2

Sample Spaces, Events, and Probabilities

building blocks for modelling random experiments

> By the end of this chapter you should:
>
> - Know what sample spaces and events are.
> - Be able to construct events given a sample space.
> - Understand the importance of the probability axioms.
> - Work with simple discrete probabilities.

Core idea in probability and statistics: **random experiment**.

A random experiment is a process whose outcome is not known in advance, but is nevertheless still subject to analysis.

Examples:

• roll a die	• bingo numbers / lotto
• flip a coin	• stock returns
• picking a card from a deck.	• outcomes from elections

To model any of these examples, we need:

coin toss $\Omega = \{H, T\}$

Draw a card $\Omega = \{K\heartsuit, Q\heartsuit, \ldots\}$

roll a die $\Omega = \{1, 2, 3, 4, 5, 6\}$

stock returns $\Omega = [-100\%, \infty)$

Omega

(i) Sample space (written Ω, all possible outcomes).

(ii) Events ($A \subseteq \Omega$, groups of outcomes).

class of events $\mathcal{F}$: $\Omega \in \mathcal{F}$, $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, $A_i \in \mathcal{F} \Rightarrow \cup_i A_i \in \mathcal{F}$

(iii) Probabilities of events (written $\mathbb{P}(A)$)

function mapping events to $[0, 1]$

Sample spaces and events

Sample space for the experiment "Winner of the 2018 World Cup":

$\Omega = \{$ Germany, France, Brazil, ..., Australia, ... $\}$

Some possible events:

$A =$ winner comes from Asian Football Confederation
$= \{$ Iraq, India, China, Australia, .. $\}$

$B =$ Australia wins
$= \{$Australia$\}$

Sample space for the experiment "views of your next Instagram photo": number of

$\Omega = \{0, 1, 2, ...\}$

$[0, \infty)$ positive real numbers

Some possible events:

$B' = 10$ or more views $= \{10, 11, 12, ...\}$
$A = 10$ views $= \{10\}$
$B =$ more than 10 views $= \{11, 12, ...\}$
$C =$ more than 2, but less than 60 views $= \{3, 4, ..., 58, 59\}$

Definition. The **sample space** Ω of a random experiment is the *set of all possible outcomes of the experiment.*

Definition. An **event** is a *subset of the sample space.* That is, a collection of some possible outcomes of the experiment.

Events will be denoted by capital letters $A, B, C, \ldots$.

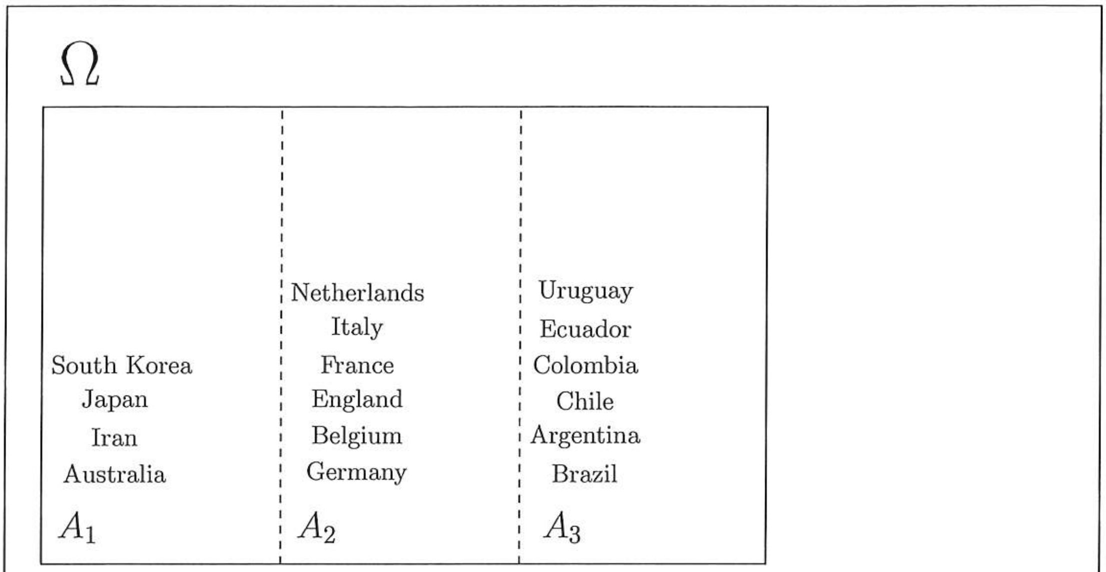
We say event A **occurs** if:

the outcome of the random experiment is in A.

