightarrow \mathbb{E}\left[X^2\right] &= 5.\end{aligned}$$

14. We flip a fair coin 10 times. How likely is it that fewer than 3 Heads appear?

    Solution: Let X be the number of heads in 10 flips of the fair coin. Therefore $X \sim \mathsf{Bin}(10, 1/2)$, giving

$$\begin{aligned}\mathbb{P}(X < 3) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) \\ &= \binom{10}{0}\left(\frac{1}{2}\right)^{10} + \binom{10}{1}\left(\frac{1}{2}\right)^{10} + \binom{10}{2}\left(\frac{1}{2}\right)^{10} \\ &= \left(\frac{1}{2}\right)^{10}\{1 + 10 + 45\} = \frac{56}{2^{10}} = \frac{56}{1024} = \frac{7}{128} \\ &= 0.0546875.\end{aligned}$$

15. We repeatedly throw two fair dice until two sixes are thrown. What is the probability that more than 20 throws are required?

    Solution: Each throw has, independently of all else, a probability of $p = 1/36$ of throwing a pair of sixes. The event can be viewed as a $\mathsf{Ber}(1/36)$ random variable. The number of throws needed, N, is thus $\mathsf{Geometric}(1/36)$, so

$$\mathbb{P}(N > 20) = \sum_{n=21}^{\infty} p(1-p)^{n-1} = (1-p)^{20} = \left(\frac{35}{36}\right)^{20} \approx 0.5693.$$

16. A random variable X takes the values $0, 2, 5$ with probabilities $1/2, 1/3, 1/6$, respectively. What is the expectation of X?

    Solution:

$$\mathbb{E}X = 0 \times \frac{1}{2} + 2 \times \frac{1}{3} + 5 \times \frac{1}{6} = \frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6} = \frac{3}{2} = 1.5.$$

17. We randomly select 3 balls from an urn with 365 balls, numbered 1, ... ,365.

    (a) How many possible outcomes of the experiment are there, if we put each ball back into the urn before we draw the next?

    (b) Answer the same question as above, but now if we *don't* put the balls back.

    (c) Calculate the probability that in case (a) we draw 3 times the same ball.

    Solution:

    (a) $(365)^3$ *(three draws, order recorded, each has 365 possible values).*

    (b) $365 \times 364 \times 363 = \frac{365!}{362!}$ *(three draws, order recorded, first has 365 possibilities, second 364, third 363).*

    *(c) For a **given** ball, the probability of drawing is* $\frac{1}{(365)^3}$. *As there are 365 balls in total, we have*

$$365 \cdot \frac{1}{(365)^3} = \frac{1}{(365)^2} \approx 7.5 \times 10^{-6}.$$

18. Prove, using the 3 axioms of probability, that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

    Solution: Note:

$$\begin{aligned} A &= (A \setminus B) \cup (A \cap B), \\ B &= (B \setminus A) \cup (A \cap B), \;\; \textit{and} \\ A \cup B &= (A \setminus B) \cup (B \setminus A) \cup (A \cap B) \end{aligned}$$

are disjoint unions, so

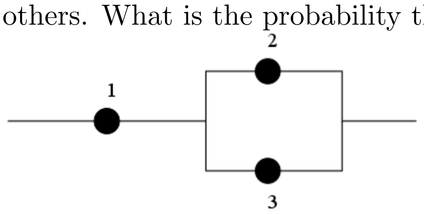
$$\begin{aligned} \mathbb{P}(A) + \mathbb{P}(B) &= \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) && \textit{(Axiom 3)} \\ &= \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) && \textit{(Axiom 3)} \\ \Rightarrow \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B). \end{aligned}$$

19. Let $\mathbb{P}(A) = 0.9$ and $\mathbb{P}(B) = 0.8$. Show that $\mathbb{P}(A \cap B) \geqslant 0.7$.

    Solution:

$$\begin{aligned} 1 \geqslant \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \geqslant 0 \\ \Rightarrow 1 &\geqslant 0.9 + 0.8 - \mathbb{P}(A \cap B) \geqslant 0 \\ \Rightarrow \mathbb{P}(A \cap B) &\geqslant 1.7 - 1 = 0.7. \end{aligned}$$

20. Consider the following system. Each component has a probability 0.1 of failing, independently of the others. What is the probability that the system works?



Solution: Let A_i denote the event that Component i works, and S denote the event that the system works. Then

$$\begin{aligned}
S &= A_1 \cap (A_2 \cup A_3) \\
\Rightarrow \mathbb{P}(S) &= \mathbb{P}(A_1 \cap (A_2 \cup A_3)) \\
&= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 \cup A_3) && \textit{(independence)} \\
&= (1 - \mathbb{P}(A_1^c)) \cdot \{\mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_2 \cap A_3)\} \\
&= (1 - 0.1) \cdot \{1 - \mathbb{P}(A_2^c) + 1 - \mathbb{P}(A_3^c) - \mathbb{P}(A_2)\mathbb{P}(A_3)\} \\
&= 0.9 \cdot \{0.9 + 0.9 - (1 - \mathbb{P}(A_2^c))(1 - \mathbb{P}(A_3^c))\} \\
&= 0.9 \left\{1.8 - 0.9^2\right\} \\
&= \frac{9}{10}\left\{\frac{180}{100} - \frac{81}{100}\right\} \\
&= \frac{9}{10} - \frac{99}{100} = \frac{891}{1000} = 0.891.
\end{aligned}$$

21. What is the probability that none of 54 people in a room share the same birthday?

Solution: Assume 365 unique birthdays. Assume each birthday equally likely. Using the Chain Rule of Probability,

$$\begin{aligned}
&\mathbb{P}(\textit{none of 54 share one of 365 birthdays}) \\
&= \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{(365 - 54 + 1)}{365} \\
&= \frac{365!}{(365 - 54)!} \cdot \frac{1}{(365)^{54}} \\
&\approx 0.016123.
\end{aligned}$$

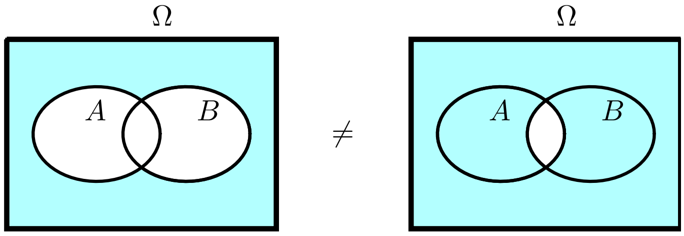
22. Answer whether the following statements are True or False. Explain your answer, e.g., by making Venn-diagrams.

 (a) $(A \cup B)^c = A^c \cup B^c$.
 (b) $(A \cap B)^c = A^c \cup B^c$.
 (c) $(A \cap \Omega) \cup (A^c \cap \Omega) = \Omega$.
 (d) $\mathbb{P}(A \cap \Omega) + \mathbb{P}(A^c \cap \Omega) = 1$.

