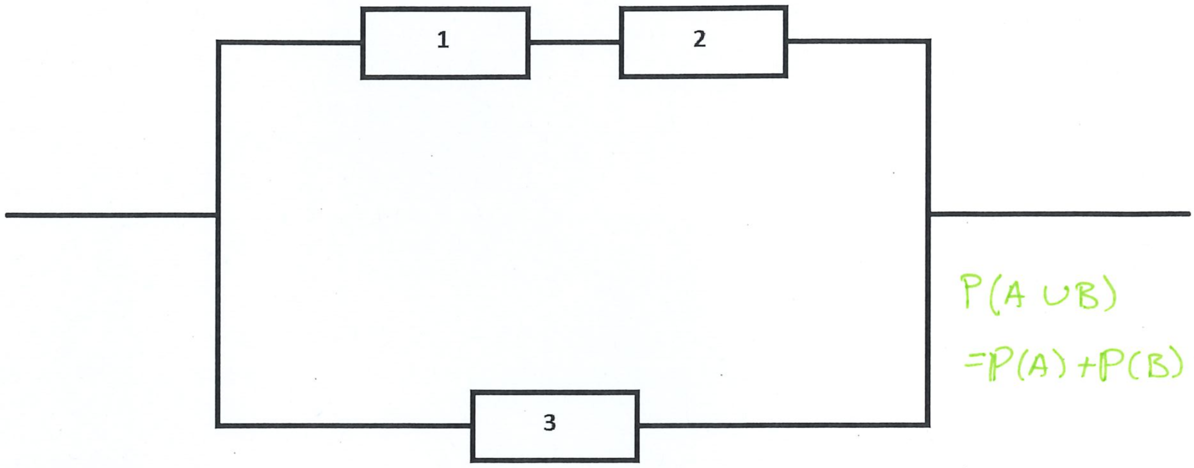
Q3. A system comprises three components as below. The system is working if there is a path from left to right through working components. After one year, component 1 has failed with probability 0.2, component 2 has failed with probability 0.4 and component 3 has failed with probability 0.5. What is the probability that the system has failed after one year?

\# Assume components fail independent of one another.



$P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$

$\mathbb{P}$(system failed) $= \mathbb{P}$({components 1 and 3 fail} $\cup$ {components 2 and 3 fail})

$= \mathbb{P}$(components 1 and 3 fail)
$+ \mathbb{P}$(components 2 and 3 fail)
$- \mathbb{P}$(components 1, 2 and 3 all fail)

$= \mathbb{P}$(component 1 fails) $\times$ $\mathbb{P}$(component 2 fails)
$+ \mathbb{P}$(component 2 fails) $\times$ $\mathbb{P}$(component 3 fails)
$- \mathbb{P}$(component 1 fail) $\times$ $\mathbb{P}$(component 2 fails) $\times$ $\mathbb{P}$(component 3 fail)

$= 0{\cdot}2 \times 0{\cdot}5 + 0{\cdot}4 \times 0{\cdot}5 - 0{\cdot}2 \times 0{\cdot}4 \times 0{\cdot}5$

$= 0{\cdot}1 + 0{\cdot}2 - 0{\cdot}04 = 0{\cdot}26$

$X \sim \text{Binomial}(2, 1/2)$ $\quad\quad$ $Y \sim \text{Binomial}(2, 3/4)$

$\mathbb{P}(X > Y) = 0$ $\quad$ and $\quad$ $\mathbb{P}(X=0, Y=2) = 1/16$

$\mathbb{P}(X=1, Y=1) = ?$ $\quad$ (a) $1/16$, (b) $1/8$, (c) $3/16$, (d) $1/4$, (e) $5/16$

		Y = 0	Y = 1	Y = 2	
X	0	1/16	1/8	1/16	$\mathbb{P}(X=0) = 1/4$
	1	0	1/4	1/4	$\mathbb{P}(X=1) = 1/2$
	2	0	0	1/4	$\mathbb{P}(X=2) = 1/4$
		1/16	6/16	9/16	

$\mathbb{P}(X=1, Y=1) = 1/4$ $\quad\quad$ (d)

$$\mathbb{P}(X = k) = \binom{2}{k} \cdot \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{2-k}$$

Q1. Let X_1, X_2, and X_3 be three independent random variables each with a *Bernoulli*(p) distribution. Define

$$Y_{ij} = X_i(1 - X_j) + X_j(1 - X_i).$$

Y_{12}	X_1	X_2
0	1	1
1	1	0
1	0	1
0	0	0

What is the joint probability mass function of Y_{12} and Y_{23}?

