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1. [10 marks] Suppose that X_1, X_2 and X_3 are independent random variables such that X_k has a Poisson(k) distribution for $k = 1, 2, 3$. Define $Y = 4(X_1 + X_2 + X_3)$.

(a) Compute $\mathbb{E}[Y]$. [2 marks]

$$\mathbb{E}[Y] = \mathbb{E}[4(X_1+X_2+X_3)] = 4\mathbb{E}(X_1) + 4\mathbb{E}(X_2) + 4\mathbb{E}(X_3)$$
$$= 4\times 1 + 4\times 2 + 4\times 3 = 24$$

(b) Compute Var(Y). [2 marks]

$$Var(Y) = Var(4(X_1+X_2+X_3)) = 4^2\, Var(X_1+X_2+X_3)$$
$$= 4^2 \times (Var(X_1) + Var(X_2) + Var(X_3))$$
$$= 4^2 \times (1 + 2 + 3) = 96$$

(c) Compute $\mathbb{E}[Y|X_1]$. [2 marks]

$$\mathbb{E}[Y|X_1] = \mathbb{E}[4(X_1+X_2+X_3)|X_1]$$
$$= 4\mathbb{E}(X_1|X_1) + 4\mathbb{E}(X_2|X_1) + 4\mathbb{E}(X_3|X_1)$$
$$= 4X_1 + 4\mathbb{E}(X_2) + 4\mathbb{E}(X_3) = 4X_1 + 20$$

(d) Compute Cov(Y, X_1). [2 marks]

$$cov(Y, X_1) = cov(4(X_1+X_2+X_3), X_1)$$
$$= 4\,cov(X_1,X_1) + 4\,cov(X_2,X_1) + 4\,cov(X_3,X_1)$$
$$= 4\,cov(X_1,X_1) = 4\,var(X_1) = 4$$

(e) Compute the moment generating function of Y. Recall the moment generating function of the Poisson(λ) distribution is $M(t) = \exp(\lambda(e^t - 1))$, $t \in \mathbb{R}$. [2 marks]

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t\cdot 4(X_1+X_2+X_3)}]$$
$$= \mathbb{E}[e^{4tX_1} \times e^{4tX_2} \times e^{4tX_3}]$$
$$= \mathbb{E}[e^{4tX_1}] \times \mathbb{E}[e^{4tX_2}] \times \mathbb{E}[e^{4tX_3}]$$

$$= \exp((e^{4t}-1)) \cdot \exp(2(e^{4t}-1))\, \exp(3(e^{4t}-1))$$
$$= \exp(6(e^{4t}-1)).$$

3. A random point (X, Y) has a joint probability distribution in which X has an $\mathrm{Exp}(1)$ distribution, and $(Y \mid X = x) \sim \mathrm{U}(0, x)$; that is, the conditional distribution of Y given $X = x$ is uniform on the interval $(0, x)$.

   (a) Formulate an algorithm to simulate a point (X, Y) from this distribution using only $\mathrm{U}(0, 1)$ random variables. [2]

~~Com~~ Write out the joint pdf of (X, Y)

$$f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)$$

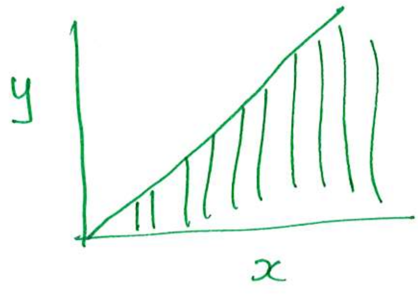
$$= \begin{cases} e^{-x} \cdot \frac{1}{x}, & x > 0,\ 0 < y < x \\ 0, & \text{else} \end{cases}$$

   (b) Sketch the typical positions of many points independently simulated from this algorithm. [1]

marginal pdf of Y

$$f_Y(y) = \int f_{X,Y}(x,y)\, dx$$

$$= \int_y^\infty e^{-x} \frac{1}{x}\, dx$$



(c) Using the formula $\mathbb{E}Y = \mathbb{E}[\mathbb{E}[Y \mid X]]$, find the expectation of Y. [2]

$$\mathbb{E}[Y|X=x] = \int y \, f_{Y|X}(y|x)\, dy = \int_0^x y \cdot \frac{1}{x}\, dy$$

$$= \frac{1}{x}\left[\tfrac{1}{2}y^2\right]_0^x = \frac{x^2}{2x} = \frac{x}{2}$$

$$\mathbb{E}[Y] = \mathbb{E}\left[\mathbb{E}[Y|X]\right] = \mathbb{E}\left[X/2\right] = \frac{1}{2}\mathbb{E}[X] = 1/2$$

(As Exp(1) has mean 1).