Solution:

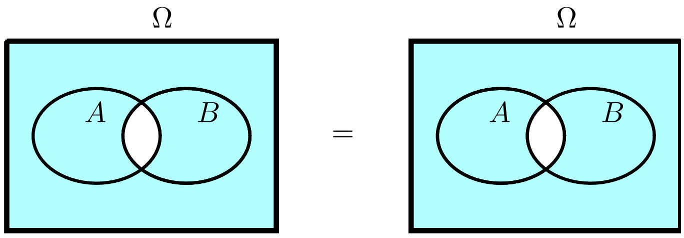
(a) False!

$$(\boldsymbol{A} \cup \boldsymbol{B})^c \neq \boldsymbol{A}^c \cup \boldsymbol{B}^c$$



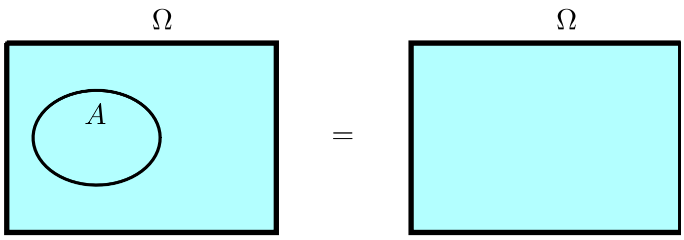
(b) *True!*

$$(\boldsymbol{A} \cap \boldsymbol{B})^c = \boldsymbol{A}^c \cup \boldsymbol{B}^c$$



(c) *True!*

$$(\boldsymbol{A} \cap \boldsymbol{\Omega}) \cup (\boldsymbol{A}^c \cap \boldsymbol{\Omega}) = \boldsymbol{\Omega}$$



(d) *True! From (c),* $\Omega = (A \cap \Omega) \cup (A^c \cap \Omega) = A \cup A^c$*, which is a disjoint union, so Axiom 3 gives*

$$1 = \mathbb{P}(\Omega) = \mathbb{P}((A \cap \Omega) \cup (A^c \cap \Omega)) = \mathbb{P}(A \cap \Omega) + \mathbb{P}(A^c \cap \Omega).$$

23. We draw 3 cards from a full deck of cards, noting the order. Number the cards from 1 to 52.

(a) Give the sample space. Is each elementary event equally likely?

(b) What is the probability that we draw 3 Aces?

(c) What is the probability that we draw 1 Ace, 1 King and 1 Queen?

(d) What is the probability that we draw no pictures (no A,K,Q,J)?

Solution:

(a)

$$\begin{aligned}\Omega = \{&(1,2,3),(1,2,4),\ldots,(1,2,52),\\&(1,3,2),(1,3,4),\ldots,(1,2,52),\\&\ldots,(50,51,52)\}.\end{aligned}$$

NB $|\Omega| = 52 \times 51 \times 50 = \frac{52!}{(52-3)!} = \frac{52!}{49!}$.

Yes. Each elementary event occurs with probability $\frac{1}{|\Omega|} = \frac{1}{52\times51\times50} = \frac{1}{132600}$.

(b) Chain Rule:

$$\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{24}{132600} = \frac{1}{5525} \approx 1.81 \times 10^{-4}.$$

(c) In a fixed order, say (A,K,Q), the probability (by the Chain Rule) would be $\frac{4}{52} \times \frac{4}{51} \times \frac{4}{50}$, *but there are* $3!$ *possible orders, so we have*

$$\left(\frac{4}{52} \times \frac{4}{51} \times \frac{4}{50}\right) \times 3! = \frac{384}{132600} = \frac{16}{5525} \approx 0.0029.$$

(d) There are 16 pictures, so $52 - 16 = 36$ *non pictures. Therefore the probability of drawing no pictures is*

$$\frac{36}{52} \times \frac{35}{51} \times \frac{34}{50} = \frac{42840}{132600} = \frac{21}{65} \approx 0.3231.$$

24. In a binary communication channel, 0s and 1s are transmitted with equal probability. The probability that a 0 is correctly received (as a 0) is 0.95. The probability that a 1 is correctly received (as a 1) is 0.99. Suppose we receive a 0, what is the probability that, in fact, a 1 was sent?

Solution: Let S *denote the event a 1 was sent and let* R *denote the event a 1 is received. We have:*

$$\begin{aligned}\mathbb{P}(S) = \mathbb{P}(S^c) &= 0.5.\\ \mathbb{P}(R|S) = 0.99 &\Rightarrow \mathbb{P}(R^c|S) = 0.01.\\ \mathbb{P}(R^c|S^c) = 0.95 &\Rightarrow \mathbb{P}(R|S^c) = 0.05.\end{aligned}$$

We seek $\mathbb{P}(S|R^c)$. *By Bayes' Rule,*

$$\begin{aligned}\mathbb{P}(S|R^c) &= \frac{\mathbb{P}(R^c|S)\mathbb{P}(S)}{\mathbb{P}(R^c|S)\mathbb{P}(S) + \mathbb{P}(R^c|S^c)\mathbb{P}(S^c)}\\ &= \frac{0.01 \times 0.5}{0.01 \times 0.5 + 0.95 \times 0.5}\\ &= \frac{0.01}{0.96} = \frac{1}{96} \approx 0.0104.\end{aligned}$$

25. Throw two fair dice one after the other.

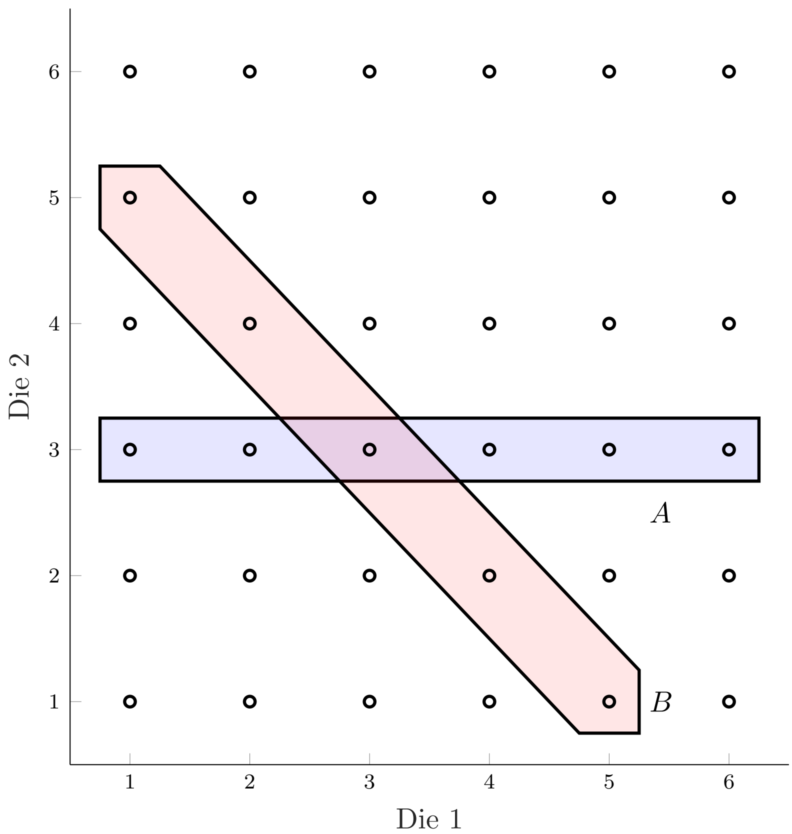
    (a) What is the probability that the second die is 3, given that the sum of the dice is 6?

    (b) What is the probability that the first die is 3 and the second not 3?

Solution:

(a) Let A be the event that the second die is 3 and let B be the event that the sum of the dice is 6.

