STAT7203: Applied Probability and Statistics
Quiz 6

1. A continuous random variable X has probability density function

$$f_X(x) = \begin{cases} cx^2, & x \in (0,1) \\ 0, & \text{else,} \end{cases}$$

for some c.

   (a) Determine the value of c.
   Solution: $c = 3$.

   (b) What is the cumulative distribution function of X ?
   Solution:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^3, & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

   (c) What is the quantile function of X?
   Solution:

$$q_X(x) = x^{1/3}, \quad x \in [0,1].$$

   (d) How would you simulate from this distribution in MATLAB? *Solution*:

```
u = rand(1);
X = u.^(1/3);

n = 10000;
u = rand(n);
X = u.^(1/3);
hist(X,100),ylabel('X Frequencies')
```

2. Consider the water cooling system schematically depicted in Figure 1. The system has four unreliable components: two identical pumps (P1 and P2) and two valves (V1 and V2). The system works if, in the diagram, there is a path from left to right traversing only working components.

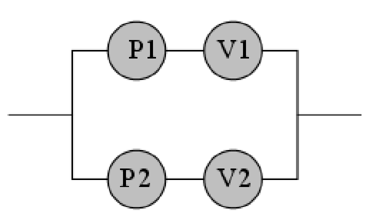

Figure 1: The system works if there is a path from left to right with working components.

We assume that the pumps and valves fail independently of each other and that the lifetime of each pump is exponentially distributed with mean 1 year, and the lifetime of each valve is exponentially distributed with mean 2 years. Calculate the probability that the system works after 1 year.

Solution: Let P_i denote the event that the i-th pump is still working after 1 year, and let V_i denote the event that the i-th valve is still working after 1 year, $i = 1, 2$. Denote by X_i the lifetime of pump i, and by Y_i the lifetime of valve i, $i = 1, 2$. Then $P_i = \{X_i > 1\}$ and $V_i = \{Y_i > 1\}$, $i = 1, 2$. Note that $\mathbb{P}(P_i) = \mathrm{e}^{-1}$ and $\mathbb{P}(V_i) = \mathrm{e}^{-1/2}$, $i = 1, 2$.

The event that the system works is $S = (P_1 \cap V_1) \cup (P_2 \cap V_2)$. Since all random variables are independent, the events are still independent, and we still may write

$$\mathbb{P}(S) = \mathbb{P}(P_1)\mathbb{P}(V_1) + \mathbb{P}(P_2)\mathbb{P}(V_2) - \mathbb{P}(P_1)\mathbb{P}(V_1)\mathbb{P}(P_2)\mathbb{P}(V_2)\,,$$

in this case yielding

$$\mathbb{P}(S) = 2\mathrm{e}^{-3/2} - \mathrm{e}^{-3} \approx 0.4463 - 0.0498 = 0.3965\,.$$

3. A manufacture produces resistors with mean resistance of 2 Ohms and standard deviation of 0.075 Ohms. Suppose that the resistance of each resistor has a Normal distribution.

   (a) What proportion of resistors have resistance in the interval $(1.9, 2.1)$ Ohms?
   Solution: Let X denote the resistance of a resistor. Then, using the standard normal table,

   $$\begin{aligned}\mathbb{P}(1.9 \leqslant X \leqslant 2.1) &= \mathbb{P}\left(\frac{1.9-2}{0.075} \leqslant Z \leqslant \frac{2.1-2}{0.075}\right) \\ &= \mathbb{P}(-1.333 \leqslant Z \leqslant 1.333) = 0.8175776,\end{aligned}$$

   where $Z \sim \mathsf{N}(0, 1)$. This can also be done using the `normcdf` function in MATLAB.

   (b) What is the probability that in a batch of 10 no more that 2 resistors have resistance outside the range $(1.9, 2.1)$ Ohms?
   Solution:

   $$\begin{aligned}&\mathbb{P}(\text{no more than 2 resistors outside range}) \\ &= \binom{10}{10}(0.8175776)^{10}(0.1824224)^0 + \binom{10}{9}(0.8175776)^9(0.1824224)^1 \\ &\quad + \binom{10}{8}(0.8175776)^8(0.1824224)^2 \\ &= 0.7301339\end{aligned}$$

   This can also be done using the `binocdf` function in MATLAB.