From "Probability and Computing" M. Mitzenmacher and E. Upfal

Exercise 1.13: A medical company touts its new test for a certain genetic disorder. The false negative rate is small: if you have the disorder, the probability that the test returns a positive result is 0.999. The false positive rate is also small: if you do not have the disorder, the probability that the test returns a positive result is only 0.005. Assume that 2% of the population has the disorder. If a person chosen uniformly at random from the population is tested and the result comes back positive, what is the probability that the person has the disorder?

$P(\text{disorder} \mid \text{test +ve}) =$

$P(\text{+ve test} \mid \text{disorder}) = 0.999$

$P(\text{+ve test} \mid \text{no disorder}) = 0.005$

$P(\text{disorder}) = 0.02$

Bayes rule

$$P(\text{disorder} \mid \text{test +ve}) = \frac{P(\text{test +ve} \mid \text{disorder})P(\text{disorder})}{P(\text{+ve test})}$$

Law of total probability

$$P(\text{+ve test}) = P(\text{+ve test} \mid \text{disorder})P(\text{disorder}) + P(\text{+ve test} \mid \text{no disorder})P(\text{no disorder})$$

$$= 0.999 \times 0.02 + 0.005 \times (1 - 0.02)$$

$$\simeq 0.02488$$

$$P(\text{disorder} \mid \text{test +ve}) = \frac{0.999 \times 0.02}{0.02488} \simeq 0.803\ldots$$

Q20. (X, Y) marginal pmf of X is Bernoulli $(1-e^{-\lambda})$

marginal pmf of Y is Poisson (λ)

$P(X=Y=0) = e^{-\lambda}$

$X \backslash Y$	0	1	2	3	4 ...	
0	$e^{-\lambda}$	0	0	0	0 ...	$e^{-\lambda}$
1	0	$\lambda e^{-\lambda}$	$\lambda^2 e^{-\lambda}/2$	$\lambda^3 e^{-\lambda}/6$	...	$1-e^{-\lambda}$
	$e^{-\lambda}$	$\lambda e^{-\lambda}$	$\frac{\lambda^2 e^{-\lambda}}{2!}$	$\frac{\lambda^3 e^{-\lambda}}{3!}$		

$$P(Y=0) = e^{-\lambda} = P(Y=0, X=0) + P(Y=0, X=1)$$
$$= e^{-\lambda} + P(Y=0, X=1)$$

$$P(X=0) = e^{-\lambda} = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2) + \cdots$$
$$= e^{-\lambda} + P(X=0, Y=1) + P(X=0, Y=2) + \cdots$$

random variables follows approximately a normal distribution. We will begin our study of the normal distribution by considering an important special case.

Definition. A continuous random variable X is said to have **standard normal distribution** if its probability density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}.$$

We will denote this by $X \sim \mathsf{Normal}(0,1)$.

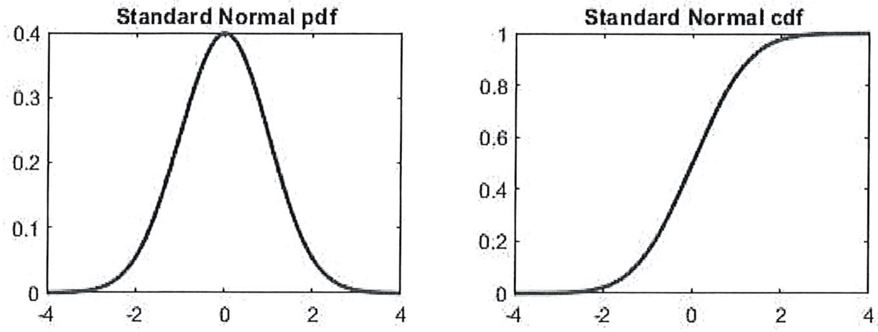
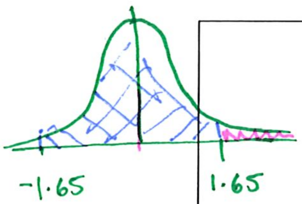


Figure 5.2: Probabilty density function and cumulative distribution function of the standard normal distribution.

There is no explicit expression for the cdf of the standard normal distribution. However, many software packages have methods for numerically evaluating the cdf, for example the `normcdf` function in MATLAB. In exams, we will use tables of the standard normal cdf like the one given at the start of these notes. Use the table of the standard normal cdf to determine the following probabilities:



$$\mathbb{P}(X \leqslant 1) = 1 - \mathbb{P}(X > 1) = 1 - 0.159 = 0.841$$
$$\mathbb{P}(X > 1.96) = 0.025$$
$$\mathbb{P}(X \leqslant -2) = \mathbb{P}(X \geqslant 2) = 0.023$$
$$\mathbb{P}(-1.65 \leqslant X \leqslant 1.65) = 1 - 2 \times 0.049 = 0.902$$
$$\mathbb{P}(X \leq 1.65) = 1 - 0.049$$

As the pdf of the standard normal distribution is symmetric around 0,

$$\mathbb{E}\,X = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = 0 \quad \text{(by symmetry)}$$

The variance of the standard normal is not so easily calculated, but can be done using integation by parts.

$$\mathrm{Var}[X] = \mathbb{E}\,[X^2] - (\mathbb{E}\,X)^2 = \mathbb{E}[X^2]$$
$$= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}}\,dx \quad \text{[Take } u = x \text{ and } v = -e^{-x^2/2}\text{]}$$
$$= \left\{ \frac{x \cdot (-e^{-x^2/2})}{\sqrt{2\pi}} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} + \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$
$$= 0 + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$$