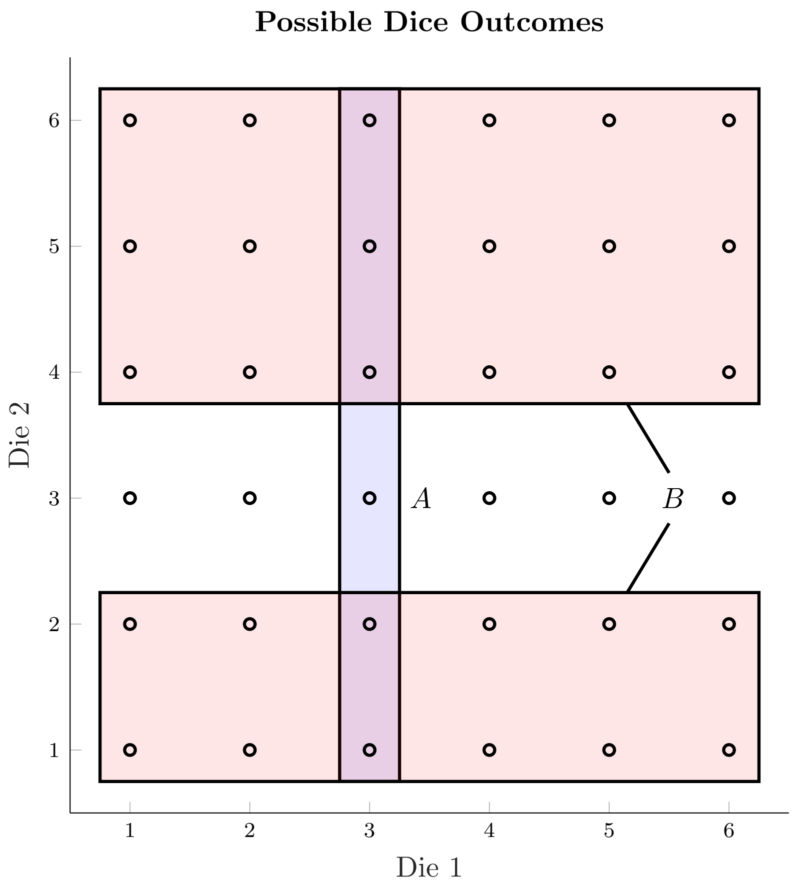
Possible Dice Outcomes



Therefore

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/36}{5/36} = \frac{1}{5} = 0.2.$$

(b) Let A be the event that the first die is 3 and let B be the event that the second die is not 3.



This gives $\mathbb{P}(A \cap B) = 5/36$.
Alternatively, by the Chain Rule,

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A) \cdot \mathbb{P}(B|A) \\ &= \frac{1}{6} \cdot \mathbb{P}(B) && (A, B \textit{ independent}) \\ &= \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}.\end{aligned}$$

26. We independently throw 10 balls into one of 3 boxes, numbered 1,2 and 3, with probabilities 1/4, 1/2 and 1/4, respectively.

    (a) What is the probability that box 1 has 2, box 2 has 5 and box 3 has 3 balls?
    (b) What is the probability that box 1 remains empty.

    Solution:

    (a) *Each* ***particular*** *combination has probability* $\left(\frac{1}{4}\right)^2 \left(\frac{1}{2}\right)^5 \left(\frac{1}{4}\right)^3$, *and there are* $\frac{10!}{2!5!3!}$ *ways in which this (choosing a particular combination) can occur, so the probability is*

    $$\frac{10!}{2!5!3!}\left(\frac{1}{4}\right)^2\left(\frac{1}{2}\right)^5\left(\frac{1}{4}\right)^3 = 0.2520 \times \frac{2^5}{4^{10}} = \frac{2520 \times 32}{1048576} = \frac{315}{4096} \approx 0.0769.$$

    (b) *The number of balls in box 1 is* $\mathsf{Bin}(10, 1/4)$, *so the probability that it is empty is*

    $$\binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} = \left(\frac{3}{4}\right)^{10} \approx 0.0563135.$$

27. We draw at random 5 numbers from 1,... 100, *with replacement* (for example, drawing number 9 twice is possible). What is the probability that exactly 3 numbers are even?

    Solution: Each number is even/odd with probability $1/2$*, independently of all others. Therefore the number of even numbers is* $\mathsf{Bin}(5, 1/2)$*, so the probability that there are 3 even numbers is*

$$\mathbb{P}(\#\ \textit{even} = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \times \frac{1}{2^5} = \frac{10}{32} = \frac{5}{16} = 0.3125.$$

28. We draw at random 5 numbers from 1,... 100, *without replacement*. What is the probability that exactly 3 numbers are even?

    Solution: Two groups: 50 even and 50 odd. For a ***particular*** *drawing (such as EEEOO), we have probability* $\frac{50}{100} \times \frac{49}{99} \times \frac{48}{98} \times \frac{50}{97} \times \frac{49}{96} = \frac{1225}{38412}$*, and there are* $\frac{5!}{3!2!}$ *to permute this, so the probability that exactly 3 of the chosen numbers are even is*

$$\frac{5!}{3!2!} \times \frac{1225}{38412} = \frac{12250}{38412} = \frac{6125}{19206} \approx 0.318911.$$

29. A radioactive source of material emits a radioactive particle with probability $1/100$ in each second. Let X be the number of particles emitted in one hour. X has approximately a Poisson distribution with what parameter?

    Solution: Each second, $p = 1/100$*. There are* $60 \times 60 = 3600$ *seconds per hour. We have*

$$\begin{aligned} X &\sim \mathsf{Bin}(3600, 1/100) \equiv \mathsf{Bin}(3600, \left(\tfrac{3600}{100}\right)/3600) \\ &\sim_{approx} \mathsf{Poisson}(3600/100) \equiv \mathsf{Poisson}(36). \end{aligned}$$

30. The sample space Ω of an experiment consists of the non-negative integers up to 10. Let A, B and C be the events defined by

$$A = \{0, 1, 2, 4, 7, 8, 9\}, \quad B = \{0, 3, 4, 5, 9, 10\}, \quad \text{and} \quad C = \{0, 2, 4, 6, 8, 10\}.$$

    List the outcomes corresponding to each of the following events: (a) $A \cup B$, (b) $B \cup C$, (c) $A \cap B^c$. Verify that the distributive law holds for the events A, B and C listed above: that is, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

    Solution:

    (a) $A \cup B = \{0, 1, 2, 3, 4, 5, 7, 8, 9, 10\}$.

    (b) $B \cup C = \{0, 2, 3, 4, 5, 6, 8, 9, 10\}$.

    (c) $B^c = \{1, 2, 6, 7, 8\} \Rightarrow A \cap B^c = \{1, 2, 7, 8\}$.

    Now

$$A \cap (B \cup C) = \{0, 2, 4, 8, 9\}$$

    while $(A \cap B) = \{0, 4, 9\}$ *and* $(A \cap C) = \{0, 2, 4, 8\}$ *give*

$$(A \cap B) \cup (A \cap C) = \{0, 2, 4, 8, 9\}$$

    Thus

$$A \cap (C \cup C) = (A \cap B) \cup (A \cap C)$$

    is verified.

31. A discrete random variable X has probability mass function

$$\mathbb{P}(X = n) = \frac{6}{\pi^2 n^2}, \quad n = 1, 2, \dots .$$

Determine the following probabilities: (a) $\mathbb{P}(X \leqslant 2)$, (b) $\mathbb{P}(X > 3)$, (c) $\mathbb{P}(1 < X \leqslant 3)$, and (d) $\mathbb{P}(1 \leqslant X \leqslant 3)$.

