SOCIAL MEDIA ANALYTICS INFS7450
Tutorial 8

Xia Xin, Xiangguo Sun

School of ITEE

The University of Queensland

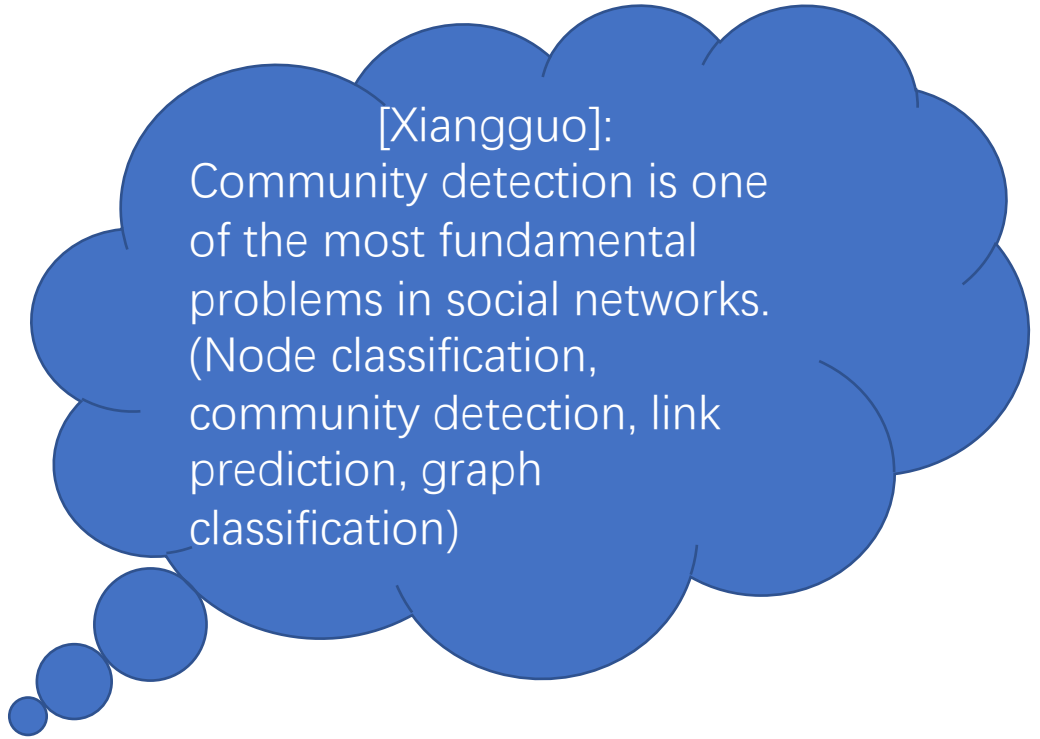
Social Community

[real-world] community

A group of individuals with common *economic*, *social*, or *political* interests or characteristics, often living in *relative proximity*.

What is Community Analysis?

- **Community detection**
  - Discovering implicit communities
- **Community evaluation**
  - Evaluating Detected Communities



What is community detection?

- The process of finding clusters of nodes (“*communities*”)
  - With **Strong** internal connections and
  - **Weak** connections between different communities

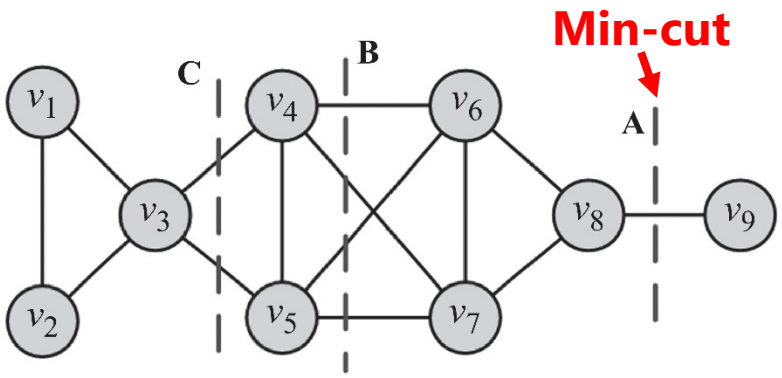
- Ideal decomposition of a large graph
  - Completely disjoint communities
  - There are no interactions between different communities.

- In practice,
  - find community partitions that are maximally decoupled.

Balanced Communities

[Xiangguo]:
If you are familiar with our tutorial week 2, you will also be familiar with this concept.