When writing down sample spaces make sure you understand the difference between *sets* $\{\ldots\}$ and *vectors* $(\ldots)$:

- *Round* brackets ( ) indicate *order*, e.g., $(1, 2, 3) \neq (3, 2, 1)$.
- *Curly* brackets { } indicate *no order*, e.g., $\{1, 2, 3\} = \{3, 2, 1\}$.

A sample space with some events marked:



$A_1 = \{$South Korea, Iran, Japan, Australia$\}$ = AFC winners
$A_2 = \{$Netherlands, Italy, France, England, Belgium, Germany, $\dots\}$ = UEFA winners
$A_3 = \{$Argentina, Brazil, Chile, Colombia, Ecuador, Uruguay$\}$ = CONMEBOL winners
$A_4 = \{$Germany, Argentina, Netherlands$\}$ = top 3 from 2014 world cup.

- The event that there are fewer than 50 defective components in a batch of 1000.

  $\boxed{A = \{0, 1, \dots, 49\}}$

- The event that a machines lives longer than 1000 days,

  $\boxed{A = \{t : t > 1000\}}$

- The event that between 10 and 20 inclusive hits occur to a web-server, during a specified time interval,

  $\boxed{A = \{10, 11, \dots, 19, 20\}}$

- The event that the sum of two dice is 10 or more (supposing the dice are thrown consecutively):

  $\boxed{A = \{(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\}}$

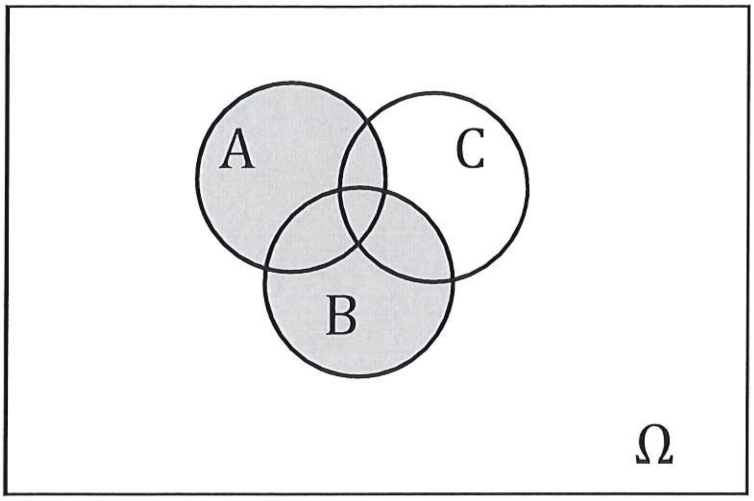
red blue

$$\Omega = \{(1,1), (1,2), \dots, (1,6), (2,1), (2,2), \dots, (2,6), \vdots (6,1), (6,2), \dots (6,6)\}$$