Solution:

(a)

$$\mathbb{P}(X \leqslant 2) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \frac{6}{\pi^2} + \frac{6}{4\pi^2} = \frac{30}{4\pi^2} = \frac{15}{2\pi^2} \approx 0.7599.$$

(b) [Corrected by Ruth, Katarina.]

$$\mathbb{P}(X > 3) = 1 - \mathbb{P}(X \leqslant 3) = 1 - \frac{6}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = 1 - \frac{49}{6\pi^2} \approx 0.17254.$$

(c)

$$\begin{aligned} \mathbb{P}(1 < X \leqslant 3) &= \mathbb{P}(X \leqslant 3) - \mathbb{P}(X \leqslant 1) \\ &= \mathbb{P}(X = 3) + \mathbb{P}(X = 2) \\ &= \frac{6}{9\pi^2} + \frac{6}{4\pi^2} = \frac{13}{6\pi^2} \approx 0.219529. \end{aligned}$$

(d)

$$\begin{aligned} \mathbb{P}(1 \leqslant X \leqslant 3) &= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) \\ &= \frac{6}{\pi^2} + \frac{6}{4\pi^2} + \frac{6}{9\pi^2} \\ &= \frac{49}{6\pi^2} \approx 0.827456. \end{aligned}$$

32. Consider the following reliability system. The system works if there is a path with working components (1, 2, 3 and 4) connecting the ends of the diagram. Let A_i be the event that the ith component is functioning, $i = 1, 2, 3, 4$. Let A be the event that the system is functioning. Express A in terms of $A_1, \dots, A_4$, using complements, intersections and/or unions.

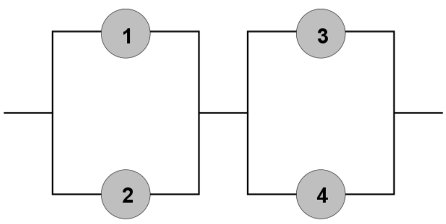


Figure B.5: Reliability system 1

Solution:

$$A = (A_1 \cup A_2) \cap (A_3 \cup A_4).$$

33. Consider the following reliability system. The system works if there is a path with working components (1, 2, 3 and 4) connecting the ends of the diagram. Let A_i be the event that the ith component is functioning, $i = 1, 2, 3, 4$. Let A be the event that the system is functioning. Express A in terms of $A_1, \ldots, A_4$, using complements, intersections and/or unions.

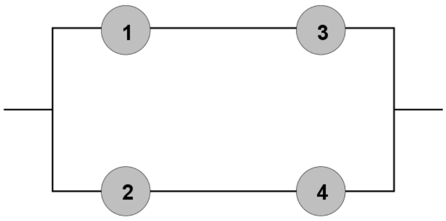


Figure B.6: Reliability system 2

Solution: [Corrected by Dabang.]

$$A = (A_1 \cap A_3) \cup (A_2 \cap A_4).$$

34. A coding scheme translates the characters $\{a, b, c, d, e\}$ into binary codes $\{1, 01, 001, 0001, 0000\}$. The efficiency of this scheme depends on the "frequency" at which the characters appear in the message that is to be transmitted. Suppose these frequencies are $\{1/2, 1/4, 1/8, 1/16, 1/16\}$. Let X be the length (in bits) of an arbitrary character (chosen in accordance with the specified frequencies).

(a) Determine the range and pmf of X.

(b) Draw the graph of the cdf of X.

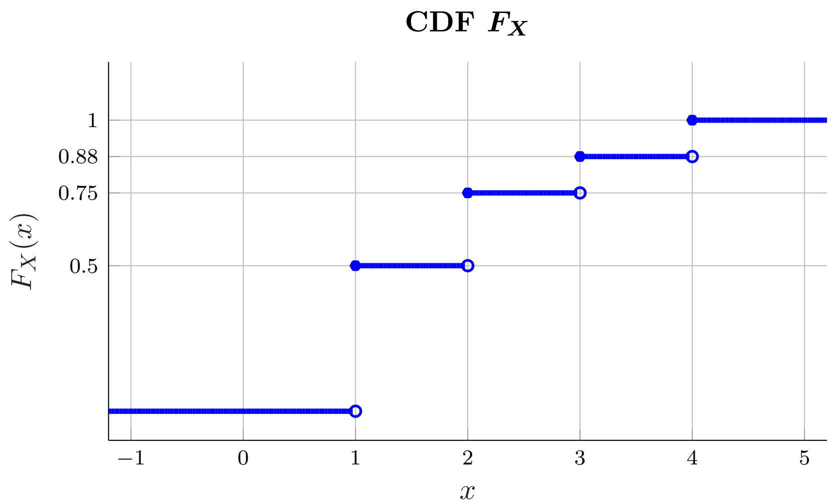
Solution:

(a) X can take on values in $\{1, 2, 3, 4\}$, with pmf:

x	*1*	*2*	*3*	*4*
$\mathbb{P}(X = x)$	1/2	1/4	1/8	1/8

Note that $\mathbb{P}(X = 4) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$.

(b) The graph of the cdf of X:



35. Consider the following model for the arrival of calls at a telephone exchange. During each of the 1-second time slots $[0,1)$, $[1,2)$, $[2,3)$, $\ldots$, either 1 call arrives (with probability 0.005) or no calls arrive. Assume that the number of calls in the time slots are independent of each other.

    (a) Let X be the number of calls in the first minute. Give the pmf of X.

    (b) What is the probability that we have to wait more than 5 minutes before the first call arrives?

    Solution:

    (a) The number of calls $X \sim \mathsf{Bin}(60, 0.005)$ *since a call arriving in each 1-second slow can be viewed as the outcome of a* $\mathsf{Ber}(0.005)$ *trial and there are 60 such trials per minute.*

    (b) Since the arrival of a call in each second is an independent $\mathsf{Ber}(0.005)$ *trial, the numbers of seconds, N, we need to wait is* $\mathsf{Geom}(0.005)$. *In 5 minutes there are* $60 \times 5 = 300$ *seconds, so*

    $$\mathbb{P}(N > 300) = (1 - 0.005)^{300} \approx 0.2223.$$

    We can also use $\mathrm{e}^{-x} = \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n$ *to get*

    $$(1-0.005)^{300} = \left(1 - \frac{300 * 0.005}{300}\right)^{300} = \left(1 - \frac{1.5}{300}\right)^{300} \approx \mathrm{e}^{-1.5} \approx 0.2231.$$

36. A discrete random variable X has probability mass function

    $$\mathbb{P}(X = n) = \frac{1}{\ln(4)\,(2(n+1)^2 - n - 1)}, \quad n = 0, 1, \ldots .$$

    Determine the following probabilities: (a) $\mathbb{P}(X \geqslant 2)$, (b) $\mathbb{P}(X \leqslant 3)$, (c) $\mathbb{P}(X = 3)$, and (d) $\mathbb{P}(0 < X \leqslant 3)$.

