STAT7203: Applied Probability and Statistics
Week 2 Questions

1. Warren Buffet challenges Bill Gates to a game of dice. Gates picks one of the following four sided dice, then Buffet chooses one of the remaining dice. Whoever rolls the highest number wins. The three dice have the following numbers on their faces:

   (die A): 12, 10, 3, 1

   (die B): 9, 8, 7, 2

   (die C): 11, 6, 5, 4

   What is the probability of Buffett winning the game, assuming he chooses his die to maximise his probability of winning?[1]

   Answer: We need to enumerate the sample space for each pair of dice selected The tables below give which die wins for particular outcomes.

	A			
B	12	10	3	1
9	A	A	B	B
8	A	A	B	B
7	A	A	B	B
2	A	A	A	B

	B			
C	9	8	7	2
11	C	C	C	C
6	B	B	B	C
5	B	B	B	C
4	B	B	B	C

	C			
A	11	6	5	4
12	A	A	A	A
10	C	A	A	A
3	C	C	C	C
1	C	C	C	C

   As the die are assumed to be fair we can use the equally likely principle. Therefore, $\mathbb{P}(A \text{ beats } B) = \mathbb{P}(B \text{ beats } C) = \mathbb{P}(C \text{ beats } A) = 9/16$. So Buffet has a probability of 9/16 of winning the game. *Aside: A set of dice with this behaviour is called non-transitive.*

2. We draw at random 5 numbers from 1,... 100, *with replacement* (for example, drawing number 9 twice is possible). What is the probability that exactly 3 numbers are even?

   Answer: We have not covered independence yet, so the answers to this question and the next will need to be done as counting problems and using the equally likely principle.

   Suppose we record the order in which the numbers are drawn. Then there are 100^5 possible sequences. We assume that each sequence is equally likely. There are 50 even numbers and 50 odd numbers between 1 and 100. Let E_i be the event that the i-th number is even and O_i be the event that the i-th number is odd. For the moment, consider only the number of sequences in the event $E_1 \cap E_2 \cap E_3 \cap O_4 \cap O_5$. There are $50^3 \times 50^2 = 50^5$ such sequences. So

$$\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap O_4 \cap O_5) = \frac{50^5}{100^5} = 0.03125$$

   Finally, there are $\binom{5}{3} = 10$ such events with 3 evens and 2 odds, namely

[1] Adapted from Janet Lowe (1998) *Bill Gates Speaks*, John Wiley & Sons

$$
\begin{array}{ll}
E_1 \cap E_2 \cap E_3 \cap O_4 \cap O_5 & E_1 \cap E_2 \cap O_3 \cap E_4 \cap O_5 \\
E_1 \cap E_2 \cap O_3 \cap O_4 \cap E_5 & E_1 \cap O_2 \cap E_3 \cap O_4 \cap E_5 \\
E_1 \cap O_2 \cap O_3 \cap E_4 \cap E_5 & E_1 \cap O_2 \cap E_3 \cap E_4 \cap O_5 \\
O_1 \cap O_2 \cap E_3 \cap E_4 \cap E_5 & O_1 \cap E_2 \cap E_3 \cap E_4 \cap O_5 \\
O_1 \cap E_2 \cap O_3 \cap E_4 \cap E_5 & O_1 \cap E_2 \cap E_3 \cap O_4 \cap O_5
\end{array}
$$

These events are disjoint and all equally probable. Therefore, the probability of exactly 3 numbers of the 5 random numbers being even is

$$\binom{5}{3} \frac{50^5}{100^5} = 0.3125.$$

3. We draw at random 5 numbers from 1,... 100, *without replacement*. What is the probability that exactly 3 numbers are even?

   Answer: We have not covered independence yet, so the answers to this question and the next will need to be done as counting problems and using the equally likely principle.