- Community detection can be viewed as *graph clustering*
- **Graph clustering:** we cut the graph into several partitions and assume these partitions represent communities
- **Cut**: partitioning (*cut*) of the graph into two (or more) sets (*cutsets*)
  - **The size of the cut** is the number of edges that are being cut
- **Minimum cut (min-cut) problem**: find a graph partition such that the number of edges between the two sets is minimized



Min-cuts can be computed efficiently using the max-flow

Min-cut often returns an imbalanced partition, with one set being a singleton

Ratio Cut and Normalized Cut

- To mitigate the min-cut problem we can change the objective function to consider community size

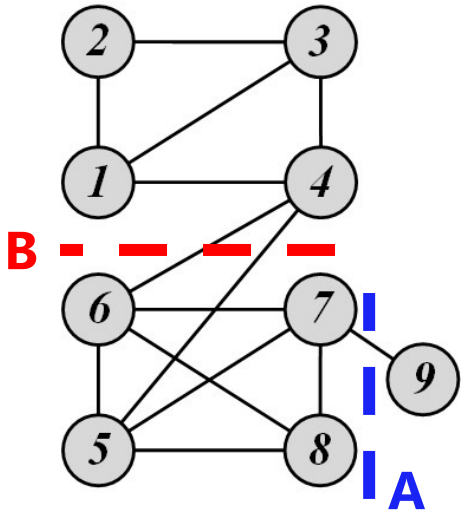
$$\text{Ratio Cut}(P) = \frac{1}{k}\sum_{i=1}^{k}\frac{\text{cut}(P_i,\bar{P}_i)}{|P_i|}$$

$$\text{Normalized Cut}(P) = \frac{1}{k}\sum_{i=1}^{k}\frac{\text{cut}(P_i,\bar{P}_i)}{\text{vol}(P_i)}$$

- $\bar{P}_i = V - P_i$ is the complement cut set
- $cut(P_i, \bar{P}_i)$ is the size of the cut
- $vol(P_i) = \sum_{v \in P_i} d_v$

Ratio Cut & Normalized Cut: Example

[Xiangguo]:
This might be in your final exam



For Cut A

$$\text{Ratio Cut}(\{1,2,3,4,5,6,7,8\},\{9\}) = \frac{1}{2}(\frac{1}{1}+\frac{1}{8}) = 9/16 = 0.56$$

$$\text{Normalized Cut}(\{1,2,3,4,5,6,7,8\},\{9\}) = \frac{1}{2}(\frac{1}{1}+\frac{1}{27}) = 14/27 = 0.52$$

For Cut B

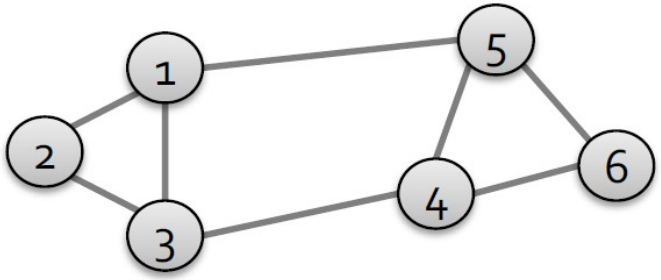
$$\text{Ratio Cut}(\{1,2,3,4\},\{5,6,7,8,9\}) = \frac{1}{2}(\frac{2}{4}+\frac{2}{5}) = 9/20 = 0.45 < 0.56$$

$$\text{Normalized Cut}(\{1,2,3,4\},\{5,6,7,8,9\}) = \frac{1}{2}(\frac{2}{12}+\frac{2}{16}) = 7/48 = 0.15 < 0.52$$

Both ratio cut and normalized cut prefer a balanced parti

More on Spectral Clustering

- **Laplacian matrix (L):**
  - $n \times n$ symmetric matrix



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

$$L = D - A$$

- **What is trivial eigenpair?**
  - $x = (1, \ldots, 1)$ then $L \cdot x = 0$ and so $\lambda = \lambda_1 = 0$
- **Important properties of L:**
  - **Eigenvalues** are non-negative real numbers
  - **Eigenvectors** are real (and always orthogonal)

Laplacian Matrix

From Miss Xin Xia' s tutorial
This won not be in your exam
If you feel difficult to understand, skip them.

Proposition 1 (Properties of L) *The matrix L satisfies the following properties:*

1. *For every vector $f \in \mathbb{R}^n$ we have*

$$f'Lf = \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}(f_i - f_j)^2.$$

2. *L is symmetric and positive semi-definite.*

3. *The smallest eigenvalue of L is 0,*
   the corresponding eigenvector is the constant one vector $\mathbb{1}$.

4. *L has n non-negative, real-valued eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.*

Laplacian Matrix

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If you feel difficult to understand, skip them.

Proof.
Part (1): By the definition of d_i,

$$f'Lf = f'Df - f'Wf = \sum_{i=1}^{n} d_i f_i^2 - \sum_{i,j=1}^{n} f_i f_j w_{ij}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} d_i f_i^2 - 2\sum_{i,j=1}^{n} f_i f_j w_{ij} + \sum_{j=1}^{n} d_j f_j^2\right) = \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}(f_i - f_j)^2.$$

Part (2): The symmetry of L follows directly from the symmetry of W and D. The positive semi-definiteness is a direct consequence of Part (1), which shows that $f'Lf \geq 0$ for all $f \in \mathbb{R}^n$.
Part (3): Obvious.
Part (4) is a direct consequence of Parts (1) - (3). □

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Let A be a linear transformation represented by a matrix $\mathbf{A}$. If there is a vector $\mathbf{X} \in \mathbb{R}^n \neq \mathbf{0}$ such

$$\mathbf{A}\mathbf{X} = \lambda \mathbf{X}$$

for some scalar λ, then λ is called the eigenvalue of $\mathbf{A}$ with corresponding (right) eigenvector $\mathbf{X}$.