    Solution:

    (a)

    $$\begin{aligned}\mathbb{P}(X \geqslant 2) &= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1)\\ &= 1 - \frac{1}{\ln(4)} - \frac{1}{6\ln(4)}\\ &= 1 - \frac{7}{6\ln(4)} \approx 0.158428.\end{aligned}$$

    (b)

    $$\begin{aligned}\mathbb{P}(X \leqslant 3) &= \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3)\\ &= \frac{7}{6\ln(4)} + \frac{1}{15\ln(4)} + \frac{1}{28\ln(4)}\\ &= \frac{533}{420\ln(4)} \approx 0.915424.\end{aligned}$$

    (c)

    $$\mathbb{P}(X = 3) = \frac{1}{28\ln(4)} \approx 0.0257624.$$

(d)

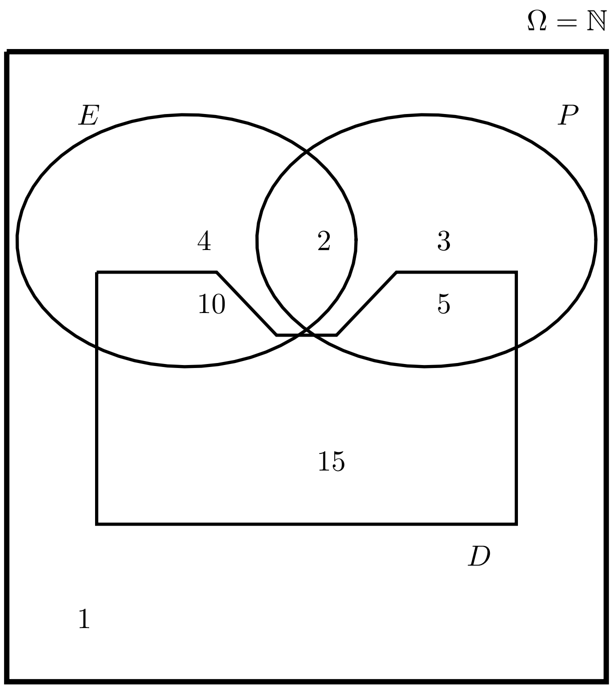
$$\begin{aligned}\mathbb{P}(0 < X \leqslant 3) &= \mathbb{P}(X \leqslant 3) - \mathbb{P}(X = 0)\\ &= \frac{533}{420\ln(4)} - \frac{1}{\ln(4)} = \frac{113}{420\ln(4)} \approx 0.194077.\end{aligned}$$

37. A positive integer (natural number) is chosen. Use a Venn diagram to illustrate the relationship between the following events: E, the event that the chosen integer is even, P, the event that the chosen integer is prime, and D, the event that the chosen integer is divisible by 5.

    Your Venn diagram should delineate seven regions corresponding to disjoint events whose union is the sample space $\Omega = \mathbb{N}$ (the natural numbers). Show each region is non-empty by indicating the smallest number that lies in each; for example, 10 is both divisible by 5 and even, and so it lies in the region represented by $E \cap D$, and 10 is clearly the smallest such number.

    Solution: Venn diagram:

E, P, and D as Subsets of $\Omega = \mathbb{N}$



38. The joint pmf of X and Y is given by the table

	y			
x	1	2	3	4
1	0.1	0.2	0.1	0
2	0.1	0.1	0	0
3	0	0.1	0.2	0.1

(a) Determine the (marginal) pmf of X and of Y.

(b) Calculate $\mathrm{Cov}(X, Y)$.

Solution:

(a) The marginal pmfs:

x	$\mathbb{P}(X = x)$
1	0.4
2	0.2
3	0.4

y	1	2	3	4
$\mathbb{P}(Y = y)$	0.2	0.4	0.3	0.1

(b) $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. *We have:*

$$\begin{aligned}
\mathbb{E}[X] &= 1 \times 0.4 + 2 \times 0.2 + 3 \times 0.4 = 5 \times 0.4 = 2, \\
\mathbb{E}[Y] &= 1 \times 0.2 + 2 \times 0.4 + 3 \times 0.3 + 4 \times 0.1 = 1 + 0.9 + 0.4 = 2.3, \ \textit{and} \\
\mathbb{E}[XY] &= 1 \times 1 \times 0.1 + 1 \times 2 \times 0.2 + 1 \times 3 \times 0.1 + 1 \times 4 \times 0 \\
&\quad + 2 \times 1 \times 0.1 + 2 \times 2 \times 0.1 + 2 \times 3 \times 0 + 2 \times 4 \times 0 \\
&\quad + 3 \times 1 \times 0 + 3 \times 2 \times 0.1 + 3 \times 3 \times 0.2 + 3 \times 4 \times 0.1 \\
&= 0.8 + 0.6 + 0.6 + 1.8 + 1.2 \\
&= 5 \\
\Rightarrow \mathrm{Cov}(X, Y) &= 5 - 2 \times 2.3 = 0.4.
\end{aligned}$$

39. Let U and V be independent random variables, with $\mathbb{P}(U = 1) = \mathbb{P}(V = 1) = 1/4$ and $\mathbb{P}(U = -1) = \mathbb{P}(V = -1) = 3/4$. Define $X = U/V$ and $Y = U + V$.

(a) Give a table for the joint pmf of X and Y.

(b) Derive the expectation and variance of X.

(c) Calculate $\mathrm{Cov}(X, Y)$.

Solution:

(a) Let's identify the possible values of X and Y (if $U = u$ and $V = v$), along with their probabilities (noting that $\mathbb{P}(U = u, V = v) = \mathbb{P}(U = u)\mathbb{P}(V = v)$ by independence):

u	v	X	Y	$\mathbb{P}(U = u, V = v) = \mathbb{P}(U = u)\mathbb{P}(V = v)$
-1	-1	1	-2	9/16
-1	1	-1	0	3/16
1	-1	-1	0	3/16
1	1	1	2	1/16

Therefore the joint pmf of X and Y is:

$x \backslash y$	-2	0	2
-1	0	6/16	0
1	9/16	0	1/16

(b) First we need the marginal pmf of X:

x	$\mathbb{P}(X = x)$
-1	$6/16$
1	$10/16$

Now we calculate

$$
\begin{aligned}
\mathbb{E}X &= -1 \times 6/16 + 1 \times 10/16 = 4/16 = 1/4 = 0.25, \\
\mathbb{E}[X^2] &= (-1)^2 \times 6/16 + (1)^2 \times 10/16 = 1 \\
\Rightarrow \mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}.
\end{aligned}
$$

(c) Marginal pmf of Y:

y	-2	0	2
$\mathbb{P}(Y = y)$	$9/16$	$6/16$	$1/16$

Now calculate:

$$
\begin{aligned}
\mathbb{E}Y &= -2 \times 9/16 + 0 \times 6/16 + 2 \times 1/16 = -1, \\
\mathbb{E}[Y^2] &= (-2)^2 \times 9/16 + (0)^2 \times 6/16 + (2)^2 \times 1/16 \\
&= 40/16 = 10/4 = 5/2 = 2.5, \\
\mathrm{Var}(Y) &= \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 2.5 - (-1)^2 = 1.5,
\end{aligned}
$$

and $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}[X]\mathbb{E}[Y]$*, so further calculations:*

$$
\begin{aligned}
\mathbb{E}(XY) &= 1 \times (-2) \times 9/16 + (-1 \times 0) \times 6/16 + 1 \times 2 \times 1/16 = -1 \\
\Rightarrow \mathrm{Cov}(X, Y) &= -1 - \left(\frac{1}{4} \times (-1)\right) = \frac{-3}{4} = -0.75.
\end{aligned}
$$

40. A certain multiple choice exams has 30 questions, each providing 3 choices. To pass the exam one needs at least 20 out of 30 correct answers. Suppose a student knows the answers to 27 questions for certain, and fills in the remaining three questions "at random". What is the probability that the student will get full marks?