$Y_{ij} \in \{0,1\}$

$$\mathbb{P}(Y_{12}=1) = \mathbb{P}(X_1=1, X_2=0) + \mathbb{P}(X_1=0, X_2=1)$$

$$= \mathbb{P}(X_1=1)\times\mathbb{P}(X_2=0) + \mathbb{P}(X_1=0)\,\mathbb{P}(X_2=1) \ *$$

$$= p\times(1-p) + (1-p)\times p = 2p(1-p)$$

$$\mathbb{P}(Y_{12}=0) = 1 - 2p(1-p).$$

Y_{23}	X_2	X_3
0	1	1
1	1	0
1	0	1
0	0	0

Y_{23} \ Y_{12}	0	1	
0	$1-3p(1-p)$	$p(1-p)$	$1-2p(1-p)$
1	$p(1-p)$	$p(1-p)$	$2p(1-p)$
	$1-2p(1-p)$	$2p(1-p)$	

When are Y_{12} and Y_{23} independent?

$$\mathbb{P}(Y_{12}=1, Y_{23}=1) = \mathbb{P}(\{X_1=1, X_2=0, X_3=1\}) + \mathbb{P}(\{X_1=0, X_2=1, X_3=0\})$$

$$= \mathbb{P}(X_1=1)\,\mathbb{P}(X_2=0)\,\mathbb{P}(X_3=1) + \mathbb{P}(X_1=0)\,\mathbb{P}(X_2=1)\,\mathbb{P}(X_3=0)$$

$$= (1-p)p^2 + (1-p)^2 p = p(1-p)$$

$$\mathbb{P}(Y_{12}=j, Y_{23}=k) = \mathbb{P}(Y_{12}=j)\,\mathbb{P}(Y_{23}=k) \qquad j,k \in \{0,1\}$$

$$\mathbb{P}(Y_{12}=1, Y_{23}=1) = p(1-p) = 2p(1-p)\times 2p(1-p)$$

$$\Rightarrow 1 = 4p(1-p)$$

$$4p(1-p) - 1 = 0 \Rightarrow (2p-1)^2 = 0 \Rightarrow p = 1/2.$$

Q2. Suppose X has a *Geometric*(p) distribution. Conditional on $\{X = x\}$, Y has a *Binomial*(x,p) distribution. What is the moment generating function of Y?

$$M_Y(s) = \mathbb{E}[e^{sY}]$$

$$= \mathbb{E}[\mathbb{E}[e^{sY} \mid X]]$$

$$\mathbb{E}[e^{sY} \mid X = x] = (1-p+pe^s)^x \qquad \text{MGF Binomial}(x,p)$$

$$\mathbb{E}[e^{sY}] = \mathbb{E}[\mathbb{E}[e^{sY} \mid X]]$$

$$= \mathbb{E}[(1-p+pe^s)^X]$$

$$= \mathbb{E}[\exp(X \log(1-p+pe^s))]$$

$$= \frac{p\exp(\log(1-p+pe^s))}{1-(1-p)\exp(\log(1-p+pe^s))}$$

[Recall MGF of Geometric(p) (see page 60)

$$\frac{pe^s}{1-(1-p)e^s}$$]

$$= \frac{p(1-p+pe^s)}{1-(1-p)(1-p+pe^s)}$$

$$= \frac{p(1-p+pe^s)}{\cancel{1}-(\cancel{1}-p+\cancel{pe^s}-p+p^2-\cancel{p^2e^s})}$$

$$= \frac{p(1-p+pe^s)}{2p-p^2-p(1-p)e^s}$$