SOLUTIONS

Practical 1

Topic 1: Probability Concepts and Discrete Distributions

Based on Chapters 4-5 of Berenson et al. (2015), *Basic Business Statistics: Concepts and Applications*, 4th Edition, Frenchs forest, N.S.W.: Pearson Australia

Executive summary:

- Mutually exclusive events (also called disjoint events): Two events that cannot occur simultaneously.
- Independent events: Occurs when one event will not have any effect on what occurs next. Because either (i) the events are unrelated, (ii) repeat an event with an item whose numbers will not change and (iii) repeat the same activity but replace the item that was removed.
- Probability: A numerical measure of the likelihood that an event will occur.
- Event: A set of outcomes to which a probability is assigned.
- An event is impossible if it has a 0 probability of occurring and an event is certain if it has a probability of 1.
- Dependent event: What happens next depends on what happened before.
- Collectively exhaustive event: One outcome in sample space must occur. All possible events are likely.
- Simple event: An outcome from sample space with one characteristic (e.g. red card from a deck of cards)
- Joint event: Events occurring simultaneously.(e.g. an ace that is also red from a deck of cards)
- Compliment event: All of the events that are not included in the event in question (e.g. if event A is the question everything not included in event A).
- A priori classical probability: The probability of an event based on prior knowledge of the process.
- Empirical classical probability: The probability of an event based on observed data.
- Subjective probability: A probability derived from an individual's personal judgement whether a specific outcome is likely to occur.

- Addition rule: Used to determine whether event both A and event B occurs or A or B occurs.
- Multiplication rule: The probability that two events both A and B occur.
- Conditional probability: The probability of one event given that another event has occurred.
- Bayes' Theorem: It is used to revise previously calculated probabilities (prior probabilities) when there is new information.
- Probability Distribution: An instrument that assigns a probability to every possible outcome of a variable.
- Discrete probability distribution: A probability distribution for a discrete random variable (e.g. binomial)
- Continuous Probability Distribution: A probability distribution for a continuous random variable (e.g. normal).
- Counting rules: Rules that facilitate the calculations of probabilities in certain situations: (i) the number of possible outcomes, (ii) the number of ways an item can be arranged, (iii) the number of ways of obtaining an ordered subset of x elements from n elements and (iv) the combination rule.
- Binomial Distribution: A discrete probability distribution that has (i) identical trials, (ii) only two possible outcomes as denoted by a success or failure, (iii) each trial is independent of the previous trials, (iv) the probability of the given success and its compliment remain constant throughout a given experiment.

A. MCQs

1. A numerical value representing the chance, likelihood, or possibility that a particular event will occur
    a. Event
    b. Probability
    c. Simple event
    d. Complement event

2. An event that has no chance of occurring
    a. Probability
    b. Certain event
    c. Impossible event
    d. None

3. Imagine rolling a six sided die, the sample space is
    a. {1, 2, 3, 4}
    b. {1, 2, 3, 4, 6}
    c. {1, 2, 3, 4, 5, 6}
    d. {1, 2}

4. A subset of sample space to which a probability is assigned
    a. Complement
    b. Experiment
    c. Event
    d. None

5. When you toss a coin, the two possible outcomes are head and tail. Each of these represents
    a. A joint event
    b. A simple event
    c. Sample space
    d. a and c

6. Getting two heads when you toss a coin twice is an example of
    a. A simple event
    b. Sample space
    c. Joint event
    d. b and c

7. Given the probability of an event, the probability of its complement can be found by subtracting the given probability from 1
    a. False
    b. Cannot be determined from the information given
    c. True

8. If A and B are mutually exclusive, then the P(A and B) is equal to
    a. 0
    b. 1
    c. 0.5
    d. 0.75

9. Consider the experiment of flipping a coin. Which statement is true?
    a. Events can be mutually exclusive but not collectively exhaustive
    b. Events can be collectively exhaustive but not mutually exclusive
    c. Cannot be determined from the information given
    d. Events can be mutually exclusive and collectively exhaustive

10. Independent events can happen when
    a. The two events are unrelated
    b. You repeat an event with an item whose numbers will not change
    c. You repeat the same activity, but you replace the item that was removed
    d. a, b and c

11. Assumption of a binomial distribution
    a. The outcome of any trial is independent of the outcome of any other trial
    b. Each trial has only two possible outcomes
    c. The experiment involves *n* identical trials
    d. b and c
    e. **a, b and c**

B. Questions requiring numerical/written answers

12. Assume A = aces; B = black cards; C = diamonds; D = hearts. For each of the following, state whether the events created are mutually exclusive and collectively exhaustive
    a. Events A, B, C and D (**not mutually exclusive, collectively exhaustive**)
    b. Events B, C and D (**mutually exclusive, collectively exhaustive**)

13.

