STAT 7203 Assignment 1

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1. (a). solution

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0.1} x \cdot 0\,dx + \int_{0.1}^{1} x \cdot \frac{1}{x \ln 10}\,dx + \int_{1}^{\infty} x \cdot 0\,dx$$

$$= 0 + \frac{1}{\ln 10} \cdot x \Big|_{0.1}^{1} + 0 = \frac{0.9}{\ln 10} = 0.390865$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-\infty}^{0.1} x^2 \cdot 0\,dx + \int_{0.1}^{1} x^2 \cdot \frac{1}{x \ln 10}\,dx + \int_{1}^{\infty} x^2 \cdot 0\,dx$$

$$= 0 + \int_{0.1}^{1} \frac{x}{\ln 10}\,dx + 0 = \frac{1}{\ln 10} \cdot \frac{x^2}{2} \Big|_{0.1}^{1} = 0.214976$$

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 0.0622$$

(b). solution

$$Y = (10X)^{-1} \Rightarrow X = \frac{1}{10Y}$$

$$0.1 \le X \le 1$$

$$0.1 \le \frac{1}{10Y} \le 1$$

$$0.1 \le Y \le 1$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{1}{10y}\right) = -\frac{1}{10y^2} \times \frac{1}{\frac{1}{10y} \ln 10} = -\frac{1}{y \ln 10}$$

$$f_Y(y) = \begin{cases} -\frac{1}{y \ln 10}, & y \in [0.1, 1] \\ 0, & \text{else} \end{cases}$$

(c) solution

$$F_X(x) = \int_{0.1}^{x} \frac{1}{x \ln 10}\, dx \qquad \cancel{x \in [0.1, 1]}$$

$$= \frac{\ln x}{\ln 10}\Bigg|_{0.1}^{x} = \frac{\ln x - \ln 0.1}{\ln 10}$$

$$= \log_{10} 10x \qquad x \in [0.1, 1]$$

$$F_X(g_X(x)) = \log_{10} 10\, g_X(x) = x$$

$$g_X(x) = 10^{x-1}$$

$$0.1 \le g_X(x) \le 1$$

$$0.1 \le 10^{x-1} \le 1$$

$$0 \le x \le 1$$

$$g_X(x) = 10^{x-1} \quad \text{when } x \in [0, 1]$$

(d) solution

```
n=100000
u=runif(n)
X=10^(u-1)
hist(X, freq=FALSE)
x=seq(from=0.1, to=1, length=10)
lines(x, 1/(x*log(10)), lwd=2)
```

(e) solution

```
n= 10000
u=runif(n)
X=10^(u-1)
Y=u
Z=X*Y
W=10^((log(Z,base=10))-ceiling(log(Z,base=10)))
hist(W, freq=FALSE)
x=seq(from=0.1, to=1, length=10)
lines(x, 1/(x*log(10)), lwd=2)
```

(f) solution

$$X_1X_2 \le 0.1 \Rightarrow X_2 \le \frac{0.1}{X_1}$$

$$P(X_1X_2 \le 0.1) = \int_{0.1}^{1}\int_{0.1}^{\frac{0.1}{x_1}} \frac{1}{x_1 \ln 10} \cdot \frac{1}{x_2 \ln 10}\, dx_1\, dx_2$$

$$= \int_{0.1}^{1} dx_1 \cdot \left.\frac{\ln x_2}{(\ln 10)^2 x_1}\right|_{0.1}^{\frac{0.1}{x_1}}$$

$$= \int_{0.1}^{1} \frac{-\ln x_1}{(\ln 10)^2 x_1}\, dx_1$$

$$= \left.\frac{-(\ln x_1)^2}{2\times(\ln 10)^2}\right|_{0.1}^{1}$$

$$= \frac{-1}{2\times(\ln 10)^2} \times ((\ln 1)^2 - (\ln 0.1)^2)$$

$$= \frac{-1}{2\times(\ln 10)^2} \times \ln 10 \times \ln 0.1$$

$$= \frac{1}{2}$$

2. (a) soluotion

$$P(B_1 \text{ is on top}) = 1^{n-1} \times \frac{4}{10}$$
$$= \frac{2}{5} \qquad n>0.$$

No, This probability doesn't depent on n.

(b) solution

$$P(B_1 \text{ is second}) \begin{cases} \frac{3}{10} + \frac{2}{10} + \frac{1}{10}, & n=1 \\ 1^{n-2} \times \frac{4}{10} \times (1-\frac{4}{10}) + (\frac{3}{10})^n + (\frac{2}{10})^n + (\frac{1}{10})^n, & n \geq 2 \end{cases}$$

$$P(B_1 \text{ is second}) \begin{cases} \frac{3}{5} & , n=1 \\ \frac{6}{25} + (\frac{3}{10})^n + (\frac{2}{10})^n + (\frac{1}{10})^n & , n \geq 2 \end{cases}$$

$$\lim_{n \to \infty} P(B_1 \text{ is second}) = \frac{6}{25}$$

(c) solution

```
bottom=c( )
for (i in 1:10000)
{
  sample = sample(c('B1', 'B2', 'B3', 'B4'), size=100,
                  replace = TRUE, prob=c(4,3,2,1))
  sample = sample[length(sample):1]
  sample = unique(sample)
  if (length(sample)==4)
```

```
{
    bottom = c(bottom, sample[4])
} else
{
    bottom = c(bottom, 'B4')
}
}
length(bottom[bottom == 'B1']) / 10000
```

(d) solution

$$(p-\frac{1}{2})^2 \geq 0$$

$$p^2 - p + \frac{1}{4} \geq 0$$

$$p(1-p) \leq \frac{1}{4}$$

$X_i \sim Bernoulli(p)$

$$Var(m^{-1}\sum_{i=1}^{m} X_i) = \frac{1}{m^2} Var(\sum_{i=1}^{m} X_i)$$

$$= \frac{1}{m^2} \times m \times p \times (1-p)$$

$$= \frac{p(1-p)}{m} \leq \frac{1}{4m} \quad (m>0)$$