Letting $\mathbf{A}$ be a $k \times k$ square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

with eigenvalue λ, then the corresponding eigenvectors satisfy

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix},$$

Positive Semidefinite Matrix

A positive semidefinite matrix is a Hermitian matrix all of whose eigenvalues are nonnegative.

Spectral Clustering

Reformulating ratio cut (or normalized cut) in matrix format

- Community membership matrix X; $X_{ij} = 1$, when node i is member of community j; 0, otherwise
- Let $D = diag(d_1, d_2, \cdots, d_n)$ be the diagonal degree matrix
- The ith entry on the diagonal of $X^T AX$ is the number of edges that are inside community i.
- The ith element on the diagonal of $X^T DX$ is the number of edges that are connected to members of community i.
- The ith element on the diagonal of $X^T(D - A)X$ is the number of edges in the cut that separates community i from other nodes.

The ith diagonal element of $X^T(D - A)X$ is equivalent to $\text{cut}(P_i, \overline{P_i})$

Spectral Clustering

So ratio cut is

$$\text{Ratio Cut}(P) = \frac{1}{k}\sum_{i=1}^{k}\frac{\text{cut}(P_i, \bar{P}_i)}{|P_i|}$$

$$= \frac{1}{k}\sum_{i=1}^{k}\frac{X_i^T(D-A)X_i}{X_i^T X_i}$$

$$= \frac{1}{k}\sum_{i=1}^{k}\hat{X}_i^T(D-A)\hat{X}_i$$

$$\hat{X}_i = X_i/(X_i^T X_i)^{1/2}$$

Spectral Clustering

Both ratio/normalized cut can be reformulated as

$$\min_{\hat{X}} \mathrm{Tr}(\hat{X}^T L \hat{X})$$

$$L = \begin{cases} D - A & \text{Ratio Cut Laplacian, i.e., Unnormalized Laplacian} \\ I - D^{-1/2} A D^{-1/2} & \text{Normalized Laplacian for Normalized Cut.} \end{cases}$$

$D = diag(d_1, d_2, \cdots, d_n)$ is a diagonal degree matrix

- It has been shown that both ratio cut and normalized cut minimization are NP-hard

NP-Hard

Approximation algorithms using relaxations are desired

Approximating RatioCut for k = 2

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- The simplest case: partitioning the graph into k=2 subgraphs
- Our goal is to solve the optimization problem:

$$\min_{A \subset V} \mathrm{RatioCut}(A, \overline{A}).$$

Given a subset $A \subset V$ we define the vector $f = (f_1, \ldots, f_n)' \in \mathbb{R}^n$ with entries

$$f_i = \begin{cases} \sqrt{|\overline{A}|/|A|} & \text{if } v_i \in A \\ -\sqrt{|A|/|\overline{A}|} & \text{if } v_i \in \overline{A}. \end{cases}$$

Approximating RatioCut for k = 2

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- Now the RatioCut objective function can be conveniently rewritten using the unnormalized graph Laplacian. This is due to the following calculation:

$$
\begin{aligned}
f'Lf &= \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}(f_i - f_j)^2 \\
&= \frac{1}{2}\sum_{i\in A, j\in \overline{A}} w_{ij}\left(\sqrt{\frac{|\overline{A}|}{|A|}} + \sqrt{\frac{|A|}{|\overline{A}|}}\right)^2 + \frac{1}{2}\sum_{i\in \overline{A}, j\in A} w_{ij}\left(-\sqrt{\frac{|\overline{A}|}{|A|}} - \sqrt{\frac{|A|}{|\overline{A}|}}\right)^2 \\
&= \mathrm{cut}(A,\overline{A})\left(\frac{|\overline{A}|}{|A|} + \frac{|A|}{|\overline{A}|} + 2\right) \\
&= \mathrm{cut}(A,\overline{A})\left(\frac{|A|+|\overline{A}|}{|A|} + \frac{|A|+|\overline{A}|}{|\overline{A}|}\right) \\
&= |V| \cdot \mathrm{RatioCut}(A,\overline{A}).
\end{aligned}
$$

Approximating RatioCut for k = 2

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Additionally, we have

$$\sum_{i=1}^{n} f_i = \sum_{i\in A}\sqrt{\frac{|\overline{A}|}{|A|}} - \sum_{i\in \overline{A}}\sqrt{\frac{|A|}{|\overline{A}|}} = |A|\sqrt{\frac{|\overline{A}|}{|A|}} - |\overline{A}|\sqrt{\frac{|A|}{|\overline{A}|}} = 0.$$

In other words, the vector f as defined in Equation (2) is orthogonal to the constant one vector $\mathbb{1}$.

