1 (a) X_1 only takes the values 0 and 1 so it is Bernoulli. The 'success' probability is $\mathbb{P}(X_1=1) = \dfrac{\#\text{ permutations with } A_1 = 1}{\#\text{ permutations in total}} = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}$.

(b) $\sum_{i=1}^{n} X_i$ does not have a Binomial(n,p) distributions. Note that if $Y \sim$ Binomial(n,p), then $\mathbb{P}(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$; On the other hand

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i = n\right) = \frac{\#\text{ permutations with } A_i = i \text{ for all } i=1,\dots,n}{\#\text{ permutations in total}}$$

$$= 1/n!.$$

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i = n-1\right) = \frac{\#\text{ permutations with } A_i = i \text{ for } (n-1) \text{ values of } i \text{ exactly}}{\#\text{permutations in total}}$$

$$= 0,$$

as having $(n-1)$ values in correct order implies the n^{th} term is also in correct order. There is no value of p s.t.

$$p^n = 1/n! \quad \text{and} \quad np^{n-1}(1-p)^1 = 0.$$

(c) $$\mathbb{E}\left(\sum_{i=1}^{n} X_i\right) \underset{\uparrow}{=} \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{n} \underbrace{\mathbb{P}(X_i=1)}_{\uparrow} = \sum_{i=1}^{n} 1/n = 1$$

linearity of expectation

This equals $\frac{1}{n}$ from part (a).

(d) $$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left[\mathbb{E}\left(\sum_{i=1}^{n} X_i\right)\right]^2$$

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n}\sum_{j\neq i} X_i X_j\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i^2\right] + \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j\neq i} X_i X_j\right] \quad \text{linearity of expectation}$$

$$= \sum_{i=1}^{n} \mathbb{E}[X_i] + \sum_{i=1}^{n}\sum_{j\neq i} \mathbb{E}[X_i X_j] \quad X_i^2 = X_i \text{ as } X_i \in \{0,1\}$$

for $i \neq j$

$$\mathbb{E}[X_i X_j] = \mathbb{P}(X_i = 1, X_j = 1) = \frac{\#\text{ permutations with } A_i = i \text{ and } A_j = j}{\#\text{ permutations in total}}.$$

$$\mathbb{E}[X_i X_j] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

As $\mathbb{E}[X_i] = 1/n$

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} \sum_{j \neq i} \frac{1}{n(n-1)} = 1 + 1 = 2$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = 2 - 1^2 = 1.$$

2(a) $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\max\{X_1, X_2, X_3\} \leq y)$ $\quad y = 1, 2, \ldots, m$

$= \mathbb{P}(X_1 \leq y, X_2 \leq y, X_3 \leq y)$

$= \mathbb{P}(X_1 \leq y)\mathbb{P}(X_2 \leq y)\mathbb{P}(X_3 \leq y)$ by independence

$= (y/m)^3$ As $\mathbb{P}(X_i \leq y) = y/m$ with $X_i \sim \mathrm{Uniform}(\{1, \ldots, m\})$

(b) $f_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(Y \leq y) - \mathbb{P}(Y \leq y-1)$ $\quad y = 1, 2, \ldots, m$

$= (y/m)^3 - \left(\frac{y-1}{m}\right)^3$

$= \dfrac{3y^2 - 3y + 1}{m^3}$

(c) $F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(\min\{X_1, X_2, X_3\} \leq z)$ $\quad z = 1, 2, \ldots, m,$

$= 1 - \mathbb{P}(\min\{X_1, X_2, X_3\} > z)$

$= 1 - \mathbb{P}(X_1 > z, X_2 > z, X_3 > z)$

$= 1 - \mathbb{P}(X_1 > z)\mathbb{P}(X_2 > z)\mathbb{P}(X_3 > z)$ by independence

$= 1 - \left(\frac{m-z}{m}\right)^3$ as $\mathbb{P}(X_i > z) = \left(\frac{m-z}{m}\right).$

$$f_Z(z) = \mathbb{P}(Z \leq z) - \mathbb{P}(Z \leq z-1)$$

$$= 1 - \left(\frac{m-z}{m}\right)^3 - \left\{1 - \left(\frac{m-z+1}{m}\right)^3\right\} = \left(\frac{m-z+1}{m}\right)^3 - \left(\frac{m-z}{m}\right)^3.$$

3.

