Summary.

	Order matters	Order does not matter
With replacement	n^r	$\binom{n+r-1}{r}$
Without replacement	${}^nP_r = \frac{n!}{(n-r)!}$	${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$

Matlab code.

```
n = 6;
r = 2;
factorial(n)
ans =
    720
nchoosek(n,r)
ans =
     15
nPr = nchoosek(n,r)*factorial(r)
nPr =
     30
```

Example. How many ways are there to order the letters in the word *INDOOROOPILLY*?

Notice that this situation does not fall into any of the above categories. In general, the number of permutations of n objects with n_1 of type 1, n_2 of type 2, et cetera, is given by

$$\frac{n!}{n_1!n_2!\ldots n_k!},$$

where k is the number of types.

2 I's 1 R
1 N 1 P
1 D 2 L's
4 O's 1 Y

Hence, the number of ways to order the letters in *INDOOROOPILLY* is

$$\boxed{\frac{13!}{2!1!1!4!1!1!2!1!} = \frac{13!}{2!4!2!}}$$

Example. Suppose that you have two red balls and three blue balls. How many *distinct* orderings of all balls are there?

$$\boxed{\frac{5!}{2!\,3!} = \frac{5\times4\times3\times2\times1}{(2\times1)(3\times2\times1)} = 10}$$

1 2 3 4 5
R R B B B
2! permutation
6 = 3!

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

why?

Take for example

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

what is the coefficient of $a^3 b^2$?

$\equiv$ How many sets of size 3 can we form from $\{1,2,3,4,5\}$? $\binom{5}{3}$

3

Conditional Probability and Independent Events

By the end of this chapter you should:

- Understand the rule for conditional probability $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$.
- Know what it means for events to be independent.
- Be able to apply Bayes' rule to simple scenarios.

Conditional Probability

How do probabilities *change* when we know *some event $B \subseteq \Omega$ has occurred?*

Suppose B *has occurred.* Thus, we know that the outcome lies in B. Then A will occur if and only if $A \cap B$ occurs, and the *relative chance of A occurring* is therefore $\mathbb{P}(A \cap B)/\mathbb{P}(B)$.

This leads to the definition of the **conditional probability** of A given B:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Remark: This only makes sense if $\mathbb{P}(B) > 0$.

Example. We throw two dice (assume consecutively). Given that the *sum of the faces is 10*, what is the probability that *one 6 is cast*?

Let B be the event that *the sum is 10*,

$\boxed{B = \{(6,4), (5,5), (4,6)\}}$

Let A be the event that *one 6 is cast*,

$\boxed{A = \{(6,1), (6,2), \dots, (6,5), (1,6), (2,6), \dots, (5,6)\}}$

Then, $\boxed{A \cap B = \{(6,4), (4,6)\}}$ Since all elementary events are equally likely, we have

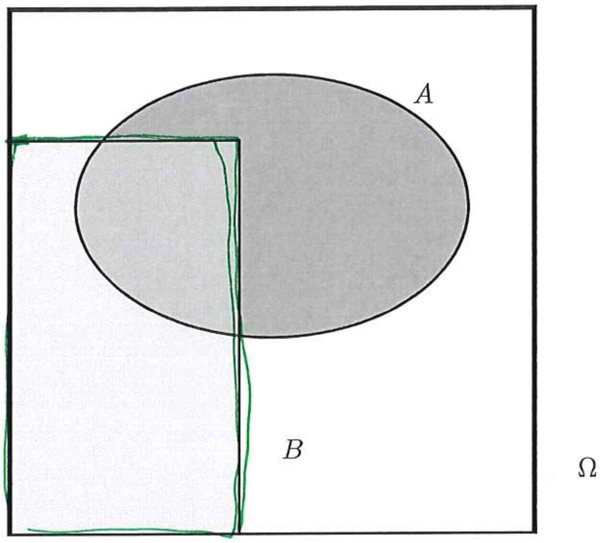


Figure 3.1: The probability of A is $\mathbb{P}(A) = \mathbb{P}(A)/\mathbb{P}(\Omega)$ However, if B has occurred then the probability of A *given* B is $\mathbb{P}(A \cap B)/\mathbb{P}(B)$

$\mathbb{P}(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{3/36} = 2/3$

Compare this with

$\mathbb{P}(A) = 10/36$

Question. Morgan (a boy) has exactly one sibling. What is the probability that Morgan has a sister?