Planned to purchase	**Actually purchased**		**Total**
	Yes (B1)	No (B2)	
Yes (A1)	200	50	250
No (A2)	100	650	750
Total	300	700	1000

Use the above information to calculate the probability of each of the following and decide whether it is simple (marginal) or joint or conditional probability.

a.	A1	250/1000 = 0.25	simple probability
b.	A2	750/1000 = 0.75	simple probability
c.	B1	300/1000 = 0.30	simple probability
d.	B2	700/1000 = 0.70	simple probability
e.	A1 and B1	200/1000 = 0.20	joint probability
f.	A1 and B2	50/1000 = 0.05	joint probability
g.	A2 and B1	100/1000 = 0.10	joint probability
h.	A2 and B2	650/1000 = 0.65	joint probability
i.	B1 | A1	(200/1000) / (250/1000) = 0.80	conditional probability

14. For the above example find
    a. P(A1 or B1) 250/1000 + 300/1000 – 200/1000 = 0.35
    b. P(A1 or B2) 250/1000 + 700/1000 – 50/1000 = 0.90

15. A sample of 500 purchasing managers was selected across Australia to determine information concerning buying behaviour. Among the questions asked was, 'Do you enjoy your role in the organisation?' Of 240 males, 136 answered YES. Of 260 females, 224 answered YES.

    a. Construct a contingency table

Purchasing managers response	**Gender**		**Total**
	Male (B1)	Female (B2)	
Yes (A1)	136	224	360
No (A2)	104	36	140
Total	240	260	500

    b. What is the probability that a respondent chosen at random enjoys his/her role in the organisation?

P(A1) = 360/500 = 0.72

    c. What is the probability that a respondent chosen at random is a female and enjoys her role in the organisation?

P(A1 and B2) = 224/500 = 0.448

or

P(A1 and B2) = P(A1) P(B2 | A1) = (360/500) (224/360) = 0.448

or

P(A1 and B2) = P(B2) P(A1 | B2) = (260/500) (224/260) = 0.448

    d. What is the probability that a respondent chosen at random is a female or enjoys her/his role in the organisation?

P(B2 or A1) = P(B2) + P(A1) – P(B2 and A1)

= 260/500 + 360/500 – 224/500 = 0.792

    e. What is the probability that a respondent chosen at random is a male or a female?

P(B1 or B2) = P(B1) + P(B2) = 240/500 + 260/500 = 1

16. A card is drawn randomly from a deck of ordinary playing cards. You win \$10 if the card is a spade or an ace. What is the probability that you will win the game?

P(spade or ace) = P(spade) + P(ace) – P(spade and ace)

= 13/52 + 4/52 – 1/52 = 16/52 = 0.3077

17. What is the probability that two tails occurs when two coins are tossed?

P(tail and tail) = P(tail) P(tail) = 1/2 x 1/2 = ¼ = 0.25

or

n (trial) = 2; X (number of successes) = 2; p = 0.5; and q = 0.5

$$P(X) = \frac{n!}{X!(n-X)!} p^X q^{n-X}$$

$$P(X = 2) = \frac{2!}{2!(2-2)!} 0.5^2 \; 0.5^0 = 0.25$$

18. The probability that a regularly scheduled flight departs on time is 0.83, the probability that it arrives on time is 0.92, and the probability that it departs and arrives on time is 0.78. Find the probability that a plane (a) arrives on time given that it departed on time, and (b) departed on time given that it has arrived on time?

Assume A = departs on time, and B = arrives on time

a. The question is to find P(B | A)

P(B | A) = P(A and B) / P(A)

= (0.78) / (0.83) = 0.9398

b. The question is to find P(A | B)

P(A | B) = P(A and B) / P(B)

= (0.78) / (0.92) = 0.8478

19. At a school, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking Technology?

Assume A = Technology and B = Spanish

The question is to find P(B | A)

P(B | A) = P(A and B) / P(A)

= (0.087) / (0.68) = 0.1279

20. Three coins are tossed. Let X be the number of heads obtained. Construct a probability distribution for X and find its mean and standard deviation.