(a) $f_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(X_1+X_2=y)$ $\qquad y = 2, 3, \ldots$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_1+X_2=y \mid X_1=k)\,\mathbb{P}(X_1=k) \qquad \text{law of total probability}$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_2=y-k \mid X_1=k)\,\mathbb{P}(X_1=k)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_2=y-k)\,\mathbb{P}(X_1=k) \qquad \text{by independence}$$

$$= \sum_{k=1}^{y-1} p\cdot(1-p)^{y-k-1} \times p(1-p)^{k-1}$$

This does not depend on k

$$= \sum_{k=1}^{y-1} \boxed{p^2(1-p)^{y-2}} = p^2(1-p)^{y-2}\left(\sum_{k=1}^{y-1} 1\right)$$

$$= (y-1)\,p^2(1-p)^{y-2} \qquad y = 2, 3, \ldots$$

(b) $\mathbb{E}[Y] = \mathbb{E}[X_1+X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 1/p + 1/p = 2/p$

Note $\mathbb{E}[X_i] = 1/p$ from formula sheet. see also pg 61 of notes where we computed mean + variance of a Geometric (p) random variable.

$\operatorname{Var}(Y) = \operatorname{Var}(X_1+X_2) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2)$ by independence

$$= \frac{1-p}{p^2} + \frac{1-p}{p^2} = \frac{2(1-p)}{p^2}.$$

(c) $\mathbb{E}[Y \mid X_1] = \mathbb{E}[X_1 + X_2 \mid X_1] = \mathbb{E}[X_1 \mid X_1] + \mathbb{E}[X_2 \mid X_1]$ (see pg 58)

$= X_1 + \mathbb{E}[X_2]$ (as X_1 and X_2 are independent)

$= X_1 + 1/p.$

(d) We first need the conditional probability mass function of X_1 given Y.

$x = 1, 2, \ldots, y-1$

$$f_{X_1|Y}(x|y) = \mathbb{P}(X_1 = x \mid Y = y) = \frac{\mathbb{P}(X_1 = x, Y = y)}{\mathbb{P}(Y=y)}$$

$$= \frac{\mathbb{P}(X_1 = x, X_1 + X_2 = y)}{(y-1)\,p^2(1-p)^{y-2}} \quad \leftarrow \text{from part (a)}$$

$$= \frac{\mathbb{P}(X_1 = x, X_2 = y - x)}{(y-1)\,p^2(1-p)^{y-2}}$$

$$= \frac{\mathbb{P}(X_1 = x)\,\mathbb{P}(X_2 = y-x)}{(y-1)\,p^2(1-p)^{y-2}} \quad \text{by independence}$$

$$= \frac{p(1-p)^{x-1} \times p(1-p)^{y-x}}{(y-1)\,p^2(1-p)^{y-2}} = \frac{1}{(y-1)}$$

$$\mathbb{E}[X_1 \mid Y] = \sum_{x=1}^{y-1} x \times \frac{1}{(y-1)} = \frac{1}{(y-1)} \cdot \frac{y(y-1)}{2} = y/2$$

4. $X \sim \text{Binomial}(n, p)$ $\qquad Y \mid X \sim \text{Binomial}(X, q)$

As Y is conditionally Binomial we can write it as the ~~sum of independent Bernoulli (q) random variables. Let Z_i be independent Bernoulli (q). Then $Y = \sum_{i=1}^{X} Z_i$. So Y is~~ the ~~sum~~ sum of a random number of random variables

$$M_Y(s) = \mathbb{E}[e^{sY}] = \mathbb{E}[\mathbb{E}[e^{sY} \mid X]]$$

$$= \mathbb{E}[(1 - q + qe^s)^X]$$

Using the fact that if $Z \sim B(n,p)$, $\mathbb{E}[e^{sZ}] = (1 - p + pe^s)^n$

$$= (1 - p + p(1 - q + qe^s))^n$$

$$= (1 - pq + pqe^s)^n$$

This is the MGF of a Binomial (n, pq) distribution

5. $M_Y(s) = \mathbb{E}[e^{sY}]$

using the fact that if $Z \sim Po(\lambda)$ then $\mathbb{E}[e^{sZ}] = \exp(\lambda(e^s - 1))$

$= \mathbb{E}[\mathbb{E}[e^{sY} | X]]$

$= \mathbb{E}[(1-q+qe^s)^X] = \mathbb{E}[e^{X\log(1-q+qe^s)}]$

$= \exp(\lambda(e^{\log(1-q+qe^s)} - 1))$

$= \exp(\lambda(1-q+qe^s-1)) = \exp(\lambda q(e^s - 1))$

So This is the MGF of a Poisson (λq) distributions.