Finally, note that f satisfies

$$\|f\|^2 = \sum_{i=1}^{n} f_i^2 = |A|\frac{|\overline{A}|}{|A|} + |\overline{A}|\frac{|A|}{|\overline{A}|} = |\overline{A}| + |A| = n.$$

Approximating RatioCut for k = 2

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Altogether we can see that the problem of minimizing (1) can be equivalently rewritten as

$$\min_{A \subset V} f' L f \text{ subject to } f \perp \mathbb{1},\ f_i \text{ as defined in Eq. (2)},\ \|f\| = \sqrt{n}.$$

This is a discrete optimization problem as the entries of the solution vector *f* are only allowed to take two particular values, and of course it is still **NP hard**.

The most obvious relaxation in this setting is to discard the discreteness condition and instead allow that f_i takes arbitrary values in $\mathbb{R}$. This leads to the relaxed optimization problem

$$\min_{f \in \mathbb{R}^n} f' L f \text{ subject to } f \perp \mathbb{1},\ \|f\| = \sqrt{n}.$$

Generalize to k = N

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The relaxation of the RatioCut minimization problem in the case of a general value *k* follows a similar principle as the one above.

Given a partition of V into k sets $A_1, \ldots, A_k$, we define k indicator vectors $h_j = (h_{1,j}, \ldots, h_{n,j})'$ by

$$h_{i,j} = \begin{cases} 1/\sqrt{|A_j|} & \text{if } v_i \in A_j \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \ldots, n;\ j = 1, \ldots, k)$$

Then we set the matrix $H \in \mathbb{R}^{n \times k}$ as the matrix containing those k indicator vectors as columns. Observe that the columns in H are orthonormal to each other that is $H'H = I$.

Generalize to k = N

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Similar to the calculations when k=2, we can see that

$$h_i' L h_i = \frac{\mathrm{cut}(A_i, \overline{A}_i)}{|A_i|}.$$

Moreover, one can check that

$$h_i' L h_i = (H' L H)_{ii}.$$

Combining those facts we get:

$$\mathrm{RatioCut}(A_1, \ldots, A_k) = \sum_{i=1}^{k} h_i' L h_i = \sum_{i=1}^{k} (H' L H)_{ii} = \mathrm{Tr}(H' L H),$$

where Tr denotes the trace of a matrix.

Generalize to k = N

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So the problem of minimizing RatioCut($A_1, \ldots, A_k$) can be rewritten as

$$\min_{H \in \mathbb{R}^{n \times k}} \operatorname{Tr}(H'LH) \text{ subject to } H'H = I.$$

Because for any A which is a symmetric matrix:

$$\forall X \in \mathbb{R}^{n}, \ \lambda_{min} X^T X \leq X^T A X \leq \lambda_{max} X^T X$$

Then minimizing the trace is equivalent to find the K smallest eigenvalues and their corresponding eigenvectors.

That is where we connect the eigen-decomposition of Laplacian matrix and minimum cut.

Summary

Given L, the top k eigenvectors corresponding to the smallest eigen values are computed and denoted as $\hat{X}$.

Each node can be represented as a feature vector of dimension k.

Run k-means on these features to separate nodes into k classes/communities.

Note that the first eigenvector is meaningless (because it is zero and its corresponding eigenvector is a constant 1 vector); hence, the rest of the eigenvectors (k-1) are often used as k-means input.

Spectral Clustering Algorithms

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Unnormalized spectral clustering

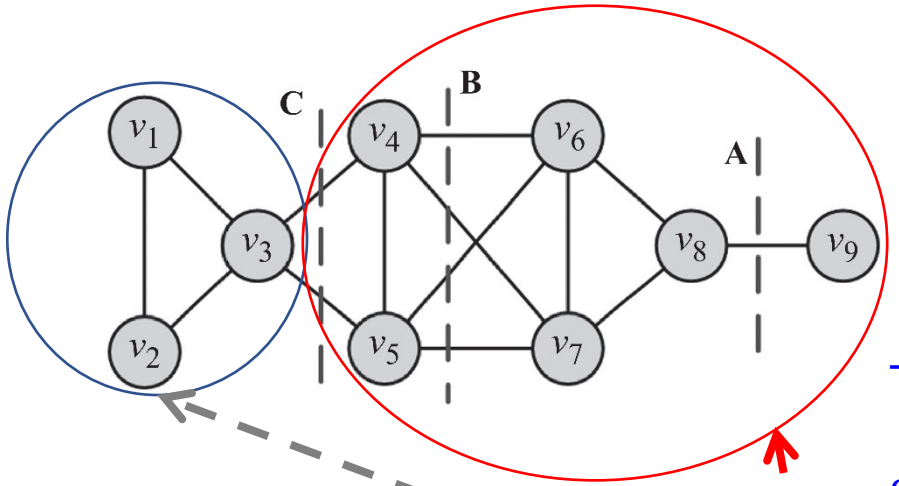
Input: Similarity matrix $S \in \mathbb{R}^{n \times n}$, number k of clusters to construct.