   Suppose we record the order in which the numbers are drawn. Then there are ${}^{100}P_5$ possible sequences. We assume that each sequence is equally likely. There are 50 even numbers and 50 odd numbers between 1 and 100. Let E_i be the event that the i-th number is even and O_i be the event that the i-th number is odd. For the moment, consider only the number of sequences in the event $E_1 \cap E_2 \cap E_3 \cap O_4 \cap O_5$. There are $50 \times 49 \times 48 \times 50 \times 49$ such sequences. So

$$\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap O_4 \cap O_5) = \frac{50 \times 49 \times 48 \times 50 \times 49}{100 \times 99 \times 98 \times 97 \times 96} \approx 0.03189.$$

   As for question 5, there are $\binom{5}{3} = 10$ such events with 3 evens and 2 odds. These events are disjoint and all equally probable. Therefore, the probability of exactly 3 numbers of the 5 random numbers being even is

$$\binom{5}{3} \frac{50 \times 49 \times 48 \times 50 \times 49}{100 \times 99 \times 98 \times 97 \times 96} \approx 0.3189.$$

4. Let Y be a number drawn at random from $\{1, 2, \ldots, 60\}$ (all numbers are equally likely to be drawn).

   (a) What is the probability that Y is divisible by 2?

   (b) What is the probability that Y is divisible by 2 or 3?

   (c) What is the probability that Y is divisible by 2 or 3 or 5?

   Answer: (a) Let A be the event that Y is divisible by 2. There are 30 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 2 so $\mathbb{P}(A) = \frac{30}{60} = \frac{1}{2}$.

   Answer: (b) Now we determine the probability that Y is divisible by 2 or 3. Let A be the event that Y is divisible by 2 and B be the event that Y is divisible by 3. Then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

There are 20 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 3 so $\mathbb{P}(B) = \frac{20}{60} = \frac{1}{3}$. The event $A \cap B$ is the event that Y is divisible by 6. (*A number is divisible by 6 if and only if it is divisible by 2 and 3.*) There are 10 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 6 so $\mathbb{P}(A \cap B) = \frac{10}{60} = \frac{1}{6}$. Putting this together gives

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}.\end{aligned}$$

Therefore, the probability that Y is not divisible by 2 or 3 is $\frac{1}{3}$.

Answer: (c) Continuing from part (b), let C be the event that Y is divisible by 5. Then

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C) \\ &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cap C) \cup (B \cap C)).\end{aligned}$$

We know $\mathbb{P}(A \cup B)$ from part (a). There are 12 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 5 so $\mathbb{P}(C) = \frac{12}{60} = \frac{1}{5}$. It remains to determine the probability $\mathbb{P}((A \cap C) \cup (B \cap C))$. Note

$$\begin{aligned}\mathbb{P}((A \cap C) \cup (B \cap C)) &= \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cap C) \cup (B \cap C)).\end{aligned}$$

The event $A \cap C$ is the event that Y is divisible by 10. There are 6 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 10 so $\mathbb{P}(A \cap C) = \frac{6}{60} = \frac{1}{10}$. The event $B \cap C$ is the event that Y is divisible by 15. There are 4 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 15 so $\mathbb{P}(B \cap C) = \frac{4}{60} = \frac{1}{15}$. The event $A \cap B \cap C$ is the event that Y is divisible by 30. There are 2 numbers in the set $\{1, 2, \ldots, 60\}$ that are divisible by 30 so $\mathbb{P}(A \cap B \cap C) = \frac{2}{60} = \frac{1}{30}$. Putting this together gives

$$\begin{aligned}\mathbb{P}((A \cap C) \cup (B \cap C)) &= \frac{1}{10} + \frac{1}{15} - \frac{1}{30} \\ &= \frac{4}{30}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \frac{2}{3} + \frac{1}{5} - \frac{4}{30} \\ &= \frac{22}{30}.\end{aligned}$$

The probability that Y is not divisible by 2 or 3 or 5 is then $\frac{8}{30}$.

5. Let $(a_1, \ldots, a_n)$ be a random permutation of the integers $\{1, 2, \ldots, n\}$ with all permuations equally likely.

   (a) What is the probability that $a_1 = 1$?

   (b) What is the probability that $a_1 = 1$ or $a_2 = 2$?

   (c) What is the probability that $a_1 = 1$ or $a_2 = 2$ or $a_3 = 3$?