Number of possible outcomes = $2^3 = 8$

HHH, HHT, HTH, THH, TTH, THT, HTT and TTT

The probability distribution

Number of heads	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

n = 3, p = 0.5 and q = 0.5

P(X = 0) = P(TTT) = $\frac{3!}{0!(3-0)!} 0.5^0\ 0.5^{3-0} = 1/8 = 0.125$

P(X = 1) = P(TTH) + P(THT) + P(HTT) = $\frac{3!}{1!(3-1)!} 0.5^1\ 0.5^{3-1} = 3/8 = 0.375$

P(X = 2) = P(HHT) + P(HTH) + P(THH) = $\frac{3!}{2!(3-2)!} 0.5^2\ 0.5^{3-2} = 3/8 = 0.375$

P(X = 3) = $\frac{3!}{3!(3-3)!} 0.5^3\ 0.5^{3-3} = 1/8 = 0.125$

Mean = 0 x (1/8) + 1 x (3/8) + 2 x (3/8) + 3 x (1/8) = 1.5

Standard deviation = Square Root [$(0 - 1.5)^2$ x (1/8) + $(1 - 1.5)^2$ x (3/8) + $(2 - 1.5)^2$ x (3/8) + $(3 - 1.5)^2$ x (1/8)] = 0.866

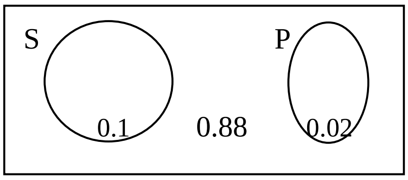
21. With each purchase of a large pizza at Tony's Pizza, the customer receives a coupon that can be scratched to see if one prize will be awarded. The chances of winning a free soft drink are 1 in 10, and the chances of winning a free large pizza are 1 in 50. You plan to eat lunch tomorrow at Tony's.

   a. Draw a Venn Diagram and construct a joint probability table.

      Let S = win free soft drink; P(S) = 0.1
      Let P = win free pizza; P(P) = 0.02

Assume each coupon only has one scratch area so that the only possibilities are S, P or not win anything. The joint probability table and Venn Diagram are below:

	P	P'	
S	0	0.1	0.1
S'	0.02	0.88	0.9
	0.02	0.98	1.0



b. What is the probability you will win a prize?

P(S or P) = P(S) + P(P) – P(S and P) => addition rule

= 0.1 + 0.02 – 0 = 0.12

c. What is the probability that you will not win a prize?

P(not win) = 1 – P(S or P) = 1 – 0.12 = 0.88 => complement

= P(S' and P')

d. What is the probability that you will not win a prize on three consecutive visits to Tony's?

On three visits: P(not win ∩ not win ∩ not win) => multiplication rule

$= (0.88)^3 = 0.6815$

22. From 4 Labours and 3 Liberals find the number of committees of 3 can be formed with 2 Labours and 1 Liberal.

Here, order doesn't matter

The number of ways of selecting 2 Labours from 4 is

$${}_nC_X = \frac{n!}{X!(n-X)!} = {}_4C_2 = \frac{4!}{2!(4-2)!} = 6$$

The number of ways of selecting 1 Liberal from 3 is

$${}_nC_X = \frac{n!}{X!(n-X)!} = {}_3C_1 = \frac{3!}{1!(3-1)!} = 3$$

Number of committees that can be formed with 2 Labours and 1 Liberal is 6 x 3 = 18

23. In a race with eight swimmers, how many ways can the swimmers finish first, second and third? (In this question order matters)

$$_{n}P_{X} = \frac{n!}{(n-X)!} = \frac{8!}{(8-3)!} = 336$$

24. At a local elementary school, a principal is making random class assignments for her 8 teachers. Each teacher must be assigned to exactly one job. In how many ways can the assignment be made?

n! = 8! = 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 40320

25. The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that exactly 5 survives?

n = 15

p = 0.4

q = (1 – p) = 0.6

Need the probability that X = 5

$$P(X) = \frac{n!}{X!\ (n-X)!} P^{X} (1-P)^{n-X}$$

$$P(X=5) = \frac{15!}{5!(15-5)!} 0.4^{5} (1-0.4)^{15-5} = 0.1859$$

26. For the above question, calculate the mean and variance.

Mean = np = 15 x 0.4 = 6

Variance = npq = 15 x 0.4 x 0.6 = 3.6

27. Use the binomial distribution table to determine P(X < 4) if n = 9 and p = 0.80

(refer to Table E.6 from the textbook)