- Construct a similarity graph by one of the ways described in Section 2. Let W be its weighted adjacency matrix.
- Compute the unnormalized Laplacian L.
- **Compute the first k eigenvectors $u_1, \ldots, u_k$ of L.**
- Let $U \in \mathbb{R}^{n \times k}$ be the matrix containing the vectors $u_1, \ldots, u_k$ as columns.
- For $i = 1, \ldots, n$, let $y_i \in \mathbb{R}^k$ be the vector corresponding to the i-th row of U.
- Cluster the points $(y_i)_{i=1,\ldots,n}$ in $\mathbb{R}^k$ with the k-means algorithm into clusters $C_1, \ldots, C_k$.

Output: Clusters $A_1, \ldots, A_k$ with $A_i = \{j | \, y_j \in C_i\}$.

Spectral Clustering: Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Two communities:
$\{v_1, v_2, v_3\}$
$\{v_4, v_5, v_6, v_7, v_8, v_9\}$

$$D = diag(2, 2, 4, 4, 4, 4, 4, 3, 1)$$

$$L = D - A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

2 Eigenvectors
i.e., we want
2 communities

$$\begin{bmatrix} 0.33 & -0.46 \\ 0.33 & -0.46 \\ 0.33 & -0.26 \\ 0.33 & 1.16 \times 10^{-16} \\ 0.33 & 1.16 \times 10^{-16} \\ 0.33 & 0.13 \\ 0.33 & 0.13 \\ 0.33 & 0.33 \\ 0.33 & 0.59 \end{bmatrix}$$

k-means, $k = 2$

Summary of Spectral Clustering

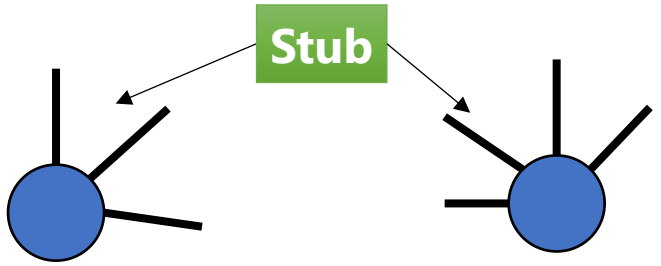
- Given L, the top k eigenvectors corresponding to the smallest eigenvalues are computed and denoted as $\hat{X}$.
  - Each node can be represented as a feature vector of dimension k.
- Run k-means on these features to separate nodes into k classes/communities.
- Note that the first eigenvector is meaningless (why?); hence, the rest of the eigenvectors (k-1) are often used as k-means input.

Community Detection

- Spectral Clustering
- Modularity Maximization

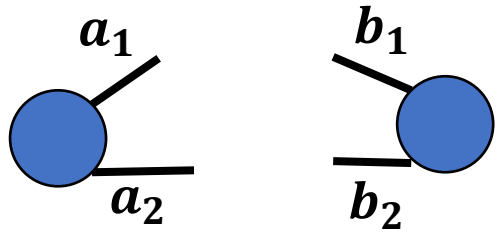
Modular Communities -Configuration Model

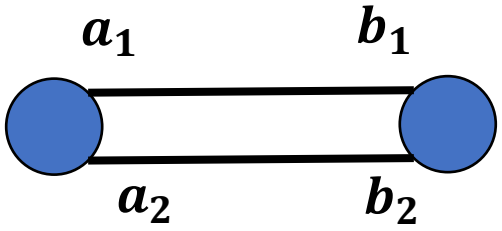
I. Given the degree sequence d_1, $d_2, \ldots, d_n$ generate n nodes, such that node i has d_i ***stubs*** (also known as ***half-edges***)

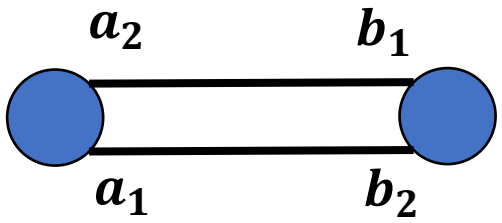


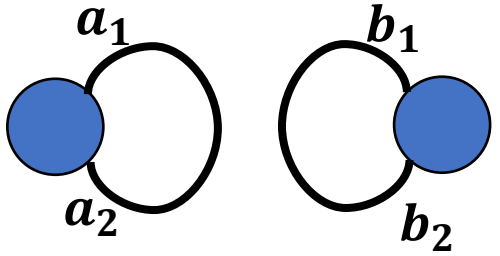
II. Randomly match stubs until there are no more stubs

- You can have loops and multiple edges
- Each **configuration** (**not each graph!**) appears with equal probability
- You can get the same graph