6. (a) $\mathbb{P}(Y > k) = \mathbb{P}(\min\{X_1, X_2\} > k)$

$= \mathbb{P}(X_1 > k, X_2 > k)$

$= \mathbb{P}(X_1 > k)\,\mathbb{P}(X_2 > k)$ by independence

$= (1-p)^k \cdot (1-p)^k = (1-p)^{2k}$ $k = 0, 1, 2, \ldots$

$f_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(Y > y-1) - \mathbb{P}(Y > y)$

$= (1-p)^{2(y-1)} - (1-p)^{2y}$

$= (1-p)^{2(y-1)}(1-(1-p)^2)$

$= (1-2p+p^2)^{y-1}(2p-p^2)$

(b) Y has a Geometric $(2p-p^2)$ distribution

c) $\mathbb{P}(X_1 = Y) = \mathbb{P}(X_1 \leq X_2)$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_1 \leq X_2 \mid X_2 = k)\,\mathbb{P}(X_2 = k)$$ law of total probability

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_1 \leq k \mid X_2 = k)\,\mathbb{P}(X_2 = k)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_1 \leq k)\,\mathbb{P}(X_2 = k)$$ by independence

$$= \sum_{k=1}^{\infty} \left(1 - (1-p)^k\right) p(1-p)^{k-1}$$

$$= \sum_{k=1}^{\infty} p(1-p)^{k-1} - \sum_{k=1}^{\infty} p(1-p)^{2k-1}$$

$$= 1 - \frac{p}{(1-p)} \times \frac{(1-p)^2}{1-(1-p)^2}$$ as $\sum_{k=1}^{\infty} (1-p)^{2k} = \frac{(1-p)^2}{1-(1-p)^2}$

using geometric series

$$= 1 - \frac{1-p}{2-p} = \frac{1}{2-p}$$

note that $\mathbb{P}(X_1 \leq k) = \sum_{j=1}^{k} p(1-p)^{j-1}$

$$= p \times \frac{1-(1-p)^k}{1-(1-p)} = 1-(1-p)^k$$

using geometric series.

7(a)

$$M_Y(s) = \mathbb{E}[e^{sY}] = \mathbb{E}[\mathbb{E}[e^{sY} \mid X]]$$

$$= \mathbb{P}(X=0)\,\mathbb{E}[e^{sY} \mid X=0] + \mathbb{P}(X=1)\,\mathbb{E}[e^{sY} \mid X=1]$$

$$= \mathbb{P}(X=0) \cdot e^{s \cdot 0} + \mathbb{P}(X=1)\exp(2(e^s - 1))$$

$$= 3/4 + 1/4 \exp(2(e^s-1))$$

(b) $f_{X|Y}(x|y) = \mathbb{P}(X=x \mid Y=y) = \dfrac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)} = \dfrac{\mathbb{P}(Y=y \mid X=x)\,\mathbb{P}(X=x)}{\mathbb{P}(Y=y)}$

for $y \geq 1$; $f_{X|Y}(0|y) = \dfrac{\mathbb{P}(Y=y \mid X=0)\,\mathbb{P}(X=0)}{\mathbb{P}(Y=y)} = 0$

as $\mathbb{P}(Y=0 \mid X=0) = 1$. Hence, $f_{X|Y}(1|y) = 1$.

for $y=0$; $f_{X|Y}(0|0) = \dfrac{\mathbb{P}(Y=0 \mid X=0)\,\mathbb{P}(X=0)}{\mathbb{P}(Y=0)}$

$\mathbb{P}(Y=0) = \mathbb{P}(Y=0 \mid X=0)\,\mathbb{P}(X=0) + \mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1)$

$= \mathbb{P}(X=0) + \mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1)$

$= 3/4 + e^{-2} \times \frac{1}{4}$

So $f_{X|Y}(0|0) = \dfrac{1 \times 3/4}{3/4 + 1/4\, e^{-2}} = \dfrac{3}{3+e^{-2}}$

$f_{X|Y}(1|0) = \dfrac{\mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1)}{\mathbb{P}(Y=0)}$

$= \dfrac{1/4 \times e^{-2}}{3/4 + 1/4\, e^{-2}} = \dfrac{e^{-2}}{3+e^{-2}}$

c) Using the MGF from part (a)

$\mathbb{E}\,Y = M_Y'(0)$. $M_Y'(s) = \frac{1}{4} \exp(2(e^s-1)) \times 2e^s$

$= 1/2$.