Now consider the following arguments:

- Assuming that the sex of siblings is independent, the other sibling is equally likely to be a girl or a boy. The answer is 1/2.
- There are four cases: BB, BG, GB, GG. The last is impossible, and in two of the remaining three cases the sibling is a girl. The answer is 2/3.

Actually this question is poorly posed, and highlights the importance of clearly specifying the model and events of interest since the answer depends on whether we know only that Morgan is one of two children, or if we know that Morgan is (for instance) the older sibling.

Suppose that a couple had two children, where each child was independently, equally likely to be a boy or a girl. Given that one of them is a boy, what is the probability that the other is a girl?

The sample space is

$$\Omega = \{BB, BG, GB, GG\}$$

and the event of interest is $A = \{BG, GB\}$
and we know that the event $B = \{BB, BG, GB\}$ occurs. Hence,

$A \cap B = A$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/4}{3/4} = 2/3$$

What is the probability that the second (to be born) child is a girl if the first one is a boy?

The sample space is

$$\Omega = \{BB, BG, GB, GG\}$$

the event of interest is $A = \{BG, GG\}$
and we know that the event $B = \{BB, BG\}$ occurs. Hence,

$A \cap B = \{BG\}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/4}{2/4} = 1/2$$

Question: Suppose that A and B are events where both $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Can we write $\mathbb{P}(B \mid A)$ in terms of $\mathbb{P}(A \mid B)$?

$A \cap B \equiv B \cap A$

$P(A \cap B) = P(B \cap A)$

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \text{ and}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \text{ so}$$

$$P(B \mid A) \cdot P(A) = P(A \cap B) = P(A \mid B) \times P(B) \text{ , or}$$

$$\mathbb{P}(B \mid A) = \frac{P(A \mid B) \times P(B)}{\mathbb{P}(A)}$$

This is a special case of *Bayes' Rule*, which we will see later in this chapter.

Example. A new COVID-19 rapid test claims that a 99% sensitivity (probability of correctly diagnosing a person with COVID-19) and 98% specificity (probability of correctly diagnosing a person who does not have COVID-19). The probability that a randomly selected person in a particular suburb has COVID-19 is 4%.

Given a randomly selected person has tested positive to COVID-19, what is the probability that they have COVID-19?

Let A be the event that a person tests positive to COVID-19 and let B be the person actually has COVID-19. By the above rule,

$$\mathbb{P}(B \mid A) = \frac{P(A \mid B) \times P(B)}{P(A)} = \frac{0.99 \times 0.04}{P(A)\,?}$$

$$P(\text{person +ve} \mid \text{test +ve}) = \frac{P(\text{+ve test} \mid \text{person +ve})\, P(\cancel{\text{+ve}}\ \text{+ve person})}{P(\text{+ve test})}$$

Product Rule

By the definition of conditional probability we have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B \mid A).$$

We can generalize this to n intersections $A_1 \cap A_2 \cap \cdots \cap A_n$ (abbreviated as $A_1 A_2 \cdots A_n$).

This gives the **product rule** (also called *chain rule*) of probability:

$$\mathbb{P}(A_1 A_2 \cdots A_n) = \overbrace{\mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)}^{P(A_1 \cap A_2)} \underbrace{\cdots}_{P(A_3 \mid A_1 \cap A_2)} \mathbb{P}(A_n \mid A_1 A_2 \cdots A_{n-1}).$$

Example. Draw five cards from a full deck of 52 cards. What is the probability of no Ace?

Let A_i be event that the i-th draw is not an ace, $i = 1, 2, 3, 4, 5$.

We are interested in $A := A_1 \cap \cdots \cap A_5$.