Set operations and events

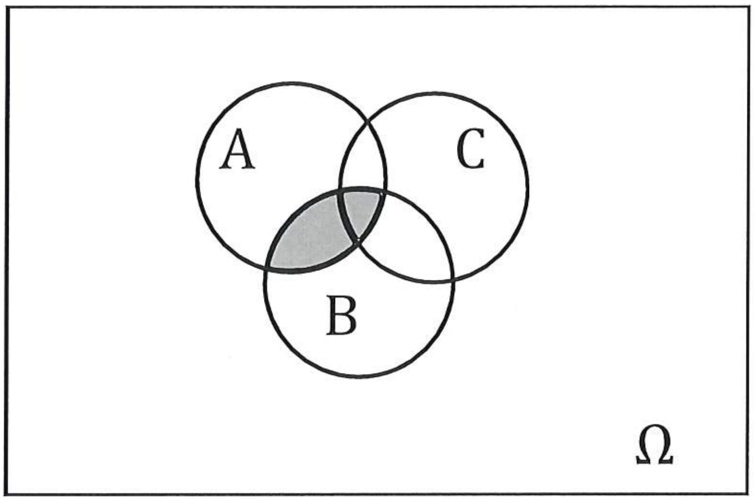
As events are *sets* of outcomes, usual set operations and definitions apply.

- The event $A \cup B$ (A **union** B) is the event that A *or* B *or both* occur.

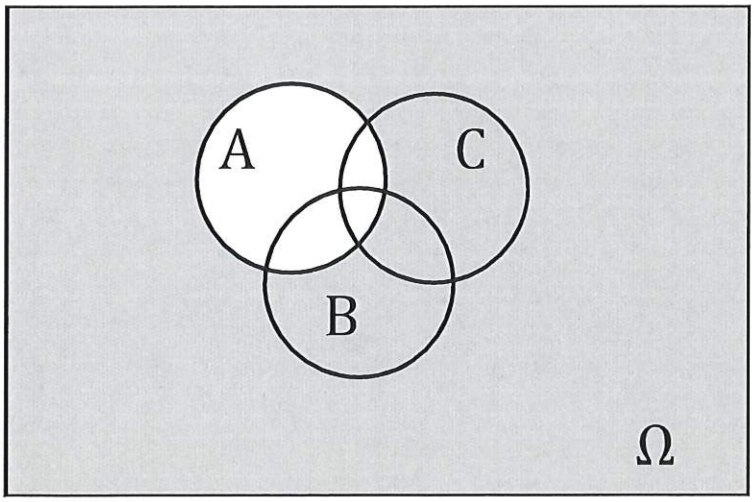


$A \cup B$

- The event $A \cap B$ (A **intersection** B) is the event that A *and* B both occur.



- The event A^c (A **complement** with respect to Ω) is the event that A does not occur.



- Two events A and B are **equal**, $A = B$, if

* some representations are easier to work with.

if A and B contain the same outcomes

- Event A **implies** event B, $A \subseteq B$, if A is a subset of B.
- The event containing no outcomes is denoted $\emptyset$ (the **empty** set or impossible event).

Example: Consider the experiment of tossing a standard, six-sided die, whose outcome is an element of the sample space Ω. Let A be the event that the outcome is an even number, and let B be the event that the outcome is an odd number. What are the following sets?

$$\Omega = \{1,2,3,4,5,6\}$$
$$A = \{2,4,6\}$$
$$B = \{1,3,5\}$$

Let $C = \{5, 6\}$ and $D = A \cap C$. What are these next sets?

$$A \cup B = \Omega$$
$$A \cap B = \emptyset$$
$$A^c = B$$
$$D = A \cap C = \{2,4,6\} \cap \{5,6\} = \{6\}$$
$$B \cap C = \{1,3,5\} \cap \{5,6\} = \{5\}$$
$$C^c = \{1,2,3,4\}$$

Are the following claims true?

$D \subseteq C$ TRUE

$C \subseteq A$ FALSE $5 \in C$ but $5 \notin A$

$D \subseteq A$ TRUE

$D \subseteq B$ FALSE

Disjoint Events

Events $A_1, A_2, \ldots, A_n$ are called **disjoint** if

for all $i \neq j$, $A_i \cap A_j = \emptyset$

A sequence $A_1, A_2, \ldots, A_n$ of disjoint events such that their union is the entire sample space Ω is called a **partition** of Ω.

Example: In the above example, $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{5,6\}$ are a partition of Ω.