$M_Y''(s) = \frac{1}{4}(2e^s)^2 \exp(2(e^s-1)) + \frac{1}{4}\exp(2(e^s-1)) \times 2$

$\mathrm{Var}(Y) = M_Y''(0) - (M_Y'(0))^2 = 1 + 1/2 - (1/2)^2 = 5/4$.

8. Let X be the number of times the stock increased. Then $X \sim \text{Binomial}(10,p)$. The number of times the stock decreased is $10-X$. Therefore, the stock price after 10 days is

$$1 \times r^{X} \times (1/r)^{10-X} = r^{2X-10}.$$

The expected value of the stock price is (Thanks Becky)

$$\mathbb{E}\left[r^{2X-10}\right] = r^{-10}\,\mathbb{E}\left[r^{2X}\right] = r^{-10}\,\mathbb{E}\left[e^{X \cdot 2\log(r)}\right] = r^{-10}\left(1-p+pe^{2\log(r)}\right)^{10}$$

$$= r^{-10}\left(1-p+pr^2\right)^{10}$$

$$= \left(\frac{1-p}{r} + pr\right)^{10}$$

This follows from the MGF of a Binomial distribution.

The variance of the stock price is

$$\mathrm{Var}\left(r^{2X-10}\right) = r^{-20}\,\mathrm{Var}\left((r^2)^X\right)$$

$$\mathrm{Var}\left((r^2)^X\right) = \mathbb{E}\left[(r^2)^{2X}\right] - \left(\mathbb{E}\left[r^{2X}\right]\right)^2$$

$$\mathbb{E}\left[(r^2)^{2X}\right] = \mathbb{E}\left[(r^4)^X\right] = \left(1-p+r^4p\right)^{10}$$ (again using the MGF)

So

$$\mathrm{Var}\left(2^{X-10}\right) = r^{-20}\left(1-p+r^4p\right)^{10} - r^{-20}\left(1-p+r^2p\right)^{20}$$

$$= \left(\frac{1-p}{r^2} + r^2p\right)^{10} - \left(\frac{1-p}{r} + rp\right)^{20}$$

Other approaches to the question are possible.

9. The probability that component i has not failed at the end of 4 years is: (Remember $X_i \sim \text{Geometric}(\frac{1}{3})$).

$$P(X_i > 4) = \mathbb{P}(X_i \geq 5) = \sum_{k=5}^{\infty}\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^{k-1} = \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4\sum_{k=0}^{\infty}\left(\tfrac{2}{3}\right)^k$$

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4}{1-2/3} = \left(\tfrac{2}{3}\right)^4.$$ (using geometric series)

The components fail independently. So the number of components working at the end of 4 years is Binomial $(5, (2/3)^4)$. The probability that at least one component is working is

$$\mathbb{P}(\text{at least one component working}) = 1 - \mathbb{P}(\text{no components working})$$

$$= 1 - \left(1 - (2/3)^4\right)^5$$

$$\simeq 0.6672$$

10. (a) $\mathbb{E}(Y) = \mathbb{E}\left[(1-X_1)(X_2 + X_3 - X_2X_3)\right]$

$= \mathbb{E}\left[(1-X_1)\right]\mathbb{E}\left[X_2 + X_3 - X_2X_3\right]$ by independence

$= (1 - \mathbb{E}(X_1))\left(\mathbb{E}X_2 + \mathbb{E}X_3 - (\mathbb{E}X_2)(\mathbb{E}X_3)\right)$ by linearity of expectation + independence of X_2 + X_3.

$= (1-p)(p + p - p^2) = p(1-p)(2-p)$ As $X_i \sim \text{Bernoulli}(p)$.

$\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$.

