Further Practice Problems for STAT2203 and STAT7203

1. Let $(A_1, A_2, \ldots, A_n)$ be a random permuation of the integers $\{1, 2, \ldots, n\}$ with all permutations equally likely. Define the random variables

$$X_i = \begin{cases} 1, & \text{if } A_i = i \\ 0, & \text{else.} \end{cases}$$

   (a) Specify the distribution of X_1.

   (b) Does $\sum_{i=1}^{n} X_i$ have a $\mathsf{Binomial}(n, p)$ distribution for some $p \in [0, 1]$?

   (c) Find the expected value of $\sum_{i=1}^{n} X_i$.

   (d) Find the variance of $\sum_{i=1}^{n} X_i$.

2. Let X_1, X_2 and X_3 be three independent random variables each having a discrete uniform distribution on the integers $\{1, 2, \ldots, m\}$.

   (a) Find the cumulative distribution function of $Y = \max\{X_1, X_2, X_3\}$.

   (b) Hence, give the probability mass function of Y.

   (c) Find the cdf and pdf of $Z = \min\{X_1, X_2, X_3\}$.

3. Let X_1 and X_2 be two independent random variables each having a $\mathsf{Geometric}(p)$ distribution.

   (a) Find the probability mass function of $Y = X_1 + X_2$.

   (b) Give $\mathbb{E}Y$ and $\mathrm{Var}(Y)$.

   (c) Compute $\mathbb{E}[Y \mid X_1]$.

   (d) Compute $\mathbb{E}[X_1 \mid Y]$.

4. Let $X \sim \mathsf{Binomial}(n, p)$. Conditional on $X = x$, Y has a $\mathsf{Binomial}(x, q)$ distribution. Using moment generating functions or otherwise, find the marginal pmf of Y. What is the name of this distribution?

5. Let $X \sim \mathsf{Poisson}(\lambda)$. Conditional on $X = x$, Y has a $\mathsf{Binomial}(x, q)$ distribution. Using moment generating functions or otherwise, find the marginal pmf of Y. What is the name of this distribution?

6. Let $X_1 \sim \mathsf{Geometric}(p)$ and $X_2 \sim \mathsf{Geometric}(p)$ be two independent random variables.

   (a) Let $Y = \min\{X_1, X_2\}$. Find $\mathbb{P}(Y > k)$ and hence determine the probability mass function of Y.

   (b) What is the name of the distribution of Y?

   (c) Find $\mathbb{P}(X_1 = Y)$.

7. Suppose $X \sim \mathsf{Bernoulli}(\frac{1}{4})$. Conditional on $X = 1$, Y has a $\mathsf{Poisson}(2)$ distribution. On the other hand, $\mathbb{P}(Y = 0 \mid X = 0) = 1$.

   (a) Give the moment generating function of Y.

   (b) Find the conditional probability mass function of X given $Y = y$.

   (c) Find the expected value and variance of Y.

8. Suppose the price of a stock increases in one day by a factor r with probabilty p and decreases by a factor $\frac{1}{r}$ with probability $1 - p$. Assuming a initial stock price of 1 and that the changes in stock price are independent between days, give the expected value and variance of the stock price after 10 days.

9. Suppose we have 5 components, the liftetimes of which are independent. Let X_i be the year in which the i-th component fails. If $X_i \sim \mathsf{Geometric}(\frac{1}{3})$, for all i, what is the probability that at the end of 4 years at least one component is still working?

10. Suppose that X_1, X_2, X_3 are independent $\mathsf{Bernoulli}(p)$ random variables. Let

$$Y = (1 - X_1)(X_2 + X_3 - X_2 X_3).$$

   (a) Find the expected value and variance of Y.

   (b) Compute $\mathbb{E}\left[Y \mid X_3\right]$.

   (c) Compute $\mathbb{E}\left[X_3 \mid Y = 1\right]$.

11. Let $X \sim \mathsf{Poisson}(1)$.

   (a) What is the moment generating function of $Y = X + 2$?

   (b) What is the moment generating function of $Y = 3X$?

12. Consider the moment generating function

$$M_X(s) = \frac{\log(1 - pe^s)}{\log(1 - p)},$$

   for $s < -\log p$. Find $\mathbb{E}X$ and $\mathrm{Var}(X)$.

13. Let (A_1, A_2, A_3) be a random permutation of the integers $\{1, 2, 3\}$ with all permutations equally likely. For $i < j$ define the random variables

$$X_{ij} = \begin{cases} 1, & \text{if } A_i > A_j \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Give the joint pmf of (X_{12}, X_{23}).

   (b) Are X_{12} and X_{23} independent?

   (c) Determine $\mathrm{Cov}(X_{12}, X_{23})$.

14. Consider the moment generating function

$$M_X(s) = \frac{e^{(e^s-1)}}{(2-e^s)}.$$

for $s \in \mathbb{R}$. Find $\mathbb{E}X$.

15. Let $X \sim \mathsf{Uniform}(\{1, 2, \ldots, n\})$ and define the random variables $Y_i,\ i = 1, 2, \ldots, n$ by

$$Y_i = \begin{cases} 1, & \text{if } X = i \\ 0, & \text{else.} \end{cases}$$

   (a) Compute $\mathrm{Cov}(Y_i, Y_j)$ for $i \neq j$.

   (b) Determine the conditional probability mass function of Y_i given Y_j for $i \neq j$.

16. Suppose $X \sim \mathsf{Binomial}(n, p)$ with $\mathbb{E}X = 8$ and $\mathrm{Var}(X) = 6$. Determine n and p.

17. Consider a system is comprised of 4 components and requires at least three of these components to be functioning in order that the system functions. The probability that a component fails in year i is $1/10$ for $i = 1, 2, \ldots, 10$. What is the probability that the system is still functioning after 8 years?

18. There are 200 academics at an international probability theory conference. During morning tea, the number of cups of coffee an academic consumes has probability mass function

Cups of coffee (x)	0	1	2
$\mathbb{P}(X = x)$	0.1	0.6	0.3

Assuming the number of cups of coffee each academic consumes is independent of the other academics, find the expected value and variance of the number of cups of coffee consumed during morning tea.

19. Consider the function $g(c) = \mathbb{E}[(X - cY)^2]$.

   (a) For what value of c is the function $g(c)$ minimised?

   (b) Suppose that X and Y are both Bernoulli random variables. Write the minimiser of $g(c)$ as a conditional probability.

20. Consider the pair of random variables (X, Y), where the marginal pmf of X is $\mathsf{Bernoulli}(1 - e^{-\lambda})$ and the marginal pmf of Y is $\mathsf{Poisson}(\lambda)$. Suppose that $P(X = Y = 0) = e^{-\lambda}$. Determine the joint pmf of (X, Y).