**Formula Sheet**

# Probability Formulas

1. **Sum rule:** $\boxed{\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)}$, when $A_1, A_2, \ldots$ are disjoint.
2. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
4. **Cdf** of $X$: $F(x) = \mathbb{P}(X \leqslant x),\ x \in \mathbb{R}$.
5. **Pmf** of $X$: (discrete r.v.) $f(x) = \mathbb{P}(X = x)$.
6. For a discrete r.v. $X$: $\boxed{\mathbb{P}(X \in B) = \sum_{x \in B} \mathbb{P}(X = x).}$
7. Marginal from joint pmf: $f_X(x) = \sum_y f_{X,Y}(x, y)$.
8. **Important discrete distributions**:

| Distr. | pmf | $x \in$ |
|---|---|---|
| $\mathsf{Ber}(p)$ | $p^x(1-p)^{1-x}$ | $\{0, 1\}$ |
| $\mathsf{Bin}(n, p)$ | $\binom{n}{x} p^x (1-p)^{n-x}$ | $\{0, 1, \ldots, n\}$ |
| $\mathsf{Poi}(\lambda)$ | $\mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}$ | $\{0, 1, \ldots\}$ |
| $\mathsf{Geom}(p)$ | $p(1-p)^{x-1}$ | $\{1, 2, \ldots\}$ |

9. **Conditional probability:** $\mathbb{P}(A \,|\, B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ .
10. **Law of total probability**:
$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A \,|\, B_i)\, \mathbb{P}(B_i)$,
where $B_1, B_2, \ldots, B_n$ is a partition of $\Omega$.
11. **Bayes' Rule:** $\mathbb{P}(B_j|A) = \frac{\mathbb{P}(B_j)\, \mathbb{P}(A|B_j)}{\sum_{i=1}^n \mathbb{P}(B_i)\, \mathbb{P}(A|B_i)}$.
12. **Product rule**:
$\boxed{\mathbb{P}(A_1 \cdots A_n) = \mathbb{P}(A_1)\, \mathbb{P}(A_2 \,|\, A_1) \cdots \mathbb{P}(A_n \,|\, A_1 \cdots A_{n-1}).}$
13. **Memoryless property** ($\mathsf{Geom}$ distribution):
$\mathbb{P}(X > s + t \,|\, X > s) = \mathbb{P}(X > t)$, $\forall s, t$.
14. **Independent events**: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\, \mathbb{P}(B)$ .
15. **Independent r.v.'s:** (discrete)
$\boxed{\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_{k=1}^n \mathbb{P}(X_k = x_k)\ .}$
16. **Expectation** (discr.): $\mathbb{E}X = \sum_x x\, \mathbb{P}(X = x)$.
17. (of function) $\mathbb{E}\, g(X) = \sum_x g(x)\, \mathbb{P}(X = x)$ .
18. **Expected sum :** $\mathbb{E}(aX + bY) = a\, \mathbb{E}X + b\, \mathbb{E}Y$ .
19. **Expected product** (if $X, Y$ independent):
$\mathbb{E}[X\, Y] = \mathbb{E}X\, \mathbb{E}Y$.
20. $\mathbb{E}X$ **and** $\mathbf{Var}(X)$ **for various distributions**:

| | $\mathbb{E}X$ | $\mathrm{Var}(X)$ |
|---|---|---|
| $\mathsf{Ber}(p)$ | $p$ | $p(1-p)$ |
| $\mathsf{Bin}(n, p)$ | $np$ | $np(1-p)$ |
| $\mathsf{Geom}(p)$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| $\mathsf{Poi}(\lambda)$ | $\lambda$ | $\lambda$ |

21. **Covariance:** $\mathrm{cov}(X, Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y)$.
22. **Properties of Var and Cov**:

| |
|---|
| $\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. |
| $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$. |
| $\mathrm{cov}(X, Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$. |
| $\mathrm{cov}(X, Y) = \mathrm{cov}(Y, X)$. |
| $\mathrm{cov}(aX + bY, Z) = a\, \mathrm{cov}(X, Z) + b\, \mathrm{cov}(Y, Z)$. |
| $\mathrm{cov}(X, X) = \mathrm{Var}(X)$. |
| $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\, \mathrm{cov}(X, Y)$. |
| $X$ and $Y$ independent $\Longrightarrow \mathrm{cov}(X, Y) = 0$. |

23. **Moment Generating Function (MGF)**:
$M_X(s) := \mathbb{E}e^{sX} = \sum_{n=0}^\infty \mathbb{P}(X = n) e^{sn}$.
24. **MGFs for various distributions:**

| | |
|---|---|
| $\mathsf{Ber}(p)$ | $1 - p + pe^s$ |
| $\mathsf{Bin}(n, p)$ | $(1 - p + pe^s)^n$ |
| $\mathsf{Geom}(p)$ | $\frac{pe^s}{1-(1-p)e^s}$ |
| $\mathsf{Poi}(\lambda)$ | $\exp(\lambda(e^s - 1))$ |

25. $\mathbb{E}[X^k] = M_X^{(k)}(0)$ In particular, $\mathbb{E}[X] = M_X'(0)$
26. **Conditional pmf**
$$f_{Y \,|\, X}(y \,|\, x) := \tfrac{f_{X,Y}(x,y)}{f_X(x)}, \quad y \in \mathbb{R}.$$
27. The corresponding **conditional expectation**:
$\mathbb{E}[Y \,|\, X = x] = \sum_y y\, f_{Y \,|\, X}(y \,|\, x)$.
28. $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$

# Other Mathematical Formulas

1. Factorial. $n! = n\,(n-1)\,(n-2)\cdots 1$. Gives the number of *permutations* (orderings) of $\{1, \ldots, n\}$.
2. Binomial coefficient. $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. Gives the number *combinations* (no order) of $k$ different numbers from $\{1, \ldots, n\}$.
3. Newton's binomial theorem: $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k\, b^{n-k}$.
4. Geometric sum: $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ $(a \neq 1)$.
If $|a| < 1$ then $1 + a + a^2 + \cdots = \frac{1}{1-a}$.
5. Logarithms:
   (a) $\log(x\, y) = \log x + \log y$.
   (b) $\mathrm{e}^{\log x} = x$.
6. Exponential:
   (a) $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.
   (b) $\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.
   (c) $\mathrm{e}^{x+y} = \mathrm{e}^x\, \mathrm{e}^y$.
7. Differentiation:
   (a) $(f + g)' = f' + g'$,
   (b) $(fg)' = f'g + fg'$,
   (c) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.
   (d) $\frac{d}{dx} x^n = n\, x^{n-1}$.
   (e) $\frac{d}{dx} \mathrm{e}^x = \mathrm{e}^x$.
   (f) $\frac{d}{dx} \log(x) = \frac{1}{x}$.
8. Chain rule: $(f(g(x)))' = f'(g(x))\, g'(x)$.