How to generate the Configuration model

1. Create a list where the **node id** for node v_i with degree d_i is repeated d_i times
2. Shuffle the list
3. Starting from the first index, join adjacent nodes

Example: Degree sequence (2,2,2)

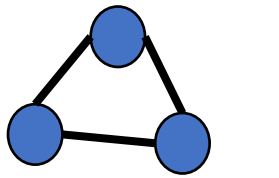
v_1	v_1	v_2	v_2	v_3	v_3

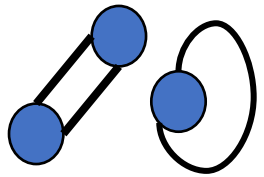
Random Shuffle 1:

v_1	v_2	v_2	v_3	v_3	v_1

Random Shuffle 2:

v_1	v_1	v_2	v_3	v_3	v_2





Modularity and Modularity Maximization

- Given a degree distribution, we know the expected number of edges between any pairs of vertices

- We assume that real-world networks should be far from random. Therefore, the more distant they are from this randomly generated network, the more structural they are.

- Modularity defines this distance and modularity maximization tries to maximize this distance

Normalized Modularity

Consider a partitioning of a graph $P = (P_1, P_2, P_3, \ldots, P_k)$

For partition P_x, this distance can be defined as

$$\sum_{i,j \in P_x} A_{ij} - \frac{d_i d_j}{2m}$$

This distance can be generalized for a partitioning P

$$\sum_{x=1}^{k} \sum_{i,j \in P_x} A_{ij} - \frac{d_i d_j}{2m}$$

The normalized version of this distance is defined as Modularity

$$Q = \frac{1}{2m} \sum_{x=1}^{k} \sum_{i,j \in P_x} A_{ij} - \frac{d_i d_j}{2m}$$

Modularity Maximization

Modularity matrix $B = A - dd^T/2m$

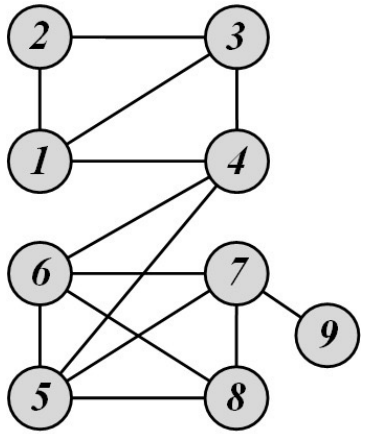
$d \in \mathbb{R}^{n\times 1}$ is the degree vector for all nodes

Reformulation of the modularity $Q = \frac{1}{2m}\mathrm{Tr}(X^T B X)$

- $X \in \mathbb{R}^{n\times k}$ is the indicator (partition membership) function:
  - $X_{ij} = 1$ iff. $v_i \in P_j$

- Similar to Spectral clustering,
  - We relax X to be real-valued matrix $\hat{X}$
  - The optimal solution for $\hat{X}$ is the top k eigenvectors of B.
  - To recover the original X, we can run k-means on $\hat{X}$

Modularity Maximization: Example



$$B = A - dd^T/2m$$
$$B_{ij} = A_{ij} - d_i d_j/2m$$

Two Communities:
{1, 2, 3, 4}
and
{5, 6, 7, 8, 9}

k-means

$$B = \begin{bmatrix} -0.32 & 0.79 & 0.68 & 0.57 & -0.43 & -0.43 & -0.43 & -0.32 & -0.11 \\ 0.79 & -0.14 & 0.79 & -0.29 & -0.29 & -0.29 & -0.29 & -0.21 & -0.07 \\ 0.68 & 0.79 & -0.32 & 0.57 & -0.43 & -0.43 & -0.43 & -0.32 & -0.11 \\ 0.57 & -0.29 & 0.57 & -0.57 & 0.43 & 0.43 & -0.57 & -0.43 & -0.14 \\ -0.43 & -0.29 & -0.43 & 0.43 & -0.57 & 0.43 & 0.43 & 0.57 & -0.14 \\ -0.43 & -0.29 & -0.43 & 0.43 & 0.43 & -0.57 & 0.43 & 0.57 & -0.14 \\ -0.43 & -0.29 & -0.43 & -0.57 & 0.43 & 0.43 & -0.57 & 0.57 & 0.86 \\ -0.32 & -0.21 & -0.32 & -0.43 & 0.57 & 0.57 & 0.57 & -0.32 & -0.11 \\ -0.11 & -0.07 & -0.11 & -0.14 & -0.14 & -0.14 & 0.86 & -0.11 & -0.04 \end{bmatrix}$$