    Solution: Each guessed question being correct can be modelled as $\{\mathsf{Bernoulli}\}(1/3)$*. Of the three guessed, the number correct can be modelled as* $X \sim \mathsf{Bin}(3, 1/3)$*, so*

    $$\mathbb{P}(X = 3) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right)^0 = \frac{1}{3^3} = \frac{1}{27} \approx 0.0370.$$

41. We draw at random 10 balls from an urn with 25 red and 75 white balls. What is the expected number of red balls amongst the 10 balls drawn? Does it matter if we draw the balls with or without replacement?

    Solution: With replacement: each drawn ball is red with probability $\frac{25}{100} = \frac{1}{4}$*, independent of all others, so the number of red is* $\mathsf{Binomial}(10, 1/4)$*, with expected value* $\frac{10}{4} = 2.5$.

Without replacement: a particular sequence of r red balls and $10-r$ white balls ($r=0,\dots,10$) has probability

$$\frac{\{\prod_{i=1}^{r}(25-i+1)\}\left\{\prod_{j=1}^{10-r}(75-j+1)\right\}}{100\times 99\times 98\times 97\times 96\times 95\times 94\times 93\times 92\times 91}$$

with $\frac{10!}{r!(10-r)!}$ distinct such sequences. If we let R be the number of red balls, then, as a formula, the probability of $R=r$ red balls (and $10-r$ white balls) can be expressed as

$$\mathbb{P}(R=r)=\frac{10!25!75!90!}{r!(10-r)!(25-r)!(65+r)!100!}.$$

The expected value is then

$$\sum_{r=0}^{10} r\cdot\mathbb{P}(R=r),$$

*which, perhaps surprisingly, evaluates to $5/2=2.5$, so, **no**, it does not matter if balls are drawn with or without replacement — we **expect** to draw 2.5 red balls of the 10 in either case.*

You may have counted these probabilities in other ways.

Solution:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2]-(\mathbb{E}X)^2.\\
\mathbb{E}X &= \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x=\int_0^1 x\cdot 1\,\mathrm{d}x=\left[\frac{x^2}{2}\right]_{x=0}^{x=1}=\frac{1}{2},\\
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x=\int_0^1 x^2\cdot 1\,\mathrm{d}x=\left[\frac{x^3}{3}\right]_{x=0}^{x=1}=\frac{1}{3},\\
\Rightarrow \mathrm{Var}(X) &= \frac{1}{3}-\left(\frac{1}{2}\right)^2=\frac{4}{12}-\frac{3}{12}=\frac{1}{12}.
\end{aligned}$$

42. We repeatedly throw two fair dice until two sixes are thrown. What is the expected number of throws required?

    Solution: As seen in problem 19, the number of throws required has a $\mathsf{Geometric}(1/36)$ *distribution, so the expected number of throws is $1/(1/36)=36$.*

43. Suppose we divide the population of Brisbane (say 2,000,001 people) randomly in groups of 3.

    (a) How many groups would you expect there to be in which all persons have the same birthday?

    (b) What is the probability that there is at least one group in which all persons have the same birthday?

    Solution: Each group of three has same birthday (assuming 365 unique birthdays, each person is equally likely to be born on any day, independent of all others) with probability $\frac{1}{365^2}$ (see also problem 21).

    Thus the event that a group of three has the same birthday can be modelled as a $\mathsf{Bernoulli}(1/365^2)$ *random variable, independent of all other groups.*

    *There are $\frac{2{,}000{,}001}{3}=666,667$ groups of three. The **number** with same birthdays is $X\sim$* $\mathsf{Binomial}(666667, 1/365^2)$ *random variable.*

(a) *We expect*

$$\frac{666667}{365^2} = \frac{666667}{133225} \approx 5.00407$$

groups to have the same birthday.

(b)

$$\begin{aligned}
\mathbb{P}(X \geqslant 1) &= 1 - \mathbb{P}(X = 0) \\
&= 1 - \binom{666667}{666667} \cdot \left(\frac{1}{365^2}\right)^0 \cdot \left(1 - \frac{1}{365^2}\right)^{666667} \\
&= 1 - \left(1 - \frac{1}{133225}\right)^{666667} \\
&= 1 - \left(1 - \frac{666667/133225}{666667}\right)^{666667} \\
&\approx 1 - \left(1 - \frac{5.00407}{666667}\right)^{666667} \\
&\approx 1 - \mathrm{e}^{-5.00407} \approx 0.993289,
\end{aligned}$$

since $\mathrm{e}^{-x} \approx \left(1 - \frac{x}{n}\right)^n$ *when* n *is large.*

44. Let X, Y be independent $\mathsf{Bernoulli}(p)$ distributed random variables. Let $U = X+Y$ and $V = |X - Y|$. Determine the joint pmf of U and V. Are they independent? Are they correlated?

Solution: Probabilities of U *and* V *being certain values. Note that* $\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$ *from independence.*

x	y	U	V	$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$
0	0	0	0	$(1-p)^2$
0	1	1	1	$p(1-p)$
1	0	1	1	$p(1-p)$
1	1	2	0	p^2

Joint pmf:

$U\backslash V$	0	1
0	$(1-p)^2$	0
1	0	$2p(1-p)$
2	p^2	0

Marginals:

u	$\mathbb{P}(U = u)$
0	$(1-p)^2$
1	$2p(1-p)$
2	p^2

v	0	1
$\mathbb{P}(V = v)$	$(1-p)^2 + p^2$	$2p(1-p)$

Not independent for any $p \in (0,1)$ since, for example,

$$\mathbb{P}(U=1, V=0) = 0 \neq \mathbb{P}(U=1)\cdot\mathbb{P}(V=0) = 2p(1-p)\cdot\left\{(1-p)^2+p^2\right\}.$$

Correlated means $\varrho(U,V) \neq 0$. Equivalently, $\mathrm{Cov}(U,V) \neq 0$.

$$\begin{aligned}
\mathrm{Cov}(U,V) &= \mathbb{E}(UV) - \mathbb{E}[U]\mathbb{E}[V]. \\
\mathbb{E}U &= 2p(1-p) + 2p^2 = 2p - 2p^2 + 2p^2 = 2p. \\
\mathbb{E}V &= 2p(1-p). \\
\mathbb{E}(UV) &= 2p(1-p). \\
\mathrm{Cov}(U,V) &= 2p(1-p) - 2p(1-p)\times 2p = 2p(1-p)\left\{1-2p\right\},
\end{aligned}$$

so they are correlated ***unless*** $p = 1/2$ *(or $p=0$ or $p=1$).*

45. A janitor with n keys wants to open a door and tries the keys independently and at random. Consider the following two selection strategies: (A) unsuccessful keys are not eliminated from further selection and (B) unsuccessful keys *are* eliminated. Let N be the number of keys tried before the correct one is found.

    (a) Give the expectation and variance of N for strategy (A).

    (b) Give the probability mass function and expectation of N for strategy (B).