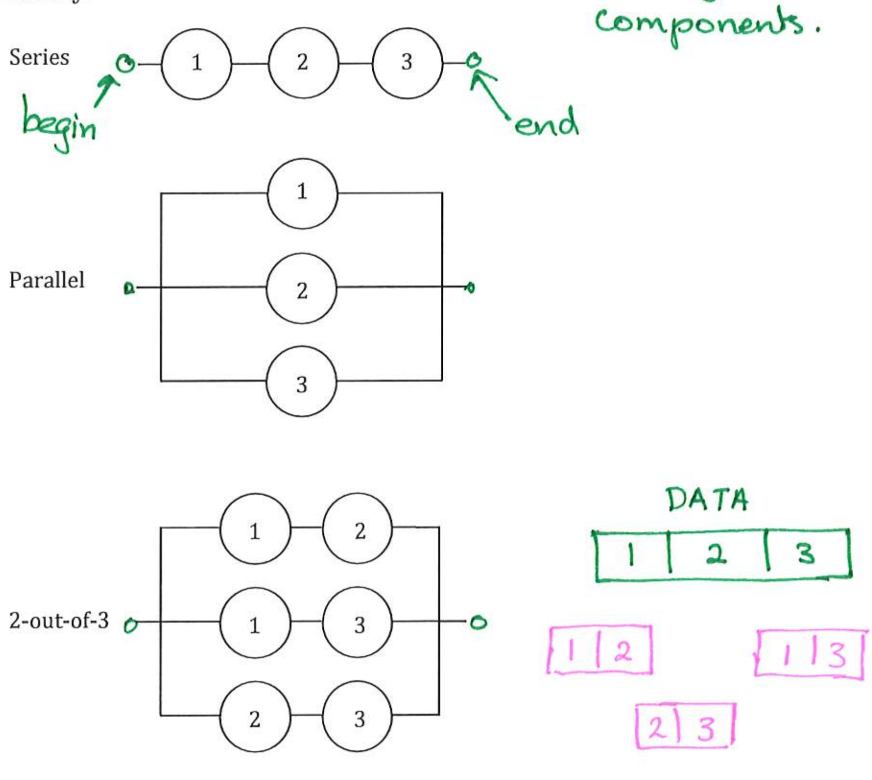
e.g. $A_1 \cap A_2 = \{1,2\} \cap \{3,4\} = \emptyset$

$A_1 \cup A_2 \cup A_3 = \{1,2\} \cup \{3,4\} \cup \{5,6\} = \{1,2,3,4,5,6\} = \Omega$

Constructing Events

system is functioning if there is a path through working components.

Example: System Reliability.



Let A_i be the event that *the i-th component is functioning*, $i = 1, 2, 3$. Note that A_i^c is the event that the i-*th* component failed.

$$D_a = \text{the series system is functioning}$$
$$= A_1 \cap A_2 \cap A_3$$

$$D_b = \text{the parallel system is functioning}$$
$$= A_1 \cup A_2 \cup A_3$$

$D_c =$ the 2-out-of-3 system is functioning

$= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3)$

Axioms and Implications

$[0,1]$

Definition. A **probability** $\mathbb{P}$ is a rule (or function) which assigns a number to each event, and which satisfies the following **axioms** (or properties):

- Axiom 1: $\mathbb{P}(A) \geqslant 0$.
- Axiom 2: $\mathbb{P}(\Omega) = 1$.
- Axiom 3: **Sum Rule**: For any disjoint $A_1, A_2, \ldots$

$$\boxed{\mathbb{P}\left(\bigcup_j A_j\right) = \sum_j \mathbb{P}(A_j).}$$

Some **consequences** of the axioms:

- Consequence 1: If $A \subseteq B$ then
- Consequence 2: $\mathbb{P}(\emptyset) =$
- Consequence 3: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
- Consequence 4: $0 \leqslant \mathbb{P}(A) \leqslant 1$.
- Consequence 5: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Proving these involves cleverly using the axioms, above.

Example: Prove Consequence 3.

$\Omega =$

$1 =$ (Axiom 2)

$=$

$=$ (Axiom 3).

Now rearrange this equation.