Answer: (a) Let A_i be the event that the random permutation $(a_1, \ldots, a_n)$ has $a_i = i$. We are interested in the probability of the event A_1. To compute $\mathbb{P}(A_1)$ note that there are $n!$ permutations of the integers $\{1, \ldots, n\}$ and there are $(n-1)!$ permutations of $\{1, \ldots, n\}$ with $a_1 = 1$. (*Note that we specify that $a_1 = 1$ but the other $n-1$ integers can go in any position except 1.*). So $\mathbb{P}(A_1) = \frac{(n-1)!}{n!} = \frac{1}{n}$.
Answer: (b) We are interested in the probability of the event $A_1 \cup A_2$. Then

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).$$

Similar to the argument for $\mathbb{P}(A_1)$, we can compute $\mathbb{P}(A_2) = \frac{1}{n}$. The event $A_1 \cap A_2$ is the event that the random permutation has $a_1 = 1$ and $a_2 = 2$. As we specify $a_1 = 1$ and $a_2 = 2$, the other $(n-2)$ integers can go in any position except 1 and 2. So $\mathbb{P}(A_1 \cap A_2) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

$$\mathbb{P}(A_1 \cup A_2) = \frac{1}{n} + \frac{1}{n} - \frac{1}{n(n-1)} = \frac{2}{n} - \frac{1}{n(n-1)}.$$

Answer: (c) Continuing from part (b),

$$\begin{aligned}
\mathbb{P}(A_1 \cup A_2 \cup A_3) &= \mathbb{P}(A_1 \cup A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cup A_2) \cap A_3) \\
&= \mathbb{P}(A_1 \cup A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cap A_3) \cup (A_2 \cap A_3))
\end{aligned}$$

We know $\mathbb{P}(A_1 \cup A_2)$ from part (a) and $\mathbb{P}(A_3) = \frac{1}{n}$. Again

$$\mathbb{P}((A_1 \cap A_3) \cup (A_2 \cap A_3)) = \mathbb{P}((A_1 \cap A_3)) + \mathbb{P}((A_2 \cap A_3) - \mathbb{P}((A_1 \cap A_2 \cap A_3).$$

Following the argument in part (a), $\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = \frac{1}{n(n-1)}$. The event $A_1 \cap A_2 \cap A_3$ is the event that the random permutation has $a_1 = 1, a_2 = 2$ and $a_3 = 3$. As we specify $a_1 = 1, a_2 = 2$ and $a_3 = 3$, the other $(n-3)$ integers can go in any position except 1, 2 and 3. So $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$. So

$$\mathbb{P}((A_1 \cap A_3) \cup (A_2 \cap A_3)) = \frac{1}{n(n-1)} + \frac{1}{n(n-1)} - \frac{1}{n(n-1)(n-2)}$$

and

$$\begin{aligned}
\mathbb{P}(A_1 \cup A_2 \cup A_3) &= \mathbb{P}(A_1 \cup A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cup A_2) \cap A_3) \\
&= \left(\frac{1}{n} + \frac{1}{n} - \frac{1}{n(n-1)}\right) + \frac{1}{n} - \left(\frac{2}{n(n-1)} - \frac{1}{n(n-1)(n-2)}\right) \\
&= \frac{3}{n} - \frac{3}{n(n-1)} + \frac{1}{n(n-1)(n-2)}.
\end{aligned}$$

6. Testing whether a multivariate polynomial is equal to zero is a standard problem in computational complexity. We will consider the following simpler setting. Let G be a polynomial of degree at most d and assume G is not identically zero. Let X be a number drawn at random from $\{1, 2, \ldots, 100d\}$, that is all numbers are equally likely to be drawn. Show that $\mathbb{P}(G(X) \neq 0) \geq 0.99$. (Hint: What does the fundamental theorem of algebra say?)

Answer: As the polynomial G has degree at most d, it is zero for at most d values. Note that $\mathbb{P}(G(X) \neq 0) = 1 - \mathbb{P}(G(X) = 0)$. Then

$$\begin{aligned}\mathbb{P}(G(X) = 0) &= \frac{\text{Number of values in } \{1, 2, \ldots, 100d\} \text{ for which } G \text{ is zero.}}{100d} \\ &\leq \frac{d}{100d} = 0.01.\end{aligned}$$

Therefore, $\mathbb{P}(G(X) \neq 0) \geq 0.99$.