    Solution:

    (a) If keys are ***not*** *eliminated then at each try, probability of selecting correct key is* $\mathsf{Bernoulli}(1/n)$ *independent of all other trials. Numbers of keys tried before success is* $\mathsf{Geometric}(1/n)$*. Therefore*

    $$\mathbb{P}(N=x) = \frac{1}{n}\left(\frac{n-1}{n}\right)^{x-1}, \quad x = 1,2,\ldots,$$

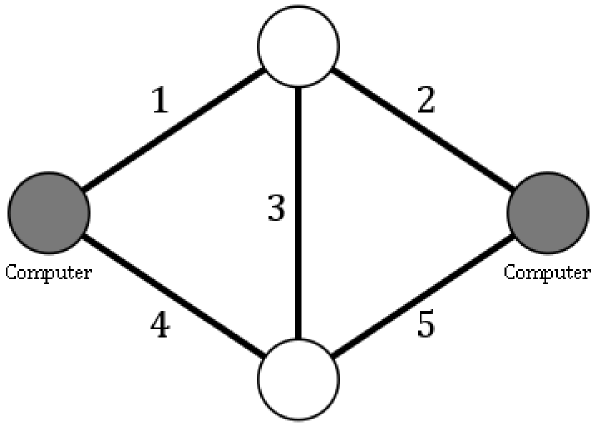
    so

    $$\begin{aligned}
    \mathbb{E}N &= \sum_{n=1}^{\infty} x\cdot\frac{1}{n}\left(\frac{n-1}{n}\right)^{x-1} = \frac{1}{(1/n)} = n, \\
    \mathbb{E}[N^2] &= \sum_{x=1}^{\infty} x^2\cdot\frac{1}{n}\left(\frac{n-1}{n}\right)^{x-1} = n(2n-1), \\
    \Rightarrow \mathrm{Var}(N) &= \mathbb{E}[N^2] - (\mathbb{E}N)^2 \\
    &= n(2n-1) - n^2 = 2n^2 - n - n^2 = n^2 - n = n(n-1).
    \end{aligned}$$

    (b) If keys ***are*** *eliminate, then we can compute the probability that* ***exactly*** *x trials are needed, for $x = 1,\ldots,n$, meaning* ***failure*** *on the first $x-1$ trials and success on the x^{th} trial. Therefore*

    $$\begin{aligned}
    \mathbb{P}(N=x) &= \frac{n-1}{n}\times\cdots\times\frac{n-(x-1)}{n-(x-2)}\times\frac{1}{n-(x-1)} \\
    &= \frac{1}{n}, \ \textit{for each } x = 1,\ldots,n, \\
    \Rightarrow \mathbb{E}N &= \sum_{x=1}^{n}\frac{x}{n} = \frac{n+1}{2}.
    \end{aligned}$$

46. Two computers are connected by five links (see Figure). Each link *works* with probability p, independently of all other links. Denote by A_i the event that link i works, for $i = 1, 2, \ldots, 5$.



(a) Write down the event B that the two computers are connected by working links in terms of the individual events $A_1, \ldots, A_5$.
(b) By using the partition $\{A_3, A_3^c\}$ and the law of total probability, determine $\mathbb{P}(B)$, the probability that the two computers are connected by working links.
(c) If $p = 0.5$, what is the probability that link 3 works, given that the two computers are connected by working links?

Solution:

(a)

$$B = (A_1 \cap A_2) \cup (A_4 \cap A_5) \cup (A_1 \cap A_3 \cap A_5) \cup (A_4 \cap A_3 \cap A_2).$$

(b)

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(B|A_3)\mathbb{P}(A_3) + \mathbb{P}(B|A_3^c)\mathbb{P}(A_3^c) \\
&= \mathbb{P}((A_1 \cup A_4) \cap (A_2 \cup A_5))\mathbb{P}(A_3) + \mathbb{P}((A_1 \cap A_2) \cup (A_4 \cap A_5))\mathbb{P}(A_3^c) \\
&= \mathbb{P}(A_1 \cup A_4)\mathbb{P}(A_2 \cup A_5)\mathbb{P}(A_3) \\
&\quad + [\mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_4 \cap A_5) - \mathbb{P}(A_1 \cap A_2 \cap A_4 \cap A_5)]\mathbb{P}(A_3^c) \\
&= (\mathbb{P}(A_1) + \mathbb{P}(A_4) - \mathbb{P}(A_1 \cap A_4))(\mathbb{P}(A_2) + \mathbb{P}(A_5) - \mathbb{P}(A_2 \cap A_5))\mathbb{P}(A_3) \\
&\quad + [\mathbb{P}(A_1)\mathbb{P}(A_2) + \mathbb{P}(A_4)\mathbb{P}(A_5) - \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_4)\mathbb{P}(A_5)](1 - \mathbb{P}(A_3)) \\
&= (p + p - \mathbb{P}(A_1)\mathbb{P}(A_4))(p + p - \mathbb{P}(A_2)\mathbb{P}(A_5))p \\
&\quad + [p^2 + p^2 - p^4](1 - p) \\
&= (2p - p^2)^2 \cdot p + [2p^2 - p^4](1 - p).
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{P}(A_3|B) &= \frac{\mathbb{P}(B|A_3)\mathbb{P}(A_3)}{\mathbb{P}(B)} \\
&= \frac{(2p - p^2)^2 p}{(2p - p^2)^2 p + [2p^2 - p^4](1 - p)} \\
&\approx 0.1678 \ \textit{(with } p = 1/2\textit{)}.
\end{aligned}$$

47. Suppose that a widget manufacturing plant has two production systems working in parallel to produce widgets. The first widget system is older, and produces 10000 widgets a day, with each widget being defective with probability 0.005, independently. The second widget system is brand new, and produces 25000 widgets a day, with each widget being defective with probability 0.002, independently. Moreover, the systems operate independently. At the end of each day, all of the day's widgets are collected together in a box.

    Let X and Y be the number of widgets the first and second system produce per day, respectively.

    (a) What are the expectations and variances of X and Y?
    (b) Determine the probability that a randomly chosen defective widget from the box was produced by the second (brand new) system.

    Solution:

    (a) $X \sim \mathsf{Bin}(10000, 0.005)$ *and* $Y \sim \mathsf{Bin}(25000, 0.002)$. *Hence*

$$\begin{aligned}\mathbb{E}X &= 10000 \times 0.005 = 50 \textit{ and} \\ \mathbb{E}Y &= 25000 \times 0.002 = 50, \\ \mathrm{Var}(X) &= 10000 \times 0.005 \times (1 - 0.005) = 49.75 \textit{ and} \\ \mathrm{Var}(Y) &= 25000 \times 0.002 \times (1 - 0.002) = 49.9.\end{aligned}$$

    (b) Let A_1 *and* A_2 *be the events that the widget was made by the first and second system, respectively. Let* D *denote the event that the widget is defective. Then, via Bayes' Rule,*

$$\mathbb{P}(A_2|D) = \frac{\mathbb{P}(D|A_2)\mathbb{P}(A_2)}{\mathbb{P}(D|A_2)\mathbb{P}(A_2) + \mathbb{P}(D|A_1)\mathbb{P}(A_1)}.$$

    Now

$$\begin{aligned}\mathbb{P}(A_1) &= \frac{10000}{(10000 + 25000)} = \frac{2}{7}, \\ \mathbb{P}(A_2) &= \frac{25000}{(10000 + 25000)} = \frac{5}{7}.\end{aligned}$$

    We have $\mathbb{P}(D|A_1) = 0.005$ *and* $\mathbb{P}(D|A_2) = 0.002$. *Hence*

$$\begin{aligned}\mathbb{P}(A_2|D) &= \frac{0.002 \times (5/7)}{0.002 \times (5/7) + 0.005 \times (2/7)} \\ &= \frac{0.01}{0.01 + 0.01} = \frac{1}{2} = 0.5.\end{aligned}$$

48. Let U and V be independent and identically distributed $\mathsf{Bernoulli}(p)$ random variables. Define $X = \min\{U, V\}$ and $Y = \max\{U, V\}$

    (a) Write down the joint pmf (as a table) of X and Y.
    (b) Determine the marginal pmf of X.
    (c) Give the conditional pmf of Y given $X = 1$.
    (d) Calculate the covariance of X and Y when $p = 1/2$. Are X and Y independent?