As stated earlier, the normal distribution is a good model for many observed quantities. However, if the normal distribution were restricted to only having expected value zero and variance one, then this would not be the case. Suppose we construct the random variable $Y = \mu + \sigma X$, where $X \sim \mathsf{Normal}(0,1)$. Then using properties of expectation and variance that we saw earlier

$\mu \in \mathbb{R}$
$\sigma > 0$

$$\mathbb{E}\,Y = \mathbb{E}[\mu + \sigma X] = \mu + \sigma\,\mathbb{E}[X] = \mu$$
$$\mathrm{Var}(Y) = \mathrm{Var}(\mu + \sigma X) = \sigma^2\,\mathrm{Var}(X) = \sigma^2$$

To determine the distribution of Y we need to study transformations of random variables.

$$\mathrm{sd}(\cancel{X}Y) = \sqrt{\mathrm{Var}(Y)} = \sqrt{\sigma^2} = \sigma .$$

Transformations of a single random variable

Many random variables are constructed by transforming one or more other random variables. Some important examples include:

- Run time of deterministic algorithms on random input (e.g. Bucket sort)
- Change of units
- Simulations // Estimators of parameters

Linear transformations

In the previous section we constructed a random variable Y from a standard normal random variable X by setting $Y = \mu + \sigma X$, where $\mu \in \mathbb{R}$ and $\sigma > 0$. We saw that $\mathbb{E}\,Y = \mu$ and $\mathrm{Var}(X) = \sigma^2$, but what is the cdf and pdf of Y? For any $y \in \mathbb{R}$,

$$F_Y(y) = \overset{\mathbb{P}}{\cancel{F_Y}}(Y \leqslant y) = \overset{\mathbb{P}}{\cancel{F_X}}(\mu + \sigma X \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-\mu}{\sigma}\right) = F_X\left(\frac{y-\mu}{\sigma}\right)$$

To get the pdf of Y we need to differentiate $F_Y(y)$ with respect to y.

$$\frac{d}{dy}F_Y(y) = \frac{d}{dy}\overset{\mathbb{P}}{\cancel{F_Y}}(Y \leqslant y) = \frac{d}{dy}\overset{\mathbb{P}}{\cancel{F_X}}(\mu + \sigma X \leqslant y)$$
$$= \frac{d}{dy} F_X\left(\frac{y-\mu}{\sigma}\right) \qquad \text{[apply the chain rule.]}$$
$$= \frac{1}{\sigma}\cdot f_X\left(\frac{y-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma} \times \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{y-\mu}{\sigma}\right)^2\Big/2\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right).$$

$y \in \mathbb{R}.$

This leads to the following definition the general normal distribution.

Definition. A continuous random variable X is said to have **normal distribution** with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ if its probability density function is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

We will denote this by $X \sim \mathsf{Normal}(\mu, \sigma^2)$. The following figure shows the pdf f_X where $X \sim \mathsf{Normal}(\mu, \sigma^2)$, for different parameters μ and σ^2.

μ - mean, expectation
σ^2 - variance (σ standard deviation).

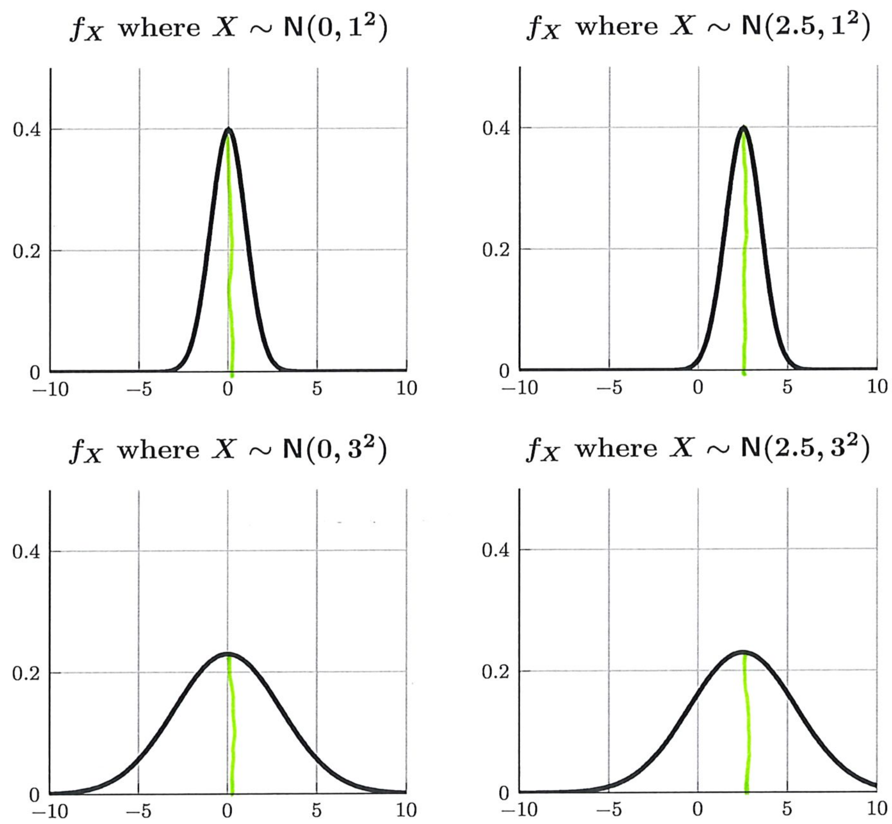


Figure 5.3: The pdfs of $X \sim \mathsf{N}(\mu, \sigma^2)$, with parameters μ and σ^2 given for each plot.