STAT7203: Applied Probability and Statistics
Assignment 1

Due by 14:00 on Tuesday the 6^{th} of October, 2020. Submission via Blackboard.

The marks for each question are indicate by the number in square brackets. There are a total of 15 marks for this assignment. Your submission to blackboard should be a single pdf file. You may prepare this however you wish, provided the result is legible.

1. A continuous random variable X has probability density function

$$f_X(x) = \begin{cases} \frac{1}{x \log 10}, & x \in [0.1, 1] \\ 0, & \text{else}, \end{cases} \tag{1}$$

where log denotes the natural logarithm.

(a) Compute the mean and variance of X. [1 mark]

(b) Determine the probability density function of $Y = (10X)^{-1}$. [1 mark]

(c) Determine the quantile function of X. [2 marks]

(d) Hence, write a small function in `R` or `MATLAB` to simulate random variables from this distribution. Supply your code. [1 mark]

(e) Let X and Y be two independent random variables where X has the probability density function (1) and Y has a continuous distribution of your choice whose support is either $(0, \infty)$ or a subset of $(0, \infty)$. [The list of built-in distributions in `R` can be found by typing `?distributions`. In `MATLAB` this can be found by typing `help stats`.] We are interested in the distribution of the mantissa of the product $Z = XY$. The mantissa of Z can be obtained as

$$W = 10^{(\log_{10} Z - \lceil \log_{10} Z \rceil)},$$

where $\lceil x \rceil$ denotes rounding up x to the nearest integer. Rounding up can be done in `R` using the function `ceiing` and in `MATLAB` using the function `ceil`. Use this procedure to generate 10 000 independent realisations of W and form a histogram of the result. Compare this with the probability density function (1). How do the two distributions compare? Provide a histogram of the outcomes with the probability density function (1) overlaid. Supply your code. [2 marks]

(f) Let X_1 and X_2 be two independent random variables with probability density function (1). Determine the probability that $X_1 X_2 \leq 0.1$. [2 marks]

2. Suppose you have four origami books which are labeled $B_1, \ldots, B_4$. The books are kept stacked on your desk. Each day you select one book from the stack and build a model. Books are selected each day at random with probabilities $\mathbb{P}(B_i \text{ is selected}) = (5 - i)/10$ and independently of past selections. Note that the probability that a

book is selected does not depend on the book's position in the stack. At the end of the day you place the selected book on top of the stack.

Suppose on day 0 the books are stacked in order of the labels. That is, book B_1 is in first position (top), book B_2 is second from the top, book B_3 is third from the top, and book B_4 is on the bottom.

(a) What is the probability that book B_1 is on top of the stack at the end of day n? Does this probability depend on n? [1 mark]

(b) Give an expression for the probability that book B_1 is second from the top of the stack at the end of day n. What is the limiting value as $n \to \infty$? [2 marks]

(c) Simulate this process 10 000 times and provide an estimate of the probability that book B_1 is on the bottom of the stack at the end of day 100. Supply your code. [2 marks]

(d) Let $X_i = 1$ if book B_1 is on the bottom of the stack in the i-th simulation of the process in part (c) and $X_i = 0$ otherwise. Assuming $X_1, X_2, \ldots, X_m$ are independent, show

$$\mathrm{Var}\left(m^{-1}\sum_{i=1}^{m} X_i\right) \leq 1/(4m).$$

[1 mark]

Total [15 marks]