Note that Y only takes the values 0 or 1. So $\mathbb{E}(Y^2) = \mathbb{E}(Y)$

Therefore,

$$\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \mathbb{E}(Y) - (\mathbb{E}(Y))^2$$

$$= p(1-p)(2-p) - p^2(1-p)^2(2-p)^2$$

(b)

Note $\mathbb{P}(Y = 1 \mid X_3 = 0) = \mathbb{P}((1-X_1)X_2 = 1 \mid X_3 = 0)$

$= \mathbb{P}((1-X_1)X_2 = 1)$ (by independence)

$= \mathbb{P}(1 - X_1 = 1, X_2 = 1)$

$= \mathbb{P}(X_1 = 0)\,\mathbb{P}(X_2 = 1)$ (by independence)

$= (1-p)p$

so $\mathbb{E}[Y \mid X_3 = 0] = 1 \times \mathbb{P}(Y=1 \mid X_3 = 0) + 0 \times \mathbb{P}(Y=0 \mid X_3 = 0)$
$= p(1-p)$

Also $\mathbb{P}(Y=1 \mid X_3 = 1) = \mathbb{P}(1 - X_1 = 1 \mid X_3 = 1)$
$= \mathbb{P}(1 - X_1 = 1)$ (by independence)
$= \mathbb{P}(X_1 = 0) = 1-p$

So $\mathbb{E}[Y \mid X_3 = 1] = 1-p$

(c) We first need $\mathbb{P}(X_3 = 1 \mid Y = 1) = \dfrac{\mathbb{P}(X_3 = 1, Y = 1)}{\mathbb{P}(Y=1)}$

$\mathbb{P}(Y=1) = \mathbb{P}((1-X_1)(X_2 + X_3 - X_2 X_3) = 1)$
$= \mathbb{P}((1-X_1) = 1)\,\mathbb{P}(X_2 + X_3 - X_2 X_3 = 1)$ (independence)
$= (1-p)\,\mathbb{P}(\{X_2 = 1\} \cup \{X_3 = 1\})$
$= (1-p)(p + p - p^2) = p(1-p)(2-p)$

$\mathbb{P}(X_3 = 1, Y = 1) = \mathbb{P}(Y = 1 \mid X_3 = 1)\,\mathbb{P}(X_3 = 1)$
$= (1-p)p$ from part (b)

$\mathbb{P}(X_3 = 1 \mid Y = 1) = \dfrac{p(1-p)}{p(1-p)(2-p)} = \dfrac{1}{2-p}$.

so $\mathbb{E}[X_3 \mid Y = 1] = (2-p)^{-1}$.

11 (a) $M_Y(s) = \mathbb{E}\left[e^{s(X+2)}\right] = \mathbb{E}\left[e^{2s} \cdot e^{sX}\right] = e^{2s}\,\mathbb{E}\left[e^{sX}\right] = e^{2s} \cdot \exp(e^s - 1)$
$= \exp(e^s + 2s - 1)$

(b) $M_Y(s) = \mathbb{E}\left[e^{s \cdot 3X}\right] = M_X(3s)$

$= \exp(e^{3s} - 1)$.

12 (a) $\cancel{G_X(s) = \exp(-(2-s-s^2))}$

$\cancel{G_X'(s) = (1+2s)\exp(-(2-s-s^2))}$

$\cancel{G_X''(s) = 2\exp(-(2-s-s^2)) + (1}$

$\cancel{\mathbb{P}(X=0) = G_X(0) = e^{-2}}$

$\cancel{\mathbb{P}(X=1) = \frac{G_X'(0)}{1!} = e^{-2}}$

$\cancel{\mathbb{P}(X=2) = \frac{G_X''(0)}{2!} = \frac{2e^{-2}+e^{-2}}{2} =}$

(b) $\mathbb{E}X = G_X'(1) = 3.$

$\cancel{\mathrm{Var}(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2}$

$\cancel{= (2+9) + 3 - 3^2 = 5.}$

$M_X(s) = \frac{\log(1-pe^s)}{\log(1-p)}$

$M_X'(s) = \frac{1}{\log(1-p)} \times \frac{1}{1-pe^s} \times (-pe^s)$

$= \frac{-pe^s}{(1-pe^s)\log(1-p)}$

$M_X'(0) = -p/((1-p)\log(1-p))$

$M_X''(s) = \frac{-1}{\log(1-p)} \times \frac{pe^s}{(1-pe^s)^2}$

$M_X''(0) = -p/((1-p)^2\log(1-p))$

$\mathrm{Var}(X) = -\frac{(p^2 + p\log(1-p))}{(1-p)^2(\log(1-p))^2}$

13. (a) There are 6 permutations of $\{1,2,3\}$, each equally likely.