Solution:

(a) *Joint pmf:*

u	v	X	Y	$\mathbb{P}(U=u,V=v)=\mathbb{P}(U=u)\mathbb{P}(V=v)$
0	0	0	0	$(1-p)^2$
0	1	0	1	$p(1-p)$
1	0	0	1	$p(1-p)$
1	1	1	1	p^2

$x\backslash y$	0	1
0	$(1-p)^2$	$2p(1-p)$
1	0	p^2

(b) *Marginal pmf of X:*

x	$\mathbb{P}(X=x)$
0	$1-p^2 \quad (=(1-p)^2+2p(1-p))$
1	p^2

(c) *Conditional pmf of Y given $X=1$:*

y	0	1
$\mathbb{P}(Y=y\|X=1)$	0	1

(d) *When $p=1/2$, $\mathbb{E}X=1\times(1/2)^2=1/4$. Marginal pmf of Y:*

y	0	1
$\mathbb{P}(Y=y)$	$(1-p)^2$	$(1-(1-p)^2)=(2p(1-p)+p^2)$

$$\begin{aligned}
\mathbb{E}Y &= 1\times\left(1-\left(1-\frac{1}{2}\right)^2\right)=1\times\left(1-\frac{1}{4}\right)=\frac{3}{4}.\\
\mathbb{E}[X^2] &= \frac{1}{4},\quad \mathbb{E}[Y^2]=\frac{3}{4}\\
\Rightarrow \operatorname{Var}(X) &= \frac{1}{4}-\left(\frac{1}{4}\right)^2=\frac{4}{16}-\frac{1}{16}=\frac{3}{16},\\
\operatorname{Var}(Y) &= \frac{3}{4}-\left(\frac{3}{4}\right)^2=\frac{12}{16}-\frac{9}{16}=\frac{3}{16}.\\
\mathbb{E}[XY] &= \frac{1}{4}\\
\Rightarrow \operatorname{Cov}(X,Y) &= \frac{1}{4}-\frac{1}{4}\cdot\frac{3}{4}=\frac{4}{16}-\frac{3}{16}=\frac{1}{16}.
\end{aligned}$$

*The variables are clearly **dependent**. For example,*

$$\mathbb{P}(X=1,Y=0)=0\neq\mathbb{P}(X=1)\mathbb{P}(Y=0)=\left(\frac{1}{2}\right)^2\times\left(1-\frac{1}{2}\right)^2=\frac{1}{16}.$$

49. Let U and V be independent and identically distributed $\mathsf{Bernoulli}(p)$ random variables. Define $X=2|U-V|-1$ and $Y=U+V-1$

(a) Write down the joint pmf (as a table) of X and Y.

(b) Determine the marginal pmf of X.

(c) Give the conditional pmf of Y given $X = 1$.

(d) Calculate the covariance of X and Y when $p = 1/2$. Are X and Y independent?

Solution:

(a) Joint pmf:

u	v	X	Y	$\mathbb{P}(U = u, V = v) = \mathbb{P}(U = u)\mathbb{P}(V = v)$
0	0	-1	-1	$(1-p)^2$
0	1	1	0	$p(1-p)$
1	0	1	0	$p(1-p)$
1	1	-1	1	p^2

$x\backslash y$	-1	0	1
-1	$(1-p)^2$	0	p^2
1	0	$2p(1-p)$	0

(b) Marginal pmf of X:

x	$\mathbb{P}(X = x)$
-1	$(1-p)^2 + p^2$
1	$2p(1-p)$

(c) Conditional pmf of Y given $X = 1$:

y	-1	0	1
$\mathbb{P}(Y = y \mid X = 1)$	0	1	0

(d) First find the expectations:

y	-1	0	1
$\mathbb{P}(Y = y)$	$(1-p)^2$	$2p(1-p)$	p^2

$$\mathbb{E}X = (-1)\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] + 1 \times \left(2 \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{-1}{2} + \frac{1}{2} = 0.$$

$$\mathbb{E}[X^2] = (-1)^2 \times \frac{1}{2} + (1)^2 \times \frac{1}{2} = 1.$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 1.$$

$$\mathbb{E}Y = (-1) \times \frac{1}{4} + 1 \times \frac{1}{4} = 0.$$

$$\mathbb{E}[Y^2] = (-1)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{4} = \frac{1}{2}.$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = \frac{1}{2}.$$

$$\mathbb{E}[XY] = (-1) \times (-1) \times \frac{1}{4} + (-1) \times 1 \times \frac{1}{4} = \frac{1}{4} - \frac{1}{4} = 0.$$

$$\Rightarrow \mathrm{Cov}(X, Y) = 0 - (\mathbb{E}X)(\mathbb{E}Y) = 0 - 0 = 0.$$

No. *X and Y are clearly dependent. For example,*

$$\mathbb{P}(X = -1, Y = 0) = 0 \neq \mathbb{P}(X = -1)\mathbb{P}(Y = 0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

50. A space shuttle has two booster rockets. Each booster rocket consists of four segments. The joint between adjoining segments is sealed by a pair of O-rings. The joint is properly (safely) sealed if at least one of the two O-rings works. Let A_i be the event that the ith joint is properly sealed, $i = 1, \ldots, 6$. The system fails if at least one of the joints is not properly sealed.

    If the 12 O-rings fail independently of each other, each with failure probability 0.1, what is the probability of system failure?

    Solution:

    *Writing p for the probability of O-ring failure, the probability of **joint** failure is $\mathbb{P}(A_i^c) = p^2$, for $i = 1, \ldots, 6$, since all O-rings fail independently. The event the system works is $B = A_1 \cap A_2 \cap \cdots \cap A_6$, so*

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(A_1 \cap \cdots \cap A_6) \\
&= \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_6) && \textit{(independence)} \\
&= (1 - \mathbb{P}(A_1^c)) \times \cdots \times (1 - \mathbb{P}(A_6^c)) \\
&= (1 - p^2) \times \cdots \times (1 - p^2) \\
&= (1 - p^2)^6.
\end{aligned}$$

    Therefore the probability of system failure is

$$\mathbb{P}(B^c) = 1 - \mathbb{P}(B) = 1 - (1 - p^2)^6 \approx 0.0585 \textit{ when } p = 0.1.$$

51. A system is comprised of two components connected in series so the system is working if and only if both components are working. The two components have independent $\mathsf{Geometric}(p)$ lifetimes. Find the probability mass function of the lifetime of the system.

    Solution: Let X_1 and X_2 be the lifetimes of the two components. The lifetime of the system is $Y = \min(X_1, X_2)$. Then, for any non-negative integer y,

$$\begin{aligned}
\mathbb{P}(Y \geqslant y) &= \mathbb{P}(X_1 \geqslant y, X_2 \geqslant y) \\
&= \mathbb{P}(X_1 \geqslant y)\mathbb{P}(X_2 \geqslant y) \\
&= (1 - p)^{y-1} \times (1 - p)^{y-1} \\
&= (1 - p)^{2(y-1)}.
\end{aligned}$$

    The probability mass function of Y is

$$\mathbb{P}(Y = y) = \mathbb{P}(Y \geqslant y) - \mathbb{P}(Y \geqslant y+1) = (1-p)^{2(y-1)} - (1-p)^{2y} = (1-p)^{2(y-1)}(1-(1-p)^2)$$

    This is the pmf of a $\mathsf{Geometric}(1 - (1 - p)^2)$ distribution.