Modularity Matrix

2 eigenvectors

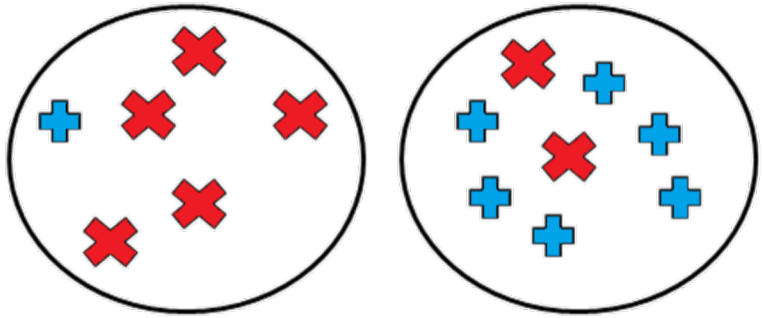
$$\begin{bmatrix} 0.44 & -0.00 \\ 0.38 & 0.23 \\ 0.44 & -0.00 \\ 0.17 & -0.48 \\ -0.29 & -0.32 \\ -0.29 & -0.32 \\ -0.38 & 0.34 \\ -0.34 & -0.08 \\ -0.14 & 0.63 \end{bmatrix}$$

Community Evaluation

Evaluating the Communities

We are given objects of two different kinds (+, ×)

- **The perfect community:** all objects inside the community are of the same type



- **Evaluation with ground truth**
- **Evaluation without ground truth**

[Xiangguo]:
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Precision and Recall

$$Precision = \frac{Relevant\ and\ retrieved}{Retrieved}$$

$$P = \frac{TP}{TP + FP}$$

$$Recall = \frac{Relevant\ and\ retrieved}{Relevant}$$

$$R = \frac{TP}{TP + FN}$$

True Positive (TP) :

- When similar member pairs are assigned to the same communities
- A **correct** decision.

False Negative (FN) :

- When similar member pairs are assigned to different communities
- An **incorrect** decision

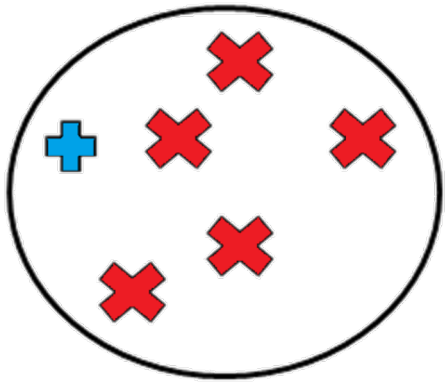
True Negative (TN) :

- When dissimilar member pairs are assigned to different communities
- A **correct** decision

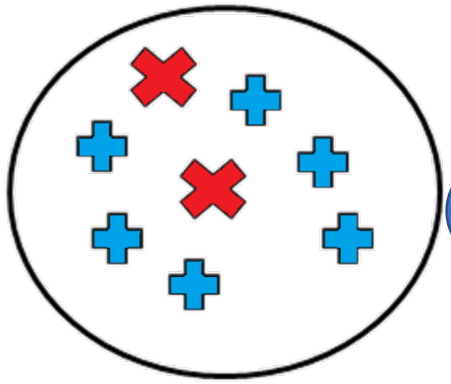
False Positive (FP) :

- When dissimilar member pairs are assigned to the same communities
- An **incorrect** decision

Precision and Recall: Example



Cluster 1



Cluster 2

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$$
\begin{aligned}
TP &= \binom{5}{2} + \binom{6}{2} + \binom{2}{2} = 26, \\
FP &= (5 \times 1) + (6 \times 2) = 17, \\
FN &= (5 \times 2) + (6 \times 1) = 16, \\
TN &= (6 \times 5) + (2 \times 1) = 32.
\end{aligned}
$$

$$P = \frac{26}{26+17} = 0.60$$

$$R = \frac{26}{26+16} = 0.61$$

F-Measure

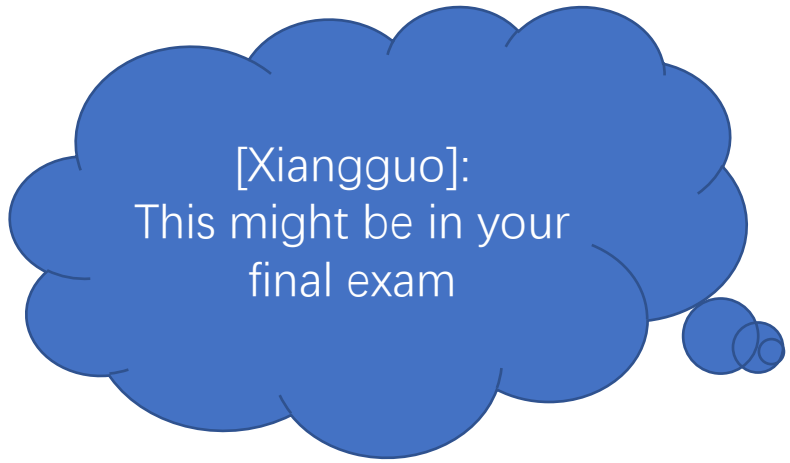
Either P or R measures one aspect of the performance,