We know

$$\mathbb{P}(A_1) = 48/52$$
$$\mathbb{P}(A_2 \mid A_1) = 47/51$$
$$\mathbb{P}(A_3 \mid A_1 \cap A_2) = 46/50$$
$$\mathbb{P}(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{45}{49}, \text{ and}$$
$$\mathbb{P}(A_5 \mid A_1 \cap A_2 \cap A_3 \cap A_4) = 44/48$$

Using the product rule, we see that

$$\mathbb{P}(A) = \frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \frac{45}{49} \times \frac{44}{48} \simeq 0.66$$

Exercise. (Birthday problem) What is the probability P that no one in a randomly selected group of $n < 365$ persons shares a birthday with someone else? (Assume 365 equally-likely birthdays.)

$$P = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{(365-(n-1))}{365}$$

Some approximations of this probability:

n	23	40	57
P	0.4927	0.1088	0.0099

Law of Total Probability

Suppose $B_1, B_2, \ldots, B_n$ is a *partition* of Ω. Then, by the *sum rule*,

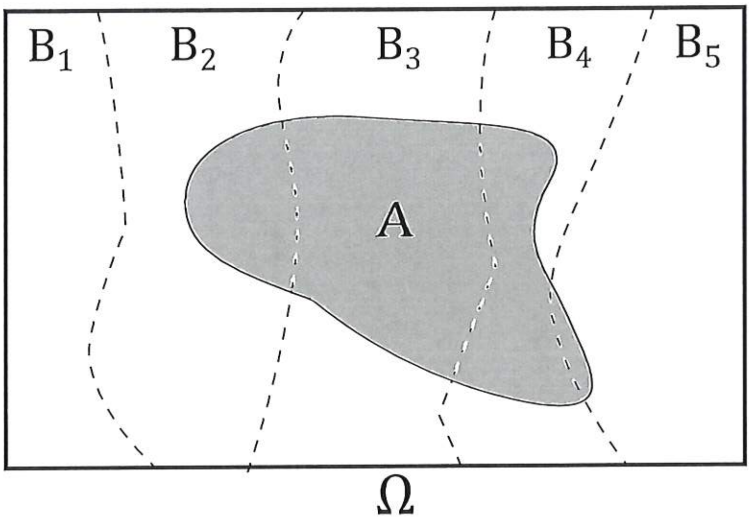
$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) \, .$$

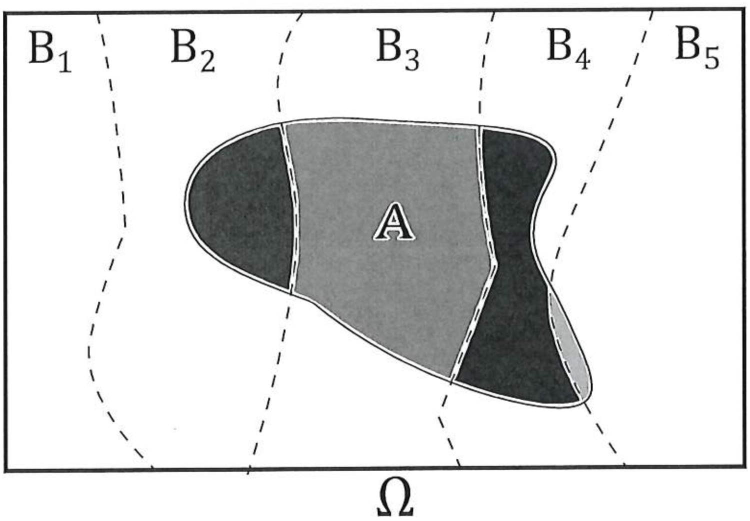
Hence,

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i) \, \mathbb{P}(B_i)$$

(as long as $\mathbb{P}(B_1) > 0$, $\mathbb{P}(B_2) > 0$, $\ldots$, $\mathbb{P}(B_n) > 0$).

This is called the **Law of Total Probability**.





Example. Draw a card from a full deck of 52 cards. What is the probability it is an Ace?

Let A be the event that the card is an ace. Also, let B_1 be the event that the card is red and B_2 be the event that the card is black. Note that $\Omega = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$ so that $\{B_1, B_2\}$ is a valid partition. Always check this!