Joint pmf

		X_{12}: 0	X_{12}: 1	
X_{23}	0	1/6	1/3	1/2
	1	1/3	1/6	1/2
		1/2	1/2	

We could enumerate all these —

(1, 2, 3) (3, 1, 2)
(1, 3, 2) (3, 2, 1)
(2, 1, 3)
(2, 3, 1)

~~For example $\mathbb{P}(X_{12}=1, X_{23}=1)$~~

The event $\{X_{12}=1\} \cap \{X_{23}=1\}$ corresponds to the permutation $(3,2,1)$ so $\mathbb{P}(X_{12}=1, X_{23}=1) = 1/6$.

(b) X_{12} and X_{23} are not independent as $\mathbb{P}(X_{12}=1)\mathbb{P}(X_{23}=1) = 1/4 \neq 1/6$.

(c) $\mathrm{Cov}(X_{12}, X_{23}) = \mathbb{E}(X_{12}X_{23}) - \mathbb{E}(X_{12})\mathbb{E}(X_{23})$

$= \mathbb{P}(X_{12}=1, X_{23}=1) - \mathbb{P}(X_{12}=1)\mathbb{P}(X_{23}=1)$

$= 1/6 - \frac{1}{2}\times\frac{1}{2} = -1/12.$

14 (a) $G_X(s) = \dfrac{p}{\exp(\lambda(1-s))-1+p}$

$M_X(s) = \dfrac{\exp(e^s-1)}{(2-e^s)}$

$G_X'(s) = \dfrac{p\lambda\exp(\lambda(1-s))}{(\exp(\lambda(1-s))-1+p)^2}$

$M_X'(s) = \dfrac{e^s\exp(e^s-1)(2-s)+e^s\exp(e^s-1)}{(2-e^s)^2}$

$\mathbb{E}X = M_X'(0) = 2.$

$G_X''(s) = \dfrac{-p\lambda^2\exp(\lambda(1-s))(\exp(\lambda(1-s))-1+p)^2 + 2p(\lambda\exp(\lambda(1-s)))^2}{(\exp(\lambda(1-s))-1+p)^4}$

$\mathbb{P}(X=0) = G_X(0) = \dfrac{p}{(\exp(\lambda)+1-p)}$

$\mathbb{P}(X=1) = \dfrac{G_X'(0)}{1!} = \dfrac{p\lambda\exp(\lambda)}{(\exp(\lambda)-1+p)^2}$

$\mathbb{P}(X=2) = \dfrac{G_X''(0)}{2!} = \dfrac{-p\lambda^2\exp(\lambda)(\exp(\lambda)-1+p)+2p\lambda^2\exp(2\lambda)}{(\exp(\lambda)-1+p)^3}$

(b) $\mathbb{E}X = G_X'(1) = \dfrac{p\lambda}{p^2} = \dfrac{\lambda}{p}$

$\mathrm{Var}(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

$= \dfrac{-p^3\lambda^2 + 2p^2\lambda^2}{p^4} + \dfrac{\lambda}{p} - \left(\dfrac{\lambda}{p}\right)^2$

$= \dfrac{\lambda^2}{p^2} + \dfrac{\lambda}{p} - \dfrac{\lambda^2}{p} = \dfrac{\lambda^2(1-p)}{p^2} + \dfrac{\lambda}{p}$

15 (a) $\mathrm{Cov}(Y_i, Y_j) = \mathbb{E}(Y_i Y_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j)$ $\quad i \neq j$

By construction, $Y_i Y_j = 0$ with probability 1 for $i \neq j$ (since we cannot have both $X = i$ and $X = j$ for $i \neq j$). So

$$\mathrm{Cov}(Y_i, Y_j) = -\mathbb{E}(Y_i)\mathbb{E}(Y_j)$$

Now $\mathbb{E}(Y_i) = \mathbb{P}(X_i = i) = 1/n$. Therefore

$$\mathrm{cov}(Y_i, Y_j) = -1/n^2 \quad \text{for } i \neq j$$

(b) $\mathbb{P}(Y_i = 1 \mid Y_j = 1) = 0$
$\mathbb{P}(Y_i = 0 \mid Y_j = 1) = 1$

$\mathbb{P}(Y_i = 1 \mid Y_j = 0) = \dfrac{\mathbb{P}(Y_i = 1, Y_j = 0)}{\mathbb{P}(Y_j = 0)}$. As $\{Y_i = 1\} \subseteq \{Y_j = 0\}$,

$$= \frac{\mathbb{P}(Y_i = 1)}{\mathbb{P}(Y_j = 0)} = \frac{1/n}{\frac{n-1}{n}} = \frac{1}{n-1}$$

$$\mathbb{P}(Y_i = 0 \mid Y_j = 0) = \frac{n-2}{n-1}$$

16. $X \sim \text{Binomial}(n, p)$, then $\mathbb{E}X = np$ and $\mathrm{Var}(X) = np(1-p)$.