- To integrate them into one measure, we can use the harmonic mean of precision of recall

$$F = 2 \cdot \frac{P \cdot R}{P + R}$$

For the example earlier,

$$F = 2 \times \frac{0.6 \times 0.61}{0.6 + 0.61} = 0.60$$



Purity

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We can assume the majority of a community represents the community

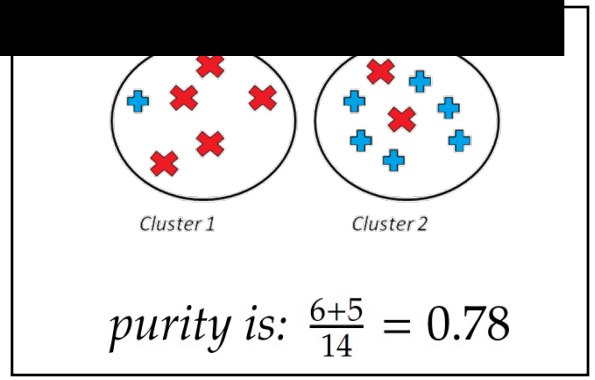
- We use the label of the majority against the label of each member to evaluate the communities

Purity. The fraction of instances that have labels equal to t

Purity can be easily **tampered** by

- Points being singleton communities (of size 1); or by
- Very large communities

$$Purity = \frac{1}{N}\sum_{i=1}^{k}\max_{j}|C_i \cap L_j|$$



- k: the number of communities
- N: total number of nodes,
- L_j: the set of instances with label j in all communities
- C_i: the set of members in community i

Mutual Information

- **Mutual information (MI).** The amount of information that two random variables share.
  - By knowing one of the variables, it measures the amount of uncertainty reduced regarding the others

$$MI = I(H, L) = \sum_{h \in H} \sum_{l \in L} \frac{n_{h,l}}{n} \log \frac{n \cdot n_{h,l}}{n_h n_l}$$

- L and H are labels and found communities;
- n_h and n_l are the number of data points in community h and with label l, respectively;
- $n_{h,l}$ is the number of nodes in community h and with label l; and n is the number of nodes

Normalizing Mutual Information (NMI)

- Mutual information (MI) is unbounded
- To address this issue, we can normalize MI

- How? We know that

$$MI \leq min(H(L), H(H)),$$
$$(MI)^2 \leq H(H)H(L).$$
$$MI \leq \sqrt{H(H)}\sqrt{H(L)}.$$

- $H(.)$ is the entropy function

$$H(L) = -\sum_{l \in L} \frac{n_l}{n} \log \frac{n_l}{n}$$
$$H(H) = -\sum_{h \in H} \frac{n_h}{n} \log \frac{n_h}{n}.$$

Normalized Mutual Information

$$NMI = \frac{MI}{\sqrt{H(L)}\sqrt{H(H)}}.$$

$$NMI = \frac{\sum_{h \in H} \sum_{l \in L} n_{h,l} \log \frac{n \cdot n_{h,l}}{n_h n_l}}{\sqrt{(\sum_{h \in H} n_h \log \frac{n_h}{n})(\sum_{l \in L} n_l \log \frac{n_l}{n})}}.$$

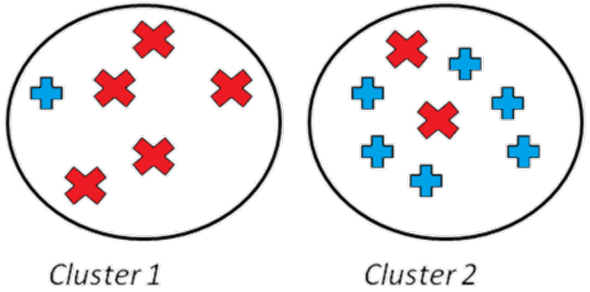
Normalized Mutual Information

$$NMI = \frac{\sum_{h,l} n_{h,l} \log \frac{n \cdot n_{h,l}}{n_h n_l}}{\sqrt{(\sum_h n_h \log \frac{n_h}{n})(\sum_l n_l \log \frac{n_l}{n})}}$$

- where l and h are known (with labels) and found communities, respectively
- n_h and n_l are the number of members in the community h and l, respectively,
- $n_{h,l}$ is the number of members in community h and labeled l,
- n is the size of the dataset

- **NMI** values close to one indicate high similarity between communities found and labels
- Values close to zero indicate high dissimilarity between them

Normalized Mutual Information: Example



[Xiangguo]:
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Found communities (H)

- [1,1,1,1,1,1, 2,2,2,2,2,2,2,2]

Actual Labels (L)

- [2,1,1,1,1,1, 2,2,2,2,2,2,1,1]

$n = 14$

	$\mathbf{n}_h$
h=1	6
h=2	8

	$\mathbf{n}_l$
$l = 1$	7
$l = 2$	7

$\mathbf{n}_{h,l}$	$l = 1$	$l = 2$
h=1	5	1
h=2	2	6