(d) In a similar way as in (c), derive the variance of Y. [3]

$$Var(Y) = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2$$

$$\mathbb{E}[Y^2] = \mathbb{E}\left[\mathbb{E}[Y^2|X]\right]$$

$$\mathbb{E}[Y^2|X=x] = \int y^2 f_{Y|X}(y|x)\,dy$$

$$= \int_0^x y^2 \cdot \frac{1}{x}\,dy = \frac{1}{x}\left[\tfrac{1}{3}y^3\right]_0^x$$

$$= \frac{x^2}{3}$$

$$\mathbb{E}[Y^2] = \mathbb{E}\left[X^2/3\right] = \frac{1}{3}\mathbb{E}[X^2]$$

$$= 2/3$$

$$Var(Y) = \frac{2}{3} - (1/2)^2 = \frac{5}{12}$$

$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$

$X \sim Exp(1)$, then

$1 = \mathbb{E}(X^2) - 1^2$

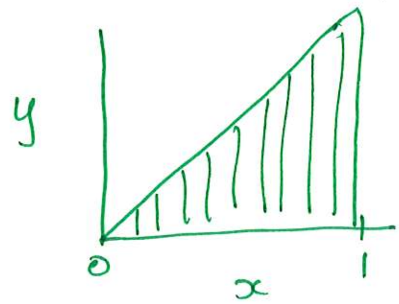
$\Rightarrow \mathbb{E}(X^2) = 2$

4. Random variables X and Y have a joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 3x & \text{if } 0 < x < 1,\ 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the marginal pdf of X. [2]

(Write your answer in the box provided and show any working below.)



$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$= \int_0^x 3x \, dy$$

$$= \begin{cases} 3x\,[y]_0^x = 3x^2, & x \in (0,1) \\ 0, & \text{else} \end{cases}$$

(b) Determine the conditional pdf of Y given $X = x$. [2]

(Write your answer in the box provided and show any working below.)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{3x}{3x^2} \qquad x \in (0,1), \quad 0 < y < x$$

$$= \begin{cases} \frac{1}{x}, & 0 < y < x \\ 0, & \text{else} \end{cases} \qquad x \in (0,1).$$

(c) Identify by name the conditional distribution of Y given $X = x$. (Be precise.) [1]

(Write your answer in the box provided and show any working below.)

$\boxed{\text{Uniform}(0,x)}$

3. Let X be a random variable with a *Weibull* distribution with parameters $\alpha > 0$ and $\lambda > 0$. This means that the cdf F of X is given by

$$F(x) = \begin{cases} 1 - \mathrm{e}^{-(\lambda x)^\alpha} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Formulate an algorithm to simulate from this distribution using the inverse-transform method, assuming that you have a method for drawing from the $\mathsf{U}(0,1)$ distribution. [4]

using quantile function

(b) Show that X can be written as $X = Z^{1/\alpha}/\lambda$, where $Z \sim \mathsf{Exp}(1)$. [3]

cdf $\mathbb{P}(Z^{1/\alpha}/\lambda \leq x)$

$= \mathbb{P}(Z^{1/\alpha} \leq \lambda x)$

$= \mathbb{P}(Z \leq (\lambda x)^\alpha) = F_Z((\lambda x)^\alpha)$

$= 1 - e^{-(\lambda x)^\alpha}$

$Z \sim Exp(1)$

$F_Z(z) = \int_0^z e^{-u}\, du$, $z > 0$

$= -e^{-u}\Big|_0^z = 1 - e^{-z}$

As the two cdfs are equal, we see that $X = Z^{1/\alpha}/\lambda$ where $Z \sim exp(1)$.