P(X = 0) = ------

P(X = 1) = 0.0000

$P(X = 2) = 0.0003$

$P(X = 3) = 0.0028$

Thus, $P(X < 4) = 0.0000 + 0.0003 + 0.0028 = 0.0031$

28. The marketing manager of a toy manufacturing company is considering the marketing of a new toy. In the past, 40% of the toys introduced by the company have been successful and 60% have been unsuccessful. Before the toy is marketed market research is conducted and a report, either favourable or unfavourable, is compiled. In the past 80% of the successful toys received a favourable market research report and 30% of the unsuccessful toys received a favourable market research report. The marketing manager wants to know (a) the probability that the toy will be successful if it receives a favourable report (b) the probability the toy will be unsuccessful if it receives a favourable report

Assume:

Event S = successful toys　　　　Event S' = unsuccessful toys

Event F = favourable report　　　　Event F' = unfavourable report

Given:

$P(S) = 0.40$　　　　$P(S') = 0.60$

$P(F \mid S) = 0.80$　　　　$P(F \mid S') = 0.30$

Questions:

(a) $P(S \mid F)$　　　　(b) $P(S' \mid F)$

Solution for (a)

First use the conditional probabilities

$P(F \mid S') = P(S' \text{ and } F)/P(S')$

$0.30 = P(S' \text{ and } F)/0.60$

$P(S' \text{ and } F) = 0.30 \times 0.60 = 0.18$

$P(F \mid S) = P(S \text{ and } F)/P(S)$

0.80 = P(S and F)/0.40

P(S and F) = 0.80 x 0.40 = 0.32

Now let's answer question (a)

P(S | F) = P(S and F)/P(F)

We know that P(F) = P(S and F) + P(S' and F)

Thus, P(S | F) = 0.32/(0.32 + 0.18) = 0.64

Solution for (b)

P(S' | F) = P(S' and F)/P(F)

= 0.18/(0.32 + 0.18) = 0.36

or

P(S' | F) = 1 – P(S | F) = 1 – 0.64 = **0.36**

Conclusion

We know that P(S) and P(S') are prior probabilities and P(F | S) and P(F | S') are conditional probabilities. So, given prior probabilities and some additional information (given by conditional probabilities) we revised the prior probabilities to what is called posterior probabilities. P(S | F) and P(S' | F) are posterior probabilities.

In other words, the prior probability values of 0.40 and 0.60 are respectively revised to 0.64 and 0.36 based on new information. The revised probabilities are called posterior probabilities.

Note: You can answer the questions in a simple way using the Bayes' Theorem

For question (a)

$$P(S|F) = \frac{P(F|S)\,P(S)}{P(F|S)\,P(S) + P(F|S')\,P(S')} = \frac{(0.80 \text{ x } 0.40)}{(0.80 \text{ x } 0.40) + (0.30 \text{ x } 0.60)} = 0.64$$

For question (b)

$$P(S'|F) = \frac{P(F|S')\,P(S')}{P(F|S')\,P(S') + P(F|S)\,P(S)} = \frac{(0.30 \text{ x } 0.60)}{(0.30 \text{ x } 0.60) + (0.80 \text{ x } 0.40)} = 0.36$$

29. If P(B) = 0.3, P(A | B) = 0.5, P(B’) = 0.7, and P(A |B’) = 0.6, find P(B | A)

First use the conditional probability

P(A | B) = P(A and B) / P(B)

0.5 = P(A and B) / 0.3

P(A and B) = 0.5 x 0.3 = 0.15

P(A | B’) = P(A and B’) / P(B’)

0.6 = P(A and B’) / 0.7

P(A and B’) = 0.6 x 0.7 = 0.42

Now,

P (B | A) = P(A and B) / P(A)

= 0.15 / (0.15 + 0.42) = 0.2632

[Note that P(A) = P(A and B) + P(A and B’)]

Or you can use the Bayes’ Theorem

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B')\,P(B')} = \frac{(0.5 \text{ x } 0.3)}{(0.5 \text{ x } 0.3) + (0.6 \text{ x } 0.7)} = 0.2632$$

30. Based on the following contingency table determine P(B | D) using **Bayes' Theorem**

	Event A	Event B	Total
Event C	9	6	15
Event D	4	21	25
Event E	7	3	10
Total	20	30	50

$$P(B|D) = \frac{P(D|B)\,P(B)}{P(D|B)\,P(B) + P(D|A)\,P(A)}$$

$$= \frac{0.7 \text{ x } 0.6}{(0.7 \text{ x } 0.6) + (0.2 \text{ x } 0.4)} = 0.84$$

Note:

P(D | B) = P (D and B) / P(B) = (21/50) / (30/50) = 0.7

P(D | A) = P(D and A) / P(A) = (4/50) / (20/50) = 0.2