So $np = 8$ and $np(1-p) = 6$
$\Rightarrow 8(1-p) = 6 \Rightarrow 1-p = 6/8 \Rightarrow p = 1/4$.

$np = 8 \Rightarrow n \times \frac{1}{4} = 8 \Rightarrow n = 32$.

17. Let $p = \mathbb{P}(\text{component working after } 8\text{yrs})$
$= \mathbb{P}(\text{component fails in yrs } 9 \text{ or } 10)$
$= 2/10 = 1/5$

$\mathbb{P}(\text{system fails after } 8 \text{ yrs}) = \mathbb{P}(\text{at least 3 out of 4 components working})$
$= p^4 + 4p^3(1-p)$
$\simeq 0.0272$

18. Let X_i be the number of cups of coffee consumed by academic i during morning tea. Let Y be the total number of cups of coffee consumed during morning tea so $Y = \sum_{i=1}^{200} X_i$.

$$\mathbb{E}Y = \mathbb{E}\left(\sum_{i=1}^{200} X_i\right) = \sum_{i=1}^{200} \mathbb{E}(X_i) = 200 \times \mathbb{E}(X_1)$$

$$Var(Y) = Var\left(\sum_{i=1}^{200} X_i\right) = \sum_{i=1}^{200} Var(X_i) = 200\, Var(X_1).$$

as the X_i are independent by assumption

We need to compute $\mathbb{E}(X_1)$ and $Var(X_1)$

$\mathbb{E}(X_1) = 0 \times 0.1 + 1 \times 0.6 + 2 \times 0.3 = 1.2 \Rightarrow \mathbb{E}(Y) = 240$

$Var(X_1) = \mathbb{E}(X_1^2) - (\mathbb{E}X_1)^2$

$\mathbb{E}(X_1^2) = 0^2 \times 0.1 + 1^2 \times 0.6 + 2^2 \times 0.3 = 1.8$

$Var(X_1) = 1.8 - (1.2)^2 = 0.36 \Rightarrow Var(Y) = 72.$

19 (a) $g(c) = \mathbb{E}[(X - cY)^2]$

$$= \mathbb{E}[X^2] - 2c\,\mathbb{E}[XY] + c^2\mathbb{E}[Y^2]$$

To minimise g we differentiate with respect to c and set the derivative to zero

$$g'(c) = -2\mathbb{E}[XY] + 2c\,\mathbb{E}[Y^2]$$

$$g'(c^*) = 0 \Rightarrow c^* = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}$$

g is minimised at $c = c^*$ since $g''(c) > 0$.

(b) If X and Y are Bernoulli random variables, then

$$\mathbb{E}[XY] = \sum_{x=0}^{1}\sum_{y=0}^{1} xy\,\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=1, Y=1)$$

$$\mathbb{E}[Y^2] = \sum_{y=0}^{1} y^2\,\mathbb{P}(Y=y) = \mathbb{P}(Y=1)$$

$$c^* = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} = \frac{\mathbb{P}(X=1, Y=1)}{\mathbb{P}(Y=1)} = \mathbb{P}(X=1 \mid Y=1)$$

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The joint pmf of (X, Y) is

		Y: 0	1	2	3	...	
X	0	$e^{-\lambda}$	0	0	0	...	$e^{-\lambda}$
	1	0	$\lambda e^{-\lambda}$	$\lambda^2 e^{-\lambda}/2$	$\lambda^3 e^{-\lambda}/6$	...	$1-e^{-\lambda}$
		$e^{-\lambda}$	$\lambda e^{-\lambda}$	$\frac{\lambda^2 e^{-\lambda}}{2}$	$\frac{\lambda^3 e^{-\lambda}}{6}$	...	

marginal pmf of X

marginal pmf of Y